SYMMETRY OF MINIMIZERS OF A GAUSSIAN ISOPERIMETRIC PROBLEM

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Abstract. We study an isoperimetric problem described by a functional that consists of the standard Gaussian perimeter and the norm of the barycenter. This second term has a repulsive effect, and it is in competition with the perimeter. Because of that, in general the solution is not the half-space. We characterize all the minimizers of this functional, when the volume is close to one, by proving that the minimizer is either the half-space or the symmetric strip, depending on the strength of the repulsive term. As a corollary, we obtain that the symmetric strip is the solution of the Gaussian isoperimetric problem among symmetric sets when the volume is close to one.

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1. Introduction

The Gaussian isoperimetric inequality (proved by Borell [7] and Sudakov-Tsirelson [29]) states that among all sets with given Gaussian measure the half-space has the smallest Gaussian perimeter. Since the half-space is not symmetric with respect to the origin, a natural question is to restrict the problem among sets which are symmetric, i.e., either central symmetric ($E = -E$) or coordinate wise symmetric (n-symmetric). This problem turns out to be rather difficult as every known method that has been used to prove the Gaussian isoperimetric inequality, such as symmetrization [14] and the Ornstein-Uhlenbeck semigroup argument [1], seems to fail. In fact, at the moment it is not even clear what the solution to this problem should be.

The Gaussian isoperimetric problem for symmetric sets or its generalization to Gaussian noise is stated as an open problem in [8, 18]. In the latter it was conjectured that the solution should be the ball or its complement, but this was recently disproved in [21]. Another natural candidate for the solution is the symmetric strip or its complement. Indeed, in [4] Barthe proved that if one replaces the standard Gaussian perimeter by a certain anisotropic perimeter, the solution of the isoperimetric problem among n-symmetric sets is the symmetric strip or its complement. We mention also a somewhat similar result by Latala and Oleszkiewicz [26, Theorem 3] who proved that the symmetric strip minimizes the Gaussian perimeter weighted with the width of the set among convex and symmetric sets with volume constraint. For the standard perimeter

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the problem is more difficult as a simple energy comparison shows (see [22]) that when the volume is exactly one half, the two-dimensional disk and the three-dimensional ball have both smaller perimeter than the symmetric strip in dimension two and three, respectively. Similar difficulty appears also in the isoperimetric problem on sphere for symmetric sets, where it is known that the union of two spherical caps does not always have the smallest surface area (see [4]). However, it might still be the case that the solution of the problem is a cylinder $B_r^k \times \mathbb{R}^{n-k}$, or its complement, for some k depending on the volume (see [22, Conjecture 1.3]). Here B_r^k denotes the k-dimensional ball with radius r. At least the results by Heilman [21, 22] and La Manna [25] seem to indicate this.

To the best of the authors knowledge there are no other results directly related to this problem. In [13] Colding and Minicozzi introduced the Gaussian entropy, which is defined for sets as

$$\Lambda(\partial E) = \sup_{x_0 \in \mathbb{R}^n, t_0 > 0} P_\gamma(t_0^{-1}(E - \{x_0\})),$$

where P_γ is the Gaussian perimeter defined below. The Gaussian entropy is important as it is decreasing under the mean curvature flow and for this reason in [13] the authors studied sets which are stable for the Gaussian entropy. It was conjectured in [12] that the sphere minimizes the entropy among closed hypersurfaces (at least in low dimensions). This was proved by Bernstein and Wang in [5] in low dimensions and more recently by Zhu [33] in every dimension. This problem is related to the symmetric Gaussian problem since the Gaussian entropy of a self-shrinker equals its Gaussian perimeter.

In this paper we prove that the symmetric strip is the solution of the Gaussian isoperimetric problem for symmetric set when the volume is close to one. (Similarly, its complement is the solution when the volume is close to zero). Our proof is direct and thus we could give an explicit estimate on how close to one the volume has to be. In particular, the bound on the volume is independent of the dimension. But as our proof is rather long and the bound on the volume is obtained after numerous inequalities, we prefer to state the result in a more qualitative way in order to avoid heavy computations.

In order to describe the main result more precisely, we introduce our setting. Given a Borel set $E \subset \mathbb{R}^n$, $\gamma(E)$ denotes its Gaussian measure, defined as

$$\gamma(E) := \frac{1}{(2\pi)^{\frac{n}{2}}} \int_E e^{-\frac{|x|^2}{2}} dx.$$

If E is an open set with Lipschitz boundary, $P_\gamma(E)$ denotes its *Gaussian perimeter*, defined as

$$P_\gamma(E) := \frac{1}{(2\pi)^{\frac{n-1}{2}}} \int_{\partial E} e^{-\frac{|x|^2}{2}} d\mathcal{H}^{n-1}(x), \tag{1}$$

where $\mathcal{H}^{n-1}$ is the $(n-1)$-dimensional Hausdorff measure. We define the (non-renormalized) barycenter of a set E as

$$b(E) := \int_E x \, d\gamma(x)$$

and define the function $\phi : \mathbb{R} \to (0, 1)$ as

$$\phi(s) := \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{s} e^{-\frac{t^2}{2}} dt.$$

Moreover, given $\omega \in \mathbb{S}^{n-1}$ and $s \in \mathbb{R}$, $H_{\omega,s}$ denotes the half-space of the form

$$H_{\omega,s} := \{x \in \mathbb{R}^n : \langle x, \omega \rangle < s\},$$

while $D_{\omega,s}$ denotes the symmetric strip

$$D_{\omega,s} := \{x \in \mathbb{R}^n : |\langle x, \omega \rangle| < a(s)\},$$

where $a(s) > 0$ is chosen such that $\gamma(H_{\omega,s}) = \gamma(D_{\omega,s})$.

We approach the problem by studying the minimizers of the functional

$$\mathcal{F}(E) := P_\gamma(E) + \varrho\sqrt{\pi/2}\,|b(E)|^2 \tag{2}$$

under the volume constraint $\gamma(E) = \phi(s)$. Note that the isoperimetric inequality implies that for $\varrho = 0$ the half-space is the only minimizer of (2), while it is easy to see that the quantity $|b(E)|$ is maximized by the half-space. Therefore the two terms in (2) are in competition and we call the barycenter term repulsive, as it prefers to balance the volume around the origin. It is proven in [2, 17] that when ϱ is small, the half-space is still the only minimizer of (2). This result implies the quantitative Gaussian isoperimetric inequality (see also [11, 30, 31, 3]). It is clear that when we keep increasing the value ϱ, there is a threshold, say ϱ_s, such that for $\varrho > \varrho_s$ the half-space $H_{\omega,s}$ is no longer the minimizer of (2). In this paper we are interested in characterizing the minimizers of (2) after this threshold. Our main result reads as follows.

Main Theorem. *There exists $s_0 > 0$ such that the following holds: when $s \geq s_0$ there is a threshold ϱ_s such that for $\varrho \in [0, \varrho_s)$ the minimizer of (2) under volume constraint $\gamma(E) = \phi(s)$ is the half-space $H_{\omega,s}$, while for $\varrho \in (\varrho_s, \infty)$ the minimizer is the symmetric strip $D_{\omega,s}$.*

As a corollary this provides the solution for the symmetric Gaussian problem, because symmetric sets have barycenter zero.

Corollary 1. *There exists $s_0 > 0$ such that for $s \geq s_0$ it holds*

$$P_\gamma(E) \geq P_\gamma(D_{\omega,s}) = \Big(1 + \frac{\ln 2}{s^2} + o(1/s^2)\Big)e^{-\frac{s^2}{2}},$$

for any symmetric set E with volume $\gamma(E) = \phi(s)$, and the equality holds if and only if $E = D_{\omega,s}$ for some $\omega \in \mathbb{S}^{n-1}$.

Another corollary of the theorem is the optimal constant in the quantitative Gaussian isoperimetric inequality (see [2, 17]) when the volume is close to one. Let us denote by $\beta(E)$ the strong asymmetry

$$\beta(E) := \min_{\omega\in\mathbb{S}^{n-1}} \big|b(E) - b(H_{\omega,s})\big|,$$

which measures the distance between a set E and the family of half-spaces.

Corollary 2. *There exists $s_0 > 0$ such that for $s \geq s_0$ it holds*

$$P_\gamma(E) - P_\gamma(H_{\omega,s}) \geq c_s\beta(E),$$

for every set E with volume $\gamma(E) = \phi(s)$. The optimal constant is given by

$$c_s = \sqrt{2\pi}\,e^{s^2/2}\,\left(P_\gamma(D_{\omega,s}) - P_\gamma(H_{\omega,s})\right) = \sqrt{2\pi}\,\frac{\ln 2}{s^2} + o(1/s^2).$$

It would be interesting to obtain a result analogous to Corollary 2 in the Euclidean setting, where the minimization problem which corresponds to (2) was introduced in [16], and on the sphere [6]. The motivation for this is that, by the result of the second author [23], the optimal constant for the quantitative Euclidean isoperimetric inequality implies an estimate on the range of volume where the ball is the minimizer of the Gamov's liquid drop model [19]. This is a classical model used in nuclear physics and has gathered a lot attention in mathematics in recent years [9, 10, 24]. We also refer to the survey paper [15] for the state-of-the-art in the quantitative isoperimetric and other functional inequalities.

The main idea of the proof is to study the functional (2) when the parameter ϱ is within a carefully chosen range $(\varrho_{s,1}, \varrho_{s,2})$, and to prove that within this range the only local minimizers, which satisfy certain perimeter bounds, are the half-space $H_{\omega,s}$ and the symmetric strip $D_{\omega,s}$. We have to choose the lower bound $\varrho_{s,1}$ large enough so that the symmetric strip is a local minimum of (2). On the other hand, we have to choose the upper bound $\varrho_{s,2}$ small enough so that no other local minimum than $H_{\omega,s}$ and $D_{\omega,s}$ exist. Naturally also the threshold value ϱ_s has to be within the range $(\varrho_{s,1}, \varrho_{s,2})$.

Our proof is based on reduction argument where we reduce the dimension of the problem from $\mathbb{R}^n$ to $\mathbb{R}$. First, we develop further our ideas from [2] to reduce the problem from $\mathbb{R}^n$ to $\mathbb{R}^2$ by a rather short argument. In this step it is crucial that we are not constrained to keep the sets symmetric. The main challenge is thus to prove the theorem in $\mathbb{R}^2$, since here we cannot apply the previous reduction argument anymore. Instead, we use an ad-hoc argument to reduce the problem from $\mathbb{R}^2$ to $\mathbb{R}$ essentially by PDE type estimates from the Euler equation and from the stability condition. We give an independent overview of this argument at the beginning of the proof of Theorem 3 in Section 4. Finally, we solve the problem in $\mathbb{R}$ by a direct argument.

2. Notation and set-up

In this section we briefly introduce our notation and discuss about preliminary results and estimates. *We remark that throughout the paper the parameter s, associated with the volume, is assumed to be large even if not explicitly mentioned. In particular, our estimates are understood to hold when s is chosen to be large enough.* C denotes a numerical constant which may vary from line to line.

We denote the $(n-1)$-dimensional Hausdorff measure with Gaussian weight by $\mathcal{H}^{n-1}_\gamma$, i.e., for every Borel set A we define

$$\mathcal{H}^{n-1}_\gamma(A) := \frac{1}{(2\pi)^{\frac{n-1}{2}}} \int_A e^{-\frac{|x|^2}{2}} d\mathcal{H}^{n-1}.$$

We minimize the functional (2) among sets with locally finite perimeter and have the existence of a minimizer for every ϱ by an argument similar to [2, Proposition 1]. If $E \subset \mathbb{R}^n$ is a set of locally finite perimeter we denote its reduced boundary by $\partial^* E$ and define its Gaussian perimeter by

$$P_\gamma(E) := \mathcal{H}^{n-1}_\gamma(\partial^* E).$$

We denote the generalized exterior normal by ν^E which is defined on $\partial^* E$. As introduction to the theory of sets of finite perimeter and perimeter minimizers we refer to [28].

If the reduced boundary $\partial^* E$ is a smooth hypersurface we denote the second fundamental form by B_E and the mean curvature by $\mathscr{H}_E$, which for us is the sum of the principle curvatures. We adopt the notation from [20] and define the tangential gradient of a function f, defined in a neighborhood of $\partial^* E$, by $\nabla_\tau f := \nabla f - (\nabla f \cdot \nu^E)\nu^E$. Similarly, we define the tangential divergence of a vector field by $\mathrm{div}_\tau X := \mathrm{div} X - \langle DX\nu^E, \nu^E\rangle$ and the Laplace-Beltrami operator as $\Delta_\tau f := \mathrm{div}_\tau(\nabla_\tau f)$. The divergence theorem on $\partial^* E$ implies that for every vector field $X \in C^1_0(\partial^* E; \mathbb{R}^n)$ it holds

$$\int_{\partial^* E} \mathrm{div}_\tau X \, d\mathcal{H}^{n-1} = \int_{\partial^* E} \mathscr{H}_E \langle X, \nu^E\rangle \, d\mathcal{H}^{n-1}.$$

If $\partial^* E$ is a smooth hypersurface, we may extend any function $f \in C^1_0(\partial^* E)$ to a neighborhood of $\partial^* E$ by the distance function. For simplicity we will omit to indicate the dependence on the set E when this is clear, by simply writing $\nu = \nu^E$, $\mathscr{H} = \mathscr{H}_E$ etc...

We denote the mean value of a function $f : \partial^* E \to \mathbb{R}$ by

$$\bar{f} := \fint_{\partial^* E} f \, \mathcal{H}^{n-1}_\gamma,$$

and its average over a subset $\Sigma \subset \partial^* E$ by

$$(f)_\Sigma := \fint_\Sigma f \, \mathcal{H}^{n-1}_\gamma.$$

We recall that for every number $a \in \mathbb{R}$ it holds

$$\int_\Sigma (f-(f)_\Sigma)^2 \, d\mathcal{H}^1_\gamma \le \int_\Sigma (f-a)^2 \, d\mathcal{H}^1_\gamma.$$

Recall that $H_{\omega,s}$ denotes the half-space $\{x \in \mathbb{R}^n : \langle x, \omega\rangle < s\}$ and $D_{\omega,s}$ denotes the symmetric strip $\{x \in \mathbb{R}^n : |\langle x,\omega\rangle| < a(s)\}$, where $a(s)$ is chosen such that $\gamma(D_{\omega,s}) = \gamma(H_{\omega,s}) = \phi(s)$. Since we are assuming that s is large, it is important to know the asymptotic behavior of the quantities $\phi(s)$, $a(s)$, and $P(D_{\omega,s})$. A simple analysis shows that

$$\phi(s) = 1 - \frac{1}{\sqrt{2\pi}} \left(\frac{1}{s} + o(1/s^2) \right) e^{-\frac{s^2}{2}}. \tag{3}$$

The asymptotic behavior of $a(s)$ is a slightly more complicated. We will show that

$$a(s) = s + \frac{\ln 2}{s} + o(1/s). \tag{4}$$

To this aim we write $a(s) = s + \delta(s)$, so that (4) is equivalent to $\lim_{s\to\infty} s\delta(s) = \ln 2$. We argue by contradiction and assume that $\lim_{s\to\infty} s\delta(s) > \ln 2$. By the volume constraint

$$2\int_{a(s)}^\infty e^{-\frac{t^2}{2}} \, dt = \int_s^\infty e^{-\frac{t^2}{2}} \, dt \tag{5}$$

and therefore

$$1 < \lim_{s\to\infty} \frac{2\int_{s+\ln 2/s}^\infty e^{-\frac{t^2}{2}} \, dt}{\int_s^\infty e^{-\frac{t^2}{2}} \, dt} = \lim_{s\to\infty} \frac{-2(1-\frac{\ln 2}{s^2})e^{-\frac{(s+\ln 2/s)^2}{2}}}{-e^{-\frac{s^2}{2}}} = 1,$$

which is a contradiction. We arrive to a similar contradiction if $\lim_{s\to\infty} s\delta(s) < \ln 2$. Therefore we have (4).

