IMPLICIT TIME DISCRETIZATION FOR THE MEAN CURVATURE FLOW OF OUTWARD MINIMIZING SETS

GUIDO DE PHILIPPIS AND TIM LAUX

Abstract. In this note we analyze the Almgren-Taylor-Wang scheme for mean curvature flow in the case of outward minimizing initial conditions. We show that the scheme preserves the outward minimizing property and, by compensated compactness techniques, that the arrival time functions converge strictly in BV. In particular, this establishes the convergence of the time-integrated perimeters of the approximations. As a corollary, the conditional convergence result of Luckhaus-Sturzenhecker becomes unconditonal in the outward minimizing case.

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1. Introduction

In 1993, Almgren-Taylor-Wang [1] proposed an implicit time discretization for mean curvature flow, which comes as a family of variational problems. Given an open subset $E_0 \subset \mathbb{R}^n$ and a time-step size $h > 0$, the sets $E_1, E_2, \ldots$ are successively obtained by solving

$$E_k \in \arg\min_E \left\{ P(E) + \frac{1}{h} \int_{E\Delta E_{k-1}} d_{E_{k-1}} \right\}, \tag{1.1}$$

where $P(E) = \sup\{\int_E \operatorname{div} \xi \colon \|\xi\|_\infty \le 1\}$ denotes the perimeter of an open subset of $\mathbb{R}^n$, d_E the distance function to the boundary of E and $E\Delta E_{k-1}$ the symmetric difference of E and E_{k-1}.

At the very heart of their idea lies the gradient-flow structure of mean curvature flow: trajectories in state space follow the steepest descent of the area functional with respect to an L^2-type metric. In fact, this scheme inspired Ennio De Giorgi [6] to define his minimizing movements for general gradient flows in metric spaces, see [3]. Given a metric dist and an energy functional E, each time step of his abstract scheme is a minimization problem of the form

$$x_k \in \arg\min_x \left\{ E(x) + \frac{1}{2h} \operatorname{dist}^2(x, x_{k-1}) \right\}.$$

In the smooth finite dimensional case when dist is the induced distance of a Riemannian metric, the Euler-Lagrange equation of the scheme boils down to the implicit Euler scheme.

In case of mean curvature flow, the metric tensor (L^2-metric on normal velocities) is completely degenerate in the sense that the induced distance vanishes identically, [18]; which explains the use of the proxy $2h \int_{E_{k+1}\Delta E_k} d_{E_k}$ for the squared distance in the minimizing movements scheme (1.1).

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The initial motivation of [1] was to define a generalized mean curvature flow through singularities as limits of the scheme (1.1). The convergence analysis as $h \downarrow 0$ has a long histrory: Compactness of the approximate solutions was already established in [1], together with the consistency of the scheme, in the sense that the approximations converge to the smooth mean curvature flow as long as the latter exists. In [5], Chambolle simplified the proof and, furthermore, proved convergence to the viscosity solution (see [9]), provided the latter is unique. More precisely, setting $E_h(t) = E_k$, $t \in [kh,(k+1)h)$ to be the piecewise constant in time interpolation of the sets E_k obtained from (1.1), then the result reads as follows, see [4] for the notion of viscosity solution in this context.

Theorem 1.1 (Convergence to viscosity solution [5, Theorem 4])**.** *Suppose $T<\infty$ and E_0 is a bounded set in $\mathbb{R}^n$ with $\mathcal{L}^n(\partial E_0)=0$ such that the viscosity solution $\mathbf{1}_{E(t)}$ starting from $\mathbf{1}_{E_0}$ is unique, then $E_h \to E$ in L^1, i.e., $\int_0^T |E_h(t)\Delta E(t)| \to 0$ as $h\downarrow 0$.*

Only shortly after [1], Luckhaus-Sturzenhecker [15] published a conditional convergence result which does not rely on the comparison principle but is purely based on the gradient-flow structure of mean curvature flow. In particular they showed that, *conditioned on the convergence of the perimeters*, the scheme converges to a BV solution of the mean curvature flow, according to the following definition.

Definition 1.2. A set of finite perimeter $E \subset \mathbb{R}_+ \times \mathbb{R}^n$ is a BV solution of the mean curvature flow if there exists $V \in L^2(0,T;L^2(\mathcal{H}^{n-1} \llcorner \partial^* E(t)))$ such that

$$\int_0^T \int_{\partial^* E(t)} (\operatorname{div}\xi - \nu\cdot D\xi\,\nu)\, d\mathcal{H}^{n-1}\,dt = -\int_0^T\int_{\partial^* E(t)} V\,\xi\cdot\nu\, d\mathcal{H}^{n-1}\,dt, \tag{1.2}$$

$$\int_0^T\int_{E(t)} \partial_t\psi(t,x)\,dx\,dt + \int_{E(0)}\psi(0,x)\,dx = -\int_0^T\int_{\partial^* E(t)} \psi(t,x) V\, d\mathcal{H}^{n-1}(x)\,dt \tag{1.3}$$

for all $\xi \in C^1_c([0,T)\times\mathbb{R}^n;\mathbb{R}^n)$ and $\psi \in C^1_c([0,T)\times\mathbb{R}^n;\mathbb{R})$. Here $E(t)$ is the time slice of E and ∂^* denotes the reduced boundary.

The main result in [15] is the following conditional convergence result:

Theorem 1.3 (Conditional convergence [15, Theorem 2.3])**.** *Let $n\le 7$ and let E_h be the (time) piecewise constant approximation built by the Almgren-Taylor-Wang scheme. Then there exists a set $E\subset \mathbb{R}_+\times\mathbb{R}^n$ and a subsequence $\{h_j\}$ such that $E_{h_j}(t)\to E$ in L^1. Moreover, if*

$$\lim_{h_j\downarrow 0}\int_0^T P(E_{h_j}(t))\,dt = \int_0^T P(E(t))\,dt, \tag{1.4}$$

then E is a BV solution of mean curvature flow.

We also refer the reader to the work of Mugnai-Seis-Spadaro [19] where the proof of [15] is revisited in the case of volume-preserving mean curvature flow.

