# Sharp regularity estimates for solutions of the continuity equation drifted by Sobolev vector fields

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**Abstract**

The aim of this note is to prove a sharp regularity estimate for solutions of the continuity equation associated to vector fields of class $W^{1,p}$ with $p > 1$. Regularity is understood with respect to a *log-Sobolev* functionals, that could be seen as a version of the Gagliardo semi-norms measuring the "logarithmic derivative" of a function.

*Key words*: Ordinary differential equations with non smooth vector fields; continuity equation; transport equation; regular Lagrangian flow; $BV$ function; log-Sobolev space; Bressan's mixing conjecture.

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## Contents



## 1 Introduction and main result

This paper is concerned with the regularity of solutions to the continuity equation over the $d$-dimensional Euclidean space:

$$\begin{cases} \partial_t u + \operatorname{div}(bu) = 0, \\ u_0 = \bar{u}, \end{cases} \qquad \text{in } [0,T] \times \mathbb{R}^d, \tag{CE}$$

where $b : [0,T] \times \mathbb{R}^d \to \mathbb{R}^d$ is a time dependent vector field, $\bar{u} : \mathbb{R}^d \to \mathbb{R}$ is the initial data and $u : [0,T] \times \mathbb{R}^d \to \mathbb{R}^d$ is the unknown of the problem. The time interval $[0,T]$ could be possibly infinite, i.e. the choice $T = \infty$ is allowed.

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We are mainly interested in the study of the Cauchy problem (CE) under the Sobolev assumption on the drift $b_t$ and the *incompressibility condition*, that is to say,

$$\int_0^T \|b_s\|_{W^{1,p}(\mathbb{R}^d)}\,\mathrm{d}s < \infty, \qquad \text{and} \quad \operatorname{div} b_s(x) = 0 \qquad \text{for-}\mathscr{L}^{d+1} \text{ a.e } (t,x) \in [0,T]\times\mathbb{R}^d,$$

for some $p > 1$. Solutions are understood in the distributional sense in the class $L^\infty([0,T]\times\mathbb{R}^d)$, more precisely we are looking for maps $t \to u_t \in L^\infty(\mathbb{R}^d)$, continuous with respect to the weak-star topology (thus they are defined *for every* $t \in [0,T]$ allowing the Cauchy formulation of (CE)), such that, for every $\varphi \in C_c^\infty(\mathbb{R}^d)$ the function $t \to \int_{\mathbb{R}^d} \varphi u_t\,\mathrm{d}x$ is absolutely continuous and fulfills

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^d} \varphi u_t\,\mathrm{d}x = \int_{\mathbb{R}^d} b_t\cdot\nabla\varphi u_t\,\mathrm{d}x \qquad \text{for a.e. } t\in[0,T].$$

The just established setting is quite natural, both from the theoretical point of view, and for its applications to the study of nonlinear partial differential equations of the mathematical physics.

The Cauchy problem (CE) is strictly linked to the system of ordinary differential equations

$$\begin{cases} \dfrac{\mathrm{d}}{\mathrm{d}t}X(t,x) = b(t,X(t,x)), \\ X(0,x) = x, \quad \forall (t,x)\in[0,T]\times\mathbb{R}^d. \end{cases} \tag{ODE}$$

Indeed, when the vector field is regular enough (for instance globally bounded and Lipschitz in the spatial variable, uniformly in time) the classical Cauchy-Lipschitz theory grants the existence of a unique flow map $X : [0,T]\times\mathbb{R}^d \to \mathbb{R}^d$, an by the formula

$$(X_t)_\# \bar{u}\mathscr{L}^d = u_t\mathscr{L}^d,\text{[1]}$$

can be recovered the unique solution $u_t$ of the Cauchy problem (CE).

Such link between the problems (CE) and (ODE) is still present out of a smooth setting, but it is very subtle. The situation is complicated by the loss of point-wise uniqueness for solutions of (ODE) when studying ordinary differential equations associated to non-Lipschitz vector fields. To overcome these difficulties Ambrosio in [A04] introduced the notion of *regular Lagrangian flow* (see Definition 2.6) and established a link between the well-posedness in $L^\infty$ of the Cauchy problem (CE) and the existence and uniqueness for regular Lagrangian flows. The author also showed the well-posedness in $L^\infty$ for (CE) when the vector field has the $BV$ spatial regularity and bounded divergence (it means $\operatorname{div} b_t \ll \mathscr{L}^d$, with density in $L^\infty$). In this manner Ambrosio extended the celebrated result in [DPL89] by DiPerna and Lions, where it has been proven the existence and uniqueness in $L^\infty$ for solutions of the Cauchy problem (CE) associated to Sobolev drifts. This theory has been recently fully established by the second author in [Nguyen1] by mean of quantitative techniques.

In the last period the problem of quantifying the *propagation of regularity* and *the rate of "mixedness"* for solutions of (CE) has been received a lot of attentions. These two questions at an informal level can be interpreted as follow: we look at the evolution of suitable norms, or functionals, measuring the regularity of a function (reasonable choices are Sobolev norms, $BV$ norms or even weaker ones) or a "mixing" level of a function (reasonable choices are negative Sobolev norms or geometrical functionals as in [Br03] or [HSSS18]) along solutions of the Cauchy problem (CE). We refer to [ACM16, IKX14, HSSS18, Se13] for an overview on the topic of mixing, while we are going to focus most on the regularity side of the problem.

Looking at the regularity problem in the smooth setting the picture is quite clear. Assume for instance a uniform Lipschitz bound on $b$

$$|b_t(x) - b_t(y)| \le L|x-y| \qquad \forall x,y\in\mathbb{R}^d,\ \forall t\in[0,T],$$

[1] This is an identity between measures, where the left hand side is defined by $(X_t)_\#\bar{u}\mathscr{L}^d(E) := (u\mathscr{L}^d)((X_t)^{-1}(E))$ for every Borel set $E\subset\mathbb{R}^d$, and (1.1) is equivalent to

$$\int_{\mathbb{R}^d}\phi(x)u_t(x)dx = \int_{\mathbb{R}^d}\phi(X(t,x))u_0(x)dx \ \ \forall\ \phi\in C_b(\mathbb{R}^d).$$

a simple Grönwall's argument yields a bi-Lipschitz estimate for the flow $X_t$ with constant $e^{tL}$. Assuming the incompressibility condition and using (1.1) we get

$$\operatorname{lip} u_t \le e^{tL} \operatorname{lip} u_0 \qquad \forall t \in [0,T], \tag{1.2}$$

where $u_t$ is the unique solution of (CE) with initial data $u_0 \in \mathrm{Lip}(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d)$ and $\operatorname{lip} u_t$ denotes the Lipschitz constant of $u_t(\cdot)$. In other worlds the Lipschitz semi-norm increases at most exponentially in time. Moreover it can be seen that this estimate is sharp building a smooth divergence-free vector field admitting a solution that increases the Lipschitz constant with exponential rate (see [ACM16]).

Assuming only a Sobolev bound on the vector field the situation is much more complicated. Every Lipschitz or Sobolev regularity (even of fractional order) of the initial data, might be instantaneously lost during the time evolution as it has been shown in [ACM18, ACM16, ACM14]. However, a very weak notion of regularity seems to be propagated also in this wild case. A first result in this direction has been established by Crippa and De Lellis in [CDL08]. They obtained a quantitative Lusin-Lipschitz estimate at the level of Lagrangian flows that implies in turn the propagation of the "Lipexp" regularity. In [BJ15] the authors proved that a suitable singular operator remains bounded during the time evolution. They used this result to deduce a compactness theorem that allowed them to built a new theory of existence of solutions to the compressible Navier-Stokes equations. By mean of very sophisticated tools from harmonic analysis in [LF16] the author studied the behavior of the following functionals

$$\int_{\mathbb{R}^d} |\log(|\xi|)| |\hat{u}_t(\xi)|^2 \, \mathrm{d}\xi, \qquad \int_{\mathbb{R}^d} \log(|\xi|)^2 |\hat{u}_t(\xi)|^2 \, \mathrm{d}\xi \tag{1.3}$$

along the solutions of the Cauchy problem (CE). He proved that, under the incompressibility condition and the uniform $W^{1,p}$ bound (with $p > 1$) on the vector field, the first functional in (1.3) increases at most linearly in time. Assuming a better regularity on the drift, i.e $W^{1,p}$ bounds with $p \ge 2$, the second functional increases at most quadratically.

The main result in the present paper is a sharp characterization of the regularity for solutions of (CE) associated to incompressible Sobolev vector fields with exponent $p > 1$. We study the propagation of regularity by mean of the what we call *log-Sobolev functionals* of order $p > 0$ associated to a function $f \in L^2(\mathbb{R}^d)$:

$$\left( \int_{B_{1/2}} \int_{\mathbb{R}^d} \frac{|f(x+h) - f(x)|^2}{|h|^d} \frac{1}{\log(1/|h|)^{1-p}} \, \mathrm{d}x \, \mathrm{d}h \right)^{1/2}. \tag{1.4}$$

Let us compare (1.4) with the well-known Gagliardo semi-norm

$$\left( \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \frac{|f(x+h) - f(x)|^2}{|h|^{d+2s}} \, \mathrm{d}x \, \mathrm{d}h \right)^{1/2} \qquad s \in (0,1),$$

that roughly speaking has the aim to measure the "size" of the $s$-derivative (i.e. derivative of order $s$) of the function $f$. At least at an intuitive level it is clear that replacing the term $|h|^{2s}$ with $\log(1/|h|)^{1-p} \mathbf{1}_{B_{1/2}}(h)$ we are taking into account the "log-derivative" of $f$, justifying the name log-Sobolev functionals. This intuitive idea is also supported by the equivalence

$$\int_{B_{1/2}} \int_{\mathbb{R}^d} \frac{|f(x+h) - f(x)|^2}{|h|^d} \frac{1}{\log(1/|h|)^{1-p}} \, \mathrm{d}x \, \mathrm{d}h \simeq_{d,p} \int_{|\xi| \ge 10} \log(|\xi|)^p |\hat{f}(\xi)|^2 \, \mathrm{d}\xi + \int_{|\xi| \le 10} |\xi|^2 |\hat{f}(\xi)|^2 \, \mathrm{d}\xi, \tag{1.5}$$

for every $f \in L^2(\mathbb{R}^d)$, that is proven in a forthcoming paper [BrNg18]. Note that our log-Sobolev functionals are comparable with the ones in (1.3) considered by Leger [LF16] when $p = 1$ and $p = 2$. Our main result is the following.