For the perimeter of the strip $D_{\omega,s}$ we have the following estimate:

$$P_\gamma(D_{\omega,s}) = \Big(1 + \frac{\ln 2}{s^2} + o(1/s^2)\Big) e^{-\frac{s^2}{2}}. \tag{6}$$

Indeed, since $P_\gamma(D_{\omega,s}) = 2e^{-a(s)^2/2}$, by differentiating (5) we get $(1+\delta')P_\gamma(D_{\omega,s}) = e^{-s^2/2}$. Moreover, by $\lim_{s\to\infty} s\delta(s) = \ln 2$ we get $\lim_{s\to\infty} s^2\delta'(s) = -\ln 2$ and thus

$$\begin{aligned} &\lim_{s\to\infty} s^2 e^{\frac{s^2}{2}} \Big(P_\gamma(D_{\omega,s}) - \Big(1+\frac{\ln 2}{s^2}\Big) e^{-\frac{s^2}{2}} \Big) \\ &= -\lim_{s\to\infty} s^2 \Big(\frac{\delta'}{1+\delta'} + \frac{\ln 2}{s^2} \Big) = 0. \end{aligned}$$

In particular, according to our main theorem the threshold value has the asymptotic behavior

$$\varrho_s = 2\ln 2 \, \frac{\sqrt{2\pi}}{s^2} e^{\frac{s^2}{2}} \left(1 + o(1)\right). \tag{7}$$

This follows from the fact that threshold value ϱ_s is the unique value of ϱ for which the functional (2) satisfies $\mathcal{F}(H_{\omega,s}) = \mathcal{F}(D_{\omega,s})$, i.e.,

$$P_\gamma(D_{\omega,s}) = e^{-\frac{s^2}{2}} + \frac{\varrho_s}{2\sqrt{2\pi}} e^{-s^2}$$

by taking into account that $|b(H_{\omega,s})| = e^{-s^2/2}/\sqrt{2\pi}$.

In order to simplify the upcoming technicalities we replace the volume constraint in the original functional (2) with a volume penalization. We redefine $\mathcal{F}$ for any set of locally finite perimeter as

$$\mathcal{F}(E) := P_\gamma(E) + \varrho\sqrt{\pi/2}\,|b(E)|^2 + \Lambda\sqrt{2\pi}\,\big|\gamma(E) - \phi(s)\big|, \tag{8}$$

where we choose

$$\Lambda = s + 1. \tag{9}$$

As with the original functional the existence of a minimizer of (8) follows from [2, Proposition 1]. It turns out that the minimizers of (8) are the same as the minimizers of (2) under the volume constraint $\gamma(E) = \phi(s)$, as proved in the last section. The advantage of a volume penalization is that it helps us to bound the Lagrange multiplier in a simple way. The constants $\sqrt{\pi/2}$ and $\sqrt{2\pi}$ in front of the last two terms are chosen to simplify the formulas of the Euler equation and the second variation.

As we explained in the introduction, the idea is to restrict the parameter ϱ in (8) within a range, which contains the threshold value (7) and such that the only local minimizers of (8), which satisfy certain perimeter bounds, are the half-space and the symmetric strip. To this aim we assume from now on that ϱ is in the range

$$\frac{6}{5}\,\frac{\sqrt{2\pi}}{s^2}e^{\frac{s^2}{2}} \leq \varrho \leq \frac{7}{5}\,\frac{\sqrt{2\pi}}{s^2}e^{\frac{s^2}{2}}. \tag{10}$$

Note that the threshold value (7) is within this interval. If we are able to show that when ϱ satisfies (10) the only local minimizers of (8) are $H_{\omega,s}$ and $D_{\omega,s}$, we obtain the main result. Indeed, when ϱ takes the lower value in (10) it holds $\mathcal{F}(H_{\omega,s}) < \mathcal{F}(D_{\omega,s})$ and the minimizer is $H_{\omega,s}$. It is then not difficult to see that for every value ϱ less than this, the minimizer is still $H_{\omega,s}$. Similarly, when ϱ takes the larger value in (10) it holds $\mathcal{F}(D_{\omega,s}) < \mathcal{F}(H_{\omega,s})$ and the minimizer is $D_{\omega,s}$. Hence, for every value ϱ larger than this, $D_{\omega,s}$ is still the minimizer of (8), since it has barycenter zero.

Next we deduce a priori perimeter bounds for the minimizer. First, we may bound the perimeter from above by the minimality as

$$P_\gamma(E) \leq \mathcal{F}(E) \leq \mathcal{F}(D_{\omega,s}) = P_\gamma(D_{\omega,s}) \leq \Big(1 + \frac{1}{s^2}\Big)e^{-\frac{s^2}{2}}. \tag{11}$$

To bound the perimeter from below is slightly more difficult. Let E be a minimizer of (8) with volume $\gamma(E) = \phi(\bar{s})$. First, it is clear that $\bar{s} \geq 0$. Let us show that $\bar{s} \leq s + \frac{1}{s}$, which by the Gaussian isoperimetric inequality implies the following perimeter lower bound

$$P_\gamma(E) \geq \frac{1}{4}e^{-\frac{s^2}{2}}. \tag{12}$$

We argue by contradiction and assume $\bar{s} > s + \frac{1}{s}$. By the Gaussian isoperimetric inequality we deduce

$$\mathcal{F}(E) \geq P_\gamma(E) + \Lambda\sqrt{2\pi}\,(\phi(s) - \phi(\bar{s})) \geq e^{-\frac{\bar{s}^2}{2}} + (s+1)\int_s^{\bar{s}} e^{-\frac{t^2}{2}}\,dt.$$

Define the function $f : [s,\infty) \to \mathbb{R}$, $f(t) := e^{-\frac{t^2}{2}} + (s+1)\int_s^t e^{-\frac{l^2}{2}}\,dl$. By differentiating we get

$$f'(t) = (-t + s + 1)e^{-\frac{t^2}{2}}.$$

The function is clearly increasing up to $t = s+1$ and then decreases to the value $\lim_{t\to\infty} f(t) = (s+1)\int_s^\infty e^{-\frac{t^2}{2}}\,dt \geq \left(1 + \frac{1}{2s}\right)e^{-\frac{s^2}{2}}$ by (3). We also deduce that $f'(t) \geq \frac{1}{4}e^{-\frac{s^2}{2}}$ for $s \leq t \leq s+1/s$ and therefore $f(s+1/s) \geq \left(1 + \frac{1}{4s}\right)e^{-\frac{s^2}{2}}$. Hence, if $\bar{s} \geq s + \frac{1}{s}$ we have that

$$\mathcal{F}(E) \geq f(\bar{s}) \geq \min\{f(s+1/s), \lim_{t\to\infty} f(t)\} \geq \left(1 + \frac{1}{4s}\right)e^{-\frac{s^2}{2}}.$$

But this contradicts $\mathcal{F}(E) \leq \mathcal{F}(D_{\omega,s}) = P(D_{\omega,s})$ by (6). Thus we have (12).

For reader's convenience we summarize the results concerning the regularity of minimizers and the first and the second variation of (8) contained in [2, Section 4] in the following theorem.

Theorem 1. *Let E be a minimizer of (8). Then the reduced boundary $\partial^* E$ is a relatively open, smooth hypersurface and satisfies the Euler equation*

$$\mathscr{H} - \langle x, \nu \rangle + \varrho \langle b, x \rangle = \lambda \qquad \text{on } \partial^* E. \tag{13}$$

The Lagrange multiplier λ can be estimated by $|\lambda| \leq \Lambda$. The singular part of the boundary $\partial E \setminus \partial^* E$ is empty when $n < 8$, while for $n \geq 8$ its Hausdorff dimension can be estimated by $\dim_{\mathcal{H}}(\partial E \setminus \partial^* E) \leq n - 8$. Moreover, the quadratic form associated with the second variation is non-negative

$$\begin{aligned} \mathcal{F}[\varphi] := \int_{\partial^* E} \left(|\nabla_\tau \varphi|^2 - |B_E|^2 \varphi^2 + \varrho \langle b, \nu \rangle \varphi^2 - \varphi^2 \right) d\mathcal{H}^{n-1}_\gamma \\ + \frac{\varrho}{\sqrt{2\pi}} \left| \int_{\partial^* E} \varphi \, x \, d\mathcal{H}^{n-1}_\gamma \right|^2 \geq 0 \end{aligned} \tag{14}$$

for every $\varphi \in C_0^\infty(\partial^* E)$ which satisfies $\int_{\partial^* E} \varphi \, d\mathcal{H}^{n-1}_\gamma = 0$.

The Euler equation (13) yields important geometric equations for the position vector x and for the Gauss map ν. For arbitrary $\omega \in \mathbb{S}^{n-1}$ we write

$$x_\omega = \langle x, \omega \rangle \qquad \text{and} \qquad \nu_\omega = \langle \nu, \omega \rangle.$$

If $\{e^{(1)}, \dots, e^{(n)}\}$ is a canonical basis of $\mathbb{R}^n$ we simply write

$$x_i = \langle x, e_i \rangle \qquad \text{and} \qquad \nu_i = \langle \nu, e_i \rangle.$$

From (13) and from the fact $\Delta_\tau x_\omega = -\mathscr{H} \nu_\omega$ [27, Proposition 1] we have

$$\Delta_\tau x_\omega - \langle \nabla_\tau x_\omega, x \rangle = -x_\omega - \lambda \nu_\omega + \varrho \langle b, x \rangle \nu_\omega. \tag{15}$$

Moreover, from (13) and from the fact $\Delta_\tau \nu_\omega = -|B_E|^2 \nu_\omega + \langle \nabla_\tau \mathscr{H}, \omega \rangle$ [20, Lemma 10.7] we get

$$\Delta_\tau \nu_\omega - \langle \nabla_\tau \nu_\omega, x \rangle = -|B_E|^2 \nu_\omega + \varrho \langle b, \nu \rangle \nu_\omega - \varrho \langle b, \omega \rangle. \tag{16}$$

By the divergence theorem on $\partial^* E$ we have that for any function $\varphi \in C_0^\infty(\partial^* E)$ and for any function $\psi \in C^1(\partial^* E)$,

$$\int_{\partial^* E} \operatorname{div}_\tau \left(e^{-\frac{|x|^2}{2}} \psi \nabla_\tau \varphi \right) d\mathcal{H}^{n-1} = \int_{\partial^* E} \mathscr{H} \langle e^{-\frac{|x|^2}{2}} \psi \nabla_\tau \varphi, \nu^E \rangle \, d\mathcal{H}^{n-1} = 0.$$

The previous equality gives us an integration by parts formula

$$\int_{\partial^* E} \psi (\Delta_\tau \varphi - \langle \nabla_\tau \varphi, x \rangle) \, d\mathcal{H}^{n-1}_\gamma = - \int_{\partial^* E} \langle \nabla_\tau \psi, \nabla_\tau \varphi \rangle \, d\mathcal{H}^{n-1}_\gamma.$$

We will use along the paper the above formula with $\varphi = x_\omega$ or $\varphi = \nu_\omega$. Also if they do not belong to $C_0^\infty(\partial^* E)$, we are allowed to do so by an approximation argument (see [2, 32]).

Remark 1. *We associate the following second order operator L with the first four terms in the quadratic form (14),*

$$L[\varphi] := -\Delta_\tau \varphi + \langle \nabla_\tau \varphi, x \rangle - |B_E|^2 \varphi + \varrho \langle b, \nu \rangle \varphi - \varphi, \tag{17}$$

where $\varphi \in C_0^\infty(\partial^* E)$. By integration by parts the inequality (14) can be written as

$$\int_{\partial^* E} L[\varphi] \varphi \, d\mathcal{H}^{n-1}_\gamma + \frac{\varrho}{\sqrt{2\pi}} \left| \int_{\partial^* E} \varphi \, x \, d\mathcal{H}^{n-1}_\gamma \right|^2 \geq 0.$$

Note that when the vector ω is orthogonal to the barycenter, i.e., $\langle \omega, b \rangle = 0$, then by (16) the function ν_ω is an eigenfunction of L and satisfies

$$L[\nu_\omega] = -\nu_\omega.$$

For every $\omega \in \mathbb{S}^{n-1}$ it holds by the divergence theorem in $\mathbb{R}^n$ that

$$\begin{aligned}\int_{\partial^* E} \nu_\omega \, d\mathcal{H}^{n-1}_\gamma(x) &= \frac{1}{(2\pi)^{\frac{n-1}{2}}} \int_E \operatorname{div}(\omega e^{-\frac{|x|^2}{2}}) \, dx \\ &= -\sqrt{2\pi} \int_E \langle x, \omega \rangle \, d\gamma(x) = -\sqrt{2\pi} \langle b, \omega \rangle.\end{aligned}$$

In particular, when $\langle \omega, b \rangle = 0$ the function $\varphi = \nu_\omega$ has zero average. Therefore by Remark 1 it is natural to use ν_ω with $\langle \omega, b \rangle = 0$ as a test function in the second variation condition (14).

The equality $\int_{\partial^* E} \nu_\omega \, d\mathcal{H}^{n-1}_\gamma = -\sqrt{2\pi}\langle b, \omega \rangle$ for every $\omega \in \mathbb{S}^{n-1}$ also implies

$$\bar{\nu} P_\gamma(E) = -\sqrt{2\pi}\, b. \tag{18}$$

In particular, we have by (10)-(12)

$$\frac{1}{4s^2}|\bar{\nu}| \le \varrho |b| \le \frac{2}{s^2}|\bar{\nu}|. \tag{19}$$

We conclude this preliminary section by providing further "regularity" properties from (15) for the minimizers of (8). We call the estimates in the following lemma "Caccioppoli inequalities" since they follow from (15) by an argument which is similar to the classical proof of Caccioppoli inequality known in elliptic PDEs. This result is an improved version of [2, Proposition 1].