To the best of our knowledge, the only case in which assumption (1.4) has been shown to be satisfied *a-priori* is in the graphical case [14], in which no singularities occur, [8]. In this note we show that if the initial data is outward minimizing, i.e., it satisfies

$$P(E) \le P(F) \qquad \text{for all } F\supset E$$

then the same property is satisfied along the discrete flows and (1.4) holds true. Note that the outward minimizing property is the variational analogue of the mean convexity condition, see Definition 2.3 and Remark 2.4.

More precisely, our main theorem reads as follows.

Theorem 1.4. *Let $E_0 \subset \mathbb{R}^n$ be a compact set with C^2 boundary and let $n \leq 7$. Assume that E_0 is outward minimizing in the sense of Definition 2.3, then* (1.4) *holds.*

Let us also remark that a similar question was raised by Ilmanen for the approximation of the mean curvature flow via the Allen-Cahn equation [13, Section 13, Question 4].

As already mentioned, along the way we establish the following natural properties of the minimizing movements scheme (1.1) for outward minimizing, which mirror Huisken's results for mean curvature flow [11]:

- The sets E_k are nested in the sense that $E_{k+1} \subset E_k$ for all $k \geq 1$.
- The scheme preserves the outward minimizing property and moreover, if $n \leq 7$, the minimum of the mean curvature of ∂E, $\min_{\partial E_k}, H_{\partial E_k}$ is increasing in k.

While Huisken's proofs are based on the maximum principle, our proofs are solely of variational nature.

Inspired by the work of Evans-Spruck [9] on mean curvature flow, we introduce the arrival time u_h of the scheme. As the name suggests, the arrival time $u(x)$ of the mean curvature flow starting from $E_0 \subset \mathbb{R}^n$ at a point $x \in E_0$ is the first time $t > 0$ at which the flow reaches x, i.e., the super level set $\{u > t\}$ is equal to $E(t)$. Similarly, as the sets E_k obtained by the scheme are nested, one may also define the arrival time u_h of the scheme so that $E_h(t) = \{u_h > t\}$. By the coarea formula the proof of Theorem 1.4 then boils down to the convergence of the total variation of the functions u_h. This is obtained by using a compensated compactness argument in line with the one in [9], together with some duality formulation of the obstacle problem established in [20].

As a direct consequence of our main theorem, the convergence result of Luckhaus-Sturzenhecker becomes unconditional in the case of outward minimizing initial data:

Corollary 1.5. *Suppose $n \leq 7$ and E_0 is mean convex, then any L^1-limit of the approximations $E_h(t)$ is a BV solution of mean curvature flow.*

The paper is organized as follows. In Section 2, we establish some basic properties of outward minimizing sets and of the minimization scheme when applied to mean convex sets. In Section 3 we define the arrival time of the scheme and prove that it solves an obstacle problem. In Section 4 we show it converges to the arrival time of the discrete evolution and eventually in Section 5, we prove Theorem 1.4.

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2. Basic properties of the scheme and mean convexity

We recall the definition and derive some first properties for the implicit time discretization scheme (1.1) when the initial set is outward minimizing. The basis of our analysis is Lemma 2.7, which states that the scheme preserves outward minimality and that $\min H_{\partial E(t)}$ is non-decreasing in t.

Let us state the minimization problem (1.1) in a more precise language: Given initial conditions $E_0 \subset \mathbb{R}^n$, obtain E_k for $k \in \mathbb{N}$ by successively minimizing $\mathcal{F}_h(E, E_{k-1})$:

$$E_k \in \arg\min \mathcal{F}_h(\,\cdot\,, E_{k-1}), \tag{2.1}$$

where the functional $\mathcal{F}_h$ is given by

$$\mathcal{F}_h(E, F) := P(E) + \frac{1}{h}\int_{E\Delta F} d_F.$$

Here and throughout the paper $d_F(x) := \mathrm{dist}(x, \partial F)$ denotes the distance function to the boundary of F. We will always work with the representative of F for which $\overline{\partial^* F} = \partial F$, $\partial^* F$ being the reduced boundary of F, see [16, Remark 15.3].

We denote by E_h the piecewise constant interpolation of the sets $E_0, E_1, E_2, \ldots$, i.e.,

$$E_h(t) = E_k \quad \text{for } t \in [kh, (k+1)h).$$

Remark 2.1. It is easy to see that the metric term $\int_{E\Delta F} d_F$ can be rewritten as

$$\int_{E\Delta F} d_F = \int_E d_F - \int_F sd_F,$$

where $sd_F := d_F - d_{\mathbb{R}^n \setminus F}$ denotes the signed distance function to the boundary ∂F. Therefore the minimization of $\mathcal{F}_h(\,\cdot\,, F)$ is equivalent to minimizing

$$P(E) + \frac{1}{h} \int_E sd_F.$$

This implies the following a priori estimate for the implicit time discretization

$$\sup_{k\geq 1} P(E_k) + \sum_{k\geq 1} \frac{1}{h} \int_{E_k \Delta E_{k-1}} d_{E_k-1} \leq P(E_0), \tag{2.2}$$

which underlies Luckhaus-Sturzenhecker's compactness and conditional convergence Theorem 1.3.

Remark 2.2. In the radially symmetric case $E_0 = B_{r_0}$, a Steiner symmetrization argument shows that the minimizers are radially symmetric. Therefore, the minimization problem (2.1) reduces to finding radii $r_0 > r_1 > r_2 > \ldots$ so that each r_k minimizes the function

$$r^{n-1} + \frac{1}{h} \int_r^{r_{k-1}} \rho^{n-1}(r_{k-1} - \rho) d\rho.$$

The Euler-Lagrange equation is

$$r_k^2 - r_{k-1} r_k + (n-1)h = 0 \text{ (or equivalently } \frac{r_k - r_{k-1}}{h} = -\frac{n-1}{r_k}),$$

so that for sufficiently small h the optimal radius is explicitly given by

$$r_k = \frac{1}{2}\Big(r_{k-1} + \sqrt{r_{k-1}^2 - 4(n-1)h} \Big).$$

Note that for fixed h, after $O(r_0^2 h^{-1})$ steps we have $r_k = 0$. Note also that, as one can easily see by induction,

$$r_k \geq \sqrt{r_0^2 - 2k(n-1)h}.$$

It is a well known fact in the study of mean curvature flow that mean-convexity of the initial condition (i.e. $H_{\partial E_0}$) is preserved, [11] and that in this setting much stronger results can be obtained, see for instance [10, 21, 22] for an incomplete list and [17] where a problem similar to ours is studied.

