A CHEEGER–KOHLER-JOBIN INEQUALITY

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Abstract. In this paper we prove the following Kohler-Jobin type inequality: for any open, bounded set $\Omega \subset \mathbb{R}^N$ and any ball $B \subset \mathbb{R}^N$ we have

$$T(\Omega)^{\frac{1}{N+2}} h_1(\Omega) \geq T(B)^{\frac{1}{N+2}} h_1(B),$$

where T denotes the torsional rigidity and h_1 the Cheeger constant. Moreover, equality holds if and only if Ω is a ball. We then exploit such inequality to provide a new proof of the sharp quantitative Cheeger inequality. Eventually, we extend these results to the nonlocal framework.

Keywords: Cheeger constant, Kohler-Jobin inequality, quantitative estimates, Poincaré-Sobolev constants

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1. Introduction

In this paper we study a class of inequalities arising from shape optimization problems involving the torsional rigidity, the eigenvalues of elliptic operators and the Cheeger constant. First of all, we introduce the mathematical objects we will deal with.

Let Ω be an open, bounded set in $\mathbb{R}^N$. For $1 \leq r < N$ and $1 \leq q < Nr/(N-r)$, or for $r \geq N$ and $1 \leq q < +\infty$, we define

$$\lambda_{r,q}(\Omega) := \inf\left\{ \frac{\int_\Omega |\nabla u|^r \mathrm{d}x}{\left(\int_\Omega |u|^q \mathrm{d}x\right)^{\frac{r}{q}}} : u \in W_0^{r,q}(\Omega) \setminus \{0\} \right\} = \inf\left\{ \frac{\int_\Omega |\nabla u|^r \mathrm{d}x}{\left(\int_\Omega |u|^q \mathrm{d}x\right)^{\frac{r}{q}}} : u \in C_c^\infty(\Omega) \setminus \{0\} \right\}, \tag{1.1}$$

which can be interpreted as the principal frequency for the nonlinear eigenvalue problem

$$-\Delta_r u = \lambda \|u\|_{L^q(\Omega)}^{r-q} |u|^{q-2} u \qquad \text{in } \Omega, \qquad\qquad u = 0 \qquad \text{on } \partial\Omega\,.$$

Alternatively, $\lambda_{r,q}(\Omega)$ can be seen as the optimal constant for the Poincaré inequality

$$\left(\int_\Omega |u|^q \mathrm{d}x\right)^{\frac{r}{q}} \leq C \int_\Omega |\nabla u|^r \mathrm{d}x\,,$$

in the Sobolev space $W_0^{r,q}(\Omega)$. Some of the functionals defined above have been intensively studied in the literature: $T_p(\cdot) := \lambda_{p,1}^{-1}(\cdot)$ is called *p-torsional rigidity* and if $p = 2$ is often defined, thanks to a homogeneity argument, in the equivalent way,

$$T_2(\Omega) = -2 \min\left\{ \frac{1}{2} \int_\Omega |\nabla u|^2 \mathrm{d}x - \int_\Omega u \,\mathrm{d}x \;:\; u \in H_0^1(\Omega) \right\}.$$

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On the other hand, $\lambda_{2,2}(\cdot)$ is the first eigenvalue of the Dirichlet-Laplacian; beware that it is often denoted by $\lambda_1(\cdot)$ in works that do not deal also with nonlinear eigenvalues. Moreover, if Ω has regular enough boundary, $\lambda_{1,1}(\cdot)$ happens to be equal to the so-called *Cheeger constant* of Ω, defined by

$$h_1(\Omega) := \inf \left\{ \frac{P(E)}{|E|} : E \subset \overline{\Omega} \right\},$$

where $P(\cdot)$ denotes the euclidean De Giorgi's perimeter, and $|\cdot|$ is the N-dimensional Lebesgue measure. As it can be rapidly checked thanks to the Pólya-Szegö inequality, or to the isoperimetric inequality if $r = q = 1$, balls minimize $\lambda_{r,q}(\cdot)$ among sets of prescribed measure, namely

$$\lambda_{r,q}(\Omega) \geq \lambda_{r,q}(B), \tag{1.2}$$

for any ball B of measure $|B| = |\Omega|$. Moreover, equality holds if and only if Ω is a ball, up to a set of null $r-$capacity. Of course this entails that balls maximize the $p-$torsional rigidity under measure constraint, that is,

$$T_p(\Omega) \leq T_p(B),$$

for any ball B with $|B| = |\Omega|$. The latter is known as *Saint-Venant inequality.*

Pólya and Szegö in [20] conjectured that the product of the torsional rigidity (raised to a suitable power) and the first eigenvalue of the Dirichlet-Laplacian was minimized by balls. Intuitively, this tells that the minimality of balls for the eigenvalue is somehow more stable compared to their maximality for the torsion. The conjecture was proved to be true by Kohler-Jobin, who showed, in [15, 16], that

$$T_2(\Omega)^{\frac{2}{N+2}} \lambda_{2,2}(\Omega) \geq T_2(B)^{\frac{2}{N+2}} \lambda_{2,2}(B), \tag{1.3}$$

with equality if and only if Ω is a ball, up to a set of null measure. The exponent $2/(N+2)$ is chosen so that the functional is scale invariant. The conjecture by Pólya and Szegö can actually be extended to the more general family of functionals $T_p^\theta(\cdot)\lambda_{p,q}(\cdot)$. This step, which is not straightforward, mostly due to the nonlinearity of the Euler-Lagrange equation governing $\lambda_{p,q}(\cdot)$ whenever $(p, q) \neq (2, 2)$, has been later accomplished by Brasco. Precisely, in [2] he shows, with somehow simplified arguments, that

$$T_p^\theta(\Omega)\lambda_{p,q}(\Omega) \geq T_p^\theta(B)\lambda_{p,q}(B)$$

whenever B is a ball, $1 < p < +\infty$, and $1 < q < Np/(N-p)$ if $p < N$, $1 \leq q < +\infty$ if $p \geq N$. Again, $\theta = \theta(N, p, q)$ is chosen so to make the inequality homogeneous and equality can hold only if Ω is a ball, up to a negligible set. We notice that the above class of parameters does not include the case $T_p^\theta(\cdot)\lambda_{r,q}(\cdot)$ with $p \neq r$ (see also Remark 1.2 below) and in particular the interesting case $p = 2$, $r = q = 1$ which involves the Cheeger constant. This latter choice of parameters has a substantial difference from the other cases: for $r > 1$ it is not difficult to show the existence of a minimizer in the definition (1.1) of $\lambda_{r,q}(\Omega)$, while for $r = 1$ this is not the case, since a minimizing sequence may relax to a function with jumps, not belonging to $W^{1,1}(\Omega)$, so that a minimizer must be searched in the space of functions with bounded variation $BV(\Omega)$. Moreover, the positive level sets of the BV minimizers turn out to be optimal sets for $h_1(\Omega)$ (more details on the Cheeger constant are given in Section 2 below).

In this note we further extend (1.3) to the functionals $T_p^\theta(\cdot)\lambda_{1,q}(\cdot)$, with $1 < p < +\infty$, $1 \leq q < N/(N-1)$, and θ the suitable "scaling" exponent. Namely, we prove the following.

