DYNAMIC PERFECT PLASTICITY AND DAMAGE IN VISCOELASTIC SOLIDS

ELISA DAVOLI, TOMÁŠ ROUBÍČEK, AND ULISSE STEFANELLI

ABSTRACT. In this paper we analyze an isothermal and isotropic model for viscoelastic media combining linearized perfect plasticity (allowing for concentration of plastic strain and development of shear bands) and damage effects in a dynamic setting. The interplay between the viscoelastic rheology with inertia, elasto-plasticity, and unidirectional rate-dependent incomplete damage affecting both the elastic and viscous response, as well as the plastic yield stress, is rigorously characterized by showing existence of weak solutions to the constitutive and balance equations of the model. The analysis relies on the notions of plastic-strain measures and bounded-deformation displacements, on sophisticated time-regularity estimates to establish a duality between acceleration and velocity of the elastic displacement, on the theory of rate-independent processes for the energy conservation in the dynamical-plastic part, and on the proof of the strong convergence of the elastic strains. Existence of a suitably defined weak solutions is proved rather constructively by using a staggered two-step time discretization scheme.

1. INTRODUCTION

Plasticity and damage are inelastic phenomena providing the macroscopical evidence of defect formation and evolution at the atomistic level. Plasticity results from the accumulation of slip defects (dislocations), which determine the behavior of a body to change from elastic and reversible to plastic and irreversible, once the magnitude of the stress reaches a certain threshold and a plastic flow develops. Damage evolution originates from the formation of cracks and voids in the microstructure of the material.

The mathematical modeling of inelastic phenomena is a very active research area, at the triple point between mathematics, physics, and materials science. A vast literature concerning damage in viscoelastic materials, both in the quasistatic and the dynamical setting is currently available. We refer, e.g., to [39, 41, 46, 51, 53] and the references therein for an overview of the main results.

The interplay between plasticity and damage has been already extensively investigated, prominently in the quasistatic framework. The interaction between damage and strain gradient plasticity is addressed in [19] whereas a perfect-plastic model has been proposed in [1], where the one-dimensional response is also studied. Existence results in general dimensions have been obtained in [18, 20], see also [21] for some recent associated lower semicontinuity results. The coupling between damage and rate-independent small-strain plasticity with hardening is the subject of [10, 44, 49]. Quasistatic perfect plasticity and damage with healing are analyzed in [48]. The identification of fracture models as limits of damage coupled with plasticity has also been considered [24, 25].

The analysis of dynamic perfect plasticity without damage has been initiated in [5]. A derivation of the equations via vanishing hardening, and vanishing viscoplasticity has been performed in [15, 16]. A generalization via the so-called cap-model approximation has been obtained in [6]. An approximation of the equations of dynamic plasticity relying on the minimization of a parameter-dependent functional defined on trajectories is the subject of [26], whereas an alternative approach based on hyperbolic conservation laws has been proposed in [7]. Dimension reduction for dynamic perfectly plastic plates has been carried on in [40]. Convergence of dynamic models to quasistatic ones has been analyzed in [23, 43].

To our best knowledge, the combination of perfect plasticity, damage, and inertia has been so far tackled in the engineering and geophysical literature (see, e.g., [27, 32, 52]), whilst a mathematical counterpart to the applicative analysis is still missing. The focus of this paper is to provide a rigorous analysis of an isothermal and isotropic model for viscoelastic media combining both small-strain perfect plasticity and damage effects in a dynamic setting.

More specifically, our main result (Theorem 2.6) shows existence of suitably weak solutions to the following system of equations and differential inclusions, complemented by suitable boundary conditions and initial data

$$\rho\ddot{u} - \operatorname{div}\sigma = f, \quad \sigma := \mathbb{C}(\alpha)e_{\rm el} + \mathbb{D}(\alpha)\dot{e}_{\rm el}, \quad e_{\rm el} = e(u) - \pi, \tag{1a}$$

1991 *Mathematics Subject Classification.* 35Q74, 37N15, 74C05, 74R05.

Key words and phrases. Perfect plasticity, inertia, cohesive damage, Kelvin-Voigt viscoelastic rheology, functions of bounded deformation, staggered time discretisation, weak solution.

.

$$\sigma_{\rm YLD}(\alpha){\rm Dir}(\dot{\pi}) \ni {\rm dev}\,\sigma, \tag{1b}$$

$$\partial\zeta(\dot{\alpha}) + \frac{1}{2}\mathbb{C}'(\alpha)e_{\rm el} : e_{\rm el} \ni \phi'(\alpha) + {\rm div}\,(\kappa|\nabla\alpha|^{p-2}\nabla\alpha), \tag{1c}$$

where u, π, and α denote the displacement, the plastic strain, and the damage variable, respectively, $\mathbb{C}(\cdot)$, $\mathbb{D}(\cdot)$, and $\sigma_{\rm YLD}(\cdot)$ are the damage-dependent elasticity tensor, viscosity tensor, and yield surface, and $e(u) = (\nabla u + \nabla u^\top)$ is the linearized strain. The notation Dir stands for the set-valued "direction" (see Subsection 2.5), dev σ identifies the deviatoric part of the stress σ, namely ${\rm dev}\,\sigma := \sigma - {\rm tr}\,(\sigma){\rm Id}/d$, ζ is the local potential of dissipative damage-driving force (see (7)), constraining the damage process to be unidirectional (no healing). Finally ϕ is the energy associated to the creation of microvoids or microcracks during the damaging process, κ is the length scale of the damage profile, and ρ the mass density. We refer to Section 2 for the precise setting of the problem, the definition of weak solution to (1a)–(1c), and the statement of Theorem 2.6.

The analysis of model (1a) presents several technical challenges. Perfect plasticity allows for plastic strain concentrations along the (possibly infinitesimally thin) slip-bands and calls for weak formulations in the spaces of bounded Radon measures for plastic strains and bounded-deformation (BD) for displacements. This requires a delicate notion of stress-strain duality (see Subsection 4.1). Considering inertia and the related kinetic energy renders the analysis quite delicate because of the interaction of possible elastic waves with nonlinearly responding slip bands, as pointed out already in [8]. Various natural extensions such as allowing healing instead of unidirectional damage, or mutually independent damage in the viscous and the elastic response (in contrast to (22b) below), or different damage behaviors in relation to compression/tension mode leading to a non-quadratic stored energy, or an enhancement by heat generation/transfer with some thermal coupling to the mechanical part, seem difficult and remain currently open.

The proof strategy relies on a staggered discretization scheme, in which at each time-step we first identify the damage variable as a solution to the damage evolution equation, and we then determine the plastic strain and elastic displacements as minimizer of a damage-dependent energy inequality (see Section 4). A standard test of (1a)–(1c) leads to the proof of a first a-priori estimate in Proposition 5. In order to ensure the strong convergence of the time-discrete elastic strains $e_{\rm el}$, needed for the limit passage in the damage flow rule, a further higher order test is performed in Proposition 5. The convergence of the elastic strains is then achieved by means of a delicate limsup estimate (see Proposition 6). Due to the failure of energy conservation under basic data qualification, the flow rule is only recovered, in the limit, in the form of an energy inequality (see Remark 2.6).

A motivation for tackling the simultaneous occurrence of dynamical perfect plasticity and damaging is the mathematical modeling of cataclasite zones in geophysics. During fast slips, lithospheric faults in elastic rocks tend to emit elastic (seismic) waves, which in turn determine the occurrence of (tectonic) earthquakes, and the local arising of cataclasis. This latter phenomenon consists in a gradual fracturing of mineral grains into core zones of lithospheric faults, which tend to arrange themselves into slip bands, sliding plastically on each other without further fracturing of the material. On the one hand, cataclasite core zone are often very narrow (sometimes centimeters wide) in comparison with the surrounding compact rocks (which typically extend for many kilometers), and can be hence modeled for rather small time scales (minutes of ongoing earthquakes or years between them, rather than millions of years) via small-strain perfect (no-gradient) plasticity. On the other hand the partially damaged area surrounding the thin cataclasite core can by relatively wide, and thus calls for a modeling via gradient-damage theories (see [45, 47]).

The novelty of our contribution is threefold. First, we extend the mathematical modeling of damage-evolution effects to an inelastic setting. Second, we characterize the interaction between damage onset and plastic slips formation in the framework of perfect plasticity, with no gradient regularization and in the absence of hardening. Third, we complement the study of dynamic perfect plasticity, by keeping track of the effects of damage both on the plastic yield surface, and on the viscoelastic behavior of the material.

The paper is organized as follows: In Section 2, we introduce some basic notation and modeling assumptions, and we state our main existence result. Section 3 highlights the formal strategy that will be employed afterward for the proof of Theorem 2.6, whereas Section 4 focuses on the formulation of our staggered two-step discretization scheme. In Section 5 we establish some a-priori energy estimates. Finally Section 6 is devoted to the proof of the main result.

2. Setting of the problem and statement of the main result

We devote this section to specify the mathematical setting of the model, and to present our main result. We first introduce some basic notation and assumptions, and we recall some notions from measure theory.

In what follows, let $\Omega \subset \mathbb{R}^d$, $d \in \{2,3\}$ be a bounded open set with C^2 boundary. In our model, the domain Ω represents the reference configuration of a linearly viscoelastic, perfectly plastic Kelvin-Voigt body subject to a possible damage in its elastic as well as in its viscous and plastic response.

We assume that the boundary $\partial\Omega =: \Gamma$ is partitioned into the union of two disjoint sets $\Gamma_{\rm D}$ and $\Gamma_{\rm N}$. In particular, we require $\Gamma_{\rm D}$ to be a connected open subset of Γ (in the relative topology of Γ) such that $\partial_\Gamma \Gamma_{\rm D}$ is a connected, $(d-2)$-dimensional, C^2 manifold, whereas $\Gamma_{\rm N}$ is defined as $\Gamma_{\rm N} := \Gamma \setminus \Gamma_{\rm D}$.

For any map $f : [0,T] \times \mathbb{R}^d \to \mathbb{R}$ we will denote by $\dot f$ its time derivative, and by ∇f its spatial gradient. We will adopt the notation $\mathbb{R}^{d\times d}$ to indicate the set of $d \times d$ matrices. Given $M, N \in \mathbb{R}^{d\times d}$, their scalar product will be denoted by $M : N := \mathrm{tr}(M^\top N)$ where tr is the trace operator, and the superscript stands for transposition. We will write $\mathrm{dev}\, M$ to identify the deviatoric part of M, namely $\mathrm{dev}\, M := M - \mathrm{tr}\,(M)\mathrm{Id}/d$, where Id is the identity matrix. The symbols $\mathbb{R}^{d\times d}_{\rm sym}$ and $\mathbb{R}^{d\times d}_{\rm dev}$ will represent the set of symmetric $d \times d$ matrices, and that of symmetric matrices having null trace, respectively.

2.1. **Function spaces, measures and functions with bounded deformation.** We use the standard notation L^p, $W^{k,p}$, and $L^p(0,T;X)$ or $W^{1,p}(0,T;X)$ for Lebesgue, Sobolev, and Bochner or Bochner-Sobolev spaces. By $C_{\rm w}(0,T;X)$ we denote the space of weakly continuous mappings with value in the Banach space X. We also use the shorthand convention $H^k := W^{k,2}$.

Given a Borel set $B \subset \mathbb{R}^d$ the symbol $\mathcal{M}_b(B;\mathbb{R}^m)$ denotes the space of bounded Borel measures on B with values in $\mathbb{R}^m$ $(m \in \mathbb{N})$. When $m = 1$ we will simply write $\mathcal{M}_b(B)$. We will endow $\mathcal{M}_b(B;\mathbb{R}^m)$ with the norm $\|\mu\|_{\mathcal{M}_b(B;\mathbb{R}^m)} := |\mu|(B)$, where $|\mu| \in \mathcal{M}_b(B)$ is the total variation of the measure μ.

If the relative topology of B is locally compact, by the Riesz representation Theorem the space $\mathcal{M}_b(B;\mathbb{R}^m)$ can be identified with the dual of $C_0(B;\mathbb{R}^m)$, which is the space of continuous functions $\varphi : B \to \mathbb{R}^m$ such that the set $\{|\varphi| \geq \delta\}$ is compact for every $\delta > 0$. The weak* topology on $\mathcal{M}_b(B;\mathbb{R}^m)$ is defined using this duality.

The space $BD(\Omega;\mathbb{R}^d)$ of functions with *bounded deformation* is the space of all functions $u \in L^1(\Omega;\mathbb{R}^d)$ whose symmetric gradient

$$e(u) := \frac{\nabla u + (\nabla u)^\top}{2}$$

(defined in the sense of distributions) belongs to $\mathcal{M}_b(\Omega;\mathbb{R}^{d\times d}_{\rm sym})$. It is easy to see that $BD(\Omega;\mathbb{R}^d)$ is a Banach space when endowed with the norm

$$\|u\|_{L^1(\Omega;\mathbb{R}^d)} + \|e(u)\|_{\mathcal{M}_b(\Omega;\mathbb{R}^{d\times d}_{\rm sym})}.$$

A sequence $\{u^k\}$ is said to converge to u weakly* in $BD(\Omega;\mathbb{R}^d)$ if $u^k \to u$ weakly in $L^1(\Omega;\mathbb{R}^d)$ and $e(u^k) \to e(u)$ weakly* in $\mathcal{M}_b(\Omega;\mathbb{R}^{d\times d}_{\rm sym})$. Every bounded sequence in $BD(\Omega;\mathbb{R}^d)$ has a weakly* converging subsequence. In our setting, since Ω is bounded and has C^2 boundary, $BD(\Omega;\mathbb{R}^d)$ can be embedded into $L^{d/(d-1)}(\Omega;\mathbb{R}^d)$ and every function $u \in BD(\Omega;\mathbb{R}^d)$ has a trace, still denoted by u, which belongs to $L^1(\Gamma;\mathbb{R}^d)$. For every nonempty subset γ of $\Gamma_{\rm D}$ which is open in the relative topology of $\Gamma_{\rm D}$, there exists a constant $C > 0$, depending on Ω and γ, such that the following Korn inequality holds true

$$\|u\|_{L^1(\Omega;\mathbb{R}^d)} \leq C\|u\|_{L^1(\gamma;\mathbb{R}^d)} + C\|e(u)\|_{\mathcal{M}_b(U;\mathbb{R}^{d\times d}_{\rm sym})} \tag{2}$$

(see [50, Chapter II, Proposition 2.4 and Remark 2.5]). For the general properties of the space $BD(\Omega;\mathbb{R}^d)$ we refer to [50].

2.2. **State of the system and admissible displacements and strains.** At each time $t \in [0,T]$, the viscoelastic perfectly-plastic behavior of the body is described by three basic state variables: the displacement $u(t) : \Omega \to \mathbb{R}^d$, the plastic strain $\pi(t) : \Omega \to \mathbb{R}^{d\times d}_{\rm dev}$, and the damage variable $\alpha(t) : \Omega \to [0,1]$. In particular, we adopt the convention (used in mathematics, in contrast to the opposite convention used in engineering and geophysics) that $\alpha = 1$ corresponds to the undamaged elastic material, whereas $\alpha = 0$ describes the situation in which the material is totally damaged. The abstract state q will be here given by the triple $q = (u,\pi,\alpha)$.

On $\Gamma_{\rm D}$ we prescribe a boundary datum $u_{\rm D} \in H^{1/2}(\Gamma_{\rm D};\mathbb{R}^d)$, later being considered to be time dependent. With a slight abuse of notation we also denote by $u_{\rm D}$ a $H^1(\Omega;\mathbb{R}^d)$-extension of the boundary condition to the set Ω.

The *set of admissible displacements and strains* for the boundary datum $u_{\rm D}$ is given by

$$\begin{aligned}\mathscr{A}(u_{\rm D}) := \Big\{ &(u, e_{\rm el}, \pi) \in \big(BD(\Omega;\mathbb{R}^d) \cap L^2(\Omega)\big) \times L^2(\Omega;\mathbb{R}^{d\times d}_{\rm sym}) \times \mathcal{M}_b(\Omega \cup \Gamma_{\rm D};\mathbb{R}^{d\times d}_{\rm dev}) : \\ &e(u) = e_{\rm el} + \pi \text{ in } \Omega, \quad \pi = (u_{\rm D} - u) \odot \nu_\Gamma \mathcal{H}^{d-1} \text{ on } \Gamma_{\rm D} \Big\},\end{aligned} \tag{3}$$

where $\odot$ stands for the symmetrized tensor product, namely

$$a \odot b := (a \otimes b + b \otimes a)/2 \quad \forall\, a, b \in \mathbb{R}^d,$$

ν_Γ is the outer unit normal to Γ, and $\mathcal{H}^{d-1}$ is the $(d-1)$-dimensional Hausdorff measure. Note that the kinematic relation $e(u) = e_{\rm el} + \pi$ in $\mathscr{A}(u_{\rm D})$ is classic in linearized elastic theories and it is usually referred to as additive strain decomposition.

We point out that the constraint

$$\pi = (u_{\rm D} - u) \odot \nu_\Gamma \mathcal{H}^{d-1} \text{ on } \Gamma_{\rm D} \tag{4}$$

is a relaxed formulation of the boundary condition $u = u_{\rm D}$ on $\Gamma_{\rm D}$; see also [42]. As remarked in [22], the mechanical meaning of (4) is that whenever the boundary datum is not attained a plastic slip develops, whose amount is directly proportional to the difference between the displacement u and the boundary condition $u_{\rm D}$.

2.3. **Stored energy.** Let $\mathcal{L}_{\rm sym}(\mathbb{R}^{d\times d}_{\rm sym})$ denote the space of linear symmetric (self-adjoint) operators $\mathbb{R}^{d\times d}_{\rm sym} \to \mathbb{R}^{d\times d}_{\rm sym}$, being understood as 4th-order symmetric tensors.

We assume the elastic tensor $\mathbb{C} : \mathbb{R} \to \mathcal{L}_{\rm sym}(\mathbb{R}^{d\times d}_{\rm sym})$ to be continuously differentiable, and nondecreasing in the sense of the Löwner ordering, i.e. the ordering of $\mathbb{R}^{d\times d}_{\rm sym}$ with respect to the cone of positive semidefinite matrices. Additionally, we require $\mathbb{C}(\alpha)$ to be positive semi-definite for every $\alpha \in \mathbb{R}$. Note that, in view of the pointwise semi-definiteness of $\mathbb{C}$, the possibility of having complete damage in the elastic part is also encoded in the model. We additionally assume that $\mathbb{C}(\alpha) = \mathbb{C}(0)$ for every $\alpha < 0$, and that $\mathbb{C}'(0) = 0$. This corresponds to the situation in which the damage is cohesive.

The *stored energy* of the model will be given by

$$\mathscr{E}(q) = \mathscr{E}(u, \pi, \alpha) = \int_\Omega \Big(\frac{1}{2}\mathbb{C}(\alpha)e_{\rm el} : e_{\rm el} - \phi(\alpha) + \frac{\kappa}{p}|\nabla\alpha|^p\Big)\,{\rm d}x \quad \text{with} \quad e_{\rm el} = e(u) - \pi, \tag{5}$$

where $\phi : \mathbb{R} \to \mathbb{R}$ stands for the specific *energy of damage*, motivated by extra energy of microvoids or microcracks created by degradation of the material during the damaging process, whereas κ represents a length scale for the damage profile. When $\phi'(\alpha) > 0$, the damage evolution is an activated processes, even if there is no activation threshold in the dissipation potential, as indeed considered in (7) below.

For the sake of allowing full generality to the choice of initial conditions, we will assume that ${\rm dev}\mathbb{C}e = \mathbb{C}{\rm dev}\, e$. Note that this is the case for isotropic materials.

2.4. **Other ingredients: dissipation and kinetic energy .** For the sake of notational simplicity, we consider isotropic materials as far as plastification is concerned.