**Theorem 1.1.** *Let $p > 1$ be fixed. Let us consider a bounded (in space and time) divergence-free vector field $b \in L^1((0,T); W^{1,p}(\mathbb{R}^d; \mathbb{R}^d))$.*

*Then for every initial data $\bar u \in BV(\mathbb{R}^d)$ with $\|\bar u\|_{L^\infty} \le 1$ the (unique) solution $u \in L^\infty((0,T)\times \mathbb{R}^d)$ of the continuity equation (CE) satisfies*

$$\int_{B_{1/2}} \int_{\mathbb{R}^d} \frac{|u_t(x+h)-u_t(x)|^2}{|h|^d} \frac{1}{\log(1/|h|)^{1-p}} \,\mathrm{d}x\,\mathrm{d}h \lesssim_{p,d} \left( \int_0^t \|\nabla b_s\|_{L^p}\,\mathrm{d}s \right)^p + \|\bar u\|_{BV}^p + \|\bar u\|_{L^1} . \tag{1.6}$$

*Moreover, there exist a divergence free vector field $b \in L^\infty((0,+\infty); W^{1,p}(\mathbb{R}^d))$ and an initial data $\bar u \in L^\infty(\mathbb{R}^d) \cap W^{1,d}(\mathbb{R}^d)$, such that the unique solution $u \in L^\infty((0,\infty)\times\mathbb{R}^d)$ of the Cauchy problem (CE) satisfies*

$$\int_{B_{1/2}} \int_{\mathbb{R}^d} \frac{|u_t(x+h)-u_t(x)|^2}{|h|^d} \frac{1}{\log(1/|h|)^{\gamma}} \,\mathrm{d}x\,\mathrm{d}h = \infty, \qquad \forall t>0, \tag{1.7}$$

*for any $\gamma < 1-p$.*

Under the additional assumption $b \in L^\infty((0,T); W^{1,p}(\mathbb{R}^d;\mathbb{R}^d))$ with $p>1$ (1.6) ensures that the log-Sobolev functional (1.4) of order $p$ increases in time at most polynomially with exponent $p$. Also this rate is sharp as we show in Theorem 3.1.

We refer to section 2 and section 3 for technical remarks around Theorem 1.1 concerning the boundedness assumption on the vector field, the regularity of the initial data and simple generalization to the case of vector fields with non-zero divergence.

Let us spend some worlds about the strategy of the proof of Theorem 1.1. For what concern the first part of the result our starting point is the Lusin-Lipschitz estimate for regular Lagrangian flows obtained by Crippa and De Lellis in [CDL08]. We prove that a suitable version of this estimate (see Proposition 2.9) implies (1.6) by mean of a general result Proposition 2.12 that has the aim to link a notion of "having a logarithm Sobolev derivative" written in term of Lusin-Lipschitz property with a quantitative estimate in term of our log-Sobolev functionals.

To achieve the second part of Theorem 1.1 we use a version of the construction proposed by Alberti Crippa and Mazzucato in [ACM16] (see also [ACM14] and [ACM18]). The main new technical tool we introduce is the interpolation inequality proved in Proposition 3.5 (see also Corollary 3.7) that links the log-Sobolev functionals (1.4) of a function $f$ with its $L^2$ and $\dot H^{-1}$ norms. As a byproduct of Theorem 1.1 and the just mentioned interpolation inequality we are able to recover the sharp bound on "mixing" for vector fields with uniformly bounded $W^{1,p}$ norm, with $p>1$. This well-known result (see for instance [CDL08, Theorem 6.2], [IKX14], [HSSS18], [Se13], [LF16]) is proved in Proposition 3.10.

The paper is organized as follow. In section 2 we deal with the first part of the Theorem 1.1, see also Theorem 2.1. The second part of the paper, that is to say section 3, is devoted to the proof of the second part of the Theorem 1.1 (see also Theorem 3.2) and of Theorem 3.1. In this section we also collect two mixing estimates (see Proposition 3.10) obtained as a byproduct of the previously developed theory.

Throughout the present paper we work in the Euclidean space of dimension $d \ge 2$ endowed with the Lebesgue measure $\mathscr{L}^d$ and the Euclidean norm $|\cdot|$. We denote by $B_r(x)$ the ball of radius $r>0$ centered at $x \in \mathbb{R}^d$. We often write $B_r$ instead of $B_r(0)$. Let us set

$$⨍_E f\,\mathrm{d}x = \frac{1}{\mathscr{L}^d(E)} \int_E f\,\mathrm{d}x, \qquad \forall\, E \subset \mathbb{R}^d \text{ Borel set},$$

and

$$Mf(x) := \sup_{r>0} ⨍_{B_r(x)} |f(y)|\,\mathrm{d}y, \qquad \forall\, x \in \mathbb{R}^d,$$

to denote the Hardy-Littlewood maximal function. We often use the expression $a \lesssim_c b$ to mean that there exists a universal constant $C$ depending only on $c$ such that $a \le Cb$. The same convention is adopted for $\gtrsim_c$ and $\simeq_c$.

## 2 Regularity result

The main result of the present section is the following.

**Theorem 2.1.** *Let $p > 1$ be fixed. Let us consider a bounded (in space and time) divergence-free vector field $b \in L^1((0,T);W^{1,p}(\mathbb{R}^d;\mathbb{R}^d))$.*

*Then for every initial data $\bar u \in BV(\mathbb{R}^d)$ with $\|\bar u\|_{L^\infty} \le 1$ the (unique) solution $u \in L^\infty((0,T)\times\mathbb{R}^d)$ of the continuity equation (CE) satisfies*

$$\int_{B_{1/2}}\int_{\mathbb{R}^d} \frac{|u_t(x+h)-u_t(x)|^2}{|h|^d}\frac{1}{\log(1/|h|)^{1-p}}\,\mathrm{d}x\,\mathrm{d}h \lesssim_{p,d} \left(\int_0^t \|\nabla b_s\|_{L^p}\,\mathrm{d}s\right)^p + \|\bar u\|_{BV}^p + \|\bar u\|_{L^1}\,. \tag{2.1}$$

Some technical remarks are in order.

*Remark* 2.2. Using standard arguments it is possible to prove that (2.1) implies

$$\sup_{h\in B_{1/2}} \log(1/|h|)^p \int_{\mathbb{R}^d} |u_t(x+h)-u_t(x)|^2\,\mathrm{d}x \lesssim_{p,d} \left(\int_0^t \|\nabla b_s\|_{L^p}\,\mathrm{d}s\right)^p + \|\bar u\|_{BV}^p + \|\bar u\|_{L^1}\,.$$

This estimate will play a role in the study of the geometric mixing norm along solutions of (CE), see Proposition 3.10.

*Remark* 2.3. The assumption $b \in L^\infty((0,T)\times\mathbb{R}^d;\mathbb{R}^d)$ can be replaced with more general growth conditions, for instance one can ask

$$\frac{b(t,x)}{1+|x|} = b_1(t,x) + b_2(t,x), \tag{2.2}$$

with $b_1 \in L^1((0,T);L^1(\mathbb{R}^d;\mathbb{R}^d))$ and $b_2 \in L^1((0,T);L^\infty(\mathbb{R}^d;\mathbb{R}^d))$ (see [CDL08, pg. 12]). Note that it contains the class $L^q((0,T);L^q(\mathbb{R}^d,\mathbb{R}^d))$ for any $q\in[1,\infty]$, since $|b(t,x)| \le |b(t,x)|^q + 1$.

Let us point out that a growth condition on $b$ is necessary to ensure the existence of a unique regular Lagrangian flow associated to the vector field, see [A04, AC14, CDL08] for a detailed discussions on this topic.

*Remark* 2.4. The divergence free condition on $b$ can be replaced with a more general

$$\exp\left\{\int_0^T \|\mathrm{div}\, b_s\|_{L^\infty}\,\mathrm{d}s\right\} = L < \infty, \tag{2.3}$$

provided we work with the *transport equation*

$$\begin{cases} \partial_t u + b\cdot\nabla u = 0, \\ u_0 = \bar u, \end{cases} \tag{TrE}$$

instead of (CE). The precise statement is the following.

Let $p > 1$ be fixed. Let us consider a bounded vector field $b \in L^1((0,T);W^{1,p}(\mathbb{R}^d;\mathbb{R}^d))$ satisfying (2.3). Then for every initial data $\bar u \in BV(\mathbb{R}^d)$ with $\|\bar u\|_{L^\infty} \le 1$ the (unique) solution $u \in L^\infty((0,T)\times\mathbb{R}^d)$ of the transport equation (TrE) satisfies

$$\int_{B_{1/2}}\int_{\mathbb{R}^d} \frac{|u_t(x+h)-u_t(x)|^2}{|h|^d}\frac{1}{\log(1/|h|)^{1-p}}\,\mathrm{d}x\,\mathrm{d}h \lesssim_{p,d} L^p \left(\int_0^t \|\nabla b_s\|_{L^p}\,\mathrm{d}s\right)^p + \|\bar u\|_{BV}^p + \|\bar u\|_{L^1}\,.$$

*Remark* 2.5. The regularity assumption on the initial data $\bar u \in BV(\mathbb{R}^d)$ is very far from being sharp, indeed it can be immediately weakened, for instance asking $u_0 \in W^{s,1}(\mathbb{R}^d)$ for $0 < s \le 1$ or $u_0$ satisfying some Lusin-Lipschitz regularity condition as (2.13). Of course we expect that the minimal assumption to ask is

$$\int_{B_{1/2}}\int_{\mathbb{R}^d} \frac{|u_0(x+h)-u_0(x)|^2}{|h|^d}\frac{1}{\log(1/|h|)^{1-p}}\,\mathrm{d}x\,\mathrm{d}h < \infty, \tag{2.4}$$

but it seams to be much more difficult to prove using our techniques. For sake of simplicity and for consistency with the counterexample in Theorem 3.2 we prefer to deal with Sobolev or $BV$ initial data.