Theorem 1.1 (Cheeger–Kohler-Jobin inequality)**.** *Let $1 < p < +\infty$, $\Omega \subset \mathbb{R}^N$ be an open, bounded set and $B \subset \mathbb{R}^N$ be any ball. Then, for any $q \in [1, \frac{N}{N-1})$, it holds*

$$T_p(\Omega)^\theta \lambda_{1,q}(\Omega) \geq T_p(B)^\theta \lambda_{1,q}(B), \tag{1.4}$$

with $\theta = \theta(N,p,q) := \frac{q-N(q-1)}{q[p+N(p-1)]}$*. In particular, when* $p=2$*,* $q=1$*, and* $P(\Omega) = \mathcal{H}^{N-1}(\partial\Omega)$*, we have*

$$T_2(\Omega)^\theta h_1(\Omega) \geq T_2(B)^\theta h_1(B)\,, \tag{1.5}$$

where $\theta = \theta(N) := \frac{1}{N+2}$.

Moreover, equality holds in (1.4) *and* (1.5) *if and only if* Ω *is equal to a ball, up to a negligible set.*

The interest of this extension is due to the choice of the first index in $\lambda_{1,q}(\cdot)$. We remark nevertheless, that in this case we get the characterization of equality cases even for $q=1$, which is an open issue for the functionals $T_p(\cdot)\lambda_{p,q}(\cdot)$ if $q=1$ and $p>1$.

Remark 1.2 (Open cases)**.** The results of the present paper, those by Brasco in [2] and those by Kohler-Jobin in [15], do not cover all the possible indexes. In particular one may consider the family of functionals $T_p(\cdot)^\theta\lambda_{r,q}(\cdot)$ when $p\neq r$, $r\in(1,\infty)$, and $q\in[1,\frac{Nr}{N-r})$ if $r<N$, $q\in[1,\infty)$ if $r\geq N$. Our strategy is not extensible to all the parameters, mainly because we can not employ the coarea formula in order to obtain the results of Step 1 in the proof of Theorem 1.1 on $\int_\Omega |\nabla u|^r \mathrm{d}x$. On the other hand one can not hope to have the ball as minimizer for any index: for instance, in the planar case, if $p=1$, $r=q=2$, this is not true, as observed by Parini in [19], who proved that balls are critical points but *not* minimizers for $T_1(\cdot)^2\lambda_{2,2}(\cdot)$.

Theorem 1.1 holds as well in the fractional setting, that is, when the Cheeger functional h_1 is replaced with its nonlocal counterpart h_s, introduced and studied in [4]. For the sake of clarity, we avoid to burden the proof of the result with the notations necessary to deal with the fractional case. For the interested reader, some details about this extension are postponed in Section 4.

We are then able to exploit Theorem 1.1 to offer a new proof of the following quantitative version of (1.2) in the case $r=q=1$.

Theorem 1.3. *There exists a dimensional constant* $\sigma(N)>0$ *such that for any* $\Omega\subset\mathbb{R}^N$ *open and bounded with* $P(\Omega)=\mathcal{H}^{N-1}(\partial\Omega)$ *and for any ball* $B\subset\mathbb{R}^N$*, we have*

$$|\Omega|^{\frac{1}{N}}h_1(\Omega) - |B|^{\frac{1}{N}}h_1(B) \geq \sigma(N)\alpha(\Omega)^2\,, \tag{1.6}$$

where $\alpha(\Omega)$ *denotes the Fraenkel asymmetry of* Ω*, see* (2.2)*. Moreover, the power* 2 *in* (1.6) *is sharp, in the sense that it can not be replaced by any lower number.*

This improvement of the Cheeger inequality was first showed by Figalli, Maggi, and Pratelli: in [10] they provide a short proof of this fact, based on the quantitative version of the isoperimetric inequality [13, 9, 8].

The approach in this paper is quite different, and borrows an idea from [3], where Brasco, De Philippis, and Velichkov show that $\lambda_{2,2}(\cdot)$ satisfies a (asimptotically sharp) quantitative estimate of the form

$$|\Omega|^{\frac{2}{N}}\lambda_{2,2}(\Omega) - |B|^{\frac{2}{N}}\lambda_{2,2}(B) \geq C(N)\alpha(\Omega)^2\,,$$

by relating the stability of $\lambda_{2,2}(\cdot)$ to that of the torsional rigidity $T_2(\cdot)$. In the very same spirit, we obtain (1.6) by combining the Cheeger–Kohler-Jobin inequality (1.5), together with the quantitative stability of the 2-torsional rigidity provided in [3].

The article is organized as follows. In Section 2 we introduce the main notions needed throughout the paper. Beside this, we shortly survey the several possible definitions of Cheeger constant, underlining some minimal regularity assumptions in order to make all of them coincide. One reason for this discussion is that it will allow us to work, in the rest of the paper, with the functional definitions of the Cheeger constant. The advantage of this choice is twofold: on one hand, it is more in the spirit of Kohler-Jobin type inequalities, where the quantities involved are defined via minimization of functional operators; on the other hand, a functional definition turns out to be easier to handle in our proofs (in particular, for the equality cases of (1.5)). Section 3 is devoted to the proof of the main results of the paper: Theorems 1.1 and 1.3. Eventually, in Section 4 we discuss the fractional case. As mentioned above, the result (but not the proof) about quantitative improvements of the Cheeger inequality just overlaps with the work of Figalli, Maggi, and Pratelli: even if the authors consider only the local case (as a matter of facts, the fractional Cheeger constant has not been defined at the time of their publication yet), it is not difficult to adapt their techniques, and consequently their result, to the fractional setting.

2. Preliminaries

In this section we collect some well known facts on geometric measure theory, which will serve our scopes later in the paper. We refer to [1, Chapter 3] for more details. Then we discuss some features and links of the several possibles definitions of the Cheeger constant.

2.1. **The perimeter and its properties.** The measure theoretic perimeter (shortly: perimeter) of a Borel set $E \subset \mathbb{R}^N$ is the quantity

$$P(E) := \sup\left\{ \int_E \nabla \cdot \phi \,\mathrm{d}x \,:\, \phi \in C^1_c(\mathbb{R}^N, \mathbb{R}^N),\ \|\phi\|_\infty \le 1 \right\}.$$

If $P(E) < +\infty$ then we say that E has finite perimeter. Equivalently, it can be defined in the setting of functions of bounded variation as the distributional derivative of characteristic functions. We recall that if $\Omega \subset \mathbb{R}^N$ is an open set, we say that $u \in L^1(\Omega)$ is a function of bounded variation, and we write $u \in BV(\Omega)$ if the distributional derivative Du of u is an $\mathbb{R}^N$-valued finite Radon measure. If $E \subset \mathbb{R}^N$ is a set of finite perimeter, then $\chi_E \in BV(\mathbb{R}^N)$ and $P(E) = |D\chi_E|(\mathbb{R}^N) =: \|D\chi_E\|_{TV(\mathbb{R}^N)}$.

Whenever it exists, the quantity

$$[0,1] \ni \theta_E(x) := \lim_{r\to 0} \frac{|E \cap B_r(x)|}{|B_r(x)|},$$

is called the density of a Borel set E at x. We denote by E^t the subset of points of $\mathbb{R}^N$ such that $\theta_E(x) = t$, and we call essential boundary the set $\partial^e E = E \setminus (E^0 \cup E^1)$. Eventually, we define the reduced boundary of E as the set $\partial^* E \subset \partial^e E$ of points of the essential boundary such that the measure theoretic inner unit normal

$$\nu_E(x) := \lim_{r\to 0} \frac{D\chi_E(B_r(x))}{|D\chi_E|(B_r(x))}$$

exists.

The geometry of the boundary of sets of finite perimeter is described in the two cornerstones of the theory of sets of finite perimeter: the De Giorgi's and the Federer's structure theorems.