Here, as in [12] we introduce the variational analogue of mean convexity:

Definition 2.3. A set $E \subset \mathbb{R}^n$ is called *outward minimizing* if

$$P(E) \leq P(F) \quad \text{for all } F \supset E. \tag{2.3}$$

Remark 2.4. Outward minimizing property as defined above is the variational formulation of the pointwise inequality

$$H \geq 0.$$

Note that being outward minimizing is a stronger condition than only $H \geq 0$ since it detects also non-local effects, the union of any two disjoint open intervals $I, J \subset \mathbb{R}$ is not outward minimizing in the sense of Definition 2.3.

It is however easy to see that if the initial set E_0 is smooth and strictly mean convex in the sense that $H_{\partial E_o} > 0$, there exists $\delta > 0$ such that E is δ-*outward minimizing*, i.e., $P(E_0) \leq P(F)$ for all sets $F \supset E_0$ with $\sup_{x\in F} \operatorname{dist}(x, E_0) < \delta$, see for instance [7, Lemma 5.12].

Since one can show that along the scheme the set does not move far away more than $O(\sqrt{h})$, see [15, Lemma 2.1,(1)], one can check that all proofs in Section 2 carry over in this case, in particular, the scheme preserves the δ-outward minimization property. However, our final argument does not trivially generalize to this setting.

The following lemma gives a characterization of outward minimality in terms of intersections with arbitrary sets.

Lemma 2.5. *E is outward minimizing if and only if*

$$P(E \cap G) \leq P(G) \quad \text{for all } G \subset \mathbb{R}^n. \tag{2.4}$$

Proof. We employ the basic inequality

$$P(E \cap F) + P(E \cup F) \leq P(E) + P(F), \tag{2.5}$$

Given any set G in $\mathbb{R}^n$, the outward minimizing property (2.3) of E tested with $F = E \cup G$ yields

$$P(E) \overset{(2.3)}{\leq} P(E \cup G) \leq P(E) + P(G) - P(E \cap G),$$

which simplifies to (2.4).

Vice versa, if $F \supset E$, we can apply (2.4) with $G = F$ to obtain (2.3). $\square$

A direct consequence of this characterization is that the outward minimizing property is stable for the L^1-convergence.

Corollary 2.6. *Let $E_h \to E$ in L^1 for some sequence $\{E_h\}_h$ of outward minimzing sets . Then E is outward minimzing.*

Proof. By Lemma 2.5 it is enough to show (2.4) instead of (2.3) for E, which in turn follows immediately from (2.4) for E_h and the lower semi-continuity of the perimeter. $\square$

If $\Sigma(t)$ is a smooth mean curvature flow then the scalar mean curvature H of $\Sigma(t)$ solves

$$\partial_t H - \Delta H = |A|^2 H,$$

where A denotes the second fundamental form of $\Sigma(t)$ and Δ the Laplace-Beltrami operator on $\Sigma(t)$, cf. [11, Corollary 3.5]. In particular, if $H \geq 0$ at $t = 0$, by the maximum principle $H \geq 0$ for $t \geq 0$ and $\min H(t)$ is non-decreasing in t. By the strong maximum principle we even have $H > 0$ for $t > 0$. The following lemma states that the same holds for the implicit time discretization (2.1).

Lemma 2.7. *Let E_0 be outward minimizing. Then the implicit time discretizations E_h are non-increasing in t:*

$$E_h(t) \subset E_h(s) \quad \text{for all } 0 \leq s \leq t, \tag{2.6}$$

$E_h(t)$ is outward minimizing for all t in the sense of Definition 2.3 and $E_h(t)$ solves the Euler-Langrange equation

$$H_{\partial E_h(t)}(x) = \frac{d_{E_h(t-h)}(x)}{h} \geq 0 \qquad x \in \partial^* E_h(t). \tag{2.7}$$

Furthermore, if $n \leq 7$, $\min H_{\partial E_h(t)}$ is non-decreasing in t.

Note that, by classical regularity for minimizers of (1.1), see e.g. [16], $\partial^* E_h(t)$ is a C^2-manifold relatively open in $\partial E_h(t)$ and $\partial E_h(t) \setminus \partial^* E_h(t)$ has Hausdorff dimension at most $n-8$. In particular (2.7) makes sense.

Proof. Let $k \geq 1$ and assume that E_{k-1} is outward minimizing. We first prove $E_k \subset E_{k-1}$ and then the outward minimizing property of E_k.

Since by assumption E_{k-1} is outward minimizing we may employ the characterization (2.4):

$$P(E_{k-1} \cap E_k) \leq P(E_k).$$

We want to use $E_{k-1} \cap E_k$ as a competitor for the minimization of $\mathcal{F}_h(\,\cdot\,, E_{k-1})$. Since

$$(E_{k-1} \cap E_k) \Delta E_{k-1} = E_{k-1} \setminus E_k \subset E_k \Delta E_{k-1}$$

we have

$$\frac{1}{h} \int_{(E_{k-1} \cap E_k)\Delta E_{k-1}} d_{E_{k-1}} \leq \frac{1}{h} \int_{E_k \Delta E_{k-1}} d_{E_{k-1}}$$

with strict inequality if $\mathcal{L}^n(E_k \setminus E_{k-1}) > 0$. Hence

$$\mathcal{F}_h(E_{k-1} \cap E_k, E_{k-1}) \leq \mathcal{F}_h(E_k, E_{k-1})$$

with strict inequality if $\mathcal{L}^n(E_k \setminus E_{k-1}) > 0$, which proves $E_k \subset E_{k-1}$ (up to Lebesgue null sets).