The proof of Theorem 2.1 strictly relies on a well-know ingredient: the quantitative Lusin-Lipschitz estimates for the Lagrangian flows associated to Sobolev vector fields, first introduced in [ALM05] and [CDL08]. The strategy of our proof is the following. First we state, in a suitable form, the afore mentioned regularity result for Lagrangian flows (see Proposition 2.9) and for sake of completeness we add a very simple proof of this fact. As a second step we turn the quantitative Lusin-Lipschitz estimate performed at the Lagrangian level to an estimate for solutions of the continuity equation (see Corollary 2.11). We point out that also this result is already present in the literature [CDL08, Theorem 5.3], in a slightly less quantitative form. Finally we establish a general result that links a suitable quantitative Lusin-Lipschitz property of a generic scalar function with an estimate on the log-Sobolev functional (1.4).

## 2.1 Regularity of Lagrangian flows

As mentioned above in this subsection we present a regularity estimate for Lagrangian flows associated to Sobolev vector fields with exponent $p > 1$. Let us begin recalling the definition of regular Lagrangian flow introduced by Ambrosio in [A04].

**Definition 2.6.** Let us fix a time dependent vector field $b \in L^1_{\rm loc}((0,T) \times \mathbb{R}^d; \mathbb{R}^d)$. We say that $X : [0,T] \times \mathbb{R}^d \to \mathbb{R}^d$ is a Regular Lagrangian flow associated to $b_t$ (RLF for short) if the following conditions hold:

(i) there exists an $\mathscr{L}^d$-negligible set $N \subset \mathbb{R}^d$ such that

$$X_t(x) = x + \int_0^t b_s(X_s(x))\,\mathrm{d}s \qquad \forall t \in [0,T],$$

for every $x \in \mathbb{R}^d \setminus N$;

(ii) there exists $L > 0$, called compressibility constant, such that

$$(X_t)_\sharp \mathscr{L}^d \leq L\mathscr{L}^d, \qquad \text{for every } t \in [0,T];$$

The regular Lagrangian flow can be thought as a "good" selection of (possible not unique) solutions of the (ODE) problem associated to a rough vector field. Is the condition (ii) that has the role to select "good" trajectories, ensuring that the flow does not concentrate too much the reference measure $\mathscr{L}^d$.

Condition (ii) plays also an important role at the technical viewpoint. Indeed it guarantees that the notion of RLF is stable under modifications of the vector field on a $\mathscr{L}^d$-negligible set. More precisely if $X$ is a regular Lagrangian flow associated to $b$ and $\bar{b}$ is such that

$$\mathscr{L}^{d+1}(\{\,(t,x) : \; |\,b(t,x) - \bar{b}(t,x)| > 0\,\}) = 0,$$

then $X$ is also a regular Lagrangian flow associated to $\bar{b}$.

*Remark* 2.7. It has been shown in [A04, Theorem 6.2, Theorem 6,4] that under the $BV$ assumption on the vector fields $\int_0^T \|b_s\|_{BV}\,\mathrm{d}s < \infty$, the uniform bound on the negative part of the divergence, i.e. $\operatorname{div} b_t \ll \mathscr{L}^d$ and $\int_0^T \|[\operatorname{div} b_s]^-\|_{L^\infty}\,\mathrm{d}s < \infty$, and a growth conditions (for instance $b \in L^\infty([0,T]\times\mathbb{R}^d)$ see also Remark 2.3 ) there exists a unique RLF.

We also remark that assuming a bound on the whole divergence

$$\exp\left\{\int_0^T \|\operatorname{div} b_s\|_{L^\infty}\,\mathrm{d}s\right\} \leq L,$$

condition (ii) in Definition 2.6 can be improved in

$$1/L\mathscr{L}^d \leq (X_t)_\sharp \mathscr{L}^d \leq L\mathscr{L}^d, \qquad \text{for every } t \in [0,T]. \tag{2.5}$$

In particular if the vector field is divergence-free $X_t$ is a measure preserving map for any $t \in [0,T]$.

*Remark* 2.8. Even though the notion of regular Lagrangian flow is stable under modification of the vector field on $\mathscr{L}^d$-negligible sets we prefer, for technical reasons, to work with a vector field *point-wise defined*, with respect to the spatial variable.

From now on when we write $b \in L^1((0,T); W^{1,p}(\mathbb{R}^d, \mathbb{R}^d))$ we are tacitly considering the representative (that we still call $b$) obtained setting $b_t(x)$ equal to $\lim_{r\to 0} \frac{1}{\omega_d r^d} \int_{B_r(x)} b_t(y)\, \mathrm{d}y$ when $x$ is a Lebesgue point of $b_t$ and 0 otherwise.

This choice allows a point-wise *Lusin-Lipschitz maximal estimate* for Sobolev vector fields:

$$|b_t(x) - b_t(y)| \lesssim_d |x-y|(M|\nabla b_t|(x) + M|\nabla b_t|(y)) \qquad \forall x, y \in \mathbb{R}^d, \text{ for } \mathscr{L}^1\text{-a.e. } t \in [0,T], \tag{2.6}$$

where $M$ is the Hardy-Littlewood maximal function. See [ST] for a proof of this result at the level of scalar Sobolev functions.

The following is the Lusin-Lipschitz regularity result for regular Lagrangian flows associated to $W^{1,p}$ vector fields with $p > 1$, compare with [CDL08, Poposition 2.3].

**Proposition 2.9.** *Assume $b \in L^1((0,T); W^{1,p}(\mathbb{R}^d; \mathbb{R}^d))$ for some $p > 1$. Let $X$ be a regular Lagrangian flow associated to $b$ with compressibility constant $L$. Then there exists a positive function $g_t : \mathbb{R}^d \to \mathbb{R} \cup \{+\infty\}$ for every $t \in [0,T]$ such that*

$$\exp\{-g_t(x) - g_t(y)\} \le \frac{|X_t(x) - X_t(y)|}{|x-y|} \le \exp\{g_t(x) + g_t(y)\}, \tag{2.7}$$

*for any $x, y \in \mathbb{R}^d$, for every $t \in [0,T]$, and*

$$\|g_t\|_{L^p} \lesssim_{p,d} L \int_0^t \|\nabla b_s\|_{L^p}\, \mathrm{d}s \qquad \forall t \in [0,T]. \tag{2.8}$$

*Moreover, if $b$ is a divergence-free drift then we can take $L = 1$.*

*Proof.* Let us take $N \subset \mathbb{R}^d$ as in point (i) of Definition 2.6, $\varepsilon > 0$, $x, y \in \mathbb{R}^d \setminus N$ and $t \in [0,T]$, we have

$$\left|\log\left(\frac{\varepsilon + |X_t(x) - X_t(y)|}{\varepsilon + |x-y|}\right)\right| = \left|\int_0^t \frac{\mathrm{d}}{\mathrm{d}s} \log\left(\varepsilon + |X_s(x) - X_s(y)|\right) \mathrm{d}s\right| \le \int_0^t \frac{|b_s(X_s(x)) - b_s(X_s(y))|}{|X_s(x) - X_s(y)|}\, \mathrm{d}s.$$

Using the Lusin maximal estimate (2.6) an letting $\varepsilon \to 0$ we get

$$\left|\log\left(\frac{|X_t(x) - X_t(y)|}{|x-y|}\right)\right| \le C_d \int_0^t M|\nabla b_s|(X_s(x))\, \mathrm{d}s + C_d \int_0^t M|\nabla b_s|(X_s(y))\, \mathrm{d}s.$$

Set $g_t(x) := C_d \int_0^t M|\nabla b_s|(X_s(x))\, \mathrm{d}s$ when $x \in \mathbb{R}^d \setminus N$ and $g_t(x) := +\infty$ otherwise. The condition (ii) in Definition 2.6, the boundness of the maximal function between $L^p$ spaces when $p > 1$, together with Minkowski's inequality (see [ST, Appendix]), yields (2.8). The proof is complete. □

*Remark* 2.10. It is clear from the proof that $g_t$ could be taken independent of $t$, simply considering $g_T := C_d \int_0^T M|\nabla b_s|(X_s(x))\, \mathrm{d}s$.

As a quite simple consequence of Proposition 2.9 we get a Lusin-Lipschitz estimate at the level of solutions to the continuity equation (CE), compare with [CDL08, Theorem 5.3].

**Corollary 2.11.** *Let us fix $p > 1$ and a bounded divergence-free vector field $b \in L^1((0,T); W^{1,p}(\mathbb{R}^d; \mathbb{R}^d))$. Then, there exists a positive function $\tilde{g}_t : \mathbb{R}^d \to \mathbb{R} \cup \{+\infty\}$ for $t \in [0,T]$, such that, for every initial data $\bar{u} \in BV(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d)$ there exists a representative $u : [0,T] \times \mathbb{R}^d \to \mathbb{R}^d$ of the (unique) solution in $L^\infty([0,T] \times \mathbb{R}^d)$ to (CE) satisfying*

$$|u_t(x) - u_t(y)| \le |x-y| \exp\{\tilde{g}_t(x) + \tilde{g}_t(y)\}, \qquad \forall x, y \in \mathbb{R}^d, \quad \forall t \in [0,T], \tag{2.9}$$

*and*

$$\|\tilde{g}_t\|_{L^p} \lesssim_{p,d} \int_0^t \|\nabla b_s\|_{L^p}\, \mathrm{d}s + \|\bar{u}\|_{BV} \qquad \forall t \in [0,T]. \tag{2.10}$$

*Proof.* Let $X$ be a unique regular Lagrangian flow associated to $b$. For every time $t \in [0,T]$ the map $x \to X_t(x)$ is essentially invertible, namely there exists $Y : [0,T] \times \mathbb{R}^d \to \mathbb{R}^d$ such that

$$X(t, Y(t,x)) = Y(t, X(t,x)) = x \qquad \text{for } \mathscr{L}^d\text{-a.e. } x \in \mathbb{R}^d, \tag{2.11}$$

see [A04, Theorem 6.2]. It is immediate to check that $Y_t$ satisfies the inequality (2.7) replacing $g_t(x)$ with $\bar{g}_t(x) := g_t(Y_t(x))$ when $x$ fulfills (2.11) and $\bar{g}_t(x) := \infty$ otherwise. Using the measure preserving property of $Y_t$ (compare with Remark 2.7) it is simple to verify that

$$\|\bar{g}_t\|_{L^p} \lesssim_{p,d} \int_0^t \|\nabla b_s\|_{L^p} \,\mathrm{d}s \qquad \forall t \in [0,T]. \tag{2.12}$$

Thus, up to modifies again $\bar{g}_t$ on a negligible set we get

$$\frac{|u_t(x) - u_t(y)|}{|x-y|} \lesssim_d \left(M|\nabla \bar{u}|(x) + M|\nabla \bar{u}|(y)\right) \exp\left\{\bar{g}_t(x) + \bar{g}_t(y)\right\}, \qquad \forall x, y \in \mathbb{R}^d, \quad \forall t \in [0,T],$$

where we used the $\mathscr{L}^d$-a.e. identity $u_t = u_0(Y_t)$ (it can be checked observing that $u_t(X_t(x)) = u_0(x)$ for $\mathscr{L}^d$-a.e. $x \in \mathbb{R}^d$) and the Lusin-Lipschitz maximal estimate (2.6) for $u_0 \in BV(\mathbb{R}^d)$.