Theorem 2.1 (De Giorgi's Structure Theorem)**.** *Let E be a set of finite perimeter. Then $\partial^* E$ is $\mathcal{H}^{N-1}-$rectifiable and $P(E) = \mathcal{H}^{N-1}(\partial^* E)$. Moreover, if $x \in \partial^* E$, then $(E-x)/r$ converges in L^1_{loc} to the hyperspace defined by the interior normal $\nu_E(x)$, as $r \to 0$. Eventually, it holds the divergence formula*

$$\int_E \nabla \cdot \phi \,\mathrm{d}x = -\int_{\partial^* E} \phi \cdot \nu_E \,\mathrm{d}\mathcal{H}^{N-1}(x),$$

for any vector field $\phi \in C^1_c(\mathbb{R}^N, \mathbb{R}^N)$.

Theorem 2.2 (Federer's Structure Theorem). *Let E be a set of finite perimeter. Then $\partial^* E \subset E^{1/2}$ and $\mathcal{H}^{N-1}(\partial^e E \setminus \partial^* E) = 0$. In particular $\partial^* E$, $E^{1/2}$, and $\partial^e E$ are the same set, up to a $\mathcal{H}^{N-1}-$null set.*

2.2. **The isoperimetric inequality and its quantitative improvement.** As it will be explained in more detail later, the Kohler-Jobin inequality is based on the simple principle of slicing the energy functionals defining $\lambda_{p,q}$ and T_p horizontally, and then rearrange the level sets of the involved functions in a suitable way. The energies before and after rearrangement are compared exploting the isoperimetric inequality, which states the following: for any set E of finite measure,

$$P(E) - P(B) \geq 0$$

whenever B is a ball of measure $|E|$, with equality if and only if E coincides with B up to a negligible set. Thanks to the rescaling property of the perimeter and of the Lebesgue measure, an equivalent version of the isoperimetric inequality is

$$P(E) - N\omega_N^{1/N}|E|^{(N-1)/N} \geq 0\,.$$

Here ω_N denotes the measure of the ball with unit radius in $\mathbb{R}^N$, and again equality holds if and only if E is a ball. While dealing with $\lambda_{1,q}$ (more precisely, with the equality cases in (1.4)), we need to exploit a stronger version of the isoperimetric inequality, proved about a decade ago in [13]: there exists a dimensional constant C_N such that, for any set $E \subset \mathbb{R}^N$ of finite measure, it holds

$$\frac{P(E) - N\omega_N^{1/N}|E|^{(N-1)/N}}{N\omega_N^{1/N}|E|^{(N-1)/N}} \geq C_N \alpha(E)^2, \tag{2.1}$$

with $\alpha(E)$ denoting the *Fraenkel asymmetry* of the set E, defined as

$$\alpha(E) := \inf_{x\in\mathbb{R}^N}\left\{\frac{|E\Delta(B+x)|}{|\Omega|}\ :\ B\subset\mathbb{R}^N \text{ is a ball, } |B| = |E|\right\}, \tag{2.2}$$

where $U\Delta V$ stands for the symmetric difference between the sets U and V.
It is worth stressing that the exponent 2 in the quantitative estimate (2.1) is *sharp*, in the sense that it can not be replaced by any lower number.

2.3. **The Cheeger constant.** The Cheeger constant was introduced in [7] to obtain lower bounds for the first eigenvalue of the Dirichlet-Laplacian. While its original definition is given on Riemannian manifolds, it has lately found many applications in the euclidean setting, where it can be defined in several ways. Here we briefly survey such different definitions (for more details, see the recent paper [17]), and we offer a criterion under which all the corresponding constants coincide. This allows us to switch from one definition to the other in the rest of the paper.

Definition 2.3. Let Ω be an open, bounded set in $\mathbb{R}^N$. Then the Cheeger constant is either

$$h_1(\Omega) := \inf\left\{\frac{P(E)}{|E|} : E \subset \overline{\Omega}\right\}, \text{ or}$$

$$h(\Omega) := \inf\left\{\frac{P(E)}{|E|} : E \Subset \Omega\right\}, \text{ or}$$

$$\lambda_1(\Omega) := \inf\left\{\frac{\|Du\|_{TV(\mathbb{R}^N)}}{\|u\|_{L^1(\Omega)}}\ :\ u \in BV(\overline{\Omega})\setminus\{0\},\ u|_{\mathbb{R}^N\setminus\overline{\Omega}} = 0\right\}, \text{ or}$$

$$\lambda_{1,1}(\Omega) := \inf\left\{ \frac{\|\nabla u\|_{L^1(\mathbb{R}^N)}}{\|u\|_{L^1(\Omega)}} : u \in C_c^\infty(\Omega)\setminus\{0\} \right\}.$$

The definition originally proposed by Cheeger [7] is the first one. Let us stress that by coarea formula, it is not difficult to show that any minimizer in the third and fourth definition has the property that each of its level set is a Cheeger set of Ω, that is, a minimizer of $h_1(\Omega)$, see [12]. Nonetheless, the constants are not the same in general.

Example 2.4. Let us consider $B(0,1)$, the ball centered at 0 of radius 1 in $\mathbb{R}^2$. Let $U = [-\frac{1}{2}, \frac{1}{2}] \times \{0\}$, and let $\Omega = B \setminus U$. Since any $E \subset \Omega$ must contain a closed curve surrounding U (whose length is greater than 2 as it can be seen by means of a projection on U), we have that $h_1(\Omega) = h_1(B)$, while

$$h(\Omega) \geq \frac{P(E)}{|E|} + 2\mathcal{H}^1(U) \geq h_1(B) + 2\,.$$

The previous example can be easily extended to higher dimensions. Moreover a very similar construction, with suitable regular functions, can be done to show that $\lambda_{1,1}(\Omega) > \lambda_1(\Omega) + 2$ (with the same Ω as above).

The feature to be underlined is that we removed a *(N-1)-dimensional manifold* (the line U) from a regular set (the ball). It is quite natural to ask if this condition is somehow sharp. The answer happens to be positive, as shown in the next proposition.

Proposition 2.5. *Let Ω be an open, bounded set of $\mathbb{R}^N$ such that*

$$P(\Omega) = \mathcal{H}^{N-1}(\partial\Omega)\,.$$

Then $h_1(\Omega) = h(\Omega) = \lambda_1(\Omega) = \lambda_{1,1}(\Omega)$.

Proof. We claim that, since $P(\Omega) = \mathcal{H}^{N-1}(\partial\Omega)$, the same happens for a minimizer E of $h_1(\Omega)$ (such a set exists since any minimizing sequence has equibounded perimeter, thus is compact in L^1, and then the perimeter is lower semicontinuous). To show this, we first split the reduced boundary of E in the following way

$$\partial^* E = (\partial^* E \cap \partial\Omega) \cup (\partial^* E \setminus \partial\Omega)\,.$$

From De Giorgi's Theorem, it is well known that outside the contact points, i.e. in $\partial^* E\setminus\partial\Omega$, the set E is regular, so that $\mathcal{H}^{N-1}(\partial^* E\setminus\partial\Omega) = \mathcal{H}^{N-1}(\partial E\setminus\partial\Omega)$. As for the contact points, we clearly have $\mathcal{H}^{N-1}(\partial^* E\cap\partial\Omega) \leq \mathcal{H}^{N-1}(\partial E\cap\partial\Omega)$. On the other hand, $\mathcal{H}^{N-1}-$a.e. $x \in \partial E\cap\partial\Omega$ belongs to $\partial E\cap\partial^*\Omega$, from our hypothesis on Ω and Federer's Theorem. Finally, thanks to [17, Prop. 3.5, point (vii)], any $x \in \partial^*\Omega\cap\partial E$ belongs to $\partial^* E$ (namely: ∂E meets $\partial\Omega$ tangentially), so that $\mathcal{H}^{N-1}(\partial^* E \cap \partial\Omega) = \mathcal{H}^{N-1}(\partial E \cap \partial\Omega)$. Summing up all the informations, we have

$$\begin{aligned} P(E) &= \mathcal{H}^{N-1}(\partial^* E) \\ &= \mathcal{H}^{N-1}(\partial^* E \cap \partial\Omega) + \mathcal{H}^{N-1}(\partial^* E \setminus \partial\Omega) = \mathcal{H}^{N-1}(\partial^* E \cap \partial\Omega) + \mathcal{H}^{N-1}(\partial E \setminus \partial\Omega) \\ &= \mathcal{H}^{N-1}(\partial E \cap \partial\Omega) + \mathcal{H}^{N-1}(\partial E \setminus \partial\Omega) = \mathcal{H}^{N-1}(\partial E)\,. \end{aligned}$$

