GENERALISED SADOWSKY THEORIES FOR RIBBONS FROM THREE-DIMENSIONAL NONLINEAR ELASTICITY

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ABSTRACT. In the 1930s Sadowsky derived an asymptotic theory for narrow ribbons. Generalised Sadowsky theories were recently obtained by the authors as a Γ-limit from two-dimensional plate models. In the present article, we provide a rigorous derivation of these generalised Sadowsky theories starting from nonlinear three-dimensional elasticity. On a technical level, this involves capturing a contribution to the asymptotic energy functional generated by a nonlinear constraint which is satisfied only approximately. It also involves the construction of fine-scale 'corrugations' capable of reaching a bending energy regime which is strictly below that of the original Sadowsky functional.

1. INTRODUCTION

In the 1930s Sadowsky [24] proposed a theory modelling the behaviour of developable ribbons, i.e., of *thin and narrow* stripes — say, rectangles of length $\ell \sim 1$ and width $h \ll 1$ cut out of an inextensible elastic sheet. According to this theory, the elastic energy of a narrow deformed ribbon, whose midline is given by a smooth arc-length parametrised space curve $y : (0, \ell) \to \mathbb{R}^3$, is

$$\int_0^\ell Q_{\text{Sadowsky}}(\mu(x_1), \tau(x_1))\, dx_1.$$

Here $\mu = |y''|$ denotes the curvature of the midline, which is assumed to be everywhere nonzero, τ denotes the torsion, and

$$Q_{\text{Sadowsky}}(\mu, \tau) = \frac{(\mu^2 + \tau^2)^2}{\mu^2}.$$

Sadowsky's ribbon model was later justified in [27, 16], where it was derived as a 'narrow strip limit' $h \to 0$ from a model which itself arises by inserting an Ansatz into the so-called Kirchhoff model for plates.

The Kirchhoff model for plates is the mathematically rigorous asymptotic model for a thin sheet of paper in the limit of zero thickness. It was proposed by Kirchhoff in [17] and justified rigorously as a Γ-limit from nonlinear three-dimensional elasticity in [11], in an energy regime corresponding to pure bending deformations. The central variable in Kirchhoff's plate theory is the deformation of the sheet of paper. The reference configuration of the (infinitesimally thin) sheet is a rectangle $(0, \ell) \times (-\frac{1}{2}, \frac{1}{2})$. Denote its deformation by $Y : (0, \ell) \times (-\frac{1}{2}, \frac{1}{2}) \to \mathbb{R}^3$. A key feature of Kirchhoff's theory is that the only admissible deformations are isometric immersions, i.e., solutions of the PDE system $(\nabla Y)^T(\nabla Y) = I$. Kirchhoff's theory assigns to such an admissible deformation the elastic energy

$$\int_0^\ell \int_{-\frac{1}{2}}^{\frac{1}{2}} |A(x)|^2\, dx,$$

where A is the second fundamental form of the immersion Y. Here and elsewhere in this introduction we restrict ourselves to isotropic materials.

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The Ansatz made in [27, 23, 16] further restricts the class of admissible deformations Y. As noted in [16], a more natural approach is to start from the Kirchhoff model without any Ansatz. This was done in [5, 6]. The asymptotic functional obtained there is

$$\int_0^\ell \overline{Q}(\mu(x_1), \tau(x_1))\, dx_1,$$

where the energy density $\overline{Q}$ is given by

$$\overline{Q}(\mu, \tau) = \begin{cases} Q_{\text{Sadowsky}}(\mu, \tau) & \text{if } |\mu| \geq |\tau|, \\ 4\tau^2 & \text{if } |\mu| < |\tau|. \end{cases} \tag{1.1}$$

In the regime $|\mu| < |\tau|$ this energy density is strictly smaller than Q_{Sadowsky}. Another key difference to Sadowsky's theory is that the natural asymptotic objects for the generalised Sadowsky theory (1.1) are not mere curves, but rather curves endowed with a normal field giving rise to a framed curve satisfying a geodesic constraint. These 'framed curves' are indeed more general objects than the regular curves to which the analysis in [16] was restricted. In particular, they allow for natural notions of curvature and torsion without the need to impose any constraints or regularity conditions on y''. For technical reasons, such restrictions had to be imposed in [16]. These restrictions, however, exclude certain (counter-intuitive but apparently mechanically relevant) fine-scale "corrugations", which in the large torsion regime have strictly lower bending energy than other more intuitive deformations. This is why $\overline{Q} < Q_{\text{Sadowsky}}$ in this regime.

The results discussed thus far took for granted that Kirchhoff's plate theory is the correct two-dimensional model to start from. But the actual physical 'ribbon' is of course a three-dimensional object, and Kirchhoff's plate theory is only one of a family of asymptotic two-dimensional models which have been derived from three-dimensional elasticity, cf. [12] for an overview. So while the results in [5, 6] provide rigorous and general asymptotics starting from Kirchhoff's theory, the mechanically more realistic initial model is not Kirchhoff's plate theory, but rather nonlinear three-dimensional elasticity. It is the purpose of this article to derive Sadowsky-type theories for ribbons from three-dimensional elasticity.

In the following, by a ribbon we mean a body whose reference configuration is a (three-dimensional) cuboid with thickness δ, width h, and length ℓ living on three different length scales. In order to fix ideas we assume that the length ℓ is of order one, while the width and the thickness are very small, with the thickness much smaller than the width, i.e.,

$$\delta \ll h \ll \ell.$$

One is interested in the asymptotic behaviour as both width and thickness go to zero simultaneously. In order to gain a first understanding of the problem, it is natural to proceed sequentially, i.e., to first let the thickness go to zero and only afterwards take the limit in the width. This is precisely the approach in the articles [5, 6] mentioned earlier.

Regarding the asymptotics where width and thickness converge to zero simultaneously, the essential 'ribbon' feature is that width h and thickness δ have different orders of magnitude. In contrast, when $\delta \sim h$, one obtains rod and string theories, which are not the topic of this article. There is a large body of literature addressing such theories, see, e.g., [1, 18, 19].

Ribbons in the sense introduced above and starting from three-dimensional nonlinear elasticity were studied in [8, 9]. It turns out that different asymptotic theories arise depending on the ratio between the elastic energy on one hand and the geometric length scales of the ribbon on the other.

In order be more precise, let us represent the midplane of the ribbon's reference configuration by

$$\sigma_h := (0, \ell) \times \Big(-\frac{h}{2}, \frac{h}{2}\Big),$$

so that the three-dimensional reference configuration of the ribbon is given by

$$\sigma_h \times \left(-\frac{\delta}{2}, \frac{\delta}{2}\right).$$

As mentioned earlier, we assume that $\delta \ll h$. We consider three-dimensional deformations

$$f : \sigma_h \times \left(-\frac{\delta}{2}, \frac{\delta}{2}\right) \to \mathbb{R}^3$$

and denote by ε^2 the scaling of their total elastic energy per unit volume, i.e.,

$$\frac{1}{h\delta} \int_{\sigma_h \times (-\frac{\delta}{2}, \frac{\delta}{2})} W(\nabla f(x))\, dx \sim \varepsilon^2.$$

The picture obtained in [8, 9] can be summarised as follows: The very high energy regime $\varepsilon \gg \delta$ was solved in [8], and the very low energy regime $\varepsilon \lesssim \delta^2$ was solved in [9]. More subtle effects are encountered in the intermediate regime

$$\delta^2 \ll \varepsilon \lesssim \delta,$$

on which our discussion focuses from now on. In this intermediate energy regime, the ratio $(\delta/h)^2$ turns out to constitute a threshold between qualitatively different theories. The low energy case $\varepsilon \ll (\delta/h)^2$ was solved in [8, 9]. It leads to 'rod'-like limiting theories. As explained in these papers, their techniques can no longer be applied to study the energy regime above the threshold $(\delta/h)^2$. Indeed, genuinely 'ribbon'-like asymptotic theories were conjectured, which preserve characteristic features of the original two-dimensional theory.