Finally observe that for $x, y \in \mathbb{R}^d$ and for every $t \in [0,T]$

$$C_d(M|\nabla \bar{u}|(x) + M|\nabla \bar{u}|(y)) \exp\left\{\bar{g}_t(x) + \bar{g}_t(y)\right\} \le \exp\left\{\tilde{g}_t(x) + \tilde{g}_t(x)\right\},$$

where $\tilde{g}_t(x) = \bar{g}_t(x) + c_p \mathbf{1}_{M|\nabla \bar{u}|(x) > \frac{1}{2C_d}} (M|\nabla \bar{u}|(x))^{1/2p}$ for some $c_p > 0$. This implies (2.9). Thanks to weak type (1,1) bound of the maximal function (see [ST]) and (2.12), we obtain (2.10). The proof is complete. ☐

## 2.2 A key lemma

This section is devoted to the proof of the following.

**Proposition 2.12.** *Let $p \ge 1$ be fixed. Let $f \in L^1(\mathbb{R}^d)$ be a function satisfying the following exponential Lusin-Lipschitz regularity estimate: there exist a positive function $g \in L^p(\mathbb{R}^d)$ such that*

$$|f(x) - f(y)| \le |x-y| \exp\left\{g(x) + g(y)\right\} \qquad \forall x, y \in \mathbb{R}^d. \tag{2.13}$$

*Then, it holds*

$$\int_{B_{1/2}} \int_{\mathbb{R}^d} \frac{1 \wedge |f(x+h) - f(x)|^2}{|h|^d} \frac{1}{\log(1/|h|)^{1-p}} \,\mathrm{d}x\,\mathrm{d}h \lesssim_{p,d} \|g\|_{L^p}^p + \|f\|_{L^1}. \tag{2.14}$$

Roughly speaking this result establishes an implication between two different notions of “having a derivative of logarithmic order”. Observe that these two conditions cannot be equivalent, the assumption (2.13) is stronger than (2.14). Indeed, the latter allows every Hölder continuous function, that in general cannot be weakly differentiable (see for instance [Nguyen2]), compare also with the following.

*Remark* 2.13. The result in Proposition 2.12 is written in a form useful for our purposes, that is very far from being sharp. For instance it can be generalized as follow. Assume that $f \in L^1(\mathbb{R}^d)$ satisfies an Hölder-Lipschitz inequality

$$|f(x) - f(y)| \le |x-y|^\alpha \exp\left\{g(x) + g(y)\right\} \qquad \forall x, y \in \mathbb{R}^d,$$

for some $\alpha \in (0,1]$ and some $g \in L^p(\mathbb{R}^d)$. Then it holds

$$\int_{B_{1/2}} \int_{\mathbb{R}^d} \frac{1 \wedge |f(x+h) - f(x)|^2}{|h|^d} \frac{1}{\log(1/|h|)^{1-p}} \,\mathrm{d}x\,\mathrm{d}h \lesssim_{p,\alpha,d} \|g\|_{L^p}^p + \|f\|_{L^1}.$$

The proof can be obtain following the proof of Proposition 2.12 with minor modifications.

Let us now prove a technical lemma.

**Lemma 2.14.** *Let $f \in L^1(\mathbb{R}^d)$ and $g \in L^p(\mathbb{R}^d)$ be as in Proposition 2.12. Then, it holds*

$$\int_{\mathbb{R}^d} 1 \wedge |f(x+h) - f(x)|^2 \,\mathrm{d}x \lesssim_d |h|^2 \int_1^{\log(1/|h|)} e^{2\lambda} \mathscr{L}^d(\{\, 2g > \lambda \,\})\,\mathrm{d}\lambda + |h| \, \|f\|_{L^1}\,, \tag{2.15}$$

*for every $h \in \mathbb{R}^d$ with $|h| \le 1/e$.*

*Proof.* Using (2.13) and the Cavalieri's summation formula we get

$$\begin{aligned}
\int_{\mathbb{R}^d} 1 \wedge &|f(x+h) - f(x)|^2 \,\mathrm{d}x \\
&= 2\int_0^{e|h|} t\mathscr{L}^d\left(\{\, x : \; |f(x+h) - f(x)| > t \,\}\right)\mathrm{d}t \\
&\quad + 2\int_{e|h|}^1 t\mathscr{L}^d\left(\{\, x : \; |f(x+h) - f(x)| > t \,\}\right)\mathrm{d}t \\
&\lesssim |h| \, \|f\|_{L^1} + \int_{e|h|}^1 t\mathscr{L}^d\left(\{\, x : \; |f(x+h) - f(x)| > t \,\}\right)\mathrm{d}t \\
&\lesssim |h| \, \|f\|_{L^1} + \int_{e|h|}^1 t\mathscr{L}^d(\{\, x : \; g(x) + g(x+h) > \log(t/\,|h|) \,\})\,\mathrm{d}t,
\end{aligned}$$

for $\mathscr{L}^d$-a.e. $h \in \mathbb{R}^d$ with $|h| \le 1/e$. Estimating $\int_{e|h|}^1 t\mathscr{L}^d(\{\, x : \; g(x) + g(x+h) > \log(t/\,|h|) \,\})\,\mathrm{d}t$ with

$$2\int_{e|h|}^1 t\mathscr{L}^d(\{\, 2g > \log(t/\,|h|) \,\})\,\mathrm{d}t, \tag{2.16}$$

setting $\lambda = \log(t/|h|)$ and changing variables in (2.16) we conclude the proof. $\square$

We are now ready to prove Proposition 2.12.

*Proof.* In order to shorten notation we set $\mu(\lambda) := \mathscr{L}^d(\{\, 2g > \lambda \,\})\,\mathrm{d}\lambda$. Using the result in Lemma 2.14 we get

$$\begin{aligned}
\int_{B_{1/e}} \int_{\mathbb{R}^d} &\frac{1 \wedge |f(x+h) - f(x)|^2}{|h|^d \log(1/|h|)^{1-p}} \,\mathrm{d}x \,\mathrm{d}h \\
&\lesssim \int_{B_{1/e}} \frac{\log(1/|h|)^{p-1}}{|h|^d} \left( |h|^2 \int_1^{\log(1/|h|)} e^{2\lambda} \,\mathrm{d}\mu(\lambda) + |h| \, \|f\|_{L^1} \right) \mathrm{d}h \\
&\lesssim_{p,d} \int_0^1 \log(1/r)^{p-1} r \int_1^{\log(1/r)} e^{2\lambda} \,\mathrm{d}\mu(\lambda) \,\mathrm{d}r + \|f\|_{L^1}\,.
\end{aligned}$$

Setting $\log(1/r) = t$, changing variables and applying Fubini theorem we get

$$\begin{aligned}
\int_0^1 \log(1/r)^{p-1} r &\int_1^{\log(1/r)} e^{2\lambda} \,\mathrm{d}\mu(\lambda) \,\mathrm{d}r \\
&= \int_0^\infty e^{-2t} t^{p-1} \int_1^t e^{2\lambda} \,\mathrm{d}\mu(\lambda) \,\mathrm{d}t \\
&= \int_1^\infty e^{2\lambda} \int_\lambda^\infty e^{-2t} t^{p-1} \,\mathrm{d}t \,\mathrm{d}\mu(\lambda).
\end{aligned}$$

Using the integration by part formula and the inequality $\lambda \ge 1$ it is elementary to check that

$$e^{2\lambda} \int_\lambda^\infty e^{-2t} t^{p-1} \,\mathrm{d}t \lesssim_p \lambda^{p-1}, \tag{2.17}$$

that together with the definition of $\mu(\lambda)$ implies

$$\int_1^\infty e^{2\lambda} \int_\lambda^\infty e^{-2t} t^{p-1} \,\mathrm{d}t \,\mathrm{d}\mu(\lambda) \lesssim_p \int_0^\infty \lambda^{p-1} \,\mathrm{d}\mu(\lambda) \lesssim_p \|g\|_{L^p}^p\,.$$

Putting all together we get

$$\int_{B_{1/e}}\int_{\mathbb{R}^d}\frac{1\wedge|f(x+h)-f(x)|^2}{|h|^d}\frac{1}{\log(1/|h|)^{1-p}}\,\mathrm{d}x\,\mathrm{d}h\lesssim_{p,d}\|g\|_{L^p}^p+\|f\|_{L^1}\,, \tag{2.18}$$

that is clearly equivalent to our thesis. □

Eventually the proof of Theorem 2.1 follows applying Proposition 2.12 with $f=u_t$, and recalling Corollary 2.11 and Remark 2.15 below.