From this fact, we can exploit [22, Theorem 1.1], which assures the existence of a sequence of smooth sets E_n compactly contained inside E, which approximate E both in L^1 and in perimeter. In particular,

$$h(\Omega) \leq h_1(\Omega) = \frac{P(E)}{|E|} \leq \lim_{n\to+\infty} \frac{P(E_n)}{|E_n|} = h(\Omega).$$

Moreover, since $E_n \Subset E$ we can construct functions $u_n \in W_0^{1,1}(\Omega)$ such that

$$\frac{\int_\Omega |\nabla u_n|\,\mathrm{d}x}{\int_\Omega |u_n|\,\mathrm{d}x} = \frac{P(E)}{|E|} + o_n(1),$$

which easily entails that $\lambda_{1,1}(\Omega) = h(\Omega) = h_1(\Omega)$. A possible construction of the sequence (u_n) above is the following: given an optimal function u for λ_1 (its existence can be easily proven by means of the direct method in the Calculus of Variations), then we define $u_n = \rho_{\varepsilon/2} * (u\chi_{E_n^\varepsilon})$, where A^ε is the set of points of A whose distance from ∂A is larger than ε, and ρ_t is a positive mollifying kernel of total mass 1. Notice that such a construction is admissible since $\mathrm{dist}(\partial E, \partial E_n) > 0$.

The proof of the fact that $\lambda_1(\Omega) = \lambda_{1,1}(\Omega)$ is analogous. The main difference in the approximation argument is that one must use [22, Theorem 1.2] instead of [22, Theorem 1.1]. We thus skip the details. □

Remark 2.6. Notice that, for sets of finite perimeter, the inequality $P(E) \le \mathcal{H}^{N-1}(\partial E)$ is always true, but the equality does not hold in general as long as the $\mathcal{H}^{N-1}-$ measure of $E^0 \cap \partial E$ and $E^1 \cap \partial E$ is non zero, as a consequence of Federer's Theorem. The condition $\mathcal{H}^{N-1}(E^0 \cap \partial E) > 0$ is quite pathological. Indeed, due to the fact that sets of finite relative perimeter satisfy density estimates on their boundary, whenever this happens to be true, then E can not even support a relative isoperimetric inequality. For a proof see [21, Lemma 3.5]. On the other hand, the condition $\mathcal{H}^{N-1}(E^1 \cap \partial E) > 0$ can hold even for self-Cheeger sets, that is those sets who are minimizer of their Cheeger constant h_1, as shown in [18, Section 2].

Remark 2.7. When dealing with the functionals $\lambda_{1,q}(\cdot)$ for $(N-1)/N > q > 1$, we still have two possible equivalent definitions. For any $\Omega \subset \mathbb{R}^N$ open and bounded, with $P(\Omega) = \mathcal{H}^{N-1}(\partial\Omega)$,

$$\lambda_{1,q}(\Omega) = \inf\left\{ \frac{\|Du\|_{TV(\mathbb{R}^N)}}{\|u\|_{L^q(\Omega)}} \,:\, u \in BV(\overline{\Omega}) \setminus \{0\},\, u|_{\mathbb{R}^N \setminus \overline{\Omega}} = 0 \right\} = \inf\left\{ \frac{\|\nabla u\|_{L^1(\mathbb{R}^N)}}{\|u\|_{L^q(\Omega)}} : u \in C_c^\infty(\Omega) \setminus \{0\} \right\}.$$

Arguing as in Proposition 2.5, one can prove this equivalence.

3. Proof of the main result

3.1. **Cheeger-Kohler-Jobin Inequality.** This paragraph is devoted to the proof of Theorem 1.1. Since our strategy is based on the Kohler-Jobin radial rearrangement technique, later extended by Brasco to the nonlinear case $p \neq 2$, we begin by a short explanation of this tool. At the same time, this allows us to introduce the needed notations for our proof.

The cornerstone of the Kohler-Jobin inequality is the following: given a non-negative function u in the usual Sobolev space $W_0^{1,p}(\Omega)$, one constructs a rearrangement u^* of u, belonging to $W_0^{1,p}(B)$ for some ball B, such that

$$\int_\Omega |\nabla u|^p \mathrm{d}x = \int_B |\nabla u^*|^p \mathrm{d}x \qquad \text{and} \qquad \int_\Omega |u|^q \mathrm{d}x \le \int_B |u^*|^q \mathrm{d}x\,. \tag{3.1}$$

A somehow natural idea, in order to obtain the Kohler-Jobin inequality, is to consider the function u^* such that any level set of u^* is a ball $B_{r(t)}$ centered at 0 with

$$T_p(B_{r(t)}) = T_p(\{u > t\})\,.$$

Unfortunately this idea is too pretentious. In particular, the second requirement in (3.1) can not hold in general: if $\{u > t\}$ is not a ball for t in a set of positive measure, then

$$\int_\Omega |u|^q \mathrm{d}x > \int_B |u^*|^q \mathrm{d}x\,,$$

from the Saint-Venant inequality (while nothing can be said on the L^p-norms of the gradients). On the other hand, even if the second condition in (3.1) holds true, the first one may fail: it is not difficult to construct a function u which is not radially decreasing, and without *plateaux* (that is such that $|\{u \geq t\} \setminus \{u > t\}| = 0$ for any t) but such that $\{u > t\}$ is a ball for any t. In this case the L^q norms of u and u^* coincide, but

$$\int_\Omega |\nabla u|^p \mathrm{d}x > \int_B |\nabla u^*|^p \mathrm{d}x\,,$$

by the characterization of equality cases of Pólya-Szegö inequality see [5]. This suggests that the rearrangement must somehow take into account the other level sets of u. The successful idea of Kohler-Jobin was to introduce the following modification of the torsional rigidity.

Definition 3.1 ([2, 15])**.** Let Ω be an open, bounded set and $1 < p < +\infty$. We say that $u \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$ is a *reference function for* Ω if $u \geq 0$ in Ω and

$$t \mapsto \frac{|\{x \in \Omega \;:\; u(x) > t\}|}{\int_{\{u=t\}} |\nabla u|^{p-1}\,\mathrm{d}\mathcal{H}^{N-1}} \in L^\infty([0, \|u\|_{L^\infty(\Omega)}])\,.$$

We call $\mathcal{A}_p(\Omega)$ the set of all reference functions for Ω.
Then, for any $u \in \mathcal{A}_p(\Omega)$, the *modified torsional rigidity* is the functional, depending on Ω and u, defined by

$$T_{p,mod}(\Omega, u) = \left(\frac{p}{p-1} \sup\left\{\int_\Omega g \circ u \,\mathrm{d}x - \frac{1}{p}\int_\Omega |\nabla g \circ u|^p \mathrm{d}x : g \in \mathrm{Lip}[0, \|u\|_{L^\infty(\Omega)}],\, g(0) = 0\right\}\right)^{p-1}.$$

The features of $T_{p,mod}$ that will be used later are collected in the next Lemma (for the proof, we refer to [2, Proposition 3.8] and [15]).