In the present paper we develop the techniques needed to address the energy regime $\varepsilon \gg (\delta/h)^2$. Within this regime we obtain the following results:

- For $\varepsilon \sim \delta$ we prove that the generalised Sadowsky model from [6] introduced earlier is indeed the correct Γ-limit arising from three-dimensional nonlinear elasticity.
- For $\varepsilon \ll \delta$ we prove that a 'small deformation' variant of the generalised Sadowsky model is the correct Γ-limit. It is the same limiting model as the one obtained in [6, 7] starting from Föppl-von Kármán theory for plates, which itself is a 'small deformation' variant of Kirchhoff's plate theory.

In hindsight, these rigorous '3d-1d' limits provide a justification to the sequential '3d-2d-1d' approach in which first thickness δ and then width h are sent to zero.

But in spite of leading to the same asymptotic functional, there are some fundamental differences between the sequential argument and the (mechanically more meaningful) simultaneous approach. Indeed, as explained earlier, the intermediate two-dimensional theory obtained in the first step (thickness $\delta \to 0$) of the sequential approach, namely Kirchhoff's plate theory, imposes an isometry constraint on the admissible deformations Y. Gauss' Theorema Egregium then implies that the second fundamental form A of Y must have zero Gauss curvature. While this constraint is no longer manifestly present in the thin-width limit, it shows up in the generalised Sadowsky functional as an energy contribution which penalises nonzero determinant. Indeed, in [5, 6] the generalised Sadowsky functional is essentially obtained as a relaxation of the functional

$$M \mapsto \begin{cases} \int_0^\ell |M(x_1)|^2\, dx_1 & \text{if } \det M = 0, \\ +\infty & \text{otherwise,} \end{cases}$$

on maps $M : (0, \ell) \to \mathbb{R}^{2\times 2}_{\mathrm{sym}}$, which enforces a hard constraint on the determinant.

In contrast to the sequential '3d-2d-1d' approach, in the rigorous '3d-1d' asymptotics there is no intermediate two-dimensional theory with such a hard constraint. The Gauss curvature constraint

$\det A = 0$ is no longer satisfied at any stage of the limiting process. Nevertheless it plays a fundament role and it must affect the asymptotic theory.

Capturing its influence is a key technical difficulty in obtaining a lower bound on the asymptotic energy. A first step to overcome it is to rewrite the essential contribution of the elastic strain as a Jacobian. Unfortunately, the usual compatibility between Jacobian structure and fine scale oscillations is spoiled due to the divergent aspect ratio ℓ/h of the midplane σ_h. A key observation is that only oscillations occurring on the length scale h are relevant. In order to keep track of them we borrow the tool of two-scale convergence from homogenization theory. It allows us to obtain a quantitative control on the deviation from an exact constraint. This control is enough to show that the generalised Sadowsky functional is a lower bound for the asymptotic energy. In this context, the threshold $(\delta/h)^2$ arises naturally.

In order to prove the optimality of the lower bound, one must construct three-dimensional deformations achieving the optimal asymptotic energy. A central ingredient is a construction related to the geometric notion of developable surfaces which are tangent to a curve on a surface, cf. [4]. In the regime $|\mu| < |\tau|$, deformations with optimal energy turn out to be wildly oscillating isometric immersions with divergent third derivatives, which resemble fine-scale corrugations. As mentioned earlier, such counter-intuitive deformations are beyond the scope of [27, 23, 16], where this sort of mechanically relevant behaviour is excluded a priori.

In closing, we mention that two-scale convergence was introduced in [22, 2] as a tool to study periodic homogenisation. More recently it has been employed in the context of thin elastic films, in problems involving the simultaneous homogenisation and thin film asymptotics. This programme has been pursued, e.g., in [21, 15, 14, 25, 3]. Notice, however, that the models studied in the present paper do not appear to be related to homogenisation in any obvious way.

The paper is organised as follows. In Section 2 we introduce the energy functionals, rescale them to a fixed domain and state the Γ-convergence results. In section 3 we recall the notion and the main properties of two-scale convergence and prove a theorem that characterises the two-scale limit of a bounded sequence of scaled gradients. In Section 4 we study the compactness properties of sequences with bounded energy and prove the Γ-liminf inequalities. In the last section we provide the recovery sequences.

Throughout the paper we will use the following notation: for every $M \in \mathbb{R}^{3\times 3}$ we denote by $\widetilde{M}$ the matrix in $\mathbb{R}^{2\times 2}$ defined by

$$\widetilde{M}_{\alpha\beta} := M_{\alpha\beta} \qquad \text{for } \alpha, \beta = 1, 2,$$

and by $\overline{M}$ the matrix in $\mathbb{R}^{3\times 2}$ given by the first two columns of M, i.e.,

$$\overline{M} := (Me_1|Me_2).$$

2. Setting of the problem and main results

We consider a sequence of parallelepipeds Ω_h of fixed length ℓ whose cross-sections are rectangles with side lengths h and δ_h. To model a ribbon we assume that

$$\lim_{h\to 0} \frac{\delta_h}{h} = 0.$$

For every $h > 0$ we denote the cross-section by

$$\omega_h := \Big\{ (z_2, z_3) : \ |z_2| < \frac{h}{2}, \, |z_3| < \frac{\delta_h}{2} \Big\},$$

and the projection of Ω_h onto the plane (z_1, z_2) by

$$\sigma_h := \Big\{ (z_1, z_2) : \ z_1 \in (0, \ell), \, |z_2| < \frac{h}{2} \Big\},$$

so that

$$\Omega_h := (0,\ell)\times\omega_h = \sigma_h \times \Big(\frac{-\delta_h}{2},\frac{\delta_h}{2}\Big) \subset \mathbb{R}^3.$$

Henceforth we shall refer to Ω_h as the reference configuration of the body and denote the elastic energy associated with a deformation $\hat f:\Omega_h\to\mathbb{R}^3$ by

$$\hat E^h(\hat f) := \int_{\Omega_h} W(\nabla \hat f(z))\,dz.$$

We assume that the stored energy density $W:\mathbb{R}^{3\times3}\to[0,+\infty]$ satisfies the following assumptions:

(H1) $W\in C^0(\mathbb{R}^{3\times3})$, W is of class C^2 in a neighborhood of $SO(3)$;

(H2) W is frame-indifferent, i.e., $W(F)=W(RF)$ for every $F\in\mathbb{R}^{3\times3}$ and $R\in SO(3)$;

(H3) there exists $C>0$ such that $W(F)\ge C\operatorname{dist}^2(F,SO(3))$ for every $F\in\mathbb{R}^{3\times3}$; moreover, $W(R)=0$ if $R\in SO(3)$.

To rewrite the energy over an h-independent domain, we set $\delta_1=1$ and define

$$\omega:=\omega_1,\qquad \sigma:=\sigma_1,\qquad \Omega:=\Omega_1,$$

and we let $p_h:\Omega\to\Omega_h$ be the scaling map defined by

$$p_h(x)=p_h(x_1,x_2,x_3)=(x_1,hx_2,\delta_h x_3).$$

We consider the scaled energy $E^h: W^{1,2}(\Omega;\mathbb{R}^3)\to[0,+\infty]$ defined by

$$E^h(f) := \frac{1}{h\delta_h}\hat E^h(\hat f)\qquad\text{with } \hat f = f\circ p_h^{-1}.$$

The scaled energy E^h takes the simple form

$$E^h(f)=\int_\Omega W(\nabla_h f(x))\,dx,$$

where the operator ∇_h is given by

$$\nabla_h f := \left(\partial_{x_1} f \,\Big|\, \frac{\partial_{x_2} f}{h} \,\Big|\, \frac{\partial_{x_3} f}{\delta_h}\right) = \nabla\hat f\circ p_h$$

and $\partial_{x_i} f$ denotes the column vector of the partial derivative of f with respect to x_i, $i=1,2,3$.