*Remark* 2.15. For every $1\le p<\infty$ the $L^p$ norm of a solution $u_t$ to (CE) is preserved in time, at least when the vector field is regular enough. For instance in the smooth setting one can simply compute

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{1}{2}\int_{\mathbb{R}^d}|u_t|^2\,\mathrm{d}x=-\int_{\mathbb{R}^d}u_t\,\mathrm{div}(b_tu_t)\,\mathrm{d}x=-\frac{1}{2}\int_{\mathbb{R}^d}\mathrm{div}(b_tu_t^2)\,\mathrm{d}x=0.$$

More in general, if $b\in L^1((0,T);BV(\mathbb{R}^d;\mathbb{R}^d))$ with bounded divergence then $u_t$ has *the renormalization property* (see [A04]) that is to say, for any $\beta\in C^1_c(\mathbb{R})$ the function $\beta(u(t,x))$ is a solution of the continuity equation as well. It implies the preservation in time of $L^p$ norms.

# 3 Counterexamples and mixing estimates

The aim of this section is to show the sharpness of Theorem 2.1 under two different viewpoints. The first result is the following.

**Theorem 3.1.** *Let $p\ge 1$ fixed. There exist a smooth divergence-free vector field $b$ belonging to $L^\infty((0,\infty);W^{1,\infty}(\mathbb{R}^d,\mathbb{R}^d))$ supported in $B_1\times[0,\infty)$, and a smooth initial data $u_0$ supported in $B_1$, such that the unique solution $u\in L^\infty((0,\infty)\times\mathbb{R}^d)$ to the continuity equation (CE) satisfies*

$$\int_{B_{1/2}}\int_{\mathbb{R}^d}\frac{|u_t(x+h)-u_t(x)|^2}{|h|^d}\frac{1}{\log(1/|h|)^{1-p}}\,\mathrm{d}x\,\mathrm{d}h\gtrsim t^p,$$

*for any $t\in(0,\infty)$.*

This result implies that the polynomial growth of order $p$ proved in (2.1) is sharp. A partial result in this direction was already obtained in [LF16] for the case $p=1$. The second and most important example reads as follow.

**Theorem 3.2.** *Let $p\ge 1$. There exist a divergence free vector field $b\in L^\infty((0,+\infty);W^{1,p}(\mathbb{R}^d))$ supported in $B_1\times[0,\infty)$, and an initial data $u_0\in L^\infty(\mathbb{R}^d)\cap W^{1,d}(\mathbb{R}^d)$ also supported in $B_1$, such that the unique solution $u\in L^\infty((0,\infty)\times\mathbb{R}^d)$ of the continuity equation (CE) satisfies*

$$\int_{B_{1/2}}\int_{\mathbb{R}^d}\frac{|u_t(x+h)-u_t(x)|^2}{|h|^d}\frac{1}{\log(1/|h|)^{\gamma}}\,\mathrm{d}x\,\mathrm{d}h=\infty,\qquad\forall\gamma<1-p, \tag{3.1}$$

*for every $t>0$.*

In other words a solution of the continuity equation associated to a divergence-free vector field belonging to $W^{1,p}$ for some $p\ge 1$ cannot preserve the log-Sobolev functional (1.4) of order $q$, when $q>p$, while it is always preserved the one of order $p>1$ as Theorem 2.1 shows.

As a consequence of Proposition 2.12 (see also Remark 2.13) and Proposition 2.9 the result in Theorem 3.2 immediately implies the following.

**Proposition 3.3.** *Let $p\ge 1$ fixed. For every $q>p$ there exists a compact supported divergence free-vector field $b\in L^\infty((0,+\infty);W^{1,p}(\mathbb{R}^d))$ whose regular Lagrangian flow $X$ satisfies the following property: for every $t>0$, for every $g\in L^q$ and for every $\alpha\in(0,1]$ there exists a set $E\subset\mathbb{R}^d$ of positive Lebesgue measure such that*

$$|X_t(x)-X_t(y)|>|x-y|^\alpha\exp\{g(x)+g(y)\}\qquad\forall x,y\in E. \tag{3.2}$$

In other words the exponential Lusin-Lipschitz regularity of order $p$ for Lagrangian flows associated to vector fields belonging to $W^{1,p}$ cannot be improved. Even an exponential Lusin Hölder regularity of order greater than $p$ cannot be reached.

The main idea behind our constructions comes from the work [ACM16] by Alberti, Crippa and Mazzucato. In this paper the authors built a solution to (CE), drifted by a divergence-free Sobolev vector field, that is smooth at time zero but it does not belong to any Sobolev space for positive times. The construction of a vector field $b$ and the solution $u_t$ is achieved by patching together a countable number of pairs $v_n$ and $\rho_n$ of velocity fields and solutions to the Cauchy problem (CE) with disjoint supports. They are obtained by rescaling in space, time and size $v$ and $\rho$, that are given by the following.

**Proposition 3.4.** *Assume $d \geq 2$ and let $Q$ be the open cube with unit side centered at the origin of $\mathbb{R}^d$. There exist a velocity field $v \in C^\infty([0,\infty)\times\mathbb{R}^d)$ and a (non trivial) solution $\rho \in L^\infty([0,\infty)\times\mathbb{R}^d)$ of the continuity equation (CE) such that*

*(i) $v_t$ is bounded, divergence-free and compactly supported in $Q$ for any $t \geq 0$;*

*(ii) $\rho_t$ has zero average and it is bounded and compactly supported in $Q$ for any $t \geq 0$;*

*(iii) $\sup_{t\geq 0} \|v_t\|_{W^{1,p}(\mathbb{R}^d)} < \infty$ for any $t \geq 0$, for any $1 \leq p \leq \infty$;*

*(iv) there exists a constant $c > 0$ such that*

$$\|\rho_t\|_{\dot{H}^{-1}(\mathbb{R}^d)} \lesssim \exp(-ct), \qquad \forall t \geq 0, \tag{3.3}$$

*where $\|\cdot\|_{\dot{H}^{-1}}$ is the negative homogeneous Sobolev norm of order $-1$.*

The result in Proposition 3.4, that is taken from [ACM18, Theorem 8] (see also [ACM14]) provides a solution of (CE) (associated to a smooth divergence-free vector field) whose $\dot{H}^{-1}$ norm decays exponentially in time. It can be shown that, under the uniform $W^{1,p}$ bound with $p > 1$ on the vector field, the rate of decay of negative Sobolev norms for solutions of (CE) is at most exponential (see Proposition 3.10 and the discussion above for more details), thus $\rho_t$ built in Proposition 3.4 saturates this rate.

By mean of Remark 2.15 and the following interpolation inequality

$$\|u_t\|_{L^2} \leq \|u_t\|_{\dot{H}^{-1}} \|u_t\|_{\dot{H}^1},$$

from (3.3) we can deduce an exponential loss of Sobolev regularity for the solution $u_t$:

$$\|u_t\|_{\dot{H}^1} \gtrsim \|u_0\|_{L^1} \exp(ct), \qquad \forall t \geq 0.$$

Moreover, using a new interpolation inequality (see Corollary 3.7) we are able to prove that the log-Sobolev functional (1.4), with exponent $p \geq 1$, evaluated on $u_t$ increases in time at least polynomially with exponent $p$, namely the solution built in Proposition 3.4 satisfies the statement of Theorem 3.1.

In order to prove Theorem 3.2 we follow a strategy very similar to the one adopted by Alberti Crippa and Mazzucato, using again our interpolation formula (see Proposition 3.5 and Corollary 3.7) as a main technical tool to deduce (3.1).

## 3.1 Interpolation inequality and proof of Theorem 3.1

The main result of this section is inspired by [DDN18, Proof of Theorem 2.4] and reads as follow.

**Proposition 3.5.** *Let us fix parameters $\gamma \in (-\infty, 1)$, $\lambda \in (0, 1/100)$ and $\delta \in (0,1]$. The following inequality holds true*

$$\|f\|_{L^2}^2 \lesssim_{d,\gamma} \frac{1}{|\log(\delta\lambda)|^{1-\gamma}} \int_{B_{\frac{1}{5\delta}}} \int_{\mathbb{R}^d} \frac{|f(x+h)-f(x)|^2}{|h|^d |\log(\delta|h|)|^\gamma} \,\mathrm{d}x\,\mathrm{d}h + |\log(\lambda)| \frac{\|f\|_{L^2}^2}{\log\left(2 + \frac{\|f\|_{L^2}}{\|f\|_{\dot{H}^{-1}}}\right)}, \tag{3.4}$$

*for every $f \in L^2(\mathbb{R}^d)$.*

Let us now describe two important corollaries of Proposition 3.5, that among other things, imply that the solution provided by Proposition 3.4 fulfills the assumption of Theorem 3.1. The first corollary give us an estimate for a scaled version of the log-Sobolev functional (1.4) of the solution built in Proposition 3.4 that would play a crucial role in the proof of Theorem 3.2.

**Corollary 3.6.** *Let $\rho$ be as in Proposition 3.4 and let $t > 0$ be fixed. For every $\gamma \in (-\infty, 1)$, $\lambda \in (0, 1/100)$ and $\gamma \in (0, 1]$, it holds*

$$\int_{B_{\frac{1}{5\delta}}} \int_{\mathbb{R}^d} \frac{|\rho_t(x+h) - \rho_t(x)|^2}{|h|^d |\log(\delta|h|)|^\gamma} \, \mathrm{d}x \, \mathrm{d}h \gtrsim_{d,\gamma} \|\rho_0\|_{L^2}^2 \, |\log(\delta\lambda)|^{1-\gamma} \left( C_\gamma - |\log(\lambda)| \frac{C(\|\rho_0\|_{L^2})}{1+ct} \right),$$

*where $c$ is the constant in Proposition 3.4, the constant $C_\gamma > 0$ depends only on $\gamma$ and $C(\|\rho_0\|_{L^2}) > 0$ depends only on $\|\rho_0\|_{L^2}$.*

It can be proved starting from (3.4) and using the property (iv) in Proposition 3.4 and Remark 2.15.
The second corollary of Proposition 3.5, that follows from (3.4) setting $\delta = 1$ and

$$|\log(\lambda)| = \left[ \frac{\log\left(2 + \frac{\|f\|_{L^2}}{\|f\|_{\dot{H}^{-1}}}\right)}{\|f\|_{L^2}^2} \int_{B_{1/5}} \int_{\mathbb{R}^d} \frac{|f(x+h) - f(x)|^2}{|h|^d \log(1/|h|)^\gamma} \, \mathrm{d}x \, \mathrm{d}h \right]^{\frac{1}{2-\gamma}}$$

is the following.