Lemma 3.2. *Let $1 < p < +\infty$, $\Omega \subset \mathbb{R}^N$ be an open, bounded set, and $u \in \mathcal{A}_p(\Omega)$. Then*
- (i) *$T_{p,mod}(\Omega, u) \leq T_p(\Omega)$;*
- (ii) *if B is a ball such that $T_{p,mod}(\Omega, u) = T_p(B)$, then $|B| \leq |\Omega|$. Equality holds in the latter if and only if $\Omega = B$ and u is a radial function.*

The idea in [15, 16, 2] is then the following: given $u \in \mathcal{A}_p(\Omega)$, define u^* as the function such that for any $t \in [0, \|u\|_{L^\infty(\Omega)}]$, the set $\{u^* > t\}$ is a ball with $p-$torsional rigidity equal to $T_{p,mod}(\{u > t\}, (u - t)_+)$, where for a function f we call $f_+ = \max\{f, 0\}$ the positive part of f. With this construction, it is possible to show a rearrangement result as follows (for a proof we refer to [2, Proposition 4.1 and Remark 4.3] and [15]).

Lemma 3.3 (Kohler-Jobin Rearrangement Theorem)**.** *Let $1 < p < +\infty$, $\Omega \subset \mathbb{R}^N$ be an open, bounded set, $u \in \mathcal{A}_p(\Omega)$, and B the origin centered ball such that $T_p(B) = T_{p,mod}(\Omega, u)$. Then, for every $q \geq 1$, there exists a radially symmetric decreasing function $u^* \in W_0^{1,p}(B)$ such that*

$$\int_\Omega |\nabla u|^p \mathrm{d}x = \int_B |\nabla u^*|^p \mathrm{d}x \quad \text{and} \quad \int_\Omega |u|^q \mathrm{d}x \leq \int_B |u^*|^q \mathrm{d}x\,.$$

Moreover, if $q > 1$, equality holds in the latter if and only if u is already a radially decreasing function.

In the sequel we will call such an u^* *Kohler-Jobin rearrangement of u.* We are now in position to prove our main result.

Proof of Theorem 1.1. Step 1. Let $u \in \mathcal{A}_p(\Omega)$, B be the origin centered ball such that $T_p(B) = T_{p,mod}(\Omega, u)$, and u^* be the Kohler-Jobin rearrangement of u. Then, by part (ii) of Lemma 3.2, we know that $|\{u^* > t\}| \le |\{u > t\}|$ for any $t \in [0, \|u\|_{L^\infty(\Omega)}]$. Thanks to the isoperimetric inequality we have then that

$$P(\{u > t\}) \ge C_N |\{u > t\}|^{\frac{N-1}{N}} \ge C_N |\{u^* > t\}|^{\frac{N-1}{N}} = P(\{u^* > t\}), \tag{3.2}$$

for a suitable dimensional constant $C_N > 0$. By integrating on $\mathbb{R}^+$ and applying the coarea formula, we obtain

$$\int_\Omega |\nabla u| \mathrm{d}x \ge \int_B |\nabla u^*| \mathrm{d}x \,.$$

Moreover, thanks to the rigidity cases of the isoperimetric inequality, equality holds if and only if $\{u > t\} = \{u^* > t\}$ for almost all $t \in \mathbb{R}$, so that, in particular, $\Omega = B$. On the other hand, by Lemma 3.3 we have that

$$\int_\Omega |u|^q \mathrm{d}x \le \int_B |u^*|^q \mathrm{d}x \,,$$

for $q \ge 1$.

Step 2. We want now to apply the first Step to functions of a minimizing sequence for $\lambda_{1,q}(\Omega)$ with $q \in [1, \frac{N}{N-1})$. By the definition of $\lambda_{1,q}(\Omega)$, see (1.1), we can find a minimizing sequence of non-negative $\varphi_h \in C_c^\infty(\Omega)$ with $h \in \mathbb{N}$, such that

$$\mathcal{R}(\varphi_h) := \frac{\int_\Omega |\nabla \varphi_h| \, \mathrm{d}x}{\left(\int_\Omega |\varphi_h|^q \mathrm{d}x\right)^{\frac{1}{q}}} \longrightarrow \lambda_{1,q}(\Omega), \qquad \text{as } h \to \infty \,. \tag{3.3}$$

It is immediate to note that $\varphi_h \in \mathcal{A}_p(\Omega)$ for all $h \in \mathbb{N}$ and for all $p \in (1, \infty)$. Hence we can apply Step 1 to $\varphi_h \in \mathcal{A}_p(\Omega)$, calling with a slight abuse of notation B_h the ball such that $T_p(B_h) = T_{p,mod}(\Omega, \varphi_h)$, and obtain, for $\theta = \theta(N, p, q) = \frac{q - N(q-1)}{q(p + N(p-1))}$,

$$T_p(\Omega)^\theta \mathcal{R}(\varphi_h) \ge T_{p,mod}(\Omega, \varphi_h)^\theta \mathcal{R}(\varphi_h) = T_p(B_h)^\theta \mathcal{R}(\varphi_h) \ge T_p(B_h)^\theta \mathcal{R}(\varphi_h^*) \ge T_p(B_h)^\theta \lambda_{1,q}(B_h) \,. \tag{3.4}$$

We highlight that the last inequality follows since the Kohler-Jobin rearrangement φ_h^* of φ_h is an admissible function for the infimum defining $\lambda_{1,q}(B_h)$. Moreover, we observe that the quantity on the right of the chain of inequalities is constant, since B_h is a ball and θ is taken so that the functional $T_p^\theta(\cdot)\lambda_{1,q}(\cdot)$ is scale invariant. Hence, passing to the limit as $h \to \infty$ on the left-hand side, we obtain

$$T_p(\Omega)^\theta \lambda_{1,q}(\Omega) = \limsup_{h \to \infty} T_p(\Omega)^\theta \mathcal{R}(\varphi_h) \ge T_p(B_h)^\theta \lambda_{1,q}(B_h) = T_p(B)^\theta \lambda_{1,q}(B) \,,$$

where B is any ball. We finally note that the case of the Cheeger constant $h_1 = \lambda_{1,1}$ follows simply taking $q = 1$, and the first part of the proof is concluded.

Step 3. Equality cases. This step is not straightforward, as in the proof of the inequality we had to pass to the limit as $h \to +\infty$. We consider $\Omega \subset \mathbb{R}^N$ that satisfies (1.4) with the equality sign and take φ_h, φ_h^*, and B_h as in Step 2. Without loss of generality, we may assume that $\|\varphi_h\|_{L^q(\Omega)} = 1$ for every h. In view of (3.3), the minimizing sequence (φ_h) is uniformly bounded in $W^{1,1}(\Omega)$; therefore, up to a subsequence (not relabeled), φ_h weakly* converges in $BV(\Omega)$ to some φ, optimal for $\lambda_{1,q}(\Omega)$, see Remark 2.7. As for φ_h^*, thanks to (3.3) and (3.4), we infer that

$$\lim_{h \to +\infty} \mathcal{R}(\varphi_h^*) = \lim_{h \to +\infty} \mathcal{R}(\varphi_h) = \lambda_{1,q}(\Omega) \,. \tag{3.5}$$