Lemma 1 (**Caccioppoli inequalities**). *Let $E \subset \mathbb{R}^n$ be a minimizer of (8). Then for any $\omega \in \mathbb{S}^{n-1}$ it holds*

$$\int_{\partial^* E} x_\omega^2 \, d\mathcal{H}^{n-1}_\gamma \le (s+2)^2 \int_{\partial^* E} \nu_\omega^2 \, d\mathcal{H}^{n-1}_\gamma + 8 P_\gamma(E) \tag{20}$$

and

$$\int_{\partial^* E} (x_\omega - \bar{x}_\omega)^2 \, d\mathcal{H}^{n-1}_\gamma \le (s+2)^2 \int_{\partial^* E} (\nu_\omega - \bar{\nu}_\omega)^2 \, d\mathcal{H}^{n-1}_\gamma + 8 P_\gamma(E). \tag{21}$$

Proof. Let us first prove (20). To simplify the notation we define

$$x_b := \begin{cases} \langle x, \frac{b}{|b|} \rangle & \text{if } b \neq 0, \\ 0 & \text{if } b = 0. \end{cases}$$

We multiply (15) by x_ω and integrate by parts over $\partial^* E$ to get

$$\int_{\partial^* E} x_\omega^2 \, d\mathcal{H}^{n-1}_\gamma = -\lambda \int_{\partial^* E} \nu_\omega x_\omega \, d\mathcal{H}^{n-1}_\gamma + \int_{\partial^* E} |\nabla_\tau x_\omega|^2 \, d\mathcal{H}^{n-1}_\gamma + \varrho |b| \int_{\partial^* E} x_b \nu_\omega x_\omega \, d\mathcal{H}^{n-1}_\gamma \tag{22}$$

We estimate the right-hand-side of (22) in the following way. We estimate the first term by Young's inequality

$$\begin{aligned}-\lambda \int_{\partial^* E} \nu_\omega x_\omega \, d\mathcal{H}^{n-1}_\gamma &\le \frac{1}{2} \int_{\partial^* E} x_\omega^2 \, d\mathcal{H}^{n-1}_\gamma + \frac{\lambda^2}{2} \int_{\partial^* E} \nu_\omega^2 \, d\mathcal{H}^{n-1}_\gamma \\ &\le \frac{1}{2} \int_{\partial^* E} x_\omega^2 \, d\mathcal{H}^{n-1}_\gamma + \frac{(s+1)^2}{2} \int_{\partial^* E} \nu_\omega^2 \, d\mathcal{H}^{n-1}_\gamma,\end{aligned}$$

where the last inequality follows from the bound on the Lagrange multiplier

$$|\lambda| \le s + 1 \tag{23}$$

given by Theorem 1 and by our choice of Λ in (9). Since $|\nabla_\tau x_\omega|^2 = 1 - \nu_\omega^2 \le 1$, we may bound the second term simply by

$$\int_{\partial^* E} |\nabla_\tau x_\omega|^2 \, d\mathcal{H}^{n-1}_\gamma \le P_\gamma(E).$$

Finally we bound the last term again by Young's inequality and by $\varrho|b| \leq \frac{2}{s^2}$ (proved in (19))

$$\varrho|b| \int_{\partial^* E} x_b \nu_\omega x_\omega \, d\mathcal{H}^{n-1}_\gamma \leq \frac{1}{s^2} \int_{\partial^* E} x_\omega^2 \, d\mathcal{H}^{n-1}_\gamma + \frac{1}{s^2} \int_{\partial^* E} x_b^2 \, d\mathcal{H}^{n-1}_\gamma.$$

By using these three estimates in (22) we obtain

$$\left(\frac{1}{2} - \frac{1}{s^2}\right) \int_{\partial^* E} x_\omega^2 \, d\mathcal{H}^{n-1}_\gamma \leq \frac{(s+1)^2}{2} \int_{\partial^* E} \nu_\omega^2 \, d\mathcal{H}^{n-1}_\gamma + P_\gamma(E) + \frac{1}{s^2} \int_{\partial^* E} x_b^2 \, d\mathcal{H}^{n-1}_\gamma. \tag{24}$$

If the barycenter is zero the claim follows immediately from (24). If $b \neq 0$, we first use (24) with $\omega = \frac{b}{|b|}$ and obtain

$$\begin{aligned}\left(\frac{1}{2} - \frac{2}{s^2}\right) \int_{\partial^* E} x_b^2 \, d\mathcal{H}^{n-1}_\gamma &\leq \frac{(s+1)^2}{2} \int_{\partial^* E} \nu_b^2 \, d\mathcal{H}^{n-1}_\gamma + P_\gamma(E) \\ &\leq \left(\frac{(s+1)^2}{2} + 1\right) P_\gamma(E).\end{aligned}$$

This implies

$$\int_{\partial^* E} x_b^2 \, d\mathcal{H}^{n-1}_\gamma \leq 2s^2 P_\gamma(E). \tag{25}$$

Therefore we have by (24)

$$\left(\frac{1}{2} - \frac{1}{s^2}\right) \int_{\partial^* E} x_\omega^2 \, d\mathcal{H}^{n-1}_\gamma \leq \frac{(s+1)^2}{2} \int_{\partial^* E} \nu_\omega^2 \, d\mathcal{H}^{n-1}_\gamma + 3P_\gamma(E).$$

This yields the claim.

The proof of the second inequality is similar. We multiply the equation (15) by $(x_\omega - \bar{x}_\omega)$ and integrate by parts over $\partial^* E$ to get

$$\begin{aligned}\int_{\partial^* E} (x_\omega - \bar{x}_\omega)^2 \, d\mathcal{H}^{n-1}_\gamma = &- \lambda \int_{\partial^* E} (x_\omega - \bar{x}_\omega)(\nu_\omega - \bar{\nu}_\omega) \, d\mathcal{H}^{n-1}_\gamma + \int_{\partial^* E} |\nabla_\tau x_\omega|^2 \, d\mathcal{H}^{n-1}_\gamma \\ &+ \varrho|b| \int_{\partial^* E} x_b \nu_\omega (x_\omega - \bar{x}_\omega) \, d\mathcal{H}^{n-1}_\gamma.\end{aligned}$$

By estimating the three terms on the right-hand-side precisely as before, we deduce

$$\begin{aligned}\left(\frac{1}{2} - \frac{1}{s^2}\right) \int_{\partial^* E} (x_\omega - \bar{x}_\omega)^2 \, d\mathcal{H}^{n-1}_\gamma &\leq \frac{(s+1)^2}{2} \int_{\partial^* E} (\nu_\omega - \bar{\nu}_\omega)^2 \, d\mathcal{H}^{n-1}_\gamma + P_\gamma(E) + \frac{1}{s^2} \int_{\partial^* E} x_b^2 \, d\mathcal{H}^{n-1}_\gamma \\ &\leq \frac{(s+1)^2}{2} \int_{\partial^* E} (\nu_\omega - \bar{\nu}_\omega)^2 \, d\mathcal{H}^{n-1}_\gamma + 3P_\gamma(E),\end{aligned}$$

where the last inequality follows from (25). This implies (21). □

3. Reduction to the two dimensional case

In this section we prove that it is enough to obtain the result in the two dimensional case. More precisely, we prove the following result.

Theorem 2. *Let E be a minimizer of (8). Then, up to a rotation, $E = F \times \mathbb{R}^{n-2}$ for some set $F \subset \mathbb{R}^2$.*

Proof. Let $\{e^{(1)}, \dots, e^{(n)}\}$ be an orthonormal basis of $\mathbb{R}^n$. We begin with a simple observation: if $i \neq j$ then by the divergence theorem

$$\int_{\partial^* E} x_i \nu_j \, d\mathcal{H}^{n-1}_\gamma = -\sqrt{2\pi} \int_E x_i x_j \, d\gamma.$$

In particular, the matrix $A_{ij} = \int_{\partial E} x_i \nu_j \, d\mathcal{H}^{n-1}_\gamma$ is symmetric. We may therefore assume that A_{ij} is diagonal, by changing the basis of $\mathbb{R}^n$ if necessary. In particular, it holds

$$\int_{\partial^* E} x_i \nu_j \, d\mathcal{H}^{n-1}_\gamma = 0 \qquad \text{for } i \neq j. \tag{26}$$

By reordering the elements of the basis we may also assume that

$$\int_{\partial^* E} x_j^2 \, d\mathcal{H}^{n-1}_\gamma \geq \int_{\partial^* E} x_{j+1}^2 \, d\mathcal{H}^{n-1}_\gamma \tag{27}$$

for $j \in \{1, \dots, n-1\}$.

Since we assume $n \geq 3$, we may choose a direction $\omega \in \mathbb{S}^{n-1}$ which is orthogonal both to the barycenter b and to $e^{(1)}$. To be more precise, we choose ω such that $\langle \omega, b \rangle = 0$ and $\omega \in \operatorname{span}\{e^{(2)}, e^{(3)}\}$. Since $\langle \omega, b \rangle = 0$, (18) yields $\bar{\nu}_\omega = 0$. In other words, the function ν_ω has zero average. We use $\varphi = \nu_\omega$ as a test function in the second variation condition (14). According to Remark 1 we may write the inequality (14) as

$$\int_{\partial^* E} L[\nu_\omega] \nu_\omega \, d\mathcal{H}^{n-1}_\gamma + \frac{\varrho}{\sqrt{2\pi}} \Big| \int_{\partial^* E} \nu_\omega \, x \, d\mathcal{H}^{n-1}_\gamma \Big|^2 \geq 0,$$

where the operator L is defined in (17). Since ω is orthogonal to b we deduce by Remark 1 that ν_ω is an eigenfunction of L and satisfies $L[\nu_\omega] = -\nu_\omega$. Therefore we get

$$-\int_{\partial^* E} \nu_\omega^2 \, d\mathcal{H}^{n-1}_\gamma + \frac{\varrho}{\sqrt{2\pi}} \Big| \int_{\partial^* E} \nu_\omega \, x \, d\mathcal{H}^{n-1}_\gamma \Big|^2 \geq 0. \tag{28}$$

The crucial step in the proof is to estimate the second term in (28), by showing that it is small enough. This is possible, because ω is orthogonal to $e^{(1)}$. Indeed, by using (26) and the fact that $\omega \in \operatorname{span}\{e^{(2)}, e^{(3)}\}$, and then Cauchy-Schwarz inequality, we get

$$\begin{aligned} \Big| \int_{\partial^* E} \nu_\omega \, x \, d\mathcal{H}^{n-1}_\gamma \Big|^2 &= \Big(\int_{\partial^* E} x_2 \nu_\omega \, d\mathcal{H}^{n-1}_\gamma \Big)^2 + \Big(\int_{\partial^* E} x_3 \nu_\omega \, d\mathcal{H}^{n-1}_\gamma \Big)^2 \\ &\leq \Big(\int_{\partial^* E} x_2^2 + x_3^2 \, d\mathcal{H}^{n-1}_\gamma \Big) \Big(\int_{\partial^* E} \nu_\omega^2 \, d\mathcal{H}^{n-1}_\gamma \Big). \end{aligned}$$

We estimate the first term on the right-hand-side first by (27), then by the Caccioppoli estimate (20) and finally by (11)

$$\begin{aligned} \int_{\partial^* E} (x_2^2 + x_3^2) \, \mathcal{H}^{n-1}_\gamma &\leq \frac{2}{3} \int_{\partial^* E} (x_1^2 + x_2^2 + x_3^2) \, \mathcal{H}^{n-1}_\gamma \\ &\leq \frac{2}{3} \Big[(s+2)^2 \int_{\partial^* E} (\nu_1^2 + \nu_2^2 + \nu_3^2) \, \mathcal{H}^{n-1}_\gamma + 24 P_\gamma(E) \Big] \\ &\leq \frac{2}{3} \big[(s+2)^2 + 24 \big] P_\gamma(E) \leq \frac{9}{13} s^2 e^{-\frac{s^2}{2}}. \end{aligned} \tag{29}$$

Since we assume $\varrho \leq \frac{7\sqrt{2\pi}}{5s^2} e^{\frac{s^2}{2}}$ (see (10)), the previous two inequalities yield

$$\frac{\varrho}{\sqrt{2\pi}} \Big| \int_{\partial^* E} \nu_\omega \, x \, d\mathcal{H}^{n-1}_\gamma \Big|^2 \leq \frac{63}{65} \int_{\partial^* E} \nu_\omega^2 \, d\mathcal{H}^{n-1}_\gamma. \tag{30}$$

Then, by collecting (28) and (30) we obtain

$$-\int_{\partial^* E} \nu_\omega^2 \, d\mathcal{H}^{n-1}_\gamma \geq 0. \tag{31}$$

This implies $\nu_\omega = 0$. We have thus reduced the problem from n to $n-1$. By repeating the previous argument we reduce the problem to the planar case. $\square$

Remark 2. *We have to be careful in our choice of direction ω, and in general we may not simply choose any direction orthogonal to the barycenter b. Indeed, if $\omega, \upsilon \in \mathbb{S}^{n-1}$ are vectors such that $\langle b, \omega \rangle = 0$ and*

$$\Big| \int_{\partial^* E} \nu_\omega \, x \, d\mathcal{H}^{n-1}_\gamma \Big| = \langle \int_{\partial^* E} \nu_\omega \, x \, d\mathcal{H}^{n-1}_\gamma, \upsilon \rangle = \int_{\partial^* E} \nu_\omega \langle x, \upsilon \rangle \, d\mathcal{H}^{n-1}_\gamma. \tag{32}$$

Then, by using Cauchy-Schwarz inequality, we may estimate the second term in (28) by

$$\frac{\varrho}{\sqrt{2\pi}} \Big| \int_{\partial^* E} \nu_\omega \, x \, d\mathcal{H}^{n-1}_\gamma \Big|^2 \le \frac{\varrho}{\sqrt{2\pi}} \left(\int_{\partial^* E} x_\upsilon^2 \, d\mathcal{H}^{n-1}_\gamma \right) \left(\int_{\partial^* E} \nu_\omega^2 \, d\mathcal{H}^{n-1}_\gamma \right).$$

We may estimate the term $\frac{\varrho}{\sqrt{2\pi}} \int_{\partial^* E} x_\upsilon^2 \, d\mathcal{H}^{n-1}_\gamma$ by the Caccioppoli estimate (20), and by (11) and (10)

$$\frac{\varrho}{\sqrt{2\pi}} \Big| \int_{\partial^* E} \nu_\omega \, x \, d\mathcal{H}^{n-1}_\gamma \Big|^2 \le \frac{8}{5} \int_{\partial^* E} \nu_\omega^2 \, d\mathcal{H}^{n-1}_\gamma$$

instead of (30). Unfortunately this estimate is not good enough. Note that we cannot shrink ϱ, since we have the constrain given by (7).