Let $F \supset E_k$; we want to verify $P(E_k) \leq P(F)$. Using the outward minimality of the predecessor E_{k-1} we have

$$P(F \cap E_{k-1}) \overset{(2.4)}{\leq} P(F)$$

and hence it is enough to prove the inequality (2.3) for sets F with $E_k \subset F \subset E_{k-1}$. Using the inclusions $E_k \subset F \subset E_{k-1}$ we have

$$F \Delta E_{k-1} = E_{k-1} \setminus F \subset E_{k-1} \setminus E_k = E_k \Delta E_{k-1}$$

and therefore

$$\frac{1}{h} \int_{F \Delta E_{k-1}} d_{E_{k-1}} \leq \frac{1}{h} \int_{E_k \Delta E_{k-1}} d_{E_{k-1}}.$$

Now the minimality $\mathcal{F}_h(E_k, E_{k-1}) \leq \mathcal{F}_h(F, E_{k-1})$ implies $P(E_k) \leq P(F)$ and hence E_k is outward minimizing. An induction over the time step k proves the mean convexity of all sets $E_h(t)$, $t \geq 0$.

Since (2.7) is classical, we now turn to prove the monotonicity of $\inf H_{\partial E_h(t)}$. Fix $k \in \mathbb{N}$ and let

$$x_0 \in \arg\min H_{\partial E_k}.$$

Since $H_{\partial E_h(t)} = \frac{1}{h} d_{E_{k-1}}$ is Lipschitz continuous and ∂E_k is compact, at least one such x_0 exists. We shift E_{k-1} by $h\, H_{\partial E_k}(x_0) = d_{E_k}(x_0)$ in the fixed direction $\nu_{\partial E_k}(x_0)$, i.e.,

$$F_{k-1} := E_{k-1} + h\, H_{\partial E_k}(x_0)\, \nu_{\partial E_k}(x_0).$$

By definition of x_0 we have $E_k \subset F_{k-1}$ and $x_0 \in \partial E_k \cap \partial F_{k-1}$ and thus

$$H_{\partial E_k}(x_0) \geq H_{\partial F_{k-1}}(x_0) \geq \min H_{\partial F_{k-1}} = \min H_{\partial E_{k-1}},$$

which is precisely our claim. $\square$

By Corollary 2.6, also the limiting set is outward minimizing. From this we can easily infer the monotonicity of the perimeters.

Corollary 2.8. *Let E_0 be mean convex and $E(t)$ an L^1-limit of the implicit time discretizations $E_h(t)$. Then $E(t)$ is outward minimizing for a.e. t and $P(E(t))$ is non-increasing in t.*

Proof. The outward minimizing property of $E(t)$ is an immediate consequence of Lemma 2.7 and Corollary 2.6. Since by Lemma 2.7 we have $E(t) \subset E(s)$ for $t \geq s$ we can use the mean convexity (2.3) of $E(t)$ to conclude $P(E(t)) \leq P(E(s))$ for $t \geq s$. □

The basic inequality (2.5) and the observation that we have the analogous equality for the distance-term in $\mathcal{F}$ yield the general inequality

$$\mathcal{F}_h(E \cap F, E_{k-1}) + \mathcal{F}_h(E \cup F, E_{k-1}) \leq \mathcal{F}_h(E, E_{k-1}) + \mathcal{F}_h(F, E_{k-1}). \tag{2.8}$$

Therefore, if E and F are minimizers, so are $E \cap F$ and $E \cup F$. In our setting, where E_{k-1} is outward minimizing , this implies the outward minimality of all these sets and we have equality in (2.5).

The following general lemma is a comparison result which holds independently of the initial conditions E_0 being mean convex and revisits Chambolle's ideas [5].

Lemma 2.9 (Comparison principle, [5])**.** *Let $E_0, F_0 \subset \mathbb{R}^n$ be two bounded open sets of finite perimeter such that E_0 is properly contained in F_0 in the sense that $E_0 \subset\subset F_0$. Let E and F be minimizers of $\mathcal{F}_h(\,\cdot\,, E_0)$ and $\mathcal{F}_h(\,\cdot\,, F_0)$, respectively, then E is properly contained in F, i.e., $E \subset\subset F$.*

Proof. The proof consists of two steps. First we prove the inclusion $E \subset F$, second we prove $\min_{x\in\partial E} d(x, \partial F) > 0$.

Inasmuch as $E_0 \subset\subset F_0$, the boundaries have a definite distance $\min_{x\in\partial E_0} d(x, \partial F_0) > 0$, which implies the strict inequality

$$sd_{F_0} < sd_{E_0} \quad \text{in } \mathbb{R}^n. \tag{2.9}$$

Probing the minimality of E and F for the modified functionals in Remark 2.1 with $E \cap F$ and $E \cup F$, respectively, yields

$$P(E) + \frac{1}{h}\int_E sd_{E_0} \leq P(E \cap F) + \frac{1}{h}\int_{E\cap F} sd_{E_0}$$

and

$$P(F) + \frac{1}{h}\int_F sd_{F_0} \leq P(E \cup F) + \frac{1}{h}\int_{E\cup F} sd_{F_0}.$$

Summing these two inequalities and using the general inequality for the perimeter of intersections and unions of sets (2.5) we obtain

$$\int_E sd_{E_0} + \int_F sd_{F_0} \leq \int_{E\cap F} sd_{E_0} + \int_{E\cup F} sd_{F_0}.$$

Rearranging the terms and using the obvious identities $\chi_{E\cap F} = \chi_E\chi_F$ and $\chi_{E\cup F} = \chi_E + \chi_F - \chi_E\chi_F$ along the way we obtain

$$0 \leq \int (sd_{E_0} - sd_{F_0})\,\chi_E\,(1 - \chi_F) = \int_{E\setminus F} (sd_{E_0} - sd_{F_0})\,.$$

Since by (2.9) the integrand is strictly negative, this means that $\mathcal{L}^n(E \setminus F) = 0$ and hence $E \subset F$.