Let (ε_h) be a sequence of positive numbers and let $(f^h)\subset W^{1,2}(\Omega;\mathbb{R}^3)$ be a sequence of deformations such that

$$E^h(f^h)\le C\varepsilon_h^2\qquad\text{for every } h>0$$

for some uniform constant $C>0$. As explained in the introduction, we are interested in the 'critical' regime

$$\lim_{h\to0}\frac{\varepsilon_h}{\delta_h}=1$$

and in the 'super-critical' regime determined by the conditions

$$\lim_{h\to0}\frac{\varepsilon_h}{\delta_h}=0\qquad\text{and}\qquad \lim_{h\to0}\frac{\varepsilon_h}{\delta_h^2}=+\infty.$$

To characterise the Γ-limits we will need the quadratic forms $Q_3:\mathbb{R}^{3\times3}\to[0,+\infty)$ defined by

$$Q_3(F):=\frac{\partial^2 W}{\partial F^2}(I)(F,F)=\sum_{i,j,k,l=1}^{3}\frac{\partial^2 W}{\partial F_{ij}\partial F_{kl}}(I)F_{ij}F_{kl}$$

for every $F\in\mathbb{R}^{3\times3}$, and $Q:\mathbb{R}^{2\times2}\to[0,+\infty)$ defined by

$$Q(M):=\min\Big\{\frac12 Q_3(F):\ \widetilde F = M\Big\} \tag{2.1}$$

for every $M \in \mathbb{R}^{2\times 2}$; according to the notation introduced above, $\widetilde{F}$ denotes the 2×2 principal minor of the matrix F corresponding to the first two lines and columns.

In order to take into account the determinant constraint we introduce the function $Q_{\rm ext} : \mathbb{R}^{2\times 2}_{\rm sym} \to [0, +\infty)$ defined by

$$Q_{\rm ext}(M) := \begin{cases} Q(M) & \text{if } \det M = 0, \\ +\infty & \text{otherwise,} \end{cases} \tag{2.2}$$

for every $M \in \mathbb{R}^{2\times 2}_{\rm sym}$ and we denote its convex envelope by $Q^{**}_{\rm ext}$. We further define $\overline{Q} : \mathbb{R}^2 \to [0, +\infty)$ as

$$\overline{Q}(\alpha, \beta) := \min_{\gamma\in\mathbb{R}} Q^{**}_{\rm ext}\left(\begin{pmatrix} \alpha & \beta \\ \beta & \gamma \end{pmatrix}\right) \tag{2.3}$$

for every $(\alpha, \beta) \in \mathbb{R}^2$.

Remark 2.1. The limiting energy density $\overline{Q}$ is defined through a two-steps procedure: first, a convexification of the function $Q_{\rm ext}$ (leading to $Q^{**}_{\rm ext}$) and then a minimisation of $Q^{**}_{\rm ext}$ in one of its arguments. The order in which these two steps are taken is irrelevant. In other words, if we define

$$\hat{Q}(\alpha, \beta) := \min_{\gamma\in\mathbb{R}} Q_{\rm ext}\begin{pmatrix} \alpha & \beta \\ \beta & \gamma \end{pmatrix}$$

for every $(\alpha, \beta) \in \mathbb{R}^2$, then $\hat{Q}^{**} = \overline{Q}$.

Indeed, for every $\alpha, \beta, \gamma \in \mathbb{R}$ we have

$$\overline{Q}(\alpha, \beta) \le Q^{**}_{\rm ext}\begin{pmatrix} \alpha & \beta \\ \beta & \gamma \end{pmatrix} \le Q_{\rm ext}\begin{pmatrix} \alpha & \beta \\ \beta & \gamma \end{pmatrix},$$

hence $\overline{Q} \le \hat{Q}$. Since $\overline{Q}$ is convex, we deduce that $\overline{Q} \le \hat{Q}^{**}$.

Conversely, for every $\alpha, \beta, \gamma \in \mathbb{R}$ we have

$$\hat{Q}^{**}(\alpha, \beta) \le \hat{Q}(\alpha, \beta) \le Q_{\rm ext}\begin{pmatrix} \alpha & \beta \\ \beta & \gamma \end{pmatrix}.$$

Since $\hat{Q}^{**}$ is convex, we deduce that

$$\hat{Q}^{**}(\alpha, \beta) \le Q^{**}_{\rm ext}\begin{pmatrix} \alpha & \beta \\ \beta & \gamma \end{pmatrix},$$

for every $\gamma \in \mathbb{R}$, hence $\hat{Q}^{**} \le \overline{Q}$.

If the elastic energy density W is isotropic, then

$$Q_3(F) = 2\mu|\operatorname{sym} F|^2 + \lambda(\operatorname{tr} F)^2 \qquad \text{for every } F \in \mathbb{R}^{3\times 3},$$

where $\lambda, \mu \in \mathbb{R}$ are the Lamé constants. In this case one can show (see, e.g., [6, Example 8]) that

$$Q^{**}_{\rm ext}(M) = \frac{2\mu(\lambda+\mu)}{\lambda+2\mu}\big(|M|^2 + 2|\det M|\big) \qquad \text{for every } M \in \mathbb{R}^{2\times 2}_{\rm sym}$$

and

$$\overline{Q}(\alpha, \beta) = \begin{cases} \dfrac{2\mu(\lambda+\mu)}{\lambda+2\mu}\,\dfrac{(\alpha^2+\beta^2)^2}{\alpha^2} & \text{if } |\alpha| \ge |\beta|, \\[2ex] \dfrac{8\mu(\lambda+\mu)}{\lambda+2\mu}\beta^2 & \text{if } |\alpha| < |\beta| \end{cases}$$

for every $(\alpha, \beta) \in \mathbb{R}^2$.

In the critical regime the domain of the Γ-limit is given by the class

$$\mathcal{A} := \{(d_1 \,|\, d_2 \,|\, d_3) \in W^{1,2}(0, \ell; SO(3)) : \; d_1' \cdot d_2 = 0 \text{ a.e. in } (0, \ell)\}.$$

The vector fields d_1, d_2, and d_3 are usually called directors: if $f:(0,\ell)\to\mathbb{R}^3$ denotes the deformation, then d_1 satisfies $d_1=f'$, that is, d_1 is tangent to the deformation, d_2 represents the transversal orientation of the ribbon, and d_3 is normal to the ribbon. The limit problem describes an inextensible rod (because of the constraint $f'=d_1$), which cannot bend within the plane of the ribbon (because of the constraint $d_1'\cdot d_2=0$). The energy depends on the out-of-plane curvature $d_1'\cdot d_3$ and on the torsion $d_2'\cdot d_3$. More precisely, we have the following result.

Theorem 2.2 (Critical regime)**.** *Let (ε_h) be a sequence of positive numbers such that*

$$\lim_{h\to 0}\frac{\varepsilon_h}{\delta_h}=1 \tag{2.4}$$

and

$$\lim_{h\to 0}\frac{\delta_h^2}{h^2\varepsilon_h}=0.$$

(i) Γ*-liminf: Let $(f^h)\subset W^{1,2}(\Omega;\mathbb{R}^3)$ be a sequence of deformations satisfying*

$$E^h(f^h)\le C\varepsilon_h^2 \qquad \text{for every } h>0 \tag{2.5}$$

for some uniform constant $C>0$. Then there exist $(d_1\,|\,d_2\,|\,d_3)\in\mathcal{A}$ and $f\in W^{2,2}(0,\ell;\mathbb{R}^3)$ with $f'=d_1$ a.e. in $(0,\ell)$, such that, up to a subsequence, $f^h\rightharpoonup f$ weakly in $W^{1,2}(\Omega;\mathbb{R}^3)$, $\nabla_h f^h\rightharpoonup(d_1\,|\,d_2\,|\,d_3)$ weakly in $L^2(\Omega;\mathbb{R}^{3\times3})$, and

$$\liminf_{h\to0}\frac{1}{\varepsilon_h^2}E^h(f^h)\ge\frac{1}{12}\int_0^\ell\overline{Q}(d_1'\cdot d_3,d_2'\cdot d_3)\,dx_1.$$

(ii) Γ*-limsup: For every $(d_1\,|\,d_2\,|\,d_3)\in\mathcal{A}$ and every $f\in W^{2,2}(0,\ell;\mathbb{R}^3)$ with $f'=d_1$ a.e. in $(0,\ell)$, there exists a sequence $(f^h)\subset W^{1,2}(\Omega;\mathbb{R}^3)$ such that $f^h\rightharpoonup f$ weakly in $W^{1,2}(\Omega;\mathbb{R}^3)$, $\nabla_h f^h\rightharpoonup(d_1\,|\,d_2\,|\,d_3)$ weakly in $L^2(\Omega;\mathbb{R}^{3\times3})$, and*

$$\limsup_{h\to0}\frac{1}{\varepsilon_h^2}E^h(f^h)\le\frac{1}{12}\int_0^\ell\overline{Q}(d_1'\cdot d_3,d_2'\cdot d_3)\,dx_1.$$

Part *(i)* of Theorem 2.2 will be proved in Proposition 4.7 and part *(ii)* in Proposition 5.5. We will see that the condition $\delta_h^2/(h^2\varepsilon_h)\to0$ is not needed in the proof of part *(ii)*.