Remark 3. *We may further reduce the problem to the one dimensional case if $b = 0$, since we may use $\omega = e^{(2)}$ in the previous argument (ν_ω has zero average and $\int_{\partial^* E} x_2^2$ is small enough). However, this is a special case and a priori nothing guaranties that $b = 0$. Because of that we have to handle the reduction to the one dimensional case in a different way.*

4. Reduction to the one dimensional case

In this section we will prove a further reduction of the problem, by showing that it is enough to obtain the result in the one dimensional case. This is technically more involved than Theorem 2 and requires more a priori information on the minimizers.

Theorem 3. *Let E be a minimizer of (8). Then, up to a rotation, $E = F \times \mathbb{R}^{n-1}$ for some set $F \subset \mathbb{R}$.*

Thanks to Theorem 2 we may assume from now on that $n = 2$. In particular, by Theorem 1 the boundary is regular and $\partial E = \partial^* E$. Moreover the Euler equation and (16) simply read as

$$k = \lambda + \langle x, \nu \rangle - \varrho \langle b, x \rangle, \tag{33}$$

$$\Delta_\tau \nu_\omega - \langle \nabla_\tau \nu_\omega, x \rangle = -k^2 \nu_\omega + \varrho \langle b, \nu \rangle \nu_\omega - \varrho \langle b, \omega \rangle, \tag{34}$$

where k is the curvature of ∂E.

The idea is to proceed by using the second variation argument once more, but this time in a direction that it is not necessarily orthogonal to the barycenter. This argument does not reduce the problem to $\mathbb{R}$, but gives us the following information on the minimizers.

Lemma 2. *Let $E \subset \mathbb{R}^2$ be a minimizer of (8). Then*

$$\fint_{\partial E} k^2 \, d\mathcal{H}^1_\gamma \le \frac{2}{s^2}. \tag{35}$$

Moreover, there exists a direction $\upsilon \in \mathbb{S}^1$ such that

$$\fint_{\partial E} (\nu_\upsilon - \bar{\nu}_\upsilon)^2 \, d\mathcal{H}^1_\gamma \le \frac{10}{s^2} \, \bar{\nu}_\upsilon^2. \tag{36}$$

Observe that the above estimate implies that ν_υ is close to a constant. In particular, this excludes the minimizers to be close to the disk.

Proof. We begin by showing that for any $\omega \in \mathbb{S}^1$ it holds

$$\begin{aligned}\bar\nu_\omega^2 \int_{\partial E} k^2\, d\mathcal{H}^1_\gamma + \int_{\partial E} |\nu_\omega - \bar\nu_\omega|^2\, d\mathcal{H}^1_\gamma \\ \leq \frac{2}{s^2}\,\bar\nu_\omega^2 P_\gamma(E) + \frac{\varrho}{\sqrt{2\pi}}\Big|\int_{\partial E} (\nu_\omega - \bar\nu_\omega)\, x\, d\mathcal{H}^1_\gamma\Big|^2.\end{aligned} \tag{37}$$

To this aim we choose $\varphi = \nu_\omega - \bar\nu_\omega$ as a test function in the second variation condition (14). We remark that because ω is not in general orthogonal to the barycenter b, neither ν_ω or $\nu_\omega - \bar\nu_\omega$ is an eigenfunction of the operator L associated with the second variation defined in Remark 1.

We multiply the equation (34) by ν_ω and integrate by parts to obtain

$$\int_{\partial E} \big(|\nabla_\tau \nu_\omega|^2 - k^2\nu_\omega^2 + \varrho\langle b,\nu\rangle \nu_\omega^2\big)\, d\mathcal{H}^1_\gamma = \varrho\langle b,\omega\rangle \bar\nu_\omega P_\gamma(E), \tag{38}$$

and simply integrate (16) over ∂E to get

$$\int_{\partial E} \big(k^2\nu_\omega - \varrho\langle b,\nu\rangle\nu_\omega\big)\, d\mathcal{H}^1_\gamma = -\varrho\langle b,\omega\rangle P_\gamma(E). \tag{39}$$

Hence, by also using $\bar\nu P_\gamma(E) = -\sqrt{2\pi}\, b$ (see (18)), we may write

$$\begin{aligned}&\int_{\partial E} \big(|\nabla_\tau \nu_\omega|^2 - k^2(\nu_\omega - \bar\nu_\omega)^2 + \varrho\langle b,\nu\rangle(\nu_\omega - \bar\nu_\omega)^2\big)\, d\mathcal{H}^1_\gamma \\ &= -\bar\nu_\omega^2 \int_{\partial E} k^2\, d\mathcal{H}^1_\gamma + \varrho\langle b,\bar\nu\rangle \bar\nu_\omega^2 P_\gamma(E) - \varrho\langle b,\omega\rangle\, \bar\nu_\omega P_\gamma(E) \\ &= -\bar\nu_\omega^2 \int_{\partial E} k^2\, d\mathcal{H}^1_\gamma + \frac{\varrho}{\sqrt{2\pi}}(1 - |\bar\nu|^2)\bar\nu_\omega^2 P_\gamma^2(E) \\ &\leq -\bar\nu_\omega^2 \int_{\partial E} k^2\, d\mathcal{H}^1_\gamma + \frac{2}{s^2}\bar\nu_\omega^2 P_\gamma(E),\end{aligned}$$

where in the last inequality we have used the estimates (10) and (11). The above inequality and the second variation condition (14) with $\varphi = \nu_\omega - \bar\nu_\omega$ imply (37).

Let us consider an orthonormal basis $\{e^{(1)}, e^{(2)}\}$ of $\mathbb{R}^2$ and assume $\int_{\partial E} x_1^2\, d\mathcal{H}^1_\gamma \geq \int_{\partial E} x_2^2\, d\mathcal{H}^1_\gamma$. As in (29), we use the Caccioppoli estimate (20) and (11) to get

$$\begin{aligned}\int_{\partial E} x_2^2\, d\mathcal{H}^1_\gamma &\leq \frac{1}{2}\int_{\partial E} (x_1^2 + x_2^2)\, \mathcal{H}^1_\gamma \\ &\leq \frac{1}{2}\big[(s+2)^2 + 16\big] P_\gamma(E) \leq \frac{1}{2}\big[(s+4)^2\big] e^{-\frac{s^2}{2}}.\end{aligned} \tag{40}$$

We choose a direction $v \in \mathbb{S}^1$ which is orthogonal to the vector $\int_{\partial E} x_1(\nu - \bar\nu)\, d\mathcal{H}^1_\gamma$. Since $\fint_{\partial E} x_1(\nu_v - \bar\nu_v)\, d\mathcal{H}^1_\gamma = \langle \fint_{\partial E} x_1(\nu - \bar\nu)\, d\mathcal{H}^1_\gamma, v\rangle = 0$, we have

$$\Big|\fint_{\partial E} x(\nu_v - \bar\nu_v)\, d\mathcal{H}^1_\gamma\Big| = \Big|\fint_{\partial E} x_2(\nu_v - \bar\nu_v)\, d\mathcal{H}^1_\gamma\Big|.$$

Then, by the above equality, by Cauchy-Schwarz inequality and by (40) we have

$$\begin{aligned}\Big|\int_{\partial E} x(\nu_v - \bar\nu_v)\, d\mathcal{H}^1_\gamma\Big|^2 &= \Big(\int_{\partial E} x_2(\nu_v - \bar\nu_v)\, d\mathcal{H}^1_\gamma\Big)^2 \\ &\leq \Big(\int_{\partial E} x_2^2\, d\mathcal{H}^1_\gamma\Big)\Big(\int_{\partial E} (\nu_v - \bar\nu_v)^2\, d\mathcal{H}^1_\gamma\Big) \\ &\leq \frac{(s+4)^2}{2} e^{-\frac{s^2}{2}} \Big(\int_{\partial E} (\nu_v - \bar\nu_v)^2\, d\mathcal{H}^1_\gamma\Big).\end{aligned}$$

With the bound $\varrho \leq \frac{7\sqrt{2\pi}}{5s^2} e^{\frac{s^2}{2}}$ (see (10)), the previous inequality yields

$$\frac{\varrho}{\sqrt{2\pi}} \left| \int_{\partial E} (\nu_v - \bar{\nu}_v)\, x \, d\mathcal{H}^1_\gamma \right|^2 \leq \frac{4}{5} \int_{\partial E} (\nu_v - \bar{\nu}_v)^2 \, d\mathcal{H}^1_\gamma.$$

Hence, the inequality (37) implies

$$\bar{\nu}_v^2 \int_{\partial E} k^2 \, d\mathcal{H}^1_\gamma + \frac{1}{5} \int_{\partial E} (\nu_v - \bar{\nu}_v)^2 \, d\mathcal{H}^1_\gamma \leq \frac{2}{s^2} \bar{\nu}_v^2 P_\gamma(E).$$

From this inequality we have immediately (36), and also (35), if $\bar{\nu}_v$ is not zero. If instead $\bar{\nu}_v = 0$, then also $\nu_v = 0$ by (36). Thus ∂E is flat, $k = 0$ and (35) holds again. □

We will also need the following auxiliary result.

Lemma 3. *Let $E \subset \mathbb{R}^2$ be a minimizer of (8). Then, for every $x \in \partial E$ it holds*

$$|x| \geq s - 1. \tag{41}$$

Proof. We argue by contradiction and assume that there exists $\tilde{x} \in \partial E$ such that $|\tilde{x}| < s - 1$. We claim that then it holds

$$\mathcal{H}^1(\partial E \cap B_{1/2}(\tilde{x})) \geq \frac{1}{s}. \tag{42}$$

We remark that $\mathcal{H}^1$ is the standard Hausdorff measure, i.e., $\mathcal{H}^1(\partial E \cap B_{1/2}(\tilde{x}))$ denotes the length of the curve. We divide the proof of (42) in two cases.

Assume first that there is a component of ∂E, say $\tilde{\Gamma}$, which is contained in the disk $B_{1/2}(\tilde{x})$. By regularity, $\tilde{\Gamma}$ is a smooth Jordan curve which encloses a bounded set $\tilde{E}$, i.e., $\tilde{\Gamma} = \partial \tilde{E}$. Note that then it holds $\tilde{E} \subset B_R$ for $R = s - 1/2$. We integrate the Euler equation (33) over $\partial \tilde{E}$ with respect to the standard Hausdorff measure and obtain by the Gauss-Bonnet formula and by the divergence theorem that

$$\begin{aligned} 2\pi = \int_{\tilde{\Gamma}} k \, d\mathcal{H}^1 &= \int_{\tilde{\Gamma}} (\langle x, \nu \rangle + \lambda - \varrho \langle b, x \rangle) \, d\mathcal{H}^1 \\ &\leq 2|\tilde{E}| + \left(|\lambda| + \frac{2}{s} \right) \mathcal{H}^1(\tilde{\Gamma}), \end{aligned} \tag{43}$$

where in the last inequality we have used $\varrho |b| \leq \frac{2}{s^2}$ (proved in (19)) and the fact that for all $x \in \tilde{E}$ it holds $|x| \leq s - 1/2$. The isoperimetric inequality in $\mathbb{R}^2$ implies

$$|\tilde{E}| \leq \frac{1}{4\pi^2} \mathcal{H}^1(\tilde{\Gamma})^2.$$

Therefore since $|\lambda| \leq s + 1$ we obtain from (43) that

$$2\pi \leq \frac{1}{2\pi^2} \mathcal{H}^1(\tilde{\Gamma})^2 + (s+2)\mathcal{H}^1(\tilde{\Gamma}).$$

This implies $\mathcal{H}^1(\tilde{\Gamma}) \geq \frac{1}{s}$ and the claim (42) follows.

Let us then assume that no component of ∂E is contained in $B_{1/2}(\tilde{x})$. In this case the boundary curve passes $\tilde{x}$ and exists the disk $B(\tilde{x}, \frac{1}{2})$. In particular, it holds $\mathcal{H}^1(\partial E \cap B_{1/2}(\tilde{x})) \geq 1/2$ which implies (42).

Since for all $x \in \partial E \cap B_{1/2}(\tilde{x})$ it holds $|x| \leq s - 1/2$, the estimate (42) implies

$$\begin{aligned} P_\gamma(E) &\geq \frac{1}{\sqrt{2\pi}} \int_{\partial E \cap B_{1/2}(\tilde{x})} e^{-\frac{|x|^2}{2}} \, d\mathcal{H}^1 \\ &\geq \frac{1}{\sqrt{2\pi}} e^{-\frac{(s-1/2)^2}{2}} \mathcal{H}^1(\partial E \cap B_{1/2}(\tilde{x})) \geq 2e^{-\frac{s^2}{2}}. \end{aligned}$$

This contradicts (11). □

For the remaining part of this section we choose a basis $\{e^{(1)}, e^{(2)}\}$ for $\mathbb{R}^2$ such that $e^{(1)} = v$, where v is the direction in Lemma 2 and $e^{(2)}$ is an orthogonal direction to that. The disadvantage of Lemma 2 is that the argument does not seem to give us any information on $\nu_2 = \langle \nu, e^{(2)} \rangle$. However, by studying closely the proof of Lemma 2 we may reduce to the case when it holds

$$\fint_{\partial E} (\nu_2 - \bar{\nu}_2)^2 \, d\mathcal{H}^1_\gamma \geq \frac{4}{7}. \tag{44}$$

Indeed, we conclude below that if (44) does not hold then the argument of the proof of Lemma 2 yields that the minimizer is one-dimensional. In fact, by the one dimensional analysis in Section 5 we deduce that if (44) does not hold then the minimizer is the half-space.