Let the *yield stress* $\sigma_{\rm YLD}$ as a function of damage $\sigma_{\rm YLD} : [0,1] \to (0, +\infty)$ be continuously differentiable and non-decreasing. For every $\pi \in \mathcal{M}_b(\Omega \cup \Gamma_{\rm D}; \mathbb{R}^{d\times d}_{\rm dev})$ let ${\rm d}\pi/{\rm d}|\pi|$ be the Radon-Nikodým derivative of π with respect to its total variation $|\pi|$. Assuming that $\alpha : [0,T] \times \Omega \to [0,1]$ is continuous, we consider the positively one-homogeneous function $M \mapsto \sigma_{\rm YLD}(\alpha)|M|$ for every $M \in \mathbb{R}^{d\times d}$, and, according to the theory of convex functions of measures [34], we introduce the functional

$$\mathcal{R}(\alpha, \pi) := \int_{\Omega\cup\Gamma_{\rm D}} \sigma_{\rm YLD}(\alpha)\frac{{\rm d}\pi}{{\rm d}|\pi|}\,{\rm d}|\pi|$$

for every $\pi \in \mathcal{M}_b(\Omega \cup \Gamma_{\rm D}; \mathbb{R}^{d\times d}_{\rm dev})$.

In what follows, we will refer to $\mathcal{R}$ as to the *damage-dependent plastic dissipation potential*. Note that, by Reshetnyak's lower semicontinuity theorem (see [2, Theorem 2.38]), the functional $\mathcal{R}$ is lower-semicontinuous in its second variable with respect to the weak* convergence in $\mathcal{M}_b(\Omega \cup \Gamma_{\rm D}; \mathbb{R}^{d\times d}_{\rm dev})$.

For α continuous and such that $\dot\alpha \le 0$ in $[0,T]\times\Omega$, we define the α-*weighted* $\mathcal{R}$-*dissipation* of a map $t \mapsto \pi(t)$ in the interval $[s_1, s_2]$ as

$$D_{\mathcal{R}}(\alpha; \pi; s_1, s_2) := \sup\Big\{\sum_{j=1}^n \mathcal{R}\big(\alpha(t_j), \pi(t_j) - \pi(t_{j-1})\big) :\ s_1 \le t_0 < t_1 < \cdots < t_n \le s_2,\ n \in \mathbb{N}\Big\}. \tag{6}$$

We will work under the assumption that the damage is unidirectional, i.e. $\dot\alpha \le 0$. Constraining the rate rather than the state itself, this constraint is to be incorporated into the dissipation potential. For a (small) damage-viscosity parameter $\eta > 0$, we define the local potential of dissipative damage-driving force as

$$\zeta(\dot\alpha) := \begin{cases} \frac{1}{2}\eta\dot\alpha^2 & \text{if } \dot\alpha \le 0, \\ +\infty. & \text{otherwise} \end{cases} \tag{7}$$

Let the viscoelastic tensor $\mathbb{D} : \mathbb{R} \to \mathcal{L}_{\rm sym}(\mathbb{R}^{d\times d}_{\rm sym})$ be given and define the overall potential of dissipative forces

$$\begin{aligned}\mathscr{R}(q; \dot q) &= \mathscr{R}(\alpha; \dot u, \dot\pi, \dot\alpha) \\ &= \int_\Omega \Big(\frac{1}{2}\mathbb{D}(\alpha)\dot e_{\rm el} : \dot e_{\rm el} + \zeta(\dot\alpha)\Big)\,{\rm d}x + \int_{\Omega\cup\Gamma_{\rm D}} \sigma_{\rm YLD}(\alpha)\frac{{\rm d}\dot\pi}{{\rm d}|\dot\pi|}\,{\rm d}|\dot\pi| \quad \text{where} \quad \dot e_{\rm el} = e(\dot u) - \dot\pi.\end{aligned} \tag{8}$$

Let $\rho \in L^\infty(\Omega)$, with $\rho > 0$ almost everywhere in Ω denote the mass density. We will additionally consider the *kinetic energy* given by

$$\mathscr{T}(\dot{u}) = \int_\Omega \frac{1}{2}\rho|\dot{u}|^2\,\mathrm{d}x. \tag{9}$$

2.5. **Governing equations by Hamilton variational principle.** We formulate the model via *Hamilton's variational principle* generalized for dissipative systems [9]. This prescribes that, among all admissible motions $q = q(t)$ on a fixed time interval $[0,T]$ given the initial and final states $q(0)$ and $q(T)$, the actual motion is a stationary point of the *action*

$$\int_0^T \mathscr{L}\big(t,q,\dot{q}\big)\,\mathrm{d}t \tag{10}$$

where $\dot{q} = \frac{\partial}{\partial t}q$ and the *Lagrangean* $\mathscr{L}(t,q,\dot{q})$ is defined as

$$\mathscr{L}\big(t,q,\dot{q}\big) := \mathscr{T}\big(\dot{q}\big) - \mathscr{E}(q) + \langle F(t),q\rangle \quad \text{with} \quad F = F_0(t) - \partial_{\dot{q}}\mathscr{R}(q,\dot{q})\,. \tag{11}$$

This corresponds to the sum of external time-dependent loading and the (negative) nonconservative force assumed for a moment fixed. In addition to $\mathscr{E}$, $\mathscr{R}$, and $\mathscr{T}$ from Sections 2.3 and 2.4, we define the outer loading F_0 as $\langle F_0(t),q\rangle = \int_\Omega f\cdot u\,\mathrm{d}x$, where f is a time-dependent external body load.

The corresponding Euler-Lagrange equations read

$$\partial_u\mathscr{L}\big(t,q,\dot{q}\big) - \frac{\mathrm{d}}{\mathrm{d}t}\partial_{\dot{q}}\mathscr{L}\big(t,q,\dot{q}\big) = 0. \tag{12}$$

This gives the abstract 2nd-order evolution equation

$$\partial^2\mathscr{T}\,\ddot{q} + \partial_{\dot{q}}\mathscr{R}(q,\dot{q}) + \mathscr{E}'(q) = F_0(t) \tag{13}$$

where ∂ indicates the (partial) Gâteaux differential. Let us now rewrite the abstract relation (13) in terms of our specific choices (5), (7)-(9). We have

the following equation/inclusion on $[0,T]\times\Omega$:

$$\rho\ddot{u} - \operatorname{div}\sigma = f, \quad \sigma := \mathbb{C}(\alpha)e_{\rm el} + \mathbb{D}(\alpha)\dot{e}_{\rm el}, \;\; e_{\rm el} = e(u) - \pi, \tag{14a}$$

$$\sigma_{\rm YLD}(\alpha)\mathrm{Dir}(\dot{\pi}) \ni \operatorname{dev}\sigma, \tag{14b}$$

$$\partial\zeta(\dot{\alpha}) + \frac{1}{2}\mathbb{C}'(\alpha)e_{\rm el} : e_{\rm el} \ni \phi'(\alpha) + \operatorname{div}\big(\kappa|\nabla\alpha|^{p-2}\nabla\alpha\big), \tag{14c}$$

complemented by the boundary conditions

$$\sigma\nu_\Gamma = 0 \quad \text{on } [0,T]\times\Gamma_{\rm N}, \quad u = u_{\rm D} \quad \text{on } [0,T]\times\Gamma_{\rm D}, \quad \nabla\alpha\cdot\nu_\Gamma = 0 \quad \text{on } [0,T]\times\Gamma. \tag{15}$$

The notation $\mathrm{Dir} : \mathbb{R}^{d\times d}_{\rm dev} \rightrightarrows \mathbb{R}^{d\times d}_{\rm dev}$ in (14b) means the set-valued "direction" mapping defined by $\mathrm{Dir}(\dot{\pi}) := [\partial|\cdot|](\dot{\pi})$. In particular

$$\mathrm{Dir}(\dot{\pi}) = \begin{cases} \dot{\pi}/|\dot{\pi}| & \text{if } \dot{\pi} \neq 0 \\ \{d \in \mathbb{R}^{d\times d}_{\rm dev} \;:\; |d| \le 1\} & \text{if } \dot{\pi} = 0 \end{cases}$$

Relations (14a), (14b), and (14c) correspond to the equilibrium equation and constitutive relation, the plastic flow rule, and the evolution law for damage, respectively.

The above boundary-value problem is complemented with initial conditions as follows ,

$$u(0) = u_0, \quad \pi(0) = \pi_0, \quad \alpha(0) = \alpha_0, \quad \dot{u}(0) = v_0. \tag{16}$$

We point out that the monotonicity of $\mathbb{C}$, combined with the unidirectionality ($\dot{\alpha} \le 0$) of damage implies that

$$\dot{\alpha}\mathbb{C}'(\alpha)e : e \le 0 \quad \text{for every } e \in \mathbb{R}^{d\times d}, \tag{17}$$

namely $\dot{\alpha}\mathbb{C}'(\alpha)$ is negative semi-definite. By the monotonicity of $\sigma_{\rm YLD}$, the unidirectionality of damage also yields that

$$\dot{\alpha}\sigma'_{\rm YLD}(\alpha) \le 0. \tag{18}$$

The *energetics* of the model (14)-(15), obtained by standard tests of (14) successively against $\dot{u}$, $\dot{\pi}$, and $\dot{\alpha}$, is formally encoded by the following energy equality

$$\underbrace{\int_\Omega \frac{\rho}{2}|\dot{u}(t)|^2\,\mathrm{d}x}_{\substack{\text{kinetic energy}\\ \text{at time } t}} \quad + \quad \underbrace{\int_\Omega \frac{1}{2}\mathbb{C}(\alpha(t))e_{\rm el}(t) : e_{\rm el}(t) - \phi(\alpha(t)) + \frac{\kappa}{p}|\nabla\alpha(t)|^p\,\mathrm{d}x}_{\text{stored energy at time } t}$$

$$
\begin{aligned}
&+ \underbrace{\int_0^t\int_\Omega \eta\dot\alpha^2 + \mathbb{D}(\alpha)\dot e_{\rm el} : \dot e_{\rm el}\,{\rm d}x\,{\rm d}s + \sigma_{\rm YLD}(\alpha)|\dot\pi|\,{\rm d}x\,{\rm d}s}_{\text{dissipation on }[0,t]}\\
&= \underbrace{\int_\Omega \frac{\rho}{2}|v_0|^2\,{\rm d}x}_{\substack{\text{kinetic energy}\\ \text{at time 0}}} + \underbrace{\int_\Omega \frac12\mathbb{C}(\alpha_0)e_{\rm el}(0) : e_{\rm el}(0) - \phi(\alpha_0) + \frac{\kappa}{p}|\nabla\alpha_0|^p\,{\rm d}x}_{\text{stored energy at time 0}}\\
&\quad + \underbrace{\int_0^t\int_\Omega f\cdot\dot u\,{\rm d}x\,{\rm d}s}_{\substack{\text{energy of}\\ \text{external bulk load}}} + \underbrace{\int_0^t\int_{\Gamma_{\rm D}} \sigma\nu_{\rm T}\cdot\dot u_{\rm D}\,{\rm d}\mathcal{H}^{d-1}\,{\rm d}s}_{\substack{\text{energy of}\\ \text{boundary condition}}}
\end{aligned}
\tag{19}
$$

where the last term has to be interpreted in the sense of (40) below. A rigorous derivation of the energy equality above will be presented in Subsection 3.1.

2.6. **Statement of the main result.** Let $p > d$ be given and assume that the data of the problem satisfy the following conditions:

$$u_0 \in L^2(\Omega;\mathbb{R}^d)\cap BD(\Omega;\mathbb{R}^d),\quad v_0\in H^1(\Omega;\mathbb{R}^d),$$
$$\pi_0\in \mathcal{M}_b(\Omega\cup\Gamma_{\rm D};\mathbb{R}^{d\times d}_{\rm dev}),\quad \dot\pi_0\in L^2(\Omega;\mathbb{R}^{d\times d}_{\rm dev}), \tag{20a}$$
$$(u_0, e(u_0)-\pi_0,\pi_0)\in\mathscr{A}(u_{\rm D}(0)),\quad (v_0, e(v_0)-\dot\pi_0,\dot\pi_0)\in\mathscr{A}(\dot u_{\rm D}(0)), \tag{20b}$$
$$\alpha_0\in W^{1,p}(\Omega),\quad 0\le\alpha_0\le 1,$$
$$\sigma_{\rm YLD}(\alpha_0){\rm Dir}(\dot\pi_0)\ni {\rm dev}\,(\mathbb{C}(\alpha_0)(e(u_0){-}\pi_0)+\mathbb{D}(\alpha_0)(e(v_0){-}\dot\pi_0)), \tag{20c}$$
$$f\in L^2(0,T;L^2(\Omega;\mathbb{R}^d)),\quad u_{\rm D}\in W^{2,\infty}(0,T;L^2(\Omega;\mathbb{R}^d))\cap H^1(0,T;H^1(\Omega;\mathbb{R}^d)). \tag{20d}$$

The regularity requirements in (20) for v_0 and $\dot\pi_0$ and the compatibility condition in (20c) are needed in order to make some higher-order estimate rigorous, see Subsection 3.2.

We now introduce the notion of weak solution to (14)–(16).

[Weak solution to (14)–(16)] A quadruple

$$
\begin{aligned}
u &\in L^\infty(0,T;BD(\Omega;\mathbb{R}^d))\cap H^2(0,T;L^2(\Omega;\mathbb{R}^d))\\
e_{\rm el} &\in H^1(0,T;L^2(\Omega;\mathbb{R}^{d\times d}_{\rm sym})),\\
\pi &\in BV(0,T;\mathcal{M}_b(\Omega\cup\Gamma_{\rm D};\mathbb{R}^{d\times d}_{\rm dev})),\\
\alpha &\in \big(H^1(0,T;L^2(\Omega))\cap C_{\rm w}(0,T;W^{1,p}(\Omega))\big)
\end{aligned}
$$

is a *weak solution* to (14)–(16) if it satisfies (16), and the following conditions are fulfilled:

(C1) $(u(t), e_{\rm el}(t),\pi(t))\in\mathscr{A}(u_{\rm D}(t))$ for every $t\in[0,T]$ (see (3));
(C2) The equilibrium equation (14a) holds almost everywhere in $\Omega\times(0,T)$;
(C3) The quadruple $(u, e_{\rm el},\pi,\alpha)$ satisfies the energy inequality

$$
\begin{aligned}
&\int_\Omega \frac{\rho}{2}|\dot u(T)|^2\,{\rm d}x + \int_0^T\int_\Omega \rho\dot u\cdot\ddot u_{\rm D}\,{\rm d}x\,{\rm d}s\\
&\quad + \int_\Omega\Big(\frac12\mathbb{C}(\alpha(T))e_{\rm el}(T) : e_{\rm el}(T) - \phi(\alpha(T)) + \frac{\kappa}{p}|\nabla\alpha(T)|^p\Big)\,{\rm d}x\\
&\quad + D_{\mathcal{R}}(\alpha;\pi;0,T) + \int_0^T\int_\Omega\Big(\mathbb{D}(\alpha)\dot e_{\rm el} : \dot e_{\rm el} + \eta\dot\alpha^2\Big)\,{\rm d}x\,{\rm d}t\\
&\le \int_\Omega\frac{\rho}{2}v_0^2\,{\rm d}x + \int_\Omega\Big(\frac12\mathbb{C}(\alpha_0)(e(u_0)-\pi_0) : (e(u_0)-\pi_0) - \phi(\alpha_0) + \frac{\kappa}{p}|\nabla\alpha_0|^p\Big)\,{\rm d}x\\
&\quad + \int_\Omega \rho\dot u(T)\cdot\dot u_{\rm D}(T)\,{\rm d}x + \int_\Omega\rho v_0\cdot\dot u_{\rm D}(0)\,{\rm d}x\\
&\quad + \int_0^T\int_\Omega\Big(\mathbb{C}(\alpha)e_{\rm el} : e(\dot u_{\rm D}) + \mathbb{D}(\alpha)\dot e_{\rm el} : e(\dot u_{\rm D}) + f\cdot(\dot u - \dot u_{\rm D})\Big)\,{\rm d}x\,{\rm d}t.
\end{aligned}
$$

(C4) The quadruple $(u, e_{\rm el},\pi,\alpha)$ satisfies the damage inequality

$$\int_0^T\int_\Omega \phi'(\alpha)\varphi - \kappa|\nabla\alpha|^{p-2}\nabla\alpha\cdot\nabla\varphi - \frac12(\varphi{-}\dot\alpha)\mathbb{C}'(\alpha)e_{\rm el} : e_{\rm el} - \eta\dot\alpha\varphi\,{\rm d}x\,{\rm d}t$$

$$\leq \int_\Omega \phi(\alpha(T)) - \phi(\alpha_0) - \frac{\kappa}{p}|\nabla\alpha(T)|^p + \frac{\kappa}{p}|\nabla\alpha_0|^p \, dx - \int_0^T\int_\Omega \eta\dot{\alpha}^2 \, dx\, dt, \tag{21}$$

for all $\varphi \in W^{1,p}(\Omega)$ with $\varphi(x) \leq 0$ for a.e. $x \in \Omega$.

The main result of the paper consists in showing existence of weak solutions to (14)–(16). Let us summarize the assumption on the data of the model:

$$\mathbb{C} : \mathbb{R} \to \mathcal{L}_{\rm sym}(\mathbb{R}^{d\times d}_{\rm sym}) \text{ continously differentiable, positive semidefinite, nondecreasing,} \tag{22a}$$
$$\mathbb{D}(\cdot) = \mathbb{D}_0 + \chi\mathbb{C}(\cdot), \mathbb{D}_0 \text{ positive definite, } \chi \geq 0, \tag{22b}$$
$$\phi : \mathbb{R} \to \mathbb{R} \text{ continuously differentiable, nondecreasing,} \tag{22c}$$
$$\sigma_{\rm YLD} : \mathbb{R} \to \mathbb{R} \text{ continuously differentiable, positive, and nondecreasing,,} \tag{22d}$$
$$\mathbb{C}'(0) = 0, \quad \phi'(0) \geq 0, \tag{22e}$$
$$\eta \in L^\infty(\Omega), \quad \eta \geq \eta_0 > 0 \text{ a.e.,} \tag{22f}$$
$$\kappa \in L^\infty(\Omega), \quad \kappa \geq \kappa_0 > 0 \text{ a.e.,} \tag{22g}$$
$$\rho \in L^\infty(\Omega), \quad \rho \geq \rho_0 > 0 \text{ a.e.} \tag{22h}$$

where $\chi > 0$ is a constant denoting a relaxation time. The structural assumption (22b) is instrumental in making our existence theory amenable. It arises naturally by assuming $\mathbb{C}(\cdot)$ and $\mathbb{D}(\cdot)$ to be pure second-order polynomials of the damage variable α, namely $\mathbb{C}(\alpha) = \alpha^2\mathbb{C}_2$ (recall (22e)) and $\mathbb{D}(\alpha) = \mathbb{D}_0 + \alpha^2\mathbb{D}_2$. By assuming the two tensors $\mathbb{C}_2$ and $\mathbb{D}_2$ to be spherical, namely $\mathbb{C}_2 = c_2 I_4$ and $\mathbb{D}_2 = d_2 I_4$ for some $c_2,\ d_2 > 0$ where I_4 is the identity 4-tensor, one can define $\chi = d_2/c_2$ in order to get (22b). Assumption (22e) ensures that α stays non-negative during the evolution even if the constraint $\alpha \geq 0$ is not explicitly included in the problem, see Remark 2.6 below.

[Existence] Under assumptions (20) on initial conditions and loading and (22) on data there exists a weak solution to (14)–(16) in the sense of Definition 2.6. Moreover, this solution has the additional regularity $(u, e_{\rm el}, \pi) \in W^{1,\infty}(0,T; BD(\Omega;\mathbb{R}^d)) \times W^{1,\infty}(0,T; L^2(\Omega;\mathbb{R}^{d\times d}_{\rm sym})) \times W^{1,\infty}(0,T; \mathcal{M}_b(\Omega\cup\Gamma_{\rm D};\mathbb{R}^{d\times d}_{\rm dev}))$.

The proof of Theorem 2.6 is postponed to Section 6, where we present a conceptually implementable, numerically stable, and convergent numerical algorithm. Instead, we conclude this section with some final remarks.

[Body and surface loads] As pointed out in [6, Introduction], for quasistatic evolution in perfect plasticity one has to impose a compatibility condition between body and surface loads, namely a *safe load* to ensure that the body is not in a free flow. In the dynamic case, under the assumption of null surface loads, this condition can be weaken for what concerns body loads; see, *e.g.*, [36].