In particular, by construction of the Kohler-Jobin rearrangement, by Lemma 3.3 and by the above formula we deduce

$$1 = \|\varphi_h\|_{L^q(\Omega)} \le \|\varphi_h^*\|_{L^q(B_h)} \le C_N \,, \tag{3.6}$$

where C_N denotes, as usual, a positive constant, depending only on the dimension N, which may vary from line to line. Improving the computations done in (3.2) with the quantitative form of the isoperimetric inequality (2.1), for $u = \varphi_h$, we obtain

$$P(\{\varphi_h > t\}) \geq P(\{\varphi_h^* > t\}) + C_N N \omega_N^{1/N} |\{\varphi_h > t\}|^{(N-1)/N} \alpha(\{\varphi_h > t\})^2 ,$$

then, integrating on $\mathbb{R}^+$ and applying the coarea formula, we have

$$\int_\Omega |\nabla \varphi_h| \, dx \geq \int_{B_h} |\nabla \varphi_h^*| \, dx + C_N \int_0^{+\infty} |\{\varphi_h > t\}|^{(N-1)/N} \alpha(\{\varphi_h > t\})^2 dt . \tag{3.7}$$

On the other hand, by combining (3.6) and (3.7) we obtain

$$\mathcal{R}(\varphi_h) \geq \mathcal{R}(\varphi_h^*) + C_N \int_0^{+\infty} |\{\varphi_h > t\}|^{(N-1)/N} \alpha(\{\varphi_h > t\})^2 dt .$$

Passing to the limit as $h \to +\infty$ and recalling (3.5), we infer that

$$\liminf_{h \to +\infty} \int_0^\infty |\{\varphi_h > t\}|^{(N-1)/N} \alpha(\{\varphi_h > t\})^2 dt = 0 ,$$

and by applying Fatou's Lemma, we have

$$\int_0^{+\infty} \liminf_{h \to +\infty} |\{\varphi_h > t\}|^{(N-1)/N} \alpha(\{\varphi_h > t\})^2 dt = 0 ,$$

hence for almost all $t \in (0, \infty)$ one has, up to pass to subsequences,

$$\lim_{h \to +\infty} |\{\varphi_h > t\}|^{(N-1)/N} \alpha(\{\varphi_h > t\})^2 = 0 .$$

Since $\varphi \neq 0$, there exists a $\bar{t} > 0$ such that $|\{\varphi > \bar{t}\}| > \varepsilon > 0$, so one has $|\{\varphi_h > \bar{t}\}| \geq \varepsilon/2 > 0$ for h large enough, since $\varphi_h \to \varphi$ in L^1 and pointwise a.e. up to subsequences. Moreover, for any $s \leq t$ and all $h \in \mathbb{N}$, it holds $\{\varphi_h > t\} \subseteq \{\varphi_h > s\}$, hence for any $t \in (0, \bar{t})$ it can not happen that $|\{\varphi_h > t\}| \to 0$ as $h \to 0$, therefore it must hold that for almost all $t \in (0, \bar{t})$,

$$\lim_{h \to +\infty} \alpha(\{\varphi_h > t\}) = 0 ,$$

entailing that $\{\varphi_h > t\} \to B_t$ in L^1 for some ball B_t. In particular we have that $\{\varphi_h > 0\} = \cup_{t>0} \{\varphi_h > t\} \to B$ in L^1 as $h \to \infty$, for some ball B. On one hand, φ is an admissible function for the variational problem defining $\lambda_{1,q}(B)$ and it is optimal for $\lambda_{1,q}(\Omega)$. Moreover we have the inclusion $B \subset \Omega$ (since any $x \in B$ is the limit of points in the support of φ_h). Thus

$$\lambda_{1,q}(\Omega) \leq \lambda_{1,q}(B) \leq \frac{\|D\varphi\|_{TV(\mathbb{R}^N)}}{\left(\int_\Omega |\varphi|^q \, dx\right)^{1/q}} = \lambda_{1,q}(\Omega) ,$$

and therefore $\lambda_{1,q}(B) = \lambda_{1,q}(\Omega)$. In view of the assumption $T_p(B)^\theta \lambda_{1,p}(B) = T_p(\Omega)^\theta \lambda_{1,q}(\Omega)$, we also have $T_p(B) = T_p(\Omega)$.

We claim now that the support of φ is Ω itself, up to a negligible set. Notice that the claim, if $q = 1$ means that Ω is self-Cheeger. If it does not hold, that is if $|\Omega \setminus B| > 0$, then, since $T_p(\cdot)$ is strictly increasing for set inclusion, we get that $T_p(\Omega) > T_p(B)$, which is clearly impossible. In conclusion we have that $\Omega = \{\varphi > 0\}$, thus we have that $|\Omega| = |B|$ and we can invoke the equality cases of the Saint-Venant or of the Faber-Krahn inequality to deduce that $\Omega = B$ up to zero measure. The proof is concluded. □

3.2. **Proof of the quantitative estimate for h_1.** We offer here the proof of Theorem 1.3. We remark that it is just a slight modification of the combination of a Kohler-Jobin type inequality with a quantitative Saint-Venant inequality proposed in [3]. We recall that the quantitative Saint-Venant inequality proved in [3, Section 5, Proof of Main Theorem] reads as

$$T_2(B)|B|^{-\frac{N+2}{N}} - T_2(\Omega)|\Omega|^{-\frac{N+2}{N}} \geq \tau(N)\alpha(\Omega)^2\,,$$

where $\tau = \tau(N) > 0$ is a dimensional constant and α the Fraenkel asymmetry (2.2).

Proof of Theorem 1.3. Since inequality (1.6) is scale invariant, we may assume without loss of generality that Ω and B have the same measure, equal to 1. Moreover, for brevity of notation, we will denote by T the 2-torsional rigidity T_2. Thanks to the Cheeger–Kohler-Jobin estimate (1.5), we have

$$\frac{h_1(\Omega)}{h_1(B)} - 1 \geq \left(\frac{T(B)}{T(\Omega)}\right)^{\theta} - 1\,.$$

We now distinguish two cases: $T(B)/T(\Omega) > 2$ and $T(B)/T(\Omega) \in [1, 2]$. In the former, exploiting the easy bound $\tau\alpha^2(\Omega) \leq T(B)$, we obtain

$$\frac{h_1(\Omega)}{h_1(B)} - 1 \geq 2^{\theta} - 1 \geq (2^{\theta} - 1)\frac{\tau\alpha^2(\Omega)}{T(B)}\,. \tag{3.8}$$

In the latter, we use the concavity of the function $x \mapsto x^{\theta}$, being $0 < \theta = 1/(N+2) < 1$. For every $x \in [1, 2]$, we have

$$x^{\theta} = ((2-x) + 2(x-1))^{\theta} \geq (2-x) + 2^{\theta}(x-1)\,, \tag{3.9}$$

since $2-x$ and $x-1$ are both in $[0, 1]$ and their sum is 1. By applying (3.9) to $x = T(B)/T(\Omega)$, we obtain

$$\begin{aligned}\frac{h_1(\Omega)}{h_1(B)} - 1 &\geq \left(2 - \frac{T(B)}{T(\Omega)}\right) + 2^{\theta}\left(\frac{T(B)}{T(\Omega)} - 1\right) - 1 = (2^{\theta}-1)\left(\frac{T(B)}{T(\Omega)} - 1\right)\\ &\geq (2^{\theta}-1)\frac{\tau\alpha^2(\Omega)}{T(\Omega)} \geq (2^{\theta}-1)\frac{\tau\alpha^2(\Omega)}{T(B)}\,.\end{aligned} \tag{3.10}$$

Finally, combining (3.8) and (3.10), we conclude the proof of (1.6) with

$$\sigma := \frac{\tau(2^{\theta}-1)h_1(B)}{T(B)}\,,$$

where B denotes an N-dimensional ball of unit measure.
Eventually, we notice that the exponent 2 is sharp. Indeed it is enough to consider the family of ellipsoids

$$\Omega_\varepsilon = \left\{(x_1, \dots, x_N) : \sum_{i=1}^{N-1} x_i^2 + (1+\varepsilon)x_N^2 \leq 1\right\}.$$