To show (44), we argue by contradiction, in which case it holds $\fint_{\partial E} (\nu_2 - \bar{\nu}_2)^2 \, d\mathcal{H}^1_\gamma < \frac{4}{7}$. Then the Caccioppoli estimate (21) yields

$$\fint_{\partial E} (x_2 - \bar{x}_2)^2 \, d\mathcal{H}^1_\gamma \leq \frac{4}{7}(s+2)^2 + 8$$

while again by (21) and by (36) from Lemma 2 we have

$$\fint_{\partial E} (x_1 - \bar{x}_1)^2 \, d\mathcal{H}^1_\gamma \leq (s+2)^2 \fint_{\partial E} (\nu_1 - \bar{\nu}_1)^2 \, d\mathcal{H}^1_\gamma + 8 \leq C.$$

Let $e \in \mathbb{S}^1$ be orthogonal to the barycenter b. We now apply the argument in the proof of Lemma 2 for the test function $\varphi = \nu_e$. By the two above inequalities and by (11) we have

$$\begin{aligned}\Big|\int_{\partial E} x \, \nu_e \, d\mathcal{H}^1_\gamma\Big|^2 =& \Big(\int_{\partial E} (x_1 - \bar{x}_1)\nu_e \, d\mathcal{H}^1_\gamma\Big)^2 + \Big(\int_{\partial E} (x_2 - \bar{x}_2)\nu_e \, d\mathcal{H}^1_\gamma\Big)^2 \\ \leq& \left(\int_{\partial E} (x_1 - \bar{x}_1)^2 + (x_2 - \bar{x}_2)^2 \, d\mathcal{H}^1_\gamma\right) \int_{\partial E} \nu_e^2 \, d\mathcal{H}^1_\gamma \\ \leq& \frac{9}{13} s^2 e^{-\frac{s^2}{2}} \int_{\partial E} \nu_e^2 \, d\mathcal{H}^1_\gamma.\end{aligned}$$

In other words, since $\varrho \leq \frac{7\sqrt{2\pi}}{5s^2} e^{\frac{s^2}{2}}$ we conclude that the crucial estimate (30) in the proof of Theorem 2 holds for a direction orthogonal to the barycenter and thus by Remark 3 we conclude that $\nu_e = 0$. Hence, we may assume from now on that (44) holds.

Let us define

$$\Sigma_+ = \{x \in \partial E : x_2 > 0\} \qquad \text{and} \qquad \Sigma_- = \{x \in \partial E : x_2 < 0\}.$$

In the next lemma we use (36) from Lemma 2 and (44) to conclude first that Σ_+ and Σ_- are flat in shape. The second estimate in the next lemma states roughly speaking that the Gaussian measure of $\{x \in \partial E : |x_2| \leq \frac{s}{3}\}$ is small. The latter estimate implies that, from measure point of view, Σ_+ and Σ_- are almost disconnected. This enables us to variate Σ_+ and Σ_- separately, which will be crucial in the proof of Theorem 3. Recall that, given a function $f : \partial E \to \mathbb{R}$, we denote $(f)_{\Sigma_+} = \fint_{\Sigma_+} f \, d\mathcal{H}^1_\gamma$ and $(f)_{\Sigma_-} := \fint_{\Sigma_-} f \, d\mathcal{H}^1_\gamma$.

Lemma 4. *Let $E \subset \mathbb{R}^2$ be a minimizer of (8) and assume (44) holds. Then we have the following:*

$$\int_{\Sigma_\pm} (x_i - (x_i)_{\Sigma_\pm})^2 \leq C P_\gamma(E), \qquad \textit{for } i = 1, 2 \tag{45}$$

$$\int_{\partial E \cap \{|x_2| \leq \frac{s}{3}\}} |x|^2 \, d\mathcal{H}^1_\gamma \leq C \int_{\partial E} \nu_1^2 \, d\mathcal{H}^1_\gamma \tag{46}$$

Proof. **Inequality** (45)**.** We first observe that the claim (45) is almost trivial for $i = 1$. Indeed, by the Caccioppoli estimate (21) and by (36) from Lemma 2 (recall that we have chosen $e^{(1)} = v$) we have

$$\begin{aligned}\int_{\Sigma_+} (x_1 - (x_1)_{\Sigma_+})^2 \, d\mathcal{H}^1_\gamma &\le \int_{\Sigma_+} (x_1 - \bar{x}_1)^2 \, d\mathcal{H}^1_\gamma \\ &\le \int_{\partial E} (x_1 - \bar{x}_1)^2 \, d\mathcal{H}^1_\gamma \\ &\le (s+2)^2 \int_{\partial E} (\nu_1 - \bar{\nu}_1)^2 \, d\mathcal{H}^1_\gamma + 8P_\gamma(E) \\ &\le CP_\gamma(E).\end{aligned}$$

Thus we need to prove (45) for $i = 2$.

We first show that

$$\int_{\Sigma_+} \left(|\nu_2| - (|\nu_2|)_{\Sigma_+}\right)^2 \, d\mathcal{H}^1_\gamma \le \frac{C}{s^2} P_\gamma(E). \tag{47}$$

Note that (44) implies $⨍_{\partial E} \nu_2^2 \, d\mathcal{H}^1_\gamma \ge \frac{4}{7}$. By Jensen's inequality we then have

$$\bar{\nu}_1^2 \le ⨍_{\partial E} \nu_1^2 \, d\mathcal{H}^1_\gamma \le \frac{3}{7}. \tag{48}$$

Therefore we deduce by (48) and by (36)

$$\begin{aligned}\int_{\Sigma_+} \left(|\nu_2| - \sqrt{1 - \bar{\nu}_1^2}\right)^2 \, d\mathcal{H}^1_\gamma &= \int_{\Sigma_+} \frac{\left(\nu_2^2 - (1 - \bar{\nu}_1^2)\right)^2}{\left(|\nu_2| + \sqrt{1 - \bar{\nu}_1^2}\right)^2} \, d\mathcal{H}^1_\gamma \\ &\le 2 \int_{\partial E} (\nu_1^2 - \bar{\nu}_1^2)^2 \, d\mathcal{H}^1_\gamma \\ &\le 8 \int_{\partial E} (\nu_1 - \bar{\nu}_1)^2 \, d\mathcal{H}^1_\gamma \\ &\le \frac{C}{s^2} P_\gamma(E).\end{aligned}$$

Since

$$\int_{\Sigma_+} \left(|\nu_2| - (|\nu_2|)_{\Sigma_+}\right)^2 \, d\mathcal{H}^1_\gamma \le \int_{\Sigma_+} \left(|\nu_2| - \sqrt{1 - \bar{\nu}_1^2}\right)^2 \, d\mathcal{H}^1_\gamma$$

we have (47).

To prove the inequality (45) for $i = 2$ we multiply the equation (15), with $\omega = e_2$, by $(x_2 + \lambda\nu_2)$ and integrate by parts

$$\int_{\partial E} (x_2 + \lambda\nu_2)^2 \, d\mathcal{H}^1_\gamma \le \int_{\partial E} \langle \nabla_\tau (x_2 + \lambda\nu_2), \nabla_\tau x_2 \rangle - \varrho \langle b, x \rangle \nu_2 (x_2 + \lambda\nu_2) \, d\mathcal{H}^1_\gamma.$$

We estimate the first term on the right-hand-side by Young's inequality and by $|\lambda| \le s + 1$

$$\langle \nabla_\tau (x_2 + \lambda\nu_2), \nabla_\tau x_2 \rangle \le 2|\nabla_\tau x_2|^2 + \lambda^2 |\nabla_\tau \nu_2|^2 \le 2 + (s+1)^2 k^2$$

and the second as

$$\varrho \langle b, x \rangle \nu_2 (x_2 + \lambda\nu_2) \le 2\varrho |b| \left(|x|^2 + (s+1)^2 \nu_2^2\right).$$

Hence, we have by $\varrho|b| \le \frac{2}{s^2}$ (proved in (19)), (21) and (35) that

$$\begin{aligned}\int_{\partial E} (x_2 + \lambda\nu_2)^2 \, d\mathcal{H}^1_\gamma &\le \int_{\partial E} \left(2 + (s+1)^2 k^2 + \frac{4}{s^2}(|x|^2 + (s+1)^2 \nu_2^2)\right) d\mathcal{H}^1_\gamma \\ &\le CP_\gamma(E).\end{aligned}$$

Therefore it holds (recall that $x_2 > 0$ on Σ_+)

$$\begin{aligned} CP_\gamma(E) &\geq \int_{\partial E} (x_2 + \lambda \nu_2)^2 \, d\mathcal{H}^1_\gamma \geq \int_{\Sigma_+} (x_2 + \lambda \nu_2)^2 \, d\mathcal{H}^1_\gamma \\ &\geq \int_{\Sigma_+} (|x_2| - \lambda |\nu_2|)^2 \, d\mathcal{H}^1_\gamma = \int_{\Sigma_+} (x_2 - \lambda |\nu_2|)^2 \, d\mathcal{H}^1_\gamma \\ &\geq \frac{1}{2} \int_{\Sigma_+} (x_2 - \lambda (|\nu_2|)_{\Sigma_+})^2 \, d\mathcal{H}^1_\gamma - 2\lambda^2 \int_{\Sigma_+} (|\nu_2| - (|\nu_2|)_{\Sigma_+})^2 \, d\mathcal{H}^1_\gamma. \end{aligned}$$

Hence, by (47) and $|\lambda| \leq s+1$ we deduce

$$\int_{\Sigma_+} (x_2 - \lambda (|\nu_2|)_{\Sigma_+})^2 \, d\mathcal{H}^1_\gamma \leq CP_\gamma(E).$$

The claim then follows from

$$\int_{\Sigma_+} (x_2 - (x_2)_{\Sigma_+})^2 \, d\mathcal{H}^1_\gamma \leq \int_{\Sigma_+} (x_2 - \lambda (|\nu_2|)_{\Sigma_+})^2 \, d\mathcal{H}^1_\gamma.$$

Inequality (46). We choose a smooth cut-off function $\zeta : \mathbb{R} \to [0,1]$ such that

$$\zeta(t) = \begin{cases} 1 & \text{for } |t| \leq \frac{s}{3} \\ 0 & \text{for } |t| \geq \frac{s}{2} \end{cases}$$

and

$$|\zeta'(t)| \leq \frac{8}{s} \qquad \text{for } t \in \mathbb{R}.$$

We multiply the equation (15), with $\omega = e_1$, by $x_1 \zeta^2(x_2)$ and integrate by parts

$$\int_{\partial E} x_1^2 \zeta^2(x_2) \, d\mathcal{H}^1_\gamma = \int_{\partial E} \big(-\lambda x_1 \nu_1 \zeta^2(x_2) + \langle \nabla_\tau x_1, \nabla_\tau (x_1 \zeta^2(x_2)) \rangle + \varrho \langle b, x \rangle \nu_1 x_1 \zeta^2(x_2) \big) \, d\mathcal{H}^1_\gamma. \tag{49}$$

We estimate the first term on right-hand-side by Young's inequality and by $|\lambda| \leq s+1$

$$-\lambda x_1 \nu_1 \zeta^2(x_2) \leq \frac{1}{2} x_1^2 \zeta^2 + \frac{(s+1)^2}{2} \nu_1^2 \zeta^2,$$

where we have written $\zeta = \zeta(x_2)$ for short. We estimate the second term by using $|\nabla_\tau \zeta(x_2)| = |\zeta'(x_2)||\nabla x_2| \leq \frac{8}{s}|\nu_1|$ as follows

$$\begin{aligned} \langle \nabla_\tau x_1, \nabla_\tau (x_1 \zeta^2(x_2)) \rangle &\leq |\nabla_\tau x_1|^2 \zeta^2 + \frac{16}{s} \zeta |x_1| |\nabla_\tau x_1| |\nu_1| \\ &\leq \zeta^2 + \frac{1}{20} x_1^2 \zeta^2 + \frac{C}{s^2} \nu_1^2. \end{aligned}$$

We estimate the third term simply by using $\varrho |b| \leq \frac{2}{s^2}$

$$\varrho \langle b, x \rangle \nu_1 x_1 \zeta^2(x_2) \leq \frac{2}{s^2} |x|^2 \zeta^2.$$

Hence, we deduce from (49) by the three above inequalities that

$$\int_{\partial E} \left(x_1^2 - \frac{1}{10} x_1^2 - \frac{4}{s^2}|x|^2 - 2 \right) \zeta^2 \, d\mathcal{H}^1_\gamma \leq \int_{\partial E} \left((s+1)^2 \nu_1^2 \zeta^2 + \frac{C}{s^2} \nu_1^2 \right) \, d\mathcal{H}^1_\gamma$$

We recall that $\zeta = 0$ when $|x_2| \geq \frac{s}{2}$ and that by (41) we have that $|x|^2 \geq (s-1)^2$ on ∂E. In particular, for every $x \in \{x \in \partial E : |x_2| \leq \frac{s}{2}\}$ it holds

$$x_1^2 = |x|^2 - x_2^2 \geq (s-1)^2 - \frac{s^2}{4} \geq \frac{3}{4}(s-2)^2 \tag{50}$$

and $|x|^2 \le 2x_1^2$. Therefore we deduce

$$\frac{4}{5}\int_{\partial E} x_1^2\zeta^2 \, d\mathcal{H}^1_\gamma \le (s+1)^2 \int_{\partial E} \nu_1^2\zeta^2 \, d\mathcal{H}^1_\gamma + \frac{C}{s^2}\int_{\partial E} \nu_1^2 \, d\mathcal{H}^1_\gamma. \tag{51}$$

We write the first term on the right-hand-side of (51) as

$$\begin{aligned}(s+1)^2 &\int_{\partial E\cap\{\nu_1^2\le 1/2\}} \nu_1^2\zeta^2 \, d\mathcal{H}^1_\gamma + (s+1)^2 \int_{\partial E\cap\{\nu_1^2> 1/2\}} \nu_1^2\zeta^2 \, d\mathcal{H}^1_\gamma \\ &\le \frac{(s+1)^2}{2}\int_{\partial E} \zeta^2 \, d\mathcal{H}^1_\gamma + (s+1)^2 \int_{\partial E\cap\{\nu_1^2> 1/2\}} \nu_1^2 \, d\mathcal{H}^1_\gamma \\ &\le \frac{3}{4}\int_{\partial E} x_1^2\zeta^2 \, d\mathcal{H}^1_\gamma + (s+1)^2 \int_{\partial E\cap\{\nu_1^2> 1/2\}} \nu_1^2 \, d\mathcal{H}^1_\gamma,\end{aligned}$$

where the last inequality follows from (50). Therefore (51) implies

$$\frac{1}{20}\int_{\partial E} x_1^2\zeta^2 \, d\mathcal{H}^1_\gamma \le (s+1)^2 \int_{\partial E\cap\{\nu_1^2> 1/2\}} \nu_1^2 \, d\mathcal{H}^1_\gamma + \frac{C}{s^2}\int_{\partial E} \nu_1^2 \, d\mathcal{H}^1_\gamma.$$

Now since $|x|^2 \le 2x_1^2$ and $\zeta(x_2) = 1$ for $|x_2| \le \frac{s}{3}$ we have

$$\int_{\partial E\cap\{|x_2|\le \frac{s}{3}\}} |x|^2 \, d\mathcal{H}^1_\gamma \le Cs^2 \int_{\partial E\cap\{\nu_1^2> 1/2\}} \nu_1^2 \, d\mathcal{H}^1_\gamma + \frac{C}{s^2}\int_{\partial E} \nu_1^2 \, d\mathcal{H}^1_\gamma.$$

Hence, we need yet to show that

$$\int_{\partial E\cap\{\nu_1^2> 1/2\}} \nu_1^2 \, d\mathcal{H}^1_\gamma \le \frac{C}{s^2}\int_{\partial E} \nu_1^2 \, d\mathcal{H}^1_\gamma \tag{52}$$

to finish the proof of (46).