[Cohesive damage assumption] We will not include in the model reaction forces to the constraint $0 \leq \alpha \leq 1$. This would be encoded by rewriting (14c) as

$$\partial\zeta(\dot{\alpha}) + \frac{1}{2}\mathbb{C}'(\alpha)e_{\rm el} : e_{\rm el} + p_{\rm R} \ni \phi'(\alpha) + {\rm div}\,(\kappa|\nabla\alpha|^{p-2}\nabla\alpha) \quad \text{where} \quad p_{\rm R} \in N_{[0,1]}(\alpha)\,;$$

here $N_{[0,1]}(\cdot)$ denote the normal cone and $p_{\rm R}$ is a "reaction pressure" to the constraints $0 \leq \alpha \leq 1$. We point out that the presence of this additional term in the damage flow rule would cause a loss of regularity for the damage variable. In order to avoid such problem we will rather enforce the constraint $0 \leq \alpha \leq 1$ by exploiting the irreversibility of damage, and by restricting our analysis to the situation in which the damage is cohesive.

[Regularity of Γ] We remark that the C^2-regularity of Γ is needed in order to apply [37, Proposition 2.5], and define a duality between stresses and plastic strains. For $d = 2$, owing to the results in [30], it is also possible to analyze the setting in which Γ is Lipschitz. The same strategy can not be applied for $d = 3$, for it would require ${\rm div}\,\sigma \in L^3(\Omega)$, whereas here we can only achieve ${\rm div}\,\sigma \in L^2(\Omega)$.

[The role of the term $\eta\dot{\alpha}$] The term $\eta\dot{\alpha}$ in (14c) guarantees strong convergence of the damage-interpolants in the time-discretization scheme to the limit damage variable. This, in turn, is a key point to ensure strong convergence of the elastic stresses, which is fundamental for the proof of the damage inequality in condition (C4). From a modeling point of view, this might be interpreted as some additional dissipation related with the speed of the damaging process contributing to the heat production, possibly leading to an increase of temperature. The microscopical idea behind it is that faster mechanical processes cause higher heat production and therefore higher dissipation.

[Phase-field fracture] Our cohesive damage with $\mathbb{C}'(0) = 0$ has the drawback that, while α approaches zero, the driving force needed for its evolution rises to infinity. This model is anyhow used in the phase-field approximation of fracture.

$$\mathscr{E}(u,\alpha) := \int_\Omega \frac{(\varepsilon/\varepsilon_0)^2 + \alpha^2}{2}\mathbb{C}_1 e(u){:}e(u) + \underbrace{G_{\rm c}\Big(\frac{1}{2\varepsilon}(1-\alpha)^2 + \frac{\varepsilon}{2}|\nabla\alpha|^2\Big)}_{\text{crack surface density}} \, dx \tag{23}$$

with with G_c denoting the energy of fracture and with ε controlling a "characteristic" width of the phase-field fracture zone(s). The physical dimension of ε_0 as well as of ε is m (meters) while the physical dimension of G_c is J/m^2. In the model (5), it means $\mathbb{C}(\alpha) = (\varepsilon^2/\varepsilon_0^2{+}\alpha^2)\mathbb{C}_1$ and $\phi(\alpha) = -G_c(1{-}\alpha)^2/(2\varepsilon)$ while $\kappa = \varepsilon$ and $p = 2$. This is known as the so-called *Ambrosio-Tortorelli functional*. Its motivation came from the static case, where this approximation was proposed by Ambrosio and Tortorelli [3, 4] and the asymptotic analysis for $\varepsilon \to 0$ was rigorously proved first for the scalar-valued case. The generalization for the vectorial case is due to Focardi [28]. Later, it was extended to the evolution situation, namely for a rate-independent cohesive damage, in [33], see also [11, 12, 14, 38, 41] where inertial forces are incorporated in the description. Note however that plasticity was not involved in all these references. Some modifications have been addressed in [13], see also [46] for various other models, and [17, 29, 31, 35] for the linearized and cohesive-fracture settings.

[Ductile damage/fracture] A combination of damage/fracture with plasticity is sometimes denoted by the adjective "ductile", in contrast to "brittle", if plasticity is not considered. There are various scenarios of combination of plastification processes with damage, that can model various phenomena in fracture mechanics. Here, we address the case of damage-dependent elastic response and the yield stress.

[Influence of damage on the energy equality] We point out that, in the absence of damage, energy conservation could be recovered. Indeed, it would be possible to prove the energy equality, which would then ensure the validity of the flow rule (14b) as well. A detailed analysis of an analogous albeit quasistationary case has been performed in [22, Section 6] in the quasistatic framework. An adaptation of the argument yields the analogous statements in the dynamic setting.

3. Some formal calculus first

We first highlight a formal strategy that will lead to the proof of Theorem 2.6, avoiding (later necessary) technicalities. In particular, we first derive the energetics of the model by performing some standard tests of (14) against the time derivatives $(\dot u, \dot\pi, \dot\alpha)$. Further a-priori estimates will be obtained by performing a test of the same equations against higher-order time-derivatives of the maps. Eventually, a direct strong-convergence argument will be presented.

All the arguments will be eventually made rigorous in Sections 5–6 by means of a time-discretization procedure, combined with a passage to the limit as the time-step vanishes. The estimates described in Subsections 3.2–3.3 will be essential to pass to the limit in the time-discrete damage equation.

3.1. Energetics of the model and first estimates.

A formal test of the equations/inclusion (14) successively against $\dot u$, $\dot\pi$, and $\dot\alpha$ yields

$$\int_\Omega \Big(\rho\ddot u(t)\cdot\dot u(t) + \sigma(t) : e(\dot u(t))\Big)\,\mathrm{d}x = \int_\Omega f(t)\cdot\dot u(t)\,\mathrm{d}x + \int_\Gamma \sigma(t)\nu_\Gamma\cdot\dot u_{\rm D}(t)\,\mathrm{d}\mathcal{H}^{d-1}, \tag{24a}$$

$$\int_\Omega \mathrm{dev}\,\sigma(t) : \dot\pi(t)\,\mathrm{d}x = \int_\Omega \sigma_{\rm YLD}(\alpha(t))|\dot\pi(t)|\,\mathrm{d}x, \tag{24b}$$

$$\int_\Omega \eta\dot\alpha(t)^2\,\mathrm{d}x = \int_\Omega \Big(\phi'(\alpha(t))\dot\alpha(t) - \frac12\mathbb{C}'(\alpha(t))\dot\alpha(t)e_{\rm el}(t) : e_{\rm el}(t) - \kappa|\nabla\alpha(t)|^{p-2}\nabla\alpha(t)\cdot\nabla\dot\alpha(t)\Big)\,\mathrm{d}x. \tag{24c}$$

Integrating (24a) in time, by (16), (24b), and by the definition of $e_{\rm el}$, we obtain

$$\begin{aligned}\int_\Omega \Big(\frac{\rho}{2}|\dot u(t)|^2 + \frac12\mathbb{C}(\alpha(t))e_{\rm el}(t) : e_{\rm el}(t)\Big)\,\mathrm{d}x - \int_0^t\!\!\int_\Omega \frac12\mathbb{C}'(\alpha)\dot\alpha e_{\rm el} : e_{\rm el}\,\mathrm{d}x\,\mathrm{d}s \\ + \int_0^t\!\!\int_\Omega \mathbb{D}(\alpha)\dot e_{\rm el} : \dot e_{\rm el}\,\mathrm{d}x\,\mathrm{d}s + \int_0^t\!\!\int_\Omega \sigma_{\rm YLD}(\alpha)|\dot\pi|\,\mathrm{d}x\,\mathrm{d}s \\ = \int_\Omega \Big(\frac{\rho}{2}|v_0|^2 + \frac12\mathbb{C}(\alpha_0)e_{\rm el}(0) : e_{\rm el}(0)\Big)\,\mathrm{d}x \\ + \int_0^t\!\!\int_\Omega f\cdot\dot u\,\mathrm{d}x\,\mathrm{d}s + \int_0^t\!\!\int_{\Gamma_{\rm D}} \sigma\nu_\Gamma\cdot\dot u_{\rm D}\,\mathrm{d}\mathcal{H}^{d-1}\,\mathrm{d}s.\end{aligned} \tag{25}$$

In view of (15) and (16), an integration in time of (24c) yields

$$\int_0^t\!\!\int_\Omega \Big(\eta\dot\alpha^2 + \frac12\dot\alpha\mathbb{C}'(\alpha)e_{\rm el} : e_{\rm el}\Big)\,\mathrm{d}x\,\mathrm{d}s + \int_\Omega \Big(\frac{\kappa}{p}|\nabla\alpha(t)|^p - \phi(\alpha(t))\Big)\,\mathrm{d}x = \int_\Omega \Big(\frac{\kappa}{p}|\nabla\alpha_0|^p - \phi(\alpha_0)\Big)\,\mathrm{d}x. \tag{26}$$

Thus, summing (25) and (26), by (15) we deduce the energy equality (19).

To see the energy-based estimates from (19), here we should use the Gronwall inequality for the term $f\cdot\dot u$ benefitting from having the kinetic energy on the left-hand side, and the by-part integration of the Dirichlet loading term. We stress that the last term in (25) can be rigorously defined as in (40). This way, we can see the estimates

$$u\in L^\infty(0,T;BD(\Omega;\mathbb{R}^d))\cap W^{1,\infty}(0,T;L^2(\Omega;\mathbb{R}^d)),\tag{27a}$$

$$e_{\rm el}\in H^1(0,T;L^2(\Omega;\mathbb{R}^{d\times d}_{\rm sym})),\tag{27b}$$

$$\pi\in BV(0,T;\mathcal{M}_b(\Omega\cup\Gamma_{\rm D};\mathbb{R}^{d\times d}_{\rm dev})),\tag{27c}$$

$$\alpha\in L^\infty(0,T;W^{1,p}(\Omega))\cap H^1(0,T;L^2(\Omega)).\tag{27d}$$

Unfortunately, these estimates do not suffice for the convergence analysis as the time step goes to 0. In particular, in relation (35) later on one needs to handle the term $\rho\ddot u_k\cdot\dot u$, which is still not integrable under (27a).

3.2. **Higher-order tests.** In this subsection we perform an extension of the regularity estimate in Subsection 3.1, relying on the unidirectionality of the damage evolution, on the fact that $\sigma_{\rm YLD}(\cdot)$ is nondecreasing, and on the monotonicity of $\mathbb{C}(\cdot)$ with respect to the Löwner ordering. We introduce the abbreviation

$$w:=u+\chi\dot u,\quad \varepsilon_{\rm el}:=e_{\rm el}+\chi\dot e_{\rm el},\quad\text{and}\quad \varpi=\pi+\chi\dot\pi,\tag{28}$$

and observe that, $\ddot u=(\dot w-\dot u)/\chi$. Hence, the equilibrium equation rewrites as

$$\rho\frac{\dot w}{\chi}-{\rm div}\,\sigma=f+\rho\frac{\dot u}{\chi}.\tag{29}$$

We first argue by testing the plastic flow rule (14b) against $\dot\varpi$. We use the (here formal) calculus

$$\sigma_{\rm YLD}(\alpha){\rm Dir}(\dot\pi){:}\,\ddot\pi=\frac{\partial}{\partial t}\Big(\sigma_{\rm YLD}(\alpha)|\dot\pi|\Big)-\dot\alpha\sigma_{\rm YLD}'(\alpha)|\dot\pi|\ge\frac{\partial}{\partial t}\Big(\sigma_{\rm YLD}(\alpha)|\dot\pi|\Big)\tag{30}$$

because $\dot\alpha\sigma'_{\rm YLD}(\alpha)|\dot\pi|\le 0$ when assuming $\sigma_{\rm YLD}(\cdot)$ nondecreasing and using $\dot\alpha\le 0$, cf. (18). This formally yields

$$\begin{aligned}\int_0^T\!\!\int_\Omega\sigma_{\rm YLD}(\alpha)|\dot\pi|\,{\rm d}x\,{\rm d}t+\chi\int_\Omega\Big(\sigma_{\rm YLD}(\alpha(t))|\dot\pi(t)|-\chi\sigma_{\rm YLD}(\alpha_0)|\dot\pi(0)|\Big)\,{\rm d}x\\ =\int_0^T\!\!\int_\Omega\sigma_{\rm YLD}(\alpha)|\dot\pi|\,{\rm d}x\,{\rm d}t+\chi\int_\Omega\sigma_{\rm YLD}(\alpha(t))|\dot\pi(t)|\,{\rm d}x-\chi\int_\Omega\sigma_{\rm YLD}(\alpha(0))|\dot\pi(0)|\,{\rm d}x\\ \le\int_0^T\!\!\int_\Omega\sigma{:}\,\dot\varpi\,{\rm d}x\,{\rm d}t.\end{aligned}\tag{31}$$

Analogously, testing (29) against $\dot w$ and integrating in time, by (15) we deduce

$$\begin{aligned}\int_0^T\!\!\int_\Omega\Big(\frac{\rho}{\chi}|\dot w|^2+\sigma:e(\dot w)\Big)\,{\rm d}x{\rm d}t=\int_0^T\!\!\int_\Omega\Big(f\cdot\dot w+\frac{\rho}{\chi}\dot u\cdot\dot w\Big)\,{\rm d}x{\rm d}t\\ +\int_0^T\!\!\int_{\Gamma_{\rm D}}\sigma\nu_\Gamma\cdot(\dot u_{\rm D}+\chi\ddot u_{\rm D})\,{\rm d}\mathcal{H}^{d-1}{\rm d}t.\end{aligned}\tag{32}$$

By the definition of the tensor $\mathbb{D}$ (see Subsection 2.3), and by (17), we infer that

$$\begin{aligned}\int_0^T\!\!\int_\Omega\sigma:e(\dot w)\,{\rm d}x\,{\rm d}t=\int_0^T\!\!\int_\Omega\Big(\mathbb{C}(\alpha)\varepsilon_{\rm el}:\dot\varepsilon_{\rm el}+\mathbb{D}_0\dot e_{\rm el}:\dot\varepsilon_{\rm el}+\sigma:\dot\varpi\Big)\,{\rm d}x\,{\rm d}t\\ \ge\int_\Omega\frac12\mathbb{C}(\alpha(t))\varepsilon_{\rm el}(t):\varepsilon_{\rm el}(t)\,{\rm d}x+\int_0^T\!\!\int_\Omega\mathbb{D}_0\dot e_{\rm el}:\dot e_{\rm el}\,{\rm d}x\,{\rm d}t-\int_\Omega\frac12\mathbb{C}(\alpha_0)\varepsilon_{\rm el}(0):\varepsilon_{\rm el}(0)\,{\rm d}x\\ +\frac{\chi}{2}\int_\Omega\mathbb{D}_0\dot e_{\rm el}(t):\dot e_{\rm el}(t)\,{\rm d}x-\frac{\chi}{2}\int_\Omega\mathbb{D}_0\dot e_{\rm el}(0):\dot e_{\rm el}(0)\,{\rm d}x+\int_0^T\!\!\int_\Omega\sigma:\dot\varpi\,{\rm d}x\,{\rm d}t.\end{aligned}\tag{33}$$

Thus, by combining (31), with (32) and (33), we obtain the inequality

$$\begin{aligned}\frac1\chi\int_0^T\!\!\int_\Omega\rho|\dot w|^2\,{\rm d}x\,{\rm d}t+\frac12\int_\Omega\mathbb{C}(\alpha(t))\varepsilon_{\rm el}(t):\varepsilon_{\rm el}(t)\,{\rm d}x\\ +\int_0^T\!\!\int_\Omega\mathbb{D}_0\dot e_{\rm el}:\dot e_{\rm el}\,{\rm d}x\,{\rm d}t+\frac{\chi}{2}\int_\Omega\mathbb{D}_0\dot e_{\rm el}(t):\dot e_{\rm el}(t)\,{\rm d}x+\int_0^T\!\!\int_\Omega\sigma_{\rm YLD}(\alpha)|\dot\pi|\,{\rm d}x\,{\rm d}t\\ +\chi\int_\Omega\sigma_{\rm YLD}(\alpha(t))|\dot\pi(t)|\,{\rm d}x\le\frac12\int_\Omega\mathbb{C}(\alpha_0)\varepsilon_{\rm el}(0):\varepsilon_{\rm el}(0)\,{\rm d}x\\ +\frac{\chi}{2}\int_\Omega\mathbb{D}_0\dot e_{\rm el}(0):\dot e_{\rm el}(0)\,{\rm d}x+\chi\int_\Omega\sigma_{\rm YLD}(\alpha_0)|\dot\pi(0)|\,{\rm d}x\end{aligned}$$

$$+\int_0^T\!\!\int_\Omega f\cdot\dot w\,\mathrm{d}x\mathrm{d}t+\int_0^T\!\!\int_{\Gamma_{\rm D}}\sigma\nu_\Gamma\cdot(\dot u_{\rm D}+\chi\ddot u_{\rm D})\,\mathrm{d}\mathcal{H}^{d-1}\mathrm{d}t+\frac{1}{\chi}\int_0^T\!\!\int_\Omega\rho\dot u\cdot\dot w\,\mathrm{d}x\mathrm{d}t.$$

Let us note that we can use (27a) in order to control $\dot u$ in the last term above. As for initial data, we need here that $\dot e_{\rm el}(0)\in L^2(\Omega;\mathbb{R}^{d\times d}_{\rm sym})$ and $\dot\pi(0)\in L^2(\Omega;\mathbb{R}^{d\times d}_{\rm dev})$, which follows under the provisions of (20). Eventually, by (19), and (28) this yields the following additional regularity for the displacement, and for the elastic and plastic strains

$$u\in W^{1,\infty}(0,T;BD(\Omega;\mathbb{R}^d))\cap H^2(0,T;L^2(\Omega;\mathbb{R}^d)),\tag{34a}$$

$$e_{\rm el}\in W^{1,\infty}(0,T;L^2(\Omega;\mathbb{M}^{d\times d})),\tag{34b}$$

$$\pi\in W^{1,\infty}(0,T;\mathcal{M}_b(\Omega\cup\Gamma_{\rm D};\mathbb{R}^{d\times d}_{\rm dev})).\tag{34c}$$

3.3. **One more estimate for the strong convergence of $e_{\rm el}$'s.** The strong convergence of the elastic strains $e_{\rm el}$ is needed for the limit passage in the damage flow rule. The failure of energy conservation (see Remark 2.6) prevents the usual "limsup-strategy", but one can estimate directly the difference between the (presently still unspecified) approximate solution (u_k,π_k) and its limit (u,π) punctually as:

$$\begin{aligned}&\int_Q\mathbb{D}(\alpha_k)(\dot e_{{\rm el},k}-\dot e_{\rm el})\colon(\dot e_{{\rm el},k}-\dot e_{\rm el})\,\mathrm{d}x\mathrm{d}t\\&\qquad\qquad+\int_\Omega\frac12\mathbb{C}(\alpha_k(T))(e_{{\rm el},k}(T)-e_{\rm el}(T))\colon(e_{{\rm el},k}(T)-e_{\rm el}(T))\,\mathrm{d}x\\&\le\int_Q\int_\Omega\big(\mathbb{D}(\alpha_k)(\dot e_{{\rm el},k}-\dot e_{\rm el})+\mathbb{C}(\alpha_k)(e_{{\rm el},k}-e_{\rm el})\big)\colon(\dot e_{{\rm el},k}-\dot e_{\rm el})\,\mathrm{d}x\mathrm{d}t\\&\le\int_Q\bigg((f-\rho\ddot u_k)\cdot(\dot u_k-\dot u)-\big(\mathbb{D}(\alpha_k)\dot e_{\rm el}+\mathbb{C}(\alpha_k)e_{\rm el}\big)\colon(\dot e_{{\rm el},k}-\dot e_{\rm el})\\&\qquad\qquad+\sigma_{\rm YLD}(\alpha_k)(|\dot\pi|-|\dot\pi_k|)\bigg)\,\mathrm{d}x\mathrm{d}t.\end{aligned}\tag{35}$$

The first inequality in (35) is due to the monotonicity of $\mathbb{C}(\cdot)$ with respect to the Löwner ordering so that, due to (17), it holds

$$\begin{aligned}&\frac{\partial}{\partial t}\left(\frac12\mathbb{C}(\alpha_k)(e_{{\rm el},k}-e_{\rm el}):(e_{{\rm el},k}-e_{\rm el})\right)\\&=\frac12\dot\alpha_k\mathbb{C}'(\alpha_k)(e_{{\rm el},k}-e_{\rm el}):(e_{{\rm el},k}-e_{\rm el})+\mathbb{C}(\alpha_k)\frac{\partial}{\partial t}\left(\frac12(e_{{\rm el},k}-e_{\rm el}):(e_{{\rm el},k}-e_{\rm el})\right)\\&\le\mathbb{C}(\alpha_k)(e_{{\rm el},k}-e_{\rm el}):(\dot e_{{\rm el},k}-\dot e_{\rm el})\,.\end{aligned}\tag{36}$$

while the second step in (35) is due to the inequality ${\rm dev}\,\sigma\colon(\dot\pi_k-\dot\pi)\ge\sigma_{\rm YLD}(\alpha_k)(|\dot\pi_k|-|\dot\pi|)$, following from the plastic flow rule $\sigma_{\rm YLD}(\alpha_k){\rm Dir}(\dot\pi_k)\ni{\rm dev}\,\sigma$, with σ from (14a).