In the supercritical regime the domain of the Γ-limit involves the class of displacements

$$\mathcal{A}^\infty:=\Big\{u\in W^{2,2}(0,\ell;\mathbb{R}^3):\ u_1'=-\frac12(u_3')^2 \text{ and } u_2=0\Big\}.$$

The constraint $u_1'=-\frac12(u_3')^2$ is a condition of inextensibility for the rod. The limit energy depends on the linearised out-of-plane curvature u_3'' and on the linearised torsion ϑ', where ϑ is the twist function of the cross-section orthogonal to the rod axis. More precisely, we have the following result.

Theorem 2.3 (Super-critical regime)**.** *Let (ε_h) be a sequence of positive numbers such that*

$$\lim_{h\to0}\frac{\varepsilon_h}{\delta_h}=0,\qquad \lim_{h\to0}\frac{\varepsilon_h}{\delta_h^2}=+\infty, \tag{2.6}$$

and

$$\lim_{h\to0}\frac{\delta_h^2}{h^2\varepsilon_h}=0.$$

(i) Γ*-liminf: Let $(f^h)\subset W^{1,2}(\Omega;\mathbb{R}^3)$ be a sequence of deformations satisfying* (2.5)*. Then there exist rotations $Q^h\in SO(3)$ and constants $c^h\in\mathbb{R}$ such that, setting $\bar f^h:=(Q^h)^Tf^h-c^h$, up to a subsequence,*

$$\bar f^h\to x_1e_1 \quad \text{in } W^{1,2}(\Omega;\mathbb{R}^3) \qquad \text{and} \qquad \nabla_h\bar f^h\to I \quad \text{in } L^2(\Omega;\mathbb{R}^{3\times3}).$$

Moreover, setting

$$u_1^h := \frac{\bar f_1^h - x_1}{(\varepsilon_h/\delta_h)^2}, \qquad u_2^h := \frac{\bar f_2^h - hx_2}{\varepsilon_h/\delta_h}, \qquad u_3^h := \frac{\bar f_3^h - \delta_h x_3}{\varepsilon_h/\delta_h}, \tag{2.7}$$

$$\vartheta^h := \frac{6}{h\varepsilon_h}\int_\omega (\delta_h x_2 \bar f_3^h - hx_3 \bar f_2^h)\, dx_2 dx_3, \tag{2.8}$$

we have that there exist $u \in \mathcal{A}^\infty$ and $\vartheta \in W^{1,2}(0,\ell)$ such that, up to a subsequence, $u^h \rightharpoonup u$ weakly in $W^{1,2}(\Omega;\mathbb{R}^3)$ and $\vartheta^h \rightharpoonup \vartheta$ weakly in $W^{1,2}(0,\ell)$, and

$$\liminf_{h\to 0} \frac{1}{\varepsilon_h^2} E^h(f^h) \geq \frac{1}{12}\int_0^\ell \overline{Q}(u_3'', \vartheta')\, dx_1.$$

(ii) Γ*-limsup: For every $u \in \mathcal{A}^\infty$ and every $\vartheta \in W^{1,2}(0,\ell)$, there exists a sequence $(f^h) \subset W^{1,2}(\Omega;\mathbb{R}^3)$ such that $f^h \to x_1 e_1$ in $W^{1,2}(\Omega;\mathbb{R}^3)$ and $\nabla_h f^h \to I$ in $L^2(\Omega;\mathbb{R}^{3\times 3})$. Moreover, defining u^h and ϑ^h as in* (2.7) *and* (2.8) *(with $\bar f^h$ replaced by f^h), we have that $u^h \rightharpoonup u$ weakly in $W^{1,2}(\Omega;\mathbb{R}^3)$, $\vartheta^h \rightharpoonup \vartheta$ weakly in $W^{1,2}(0,\ell)$, and*

$$\limsup_{h\to 0} \frac{1}{\varepsilon_h^2} E^h(f^h) \leq \frac{1}{12}\int_0^\ell \overline{Q}(u_3'', \vartheta')\, dx_1.$$

Part *(i)* of Theorem 2.3 will be proved in Proposition 4.8 and part *(ii)* in Proposition 5.6. Again, the condition $\delta_h^2/(h^2\varepsilon_h) \to 0$ is not needed in the proof of part *(ii)*.

3. Two-scale convergence

Following the notation introduced at the end of the introduction, we set

$$\overline{\nabla}_h := \Big(\partial_{x_1}\Big|\frac{\partial_{x_2}}{h}\Big).$$

In order to capture the asymptotic behaviour of the strain, we will apply the tool of two-scale convergence. We recall here the definition and some of its properties, and we refer to [2, 22, 21] for more details.

Let $Y := [0,1)$ and $N \in \mathbb{N}$. For every $h > 0$ the partial unfolding operator $\mathcal{T}_h$ is defined as follows: for every Lebesgue measurable function $\varphi : \Omega \to \mathbb{R}^N$ we define $\mathcal{T}_h(\varphi) : \Omega \times Y \to \mathbb{R}^N$ as

$$\mathcal{T}_h(\varphi)(x, y_1) := \begin{cases} \varphi(h[x_1/h] + hy_1, x_2, x_3) & \text{if } (h[x_1/h] + hY, x_2, x_3) \subset \Omega, \\ 0 & \text{otherwise,} \end{cases}$$

for $(x, y_1) \in \Omega \times Y$.

If φ is a differentiable function, we have that

$$\mathcal{T}_h(\partial_{x_1}\varphi) = \frac{1}{h}\partial_{y_1}\mathcal{T}_h(\varphi) \qquad \text{and} \qquad \mathcal{T}_h(\partial_{x_i}\varphi) = \partial_{x_i}\mathcal{T}_h(\varphi) \text{ for } i = 2, 3. \tag{3.1}$$

Hereafter, we shall denote by

$$\nabla := (\partial_{y_1}|\partial_{x_2}|\partial_{x_3}) \qquad \text{and} \qquad \overline{\nabla} := (\partial_{y_1}|\partial_{x_2}).$$

With this notation, by (3.1) we have that

$$\mathcal{T}_h(\overline{\nabla}_h\varphi) = \frac{1}{h}\overline{\nabla}\mathcal{T}_h(\varphi). \tag{3.2}$$

Our use of the unfolding operator, and the effectiveness of two-scale convergence, relies on the identity above: the unfolding operator transforms a scaled gradient into an ordinary gradient.

Let $1 \leq p < \infty$ and let (φ^h) be a sequence in $L^p(\Omega;\mathbb{R}^N)$. We say that φ^h strongly two-scale converges to φ in $L^p(\Omega \times Y;\mathbb{R}^N)$, as $h \to 0$, and we write

$$\varphi^h \xrightarrow{2} \varphi \quad \text{in } L^p(\Omega \times Y;\mathbb{R}^N),$$

if $\mathcal{T}_h(\varphi^h)$ strongly converges to φ in $L^p(\Omega\times Y;\mathbb{R}^N)$. Analogously, we say that φ^h weakly two-scale converges to φ in $L^p(\Omega\times Y;\mathbb{R}^N)$, and we write

$$\varphi^h \overset{2}{\rightharpoonup} \varphi \quad \text{weakly in } L^p(\Omega\times Y;\mathbb{R}^N),$$

if $\mathcal{T}_h(\varphi^h)$ weakly converges to φ in $L^p(\Omega\times Y;\mathbb{R}^N)$.