Now assume for a contradiction $\partial E \cap \partial F \neq \emptyset$. Let $x_0 \in \partial E \cap \partial F$ be a point in the intersection. Since $E \subset F$ we have $H_{\partial E} \geq H_{\partial F}$ at that point x_0 and therefore

$$\frac{1}{h} sd_{E_0} = -H_{\partial E} \leq -H_{\partial F} = \frac{1}{h} sd_{F_0},$$

a contradiction to (2.9). □

3. The arrival time for the implicit time discretization

Since by Lemma 2.7 the sets $E_h(t)$ are nested, it makes sense to define the (discrete) arrival time u_h for the scheme. In this section we show that, up to subsequences, u_h converges uniformly to some continuous function u. In the next section we will identify u as the arrival time for the limiting evolution starting from E_0.

Definition 3.1. Let E_0 be outward minimizing in the sense of Definition 2.3, let E_k, $k \geq 1$, be given by (2.1) and let E_h denote their piecewise constant interpolation in time. We define the *arrival time* $u_h \colon \mathbb{R}^n \to [0, \infty)$ by

$$u_h(x) := h \sum_{k \geq 0} \chi_{E_k}(x) = \int_0^\infty \chi_{E_h(t)}(x)\, dt \quad (x \in \mathbb{R}^n). \tag{3.1}$$

Let us first note that $u_h \in BV(\mathbb{R}^n)$ since the a priori estimate (2.2) implies

$$\int_{\mathbb{R}^n} |Du_h| = \int_0^{T_h} P(E_h(t))\, dt \leq T_h P(E_0), \tag{3.2}$$

where T_h denotes the extinction time of $(E_h(t))_{t \geq 0}$. Note that the extinction time is finite: If $R > 0$ is sufficiently large such that $E_0 \subset B_R$, then by Lemma 2.9 we have $E_h(t) \subset B_{r_h(t)}$, where r_h is given in Remark 2.2. Clearly $r_h(t) = 0$ for t larger than, say twice the extinction time of the mean curvature flow starting from B_R.

The following lemma states that under our outward minimality assumption on the initial condition, the arrival time solves a (one-sided) variational problem.

Lemma 3.2. *Let E_0 be mean convex in the sense of Definition 2.3. Then the arrival time is outward minimizing in the sense that*

$$\int_{\mathbb{R}^n} |Du_h| \leq \int_{\mathbb{R}^n} |Dv| \quad \textit{for all } v \in BV(\mathbb{R}^n) \textit{ s.t. } v \geq u_h. \tag{3.3}$$

Proof. Given $v \in BV(\mathbb{R}^n)$ with $v \geq u_h$ we employ the coarea formula, cf. [2, Theorem 3.40], to manipulate the total variation of v:

$$\int_{\mathbb{R}^n} |Dv| = \int_0^\infty P(\{x \in \mathbb{R}^n \colon v(x) > t\})\, dt.$$

Since $v \geq u_h$ implies

$$E_h(t) = \{x \in \mathbb{R}^n \colon u_h(x) > t\} \subset \{x \in \mathbb{R}^n \colon v(x) > t\},$$

we obtain

$$\int_{\mathbb{R}^n} |Du_h| = \int_0^\infty P(E_h(t))\, dt \leq \int_0^\infty P(\{x \in \mathbb{R}^n \colon v(x) > t\})\, dt = \int_{\mathbb{R}^n} |Dv|\,. \qquad \square$$

The next lemma states that if E_0 is strictly outward minimizing, we have a uniform estimate on the modulus of continuity of u_h except for fluctuations on scales below h; and hence after passing to a subsequence, we obtain uniform convergence to a continuous function.

Lemma 3.3. *Let E_0 be a smooth bounded open set which is strictly outward minimizing in the sense of Definition 2.3 and such that $\min H_{\partial E_0} > 0$. Then there exists a subsequence $h_j \downarrow 0$ and a continuous function $u\colon \mathbb{R}^n \to [0,\infty)$ with $\operatorname{supp} u \subset E_0$ such that*

$$u_{h_j} \to u \qquad \textit{uniformly} \tag{3.4}$$

$$Du_{h_j} \rightharpoonup Du \qquad \textit{as measures} \tag{3.5}$$

Proof. Let $H_0 := \min H_{\partial E_0} > 0$, which by Lemma 2.7 implies $\min H_{\partial E_k} \geq H_0$ for all $k \geq 0$.

We claim that we have a uniform bound on the modulus of continuity up to fluctuations on scales below h, i.e.,

$$|u_h(x) - u_h(y)| \leq \frac{1}{H_0}|x-y| + h \qquad \text{for all } x, y \in \mathbb{R}^n. \tag{3.6}$$

In order to prove (3.6) let $x, y \in E_0$ be given. Without loss of generality we may assume $x \in E_n$ and $y \in E_m$ with $-1 \leq m < n$, where we have set $E_{-1} := \mathbb{R}^n \setminus E_0$. Since the sets E_k, $k \geq 0$ are nested, the segment $[x,y]$ intersects the intermediate boundaries non-trivially: There are points z_k, $k = m+1, \ldots, n$, such that $z_k \in \partial E_k \cap [x,y]$. Using the Euler-Lagrange equation (2.7) along these points we obtain

$$|x-y| \geq |z_n - z_{m+1}| = \sum_{k=m+2}^{n} |z_k - z_{k-1}| \geq \sum_{k=m+2}^{n} d(z_k, \partial E_{k-1}) \geq (m-n-1)hH_0.$$

Since $|u(x) - u(y)| = (m-n)h$, this is precisely our claim (3.6). Therefore, by Arzelà-Ascoli, we obtain the compactness (3.4). The weak convergence of the gradients (3.5) follows immediately from (3.2). $\square$

4. Convergence to the continuous arrival time

Let E_0 be an outward minimizing set such that $H_{\partial E_0} > 0$. According to the previous section the arrival times u_h of the discrete scheme converge, up to subsequences, to a limiting function u. In this section we identify this function as the arrival time of the limiting equation. We start by recalling the following

Theorem 4.1 (Evans-Spruck [9])**.** *Let E_0 be a bounded C^2 open set with $H_{\partial E_0} >$. Then there exists a unique continuous viscosity solution u of*

$$\begin{cases} |Du| \operatorname{div}\left(\dfrac{Du}{|Du|}\right) = -1 & \textit{in } E_0 \\ u = 0 & \textit{on } \partial E_0. \end{cases} \tag{4.1}$$

Moreover, for all $t \in [0, \sup u]$ the set $\{u \geq t\}$ is the evolution of $\overline{E_0} = \{u \geq 0\}$ via mean curvature flow.