By using weak* upper semicontinuity and the uniform convergence $\alpha_k\to\alpha$ one checks that the limit superior of the right-hand side in (35) can be estimated from above by zero (so that, in fact, the limit does exist and equals to zero). We refer to Proposition 6 for the rigorous implementation of the above argument.

4. Staggered two-step time-discretization scheme

This section is devoted to the solution of a discrete counterpart of the system of equations (14)–(16), and to the proof of a-priori estimates for the associated piecewise constant, piecewise affine, and piecewise quadratic in-time interpolants.

Fix $n\in\mathbb{N}$, set $\tau:=T/n$, and consider the equidistant time partition of the interval $[0,T]$ with step τ. We define the discrete body-forces by setting $f_\tau^k:=\int_{(k-1)\tau}^{k\tau}f(t)\,\mathrm{d}t$ for all $k\in\{1,\dots,T/\tau\}$. We consider the following time-discretization scheme:

$$\rho\delta^2u_\tau^k-{\rm div}\big(\mathbb{C}(\alpha_\tau^{k-1})e_{{\rm el},\tau}^k+\mathbb{D}(\alpha_\tau^{k-1})\delta e_{{\rm el},\tau}^k\big)=f_\tau^k,\tag{37a}$$

$$\sigma_{\rm YLD}(\alpha_\tau^{k-1}){\rm Dir}(\delta\pi_\tau^k)\ni{\rm dev}\big(\mathbb{C}(\alpha_\tau^{k-1})e_{{\rm el},\tau}^k+\mathbb{D}(\alpha_\tau^{k-1})\delta e_{{\rm el},\tau}^k\big),\tag{37b}$$

$$\partial\zeta(\delta\alpha_\tau^k)+\frac12\mathbb{C}^\circ(\alpha_\tau^k,\alpha_\tau^{k-1})e_{{\rm el},\tau}^k:e_{{\rm el},\tau}^k\ni\phi^\circ(\alpha_\tau^k,\alpha_\tau^{k-1})+{\rm div}\,(\kappa|\nabla\alpha_\tau^k|^{p-2}\nabla\alpha_\tau^k),\tag{37c}$$

to be complemented with the boundary conditions

$$\big(\mathbb{C}(\alpha_\tau^{k-1})e_{{\rm el},\tau}^k+\mathbb{D}(\alpha_\tau^{k-1})\delta e_{{\rm el},\tau}^k\big)\,\nu_\Gamma=0\quad\text{on }\Gamma_{\rm N}\tag{38a}$$

$$\kappa|\nabla\alpha_\tau^k|^{p-2}\nabla\alpha_\tau^k\cdot\nu_\Gamma=0\quad\text{on }\Gamma.\tag{38b}$$

Here, δ and δ^2 denote the first and second order finite-difference operator, that is

$$\delta u_\tau^k := \frac{u_\tau^k - u_\tau^{k-1}}{\tau} \quad \text{and} \quad \delta^2 u_\tau^k := \delta\big[\delta u_\tau^k\big] = \frac{u_\tau^k - 2u_\tau^{k-1} + u_\tau^{k-2}}{\tau^2},$$

and where the tensor $\mathbb{C}^\circ(\alpha,\tilde\alpha)$ and the scalar $\phi^\circ(\alpha,\tilde\alpha)$ are defined for all $\alpha,\ \tilde\alpha \in \mathbb{R}$ as

$$\mathbb{C}^\circ(\alpha,\tilde\alpha) := \begin{cases} \dfrac{\mathbb{C}(\alpha)-\mathbb{C}(\tilde\alpha)}{\alpha-\tilde\alpha} & \text{if } \alpha \neq \tilde\alpha \\ \mathbb{C}'(\alpha) = \mathbb{C}'(\tilde\alpha) & \text{if } \alpha = \tilde\alpha, \end{cases}$$

$$\phi^\circ(\alpha,\tilde\alpha) := \begin{cases} \dfrac{\phi(\alpha)-\phi(\tilde\alpha)}{\alpha-\tilde\alpha} & \text{if } \alpha \neq \tilde\alpha\,, \\ \phi'(\alpha) = \phi'(\tilde\alpha) & \text{if } \alpha = \tilde\alpha\,. \end{cases}$$

Let us note that, if $\phi(\cdot)$ is affine, then simply $\phi^\circ(\alpha_\tau^k,\alpha_\tau^{k-1}) = \phi'$. Similarly, if $\mathbb{C}(\cdot)$ were affine, then $\mathbb{C}^\circ(\alpha_\tau^k,\alpha_\tau^{k-1}) = \mathbb{C}'$. We point out that this case would be in conflict with (22e) unless $\mathbb{C}$ would be independent of damage.

4.1. **Weak solutions to the time-discretization scheme.** In order to define a notion of weak solutions to (37b), we need to preliminary introduce a duality between stresses and plastic strains. We work along the footsteps of [37] and [22, Subsection 2.3]. We first define the set

$$\Sigma(\Omega) := \big\{\sigma \in L^2(\Omega;\mathbb{R}^{d\times d}_{\mathrm{sym}}) : \ \operatorname{dev}\sigma \in L^\infty(\Omega;\mathbb{R}^{d\times d}_{\mathrm{dev}}) \ \text{ and } \ \operatorname{div}\sigma \in L^2(\Omega;\mathbb{R}^d)\big\}. \tag{39}$$

By [37, Proposition 2.5 and Corollary 2.6], for every $\sigma \in \Sigma(\Omega)$ there holds

$$\sigma \in L^\infty(\Omega;\mathbb{R}^{d\times d}_{\mathrm{sym}}).$$

In addition, we can introduce the trace $[\sigma\nu_\Gamma] \in H^{-1/2}(\Gamma;\mathbb{R}^d)$ (see e.g. [50, Theorem 1.2, Chapter I]) as

$$\langle[\sigma\nu_\Gamma],\ \psi\rangle_\Gamma := \int_\Omega \operatorname{div}\sigma\cdot\psi\,\mathrm{d}x + \int_\Omega \sigma : e(\psi)\,\mathrm{d}x \tag{40}$$

for every $\psi \in H^1(\Omega;\mathbb{R}^d)$. Defining the normal and the tangential part of $[\sigma\nu_\Gamma]$ as

$$[\sigma\nu_\Gamma]_\nu := ([\sigma\nu_\Gamma]\cdot\nu_\Gamma)\nu_\Gamma \quad \text{and} \quad [\sigma\nu_\Gamma]_\nu^\perp := [\sigma\nu_\Gamma] - ([\sigma\nu_\Gamma]\cdot\nu_\Gamma)\nu_\Gamma,$$

by [37, Lemma 2.4] we have that $[\sigma\nu_\Gamma]_\nu^\perp \in L^\infty(\Gamma;\mathbb{R}^d)$, and

$$\|[\sigma\nu_\Gamma]_\nu^\perp\|_{L^\infty(\Gamma;\mathbb{R}^d)} \le \frac{1}{\sqrt{2}}\|\operatorname{dev}\sigma\|_{L^\infty(\Omega;\mathbb{R}^{d\times d}_{\mathrm{dev}})}.$$

Let $\sigma \in \Sigma(\Omega)$ and let $u \in BD(\Omega;\mathbb{R}^d) \cap L^2(\Omega;\mathbb{R}^d)$, with $\operatorname{div} u \in L^2(\Omega)$. We define the distribution $[\operatorname{dev}\sigma : \operatorname{dev} e(u)]$ on Ω as

$$\langle[\operatorname{dev}\sigma : \operatorname{dev} e(u)],\ \varphi\rangle := -\int_\Omega \varphi\operatorname{div}\sigma\cdot u\,\mathrm{d}x - \frac{1}{d}\int_\Omega \varphi\operatorname{tr}\sigma\cdot\operatorname{div} u\,\mathrm{d}x - \int_\Omega \sigma : (u\odot\nabla\varphi)\,\mathrm{d}x \tag{41}$$

for every $\varphi \in C_c^\infty(\Omega)$. By [37, Theorem 3.2] it follows that $[\operatorname{dev}\sigma : \operatorname{dev} e(u)]$ is a bounded Radon measure on Ω, whose variation satisfies

$$|[\operatorname{dev}\sigma : \operatorname{dev} e(u)]| \le \|\operatorname{dev}\sigma\|_{L^\infty(\Omega;\mathbb{R}^{d\times d}_{\mathrm{dev}})}|\operatorname{dev} e(u)| \quad \text{in } \Omega.$$

Let $\Pi_{\Gamma_{\mathrm{D}}}(\Omega)$ be the set of admissible plastic strains, namely the set of maps $\pi \in \mathcal{M}_b(\Omega\cup\Gamma_{\mathrm{D}};\mathbb{R}^{d\times d}_{\mathrm{dev}})$ such that there exist $u \in BD(\Omega;\mathbb{R}^d)\cap L^2(\Omega;\mathbb{R}^d)$, $e \in L^2(\Omega;\mathbb{R}^{d\times d}_{\mathrm{sym}})$, and $w \in W^{1,2}(\Omega;\mathbb{R}^d)$ with $(u,e,\pi) \in \mathscr{A}(w)$. Note that the additive decomposition $e(u) = e + \pi$ implies that $\operatorname{div} u \in L^2(\Omega)$.

It is possible to define a duality between elements of $\Sigma(\Omega)$ and $\Pi_{\Gamma_{\mathrm{D}}}(\Omega)$. To be precise, given $\pi \in \Pi_{\Gamma_{\mathrm{D}}}(\Omega)$, and $\sigma \in \Sigma(\Omega)$, we fix (u,e,w) such that $(u,e,\pi) \in \mathscr{A}(w)$, with $u \in L^2(\Omega;\mathbb{R}^d)$, and we define the measure $[\operatorname{dev}\sigma : \pi] \in \mathcal{M}_b(\Omega\cup\Gamma_{\mathrm{D}};\mathbb{R}^{d\times d}_{\mathrm{dev}})$ as

$$[\operatorname{dev}\sigma : \pi] := \begin{cases} [\operatorname{dev}\sigma : \operatorname{dev} e(u)] - \operatorname{dev}\sigma : \operatorname{dev} e & \text{in } \Omega \\ [\sigma\nu_\Gamma]_\nu^\perp\cdot(w-u)\mathcal{H}^{d-1} & \text{on } \Gamma_{\mathrm{D}}, \end{cases}$$

so that

$$\int_{\Omega\cup\Gamma_{\mathrm{D}}} \varphi\,\mathrm{d}[\operatorname{dev}\sigma : \pi] = \int_\Omega \varphi\,\mathrm{d}[\operatorname{dev}\sigma : \operatorname{dev} e(u)] - \int_\Omega \varphi\operatorname{dev}\sigma : \operatorname{dev} e\,\mathrm{d}x + \int_{\Gamma_{\mathrm{D}}} \varphi[\sigma\nu]_\nu^\perp\cdot(w-u)\,\mathrm{d}\mathcal{H}^{d-1}$$

for every $\varphi \in C(\bar\Omega)$. Arguing as in [22, Section 2], one can prove that the definition of $[\operatorname{dev}\sigma : \pi]$ is independent of the choice of (u,e,w), and that if $\operatorname{dev}\sigma \in C(\bar\Omega;\mathbb{R}^{d\times d}_{\mathrm{dev}})$ and $\varphi \in C(\bar\Omega)$, then

$$\int_{\Omega\cup\Gamma_{\mathrm{D}}} \varphi\,\mathrm{d}[\operatorname{dev}\sigma : \pi] = \int_{\Omega\cup\Gamma_{\mathrm{D}}} \varphi\operatorname{dev}\sigma : \mathrm{d}\pi.$$

We are now in a position to state the definition of weak solutions to the time-discretization scheme.

[Weak discrete solutions] For every $k \in \{1,\dots,T/\tau\}$, a quadruple $(u_\tau^k, e_{{\rm el},\tau}^k, \pi_\tau^k, \alpha_\tau^k)$ is a weak solution to (37) if $(u_\tau^k, e_\tau^k, \pi_\tau^k) \in \mathscr{A}(u_{{\rm D},\tau}^k)$, $\alpha_\tau^k \in W^{1,p}(\Omega)\cap L^\infty(\Omega)$ satisfies $0 \le \alpha_\tau^k \le 1$, the quadruple solves (37c) and (38), property (37a) holds almost everywhere, and the following discrete flow-rule is fulfilled

$$[{\rm dev}\,\sigma_\tau^k : \delta\pi_\tau^k](\Omega\cup\Gamma_{\rm D}) = \mathcal{R}(\alpha_\tau^{k-1}, \delta\pi_\tau^k), \quad \text{with} \quad \sigma_\tau^k := \mathbb{C}(\alpha_\tau^{k-1})e_{{\rm el},\tau}^k + \mathbb{D}(\alpha_\tau^{k-1})\delta e_{{\rm el},\tau}^k. \tag{42}$$

[The discrete flow-rule] A crucial difference with respect to the results in [6, Proposition 3.3] is the fact that condition (42) is much weaker than the differential inclusion (37b). This is due to a key peculiarity of our model, for we consider a viscous contribution involving only the elastic strain, but we still allow for perfect plasticity. In fact, in our setting (37b) is only formal, as for every τ and k, the map $\delta\pi_\tau^k$ is a bounded Radon measure. In particular the quantity $\sigma_{\rm YLD}(\alpha_\tau^{k-1}){\rm Dir}(\delta\pi_\tau^k)$ does not have a pointwise meaning. As customary in the setting of perfect plasticity, the discrete flow-rule is thus only recovered in a weaker form.

4.2. **Existence of weak solutions.** Let us start by specifying the discretization of the boundary Dirichlet data as system

$$u_{{\rm D},\tau}^0 := u_{\rm D}(0), \quad u_{{\rm D},\tau}^{-1} := u_{\rm D}(0) - \tau\dot{u}_{\rm D}(0), \quad u_{{\rm D},\tau}^k := u_{\rm D}(k\tau) \quad \text{for every } k \in \{1,\dots,T/\tau\}.$$

As for initial data, we recall (20) and prescribe

$$u_\tau^0 := u_0, \ \ \pi_\tau^0 := \pi_0, \ \ \alpha_\tau^0 := \alpha_0, \ \ e_{{\rm el},\tau}^0 = e(u_0) - \pi_0.$$

In order to reproduce the higher-order tests of Subsection 3.2 at the discrete level we need to specify additionally the following

$$u_\tau^{-1} := u_0 - \tau v_0, \ \ \pi_\tau^{-1} := \pi_0 - \tau\dot{\pi}_0, \ \ \alpha_\tau^{-1} := \alpha_\tau^0, \ \ e_{{\rm el},\tau}^{-1} = e(u_0) - \tau(e(v_0) - \dot{\pi}_0).$$

In particular, the last condition in (20c) ensures that the discrete flow rule (37b) holds at level $k=0$ as well.

In order to check for the solvability of the discrete system (37) we proceed in two steps. For given $\alpha_\tau^{k-1} \in W^{1,p}(\Omega)\cap L^\infty(\Omega)$ with $0 \le \alpha_\tau^{k-1} \le 1$ we look for the triple $(u_\tau^k, e_{{\rm el},\tau}^k, \pi_\tau^k)$ given by the unique solution to the minimum problem

$$\begin{aligned}\min\bigg\{ &\int_\Omega \Big(\frac12\mathbb{C}(\alpha_\tau^{k-1})e:e + \frac{1}{2\tau}\mathbb{D}(\alpha_\tau^{k-1})(e - e_{{\rm el},\tau}^{k-1}):(e - e_{{\rm el},\tau}^{k-1}) - f_\tau^k\cdot u\Big)\,{\rm d}x \\ &+ \frac{\rho}{2\tau^2}\|u - 2u_\tau^{k-1} + u_\tau^{k-2}\|^2_{L^2(\Omega;\mathbb{R}^d)} + \mathcal{R}(\alpha_\tau^{k-1}, \pi - \pi_\tau^{k-1}) \ : \ (u,e,\pi) \in \mathscr{A}(u_{{\rm D},\tau}^k)\bigg\}.\end{aligned} \tag{43}$$

where $\mathscr{A}(\cdot)$ is defined in (3). The existence and uniqueness of solutions to (43) is ensured by Lemma 4.2 below.

Once $(u_\tau^k, e_{{\rm el},\tau}^k, \pi_\tau^k)$ are found, we determine α_τ^k by solving

$$\begin{aligned}\min\bigg\{ &\int_\Omega \bigg(\tau\zeta\Big(\frac{\alpha - \alpha_\tau^{k-1}}{\tau}\Big) + \frac{\kappa}{p}|\nabla\alpha|^p \\ &+ \int_0^{\alpha(x)} \frac12\mathbb{C}^\circ(s,\alpha_\tau^{k-1}(x))\,e_{{\rm el},\tau}^k(x) : e_{{\rm el},\tau}^k(x) - \phi^\circ(s,\alpha_\tau^{k-1}(x))\,{\rm d}s\bigg)\,{\rm d}x \ : \quad (44) \\ &\alpha \in W^{1,p}(\Omega), \ 0 \le \alpha \le 1\bigg\} \quad (45)\end{aligned}$$

in Lemma 4.2 below.

[Existence of time-discrete displacements and strains] Let $\alpha_\tau^{k-1} \in W^{1,p}(\Omega)\cap L^\infty(\Omega)$, with $0 \le \alpha_\tau^{k-1} \le 1$, be given. Then, there exists a unique triple $(u_\tau^k, e_{{\rm el},\tau}^k, \pi_\tau^k) \in \mathscr{A}(u_{{\rm D},\tau}^k)$ solving (43).

Proof. The result follows by compactness, lower-semicontinuity, and by Korn's inequality (2). The uniqueness of the solution is a consequence of the strict convexity of the functional, and the fact that $\mathscr{A}(u_{{\rm D},\tau}^k)$ is affine. ☐

Minimizers of (43) satisfy the following first order optimality conditions.

[Time-discrete Euler-Lagrange equations for displacement and strains] Let $\alpha_\tau^{k-1} \in W^{1,p}(\Omega)\cap L^\infty(\Omega)$ be a solution to (37c) satisfying $0 \le \alpha_\tau^{k-1} \le 1$. Let $(u_\tau^k, e_{{\rm el},\tau}^k, \pi_\tau^k)$ be the minimizing triple of (43). Then, $(u_\tau^k, e_{{\rm el},\tau}^k, \pi_\tau^k)$ solves (37a) and (42), ${\rm div}\,\sigma_\tau^k \in L^2(\Omega;\mathbb{R}^d)$, and $[\sigma_\tau^k\nu_\Gamma] = 0$ on $\Gamma_{\rm N}$.

Proof. We omit the proof of (37a), as it follows closely the argument in [6, Proposition 3.3]. The proof of (42) is postponed to Corollary 5. ☐

We conclude this subsection by showing existence of solutions to (37c).

[Existence of admissible time-discrete damage variables] Let $k \in \{1, \dots, T/\tau\}$, and assume that $\alpha_\tau^{k-1} \in W^{1,p}(\Omega) \cap L^\infty(\Omega)$, with $0 \leq \alpha_\tau^{k-1} \leq 1$, is given. Let $(u_\tau^k, e_{\mathrm{el},\tau}^k, \pi_\tau^k)$ be the minimizing triple of (43). Then there exists $\alpha_\tau^k \in W^{1,p}(\Omega) \cap L^\infty(\Omega)$ solving (37c), and satisfying $0 \leq \alpha_\tau^k \leq 1$.