We note that $\mathcal{T}_h(\varphi\psi)=\mathcal{T}_h(\varphi)\mathcal{T}_h(\psi)$ and recall that

- $\|\mathcal{T}_h(\varphi)\|_{L^p}\le\|\varphi\|_{L^p}$ for every $h>0$;
- if $\varphi^h\to\varphi$ in $L^p(\Omega;\mathbb{R}^3)$, then $\varphi^h \overset{2}{\longrightarrow} \varphi$ in $L^p(\Omega\times Y;\mathbb{R}^3)$;
- if $p>1$ and $\sup_h\|\varphi^h\|_{L^p}<+\infty$, then there exists a subsequence of (φ^h) that converges weakly two-scale.

Hereafter, we set

$$\mathcal{H}:=\{v\in W^{1,2}((-1/2,1/2)^2\times Y;\mathbb{R}^3):\ v(\cdot,\cdot,0)=v(\cdot,\cdot,1)\}. \tag{3.3}$$

The following theorem characterises the two-scale limit of a bounded sequence of scaled gradients.

Theorem 3.1. *Let $\varphi^h,\varphi\in W^{1,2}(\Omega;\mathbb{R}^3)$ be such that $\varphi^h\rightharpoonup\varphi$ in $W^{1,2}(\Omega;\mathbb{R}^3)$ and*

$$\sup_h\|\nabla_h\varphi^h\|_{L^2}<+\infty. \tag{3.4}$$

Then there exists a function $\varphi^1\in L^2(0,\ell;\mathcal{H})$ such that, up to a subsequence,

$$\overline{\nabla}_h\varphi^h \overset{2}{\rightharpoonup} \partial_{x_1}\varphi\otimes e_1+\overline{\nabla}\varphi^1 \quad \textit{weakly in } L^2(\Omega\times Y;\mathbb{R}^{3\times2}). \tag{3.5}$$

Moreover,

$$\partial_{x_2}\varphi=\partial_{x_3}\varphi=0 \qquad \textit{and} \qquad \partial_{x_3}\varphi^1=0.$$

Proof. Set

$$z^h:=\frac{1}{h}\Big(\mathcal{T}_h(\varphi^h)-\int_{-1/2}^{1/2}\int_{-1/2}^{1/2}\int_0^1\mathcal{T}_h(\varphi^h)\,dy_1dx_2dx_3\Big).$$

Then, by (3.1) and (3.2), we have

$$\overline{\nabla}z^h=\mathcal{T}_h(\overline{\nabla}_h\varphi^h) \qquad \text{and} \qquad \partial_{x_3}z^h=\frac{\delta_h}{h}\mathcal{T}_h\Big(\frac{1}{\delta_h}\partial_{x_3}\varphi^h\Big).$$

By (3.4) and a partial Poincaré's inequality (see, e.g., [10, Theorem 4.1]) we have that, up to a subsequence,

$$z^h\rightharpoonup z \quad \text{in } L^2(0,\ell;W^{1,2}((-1/2,1/2)^2\times Y;\mathbb{R}^3))$$

and, since $\delta_h/h\to0$, we deduce that $\partial_{x_3}z=0$. In particular, this implies that

$$\overline{\nabla}_h\varphi^h \overset{2}{\rightharpoonup} \overline{\nabla}z \quad \text{in } L^2(\Omega\times Y;\mathbb{R}^{3\times2}).$$

That φ depends only on x_1 follows immediately from (3.4). Denoting by

$$\varphi^1(x,y_1):=z(x,y_1)-\Big(y_1-\frac12\Big)\partial_{x_1}\varphi(x_1)$$

we obtain (3.5) and $\partial_{x_3}\varphi^1=0$.

To conclude the proof it remains to show that φ^1 is periodic in the variable y_1. Let $\psi \in C_0^\infty(\Omega)$. Then, for every h small enough, we have that

$$\begin{aligned}\int_\Omega \big(z^h(x,1) - z^h(x,0)\big)\psi(x)\,dx &= \int_\Omega \frac{\mathcal{T}_h(\varphi^h)(x,1) - \mathcal{T}_h(\varphi^h)(x,0)}{h}\psi(x)\,dx \\ &= \int_\Omega \frac{\varphi^h(h[x_1/h]+h, x_2, x_3) - \varphi^h(h[x_1/h], x_2, x_3)}{h}\psi(x)\,dx \\ &= \int_\Omega \varphi^h(h[x_1/h], x_2, x_3)\frac{\psi(x_1-h, x_2, x_3) - \psi(x)}{h}\,dx \\ &= \int_\Omega \mathcal{T}_h(\varphi^h)(x,0)\frac{\psi(x_1-h, x_2, x_3) - \psi(x)}{h}\,dx,\end{aligned}$$

and by passing to the limit we obtain

$$\int_\Omega \big(z(x,1) - z(x,0)\big)\psi(x)\,dx = -\int_\Omega \varphi(x_1)\partial_{x_1}\psi(x)\,dx = \int_\Omega \partial_{x_1}\varphi(x_1)\psi(x)\,dx,$$

where we used the fact that $\mathcal{T}_h(\varphi^h) \to \varphi$ in $L^2(0,\ell; W^{1,2}((-1/2,1/2)^2 \times Y; \mathbb{R}^3))$, as it can be easily checked by means of (3.1). Since ψ is arbitrary, the last identity implies that

$$z(x,1) - z(x,0) = \partial_{x_1}\varphi(x_1)$$

which is equivalent to $\varphi^1(x,0) = \varphi^1(x,1)$. □

4. Compactness and lower bounds

In this section we prove compactness and the Γ-liminf inequalities. These results are based on a characterisation of the limiting behaviour of the nonlinear three-dimensional strains. By the rigidity estimate we first deduce that deformations whose energy is of order ε_h^2, are close to isometries (Lemma 4.1). We then apply two-scale convergence to capture the correct asymptotics of the zero Gauss curvature constraint (Lemma 4.3). This is crucial to deduce the optimal lower bounds (Propositions 4.7 and 4.8).

We begin by introducing a useful decomposition of the three-dimensional deformations.

Lemma 4.1. *Let $(f^h) \subset W^{1,2}(\Omega;\mathbb{R}^3)$ satisfy the energy bound (2.5). Assume that either (2.4) or (2.6) holds. Then there exists $(R^h) \subset C^\infty(\overline{\sigma}; SO(3))$ such that, for every $h > 0$,*

$$\|\nabla_h f^h - R^h\|_{L^2} \le C\varepsilon_h, \tag{4.1}$$

$$\|\partial_{x_1} R^h\|_{L^2} \le C\frac{\varepsilon_h}{\delta_h}, \qquad \|\partial_{x_2} R^h\|_{L^2} \le Ch\frac{\varepsilon_h}{\delta_h}. \tag{4.2}$$

Moreover, setting

$$\check{f}^h(x_1,x_2) := \int_{-1/2}^{1/2} f^h(x_1,x_2,x_3)\,dx_3 \qquad \text{and} \qquad w^h := f^h - \check{f}^h - \delta_h x_3 R^h e_3,$$

we have

$$\|w^h\|_{L^2} \le C\varepsilon_h\delta_h \qquad \text{and} \qquad \|\nabla_h w^h\|_{L^2} \le C\varepsilon_h. \tag{4.3}$$

Finally, under assumption (2.6) we also have that, up to a frame rotation,

$$\|R^h - I\|_{L^2} \le C\frac{\varepsilon_h}{\delta_h}. \tag{4.4}$$

Proof. For the proof of (4.1), (4.2), and (4.4) see [9, Lemma 3.1].

We now prove (4.3). Since

$$f^h = \check{f}^h + \delta_h x_3 R^h e_3 + w^h, \tag{4.5}$$

we have that

$$\frac{\partial_{x_3} f^h}{\delta_h} = R^h e_3 + \frac{\partial_{x_3} w^h}{\delta_h},$$

which leads, by means of (4.1), to

$$\Big\| \frac{\partial_{x_3} w^h}{\delta_h} \Big\|_{L^2} \leq C\varepsilon_h. \tag{4.6}$$

By definition

$$\int_{-1/2}^{1/2} w^h(x_1, x_2, x_3)\, dx_3 = 0,$$

and hence, by a partial Poincaré inequality (see, e.g., [10, Theorem 4.1]) we obtain the first estimate in (4.3).