Here a solution of (4.1) is understood in the viscosity sense, that is for all $x \in E_0$ and all $\varphi \in C^2(E_0)$ such that $u - \varphi$ has a minimum at x (resp. a maximum) then

$$\Delta\varphi(x) - \frac{D^2\varphi(x)[D\varphi(x), D\varphi(x)]}{|D\varphi(x)|^2} \leq -1 \quad (\geq -1) \qquad \text{if } D\varphi(x) \neq 0 \tag{4.2}$$

$$\exists \eta \in \mathbb{S}^{n-1} \text{ such that } \Delta\varphi(x) - D^2\varphi(x)[\eta,\eta] \leq -1 \quad (\geq -1) \qquad \text{if } D\varphi(x) = 0. \tag{4.3}$$

Our aim is to prove that any limit point of the sequence of the discrete arrival times is a viscosity solution of (4.1).

Proposition 4.2. *Let E_0 be as in Theorem 4.1 and let u_h be as in Definition 3.1. Then every limit point of u of u_h is a viscosity solution of* (4.1)*. In particular all the sequnece u_h converges to u.*

Proof. Let u be such that (up to subsequences) $u_h \to u$ uniformly. Let $x \in E_0$ and $\varphi \in C^2(E_0)$ be such that $u - \varphi$ has a minimum at x. By changing coordinates we may assume without loss of generality that $x = 0$, moreover, by replacing φ by $\varphi - |x|^4$ we may assume that the minimum is global and strict:

$$u(x) - \varphi(x) > u(0) - \varphi(0) \qquad \text{for all } x \in E_0 \setminus \{0\}. \tag{4.4}$$

By classical arguments we can find a sequence of points x_h such that $x_h \to 0$ and

$$(u_h)_*(x) - \varphi(x) \geq (u_h)_*(x_h) - \varphi(x_h)$$

where $(u_h)_*$ is the lower semicontinuous envelop of u_h, namely

$$(u_h)_* = \sum_{k=1}^{T_h/h} h\chi_{\mathrm{Int}(E_k)}.$$

Here T_h is the extinction time of the scheme. Note in particular that $(u_h)_* \to 0$ uniformly. For simplicity, from now on we assume that the sets E_k are open and that u_h is already lower-semicontinuous (observe that by the regularity theory for almost minimizers of the perimeter $|\overline{E_k} \setminus \mathrm{Int}(E_k)| = 0$ which allows us to choose such a representative). We also let $k_h \in \mathbb{N}$ be the unique integer such that $u_h(x_h) = k_h h$, in particular $x_h \in E_{k_h}$

We now distinguish two cases.

Case 1: $D\varphi(0) \neq 0$. Since $x_h \to 0$ we have $D\varphi(x_h) \neq 0$ if h is sufficiently small. Hence, u_h cannot be flat constant in a neighborhood of x_h, so $x_h \notin \mathrm{Int}(E_{k_h} \setminus E_{k_h+1})$ and thus, since E_{k_h} is open,

$$x_h \in \partial E_{k_h+1}.$$

In particular

$$U := \{\varphi > \varphi(x_h)\} \subset E_{k_h+1} \qquad \text{and} \qquad x_h \in \partial U \cap \partial E_{k_h+1}.$$

Since both ∂U and ∂E_{k_h+1} are smooth in a neighborhood of x_h, the comparison principle and the Euler-Lagrange equation (2.7) yield

$$\begin{aligned} \operatorname{div}\left(\frac{D\varphi(x_h)}{|D\varphi(x_h)|}\right) &= -H_{\partial U}(x_h) \leq -H_{\partial E_{k_h+1}}(x_h) \\ &= -\frac{\mathrm{dist}(x_h, \partial E_{k_h})}{h} \leq -\frac{\mathrm{dist}(x_h, \partial\{\varphi > \varphi(x_h) - h\})}{h}, \end{aligned} \tag{4.5}$$

where in the last inequality we have used that

$$E_{k_h}^c = \{u \leq u(y_h) - h\} \subset \{\varphi \leq \varphi(x_h) - h\}.$$

Moreover, by Taylor expansion, one easily verifies

$$\frac{\mathrm{dist}(x_h, \partial\{\varphi > \varphi(x_h) - h\})|D\varphi(x_h)|}{h} \to 1 \qquad \text{as } h \to 0. \tag{4.6}$$

Combining (4.5) and (4.6) we conclude the validity of (4.2).

Case 2: $D\varphi(0) = 0$. This time we can not assume a priori that $D\varphi(x_h) = 0$. To overcome this difficulty we exploit Jensen's inf-convolution (on a fixed scale of order h). To this aim let us define

$$v_h(x) := \inf_{y \in \overline{E_0}} \left\{ u_h(y) + \frac{|x-y|^4}{2c_n^4 h} \right\} \quad \text{for } x \in E_0,$$

where c_n is a constant that will be fixed later in dependence only on the dimension n. We also let z_h be a minimum point of $v_h - \varphi$, namely

$$v_h(x) - \varphi(x) \geq v_h(z_h) - \varphi(z_h) \quad \text{for all } x \in \bar{E}_0$$

and let $y_h \in \overline{E_0}$ be such that

$$v_h(z_h) = u_h(y_h) + \frac{|z_h - y_h|^4}{2c_n^4 h}.$$

Note that the existence of y_h is ensured by the lower semicontinuity of u_h.