Proof. We preliminary observe that α_τ^k solves (37c) if and only if it minimizes the functional in (45). The existence of a minimizer $\alpha_\tau^k \in W^{1,p}(\Omega)$ follows by the continuity of $\phi(\cdot)$ and $\mathbb{C}(\cdot)$, by lower-semicontinuity, and by the Dominated Convergence Theorem. The fact that $\alpha_\tau^k(x) \leq 1$ for every $x \in \Omega$ is a consequence of the assumption that $0 \leq \alpha_\tau^{k-1} \leq 1$ in Ω, and of the constraint $\alpha_\tau^k \leq \alpha_\tau^{k-1}$. The constraint $0 \leq \alpha_\tau^k$ instead is satisfied due to the assumptions on $\mathbb{C}(\cdot)$ and $\phi(\cdot)$ (see Subsections 2.3 and 2.5), and owing to a truncation argument. □

5. A-PRIORI ENERGY ESTIMATES

In order to pass to the limit in the discrete scheme as the fineness τ of the partition goes to 0 we establish a few a-priori estimates on time interpolants between the quadruple identified via the time-discretization scheme of Section 4. We first rewrite [22, Proposition 2.2] in our framework.

[Integration by parts] Let $\sigma \in \Sigma(\Omega)$, $u_{\mathrm{D}} \in H^1(\Omega;\mathbb{R}^d)$, and $(u, e_{\mathrm{el}}, \pi) \in \mathscr{A}(u_{\mathrm{D}})$ with $\mathscr{A}(\cdot)$ from (3), with $u \in L^2(\Omega;\mathbb{R}^d)$. Assume that $[\sigma\nu_\Gamma] = 0$ on Γ_{N}. Then

$$[\operatorname{dev}\sigma : \pi](\Omega \cup \Gamma_{\mathrm{D}}) + \int_\Omega \sigma : (e_{\mathrm{el}} - e(u_{\mathrm{D}}))\,\mathrm{d}x = -\int_\Omega \operatorname{div}\sigma \cdot (u - u_{\mathrm{D}})\,\mathrm{d}x.$$

Note that the above lemma serves as definition of $[\operatorname{dev}\sigma : \pi](\Omega \cup \Gamma_{\mathrm{D}})$, which is a priori not defined for $\operatorname{dev}\sigma \in L^2(\Omega;\mathbb{R}^{d\times d}_{\mathrm{dev}})$ and $\pi \in \mathcal{M}_b(\Omega \cup \Gamma_{\mathrm{D}};\mathbb{R}^{d\times d}_{\mathrm{dev}})$.

We are now in a position of providing, in the following lemmas and corollary, further optimality conditions for triples $(u_\tau^k, e_\tau^k, \pi_\tau^k)$ solving (43).

[Discrete Euler-Lagrange equations for the plastic strain] Let $(u_\tau^k, e_{\mathrm{el},\tau}^k, \pi_\tau^k)$ be the minimizing triple of (43), and let σ_τ^k be the quantity defined in (42). Then, there holds

$$\mathcal{R}(\alpha_\tau^{k-1}, \tau\delta\pi_\tau^k + \pi) - \mathcal{R}(\alpha_\tau^{k-1}, \tau\delta\pi_\tau^k) - [\operatorname{dev}\sigma_\tau^k : \pi](\Omega \cup \Gamma_{\mathrm{D}}) \geq 0 \tag{46}$$

for every $\pi \in \mathcal{M}_b(\Omega \cup \Gamma_{\mathrm{D}};\mathbb{R}^{d\times d}_{\mathrm{dev}})$ such that there exist $u \in BD(\Omega;\mathbb{R}^d) \cap L^2(\Omega;\mathbb{R}^d)$, and $e \in L^2(\Omega;\mathbb{R}^{d\times d}_{\mathrm{sym}})$ with $(u, e, \pi) \in \mathscr{A}(0)$.

Proof. Considering variations of the form $(u_\tau^k, e_\tau^k, \pi_\tau^k) + \lambda(u, e, \pi)$ for $\lambda \geq 0$ and $(u, e, \pi) \in \mathscr{A}(0)$ in (43), by the convexity of $\mathcal{R}$ in its second variable we obtain

$$\frac{1}{\lambda}\big(\mathcal{R}(\alpha_\tau^{k-1}, \tau\delta\pi_\tau^k + \lambda\pi) - \mathcal{R}(\alpha_\tau^{k-1}, \tau\delta\pi_\tau^k)\big) \leq \mathcal{R}(\alpha_\tau^{k-1}, \tau\delta\pi_\tau^k + \pi) - \mathcal{R}(\alpha_\tau^{k-1}, \tau\delta\pi_\tau^k),$$

which yields

$$\int_\Omega \sigma_\tau^k : e\,\mathrm{d}x + \int_\Omega \rho\delta^2 u_\tau^k \cdot u\,\mathrm{d}x + \mathcal{R}(\alpha_\tau^{k-1}, \tau\delta\pi_\tau^k + \pi) - \mathcal{R}(\alpha_\tau^{k-1}, \tau\delta\pi_\tau^k) - \int_\Omega f_\tau^k \cdot u\,\mathrm{d}x \geq 0, \tag{47}$$

for every $u \in BD(\Omega;\mathbb{R}^d) \cap L^2(\Omega;\mathbb{R}^d)$, $e \in L^2(\Omega;\mathbb{R}^{d\times d}_{\mathrm{sym}})$, and $\pi \in \mathcal{M}_b(\Omega \cup \Gamma_{\mathrm{D}};\mathbb{R}^{d\times d}_{\mathrm{dev}})$ such that $(u, e, \pi) \in \mathscr{A}(0)$. In view of Lemma 5, and by (37a) the previous inequality implies (46). □

[Discrete flow-rule] Let $(u_\tau^k, e_{\mathrm{el},\tau}^k, \pi_\tau^k)$ be the minimizing triple of (43), let α_τ^k be the solution to (37c) provided by Lemma 4.2, and let σ_τ^k be the quantity defined in (42). Then, $(u_\tau^k, e_{\mathrm{el},\tau}^k, \pi_\tau^k, \alpha_\tau^k)$ solve the discrete flow-rule (42).

Proof. The assert follows by choosing $\pi = \tau\delta\pi_\tau^k$, and $\pi = -\tau\delta\pi_\tau^k$ in (46). □

For $k \in \{1, \dots T/\tau\}$, let $(u_\tau^k, e_{\mathrm{el},\tau}^k, \pi_\tau^k)$ be the minimizing triple of (43), and let σ_τ^k be the quantity defined in (42). Then, there holds

$$\begin{aligned}\mathcal{R}(\alpha_\tau^{k-1}, \tau\delta\pi_\tau^k + \pi) + \mathcal{R}(\alpha_\tau^{k-2}, \tau\delta\pi_\tau^{k-1} - \pi) - \mathcal{R}(\alpha_\tau^{k-1}, \tau\delta\pi_\tau^k)&\\ - \mathcal{R}(\alpha_\tau^{k-2}, \tau\delta\pi_\tau^{k-1}) - \tau[\operatorname{dev}\delta\sigma_\tau^k : \pi](\Omega \cup \Gamma_{\mathrm{D}}) &\geq 0\end{aligned}$$

for every $\pi \in \mathcal{M}_b(\Omega \cup \Gamma_{\mathrm{D}};\mathbb{R}^{d\times d}_{\mathrm{dev}})$ such that there exist $u \in BD(\Omega;\mathbb{R}^d) \cap L^2(\Omega;\mathbb{R}^d)$, and $e \in L^2(\Omega;\mathbb{R}^{d\times d}_{\mathrm{sym}})$ with $(u, e, \pi) \in \mathscr{A}(0)$.

Proof. Considering variations of the form $(u_\tau^{k-1}, e_\tau^{k-1}, \pi_\tau^{k-1}) - \lambda(u, e, \pi)$ for $\lambda \geq 0$ and $(u, e, \pi) \in \mathscr{A}(0)$ in (43) at level $i-1$, the convexity of $\mathcal{R}$ in its second variable yields

$$\mathcal{R}(\alpha_\tau^{k-2}, \tau\delta\pi_\tau^{k-1} - \pi) - \mathcal{R}(\alpha_\tau^{k-1}, \tau\delta\pi_\tau^{k-1}) - \int_\Omega \Big(\sigma_\tau^{k-1} : e + \rho\delta^2 u_\tau^{k-1} \cdot u - f_\tau^{k-1} \cdot u\Big)\,\mathrm{d}x \geq 0, \tag{48}$$

for every $u \in BD(\Omega;\mathbb{R}^d) \cap L^2(\Omega;\mathbb{R}^d)$, $e \in L^2(\Omega;\mathbb{R}^{d\times d}_{\rm sym})$, and $\pi \in \mathcal{M}_b(\Omega\cup\Gamma_{\rm D};\mathbb{R}^{d\times d}_{\rm dev})$ such that $(u, e, \pi) \in \mathscr{A}(0)$. The assert follows by summing (47) and (48), and by applying Lemma 5, and (37a). □

Let now $\underline{u}_\tau$, and $\bar{u}_\tau$ be the backward- and forward- piecewise constant in-time interpolants associated to the maps u_τ^k, namely

$$\underline{u}_\tau(0) := u_0, \quad \underline{u}_\tau(t) := u_\tau^{k-1} \quad \text{for every } t \in [(k-1)\tau, k\tau),\ k \in \{1, \ldots, T/\tau\}, \tag{49}$$

and

$$\bar{u}_\tau(0) := u_0, \quad \bar{u}_\tau(t) := u_\tau^k \quad \text{for every } t \in ((k-1)\tau, k\tau],\ k \in \{1, \ldots, T/\tau\}. \tag{50}$$

Denote by u_τ the associated piecewise affine in-time interpolant, that is

$$u_\tau(0) := u_0, \quad u_\tau(t) := \frac{(t-(k-1)\tau)}{\tau} u_\tau^k + \Big(1 - \frac{(t-(k-1)\tau)}{\tau}\Big) u_\tau^{k-1}, \tag{51}$$

for every $t \in ((k-1)\tau, k\tau]$, $k \in \{1, \ldots, T/\tau\}$, and let finally $\widetilde{u}_\tau$ be the piecewise quadratic interpolant satisfying $\widetilde{u}(k\tau) = u_\tau^k$, and

$$\ddot{\widetilde{u}}_\tau(t) = \delta^2 u_\tau^k \quad \text{for every } t \in ((k-1)\tau, k\tau],\ k \in \{1, \ldots, T/\tau\}.$$

Let $\underline{\alpha}_\tau, \bar{\pi}_\tau, \bar{e}_{\mathrm{el},\tau}, \bar{\alpha}_\tau, \pi_\tau, e_\tau$, and α_τ be defined analogously. The following proposition provides a first uniform estimate for the above quantities.

[Discrete energy inequality] Under assumptions (20), the following energy inequality holds true

$$\begin{aligned}
&\int_\Omega \frac{\rho}{2}|\dot{u}_\tau(T)|^2\,\mathrm{d}x + \frac{\tau}{2}\int_0^T\!\!\int_\Omega \rho|\ddot{\widetilde{u}}_\tau|^2\,\mathrm{d}x\,\mathrm{d}s + \int_\tau^T\!\!\int_\Omega \rho\dot{u}_\tau(\cdot-\tau)\cdot\ddot{\widetilde{u}}_{\mathrm{D},\tau}\,\mathrm{d}x\,\mathrm{d}s\\
&\quad + D_{\mathcal{R}}(\alpha_\tau;\pi_\tau;0,T) + \int_\Omega \Big(\frac{1}{2}\mathbb{C}(\underline{\alpha}_\tau(T))\bar{e}_{\mathrm{el},\tau}(T) : \bar{e}_{\mathrm{el},\tau}(T) - \phi(\alpha_\tau(T))\,\mathrm{d}x + \frac{\kappa}{p}|\nabla\alpha_\tau(T)|^p\Big)\,\mathrm{d}x\\
&\quad + \int_0^T\!\!\int_\Omega \mathbb{D}(\underline{\alpha}_\tau)\dot{e}_{\mathrm{el},\tau} : \dot{e}_{\mathrm{el},\tau}\,\mathrm{d}x\,\mathrm{d}s + \int_\Omega \eta\dot{\alpha}_\tau(T)^2\,\mathrm{d}x\\
&\leq \int_\Omega \Big(\frac{\rho}{2}v_0^2 + \rho\dot{u}_\tau(T)\cdot\dot{u}_{\mathrm{D},\tau}(t) + \rho v_0\cdot\delta u^1_{\mathrm{D},\tau}\Big)\,\mathrm{d}x\\
&\quad + \int_\Omega \frac{1}{2}\mathbb{C}(\alpha_0)(e(u_0)-\pi_0) : (e(u_0)-\pi_0) - \phi(\alpha_0)\,\mathrm{d}x + \frac{\kappa}{p}|\nabla\alpha_0|^p\,\mathrm{d}x\\
&\quad + \int_0^T\!\!\int_\Omega \Big(\mathbb{C}(\underline{\alpha}_\tau)\bar{e}_{\mathrm{el},\tau} : e(\dot{u}_{\mathrm{D},\tau}) + \mathbb{D}(\underline{\alpha}_\tau)\dot{e}_{\mathrm{el},\tau} : e(\dot{u}_{\mathrm{D},\tau}) + \bar{f}_\tau\cdot(\dot{u}_\tau - \dot{u}_{\mathrm{D},\tau})\Big)\,\mathrm{d}x\,\mathrm{d}s.
\end{aligned} \tag{52}$$

Proof. Fix $k \in \{1, \ldots, T/\tau\}$. Testing (37c) against $\delta\alpha_\tau^k$, we deduce the equality

$$\begin{aligned}
&\int_\Omega \eta|\delta\alpha_\tau^k|^2\,\mathrm{d}x + \int_\Omega \frac{1}{2}\mathbb{C}^\circ(\alpha_\tau^k, \alpha_\tau^{k-1})\delta\alpha_\tau^k e^k_{\mathrm{el},\tau} : e^k_{\mathrm{el},\tau}\,\mathrm{d}x\\
&\quad + \int_\Omega \Big(-\phi^\circ(\alpha_\tau^k, \alpha_\tau^{k-1})\delta\alpha_\tau^k + |\nabla\alpha_\tau^k|^{p-2}\nabla\alpha_\tau^k\cdot\nabla(\delta\alpha_\tau^k)\Big)\,\mathrm{d}x = 0.
\end{aligned} \tag{53}$$

Taking $\delta u_\tau^k - \delta u^k_{\mathrm{D},\tau}$ as test function in (37a), we have

$$\int_\Omega \rho\delta^2 u_\tau^k\cdot(\delta u_\tau^k - \delta u^k_{\mathrm{D},\tau})\,\mathrm{d}x - \int_\Omega \operatorname{div}\sigma_\tau^k\cdot(\delta u_\tau^k - \delta u^k_{\mathrm{D},\tau})\,\mathrm{d}x = \int_\Omega f_\tau^k\cdot(\delta u_\tau^k - \delta u^k_{\mathrm{D},\tau})\,\mathrm{d}x,$$

which by Lemma 5, and by the fact that $[\sigma_\tau^k\nu_\Gamma] = 0$ on $\Gamma_{\rm N}$ (see Proposition 4.2), yields

$$\begin{aligned}
&\int_\Omega \rho\delta^2 u_\tau^k\cdot(\delta u_\tau^k - \delta u^k_{\mathrm{D},\tau})\,\mathrm{d}x + [\operatorname{dev}\sigma_\tau^k : \delta\pi_\tau^k](\Omega\cup\Gamma_{\rm D}) + \int_\Omega \sigma_\tau^k : (\delta e^k_{\mathrm{el},\tau} - e(\delta u^k_{\mathrm{D},\tau}))\,\mathrm{d}x\\
&= \int_\Omega f_\tau^k\cdot(\delta u_\tau^k - \delta u^k_{\mathrm{D},\tau})\,\mathrm{d}x.
\end{aligned}$$

In view of Corollary 5, we obtain

$$\int_\Omega \rho\delta^2 u_\tau^k\cdot(\delta u_\tau^k - \delta u^k_{\mathrm{D},\tau})\,\mathrm{d}x + \mathcal{R}(\alpha_\tau^{k-1}, \delta\pi_\tau^k) + \int_\Omega \sigma_\tau^k : (\delta e^k_{\mathrm{el},\tau} - e(\delta u^k_{\mathrm{D},\tau}))\,\mathrm{d}x$$

$$= \int_\Omega f_\tau^k \cdot (\delta u_\tau^k - \delta u_{\mathrm{D},\tau}^k)\,\mathrm{d}x. \tag{54}$$

For $n \in \{1, \ldots, T/\tau\}$, a discrete integration by parts in time yields

$$\tau \sum_{k=1}^n \rho\delta^2 u_\tau^k \cdot \delta u_\tau^k = \sum_{k=1}^n \rho\big((\delta u_\tau^k)^2 - \delta u_\tau^k \cdot \delta u_\tau^{k-1}\big) = \frac{1}{2}\rho(\delta u_\tau^n)^2 - \frac{1}{2}\rho v_0^2 + \frac{\tau^2}{2}\sum_{k=1}^n \rho(\delta^2 u_\tau^k)^2 \tag{55}$$

a.e. on Ω. Analogously, we deduce that

$$-\tau \sum_{k=1}^n \rho\delta^2 u_\tau^k \cdot \delta u_{\mathrm{D},\tau}^k = \tau \sum_{k=1}^n \rho\delta u_\tau^{k-1} \cdot \delta^2 u_{\mathrm{D},\tau}^k - \rho\delta u_\tau^n \cdot \delta u_{\mathrm{D},\tau}^n - \rho v_0 \cdot \delta u_{\mathrm{D},\tau}^0 \tag{56}$$

a.e. on Ω. Additionally, by the monotonicity of $\mathbb{C}$ in the Löwner ordering, and (22b), we have

$$\begin{aligned}
\tau \sum_{k=1}^n \int_\Omega \sigma_\tau^k : \delta e_{\mathrm{el},\tau}^k\,\mathrm{d}x &= \tau \sum_{k=1}^n \int_\Omega \mathbb{C}(\alpha_\tau^{k-1}) e_{\mathrm{el},\tau}^k : \delta e_{\mathrm{el},\tau}^k\,\mathrm{d}x + \tau \sum_{k=1}^n \int_\Omega \mathbb{D}(\alpha_\tau^{k-1})\delta e_{\mathrm{el},\tau}^k : \delta e_{\mathrm{el},\tau}^k\,\mathrm{d}x \\
&\geq \int_\Omega \frac{1}{2}\mathbb{C}(\alpha_\tau^n) e_{\mathrm{el},\tau}^n : e_{\mathrm{el},\tau}^n\,\mathrm{d}x - \int_\Omega \frac{1}{2}\mathbb{C}(\alpha_0)(e(u_0) - \pi_0) : (e(u_0) - \pi_0)\,\mathrm{d}x \\
&\quad - \tau \sum_{k=1}^n \int_\Omega \frac{1}{2}\delta[\mathbb{C}(\alpha_\tau^k)] e_{\mathrm{el},\tau}^k : e_{\mathrm{el},\tau}^k\,\mathrm{d}x + \tau \sum_{k=1}^n \int_\Omega \mathbb{D}(\alpha_\tau^k)\delta e_{\mathrm{el},\tau}^k : \delta e_{\mathrm{el},\tau}^k\,\mathrm{d}x,
\end{aligned} \tag{57}$$

and

$$\frac{\tau}{2}\sum_{k=1}^n \int_\Omega \underbrace{\big(\mathbb{C}^\circ(\alpha_\tau^k, \alpha_\tau^{k-1})\delta\alpha_\tau^k - \delta[\mathbb{C}(\alpha_\tau^k)]\big)}_{=0} \; e_{\mathrm{el},\tau}^k : e_{\mathrm{el},\tau}^k\,\mathrm{d}x = 0. \tag{58}$$