We obtain by (48) and (36) that

$$\begin{aligned}\int_{\partial E\cap\{\nu_1^2> 1/2\}} \left||\nu_1| - \frac{\sqrt{3}}{\sqrt{7}}\right|^2 d\mathcal{H}^1_\gamma &\le \int_{\partial E\cap\{\nu_1^2> 1/2\}} \big||\nu_1| - |\bar\nu_1|\big|^2 \, d\mathcal{H}^1_\gamma \\ &\le \int_{\partial E} \big||\nu_1| - |\bar\nu_1|\big|^2 \, d\mathcal{H}^1_\gamma \le \frac{C}{s^2}\int_{\partial E} \nu_1^2 \, d\mathcal{H}^1_\gamma.\end{aligned}$$

Thus we have

$$\mathcal{H}^1_\gamma\left(\partial E\cap\{\nu_1^2>1/2\}\right) \le C\int_{\partial E\cap\{\nu_1^2> 1/2\}} \left||\nu_1| - \frac{\sqrt{3}}{\sqrt{7}}\right|^2 d\mathcal{H}^1_\gamma \le \frac{C}{s^2}\int_{\partial E} \nu_1^2 \, d\mathcal{H}^1_\gamma.$$

This proves (52) and concludes the proof of (46). □

We are now ready to prove the reduction to the one dimensional case.

Proof of Theorem 3. We recall that

$$\Sigma_+ = \{x\in\partial E : x_2 > 0\} \qquad \text{and} \qquad \Sigma_- = \{x\in\partial E : x_2 < 0\}.$$

As we mentioned in Remark 2, using $\varphi = \nu_e$ with $e\in\mathbb{S}^1$ orthogonal to the barycenter as a test function in the second variation inequality (14), does not provide any information on the minimizer since the term $|\int_{\partial E} \nu_e \, x \, d\mathcal{H}^1_\gamma|$ can be too large and thus (28) becomes trivial inequality. We overcome this problem by essentially variating only Σ_+ while keeping Σ_- unchanged, and vice-versa (see Figure 1). To be more precise, we restrict the class of test function by assuming $\varphi\in C^\infty(\partial E)$ to have zero average and to satisfy $\varphi(x) = 0$ for every $x\in\partial E\cap\{x_2\le -\frac{s}{3}\}$ (or

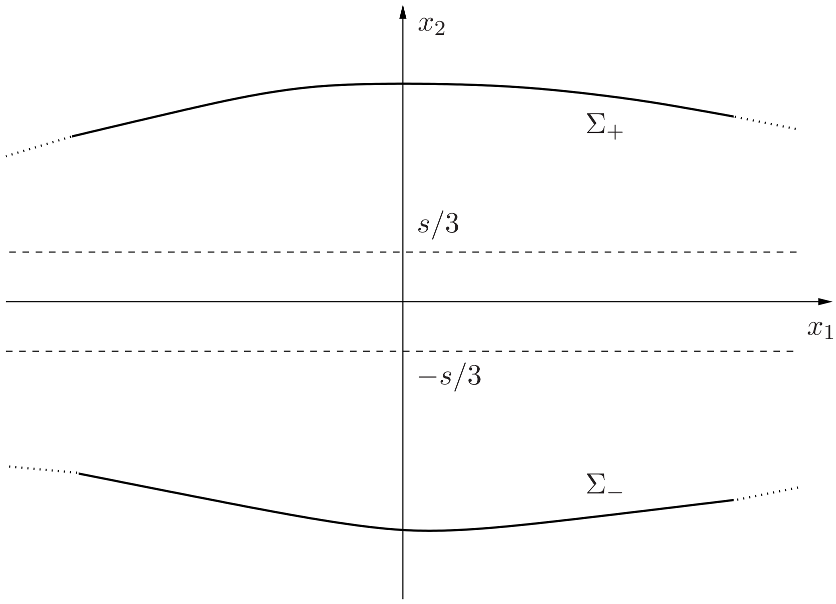

FIGURE 1. The sets Σ_+ and Σ_-.

$\varphi(x) = 0$ for every $x \in \partial E \cap \{x_2 \geq \frac{s}{3}\}$). The point is that for these test function an estimate similar to (30) holds,

$$\frac{\varrho}{\sqrt{2\pi}} \Big| \int_{\partial E} \varphi\, x \, d\mathcal{H}^1_\gamma \Big|^2 \leq \frac{1}{2} \int_{\partial E} \varphi^2 \, d\mathcal{H}^1_\gamma. \tag{53}$$

Indeed, by writing

$$\begin{aligned} \Big| \int_{\partial E} \varphi\, x \, d\mathcal{H}^1_\gamma \Big|^2 &= \Big(\int_{\partial E} x_1 \varphi \, d\mathcal{H}^1_\gamma \Big)^2 + \Big(\int_{\partial E} x_2 \varphi \, d\mathcal{H}^1_\gamma \Big)^2 \\ &= \Big(\int_{\partial E} (x_1 - (x_1)_{\Sigma_+}) \varphi \, d\mathcal{H}^1_\gamma \Big)^2 + \Big(\int_{\partial E} (x_2 - (x_2)_{\Sigma_+}) \varphi \, d\mathcal{H}^1_\gamma \Big)^2 \end{aligned}$$

and estimating both the terms by (45) and (46) we have

$$\begin{aligned} \Big(\int_{\partial E} (x_i - (x_i)_{\Sigma_+}) \varphi \, d\mathcal{H}^1_\gamma \Big)^2 &= \Big(\int_{\Sigma_+ \cup \{|x_2| \leq \frac{s}{3}\}} (x_i - (x_i)_{\Sigma_+}) \varphi \, d\mathcal{H}^1_\gamma \Big)^2 \\ &\leq 8 \Big(\int_{\Sigma_+} (x_i - (x_i)_{\Sigma_+})^2 + \int_{\partial E \cap \{|x_2| \leq \frac{s}{3}\}} |x|^2 \, d\mathcal{H}^1_\gamma \Big) \Big(\int_{\partial E} \varphi^2 \, d\mathcal{H}^1_\gamma \Big) \\ &\leq C P_\gamma(E) \Big(\int_{\partial E} \varphi^2 \, d\mathcal{H}^1_\gamma \Big), \qquad \text{for } i = 1, 2. \end{aligned}$$

Hence, we get (53) thanks to (11) and $\varrho \leq \frac{7\sqrt{2\pi}}{5s^2} e^{\frac{s^2}{2}}$ from (10).

In order to explain the idea of the proof, we assume first that Σ_+ and Σ_- are different components of ∂E. This is of course a major simplification but it will hopefully help the reader to follow the actual proof below. In this case we may use the following test functions in the second variation condition,

$$\varphi_i := \begin{cases} \nu_i - (\nu_i)_{\Sigma_+} & \text{on } \Sigma_+ \\ 0 & \text{on } \Sigma_- \end{cases} \tag{54}$$

for $i = 1, 2$, where $(\nu_i)_{\Sigma_+}$ is the average of ν_i on Σ_+. We use φ_i as a test functions in the second variation condition (14) and use (53) to obtain

$$\int_{\partial E} \left(|\nabla_\tau \varphi_i|^2 - k^2 \varphi_i^2 + \varrho \langle b, \nu \rangle \varphi_i^2 - \frac{1}{2} \varphi_i^2 \right) d\mathcal{H}^1_\gamma(x) \geq 0.$$

By using equalities (38) and (39), rewritten on Σ_+, we get after straightforward calculations

$$(\nu_i)^2_{\Sigma_+} \int_{\Sigma_+} (k^2 - \varrho \langle b, \nu \rangle) \, d\mathcal{H}^1_\gamma + \frac{1}{2} \int_{\Sigma_+} (\nu_i - (\nu_i)_{\Sigma_+})^2 \, d\mathcal{H}^1_\gamma \leq -\varrho \langle b, e_i \rangle \int_{\Sigma_+} \nu_i \, d\mathcal{H}^1_\gamma, \quad i = 1, 2. \tag{55}$$

By summing up the previous inequality for $i = 1, 2$ we get

$$[(\nu_1)^2_{\Sigma_+} + (\nu_2)^2_{\Sigma_+}] \int_{\Sigma_+} (k^2 - \varrho \langle b, \nu \rangle) \, d\mathcal{H}^1_\gamma + \frac{1}{2} \int_{\Sigma_+} [1 - (\nu_1)^2_{\Sigma_+} - (\nu_1)^2_{\Sigma_+}] \, d\mathcal{H}^1_\gamma \leq -\varrho \int_{\Sigma_+} \langle b, \nu \rangle \, d\mathcal{H}^1_\gamma.$$

This can be rewritten as

$$[1 - (\nu_1)^2_{\Sigma_+} - (\nu_2)^2_{\Sigma_+}] \int_{\Sigma_+} \left(\varrho \langle b, \nu \rangle + \frac{1}{2} \right) d\mathcal{H}^1_\gamma + [(\nu_1)^2_{\Sigma_+} + (\nu_2)^2_{\Sigma_+}] \int_{\Sigma_+} k^2 \, d\mathcal{H}^1_\gamma \leq 0.$$

By Jensen inequality $1 - (\nu_1)^2_{\Sigma_+} - (\nu_2)^2_{\Sigma_+} \geq 0$, while $|\varrho \langle b, \nu \rangle| \leq \frac{2}{s^2}$ which follows from (19). Therefore $k = 0$ and Σ_+ is a line. It is clear that a similar conclusion holds also for in Σ_-.

When Σ_+ and Σ_- are connected the argument is more involved, since we need a cut-off argument in order to "separate" Σ_+ and Σ_-. This is possible due to (46), which implies that the perimeter of the minimizer in the strip $\{|x_2| \leq s/3\}$ is small. Therefore the cut-off argument produces an error term, which by (46) is small enough so that we may apply the previous argument. However, the presence of the cut-off function makes the equations more tangled and the estimates more complicated. Since the argument is technically involved we split the rest of the proof in two steps.

Step 1. In the first step we prove

$$\left((\nu_1)^2_{\Sigma_+} + (\nu_2)^2_{\Sigma_+} \right) \int_{\Sigma_+} k^2 \, d\mathcal{H}^1_\gamma + \int_{\Sigma_+} \left(1 - (\nu_1)^2_{\Sigma_+} - (\nu_2)^2_{\Sigma_+} \right) d\mathcal{H}^1_\gamma \leq R, \tag{56}$$

where the remeinder term satisfies

$$R \leq \frac{C}{s^4} \left(\frac{P_\gamma(E)^2}{\mathcal{H}^1_\gamma(\Sigma_+)} \right) \bar{\nu}_1^2. \tag{57}$$

We do this by proving the counterpart of (55), which now reads as

$$(\nu_i)^2_{\Sigma_+} \int_{\Sigma_+} \left(\frac{k^2}{2} - \varrho \langle b, \nu \rangle \right) d\mathcal{H}^1_\gamma + \frac{1}{2} \int_{\Sigma_+} (\nu_i - (\nu_i)_{\Sigma_+})^2 \, d\mathcal{H}^1_\gamma \leq -\varrho \langle b, e_i \rangle \int_{\Sigma_+} \nu_i \, d\mathcal{H}^1_\gamma + R, \tag{58}$$

for $i = 1, 2$, where the reminder R satisfies (57). Let us show first how (56) follows from (58).
Indeed, by $\varrho |b| \leq \frac{2}{s^2}$ given by (19) we have

$$-(\nu_i)^2_{\Sigma_+} \int_{\Sigma_+} \varrho \langle b, \nu \rangle \, d\mathcal{H}^1_\gamma \geq -\left(\fint_{\Sigma_+} \nu_i^2 \, d\mathcal{H}^1_\gamma \right) \int_{\Sigma_+} \varrho \langle b, \nu \rangle \, d\mathcal{H}^1_\gamma - \frac{2}{s^2} \int_{\Sigma_+} (\nu_i - (\nu_i)_{\Sigma_+})^2 \, d\mathcal{H}^1_\gamma.$$

Therefore we have

$$-(\nu_i)^2_{\Sigma_+} \int_{\Sigma_+} \varrho \langle b, \nu \rangle \, d\mathcal{H}^1_\gamma + \frac{1}{4} \int_{\Sigma_+} (\nu_i - (\nu_i)_{\Sigma_+})^2 \, d\mathcal{H}^1_\gamma \geq -\left(\fint_{\Sigma_+} \nu_i^2 \, d\mathcal{H}^1_\gamma \right) \int_{\Sigma_+} \varrho \langle b, \nu \rangle \, d\mathcal{H}^1_\gamma.$$

Thus we obtain from (58)

$$(\nu_i)_{\Sigma_+}^2 \int_{\Sigma_+} \frac{k^2}{2}\, d\mathcal{H}^1_\gamma - \left(\fint_{\Sigma_+} \nu_i^2 \, d\mathcal{H}^1_\gamma \right) \int_{\Sigma_+} \varrho \langle b, \nu \rangle \, d\mathcal{H}^1_\gamma + \frac{1}{4} \int_{\Sigma_+} (\nu_i - (\nu_i)_{\Sigma_+})^2 \, d\mathcal{H}^1_\gamma$$
$$\leq -\varrho \langle b, e_i \rangle \int_{\Sigma_+} \nu_i \, d\mathcal{H}^1_\gamma + R.$$

Note that $\sum_{i=1}^2 \int_{\Sigma_+} \langle b, e_i \rangle \nu_i \, d\mathcal{H}^1_\gamma = \int_{\Sigma_+} \langle b, \nu \rangle \, d\mathcal{H}^1_\gamma$. Therefore, by adding the above estimate with $i = 1, 2$ we obtain

$$\left((\nu_1)_{\Sigma_+}^2 + (\nu_2)_{\Sigma_+}^2 \right) \int_{\Sigma_+} \frac{k^2}{2}\, d\mathcal{H}^1_\gamma - \int_{\Sigma_+} \varrho \langle b, \nu \rangle \, d\mathcal{H}^1_\gamma + \frac{1}{4} \int_{\Sigma_+} \left(\nu_1^2 - (\nu_1)_{\Sigma_+}^2 + \nu_2^2 - (\nu_2)_{\Sigma_+}^2 \right) d\mathcal{H}^1_\gamma$$
$$\leq - \int_{\Sigma_+} \varrho \langle b, \nu \rangle \, d\mathcal{H}^1_\gamma + R,$$

which implies (56). Hence, we need to prove (58).