We now divide the proof in some steps:

Step 1: $|z_h - y_h| \to 0$. Indeed, since $v_h \leq u_h \leq 2\|u\|_\infty$ we obtain

$$|z_h - y_h|^4 \leq 8c_n^4 h\|u\|_\infty \to 0 \quad \text{as } h \to 0.$$

Step 2: $z_h \to 0$. Indeed, by $v_h \leq u_h$ and the definition of z_h,

$$u_h(x_h) - \varphi(x_h) \geq v_h(z_h) - \varphi(z_h) \geq u_h(y_h) - \varphi(y_h) + \varphi(y_h) - \varphi(z_h).$$

If we let $\bar{z} \in \overline{E_0}$ be an accumulation point of z_h (and hence of y_h) we deduce from the above inequality and the uniform convergence of u_h to u that

$$u(0) - \varphi(0) \geq u(\bar{z}) - \varphi(\bar{z})$$

which in view of (4.4) forces $\bar{z} = 0$.

Step 3: $z_h \neq y_h$. Let us assume by contradiction that $z_h = y_h$. By the very definition of v_h this means that

$$u_h(z_h) = v_h(z_h) \leq u(y) + \frac{|y - z_h|^4}{c_n^4 h} \quad \text{for all } y \in \bar{E}_0. \tag{4.7}$$

Let also $j_h \in \mathbb{N}$ be such that $u_h(z_h) = j_h h$. Note that since $u > 0$ in E_0 and $u_h(z_h) \to u(0) > 0$ we may assume that $j_h \gg 1$. In particular

$$z_h \in E_{j_h} \setminus E_{j_h+1}.$$

We now note that (4.7) implies

$$F_0 := B\big(z_h, c_n\sqrt{h}\big) \subset\subset \big\{u_h \geq (j_h - 1)h\big\} = E_{j_h-1}. \tag{4.8}$$

If we let F_1 and F_2 be minimizers of (2.1) starting from F_0 and F_1, respectively, Remark 2.2 ensures that

$$F_2 = B\big(z_h, r_h\big) \qquad \text{with} \qquad r_h \geq \sqrt{c_n - 4(n-1)} > 0$$

provided c_n is chosen sufficiently large. However, by Lemma 2.9 and (4.8)

$$z_h \in F_2 \subset\subset E_{j_h+1},$$

a contradiction.

Step 4: Conclusion. By the very definitions of v_h, y_h and z_h we have

$$u_h(y_h) + \frac{|z_h - y_h|^4}{2c_n^4 h} - \varphi(z_h) \leq u_h(y) + \frac{|x - y|^4}{2c_n^4 h} - \varphi(x) \quad \text{for all } x, y \in \bar{E}_0. \tag{4.9}$$

In particular, the optimality condition in the x-variable implies

$$D\varphi(z_h) = \frac{2|z_h - y_h|^2(z_h - y_h)}{c_n^4 h} \neq 0.$$

Moreover, if we set

$$\psi_h(x) := \varphi(x + (z_h - y_h)) + \frac{|z_h - y_h|^4}{2c_n^4 h},$$

the function $u - \psi_h$ has a minimum at y_h with $D\psi_h(z_h) \neq 0$. By the very same arguments of Case 1 we obtain that

$$\Delta\varphi(z_h) - \frac{D^2\varphi(z_h)[D\varphi(z_h), D\varphi(z_h)]}{|D\varphi(z_h)|^2} \leq -1 + o(1),$$

which gives (4.3) with η being any limiting point of the sequence $\frac{D\varphi(z_h)}{|D\varphi(z_h)|}$.

Since the case in which $u - \varphi$ has a maximum at some $x \in E_0$ can be treated analogously, this completes the proof. □

5. Compensated compactness for the arrival time and proof of Theorem 1.4

In this section we establish the convergence of the total variations of the arrival times u_h and we prove Theorem 1.4 The proof is based on the compensated compactness argument of Evans-Spruck [9] together with the outward minimality of the u_h established in Lemma 3.2 and the dual formulation of the obstacle problem for BV functions established in [20].

Proposition 5.1. *Let E_0 be strictly mean convex in the sense of Definition 2.3 and let u_{h_j} defined by* (3.1) *satisfy* (3.4) *and* (3.5)*. Then,*

$$|Du_{h_j}| \rightharpoonup |Du| \quad \textit{as measures.}$$

In particular it holds

$$\int |Du_{h_j}| \to \int |Du|.$$

While the compensated compactness argument of Evans-Spruck is based on the curious estimate

$$\sup_{\varepsilon>0} \int_{\mathbb{R}^n} |H_\varepsilon(x)|\, dx < \infty, \tag{5.1}$$

which miraculously holds true for the elliptic regularizations u_ε of the level set formulation, this estimate is very intuitive in our situation:

Informally, the Euler-Lagrange equation of the minimization problem in Lemma 3.2 reads

$$\operatorname{div}\left(\frac{Du_h}{|Du_h|}\right) \geq 0.$$

This means that these distributions are in fact measures, for which it should be reasonable to get appropriate bounds. This resembles the L^1-bound (5.1) and would allow us to pass to the limit in

$$\int \zeta\, |Du_h| = \int \zeta Du_h \cdot \frac{Du_h}{|Du_h|} = -\int \zeta u_h \operatorname{div}\left(\frac{Du_h}{|Du_h|}\right) - \int u_h \frac{Du_h}{|Du_h|} \cdot D\zeta. \tag{5.2}$$

The following proof makes this argument rigorous by exploiting the dual formulation of the minimization problem for u_h in Lemma 3.2.