Thus, multiplying (53) and (54) by τ, and summing for $k = 1, \ldots, T/\tau$, in view of (55), (56), (57), and (58) we deduce

$$\begin{aligned}
&\int_\Omega \frac{\rho}{2}|\dot{u}_\tau(T)|^2\,\mathrm{d}x + \frac{\tau}{2}\int_0^T \int_\Omega \rho|\ddot{\tilde{u}}_\tau|^2\,\mathrm{d}x\,\mathrm{d}t + \int_\tau^T \int_\Omega \rho\dot{u}_\tau(\cdot - \tau) \cdot \ddot{\tilde{u}}_{\mathrm{D},\tau}\,\mathrm{d}x\,\mathrm{d}t \\
&\quad + \tau \sum_{k=1}^{T/\tau} \mathcal{R}(\alpha_\tau^{k-1}, \delta\pi_\tau^k) + \frac{1}{2}\int_\Omega \mathbb{C}(\overline{\alpha}_\tau(T))\bar{e}_{\mathrm{el},\tau}(T) : \bar{e}_{\mathrm{el},\tau}(T)\,\mathrm{d}x \\
&\quad + \int_0^T \int_\Omega \mathbb{D}(\overline{\alpha}_\tau)\dot{e}_{\mathrm{el},\tau} : \dot{e}_{\mathrm{el},\tau}\,\mathrm{d}x\,\mathrm{d}t + \int_\Omega \Big(\eta\dot{\alpha}_\tau(T)^2 - \phi(\alpha_\tau(T)) + \frac{\kappa}{p}|\nabla\alpha_\tau(T)|^p\Big)\,\mathrm{d}x \\
&\leq \int_\Omega \Big(\frac{\rho}{2}v_0^2 + \rho\dot{u}_\tau(T) \cdot \dot{u}_{\mathrm{D},\tau}(T) + \rho v_0 \cdot \delta u_{\mathrm{D},\tau}^0\Big)\,\mathrm{d}x \\
&\quad + \int_\Omega \Big(\frac{1}{2}\mathbb{C}(\alpha_0)(e(u_0) - \pi_0) : (e(u_0) - \pi_0) - \phi(\alpha_0) + \frac{\kappa}{p}|\nabla\alpha_0|^p\Big)\,\mathrm{d}x \\
&\quad + \int_0^T \int_\Omega \mathbb{C}(\underline{\alpha}_\tau)\bar{e}_{\mathrm{el},\tau} : e(\dot{u}_{\mathrm{D},\tau})\,\mathrm{d}x\,\mathrm{d}s \\
&\quad + \int_0^T \int_\Omega \mathbb{D}(\underline{\alpha}_\tau)\dot{e}_{\mathrm{el},\tau} : e(\dot{u}_{\mathrm{D},\tau}) + \bar{f}_\tau \cdot (\dot{u}_\tau - \dot{u}_{\mathrm{D},\tau})\,\mathrm{d}x\,\mathrm{d}s.
\end{aligned} \tag{59}$$

Additionally, recalling definition (6), and observing that π_τ jumps exactly only in the points τk, $k \in \{1, \ldots, T/\tau\}$, by the monotonicity of the maps α_τ (see Subsection 2.4), we have

$$D_{\mathcal{R}}(\alpha_\tau; \pi_\tau; 0, T) = \tau \sum_{k=1}^{T/\tau} \mathcal{R}(\alpha_\tau^{k-1}, \delta\pi_\tau^k). \tag{60}$$

This concludes the proof of the energy inequality (52) and of the proposition. □

Owing to the previous discrete energy inequality, we are now in a position to deduce some first a-priori estimates for the piecewise affine interpolants.

[A-priori estimates] Under assumptions (20), for τ small enough there exists a constant C, dependent only on the initial conditions, on f, and on u_{D}, such that

$$\begin{aligned}
&\|\alpha_\tau\|_{H^1(0,T;L^2(\Omega))} + \|\alpha_\tau\|_{L^\infty(0,T;W^{1,p}(\Omega))} + \|e_{\mathrm{el},\tau}\|_{H^1(0,T;L^2(\Omega;\mathbb{R}^{d\times d}_{\mathrm{sym}}))} \\
&\quad + \|u_\tau\|_{W^{1,\infty}(0,T;L^2(\Omega;\mathbb{R}^d))} + \|u_\tau\|_{BV(0,T;BD(\Omega;\mathbb{R}^d))} + \|\pi_\tau\|_{BV(0,T;\mathcal{M}_b(\Omega\cup\Gamma_{\mathrm{D}};\mathbb{R}^{d\times d}_{\mathrm{dev}}))} \\
&\quad + \|\underline{\alpha}_\tau\|_{L^\infty((0,T)\times\Omega)} + \|\bar{\alpha}_\tau\|_{L^\infty((0,T)\times\Omega)} + \|\bar{e}_{\mathrm{el},\tau}\|_{L^\infty(0,T;L^2(\Omega;\mathbb{R}^{d\times d}_{\mathrm{sym}}))} \leq C.
\end{aligned} \tag{61}$$

Proof. The assert follows by Proposition 5, by the regularity of the applied force f and of the boundary datum $u_{\rm D}$, and by applying Hölder's and discrete Gronwall's inequalities, for τ small enough. □

We proceed by performing at the discrete level the higher-order test with the strategy formally sketched in Subsection 3.2.

[Second a-priori estimates] Under assumptions (20), for τ small enough there exists a constant C, dependent only on the initial conditions, on f, and on $u_{\rm D}$, such that

$$\|\widetilde{u}_\tau\|_{H^2(0,T;L^2(\Omega;\mathbb{R}^d))}+\|u_\tau\|_{W^{1,\infty}(0,T;BD(\Omega;\mathbb{R}^d))}\\ +\|\pi_\tau\|_{W^{1,\infty}(0,T;\mathcal{M}_b(\Omega\cup\Gamma_{\rm D};\mathbb{R}^{d\times d}_{\rm dev}))}+\|e_{{\rm el},\tau}\|_{W^{1,\infty}(0,T;L^2(\Omega;\mathbb{R}^{d\times d}_{\rm sym}))}\le C.$$

Proof. Fix $k\in\{1,\dots,T/\tau\}$, and consider the map $w^k_\tau:=u^k_\tau+\chi\delta u^k_\tau$, where $\chi>0$ is the constant introduced in Subsection 2.3. Since $\delta^2u^k_\tau=(\delta w^k_\tau-\delta u^k_\tau)/\chi$, equation (37a) rewrites as

$$\rho\Big(\frac{\delta w^k_\tau}{\chi}\Big)-{\rm div}\,\sigma^k_\tau=f^k_\tau+\rho\Big(\frac{\delta u^k_\tau}{\chi}\Big). \tag{62}$$

Now, testing (62) against $\delta w^k_\tau-(\delta u^k_{{\rm D},\tau}+\chi\delta^2u^k_{{\rm D},\tau})$, by Lemma 5 we deduce the estimate

$$\begin{aligned}&\frac{1}{\chi}\int_\Omega\rho|\delta w^\tau_k|^2\,{\rm d}x+[{\rm dev}\,\sigma^k_\tau:(\delta\pi^k_\tau+\chi\delta^2\pi^k_\tau)](\Omega\cup\Gamma_{\rm D})\\&\quad+\int_\Omega\sigma^k_\tau:(\delta e^k_{{\rm el},\tau}+\chi\delta^2e^k_{{\rm el},\tau}-e(\delta u^k_{{\rm D},\tau})-\chi e(\delta^2u^k_{{\rm D},\tau}))\,{\rm d}x\\&=\int_\Omega f^k_\tau\cdot(\delta w^k_\tau-\delta u^k_{{\rm D},\tau}-\chi\delta^2u^k_{{\rm D},\tau})\,{\rm d}x+\frac{1}{\chi}\int_\Omega\rho\delta u^k_\tau\cdot(\delta w^k_\tau-\delta u^k_{{\rm D},\tau}-\chi\delta^2u^k_{{\rm D},\tau})\,{\rm d}x\\&\quad+\frac{1}{\chi}\int_\Omega\rho\delta w^k_\tau\cdot(\delta u^k_{{\rm D},\tau}+\chi\delta^2u^k_{{\rm D},\tau})\,{\rm d}x.\end{aligned} \tag{63}$$

In view of Lemma 5 we have

$$[{\rm dev}\,\sigma^k_\tau:(\delta\pi^k_\tau+\chi\delta^2\pi^k_\tau)](\Omega\cup\Gamma_{\rm D})=[{\rm dev}\,\sigma^k_\tau:\delta\pi^k_\tau](\Omega\cup\Gamma_{\rm D})+\chi[{\rm dev}\,\sigma^k_\tau:\delta^2\pi^k_\tau](\Omega\cup\Gamma_{\rm D}).$$

Now, Corollary 5 yields

$$[{\rm dev}\,\sigma^k_\tau:\delta\pi^k_\tau](\Omega\cup\Gamma_{\rm D})=\mathcal{R}(\alpha^{k-1}_\tau,\delta\pi^k_\tau), \tag{64}$$

whereas Lemma 5 entails

$$\begin{aligned}&\chi[{\rm dev}\,\sigma^k_\tau:\delta^2\pi^k_\tau)](\Omega\cup\Gamma_{\rm D})=\chi\delta\{[{\rm dev}\,\sigma^k_\tau:\delta\pi^k_\tau](\Omega\cup\Gamma_{\rm D})\}-\chi[{\rm dev}\,\delta\sigma^k_\tau:\delta\pi^{k-1}_\tau](\Omega\cup\Gamma_{\rm D})\\&\quad\ge\chi\delta\{[{\rm dev}\,\sigma^k_\tau:\delta\pi^k_\tau](\Omega\cup\Gamma_{\rm D})\}+\mathcal{R}(\alpha^{k-1}_\tau,\delta\pi^k_\tau)+\mathcal{R}(\alpha^{k-2}_\tau,\delta\pi^{k-1}_\tau)-\mathcal{R}(\alpha^{k-1}_\tau,\delta\pi^k_\tau+\delta\pi^{k-1}_\tau)\\&\quad\ge\chi\delta\{[{\rm dev}\,\sigma^k_\tau:\delta\pi^k_\tau](\Omega\cup\Gamma_{\rm D})\}+\mathcal{R}(\alpha^{k-1}_\tau,\delta\pi^k_\tau)+\mathcal{R}(\alpha^{k-1}_\tau,\delta\pi^{k-1}_\tau)-\mathcal{R}(\alpha^{k-1}_\tau,\delta\pi^k_\tau+\delta\pi^{k-1}_\tau)\\&\quad\ge\chi\delta\{[{\rm dev}\,\sigma^k_\tau:\delta\pi^k_\tau](\Omega\cup\Gamma_{\rm D})\},\end{aligned} \tag{65}$$

where the second-to-last step follows by the fact that $\sigma_{\rm YLD}$ is nondecreasing (see Subsection 2.4), and the last step is a consequence of the triangle inequality. By combining (63), (64), and (65), we obtain

$$\begin{aligned}&\frac{1}{\chi}\int_\Omega\rho|\delta w^k_\tau|^2\,{\rm d}x+\chi\delta\{[{\rm dev}\,\sigma^k_\tau:\delta\pi^k_\tau](\Omega\cup\Gamma_{\rm D})\}+\mathcal{R}(\alpha^{k-1}_\tau,\delta\pi^k_\tau)\\&\quad+\int_\Omega\sigma^k_\tau:(\delta e^k_{{\rm el},\tau}+\chi\delta^2e^k_{{\rm el},\tau}-e(\delta u^k_{{\rm D},\tau})-\chi e(\delta^2u^k_{{\rm D},\tau}))\,{\rm d}x\\&\le\int_\Omega f^k_\tau\cdot(\delta w^k_\tau-\delta u^k_{{\rm D},\tau}-\chi\delta^2u^k_{{\rm D},\tau})\,{\rm d}x+\frac{1}{\chi}\int_\Omega\rho\delta u^k_\tau\cdot(\delta w^k_\tau-\delta u^k_{{\rm D},\tau}-\chi\delta^2u^k_{{\rm D},\tau})\,{\rm d}x\\&\quad+\frac{1}{\chi}\int_\Omega\rho\delta w^k_\tau\cdot(\delta u^k_{{\rm D},\tau}+\chi\delta^2u^k_{{\rm D},\tau})\,{\rm d}x.\end{aligned} \tag{66}$$

Multiplying (66) by τ, summing for $k=1,\dots,n$, with $n\in\{1,\dots,T/\tau\}$, and using again (64) with $k=n$, we infer that

$$\begin{aligned}&\frac{\tau}{\chi}\sum_{k=1}^n\int_\Omega\rho|\delta w^k_\tau|^2\,{\rm d}x+\chi\mathcal{R}(\alpha^{n-1}_\tau,\delta\pi^n_\tau)-\chi\mathcal{R}(\alpha^{-1}_\tau,\delta\pi^0_\tau)\\&\quad+\tau\sum_{k=1}^n\mathcal{R}(\alpha^{k-1}_\tau,\delta\pi^k_\tau)+\tau\sum_{k=1}^n\int_\Omega\sigma^k_\tau:(\delta e^k_{{\rm el},\tau}+\chi\delta^2e^k_{{\rm el},\tau})\,{\rm d}x\end{aligned}$$

$$\leq \tau \sum_{k=1}^{n} \int_{\Omega} f_{\tau}^{k} \cdot (\delta w_{\tau}^{k} - \delta u_{\mathrm{D},\tau}^{k} - \chi \delta^2 u_{\mathrm{D},\tau}^{k})\,\mathrm{d}x + \frac{\tau}{\chi} \sum_{k=1}^{n} \int_{\Omega} \rho \delta u_{\tau}^{k} \cdot (\delta w_{\tau}^{k} - \delta u_{\mathrm{D},\tau}^{k} - \chi \delta^2 u_{\mathrm{D},\tau}^{k})\,\mathrm{d}x$$
$$+ \frac{\tau}{\chi} \sum_{k=1}^{n} \int_{\Omega} \rho \delta w_{\tau}^{k} \cdot (\delta u_{\mathrm{D},\tau}^{k} + \chi \delta^2 u_{\mathrm{D},\tau}^{k})\,\mathrm{d}x + \tau \sum_{k=1}^{n} \int_{\Omega} \sigma_{\tau}^{k} : (e(\delta u_{\mathrm{D},\tau}^{k}) + \chi e(\delta^2 u_{\mathrm{D},\tau}^{k}))\,\mathrm{d}x. \tag{67}$$

By (22b),

$$\tau \sum_{k=1}^{n} \int_{\Omega} \sigma_{\tau}^{k} : (\delta e_{\mathrm{el},\tau}^{k} + \chi \delta^2 e_{\mathrm{el},\tau}^{k})\,\mathrm{d}x = \tau \sum_{k=1}^{n} \int_{\Omega} \mathbb{C}(\alpha_{\tau}^{k-1})(e_{\mathrm{el},\tau}^{k} + \chi \delta e_{\mathrm{el},\tau}^{k}) : (\delta e_{\mathrm{el},\tau}^{k} + \chi \delta^2 e_{\mathrm{el},\tau}^{k})\,\mathrm{d}x$$
$$+ \tau \sum_{k=1}^{n} \int_{\Omega} \mathbb{D}_0 \delta e_{\mathrm{el},\tau}^{k} : (\delta e_{\mathrm{el},\tau}^{k} + \chi \delta^2 e_{\mathrm{el},\tau}^{k})\,\mathrm{d}x.$$

Thus, arguing as in (57), we have

$$\tau \sum_{k=1}^{n} \int_{\Omega} \sigma_{\tau}^{k} : (\delta e_{\mathrm{el},\tau}^{k} + \chi \delta^2 e_{\mathrm{el},\tau}^{k})\,\mathrm{d}x \geq \int_{\Omega} \frac{1}{2} \mathbb{C}(\alpha_{\tau}^{n})(e_{\mathrm{el},\tau}^{n} + \chi \delta e_{\mathrm{el},\tau}^{n}) : (e_{\mathrm{el},\tau}^{n} + \chi \delta e_{\mathrm{el},\tau}^{n})\,\mathrm{d}x$$
$$- \int_{\Omega} \frac{1}{2} \mathbb{C}(\alpha_0)(e(u_0) - \pi_0 + \chi(e(v_0) - \dot{\pi}_0)) : (e(u_0) - \pi_0 + \chi(e(v_0) - \dot{\pi}_0)\,\mathrm{d}x$$
$$+ \int_{\Omega} \Big(\tau \sum_{k=1}^{n} \mathbb{D}_0 \delta e_{\mathrm{el},\tau}^{k} : \delta e_{\mathrm{el},\tau}^{k} + \frac{\chi}{2} \mathbb{D}_0 \delta e_{\mathrm{el},\tau}^{n} : \delta e_{\mathrm{el},\tau}^{n}\,\mathrm{d}x - \frac{\chi}{2} \mathbb{D}_0 e(v_0) : e(v_0) \Big)\,\mathrm{d}x. \tag{68}$$

Eventually, by (67) and (68), and by recalling (60), for every $t = k\tau$,

$$\frac{1}{\chi} \int_0^t \int_{\Omega} \rho |\dot{w}_{\tau}|^2\,\mathrm{d}x\,\mathrm{d}s + \chi \mathcal{R}(\bar{\alpha}_{\tau}(t), \dot{\pi}_{\tau}(t)) + D_{\mathcal{R}}(\alpha_{\tau}; \pi_{\tau}; 0, t) + \frac{\chi}{2} \int_{\Omega} \mathbb{D}_0 \dot{e}_{\mathrm{el},\tau}(t) : \dot{e}_{\mathrm{el},\tau}(t)\,\mathrm{d}x$$
$$+ \int_{\Omega} \mathbb{C}(\underline{\alpha}(t))(\bar{e}_{\mathrm{el},\tau}(t) + \chi \dot{e}_{\mathrm{el},\tau}(t)) : (\bar{e}_{\mathrm{el},\tau}(t) + \chi \dot{e}_{\mathrm{el},\tau}(t))\,\mathrm{d}x + \int_0^t \int_{\Omega} \mathbb{D}_0 \dot{e}_{\mathrm{el},\tau} : \dot{e}_{\mathrm{el},\tau}\,\mathrm{d}x\,\mathrm{d}s$$
$$\leq \int_{\Omega} \Big(\frac{1}{2} \mathbb{C}(\alpha_0)(e(u_0) - \pi_0 + \chi e(v_0) - \chi \dot{\pi}_0) : (e(u_0) - \pi_0 + \chi e(v_0) - \chi \dot{\pi}_0) + \frac{\chi}{2} \mathbb{D}_0 e(v_0) : e(v_0) \Big)\,\mathrm{d}x$$
$$+ \int_0^t \int_{\Omega} \bar{f}_{\tau} \cdot (\dot{w}_{\tau} - \dot{u}_{\mathrm{D},\tau} - \chi \ddot{\tilde{u}}_{\mathrm{D},\tau})\,\mathrm{d}x\,\mathrm{d}s + \frac{1}{\chi} \int_0^t \int_{\Omega} \dot{u}_{\tau} \cdot (\dot{w}_{\tau} - \dot{u}_{\mathrm{D},\tau} - \chi \ddot{\tilde{u}}_{\mathrm{D},\tau})\,\mathrm{d}x\,\mathrm{d}s$$
$$+ \frac{1}{\chi} \int_0^t \int_{\Omega} \dot{w}_{\tau} \cdot (\dot{u}_{\mathrm{D},\tau} + \chi \ddot{\tilde{u}}_{\mathrm{D},\tau})\,\mathrm{d}x\,\mathrm{d}s + \chi \mathcal{R}(\alpha_0, \dot{\pi}_0)$$
$$+ \int_0^t \int_{\Omega} \big(\mathbb{C}(\underline{\alpha}_{\tau}) \bar{e}_{\mathrm{el},\tau} + \mathbb{D}(\underline{\alpha}_{\tau}) \dot{e}_{\mathrm{el},\tau} \big) : (e(\dot{u}_{\mathrm{D},\tau}) + \chi e(\ddot{\tilde{u}}_{\mathrm{D},\tau})))\,\mathrm{d}x\,\mathrm{d}s.$$

The assert follows by Hölder's inequality, Proposition 5, and the assumptions on σ_{YLD} (see Subsection 2.4). □