A simple computation shows that $|\Omega_\varepsilon|^{\frac{N-1}{N}}P(\Omega_\varepsilon) - |B|^{\frac{N-1}{N}}P(B) \simeq \varepsilon^2$ while $\alpha(\Omega_\varepsilon) \simeq \varepsilon$ as $\varepsilon \to 0$. On the other hand, since $h_1(B) = N$,

$$h_1(\Omega_\varepsilon) - h_1(B) \leq \frac{P(\Omega_\varepsilon) - P(B)}{|\Omega_\varepsilon|} \simeq P(\Omega_\varepsilon) - P(B)\,.$$

This concludes the proof. □

4. The fractional case

4.1. **Preliminaries on fractional Sobolev spaces.** In this section we introduce the fractional Sobolev spaces $W^{s,p}$, for $s \in (0,1)$ and $p \in [1,\infty)$. The $W^{s,p}$ Gagliardo seminorm on $\mathbb{R}^N$ is defined by

$$[u]^p_{W^{s,p}(\mathbb{R}^N)} := \iint_{\mathbb{R}^N\times\mathbb{R}^N} \frac{|u(x)-u(y)|^p}{|x-y|^{N+sp}}\,\mathrm{d}x\mathrm{d}y\,.$$

We then call

$$W^{s,p}(\mathbb{R}^N) := \left\{u \in L^p(\mathbb{R}^N)\ :\ [u]_{W^{s,p}(\mathbb{R}^N)} < \infty\right\}\,,$$

which is a Banach space (Hilbert for $p=2$) once endowed with the norm

$$\|u\|^p_{W^{s,p}(\mathbb{R}^N)} := \|u\|^p_{L^p(\mathbb{R}^N)} + [u]^p_{W^{s,p}(\mathbb{R}^N)}\,.$$

Since our aim is to study functionals with Dirichlet boundary conditions, it is natural to define, for an open, bounded set $\Omega \subset \mathbb{R}^N$, the Banach space $\widetilde{W}^{s,p}_0(\Omega)$ as the completion of $C^\infty_c(\Omega)$ with respect to the norm

$$u \mapsto \|u\|_{L^p(\Omega)} + [u]_{W^{s,p}(\mathbb{R}^N)}\,.$$

We note that, in $\widetilde{W}^{s,p}_0(\Omega)$, the norm $\|\cdot\|_{L^p(\Omega)} + [\cdot]_{W^{s,p}(\mathbb{R}^N)}$ is equivalent to the seminorm $[\cdot]_{W^{s,p}(\mathbb{R}^N)}$. This space is reflexive for $p \in (1,\infty)$, while $\widetilde{W}^{s,1}_0(\Omega)$ is not reflexive, hence it is often substituted by the (larger) space

$$W^{s,1}_0(\Omega) := \left\{u \in L^1(\Omega)\ :\ [u]_{W^{s,1}(\mathbb{R}^N)} < \infty,\ u = 0 \text{ a.e. in } \mathbb{R}^N\setminus\Omega\right\}.$$

Observe that $\widetilde{W}^{s,1}_0(\Omega) \subset W^{s,1}_0(\Omega)$.

There are other possible definitions of fractional Sobolev spaces, but we do not use them and so we just refer the interested reader to [4] and the references therein for a broader discussion.

We recall that the fractional perimeter is defined, for measurable sets $E \subset \mathbb{R}^N$ and $s \in (0,1)$, as

$$P_s(E) := [\chi_E]_{W^{s,1}(\mathbb{R}^N)} = \iint_{\mathbb{R}^N\times\mathbb{R}^N} \frac{|\chi_E(x)-\chi_E(y)|}{|x-y|^{N+s}}\,\mathrm{d}x\mathrm{d}y = 2\iint_{\mathbb{R}^N\times\mathbb{R}^N} \frac{\chi_E(x)\,\chi_{E^c}(y)}{|x-y|^{N+s}}\,\mathrm{d}x\mathrm{d}y\,,$$

and we note that the functional $P_s(\cdot)$ is $N-s$ positively homogeneous. In [4], Brasco, Lindgren, and Parini study several properties of the the nonlocal counterpart of the Cheeger constant of an open, bounded set $\Omega \subset \mathbb{R}^N$, which they define, for $s \in (0,1)$, as

$$h_s(\Omega) = \inf\left\{\frac{P_s(E)}{|E|} : E \subset \Omega\right\}.$$

In the spirit of the multiple definitions available in the local case, if $\Omega \subset \mathbb{R}^N$ is open, bounded, and Lipschitz (see [4, Theorem 5.8]), the fractional Cheeger constant can be equivalently characterized as

$$h_s(\Omega) = \lambda^s_{1,1}(\Omega) := \inf\left\{\frac{[u]_{W^{s,1}(\mathbb{R}^N)}}{\int_\Omega |u|\,\mathrm{d}x} : u \in W^{s,1}_0(\Omega)\setminus\{0\}\right\} = \inf\left\{\frac{[u]_{W^{s,1}(\mathbb{R}^N)}}{\int_\Omega |u|\,\mathrm{d}x} : u \in C^\infty_c(\Omega)\setminus\{0\}\right\}\,;$$

moreover, Brasco, Lingdren, and Parini prove the following fractional Cheeger inequality: for $s \in (0,1)$, it holds

$$|\Omega|^{\frac{s}{N}} h_s(\Omega) \geq |B|^{\frac{s}{N}} h_s(B)\,,$$

where B is any N-dimensional ball; moreover, equality holds true if and only if Ω is a ball. Since we have the equivalence between the two definitions of Cheeger constant only among bounded Lipschitz domains, from now on we restrict ourselves to this class of sets in the fractional setting. We note that the hypothesis of Lipschitz regularity for $\partial\Omega$ is a lot stronger than the assumption $P(\Omega)=\mathcal{H}^{N-1}(\partial\Omega)$ that we considered in the local case $s=1$.

4.2. **Proof of the main results for the fractional case.** We now state the results analogous to Theorem 1.1 and 1.3 for the nonlocal case. Since the strategy of the proofs is precisely the same of the local case, we just highlight the points where some technical differences arise.

Theorem 4.1. *Let $p\in(1,\infty)$, $s\in(0,1)$, and $q\in[1,\frac{N}{N-s})$. Then for any open, bounded and Lipschitz set $\Omega\subset\mathbb{R}^N$ it holds*

$$T_p(\Omega)^\theta\lambda^s_{1,q}(\Omega)\geq T_p(B)^\theta\lambda^s_{1,q}(B), \tag{4.1}$$

where $\theta=\theta(N,s,q,p)=\frac{\frac{N}{q}+s-N}{Np-N+p}$ and

$$\lambda^s_{1,q}(\Omega):=\inf\left\{\frac{[u]_{W^{s,1}(\mathbb{R}^N)}}{\left(\int_\Omega|u|^q\mathrm{d}x\right)^{1/q}}:u\in W^{s,1}_0(\Omega)\setminus\{0\}\right\}=\inf\left\{\frac{[u]_{W^{s,1}(\mathbb{R}^N)}}{\left(\int_\Omega|u|^q\mathrm{d}x\right)^{1/q}}:u\in C^\infty_c(\Omega)\setminus\{0\}\right\}. \tag{4.2}$$

In particular,

$$T_2(\Omega)^\theta h_s(\Omega)\geq T_2(B)^\theta h_s(B), \tag{4.3}$$

with $\theta=\theta(N,s)=\frac{s}{N+2}$.