We prove (58) by using the second variation condition (14) with test function

$$\varphi_i := (\nu_i - \alpha_i) \zeta(x_2)$$

for $i = 1, 2$. Here $\zeta : \mathbb{R} \to [0, 1]$ is a smooth cut-off function such that

$$\zeta(t) = \begin{cases} 1 & \text{for } t \geq 0, \\ 0 & \text{for } t \leq -s/3, \end{cases} \qquad \text{and} \qquad |\zeta'(t)| \leq \frac{4}{s} \quad \text{for all } t \in \mathbb{R}$$

and α_i is chosen so that φ_i has zero average. This choice is the counterpart of (54) in the case when ∂E is connected. In particular, the cut-off function ζ guarantees that $\varphi_i(x) = 0$, for $x \in \partial E \cap \{x_2 \leq -\frac{s}{3}\}$. Therefore the estimate (53) holds and the second variation condition (14) yields

$$\int_{\partial E} \left(|\nabla_\tau \varphi_i|^2 - k^2 \varphi_i^2 + \varrho \langle b, \nu \rangle \varphi_i^2 - \frac{1}{2} \varphi_i^2 \right) d\mathcal{H}^1_\gamma(x) \geq 0. \tag{59}$$

Let us simplify the above expression. Recall that the test function is $\varphi = (\nu_i - \alpha_i)\zeta$, where $\zeta = \zeta(x_2)$. By straightforward calculation

$$\int_{\partial E} |\nabla_\tau \varphi_i|^2 \, d\mathcal{H}^1_\gamma = \int_{\partial E} \left(\varphi_i (-\Delta_\tau \varphi_i + \langle \nabla \varphi_i, x \rangle \right) d\mathcal{H}^1_\gamma$$
$$= \int_{\partial E} \left(\varphi_i \zeta (-\Delta_\tau \nu_i + \langle \nabla_\tau \nu_i, x \rangle) + (\nu_i - \alpha_i)^2 |\nabla_\tau \zeta|^2 \right) d\mathcal{H}^1_\gamma.$$

Therefore we have by the above equality and by multiplying the equation (34) with φ_i and integrating by parts

$$\int_{\partial E} |\nabla_\tau \varphi_i|^2 \, d\mathcal{H}^1_\gamma = \int_{\partial E} (k^2 - \varrho \langle b, \nu \rangle) \zeta^2 \nu_i (\nu_i - \alpha_i) \, d\mathcal{H}^1_\gamma + R_1, \tag{60}$$

where the remainder term is

$$R_1 = \int_{\partial E} \left(\varrho \langle b, e_i \rangle \, \varphi_i \zeta + (\nu_i - \alpha_i)^2 |\nabla_\tau \zeta|^2 \right) d\mathcal{H}^1_\gamma. \tag{61}$$

On the other hand, multiplying (34) with ζ^2 and integrating by parts yields

$$\begin{aligned} \alpha_i \int_{\partial E} \left((k^2 - \varrho \langle b, \nu \rangle) \nu_i \zeta^2 \right) d\mathcal{H}^1_\gamma &= \alpha_i \int_{\partial E} \left((-\Delta_\tau \nu_i + \langle \nabla_\tau \nu_i, x \rangle) \zeta^2 - \varrho \langle b, e_i \rangle \zeta^2 \right) d\mathcal{H}^1_\gamma \\ &= -\alpha_i \int_{\partial E} \varrho \langle b, e_i \rangle \zeta^2 \, d\mathcal{H}^1_\gamma + R_2, \end{aligned} \tag{62}$$

where the remainder term is

$$R_2 = 2\alpha_i \int_{\partial E} \zeta \langle \nabla_\tau \nu_i, \nabla_\tau \zeta \rangle \, d\mathcal{H}^1_\gamma. \tag{63}$$

Collecting (59), (60), (62) yields

$$\int_{\partial E} \left(\alpha_i^2 (k^2 - \varrho\langle b, \nu\rangle) + \frac{1}{2}|\nu_i - \alpha_i|^2 \right) \zeta^2 \, d\mathcal{H}^1_\gamma \leq -\alpha_i \int_{\partial E} \varrho\langle b, e_i\rangle \zeta^2 \, d\mathcal{H}^1_\gamma + R_1 + R_2, \tag{64}$$

where the remainder terms R_1 and R_2 are given by (61) and (63) respectively.

Let us next estimate the remainder terms in (64). We note that (36) (recall that $\nu_v = \nu_1$) implies $⨍_{\partial E} \nu_1^2 \, d\mathcal{H}^1_\gamma \leq \left(1 + \frac{10}{s^2}\right) \bar{\nu}_1^2$. Therefore we deduce from (41) and (46) that

$$\mathcal{H}^1_\gamma\big(\{x \in \partial E : |x_2| \leq s/3\}\big) \leq \frac{C}{s^2} P_\gamma(E)\, \bar{\nu}_1^2. \tag{65}$$

Therefore since $|\nabla_\tau \zeta(x)| \leq 4/s$, for $|x_2| \leq s/3$, and $\nabla_\tau \zeta(x) = 0$ otherwise, (65) yields

$$\int_{\partial E} |\nabla \zeta|^2 \, d\mathcal{H}^1_\gamma \leq \frac{C}{s^4} P_\gamma(E)\, \bar{\nu}_1^2. \tag{66}$$

We may therefore estimate R_2 (given by (63)) by Young's inequality and by (66) as

$$\begin{aligned} R_2 &\leq \frac{\alpha_i^2}{2} \int_{\partial E} |\nabla_\tau \nu_i|^2 \zeta^2 \, d\mathcal{H}^1_\gamma + 2 \int_{\partial E} |\nabla_\tau \zeta|^2 \, d\mathcal{H}^1_\gamma \\ &\leq \frac{\alpha_i^2}{2} \int_{\partial E} k^2 \zeta^2 \, d\mathcal{H}^1_\gamma + \frac{C}{s^4} P_\gamma(E)\, \bar{\nu}_1^2. \end{aligned}$$

Similarly we may estimate (61) as

$$R_1 \leq \varrho\langle b, e_i\rangle \int_{\partial E} \varphi_i \zeta \, d\mathcal{H}^1_\gamma + \frac{C}{s^4} P_\gamma(E)\, \bar{\nu}_1^2.$$

To estimate the first term in R_1 we recall that $\int_{\partial E} \varphi_i \, d\mathcal{H}^1_\gamma = 0$ and therefore $\int_{\partial E} \varphi_i \zeta \, d\mathcal{H}^1_\gamma = \int_{\partial E} \varphi_i (\zeta - 1) \, d\mathcal{H}^1_\gamma$. Since $\varphi_i(\zeta - 1) = 0$ on $\partial E \cap \{|x_2| > s/3\}$, we deduce by $\varrho|b| \leq \frac{2}{s^2}$ and by (65) that

$$\varrho\langle b, e_i\rangle \int_{\partial E} \varphi_i \zeta \, d\mathcal{H}^1_\gamma \leq \frac{C}{s^4} P_\gamma(E)\, \bar{\nu}_1^2.$$

Hence, we may write (64) as

$$\int_{\partial E} \left(\alpha_i^2 \left(\frac{k^2}{2} - \varrho\langle b, \nu\rangle \right) + \frac{1}{2}|\nu_i - \alpha_i|^2 \right) \zeta^2 \, d\mathcal{H}^1_\gamma \leq -\alpha_i \int_{\partial E} \varrho\langle b, e_i\rangle \zeta^2 \, d\mathcal{H}^1_\gamma + \tilde{R}, \tag{67}$$

where the remainder term $\tilde{R}$ satisfies

$$\tilde{R} \leq \frac{C}{s^4} P_\gamma(E)\, \bar{\nu}_1^2. \tag{68}$$

By a similar argument we may also get rid of the cut-off function in (67). Indeed by $\varrho|b| \leq 2/s^2$ and (65) we have $-\int_{\partial E} \varrho\langle b, \nu\rangle \zeta^2 \, d\mathcal{H}^1_\gamma \geq -\int_{\Sigma_+} \varrho\langle b, \nu\rangle \, d\mathcal{H}^1_\gamma - \tilde{R}$, where $\tilde{R}$ satisfies (68). Similarly we get $-\alpha_i \int_{\partial E} \varrho\langle b, e_i\rangle \zeta^2 \, d\mathcal{H}^1_\gamma \leq -\alpha_i \int_{\Sigma_+} \varrho\langle b, e_i\rangle \, d\mathcal{H}^1_\gamma + \tilde{R}$. Therefore we obtain from (67)

$$\int_{\Sigma_+} \left(\alpha_i^2 \left(\frac{k^2}{2} - \varrho\langle b, \nu\rangle \right) + \frac{1}{2}|\nu_i - \alpha_i|^2 \right) d\mathcal{H}^1_\gamma \leq -\alpha_i \int_{\Sigma_+} \varrho\langle b, e_i\rangle \, d\mathcal{H}^1_\gamma + \tilde{R}, \tag{69}$$

where the remainder term $\tilde{R}$ satisfies (68).

We need yet to replace α_i by $(\nu_i)_{\Sigma_+}$ in order to obtain (58). We do this by showing that α_i is close the average $(\nu_i)_{\Sigma_+}$. To be more precise we show that

$$|\alpha_i - (\nu_i)_{\Sigma_+}| \leq \frac{C}{s^2}\left(\frac{P_\gamma(E)}{\mathcal{H}^1_\gamma(\Sigma_+)}\right)\bar{\nu}_1^2. \tag{70}$$

Indeed, since $\zeta = 1$ on Σ_+ we may write

$$\mathcal{H}^1_\gamma(\Sigma_+)(\alpha_i - (\nu_i)_{\Sigma_+}) = \int_{\Sigma_+}(\alpha_i - (\nu_i)_{\Sigma_+})\zeta\, d\mathcal{H}^1_\gamma.$$

Since $\zeta = 0$ when $x_2 \leq -s/3$ we may estimate

$$\mathcal{H}^1_\gamma(\Sigma_+)\big|\alpha_i - (\nu_i)_{\Sigma_+}\big| \leq \left|\int_{\partial E}(\alpha_i - (\nu_i)_{\Sigma_+})\zeta\, d\mathcal{H}^1_\gamma\right| + 2\mathcal{H}^1_\gamma\big(\{x \in \partial E : |x_2| \leq s/3\}\big).$$

The inequality (70) then follows from $\int_{\partial E}(\alpha_i - (\nu_i)_{\Sigma_+})\zeta\, d\mathcal{H}^1_\gamma = -\int_{\partial E}\varphi_i\, d\mathcal{H}^1_\gamma = 0$ and from (65).

We use (35) and $\varrho|b| \leq \frac{2}{s^2}$ to conclude that $\int_{\Sigma_+} k^2 + |\varrho\langle b,\nu\rangle|\, d\mathcal{H}^1_\gamma \leq \frac{C}{s^2}P_\gamma(E)$. Therefore we may estimate (69) by (70) and get

$$\int_{\Sigma_+}\left((\nu_i)_{\Sigma_+}^2\left(\frac{k^2}{2} - \varrho\langle b,\nu\rangle\right) + \frac{1}{2}|\nu_i - \alpha_i|^2\right) d\mathcal{H}^1_\gamma \leq -(\nu_i)_{\Sigma_+}\int_{\Sigma_+}\varrho\langle b, e_i\rangle\, d\mathcal{H}^1_\gamma + R,$$

where the remainder term R satisfies (57). Finally the inequality (58) follows from

$$\int_{\Sigma_+}|\nu_i - (\nu_i)_{\Sigma_+}|^2\, d\mathcal{H}^1_\gamma \leq \int_{\Sigma_+}|\nu_i - \alpha_i|^2\, d\mathcal{H}^1_\gamma.$$

Step 2. Precisely similar argument as in the previous step, gives the estimate (68) also for Σ_-, i.e.,

$$\left((\nu_1)_{\Sigma_-}^2 + (\nu_2)_{\Sigma_-}^2\right)\int_{\Sigma_-}k^2\, d\mathcal{H}^1_\gamma + \int_{\Sigma_-}\left(1 - (\nu_1)_{\Sigma_-}^2 - (\nu_2)_{\Sigma_+}^2\right) d\mathcal{H}^1_\gamma \leq \tilde{R}, \tag{71}$$

where the remainder satisfies

$$\tilde{R} \leq \frac{C}{s^4}\left(\frac{P_\gamma(E)^2}{\mathcal{H}^1_\gamma(\Sigma_-)}\right)\bar{\nu}_1^2. \tag{72}$$

Let us next prove that

$$\mathcal{H}^1_\gamma(\Sigma_+) \geq \frac{1}{10}P_\gamma(E) \qquad \text{and} \qquad \mathcal{H}^1_\gamma(\Sigma_-) \geq \frac{1}{10}P_\gamma(E). \tag{73}$$

Without loss of generality we may assume that $\mathcal{H}^1_\gamma(\Sigma_-) \geq \mathcal{H}^1_\gamma(\Sigma_+)$. In particular, we have $\mathcal{H}^1_\gamma(\Sigma_-) \geq \frac{1}{2}P_\gamma(E)$ and therefore (71) and (72) imply

$$\int_{\Sigma_-}(\nu_2 - (\nu_2)_{\Sigma_-})^2\, d\mathcal{H}^1_\gamma \leq \frac{C}{s^4}P_\gamma(E). \tag{74}$$

We need to show the first inequality in (73). We use (44) and (74) to deduce

$$\begin{aligned}\frac{4}{7}P_\gamma(E) &\leq \int_{\partial E}(\nu_2 - \bar{\nu}_2)^2\, d\mathcal{H}^1_\gamma \leq \int_{\partial E}(\nu_2 - (\nu_2)_{\Sigma_-})^2\, d\mathcal{H}^1_\gamma \\ &= \int_{\Sigma_-}(\nu_2 - (\nu_2)_{\Sigma_-})^2\, d\mathcal{H}^1_\gamma + \int_{\Sigma_+}(\nu_2 - (\nu_2)_{\Sigma_-})^2\, d\mathcal{H}^1_\gamma \\ &\leq \frac{C}{s^4}P_\gamma(E) + 4\mathcal{H}^1_\gamma(\Sigma_+).\end{aligned}$$

Hence we obtain $\mathcal{H}^1_\gamma(\Sigma_+) \geq \frac{1}{10}P_\gamma(E)$. Thus we have (73).