6. Convergence and proof of Theorem 2.6

[Compactness] Under the assumptions of Theorem 2.6, there exist α, e_{el}, π, and u such that $(u(t), e_{\mathrm{el}}(t), \pi(t)) \in \mathscr{A}(u_{\mathrm{D}}(t))$ for every $t \in [0, T]$ (see (3)), the initial conditions (16) are satisfied, and up to the extraction of a (non-relabeled) subsequence, there holds

$$\alpha_{\tau} \to \alpha \quad \text{weakly* in } H^1(0,T; L^2(\Omega)) \cap L^{\infty}(0,T; W^{1,p}(\Omega)), \tag{69a}$$
$$e_{\mathrm{el},\tau} \to e_{\mathrm{el}} \quad \text{weakly* in } W^{1,\infty}(0,T; L^2(\Omega; \mathbb{R}^{d\times d}_{\mathrm{sym}})), \tag{69b}$$
$$\pi_{\tau} \to \pi \quad \text{weakly* in } W^{1,\infty}(0,T; \mathcal{M}_b(\Omega \cup \Gamma_{\mathrm{D}}; \mathbb{R}^{d\times d}_{\mathrm{dev}})), \tag{69c}$$
$$u_{\tau} \to u \quad \text{weakly* in } W^{1,\infty}(0,T; BD(\Omega; \mathbb{R}^d)) \cap H^1(0,T; L^2(\Omega; \mathbb{R}^d)), \tag{69d}$$
$$\underline{\alpha}_{\tau} \to \alpha \text{ and } \bar{\alpha}_{\tau} \to \alpha \quad \text{weakly* in } L^{\infty}((0,T) \times \Omega), \tag{69e}$$
$$\bar{e}_{\mathrm{el},\tau} \to e_{\mathrm{el}} \quad \text{weakly* in } L^{\infty}(0,T; L^2(\Omega; \mathbb{R}^{d\times d}_{\mathrm{sym}})), \tag{69f}$$
$$\tilde{u}_{\tau} \to u \quad \text{weakly in } H^2(0,T; L^2(\Omega; \mathbb{R}^d)). \tag{69g}$$

Proof. Properties (69a)–(69d) are a consequence of Propositions 5 and 5. The admissibility condition (C1) (see Definition 2.6) follows by the same argument as in [22, Lemma 2.1]. Additionally, by Proposition 5 there holds

$$u_\tau \to u \quad \text{weakly* in } W^{1,\infty}(0,T;L^2(\Omega;\mathbb{R}^d)), \tag{70}$$

and there exist $\check{\alpha}, \hat{\alpha} \in L^\infty((0,T)\times\Omega)$, and $\hat{e} \in L^\infty(0,T;L^2(\Omega;\mathbb{R}^{d\times d}_{\rm sym}))$ such that

$$\bar{\alpha}_\tau \to \check{\alpha} \ \text{ and } \ \underline{\alpha}_\tau \to \hat{\alpha} \quad \text{weakly* in } L^\infty((0,T)\times\Omega)$$

and

$$\bar{e}_{{\rm el},\tau} \to \hat{e} \quad \text{weakly* in } L^\infty(0,T;L^2(\Omega;\mathbb{R}^{d\times d}_{\rm sym})).$$

Additionally by Proposition 5 there exists a map $\hat{u} \in H^2(0,T;L^2(\Omega;\mathbb{R}^d))$ such that, up to the extraction of a (non-relabeled) subsequence,

$$\widetilde{u}_\tau \to \hat{u} \quad \text{weakly in } H^2(0,T;L^2(\Omega;\mathbb{R}^d)). \tag{71}$$

By the compact embeddings of $W^{1,\infty}(0,T;L^2(\Omega;\mathbb{R}^d))$ and $H^2(0,T;L^2(\Omega;\mathbb{R}^d))$ into $C_{\rm w}(0,T;L^2(\Omega;\mathbb{R}^d))$, we deduce

$$u_\tau(t) \to u(t) \quad \text{weakly in } L^2(\Omega;\mathbb{R}^d), \tag{72}$$

and

$$\widetilde{u}_\tau(t) \to \hat{u}(t) \quad \text{weakly in } L^2(\Omega;\mathbb{R}^d), \tag{73}$$

for every $t\in[0,T]$. To complete the proof of (69) , it remains to show that $\check{\alpha} = \hat{\alpha} = \alpha$, $\hat{e} = e_{\rm el}$, and $\hat{u} = u$.

We proceed by showing this last equality; the proof of the other two identities is analogous. Fix $k \in \{1,\dots,T/\tau\}$, and $t \in ((k-1)\tau, k\tau]$. Then, using the fact that

$$\dot{\widetilde{u}}_\tau(t) = \frac{(t-(k-1)\tau)}{\tau}\delta u_\tau^k + \Big(1 - \frac{(t-(k-1)\tau)}{\tau}\Big)\delta u_\tau^{k-1}$$

for every $t \in ((k-1)\tau, k\tau]$, we have

$$\begin{aligned}\int_0^T \|\dot{\widetilde{u}}_\tau(t) - \dot{u}_\tau(t)\|^2_{L^2(\Omega;\mathbb{R}^d)}{\rm d}t &= \sum_{k=1}^N \int_{(k-1)\tau}^{k\tau} \|\dot{\widetilde{u}}_\tau(t) - \dot{u}_\tau(t)\|^2_{L^2(\Omega;\mathbb{R}^d)}{\rm d}t \\ = \tau^2\sum_{k=1}^N \int_{(k-1)\tau}^{k\tau} (1-\overline{\alpha}_\tau(t))^2\,{\rm d}t \left\|\frac{\dot{u}_\tau - \dot{u}_\tau(\cdot-\tau)}{\tau}\right\|^2 &= \frac{\tau^2}{3}\sum_{k=1}^N \tau\|\delta^2 u_k\|^2_{L^2(\Omega;\mathbb{R}^d)} \le C\tau^2,\end{aligned} \tag{74}$$

where the last inequality follows by Proposition 5. The assert follows then by combining (72), (73), and (74). □

[Strong convergence of the elastic strains] Let $e_{\rm el}$ be the map identified in Proposition 6. Under the assumptions of Theorem 2.6, there holds

$$e_{{\rm el},\tau} \to e_{\rm el} \quad \text{strongly in } H^1(0,T;L^2(\Omega;\mathbb{R}^{d\times d}_{\rm sym})), \tag{75}$$

and

$$\bar{e}_{{\rm el},\tau}(t) \to e_{\rm el}(t) \quad \text{strongly in } L^2(\Omega;\mathbb{R}^{d\times d}_{\rm sym})) \quad \text{for a.e. } t\in[0,T]. \tag{76}$$

Proof. For $k \in \{1,\dots,T/\tau\}$, denote by $\delta e_{\rm el}(k\tau)$ the quantity

$$\delta e_{\rm el}(k\tau) := \frac{e_{\rm el}(k\tau) - e_{\rm el}((k-1)\tau)}{\tau},$$

and by $\bar{e}^\tau_{\rm el}$, $e^\tau_{\rm el}$ the forward-piecewise constant and the affine interpolants between the values $\{e(k\tau)\}_{k=1,\dots,T/\tau}$ (see (49) and (51)). Let $\delta u(k\tau)$, $\delta\pi(k\tau)$, $\bar{u}^\tau$, u^τ, $\bar{\pi}^\tau$, and π^τ be defined analogously. Note that here we cannot directly use the values at time t, for this would prevent relation (81) to hold. Here, the pointwise value of π is simply that of its right-continuous representative.

Fix $k \in \{1,\dots,T/\tau\}$. We proceed by testing the time-discrete equilibrium equation (37a) by $\delta u_\tau^k - \delta u(k\tau)$. On the one hand, by Lemma 5, we have

$$\begin{aligned}&\int_\Omega \rho\delta^2 u_\tau^k\cdot(\delta u_\tau^k - \delta u(k\tau))\,{\rm d}x + [{\rm dev}\,\sigma_\tau^k : (\delta\pi_\tau^k - \delta\pi(k\tau))](\Omega\cup\Gamma_{\rm D}) \\ &\quad + \int_\Omega \sigma_\tau^k : (\delta e^k_{{\rm el},\tau} - \delta e_{\rm el}(k\tau))\,{\rm d}x - \int_\Omega f_\tau^k\cdot(\delta u_\tau^k - \delta u(k\tau))\,{\rm d}x = 0.\end{aligned} \tag{77}$$

On the other hand, Lemma 5 yields

$$[{\rm dev}\,\sigma_\tau^k : (\delta\pi_\tau^k - \delta\pi(k\tau))](\Omega\cup\Gamma_{\rm D}) \ge \mathcal{R}(\alpha_\tau^{k-1}, \delta\pi_\tau^k) - \mathcal{R}(\alpha_\tau^{k-1}, \delta\pi(k\tau)). \tag{78}$$

By combining (77) and (78), we obtain

$$\int_\Omega \sigma_\tau^k : (\delta e_{\mathrm{el},\tau}^k - \delta e(k\tau))\,\mathrm{d}x \le \mathcal{R}(\alpha_\tau^{k-1}, \delta\pi(k\tau)) - \mathcal{R}(\alpha_\tau^{k-1}, \delta\pi_\tau^k) + \int_\Omega (f_\tau^k - \rho\delta^2 u_\tau^k)\cdot(\delta u_\tau^k - \delta u(k\tau))\,\mathrm{d}x. \tag{79}$$

In view of the definition of σ_k there holds

$$\begin{aligned}\int_\Omega \sigma_\tau^k : (\delta e_{\mathrm{el},\tau}^k - \delta e_{\mathrm{el}}(k\tau))\,\mathrm{d}x &= \int_\Omega \mathbb{C}(\alpha_\tau^{k-1})(e_{\mathrm{el},\tau}^k - e_{\mathrm{el}}(k\tau)) : (\delta e_{\mathrm{el},\tau}^k - \delta e_{\mathrm{el}}(k\tau))\,\mathrm{d}x\\ &+ \int_\Omega \mathbb{D}(\alpha_\tau^{k-1})(\delta e_{\mathrm{el},\tau}^k - \delta e_{\mathrm{el}}(k\tau)):(\delta e_{\mathrm{el},\tau}^k - \delta e_{\mathrm{el}}(k\tau)) + \mathbb{D}(\alpha_\tau^{k-1})\delta e_{\mathrm{el}}(k\tau):(\delta e_{\mathrm{el},\tau}^k - \delta e_{\mathrm{el}}(k\tau))\,\mathrm{d}x\\ &+ \int_\Omega \mathbb{C}(\alpha(k\tau)) e_{\mathrm{el}}(k\tau) : (\delta e_{\mathrm{el},\tau}^k - \delta e_{\mathrm{el}}(k\tau))\,\mathrm{d}x\\ &- \int_\Omega (\mathbb{C}(\alpha(k\tau)) - \mathbb{C}(\alpha_\tau^{k-1})) e_{\mathrm{el}}(k\tau) : (\delta e_{\mathrm{el},\tau}^k - \delta e_{\mathrm{el}}(k\tau))\,\mathrm{d}x.\end{aligned} \tag{80}$$

Let now $n \in \{1, \dots, T/\tau\}$. By the monotonicity of $\mathbb{C}$ in the Löwner order, arguing as in the proof of (57), we deduce

$$\tau\sum_{k=1}^n \int_\Omega \mathbb{C}(\alpha_\tau^{k-1})(e_{\mathrm{el},\tau}^k - e_{\mathrm{el}}(k\tau)) : (\delta e_{\mathrm{el},\tau}^k - \delta e_{\mathrm{el}}(k\tau))\,\mathrm{d}x \ge \int_\Omega \frac{1}{2}\mathbb{C}(\alpha_\tau^n)(e_{\mathrm{el},\tau}^n - e_{\mathrm{el}}(n\tau)) : (e_{\mathrm{el},\tau}^n - \delta e_{\mathrm{el}}(n\tau))\,\mathrm{d}x. \tag{81}$$

Multiplying (79) by τ, and summing for $k = 1, \dots, T/\tau$, in view of (80) and (81), we obtain the estimate

$$\begin{aligned}&\int_\Omega \frac{1}{2}\mathbb{C}(\underline{\alpha}_\tau(T))(\bar e_{\mathrm{el},\tau}(T) - \bar e_{\mathrm{el}}^\tau(T)) : (\bar e_{\mathrm{el},\tau}(T) - \bar e_{\mathrm{el}}^\tau(T))\,\mathrm{d}x\\ &\quad + \int_0^T\int_\Omega \mathbb{D}(\underline{\alpha}_\tau)(\dot e_{\mathrm{el},\tau} - \dot e_{\mathrm{el}}^\tau) : (\dot e_{\mathrm{el},\tau} - \dot e_{\mathrm{el}}^\tau)\,\mathrm{d}x\,\mathrm{d}s\\ &\le \tau\sum_{k=1}^{T/\tau} \mathcal{R}(\alpha_\tau^{k-1}, \delta\pi(k\tau)) - D_{\mathcal{R}}(\alpha_\tau;\pi_\tau;0,T) + \int_0^T\int_\Omega (\bar f_\tau - \ddot{\tilde u}_\tau)\cdot(\dot u_\tau - \dot u^\tau)\,\mathrm{d}x\,\mathrm{d}s\\ &\quad - \int_0^T\int_\Omega \mathbb{D}(\underline{\alpha}_\tau)\dot e_{\mathrm{el}}^\tau : (\dot e_{\mathrm{el},\tau} - \dot e_{\mathrm{el}}^\tau)\,\mathrm{d}x\,\mathrm{d}s - \int_0^T\int_\Omega \mathbb{C}(\overline{\alpha}^\tau)\bar e_{\mathrm{el}}^\tau : (\dot e_{\mathrm{el},\tau} - \dot e_{\mathrm{el}}^\tau)\,\mathrm{d}x\,\mathrm{d}s\\ &\quad + \int_0^T\int_\Omega \big(\mathbb{C}(\overline{\alpha}^\tau) - \mathbb{C}(\underline{\alpha}_\tau)\big)\bar e_{\mathrm{el}}^\tau : (\dot e_{\mathrm{el},\tau} - \dot e_{\mathrm{el}}^\tau)\big)\,\mathrm{d}x\,\mathrm{d}s.\end{aligned}$$

By Proposition 6 we infer that

$$\begin{aligned}&\limsup_{\tau\to0}\int_0^T\int_\Omega \mathbb{D}(\overline{\alpha}_\tau)(\dot e_{\mathrm{el},\tau} - \dot e_{\mathrm{el}}) : (\dot e_{\mathrm{el},\tau} - \dot e_{\mathrm{el}})\,\mathrm{d}x\,\mathrm{d}s\\ &+ \limsup_{\tau\to0}\Big\{\int_0^T \mathcal{R}(\underline{\alpha}_\tau, \dot\pi^\tau)\,\mathrm{d}s - D_{\mathcal{R}}(\alpha_\tau;\pi_\tau;0,T) + \int_0^T\int_\Omega (\bar f_\tau - \ddot{\tilde u}_\tau)\cdot(\dot u_\tau - \dot u^\tau)\,\mathrm{d}x\,\mathrm{d}s\\ &\quad - \int_0^T\int_\Omega \mathbb{D}(\underline{\alpha}_\tau)\dot e_{\mathrm{el}}^\tau : (\dot e_{\mathrm{el},\tau} - \dot e_{\mathrm{el}}^\tau)\,\mathrm{d}x\,\mathrm{d}s - \int_0^T\int_\Omega \mathbb{C}(\underline{\alpha}^\tau)\bar e_{\mathrm{el}}^\tau : (\dot e_{\mathrm{el},\tau} - \dot e_{\mathrm{el}}^\tau)\,\mathrm{d}x\,\mathrm{d}s\\ &\quad + \int_0^T\int_\Omega \big(\mathbb{C}(\bar\alpha^\tau) - \mathbb{C}(\underline{\alpha}_\tau)\big)\bar e_{\mathrm{el}}^\tau : (\dot e_{\mathrm{el},\tau} - \dot e_{\mathrm{el}}^\tau)\big)\,\mathrm{d}x\,\mathrm{d}s\Big\}.\end{aligned}$$

Since $u \in H^2(0,T;L^2(\Omega;\mathbb{R}^d))$ and $e_{\mathrm{el}} \in W^{1,\infty}(0,T;L^2(\Omega;\mathbb{R}^{d\times d}_{\mathrm{sym}}))$, it follows that

$$u^\tau \to u \quad \text{strongly in } L^2((0,T)\times\Omega;\mathbb{R}^d), \tag{82}$$

and

$$\bar e_{\mathrm{el}}^\tau \to \bar e_{\mathrm{el}} \quad \text{strongly in } L^2((0,T)\times\Omega;\mathbb{R}^{d\times d}_{\mathrm{sym}}). \tag{83}$$

Additionally, by the definition of the affine interpolants,

$$\dot e_{\mathrm{el}}^\tau \to \dot e_{\mathrm{el}} \quad \text{strongly in } L^2((0,T)\times\Omega;\mathbb{R}^{d\times d}_{\mathrm{sym}}), \tag{84}$$

$$\dot\pi^\tau \to \dot\pi \quad \text{strongly in } L^1(0,T;\mathcal{M}_b(\Omega\cup\Gamma_{\mathrm{D}};\mathbb{R}^{d\times d}_{\mathrm{dev}})). \tag{85}$$

By (69a) and by the Aubin-Lions Lemma, up to the extraction of a (non-relabeled) subsequence,

$$\alpha_\tau \to \alpha \quad \text{strongly in } C([0,T]\times\bar{\Omega}). \tag{86}$$

Since $\alpha \in H^1(0,T;L^2(\Omega)) \cap L^\infty(0,T;W^{1,p}(\Omega))$,

$$\bar{\alpha}^\tau,\, \underline{\alpha}^\tau \to \alpha \quad \text{strongly in } L^2((0,T)\times\Omega). \tag{87}$$

Thus, by the Dominated Convergence Theorem, we deduce that

$$\mathbb{C}(\bar{\alpha}^\tau)\bar{e}_{\rm el}^\tau \to \mathbb{C}(\alpha)e_{\rm el} \quad \text{strongly in } L^2((0,T)\times\Omega;\mathbb{R}^{d\times d}_{\rm sym}), \tag{88}$$

$$\big(\mathbb{C}(\bar{\alpha}^\tau) - \mathbb{C}(\underline{\alpha}_\tau)\big)\bar{e}_{\rm el}^\tau \to 0 \quad \text{strongly in } L^2((0,T)\times\Omega;\mathbb{R}^{d\times d}_{\rm sym}), \tag{89}$$

$$\mathbb{D}(\underline{\alpha}_\tau)\dot{e}_{\rm el}^\tau \to \mathbb{D}(\alpha)\dot{e}_{\rm el} \quad \text{strongly in } L^2((0,T)\times\Omega;\mathbb{R}^{d\times d}_{\rm sym}). \tag{90}$$

Finally, by the assumptions on f, we have

$$\bar{f}_\tau \to f \quad \text{strongly in } L^2((0,T)\times\Omega;\mathbb{R}^d). \tag{91}$$

By combining (82)–(91) we conclude that

$$\begin{aligned}\limsup_{\tau\to 0}\int_0^T\!\!\int_\Omega \mathbb{D}(\underline{\alpha}_\tau)(\dot{e}_{{\rm el},\tau}-\dot{e}_{\rm el}) : (\dot{e}_{{\rm el},\tau}-\dot{e}_{\rm el})\,{\rm d}x\,{\rm d}s \\ \le \limsup_{\tau\to 0}\int_0^T \mathcal{R}(\bar{\alpha}_\tau,\dot{\pi}^\tau)\,{\rm d}s - \liminf_{\tau\to 0} D_{\mathcal{R}}(\alpha_\tau;\pi_\tau;0,T).\end{aligned} \tag{92}$$

Arguing as in [22, Theorem 7.1], since $\pi \in W^{1,\infty}(0,T;\mathcal{M}_b(\Omega\cup\Gamma_{\rm D};\mathbb{R}^{d\times d}_{\rm dev}))$ we deduce the uniform bound

$$\int_0^T \|\dot{\pi}^\tau\|_{\mathcal{M}_b(\Omega\cup\Gamma_{\rm D};\mathbb{R}^{d\times d}_{\rm dev})} = \tau\sum_{k=1}^{T/\tau}\|\delta\pi(k\tau)\|_{\mathcal{M}_b(\Omega\cup\Gamma_{\rm D};\mathbb{R}^{d\times d}_{\rm dev})} \le \int_0^T \|\dot{\pi}\|_{\mathcal{M}_b(\Omega\cup\Gamma_{\rm D};\mathbb{R}^{d\times d}_{\rm dev})}\,{\rm d}s \le C. \tag{93}$$

Hence, by (93) and by the continuity and monotonicity of $\sigma_{\rm YLD}(\cdot)$ (see Subsection 2.4), there holds

$$\begin{aligned}&\limsup_{\tau\to 0}\int_0^T \mathcal{R}(\bar{\alpha}_\tau,\dot{\pi}^\tau)\,{\rm d}s \le \limsup_{\tau\to 0}\int_0^T \mathcal{R}(\alpha_\tau,\dot{\pi}^\tau)\,{\rm d}s \\ &\le \limsup_{\tau\to 0}\Big\{\int_0^T \mathcal{R}(\alpha,\dot{\pi}^\tau)\,{\rm d}s + \Big|\int_0^T (\mathcal{R}(\alpha_\tau,\dot{\pi}^\tau)-\mathcal{R}(\alpha,\dot{\pi}^\tau))\,{\rm d}s\Big|\Big\} \\ &\le \limsup_{\tau\to 0}\int_0^T \mathcal{R}(\alpha,\dot{\pi}^\tau)\,{\rm d}s + C\limsup_{\tau\to 0}\|\sigma_{\rm YLD}(\alpha_\tau)-\sigma_{\rm YLD}(\alpha)\|_{L^\infty((0,T)\times\Omega)} = \int_0^T \mathcal{R}(\alpha,\dot{\pi})\,{\rm d}s,\end{aligned} \tag{94}$$

where the last step follows by (85).