Proof of Theorem 4.1. Step 1. Let $u\in\mathcal{A}_p(\Omega)$, B be the origin centered ball such that $T_p(B)=T_{p,mod}(\Omega,u)$, and u^* be the Kohler-Jobin rearrangement of u. Then, by part *(ii)* of Lemma 3.2, we know that $|\{u^*>t\}|\leq|\{u>t\}|$ for any $t\in[0,\|u\|_{L^\infty(\Omega)}]$. Applying the fractional isoperimetric inequality for the s-perimeter (see [11, 14]) we have then that

$$P_s(\{u>t\})\geq C(N,s)|\{u>t\}|^{\frac{N-s}{N}}\geq C(N,s)|\{u^*>t\}|^{\frac{N-s}{N}}=P_s(\{u^*>t\}),$$

for some constant $C(N,s)>0$. Integrating over $\mathbb{R}^+$ and using a suitable coarea formula for the fractional setting, which can be found in [4, Lemma 4.7] or in [6], we have

$$\begin{aligned}[u]_{W^{s,1}(\mathbb{R}^N)}&=\iint_{\mathbb{R}^N\times\mathbb{R}^N}\frac{|u(x)-u(y)|}{|x-y|^{N+s}}\,\mathrm{d}x\mathrm{d}y=2\int_{t=0}^\infty\left(\iint_{\{u(x)>t\}\times\{u(y)\leq t\}}\frac{\mathrm{d}x\mathrm{d}y}{|x-y|^{N+s}}\right)\mathrm{d}t\\&=\int_{t=0}^\infty P_s(\{u>t\})\,\mathrm{d}t\geq\int_{t=0}^\infty P_s(\{u^*>t\})\,\mathrm{d}t=[u^*]_{W^{s,1}(\mathbb{R}^N)}.\end{aligned} \tag{4.4}$$

Moreover, thanks to the rigidity cases of the fractional isoperimetric inequality, equality holds if and only if $\{u>t\}=\{u^*>t\}$ for almost all $t\in\mathbb{R}$, so that, in particular, $\Omega=B$. On the other hand, by Lemma 3.3 we have that

$$\int_\Omega|u|^q\mathrm{d}x\leq\int_B|u^*|^q\mathrm{d}x$$

for $q\geq1$.

Step 2. We want now to apply the facts above to the elements of a minimizing sequence for $\lambda^s_{1,q}(\Omega)$, with $q\in[1,\frac{N}{N-s})$. By the definition (4.2) of $\lambda^s_{1,q}(\Omega)$, we can find a minimizing sequence of non-negative $\varphi_h\in C^\infty_c(\Omega)$ with $h\in\mathbb{N}$, such that

$$\mathcal{R}(\varphi_h):=\frac{[\varphi_h]_{W^{s,1}(\mathbb{R}^N)}}{\left(\int_\Omega|\varphi_h|^q\mathrm{d}x\right)^{\frac{1}{q}}}\longrightarrow\lambda^s_{1,q}(\Omega),\qquad\text{as }h\to\infty.$$

Since $\varphi_h\in\mathcal{A}_p(\Omega)$ for all h and $p\in(1,\infty)$, we can define φ_h^* its Kohler-Jobin rearrangement, and then the remainder of the proof follows word by word as in Step 2 of the proof of

Theorem 1.1. We note that the case of the Cheeger constant $h_s = \lambda_{1,1}^s$ follows simply taking $q = 1$.

Step 3. Equality cases. Also this part of the proof can be done similarly to Step 3 in the proof of Theorem 1.1, using the suitable fractional perimeter and the coarea type formula described in Step 1. It is important to highlight that this step is actually easier in the fractional setting, since we do not need a relaxation into BV functions.

We consider $\Omega \subset \mathbb{R}^N$ that satisfies (4.1) with the equality sign, we take φ_h, φ_h^* as in Step 2 and B_h the ball such that $T_p(B_h) = T_{p,mod}(\Omega, \varphi_h)$. Then, since the functions $\varphi_h \in C_c^\infty(\Omega)$ are bounded in $W^{s,1}(\Omega)$, one can find a subsequence (not relabeled) strongly converging in L^q, for any $q \in [1, \frac{N}{N-s})$, to some function φ. Moreover, up to pass to a subsequence converging pointwise a.e., one can also see that

$$[\varphi]_{W^{s,1}(\mathbb{R}^N)} \leq \liminf_{h\to\infty} [\varphi_h]_{W^{s,1}(\mathbb{R}^N)},$$

using Fatou's Lemma. In conclusion we have that

$$\frac{[\varphi]_{W^{s,1}(\mathbb{R}^N)}}{\left(\int_\Omega |\varphi|^q \mathrm{d}x\right)^{\frac{1}{q}}} \leq \liminf_{h\to\infty} \frac{[\varphi_h]_{W^{s,1}(\mathbb{R}^N)}}{\left(\int_\Omega |\varphi_h|^q \mathrm{d}x\right)^{\frac{1}{q}}} = \lambda_{1,q}^s(\Omega)\,,$$

and hence, noting that the pointwise convergence entails $\varphi = 0$ a.e. in $\mathbb{R}^N \setminus \Omega$, we have that $\varphi \in W_0^{s,1}(\Omega)$ is an optimal function for $\lambda_{1,q}^s(\Omega)$. On the other hand, by hypothesis and construction of the Kohler-Jobin rearrangement, one also has

$$\lim_{h\to\infty} \frac{[\varphi_h^*]_{W^{s,1}(\mathbb{R}^N)}}{\left(\int_\Omega |\varphi_h^*|^q \mathrm{d}x\right)^{\frac{1}{q}}} = \lambda_{1,q}^s(\Omega), \qquad \|\varphi_h^*\|_{L^q} \leq C_N\,.$$

At this point it is enough to repeat the arguments already detailed in the local case with the due changes. First of all, we use the fractional quantitative isoperimetric inequality [14, 11] with the fractional perimeter and coarea type formula [4, Lemma 4.7] in place of their local counterparts in order to improve the estimates in (4.4).

As a consequence, we have that $\{\varphi_h > 0\}$ converges in L^1 to a ball B, hence φ is an admissible function for the variational problem defining $\lambda_{1,q}^s(B)$ and optimal for $\lambda_{1,q}^s(\Omega)$. On the other hand $B \subset \Omega$, due to the L^1-convergence of the supports of φ_h. Thus we deduce $\lambda_{1,q}^s(B) = \lambda_{1,q}^s(\Omega)$, which together with the assumption that (4.1) holds with the equality, entails that $T_p(\Omega) = T_p(B)$. Eventually we conclude that $\Omega = B$ a.e. thanks to the strict monotonicity of the torsional rigidity and the characterization of equality cases for the Saint-Venant inequality. □

We are now able to exploit the fractional Cheeger–Kohler-Jobin inequality in order to obtain a quantitative estimate for the nonlocal Cheeger constant.

Theorem 4.2. *There exists a dimensional constant $\sigma(N) > 0$ such that for any open, bounded and Lipschitz set $\Omega \subset \mathbb{R}^N$ and for any ball $B \subset \mathbb{R}^N$, we have*

$$|\Omega|^{\frac{s}{N}} h_s(\Omega) - |B|^{\frac{s}{N}} h_s(B) \geq \sigma(N)\alpha(\Omega)^2\,, \tag{4.5}$$

where α is the Fraenkel asymmetry. Moreover, the power 2 in (4.5) is sharp, in the sense that it can not be replaced by any lower number.

Proof of Theorem 4.2. The argument follows precisely as the proof of Theorem 1.3, using the fractional Cheeger–Kohler-Jobin inequality (4.3) proved in Theorem 4.1 instead of the local version (1.5). We stress that, also for the proof of the fractional quantitative Cheeger inequality, we rely on the quantitative result for the *local* torsional rigidity T_2 proved in [3]. □

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