We conclude from (73) and from (35) that $\fint_{\Sigma_+} k^2 \, d\mathcal{H}^1_\gamma \leq \frac{C}{s^2}$. Therefore we have by (56) and (73) that

$$\int_{\Sigma_+} k^2 \, d\mathcal{H}^1_\gamma \leq \frac{C}{s^4} P_\gamma(E) \, \bar{\nu}_1^2.$$

Similarly we get

$$\int_{\Sigma_-} k^2 \, d\mathcal{H}^1_\gamma \leq \frac{C}{s^4} P_\gamma(E) \, \bar{\nu}_1^2$$

and therefore

$$\fint_{\partial E} k^2 \, d\mathcal{H}^1_\gamma \leq \frac{C}{s^4} \, \bar{\nu}_1^2. \tag{75}$$

We are now close to finish the proof. We proceed by recalling the equation (34) for ν_1, i.e.,

$$\Delta \nu_1 - \langle \nabla \nu_1, x\rangle = -k^2 \nu_1 + \varrho \langle b, \nu\rangle \nu_1 - \varrho \langle b, e^{(1)}\rangle.$$

We integrate this over ∂E, use (75) and get

$$-\varrho \langle b, e^{(1)}\rangle + \varrho \fint_{\partial E} \langle b, \nu\rangle \nu_1 \, d\mathcal{H}^1_\gamma \leq \fint_{\partial E} k^2 \, d\mathcal{H}^1_\gamma \leq \frac{C}{s^4} \bar{\nu}_1^2.$$

Note that by $\bar{\nu} P_\gamma(E) = -\sqrt{2\pi}\, b$ (proved in (18)) we have $|\bar{\nu}| \, \langle b, e^{(1)}\rangle = -|b| \bar{\nu}_1$. Thus we deduce from the above inequality that

$$\varrho |b| \, |\bar{\nu}_1| \leq \varrho |b| \, |\bar{\nu}| \fint_{\partial E} |\nu_1| \, d\mathcal{H}^1_\gamma + \frac{C}{s^4} \bar{\nu}_1^2 \, |\bar{\nu}|. \tag{76}$$

We proceed by concluding from (44) that

$$\frac{4}{7} \leq \fint_{\partial E} (\nu_2 - \bar{\nu}_2)^2 \, d\mathcal{H}^1_\gamma \leq \fint_{\partial E} \left((\nu_1 - \bar{\nu}_1)^2 + (\nu_2 - \bar{\nu}_2)^2\right) d\mathcal{H}^1_\gamma = 1 - \bar{\nu}_1^2 - \bar{\nu}_2^2.$$

This implies

$$|\bar{\nu}|^2 = \bar{\nu}_1^2 + \bar{\nu}_2^2 \leq \frac{3}{7}.$$

Using this and the inequality $\fint_{\partial E} \nu_1^2 \, d\mathcal{H}^1_\gamma \leq \left(1 + \frac{10}{s^2}\right) \bar{\nu}_1^2$ (given by (36)) we estimate

$$\varrho |b| \, |\bar{\nu}| \fint_{\partial E} |\nu_1| \, d\mathcal{H}^1_\gamma \leq \frac{\sqrt{3}}{\sqrt{7}} \varrho |b| \left(\fint_{\partial E} \nu_1^2 \, d\mathcal{H}^1_\gamma \right)^{1/2} \leq \frac{3}{4} \varrho |b| \, |\bar{\nu}_1|.$$

Therefore we deduce from (76)

$$\frac{1}{4} \varrho |b| |\bar{\nu}_1| \leq \frac{C}{s^4} \bar{\nu}_1^2 \, |\bar{\nu}|.$$

We use $\varrho |b| \geq \frac{1}{4s^2} |\bar{\nu}|$ (from (19)) to conclude

$$\frac{1}{s^2} |\bar{\nu}_1| |\bar{\nu}| \leq \frac{C}{s^4} \bar{\nu}_1^2 \, |\bar{\nu}|.$$

This yields $\bar{\nu}_1 = 0$. But then (36) implies

$$\nu_1 = 0$$

and we have reduced the problem to the one dimensional case. □

5. The one dimensional case

In this short section we finish the proof of the main theorem which states that the minimizer of (8) is either the half-space $H_{\omega,s}$ or the symmetric strip $D_{\omega,s}$. By the previous results it is enough to solve the problem in the one-dimensional case.

Theorem 4. *When s is large enough the minimizer $E \subset \mathbb{R}$ of (8) is either $(-\infty, s)$, $(-s,\infty)$ or $(-a(s), a(s))$.*

Proof. As we explained in Section 2, we have to prove that, when ϱ is in the interval (10), the only local minimizers of (8) are $(-\infty, s)$, $(-s,\infty)$ and $(-a(s), a(s))$.

Let us first show that the minimizer E is an interval. Recall that since $E \subset \mathbb{R}$ is a set of locally finite perimeter it has locally finite number of boundary points. Moreover, since there is no curvature in dimension one the Euler equation (13) reads as

$$-x\nu(x) + \varrho bx = \lambda. \tag{77}$$

By (41) we have that $(-s+1, s-1) \subset E$. It is therefore enough to prove that the boundary ∂E has at most one positive and one negative point. Assume by contradiction that ∂E has at least two positive points (the case of two negative points is similar).

If x is a positive point which is closest to the origin on ∂E then $\nu(x) = 1$. On the other hand, if y is the next boundary point, then $\nu(y) = -1$. Then the Euler equation yields

$$-x + \varrho bx = y + \varrho by.$$

By $\varrho|b| \leq \frac{2}{s^2}$ (proved in (19)) we conclude that

$$\left(1 - \frac{2}{s^2}\right) y \leq -\left(1 - \frac{2}{s^2}\right) x,$$

which is a contradiction when since $x, y > 0$.

The minimizer of (8) is thus an interval of the form

$$E = (-x, y),$$

where $s - 1 \leq x, y \leq \infty$. Without loss of generality we may assume that $x \leq y$. Therefore we have

$$e^{-\frac{x^2}{2}} \leq P_\gamma(E) \leq 2e^{-\frac{x^2}{2}}.$$

Using the bounds on the perimeter (11) and (12) we conclude that $s - 1/s \leq x \leq s + 3/s$. The Euler equation (77) yields

$$x + \varrho bx = y - \varrho by.$$

Hence, we conclude from $\varrho|b| \leq \frac{2}{s^2}$ that

$$s - \frac{1}{s} \leq x \leq y \leq s + \frac{8}{s}. \tag{78}$$

Let us next prove that the minimizer has the volume $\gamma(E) = \phi(s)$. Indeed, it is not possible that $\gamma(E) < \phi(s)$, because by enlarging E we can decrease its perimeter, barycenter and the volume penalization term in (8). Also $\gamma(E) > \phi(s)$ is not possible. Indeed, in this case we can perturb the set E by

$$E_t = (-x + t, y), \qquad t > 0.$$

Then $\phi(s) \leq \gamma(E_t) < \gamma(E)$ and

$$\begin{aligned} \frac{d}{dt}\mathcal{F}(E_t)\big|_{t=0} &= xe^{-\frac{x^2}{2}} + \varrho\, b(E)\, xe^{-\frac{x^2}{2}} - (s+1)e^{-\frac{x^2}{2}} \\ &\leq \left(1 + \frac{2}{s^2}\right) xe^{-\frac{x^2}{2}} - (s+1)e^{-\frac{x^2}{2}}, \end{aligned}$$

taking again into account that $\varrho|b(E)| \leq \frac{2}{s^2}$. But since $x \leq s + 3/s$ the above inequality yields $\frac{d}{dt}\mathcal{F}(E_t)\big|_{t=0} < 0$, which contradicts the minimality of E.

Let us finally show that if a local minimizer is a finite interval $E = (-x, y)$ for $x \leq y < \infty$, then necessarily $x = y = a(s)$. We study the value of the functional (8) for intervals $E_t = (-\alpha(t), t)$, which have the volume $\gamma(E_t) = \phi(s)$. By the inequality (78) we need to only study the case when $a(s) \leq t \leq s + \frac{8}{s}$. This leads us to study the function $f : [a(s), s + \frac{8}{s}] \to \mathbb{R}$,

$$f(t) := \mathcal{F}(E_t) = e^{-\frac{t^2}{2}} + e^{-\frac{\alpha^2(t)}{2}} + \frac{\varrho}{2\sqrt{2\pi}}\left(e^{-\frac{\alpha^2(t)}{2}} - e^{-\frac{t^2}{2}}\right)^2.$$

The volume constraint reads as $\int_{-\alpha(t)}^{t} e^{-\frac{l^2}{2}}\, dl = \sqrt{2\pi}\, \phi(s)$. By differentiating this we obtain

$$\alpha'(t)e^{-\frac{\alpha^2(t)}{2}} = -e^{-\frac{t^2}{2}}. \tag{79}$$

From (79) we conclude that for $t \geq \alpha(t)$ it holds $0 > \alpha'(t) > -1$.

By differentiating f once and by using (79) we get

$$f'(t) = \left(-t + \alpha(t) + \frac{\varrho}{\sqrt{2\pi}}(t + \alpha(t))\left(e^{-\frac{\alpha^2(t)}{2}} - e^{-\frac{t^2}{2}}\right)\right)e^{-\frac{t^2}{2}}.$$

Therefore at a critical point it holds

$$\frac{\varrho}{\sqrt{2\pi}}(t + \alpha(t))\left(e^{-\frac{\alpha^2(t)}{2}} - e^{-\frac{t^2}{2}}\right) = t - \alpha(t). \tag{80}$$

We are interested in the sign of $f''(t)$ at critical points on the interval $t \in [a(s), s + \frac{8}{s}]$. Let us denote the barycenter of E_t by

$$b_t := b(E_t) = \frac{1}{\sqrt{2\pi}}\left(e^{-\frac{\alpha^2(t)}{2}} - e^{-\frac{t^2}{2}}\right).$$

By differentiating f twice and by using (79) and (80) we obtain

$$f''(t) = \left(-(1 - \varrho b_t) + \alpha'(t)(1 + \varrho b_t) + \frac{\varrho}{\sqrt{2\pi}}(t + \alpha(t))^2 e^{-\frac{t^2}{2}}\right)e^{-\frac{t^2}{2}}$$

at a critical point t. Let us write $\varrho = \frac{\varrho_0\sqrt{2\pi}}{s^2}e^{\frac{s^2}{2}}$, where $\frac{6}{5} \leq \varrho_0 \leq \frac{7}{5}$. In order to analyze the sign of $f''(t)$ at critical points we define $g : [a(s), s + \frac{8}{s}] \to \mathbb{R}$ as

$$g(t) := -(1 - \varrho b_t) + \alpha'(t)(1 + \varrho b_t) + \frac{\varrho_0}{s^2}(t + \alpha(t))^2 e^{-\frac{t^2}{2}} e^{\frac{s^2}{2}}.$$

By recalling that by (78) $\alpha(t) \geq s - 1/s$, we have $\varrho|b_t| \leq \frac{\varrho}{\sqrt{2\pi}}e^{-\frac{\alpha^2(t)}{2}} \leq \frac{4}{s^2}$.

Note that the end point $t = \alpha(t) = a(s)$ is of course a critical point of f. Let us check that it is a local minimum. We have for the barycenter $b_{a(s)} = 0$, $\alpha'(a(s)) = -1$ by (79), $a(s) = s + \frac{\ln 2}{s} + o(\frac{1}{s})$ by (4) and $e^{-\frac{a(s)^2}{2}} = \frac{1}{2}\Big(1 + \frac{\ln 2}{s^2} + o(1/s^2)\Big)e^{-\frac{s^2}{2}}$ by (6). Therefore it holds

$$g(a(s)) \geq -2 + 2\varrho_0 - \frac{C}{s^2} > 0$$

when s is large. In particular, we deduce that $t = a(s)$ is a local minimum of f.

Let us next show that g is strictly decreasing. Let us first fix a small number $\delta > 0$, which value will be clear later. We obtain by differentiating (79) that

$$\alpha'' = \alpha'(\alpha\alpha' - t).$$

By recalling that $|\alpha'(t)| \leq 1$ and that by (78) $\alpha(t) \leq s + 8/s$, we get that $|\alpha''(t)| \leq 2s\,|\alpha'(t)| + 16/s$ for $t \in [a(s), s + \frac{8}{s}]$. Moreover, we estimate $\left|\varrho\, b_t'\right| \leq C/s$, where $b_t' = \frac{d}{dt} b_t$. We may then estimate the derivative of g as

$$
\begin{aligned}
g'(t) \leq & \alpha''(t)(1 + \varrho b_t) + (1 + \alpha'(t))\,\varrho\, b_t' \\
& - \frac{\varrho_0}{s^2}\, t(t + \alpha(t))^2 e^{-\frac{t^2}{2}} e^{\frac{s^2}{2}} + \frac{2\varrho_0}{s^2}(t + \alpha(t))(1 + \alpha'(t)) e^{-\frac{t^2}{2}} e^{\frac{s^2}{2}} \\
\leq & 2s\,|\alpha'(t)| - 4\varrho_0\, s\, e^{-\frac{t^2}{2}} e^{\frac{s^2}{2}} + \delta
\end{aligned}
\tag{81}
$$

when $t \in [a(s), s + \frac{8}{s}]$ and $\alpha \in [s - \frac{1}{s}, a(s)]$. To study (81) it is convenient to write

$$
t = s + \frac{\ln z}{s}
$$

where $2 - \delta \leq z \leq e^8$. We obtain from the volume condition $\int_{-\alpha(t)}^{t} e^{-\frac{l^2}{2}}\, dl = \sqrt{2\pi}\,\phi(s)$ arguing similarly as in (4) we obtain

$$
\alpha(t) = s + \frac{1}{s} \ln\left(\frac{z}{z-1}\right) + \frac{\varepsilon(z)}{s}
$$

and from (79) that

$$
\alpha'(t) = -\frac{1}{z-1} + \varepsilon(z),
$$

where $\varepsilon(z)$ is a function which converges uniformly to zero as $s \to \infty$. Keeping these in mind we may estimate (81) as

$$
g'(t) \leq \frac{2s}{z-1} - \frac{4\varrho_0\, s}{z} + \delta s \leq \frac{2s(12 - 7z)}{5z(z-1)} + \delta s.
$$

Since $2 - \delta \leq z \leq e^8$, the above inequality shows that $g'(t) < 0$ when δ is chosen small enough. Hence, we conclude that g is strictly decreasing.

Recall that $g(a(s)) > 0$. Since g is strictly decreasing, there is $t_0 \in (a(s), s + \frac{8}{s})$ such that $g(t) > 0$ for $t \in [a(s), t_0)$ and $g(t) < 0$ for $t \in (t_0, s + \frac{8}{s}]$. Therefore the function f has no other local minimum on $[a(s), s + \frac{8}{s}]$ than the end point $t = a(s)$. Indeed, if there were another local minimum on $(a(s), t_0]$ there would be at least one local maximum on $(a(s), t_0)$. This is impossible as the previous argument shows that $f''(t) > 0$ at every critical point on $(a(s), t_0)$. Moreover, from $g(t) < 0$ for $t \in (t_0, s + \frac{8}{s}]$ we conclude that there are no local minimum points on $(t_0, s + \frac{8}{s}]$. This completes the proof. □

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