Proof of Proposition 5.1. To make the above argument rigorous, we interpret the minimization problem in Lemma 3.2 as an obstacle problem on a bounded set $\Omega \supset\supset E_0$ with homogeneous Dirichlet boundary conditions. Here the obstacle is of class BV and happens to be our minimizer u_h itself. This allows us to use the general theory for dual formulations of obstacle problems: By [20, Theorem 3.6, Remark 3.8] the dual problem reads

$$\max [\![\sigma, Du_h^+]\!](\bar{\Omega}).$$

where the maximum runs over all measurable vector fields $\sigma\colon \Omega \to \mathbb{R}^n$ with $|\sigma| \leq 1$ a.e. in Ω and $\operatorname{div}\sigma \leq 0$ distributionally in Ω. Note that this implies that $\operatorname{div}\sigma$ is a measure on $\bar{\Omega}$ and

$$(-\operatorname{div}\sigma)(\Omega) \leq C(n)\mathcal{H}^{n-1}(\partial\Omega). \tag{5.3}$$

Here

$$u_h^+(x) = \text{ap-limsup}_{y\to x}\, u_h(x)$$

denotes the largest representative of u_h, see [20], and the measure $[\![\sigma, Du_h^+]\!]$ is defined as

$$[\![\sigma, Du_h^+]\!](\zeta) := -\int_\Omega \zeta u_h^+ \operatorname{div}(\sigma)\, dx - \int_\Omega u_h\, (\sigma\cdot D\zeta)\, dx, \tag{5.4}$$

for test functions $\zeta \in C^1(\bar{\Omega})$. This yields a vector field σ_h for any $h > 0$ with the above mentioned properties and such that

$$\int |Du_h| = [\![\sigma, Du_h^+]\!](\bar{\Omega}) = [\![\sigma, Du_h^+]\!](\mathbb{R}^n). \tag{5.5}$$

Here we used the fact that u_h vanishes away from $E_0 \subset\subset \Omega$ so that we may think of σ_h extended by zero on $\mathbb{R}^n \setminus \Omega$. The sequence $(\sigma_{h_j})_{h_j}$ is precompact: there exists a subsequence, which we do not relabel, and a measure μ such that

$$\operatorname{div}\sigma_{h_j} \rightharpoonup \mu \quad \text{as measures.} \tag{5.6}$$

Since $|\sigma_h| \leq 1$, we may assume that there exists a measurable vector field σ with $|\sigma| \leq 1$ such that

$$\sigma_{h_j} \overset{*}{\rightharpoonup} \sigma \quad \text{in } L^\infty. \tag{5.7}$$

In particular

$$\operatorname{div}\sigma = \mu \quad \text{in } \Omega. \tag{5.8}$$

Indeed, for any test function $\zeta \in C_0^1(\Omega)$ we have

$$\int \zeta\, d\mu = \lim_{h_j\downarrow 0} \int \zeta \operatorname{div}\sigma_{h_j}\, dx = \lim_{h_j\downarrow 0} -\int D\zeta\cdot\sigma_{h_j}\, dx = \int -\int D\zeta\cdot\sigma\, dx.$$

Now we can make the idea of the aforementioned compensated compactness argument rigorous. By (5.5) we have

$$\int \zeta\, |Du_h| = -\int \zeta u_h^+ \operatorname{div}(\sigma_h)\, dx - \int u_h\, (\sigma_h\cdot D\zeta)\, dx, \tag{5.9}$$

which is precisely the analogue of (5.2) with the important difference that we can give a meaning to (and have precise estimates for) all products appearing on the right. Along the subsequence $h_j \downarrow 0$, on the one hand, since $u = \lim u_{h_j}$ is continuous, we have

$$\lim_{h_j\downarrow 0} -\int \zeta u \operatorname{div}(\sigma_{h_j})\, dx \overset{(5.6)}{=} -\int \zeta u\, d\mu.$$

On the other hand, by the uniform convergence (3.4), we have

$$\left| -\int \zeta \left(u_{h_j}^+ - u\right) \operatorname{div}(\sigma_{h_j})\, dx\right| \overset{(5.3)}{\leq} \|\zeta\|_\infty \|u_{h_j}^+ - u\|_\infty C(n)\mathcal{H}^{n-1}(\partial\Omega) \to 0.$$

Therefore, we can pass to the limit in the first right-hand side product of (5.9):

$$\lim_{h_j\downarrow 0} -\int \zeta u_h^+ \operatorname{div}(\sigma_h)\,dx = -\int \zeta u\,d\mu = \int D(\zeta u)\cdot\sigma\,dx. \tag{5.10}$$

Since $\operatorname{supp} u_h \subset \Omega$ is equibounded, the convergence $u_{h_j} \to u$ is strong in L^1 and hence we may pass to the limit in the second right-hand side product of (5.9). Therefore, for any non-negative test function $\zeta \in C^1(\mathbb{R}^n)$ we obtain

$$\lim_{h_j\downarrow 0}\int \zeta\left|Du_{h_j}\right| = \int D(\zeta u)\cdot\sigma\,dx - \int u\,(\sigma\cdot D\zeta)\,dx = \int \zeta\,\sigma\cdot Du \le \int \zeta\,|Du|\,,$$

where we used the pointwise bound $|\sigma| \le 1$ a.e. in the last inequality. The lower semi-continuity of the total variation implies

$$\int \zeta\,|Du| \le \liminf_{h_j\downarrow 0}\int \zeta\left|Du_{h_j}\right|$$

for all non-negative test function $\zeta \in C^1(\mathbb{R}^n)$. Therefore

$$\lim_{h_j\downarrow 0}\int \zeta\left|Du_{h_j}\right| = \int \zeta\,|Du|$$

holds for all non-negative test functions $\zeta \in C^1(\mathbb{R}^n)$. By linearity and continuity in ζ the convergence holds for all continuous test functions $\zeta \in C(\mathbb{R}^n)$ without restriction on the sign, which proves $\left|Du_{h_j}\right| \rightharpoonup |Du|$ as measures. □

We are now ready to prove Theorem 1.4:

Proof of Theorem 1.4. Passing to a subsequence, we may assume $E_{h_j} \to E$ in L^1. By Proposition 4.2 u_{h_j} converges to the arrival time of the limiting evolution u. By the co-area formula and Proposition 5.1

$$\lim_{h\to 0}\int_0^\infty P(E_h(t))\,dt = \lim_{h\to 0}\int |Du_h| = \int |Du| = \int_0^\infty P(E(t))\,dt,$$

which proves (1.4). □

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Scuola Internazionale Superiore di Studi Avazanti, Via Bonomea 265, 34136 Trieste, Italy
E-mail address: guido.dephilippis@sissa.it

Department of Mathematics, University of California, Berkeley, CA 94720-3840 USA
E-mail address: tim.laux@math.berkeley.edu