By differentiating (4.5) we get

$$\overline{\nabla}_h f^h = \overline{\nabla}_h \check{f}^h + \delta_h x_3 \overline{\nabla}_h (R^h e_3) + \overline{\nabla}_h w^h,$$

hence

$$\overline{\nabla}_h w^h = \overline{\nabla}_h f^h - \bar{R}^h - (\overline{\nabla}_h \check{f}^h - \bar{R}^h) - \delta_h x_3 \Big(\partial_1 R^h e_3 \Big| \frac{\partial_2 R^h}{h} e_3 \Big). \tag{4.7}$$

By Jensen's inequality and Lemma 4.1, we have

$$\|\overline{\nabla}_h \check{f}^h - \bar{R}^h\|_{L^2(\omega)} \leq \|\overline{\nabla}_h f^h - \bar{R}^h\|_{L^2(\Omega)} \leq C\varepsilon_h,$$

and from this inequality, Lemma 4.1, and (4.7), it follows that

$$\|\overline{\nabla}_h w^h\|_{L^2} \leq C\varepsilon_h$$

and therefore, taking into account (4.6), we obtain the second estimate in (4.3). □

We introduce the strain tensor G^h defined as

$$G^h := \frac{(R^h)^T \nabla_h f^h - I}{\varepsilon_h}. \tag{4.8}$$

The definition of w^h, given in Lemma 4.1, allows us to decompose the strain G^h into three components. Indeed, the scaled gradient of f^h can be written as

$$\begin{aligned} \nabla_h f^h &= \Big(\overline{\nabla}_h f^h \Big| \frac{\partial_{x_3} f^h}{\delta_h} \Big) \\ &= (\overline{\nabla}_h \check{f}^h + \delta_h x_3 \overline{\nabla}_h (R^h e_3) + \overline{\nabla}_h w^h | R^h e_3 + \partial_{x_3} w^h/\delta_h) \\ &= (\overline{\nabla}_h \check{f}^h | R^h e_3) + (\delta_h x_3 \overline{\nabla}_h (R^h e_3) | 0) + \nabla_h w^h. \end{aligned}$$

From this equality we deduce that G^h admits the following decomposition:

$$G^h = G^h_{\mathrm{I}} + x_3 G^h_{\mathrm{II}} + G^h_{\mathrm{III}}, \tag{4.9}$$

where

$$\begin{aligned} G^h_{\mathrm{I}} &:= \frac{(R^h)^T (\overline{\nabla}_h \check{f}^h | R^h e_3) - I}{\varepsilon_h}, \\ G^h_{\mathrm{II}} &:= \frac{\delta_h}{\varepsilon_h} (R^h)^T (\overline{\nabla}_h (R^h e_3) | 0), \\ G^h_{\mathrm{III}} &:= \frac{(R^h)^T \nabla_h w^h}{\varepsilon_h}. \end{aligned}$$

We note that G^h_{I} and G^h_{II} are independent of x_3. This decomposition will play an important role in the identification of a lower bound for the Γ-limit.

We start by characterising the limit behaviour of the sequence of rotations (R^h) introduced in Lemma 4.1. To keep the notation compact, hereafter, we denote the i-th column of the generic matrix F by

$$F_i := Fe_i$$

for $i = 1, 2, 3$.

Lemma 4.2. *Assume that the hypotheses of Lemma 4.1 are satisfied and let* $(R^h) \subset C^\infty(\overline{\sigma}; SO(3))$ *be as in the conclusion of that lemma. Then there exist* $R \in W^{1,2}(0,\ell; SO(3))$ *and* $A \in W^{1,2}(0,\ell;\mathbb{R}^{3\times 3}_{\rm skew})$ *such that, up to a subsequence,*

$$R^h \rightharpoonup R \quad \textit{weakly in } W^{1,2}(\sigma;\mathbb{R}^{3\times 3}), \qquad \frac{R^h - I}{\varepsilon_h/\delta_h} \rightharpoonup A \quad \textit{weakly in } W^{1,2}(\sigma;\mathbb{R}^{3\times 3}).$$

If (2.4) *holds, then* $A = R - I$*; if* (2.6) *holds, then* $R = I$*. Moreover, with* $\mathcal{H}$ *as in* (3.3)*, for* $\alpha = 1, 2$ *there exists* $A^1_\alpha \in L^2(0,\ell;\mathcal{H})$ *with* $\partial_{x_3} A^1_\alpha = 0$ *and such that, up to a subsequence,*

$$\frac{\delta_h}{\varepsilon_h}\overline{\nabla}_h R^h_\alpha \xrightarrow{2} \partial_{x_1} A_\alpha \otimes e_1 + \overline{\nabla} A^1_\alpha \quad \textit{weakly in } L^2(\Omega\times Y;\mathbb{R}^{3\times 2}).$$

Proof. The first two convergence results follow directly from the estimates of Lemma 4.1. The statement on two-scale convergence is a consequence of Theorem 3.1. □

The following key lemma characterises the limiting behaviour of the strain tensors introduced in (4.9). To this aim we introduce the following space:

$$\mathcal{H}^2 := \Big\{ g \in W^{2,2}((-1/2,1/2)\times Y) : \ g(\cdot,0) = g(\cdot,1), \ \overline{\nabla} g(\cdot,0) = \overline{\nabla} g(\cdot,1), \\ \int_{(-1/2,1/2)\times Y} \partial_{x_2}\partial_{x_2} g \, dx_2 dy_1 = 0 \Big\}.$$

Lemma 4.3. *Under the assumptions of Lemma 4.2 and with* A *and* R *as in the conclusion of that lemma, we have that, up to a subsequence,*

$$\widetilde{G}^h_{\rm I} \xrightarrow{2} \widetilde{G}_{\rm I}, \qquad \widetilde{G}^h_{\rm II} \xrightarrow{2} \widetilde{G}_{\rm II}, \qquad \widetilde{G}^h_{\rm III} \xrightarrow{2} 0 \quad \textit{weakly in } L^2(\Omega\times Y;\mathbb{R}^{2\times 2}).$$

The map $\widetilde{G}_{\rm I}$ *does not depend on* x_3 *and there exist* $\gamma \in L^2(0,\ell)$ *and* $g \in L^2(0,\ell;\mathcal{H}^2)$ *such that*

$$\widetilde{G}_{\rm II} := -\begin{pmatrix} \partial_{x_1} A_1 \cdot R_3 & \partial_{x_1} A_2 \cdot R_3 \\ \partial_{x_1} A_2 \cdot R_3 & \gamma \end{pmatrix} - \overline{\nabla}^2 g,$$

where $\overline{\nabla}^2 g$ *denotes the Hessian of* g *with respect to the variables* y_1 *and* x_2*. Moreover, we have that*

$$\mathcal{T}_h(\det \widetilde{G}^h_{\rm II}) \rightharpoonup \det \widetilde{G}_{\rm II} \tag{4.10}$$

in the sense of distributions and, if

$$\lim_{h\to 0} \frac{\delta_h^2}{h^2 \varepsilon_h} = 0, \tag{4.11}$$

then

$$\det \widetilde{G}_{\rm II} = 0 \quad \textit{a.e. in } \Omega\times Y. \tag{4.12}$$

Proof. By Lemma 4.1 the sequences $(\widetilde{G}^h_{\rm I})$, $(\widetilde{G}^h_{\rm II})$, and $(\widetilde{G}^h_{\rm III})$ are bounded in $L^2(\Omega;\mathbb{R}^{3\times 3})$; therefore, up to a subsequence, they admit a weak two-scale limit. Since $G^h_{\rm I}$ does not depend on x_3 for every h, the same is true for its two-scale limit $\widetilde{G}_{\rm I}$.