To complete the proof of (75), it remains to show that

$$D_{\mathcal{R}}(\alpha;\pi;0,T) \le \liminf_{\tau\to 0} D_{\mathcal{R}}(\alpha_\tau;\pi_\tau;0,T). \tag{95}$$

Let $0 < t_0 < t_1 < \cdots < t_n \le T$. By the definition of $D_{\mathcal{R}}$, we have

$$\begin{aligned}D_{\mathcal{R}}(\alpha_\tau;\pi_\tau;0,T) &\ge \sum_{j=1}^{T/\tau}\mathcal{R}(\alpha_\tau(t_j),\pi_\tau(t_j)-\pi_\tau(t_{j-1})) \ge \sum_{j=1}^{T/\tau}\mathcal{R}(\alpha(t_j),\pi_\tau(t_j)-\pi_\tau(t_{j-1})) \\ &\quad - \tau\sum_{j=1}^{T/\tau}\|\sigma_{\rm YLD}(\alpha_\tau(t_j))-\sigma_{\rm YLD}(\alpha(t_j))\|_{L^\infty(\Omega)}\|\dot{\pi}_\tau\|_{L^\infty(0,T;\mathcal{M}_b(\Omega\cup\Gamma_{\rm D};\mathbb{R}^{d\times d}_{\rm dev}))}.\end{aligned}$$

Now, by (69c) and (86),

$$\lim_{\tau\to 0}\tau\sum_{j=1}^{T/\tau}\|\sigma_{\rm YLD}(\alpha_\tau(t_j))-\sigma_{\rm YLD}(\alpha(t_j))\|_{L^\infty(\Omega)}\|\dot{\pi}_\tau\|_{L^\infty(0,T;\mathcal{M}_b(\Omega\cup\Gamma_{\rm D};\mathbb{R}^{d\times d}_{\rm dev}))} = 0.$$

Thus, by (69c),

$$\liminf_{\tau\to 0} D_{\mathcal{R}}(\alpha_\tau;\pi_\tau;0,T) \ge \liminf_{\tau\to 0}\sum_{j=1}^{T/\tau}\mathcal{R}(\alpha(t_j),\pi_\tau(t_j)-\pi_\tau(t_{j-1})) \ge \sum_{j=1}^{T/\tau}\mathcal{R}(\alpha(t_j),\pi(t_j)-\pi(t_{j-1})).$$

By the arbitrariness of the partition, we deduce (95), which in turn yields (75).

Property (76) follows arguing exactly as in the proof of (74). □

Let us now conclude the proof of Theorem 2.6.

Proof of Theorem 2.6. Let $(u, e_{\rm el}, \pi, \alpha)$ be the limit quadruple identified in Proposition 6. By Proposition 6 we already know that condition (C1) in Definition 2.6 is fulfilled. For convenience of the reader we subdivide the proof of the remaining conditions into three steps.
Step 1: We first show that the limit quadruple satisfies the equilibrium equation (14a). In view of (37a) we have

$$\rho\ddot{\tilde{u}}_\tau - \operatorname{div}\left(\mathbb{C}(\underline{\alpha}_\tau)\bar{e}_{{\rm el},\tau} + \mathbb{D}(\underline{\alpha}_\tau)\dot{e}_{{\rm el},\tau}\right) = \bar{f}_\tau$$

for a.e. $x \in \Omega$ and $t \in [0, T]$, and for all $\tau > 0$. In particular, for all $\varphi \in C^\infty_c(0, T; C^\infty_c(\Omega))$ there holds

$$\int_0^T\int_\Omega \rho\ddot{\tilde{u}}_\tau\cdot\varphi + (\mathbb{C}(\underline{\alpha}_\tau)\bar{e}_{{\rm el},\tau} + \mathbb{D}(\underline{\alpha}_\tau)\dot{e}_{{\rm el},\tau}) : e(\varphi)\,{\rm d}x\,{\rm d}t = \int_0^T\int_\Omega \bar{f}_\tau\cdot\varphi\,{\rm d}x\,{\rm d}t.$$

By (69e-g) and (91), we infer that

$$\int_0^T\int_\Omega \rho\ddot{u}\cdot\varphi + (\mathbb{C}(\alpha)e_{\rm el} + \mathbb{D}(\alpha)\dot{e}_{\rm el}) : e(\varphi)\,{\rm d}x\,{\rm d}t = \int_0^T\int_\Omega f\cdot\varphi\,{\rm d}x\,{\rm d}t,$$

which in turn yields (14a) for a.e. $x \in \Omega$, and $t \in [0, T]$. In particular, (69g) guarantees that $u(0) = u^0$, and $\dot{u}(0) = v_0$.

Step 2: The limit energy inequality is a direct consequence of (52), Propositions 6 and 6, and (95).
Step 3: We now pass to the limit in the discrete damage law. In view of (37c), for every $k \in \{1, \ldots, T/\tau\}$ we deduce the inequality

$$\int_\Omega \Big(\phi^\circ(\alpha^k_\tau, \alpha^{k-1}_\tau) + \operatorname{div}\left(|\nabla\alpha^k_\tau|^{p-2}\nabla\alpha^k_\tau\right) - \frac{1}{2}\mathbb{C}^\circ(\alpha^k_\tau, \alpha^{k-1}_\tau)e^k_{{\rm el},\tau} : e^k_{{\rm el},\tau} - \eta\delta\alpha^k_\tau\Big)(\varphi - \delta\alpha^k_\tau)\,{\rm d}x = 0$$

for all $\varphi \in W^{1,p}(\Omega)$ such that $\varphi(x) \le 0$ for a.e. $x \in \Omega$. Thus, summing in k, we conclude that

$$\begin{aligned}
&\int_0^T\int_\Omega \Big(\phi^\circ(\bar{\alpha}_\tau, \underline{\alpha}_\tau)\varphi - |\nabla\bar{\alpha}_\tau|^{p-2}\nabla\bar{\alpha}_\tau\cdot\nabla\varphi - \frac{1}{2}\mathbb{C}^\circ(\bar{\alpha}_\tau, \underline{\alpha}_\tau)\bar{e}_{{\rm el},\tau} : \bar{e}_{{\rm el},\tau}\varphi - \eta\dot{\alpha}_\tau\varphi\Big)\,{\rm d}x\,{\rm d}t \\
&\quad\le \int_\Omega \Big(\phi(\alpha_\tau(T)) - \phi(\alpha_0) - \frac{\kappa}{p}|\nabla\bar{\alpha}_\tau(T)|^p + \frac{\kappa}{p}|\nabla\alpha_0|^p\Big)\,{\rm d}x \\
&\qquad - \int_0^T\int_\Omega \frac{1}{2}\Big(\mathbb{C}^\circ(\bar{\alpha}_\tau, \underline{\alpha}_\tau)\bar{e}_{{\rm el},\tau} : \bar{e}_{{\rm el},\tau}\Big)\dot{\alpha}_\tau\,{\rm d}x\,{\rm d}t - \int_0^T\int_\Omega \eta\dot{\alpha}^2_\tau\,{\rm d}x\,{\rm d}t\,.
\end{aligned}$$

Condition (C4) in Definition 2.6 follows then in view of Propositions 6 and 6. □

Acknowledgments: This research received a partial support from from the Vienna Science and Technology Fund (WWTF) project MA14-009, and from the Austrian Science Fund (FWF) projects F 65 and P 27052, from the Czech Sci. Foundation (CSF) projects 17-04301S and 18-03834S, from the Austrian-Czech project I 2375/16-34894L (FWF/CSF), as well as from the institutional support RVO:61388998 (ČR).

References

[1] R. Alessi, J. J. Marigo, and S. Vidoli, Gradient damage models coupled with plasticity and nucleation of cohesive cracks, Arch. Ration. Mech. Anal. **214**(2), 575–615 (2014).

[2] L. Ambrosio, N. Fusco, and D. Pallara, Functions of bounded variation and free discontinuity problems (The Clarendon Press, Oxford Univ. Press, New York, 2000).

[3] L. Ambrosio and V. M. Tortorelli, Approximation of functional depending on jumps via by elliptic functionals via Γ-convergence, Comm. Pure Appl. Math. **43**, 999–1036 (1990).

[4] L. Ambrosio and V. M. Tortorelli, On the approximation of free discontinuity problems, Boll. Unione Mat. Ital. **7**, 105–123 (1992).

[5] G. Anzellotti and S. Luckhaus, Dynamical evolution of elasto-perfectly plastic bodies, Appl. Math. Optim. **15**, 121–140 (1987).

[6] J. F. Babadjian and M. G. Mora, Approximation of dynamic and quasi-static evolution problems in elasto-plasticity by cap models, Quart. Appl. Math. **73**, 265–316 (2015).

[7] J. Babadjian and C. Mifsud, Hyperbolic structure for a simplified model of dynamical perfect plasticity, Arch. Ration. Mech. Anal. **223**, 761–815 (2017).

[8] S. Bartels and T. Roubíček, Numerical approaches to thermally coupled perfect plasticity, Numer. Meth. Partial Diff. Equations **29**, 1837–1863 (2013).

[9] A. Bedford, Hamilton's Principle in Continuum Mechanics (Pitman, Boston, 1985).

[10] E. Bonetti, E. Rocca, R. Rossi, and M. Thomas, A rate-independent gradient system in damage coupled with plasticity via structured strains, in: Gradient flows: from theory to application, edited by B. Düring, C. B. Schönlieb, and M. T. Wolfram, ESAIM Proc. Surveys Vol. 54 (EDP Sci., Les Ulis, 2016), pp. 54–69.

[11] B. Bourdin, G. A. Francfort, and J. J. Marigo, The variational approach to fracture, J. Elasticity **91**, 5–148 (2008).

[12] B. Bourdin, C. J. Larsen, and C. L. Richardson, A time-discrete model for dynamic fracture based on crack regularization, Int. J. Fract. **10**, 133–143 (2011).

[13] B. Bourdin, J. J. Marigo, C. Maurini, and P. Sicsic, Morphogenesis and propagation of complex cracks induced by thermal shocks, Phys. Rev. Lett. **112**, 014301 (2014).

[14] M. Caponi, Existence of solutions to a phase-field model of dynamic fracture with a crack-dependent dissipation, Preprint SISSA 06/2018/MATE.
[15] K. Chełmiński, Coercive approximation of viscoplasticity and plasticity, Asympt. Anal. **26**, 105–133 (2001).
[16] K. Chełmiński, Perfect plasticity as a zero relaxation limit of plasticity with isotropic hardening, Math. Meth. Appl. Sci. **24**, 117–136 (2001).
[17] S. Conti, M. Focardi, and F. Iurlano, Phase field approximation of cohesive fracture models, Ann. Inst. H. Poincaré Anal. Non Linéaire **33**(4), 1033–1067 (2016).
[18] V. Crismale, Globally stable quasistatic evolution for a coupled elastoplastic-damage model, ESAIM Control Optim. Calc. Var. **22**(3), 883–912 (2016).
[19] V. Crismale, Globally stable quasistatic evolution for strain gradient plasticity coupled with damage, Ann. Mat. Pura Appl. (4) **196**(2), 641–685 (2017).
[20] V. Crismale and G. Lazzaroni, Viscous approximation of quasistatic evolutions for a coupled elastoplastic-damage model, Calc. Var. Partial Differential Equations **55**(1), Art. 17, 54 (2016).
[21] V. Crismale and G. Orlando, A Reshetnyak-type lower semicontinuity result for linearised elasto-plasticity coupled with damage in $W^{1,n}$, NoDEA Nonlinear Differential Equations Appl. **25**(2), Art. 16, 20 (2018).
[22] G. Dal Maso, A. DeSimone, and M. G. Mora, Quasistatic evolution problems for linearly elastic-perfectly plastic materials, Arch. Ration. Mech. Anal. **180**, 237–291 (2006).
[23] G. Dal Maso and R. Scala, Quasistatic evolution in perfect plasticity as limit of dynamic processes, J. Dynam. Differential Equations **26**(4), 915–954 (2014).
[24] G. Dal Maso and F. Iurlano, Fracture models as Γ-limits of damage models, Commun. Pure Appl. Anal. **12**(4), 1657–1686 (2013).
[25] G. Dal Maso, G. Orlando, and R. Toader, Fracture models for elasto-plastic materials as limits of gradient damage models coupled with plasticity: the antiplane case, Calc. Var. Partial Differential Equations **55**(3), Art. 45, 39 (2016).
[26] E. Davoli and U. Stefanelli, Dynamic perfect plasticity as convex minimization, Submitted. (2018), Preprint available at `http://cvgmt.sns.it/paper/3229/`.
[27] B. Duan and S. Day, Inelastic strain distribution and seismic radiation from rupture of a fault kink, Journal of Geophysical Research: Solid Earth **113**(B12) (2008).
[28] M. Focardi, On the variational approximation of free-discontinuity problems in the vectorial case, Math. Models Methods Appl. Sci. **11**, 663–684 (2001).
[29] M. Focardi and F. Iurlano, Asymptotic analysis of Ambrosio-Tortorelli energies in linearized elasticity, SIAM J. Math. Anal. **46**(4), 2936–2955 (2014).
[30] G. Francfort and A. Giacomini, On periodic homogenization in perfect elasto-plasticity, J. Eur. Math. Soc. (JEMS) **16**, 409–461 (2014).
[31] F. Freddi and F. Iurlano, Numerical insight of a variational smeared approach to cohesive fracture, J. Mech. Phys. Solids **98**, 156–171 (2017).
[32] F. Gatuingt and G. Pijaudier-Cabot, Coupled damage and plasticity modelling in transient dynamic analysis of concrete, International Journal for Numerical and Analytical Methods in Geomechanics **26**(1), 1–24 (2002).
[33] A. Giacomini, Ambrosio-Tortorelli approximation of quasi-static evolution of brittle fractures, Calc. Var. Partial Differential Equations **22**, 129–172 (2005).
[34] C. Goffman and J. Serrin, Sublinear functions of measures and variational integrals, Duke Math. J. **31**, 159–178 (1964).
[35] F. Iurlano, A density result for GSBD and its application to the approximation of brittle fracture energies, Calc. Var. Partial Differential Equations **51**(1-2), 315–342 (2014).
[36] K. Kisiel, Dynamical model of viscoplasticity, in: Equadiff 2017, edited by D. Ševčovič and J. Urbán (Spectrum, 2017), pp. 29–36.
[37] R. Kohn and R. Temam, Dual spaces of stresses and strains, with applications to Hencky plasticity, Appl. Math. Optim. **10**, 1–35 (1983).
[38] C. J. Larsen, C. Ortner, and E. Süli, Existence of solution to a regularized model of dynamic fracture, Math. Models Meth. Appl. Sci. **20**, 1021–1048 (2010).
[39] G. Lazzaroni, R. Rossi, M. Thomas, and R. Toader, Some remarks on a model for rate-independent damage in thermo-visco-elastodynamics, J. Phys. Conf. Ser. **727**, 012009, 20 (2016).
[40] G. B. Maggiani and M. G. Mora, A dynamic evolution model for perfectly plastic plates, Math. Models Methods Appl. Sci. **26**, 1825–1864 (2016).
[41] A. Mielke and T. Roubíček, Rate-Independent Systems – Theory and Application (Springer, New York, 2015).
[42] M. G. Mora, Relaxation of the Hencky model in perfect plasticity, J. Math. Pures Appl. **106**, 725–743 (2016).
[43] R. Rossi, From visco to perfect plasticity in thermoviscoelastic materials, Preprint: arXiv:1609.07232.
[44] R. Rossi and M. Thomas, Coupling rate-independent and rate-dependent processes: existence results, SIAM J. Math. Anal. **49**(2), 1419–1494 (2017).
[45] T. Roubíček, Geophysical models of heat and fluid flow in damageable poro-elastic continua, Contin. Mech. Thermodyn. **29**, 625–646 (2017).
[46] T. Roubíček, Models of dynamic damage and phase-field fracture, and their various time discretisations, in: Topics in Applied Analysis and Optimisation, edited by J. F. Rodrigues and M. Hintermüller. CIM Series in Math. Sci. (Springer, submitted).
[47] T. Roubíček and U. Stefanelli, Thermodynamics of elastoplastic porous rocks at large strains towards earthquake modeling, Submitted (2018).
[48] T. Roubíček and J. Valdman, Perfect plasticity with damage and healing at small strains, its modeling, analysis, and computer implementation, SIAM J. Appl. Math. **76**(1), 314–340 (2016).
[49] T. Roubíček and J. Valdman, Stress-driven solution to rate-independent elasto-plasticity with damage at small strains and its computer implementation, Math. Mech. Solids **22**(6), 1267–1287 (2017).
[50] R. Temam, Problèmes mathématiques en plasticité (Montrouge, Gauthier-Villars, 1983).
[51] M. Thomas and A. Mielke, Damage of nonlinearly elastic materials at small strain: Existence and regularity results, Z. Angew. Math. Mech. **90**(2), 88–112.
[52] B. Xu, D. Zou, X. Kong, Z. Hu, and Y. Zhou, Dynamic damage evaluation on the slabs of the concrete faced rockfill dam with the plastic-damage model, Computers and Geotechnics **65**, 258–265 (2015).
[53] C. H. Zhang and D. Gross, Dynamic behavior of damaged brittle solids, Z. Angew. Math. Mech. **79**(S1), 131–134.

(E. Davoli) FACULTY OF MATHEMATICS, UNIVERSITY OF VIENNA, OSKAR-MORGENSTERN-PLATZ 1, A-1090 VIENNA, AUSTRIA
E-mail address: elisa.davoli@univie.ac.at

(T. Roubíček) MATHEMATICAL INSTITUTE, CHARLES UNIVERSITY, SOKOLOVSKÁ 83, CZ-186 75 PRAHA 8, CZECH REPUBLIC, AND INSTITUTE OF THERMOMECHANICS, CZECH ACADEMY OF SCIENCES, DOLEJŠKOVA 5, CZ-182 00 PRAHA 8, CZECH REPUBLIC
E-mail address: tomas.roubicek@mff.cuni.cz

(U. Stefanelli) FACULTY OF MATHEMATICS, UNIVERSITY OF VIENNA & ISTITUTO DI MATEMATICA APPLICATA E TECNOLOGIE INFORMATICHE *E. Magenes*, CONSIGLIO NAZIONALE DELLE RICERCHE, VIA FERRATA 1, I-27100 PAVIA, ITALY, AND OSKAR-MORGENSTERN-PLATZ 1, A-1090 VIENNA, AUSTRIA
E-mail address: ulisse.stefanelli@univie.ac.at