**Corollary 3.7.** *For every parameter $\gamma \in (-\infty, 1)$ it holds*

$$\log\left(2 + \frac{\|f\|_{L^2}}{\|f\|_{\dot{H}^{-1}}}\right)^{1-\gamma} \|f\|_{L^2} \lesssim_{d,\gamma} \int_{B_{1/5}} \int_{\mathbb{R}^d} \frac{|f(x+h) - f(x)|^2}{|h|^d} \frac{1}{\log(1/|h|)^\gamma} \, \mathrm{d}x \, \mathrm{d}h, \tag{3.5}$$

*for every $f \in L^2(\mathbb{R}^d)$.*

It easily implies the proof Theorem 3.1. Indeed, we can consider a vector field $v$ and the solution $u_t$ of (CE) as in the statement of Proposition 3.4. Using (3.5) with $\gamma = 1 - p$, the property (iv) in Proposition 3.4 and Remark 2.15 we get the desired result.

The remaining part of this section is devoted to the proof of Proposition 3.5.

*Proof of Proposition 3.5.* Fix $\varepsilon > 0$. Let us fix $\varphi \in C_c^\infty(\mathbb{R}^d)$ such that $\varphi = 1$ in $B_1 \setminus B_{1/2}$, $\varphi = 0$ in $\left(B_{5/4} \setminus B_{1/4}\right)^c$ and $\int_{\mathbb{R}^d} \varphi = 1$. Set $\varphi_\varepsilon(x) := \varepsilon^{-d} \varphi(x/\varepsilon)$. Thus, we have

$$\begin{aligned} \|f * \varphi_\varepsilon\|_{L^2}^2 = \left\| \hat{f} \hat{\varphi}_\varepsilon \right\|_{L^2}^2 &\le \left\| \log(2 + |\cdot|) |\hat{\varphi}_\varepsilon(\cdot)|^2 \right\|_{L^\infty} \int_{\mathbb{R}^d} \frac{|\hat{f}(\xi)|^2}{\log(2 + |\xi|)} \, \mathrm{d}\xi \\ &= \left\| \log(2 + \varepsilon^{-1}|\cdot|) |\hat{\varphi}(\cdot)|^2 \right\|_{L^\infty} \int_{\mathbb{R}^d} \frac{|\hat{f}(\xi)|^2}{\log(2 + |\xi|)} \, \mathrm{d}\xi \\ &\lesssim_d \left| \log\left(\varepsilon \wedge \frac{1}{2}\right) \right| \int_{\mathbb{R}^d} \frac{|\hat{f}(\xi)|^2}{\log(2 + |\xi|)} \, \mathrm{d}\xi, \end{aligned}$$

and

$$\|f - f * \varphi_\varepsilon\|_{L^2}^2 \lesssim \int_{\varepsilon \le 4|h| \le 5\varepsilon} \int_{\mathbb{R}^d} \frac{|f(x+h) - f(x)|^2}{|h|^d} \, \mathrm{d}x \, \mathrm{d}h.$$

Thus,

$$\|f\|_{L^2}^2 \lesssim_d \int_{\varepsilon \le 4|h| \le 5\varepsilon} \int_{\mathbb{R}^d} \frac{|f(x+h) - f(x)|^2}{|h|^d} \, \mathrm{d}x \, \mathrm{d}h + \left| \log\left(\varepsilon \wedge \frac{1}{2}\right) \right| \int_{\mathbb{R}^d} \frac{|\hat{f}(\xi)|^2}{\log(2 + |\xi|)} \, \mathrm{d}\xi. \tag{3.6}$$

Now we integrate (3.6) with respect to a variable $\varepsilon$ against a suitable kernel obtaining the following.

$$\int_{\lambda}^{\frac{1}{10\delta}} \frac{1}{|\log(\delta\varepsilon)|^{\gamma}} \frac{\mathrm{d}\varepsilon}{\varepsilon} \|f\|^2_{L^2(\mathbb{R}^d)} \lesssim_d \int_{\lambda}^{\frac{1}{10\delta}} \int_{\varepsilon\le 4|h|\le 5\varepsilon} \int_{\mathbb{R}^d} \frac{|f(x+h)-f(x)|^2}{|h|^d} \,\mathrm{d}x\,\mathrm{d}h \frac{1}{|\log(\delta\varepsilon)|^{\gamma}} \frac{\mathrm{d}\varepsilon}{\varepsilon}$$
$$+ \int_{\lambda}^{\frac{1}{10\delta}} \frac{\left|\log\left(\varepsilon\wedge\frac{1}{2}\right)\right|}{|\log(\delta\varepsilon)|^{\gamma}} \frac{\mathrm{d}\varepsilon}{\varepsilon} \int_{\mathbb{R}^d} \frac{|\hat{f}(\xi)|^2}{\log(2+|\xi|)}\,\mathrm{d}\xi. \tag{3.7}$$

Starting from the elementary inequalities

$$-\frac{1}{1-\gamma}\frac{\mathrm{d}}{\mathrm{d}\varepsilon}|\log(\delta\varepsilon)|^{1-\gamma} = \frac{1}{|\log(\delta\varepsilon)|^{\gamma}}\frac{1}{\varepsilon} \qquad \text{for } \varepsilon<\frac{1}{\delta},$$

and

$$-\frac{\mathrm{d}}{\mathrm{d}\varepsilon}\frac{|\log(\varepsilon)|^2}{|\log(\delta\varepsilon)|^{\gamma}} = \frac{|\log(\varepsilon)|}{|\log(\delta\varepsilon)|^{\gamma}}\frac{1}{\varepsilon}\left(2-\gamma\frac{|\log(\varepsilon)|}{|\log(\delta\varepsilon)|}\right) \ge \frac{|\log(\varepsilon)|}{|\log(\delta\varepsilon)|^{\gamma}}\frac{1}{\varepsilon} \qquad \text{for } \varepsilon<1,$$

we deduce

$$\int_{\lambda}^{\frac{1}{10\delta}} \frac{1}{|\log(\delta\varepsilon)|^{\gamma}}\frac{\mathrm{d}\varepsilon}{\varepsilon} \simeq_{\gamma} |\log(\delta\lambda)|^{1-\gamma}, \tag{3.8}$$

when $\delta\lambda$ is small enough (for instance we can ask $\delta\lambda<1/100$, that is verified under our assumption), and

$$\int_{\lambda}^{\frac{1}{10\delta}} \frac{\left|\log\left(\varepsilon\wedge\frac{1}{2}\right)\right|}{|\log(\delta\varepsilon)|^{\gamma}}\frac{\mathrm{d}\varepsilon}{\varepsilon} \lesssim_{\gamma} |\log(\delta\lambda)|^{1-\gamma}\left(\frac{|\log(\lambda)^2}{\log(\delta\lambda)} + \left(\frac{|\log(\delta)|}{|\log(\delta\lambda)|}\right)^{1-\gamma}\right) \lesssim_{\gamma} |\log(\delta\lambda)|^{1-\gamma}|\log(\lambda)|, \tag{3.9}$$

for every $\delta>0$. Putting (3.7), (3.8) and (3.9) together we get

$$\|f\|^2_{L^2} \lesssim_{d,\gamma} \frac{1}{|\log(\delta\lambda)|^{1-\gamma}} \int_{B_{\frac{1}{5\delta}}} \int_{\mathbb{R}^d} \frac{|f(x+h)-f(x)|^2}{|h|^d|\log(\delta|h|)|^{\gamma}}\,\mathrm{d}x\,\mathrm{d}h + |\log(\lambda)| \int_{\mathbb{R}^d} \frac{|\hat{f}(\xi)|^2}{\log(2+|\xi|)}\,\mathrm{d}\xi.$$

In order to conclude the proof remain only to show

$$\int_{\mathbb{R}^d} \frac{|\hat{f}(\xi)|^2}{\log(2+|\xi|)}\,\mathrm{d}\xi \le \frac{2\,\|f\|^2_{L^2}}{\log(2+\frac{\|f\|_{L^2}}{\|f\|_{\dot{H}^{-1}}})}. \tag{3.10}$$

To this aim we fix a parameter $\nu>0$, and we estimate

$$\begin{aligned}
\int_{\mathbb{R}^d} \frac{|\hat{f}(\xi)|^2}{\log(2+|\xi|)}\,\mathrm{d}\xi &= \int_{|\xi|\le\nu} \frac{|\xi|^2}{\log(2+|\xi|)}|\xi|^{-2}|\hat{f}(\xi)|^2\,\mathrm{d}\xi + \int_{|\xi|\ge\nu} \frac{1}{\log(2+|\xi|)}|\hat{f}(\xi)|^2\,\mathrm{d}\xi \\
&\le \frac{\nu^2}{\log(2+\nu)}\int_{|\xi|\le\nu} |\xi|^{-2}|\hat{f}(\xi)|^2\,\mathrm{d}\xi + \frac{1}{\log(2+\nu)}\int_{|\xi|\ge\nu} |\hat{f}(\xi)|^2\,\mathrm{d}\xi \\
&\le \frac{\nu^2}{\log(2+\nu)}\|f\|^2_{\dot{H}^{-1}} + \frac{1}{\log(2+\nu)}\|f\|^2_{L^2}.
\end{aligned}$$

Choosing $\nu=\frac{\|f\|_{L^2}}{\|f\|_{\dot{H}^{-1}}}$, one gets (3.10). The thesis is now proved. □

## 3.2 Proof of Theorem 3.2

Before going into details with the proof of Theorem 3.2 we present the last technical ingredient. It can be seen as an orthogonality property, with respect to the log-Sobolev functional (1.4), for functions with disjoint supports.