The second bound in (4.3) implies that there exists $\overline{W} \in L^2(\Omega\times Y;\mathbb{R}^{3\times 2})$ such that, up to a subsequence,

$$\overline{\nabla}_h(w^h/\varepsilon_h) \xrightarrow{2} \overline{W} \quad \text{weakly in } L^2(\Omega\times Y;\mathbb{R}^{3\times 2}).$$

On the other hand, for $\psi \in C_0^\infty(\Omega \times Y)$ we have by (3.2) that

$$\begin{aligned}\Big|\int_{\Omega\times Y}\mathcal{T}_h(\overline{\nabla}_h(w^h/\varepsilon_h))\psi\,dxdy_1\Big| &= \Big|\int_{\Omega\times Y}\frac{1}{h}\overline{\nabla}\mathcal{T}_h(w^h/\varepsilon_h)\psi\,dxdy_1\Big|\\ &= \Big|\int_{\Omega\times Y}\frac{1}{h}\mathcal{T}_h(w^h/\varepsilon_h)\otimes\overline{\nabla}\psi\,dxdy_1\Big|\\ &\le C\frac{\delta_h}{h},\end{aligned}$$

where the last inequality follows from the first bound in (4.3). Thus, $\overline{W}=0$. Since

$$\widetilde{G}^h_{\rm III} = (\overline{R}^h)^T\overline{\nabla}_h(w^h/\varepsilon_h),$$

and $\overline{R}^h \xrightarrow{2} \overline{R}$, we conclude that $\widetilde{G}^h_{\rm III} \xrightharpoonup{2} 0$ weakly in $L^2(\Omega\times Y;\mathbb{R}^{2\times 2})$.

By definition

$$\begin{aligned}\widetilde{G}^h_{\rm II} = \frac{\delta_h}{\varepsilon_h}(\overline{R}^h)^T\overline{\nabla}_h R^h_3 &= \frac{\delta_h}{\varepsilon_h}\begin{pmatrix} R^h_1\cdot\partial_{x_1}R^h_3 & R^h_1\cdot\partial_{x_2}R^h_3/h\\ R^h_2\cdot\partial_{x_1}R^h_3 & R^h_2\cdot\partial_{x_2}R^h_3/h\end{pmatrix}\\ &= -\frac{\delta_h}{\varepsilon_h}\begin{pmatrix}\partial_{x_1}R^h_1\cdot R^h_3 & \partial_{x_2}R^h_1\cdot R^h_3/h\\ \partial_{x_1}R^h_2\cdot R^h_3 & \partial_{x_2}R^h_2\cdot R^h_3/h\end{pmatrix}.\end{aligned} \tag{4.13}$$

Since $R^h_3 \to R_3$ in $L^2(\Omega;\mathbb{R}^3)$, we have that $R^h_3 \xrightarrow{2} R_3$ in $L^2(\Omega\times Y;\mathbb{R}^3)$. By Lemma 4.2 we can pass to the limit in the identity above and deduce that

$$\widetilde{G}_{\rm II} = -\begin{pmatrix}(\partial_{x_1}A_1+\partial_{y_1}A^1_1)\cdot R_3 & \partial_{x_2}A^1_1\cdot R_3\\ (\partial_{x_1}A_2+\partial_{y_1}A^1_2)\cdot R_3 & \partial_{x_2}A^1_2\cdot R_3\end{pmatrix}. \tag{4.14}$$

We now show that $\widetilde{G}_{\rm II}$ is symmetric. Indeed, note that

$$\partial_{x_1}R^h_2 - \frac{\partial_{x_2}R^h_1}{h} = \partial_{x_1}\Big(R^h_2 - \frac{\partial_{x_2}f^h}{h}\Big) - \frac{\partial_{x_2}(R^h_1-\partial_{x_1}f^h)}{h}$$

in the sense of distributions. Hence, for $\psi\in C_0^\infty(\Omega\times Y)$ we have that

$$\Big|\int_{\Omega\times Y}\Big(\mathcal{T}_h\Big(\frac{\delta_h}{\varepsilon_h}\partial_{x_1}R^h_2\Big)-\mathcal{T}_h\Big(\frac{\delta_h}{\varepsilon_h}\frac{\partial_{x_2}R^h_1}{h}\Big)\Big)\psi\,dxdy_1\Big| \le \frac{\delta_h}{h\varepsilon_h}\|\overline{R}^h-\overline{\nabla}_h f^h\|_{L^2}\|\overline{\nabla}\psi\|_{L^2}\le C\frac{\delta_h}{h},$$

owing to Lemma 4.1. From the above inequality it follows that the two-scale weak limits of $\frac{\delta_h}{\varepsilon_h}\partial_{x_1}R^h_2$ and $\frac{\delta_h}{\varepsilon_h}\frac{\partial_{x_2}R^h_1}{h}$ coincide. Hence, by Lemma 4.2 we have that

$$\partial_{x_1}A_2 + \partial_{y_1}A^1_2 = \partial_{x_2}A^1_1. \tag{4.15}$$

This identity and (4.14) imply that $\widetilde{G}_{\rm II}$ is symmetric.

Now, since A_2 is independent of x_2, we can rewrite (4.15) as

$$\partial_{y_1}A^1_2 = \partial_{x_2}\big(A^1_1 - x_2\partial_{x_1}A_2\big). \tag{4.16}$$

Taking into account that A^1_2 is Y-periodic we obtain that

$$\partial_{x_2}\int_Y (A^1_1 - x_2\partial_{x_1}A_2)\,dy_1 = 0. \tag{4.17}$$

Thus, (4.16) is equivalent to

$$\partial_{y_1}\Big(A^1_2 - x_2\int_{(-1/2,1/2)\times Y}\partial_{x_2}A^1_2\,dx_2dy_1\Big) = \partial_{x_2}\Big(A^1_1 - x_2\partial_{x_1}A_2 - \int_Y (A^1_1 - x_2\partial_{x_1}A_2)\,dy_1\Big).$$

Since the expressions above do not depend on x_3, by [13, Chapter I, Theorem 3.4] there exists a function $r \in L^2(0,\ell; W^{1,2}((-1/2,1/2)\times Y;\mathbb{R}^3))$ such that

$$A_2^1 - x_2 \int_{(-1/2,1/2)\times Y} \partial_{x_2} A_2^1 \, dx_2 dy_1 = \partial_{x_2} r, \tag{4.18}$$

$$A_1^1 - x_2 \partial_{x_1} A_2 - \int_Y (A_1^1 - x_2\partial_{x_1} A_2)\, dy_1 = \partial_{y_1} r. \tag{4.19}$$

From the above identities it immediately follows that $\overline{\nabla} r$ is Y-periodic. By integrating (4.19) we deduce that also r is Y-periodic, and by differentiating (4.18) we deduce that

$$\int_{\Omega\times Y} \partial_{x_2}\partial_{x_2} r \, dx dy_1 = 0.$$

We set

$$\gamma := R_3 \cdot \int_{(-1/2,1/2)\times Y} \partial_{x_2} A_2^1 \, dx_2 dy_1 \qquad \text{and} \qquad g := R_3 \cdot r,$$

so that $\gamma \in L^2(0,\ell)$ and $g \in L^2(0,\ell;\mathcal{H}^2)$. With these definitions, using (4.18) and (4.19) and recalling (4.17), we may rewrite (4.14) as

$$\widetilde{G}_{\mathrm{II}} = -\begin{pmatrix} \partial_{x_1} A_1 \cdot R_3 + \partial_{y_1}\partial_{y_1} g & \partial_{x_1} A_2 \cdot R_3 + \partial_{y_1}\partial_{x_2} g \\ \partial_{x_1} A_2 \cdot R_3 + \partial_{y_1}\partial_{x_2} g & \gamma + \partial_{x_2}\partial_{x_2} g \end{pmatrix},$$

which coincides with the expression given in the statement of the lemma.