**Lemma 3.8.** *Let $\gamma\in(-\infty,1)$ be fixed. For every $n\in\mathbb{N}$ consider an open set $\Omega_n$, a function $f_n\in L^2(\mathbb{R}^d)$ and a parameter $0<\lambda_n<1/4$. Assume that the family $\{\,\Omega_n\,\}_{n\in\mathbb{N}}$ is disjoint and that the distance between* supp $f_n$ *and $\mathbb{R}^d\setminus\Omega_n$ is bigger than $\lambda_n$ for every $n\in\mathbb{N}$.*

*Then it holds*

$$\int_{B_{1/2}} \int_{\mathbb{R}^d} \frac{|\sum_n f_n(x+h) - \sum_n f_n(x)|^2}{|h|^d} \frac{1}{\log(1/|h|)^\gamma} \,\mathrm{d}x\,\mathrm{d}h$$
$$\geq \limsup_{N\to\infty} \sum_{n=1}^{N} \left( \int_{B_{1/2}} \int_{\mathbb{R}^d} \frac{|f_n(x+h) - f_n(x)|^2}{|h|^d \log(1/|h|)^\gamma} \,\mathrm{d}x\,\mathrm{d}h - \frac{4\,\|f_n\|_{L^2}^2}{1-\gamma} |\log(\lambda_n)|^{1-\gamma} \right). \tag{3.11}$$

*Proof.* Let us call $\bar{\Omega}_n \subset \Omega_n$ the set of $x \in \mathbb{R}^d$ whose distance from $\operatorname{supp} f_n$ is smaller than $\lambda_n/2$. Observe that

$$\int_{B_1} \int_{\mathbb{R}^d} \frac{|\sum_n f_n(x+h) - \sum_n f_n(x)|^2}{|h|^d} \frac{1}{\log(1/|h|)^\gamma} \,\mathrm{d}x\,\mathrm{d}h$$
$$\geq \limsup_{N\to\infty} \sum_{n=1}^{N} \int_{B_{\lambda_n/2}} \int_{\bar{\Omega}_n} \frac{|f_n(x+h) - f_n(x)|^2}{|h|^d} \frac{1}{\log(1/|h|)^\gamma} \,\mathrm{d}x\,\mathrm{d}h$$
$$= \limsup_{N\to\infty} \sum_{n=1}^{N} \Big( \int_{B_{1/2}} \int_{\mathbb{R}^d} \frac{|f_n(x+h) - f_n(x)|^2}{|h|^d} \frac{1}{\log(1/|h|)^\gamma} \,\mathrm{d}x\,\mathrm{d}h$$
$$- \int_{B_{1/2}\setminus B_{\lambda_n/2}} \int_{\mathbb{R}^d} \frac{|f_n(x+h) - f_n(x)|^2}{|h|^d} \frac{1}{\log(1/|h|)^\gamma} \,\mathrm{d}x\,\mathrm{d}h \Big).$$

On the other hand,

$$\int_{B_{1/2}\setminus B_{\lambda_n/2}} \int_{\mathbb{R}^d} \frac{|f_n(x+h) - f_n(x)|^2}{|h|^d} \frac{1}{\log(1/|h|)^\gamma} \,\mathrm{d}x\,\mathrm{d}h$$
$$\leq 2 \int_{B_{1/2}\setminus B_{\lambda_n/2}} \int_{\mathbb{R}^d} \frac{|f_n(x)|^2}{|h|^d \log(1/|h|)^\gamma} \,\mathrm{d}x\,\mathrm{d}h + 2 \int_{B_{1/2}\setminus B_{\lambda_n}} \int_{\mathbb{R}^d} \frac{|f_n(x+h)|^2}{|h|^d \log(1/|h|)^\gamma} \,\mathrm{d}x\,\mathrm{d}h$$
$$\leq 4\,\|f_n\|_{L^2}^2 \int_{B_{1/2}\setminus B_{\lambda_n/2}} \frac{1}{|h|^d} \frac{1}{\log(1/|h|)^\gamma} \,\mathrm{d}h \leq \frac{4\,\|f_n\|_{L^2}^2}{1-\gamma} |\log(\lambda_n)|^{1-\gamma}.$$

Combining these inequalities the thesis follows. □

We are now in position to prove Theorem 3.2.

*Proof.* Let $p \geq 1$ be fixed. We consider $v$ and $\rho$ as in Proposition 3.4, and a family of disjoint open cubes $\{ Q_n \}_{n\in\mathbb{N}}$ contained in $B_1$. Let us define sequences

$$\lambda_n = e^{-n}, \qquad \gamma_n = \frac{1}{n^2}, \qquad \tau_n = \left(n^2 e^{-dn}\right)^{1/p}. \tag{3.12}$$

Assuming that the cube $Q_n$ has side of length $3\lambda_n$ and center at $x_n \in B_1$, we set

$$v_n(t,x) := \frac{\lambda_n}{\tau_n} v\left(\frac{t}{\tau_n}, \frac{x-x_n}{\lambda_n}\right), \qquad \rho_n(t,x) := \gamma_n \rho\left(\frac{t}{\tau_n}, \frac{x-x_n}{\lambda_n}\right),$$

for every $x \in \mathbb{R}^d$, $t \geq 0$ and $n \in \mathbb{N}$. Observe that $u_n$ is supported in $Q_n$ and $\operatorname{dist}(\operatorname{supp} u_n, \mathbb{R}^d \setminus Q_n) \geq \lambda_n$ for every $n \in \mathbb{N}$.

Setting

$$b(t,x) := \sum_n v_n(t,x), \qquad u(t,x) := \sum_n \rho_n(t,x) \qquad \forall x \in \mathbb{R}^d, \quad \forall t > 0,$$

the following facts hold true.

(i) $b$ is a divergence free vector field supported in $B_1 \times [0,\infty]$ and belonging to $L^\infty((0,\infty); W^{1,p}(\mathbb{R}^d))$;

(ii) $u$ is supported in $B_1 \times [0,\infty)$ and $u \in L^\infty((0,\infty) \times \mathbb{R}^d)$;

(iii) the initial data $u_0$ belongs to $W^{1,d}(\mathbb{R}^d)$;

(iv) $u$ is a solution of the continuity equation (CE) with vector field $b$.

Let us fix $t \geq 0$. We have

$$\begin{aligned}\|b_t\|_{W^{1,p}}^p &\leq \sum_n (\|v_n(t,\cdot)\|_{L^p}^p + \|\nabla v_n(t,\cdot)\|_{L^p}^p) \\ &= \sum_n \left(\frac{\lambda_n}{\tau_n}\right)^p (\lambda_n^d \|v_{t/\tau_n}\|_{L^p}^p + \lambda_n^{d-p} \|\nabla v_{t/\tau_n}\|_{L^p}^p) \\ &\leq \sup_{s\geq 0} \|v_s\|_{W^{1,p}}^p \sum_n \frac{\lambda_n^d}{\tau_n^p} = \sup_{s\geq 0}\|v_s\|_{W^{1,p}}^p \sum_n \frac{1}{n^2} < \infty,\end{aligned}$$

this proves the non-trivial part of (i). The point (ii) follows observing that $\sup_n \gamma_n < \infty$. In order to prove (iii) we estimate

$$\begin{aligned}\|u_0\|_{W^{1,d}}^d &\leq \sum_n \left(\|\rho_n(0,\cdot)\|_{L^d}^d + \|\nabla\rho_n(0,\cdot)\|_{L^d}^d\right) \\ &= \sum_n \gamma_n^d \left(\lambda_n^d \|\rho_0\|_{L^d}^d + \|\nabla \rho_0\|_{L^d}^d\right) \\ &\leq \|\rho_0\|_{W^{1,d}}^d \sum_n \gamma_n^d < \infty.\end{aligned}$$

The last point follows from the construction.

We are now ready to prove (3.1). Fix a time $t > 0$ and $\gamma \in (-\infty, 1)$. Thanks to Lemma 3.8 and Remark 2.15 we have

$$\begin{aligned}&\int_{B_1}\int_{\mathbb{R}^d} \frac{|u(t,x+h)-u(t,x)|^2}{|h|^d \log(1/|h|)^\gamma}\,\mathrm{d}x\,\mathrm{d}h \\ &\geq \limsup_{N\to\infty} \sum_{n=1}^N \left(\int_{B_{1/2}}\int_{\mathbb{R}^d} \frac{|\rho_n(t,x+h)-\rho_n(t,x)|^2}{|h|^d\log(1/|h|)^\gamma}\,\mathrm{d}x\,\mathrm{d}h - \frac{4\,\|\rho_n(t,\cdot)\|_{L^2}^2}{1-\gamma}|\log(\lambda_n)|^{1-\gamma}\right) \\ &= \limsup_{N\to\infty}\sum_{n=1}^N \gamma_n^2 \left(\int_{B_{1/2}}\int_{\mathbb{R}^d} \frac{|\rho\left(\frac{t}{\tau_n}, \frac{x+h-x_n}{\lambda_n}\right) - \rho\left(\frac{t}{\tau_n},\frac{x-x_n}{\lambda_n}\right)|^2}{|h|^d\log(1/|h|)^\gamma}\,\mathrm{d}x\,\mathrm{d}h - \frac{4\left\|\rho\left(\frac{t}{\tau_n},\frac{\cdot}{\lambda_n}\right)\right\|_{L^2}^2}{1-\gamma}|\log(\lambda_n)|^{1-\gamma}\right) \\ &= \limsup_{N\to\infty}\sum_{n=1}^N \gamma_n^2\lambda_n^d \left(\int_{B_{\frac{1}{2\lambda_n}}}\int_{\mathbb{R}^d} \frac{|\rho\left(\frac{t}{\tau_n},x+h\right)-\rho\left(\frac{t}{\tau_n},x\right)|^2}{|h|^d|\log(|\lambda_n h|)|^\gamma}\,\mathrm{d}x\,\mathrm{d}h - \frac{4\,\|\rho_0\|_{L^2}^2}{1-\gamma}|\log(\lambda_n)|^{1-\gamma}\right).\end{aligned}$$

Let us fix $n \in \mathbb{N}$ and a parameter $\lambda \in (0, 1/100)$ to be specified later. Applying Corollary 3.6 with parameters $\gamma$, $\lambda$, and $\delta = \lambda_n$ (we need to consider $n$ bigger than a suitable integer $n_\gamma$ depending only on $\gamma$) we get

$$\begin{aligned}&\int_{B_{\frac{1}{2\lambda_n}}}\int_{\mathbb{R}^d} \frac{|\rho\left(\frac{t}{\tau_n},x+h\right)-\rho\left(\frac{t}{\tau_n},x\right)|^2}{|h|^d|\log(|\lambda_n h|)|^\gamma}\,\mathrm{d}x\,\mathrm{d}h \\ &\qquad\gtrsim_\gamma \|\rho_0\|_{L^2}^2\,|\log(\lambda_n\lambda)|^{1-\gamma}\left(C_\gamma - |\log(\lambda)|\frac{\tau_n C(\|\rho_0\|_{L^2})}{\tau_n+ct}\right).\end{aligned}$$

We now take $\lambda$ such that $a_t\tau_n^{-1} = |\log(\lambda)|$ (at least for $n$ big enough), where

$$a_t := \frac{C_\gamma}{2C(\|\rho_0\|_{L^2})}ct,$$

with this choice we obtain

$$|\log(\lambda_n\lambda)|^{1-\gamma}\left(C_\gamma - |\log(\lambda)|\frac{\tau_n C(\|\rho_0\|_{L^2})}{\tau_n+ct}\right) \geq (|\log(\lambda_n)| + a_t(\tau_n)^{-1})^{1-\gamma}\frac{C_\gamma}{2} \geq \bar{C}t^{1-\gamma}\tau_n^{\gamma-1},$$

for every $n \geq n_\gamma$, where $\bar{C}$ is a positive constant depending only on, $c$ (see Proposition 3.4), $\|\rho_0\|_{L^2}$ and $\gamma$.

Putting all together and recalling (3.12) we get

$$
\begin{aligned}
&\int_{B_{1/2}} \int_{\mathbb{R}^d} \frac{|u(t,x+h)-u(t,x)|^2}{|h|^d} \frac{1}{\log(1/|h|)^\gamma} \,\mathrm{d}x \,\mathrm{d}h \\
&\geq \bar{C} t^{1-\gamma} \sum_{n=n_\gamma}^{\infty} \gamma_n^2 \lambda_n^d \tau_n^{\gamma-1} - \frac{4\,\|\rho_0\|_{L^2}}{1-\gamma} \sum_{n=1}^{\infty} \gamma_n^2 \lambda_n^d |\log(\lambda_n)|^{1-\gamma} \\
&= \bar{C} t^{1-\gamma} \sum_{n=n_\gamma}^{\infty} n^{\frac{2(\gamma-1)}{p}-4} e^{-dn\frac{\gamma+p-1}{p}} - \frac{4\,\|\rho_0\|_{L^2}}{1-\gamma} \sum_{n=1}^{\infty} n^{-\gamma-3} e^{-dn},
\end{aligned}
$$

that is equal to $+\infty$ when $\gamma < 1-p$ and $t > 0$. The thesis is proved. □

## 3.3 Mixing estimates

As a simple byproduct of our results in Theorem 2.1 and Corollary 3.7 we get two estimates on the mixing rate for solutions of (CE) drifted by divergence-free vector fields that are bounded in $W^{1,p}$, uniformly in time, for $p > 1$.

These results are already present in the literature (see [CDL08, Theorem 6.2], [IKX14], [Se13], [LF16]), the extension to the case $p = 1$ is an important open problem related to the so-called Bressan's mixing conjecture (see [Br03]).

Let us begin with a simple estimate involving the geometric mixing scale (see [Br03]).

**Lemma 3.9.** *Let $\sigma > 0$, $f \in L^\infty(\mathbb{R}^d)$ be such that $\|f\|_{L^\infty} = 1$ and $\mathscr{L}^d(\{|f| = 1\}) \geq c_0 > 0$. Then, for any $\kappa \in (0,1)$, and $\varepsilon \in (0, 1/2)$, it holds*

$$
\sup_{x \in \mathbb{R}^d} \left| \fint_{B_\varepsilon} f(x+y)\,\mathrm{d}y \right| < \kappa \Longrightarrow \varepsilon \geq \exp\left( -C \left[ \sup_{h \in B_{1/2}} \log(1/|h|)^\sigma \int_{\mathbb{R}^d} |f(x+h) - f(x)|\,\mathrm{d}x \right]^{1/\sigma} \right), \tag{3.13}
$$

*where $C = (c_0(1-\kappa))^{-1/\sigma}$.*

*Proof.* For any $x \in \{|f| = 1\}$, we have $1 - \kappa < \left| \fint_{B_\varepsilon} (f(x) - f(x+y))\,\mathrm{d}y \right|$. So,

$$
\begin{aligned}
c_0(1-\kappa) &< \fint_{B_\varepsilon} \int_{\mathbb{R}^d} |f(x) - f(x+y)|\,\mathrm{d}x\,\mathrm{d}y \\
&\leq \left[ \sup_{h \in B_{1/2}} \log(1/|h|)^\sigma \int_{\mathbb{R}^d} |f(x) - f(x+h)| \mathrm{d}x \right] \fint_{B_\varepsilon} \log(1/|y|)^{-\sigma}\,\mathrm{d}y \\
&\leq \left[ \sup_{h \in B_{1/2}} \log(1/|h|)^\sigma \int_{\mathbb{R}^d} |f(x) - f(x+h)|\,\mathrm{d}x \right] |\log(\varepsilon)|^{-\sigma}.
\end{aligned}
$$

This implies (3.13). The proof is complete.

□

We are now ready to state and prove the aforementioned mixing estimates.

**Proposition 3.10.** *Let $p > 1$ be fixed. Let us consider a bounded divergence-free vector field $b$ such that*

$$
\|\nabla b_t\|_{L^p} \leq B < \infty \qquad \text{for a.e. } t \geq 0.
$$

*Then for every initial data $u_0 \in BV(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d)$ the (unique) solution $u \in L^\infty((0,T) \times \mathbb{R}^d)$ of the continuity equation (CE) satisfies*

$$
\|u_t\|_{\dot{H}^{-1}} \geq C \exp(-cBt), \qquad t \geq 0, \tag{3.14}
$$

*where $C > 0$ and $c > 0$ depend on $\|u_0\|_{L^2}$, $\|u_0\|_{BV}$, $p$ and $d$.*

*Furthermore, assume $u_0(x) \in \{1, -1, 0\}$ for every $x \in \mathbb{R}^d$ and $\int_{\mathbb{R}^d} |u_0| \, dx \geq c_0 > 0$. Then, for any $\kappa \in (0,1)$, and $\varepsilon \in (0, 1/2)$, it holds*

$$\sup_{x \in \mathbb{R}^d} \left| \fint_{B_\varepsilon(x)} u_t(y) \, dy \right| < \kappa \Longrightarrow \varepsilon \geq \exp(-CBt), \tag{3.15}$$

*where $C > 0$ depends on $p$, $d$, $\kappa$, $c_0$ and $\|u_0\|_{BV}$.*

*Proof.* The first part of the statement follows immediately applying Corollary 3.7 with $\gamma = 1-p$ and Theorem 2.1. The second part is a consequence of Lemma 3.9, applied with $\sigma = p$, and Remark 2.2 together with the following elementary observation: if $f$ is a measurable function that takes only the values 1, 0 and $-1$ then

$$\int_{\mathbb{R}^d} |f(x+h) - f(x)| \, dx \leq \int_{\mathbb{R}^d} |f(x+h) - f(x)|^2 \, dx,$$

for every $h \in \mathbb{R}^d$. □

Two remarks are in order.

*Remark* 3.11. The mixing estimate (3.14) is still true considering the $\dot{H}^{-s}$ semi-norm with $s > 0$, indeed we have

$$\|u_t\|_{\dot{H}^{-s}} \geq C \exp(-cBst), \qquad t \geq 0.$$

It can be proved modifying slightly the interpolation inequality (3.4).

*Remark* 3.12. Let us assume $b_t$ to be smooth and compactly supported in $Q = [0,1]^d \subset \mathbb{R}^d$. Call $X_t$ its flow. Setting $u_0 = \mathbf{1}_A - \mathbf{1}_{Q \setminus A} \in BV(\mathbb{R}^d)$ with $A \subset [0,1]^d$ and $\mathscr{L}^d(A) = \frac{1}{2}$. It is immediate to see that $u_t := \mathbf{1}_{A_t} - \mathbf{1}_{Q \setminus A_t}$ where $A_t = X_t(A)$. Setting $\kappa = 1/2$ and fixing $p > 1$ the implication in (3.15) implies

$$\frac{1}{4} \leq \frac{\mathscr{L}^d(B(x,\varepsilon) \cap A_t)}{\omega_d \varepsilon^d} \leq \frac{3}{4} \implies \int_0^t \|\nabla b_s\|_{L^p} \, ds \geq C |\log(\varepsilon)|, \tag{3.16}$$

where $C$ depends on $p, d$ and $\|u_0\|_{BV}$.

Note that (3.16) is exactly the statement of the Bressan's conjecture for $p > 1$ (see [Br03]) that has been proved for a first time in [CDL08].

Let us conclude the section with an open question.

**Open Question 3.13.** Let $b \in L^\infty((0,+\infty); W^{1,1}(\mathbb{R}^d, \mathbb{R}^d))$ be a divergence-free vector field with compact support. Fix an initial data $\bar{u} \in C_c^\infty(\mathbb{R}^d)$ and consider $u_t$ the unique solution in $L^\infty([0,T] \times \mathbb{R}^d)$ of the continuity equation (CE). Is there an increasing function $\psi : (0,\infty) \to (0,\infty)$ with $\lim_{a \to \infty} \psi(a) = \infty$ and $\psi^{-1}(2t) \leq C\psi^{-1}(t)$ such that

$$\sup_{h \in B_{1/2}} \psi\left(\log(1/|h|)\right) \int_{\mathbb{R}^d} |u_t(x+h) - u_t(x)|^2 \, dx \leq C\psi(t) < \infty,$$

for every time $t$ big enough?

Let us remark that a positive answer of (3.13), together with the proof of Lemma 3.9 implies an exponential bound on mixing in the case $p = 1$ (see Proposition 3.10), proving the complete version of the conjecture by Bressan (see [Br03]).

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