We now prove the statements concerning the determinants. By (4.13) we deduce that

$$\begin{aligned} \frac{\varepsilon_h^2}{\delta_h^2} \det \widetilde{G}^h_{\mathrm{II}} &= (\partial_{x_1} R_1^h \cdot R_3^h)\Big(\frac{\partial_{x_2} R_2^h}{h} \cdot R_3^h\Big) - (\partial_{x_1} R_2^h \cdot R_3^h)\Big(\frac{\partial_{x_2} R_1^h}{h} \cdot R_3^h\Big) \\ &= \partial_{x_1} R_1^h \cdot \frac{\partial_{x_2} R_2^h}{h} - \partial_{x_1} R_2^h \cdot \frac{\partial_{x_2} R_1^h}{h}, \end{aligned} \tag{4.20}$$

where the second equality follows by noticing that

$$\partial_{x_1} R_1^h = (\partial_{x_1} R_1^h \cdot R_2^h) R_2^h + (\partial_{x_1} R_1^h \cdot R_3^h) R_3^h,$$
$$\partial_{x_2} R_2^h = (\partial_{x_2} R_2^h \cdot R_1^h) R_1^h + (\partial_{x_2} R_2^h \cdot R_3^h) R_3^h,$$

and thus

$$\partial_{x_1} R_1^h \cdot \partial_{x_2} R_2^h = (\partial_{x_1} R_1^h \cdot R_3^h)\,(\partial_{x_2} R_2^h \cdot R_3^h). \tag{4.21}$$

Similarly,

$$\partial_{x_1} R_2^h \cdot \partial_{x_2} R_1^h = (\partial_{x_1} R_2^h \cdot R_3^h)\,(\partial_{x_2} R_1^h \cdot R_3^h).$$

Therefore, identity (4.20) is equivalent to

$$\frac{\varepsilon_h^2}{\delta_h^2} \det \widetilde{G}^h_{\mathrm{II}} = \overline{\nabla}_h R_1^h \cdot \Big(\frac{\partial_{x_2} R_2^h}{h} \Big| - \partial_{x_1} R_2^h\Big),$$

so that

$$\mathcal{T}_h(\det \widetilde{G}^h_{\mathrm{II}}) = \mathcal{T}_h\Big(\frac{\delta_h}{\varepsilon_h} \overline{\nabla}_h R_1^h\Big) \cdot \mathcal{T}_h\Big(\frac{\delta_h}{\varepsilon_h}\Big(\frac{\partial_{x_2} R_2^h}{h}\Big| - \partial_{x_1} R_2^h\Big)\Big). \tag{4.22}$$

By (3.1) we obtain that

$$\begin{aligned} \mathrm{curl}_{(y_1,x_2)} \mathcal{T}_h\Big(\frac{\delta_h}{\varepsilon_h}\overline{\nabla}_h R_1^h\Big) &= \frac{\delta_h}{\varepsilon_h}\Big(\partial_{y_1}\mathcal{T}_h\Big(\frac{\partial_{x_2} R_1^h}{h}\Big) - \partial_{x_2}\mathcal{T}_h(\partial_{x_1} R_1^h)\Big) \\ &= \frac{\delta_h}{\varepsilon_h}\frac{\partial_{y_1}\partial_{x_2}\mathcal{T}_h(R_1^h) - \partial_{x_2}\partial_{y_1}\mathcal{T}_h(R_1^h)}{h} = 0 \end{aligned}$$

and

$$\operatorname{div}_{(y_1,x_2)}\mathcal{T}_h\Big(\frac{\delta_h}{\varepsilon_h}\Big(\frac{\partial_{x_2}R_2^h}{h}\Big| -\partial_{x_1}R_2^h\Big)\Big) = \frac{\delta_h}{\varepsilon_h}\left(\partial_{y_1}\mathcal{T}_h\Big(\frac{\partial_{x_2}R_2^h}{h}\Big) - \partial_{x_2}\mathcal{T}_h(\partial_{x_1}R_2^h)\right)$$
$$= \frac{\delta_h}{\varepsilon_h}\frac{\partial_{y_1}\partial_{x_2}\mathcal{T}_h(R_2^h) - \partial_{x_2}\partial_{y_1}\mathcal{T}_h(R_2^h)}{h} = 0$$

in the sense of distributions. On the other hand, by Lemma 4.2, we have that

$$\mathcal{T}_h\Big(\frac{\delta_h}{\varepsilon_h}\overline{\nabla}_h R_1^h\Big) \rightharpoonup \partial_{x_1}A_1 \otimes e_1 + \overline{\nabla}A_1^1 \quad \text{weakly in } L^2(\Omega\times Y;\mathbb{R}^{3\times 2})$$

and

$$\mathcal{T}_h\Big(\frac{\delta_h}{\varepsilon_h}\Big(\frac{\partial_{x_2}R_2^h}{h}\Big| -\partial_{x_1}R_2^h\Big)\Big) \rightharpoonup (\partial_{x_2}A_2^1| -\partial_{x_1}A_2 - \partial_{y_1}A_2^1) \quad \text{weakly in } L^2(\Omega\times Y;\mathbb{R}^{3\times 2}).$$

Therefore, we can apply the div-curl lemma (see, e.g., [20]) and pass to the limit in (4.22); in this way, we obtain that

$$\mathcal{T}_h(\det \widetilde{G}_{\mathrm{II}}^h) \rightharpoonup (\partial_{x_1}A_1 + \partial_{y_1}A_1^1)\cdot\partial_{x_2}A_2^1 - (\partial_{x_1}A_2 + \partial_{y_1}A_2^1)\cdot\partial_{x_2}A_1^1 \tag{4.23}$$

in the sense of distributions.

By passing to the limit in the identities

$$\frac{\delta_h}{\varepsilon_h}\partial_{x_1}R_1^h\cdot R_1^h = 0 \qquad \text{and} \qquad \frac{\delta_h}{\varepsilon_h}\frac{\partial_{x_2}R_2^h}{h}\cdot R_2^h = 0,$$

we deduce that

$$(\partial_{x_1}A_1 + \partial_{y_1}A_1^1)\cdot R_1 = 0 \qquad \text{and} \qquad \partial_{x_2}A_2^1\cdot R_2 = 0.$$

Hence, with an argument similar to that used to obtain (4.21) we find

$$(\partial_{x_1}A_1 + \partial_{y_1}A_1^1)\cdot\partial_{x_2}A_2^1 = ((\partial_{x_1}A_1 + \partial_{y_1}A_1^1)\cdot R_3)(\partial_{x_2}A_2^1\cdot R_3).$$

A similar identity holds for the second term in the right-hand side of (4.23). Thus, we can rewrite (4.23) as

$$\mathcal{T}_h(\det \widetilde{G}_{\mathrm{II}}^h) \rightharpoonup ((\partial_{x_1}A_1 + \partial_{y_1}A_1^1)\cdot R_3)(\partial_{x_2}A_2^1\cdot R_3) - ((\partial_{x_1}A_2 + \partial_{y_1}A_2^1)\cdot R_3)(\partial_{x_2}A_1^1\cdot R_3)$$

in the sense of distributions. By (4.14) this is equivalent to (4.10).

To conclude the proof of the lemma, we now show that

$$\Big|\int_{\Omega\times Y}\mathcal{T}_h(\det \widetilde{G}_{\mathrm{II}}^h)\psi\,dx dy_1\Big| \le C\frac{\delta_h^2}{h^2\varepsilon_h} \tag{4.24}$$

for every $\psi \in C_0^\infty(\Omega\times Y)$. This immediately implies (4.12) when (4.11) is satisfied.

Setting

$$\operatorname{curl}_h \overline{R}^h := \partial_{x_1}R_2^h - \frac{\partial_{x_2}R_1^h}{h},$$

and using the identity

$$\partial_{x_\gamma}R_\alpha^h\cdot R_\beta^h + R_\alpha^h\cdot\partial_{x_\gamma}R_\beta^h = 0$$

which holds for $\alpha,\beta,\gamma = 1,2$, we rewrite (4.20) as

$$\frac{\varepsilon_h^2}{\delta_h^2}\det \widetilde{G}_{\mathrm{II}}^h = \partial_{x_1}\Big(R_1^h\cdot\frac{\partial_{x_2}R_2^h}{h}\Big) - \frac{\partial_{x_2}(R_1^h\cdot\partial_{x_1}R_2^h)}{h}$$
$$= -\partial_{x_1}\Big(R_2^h\cdot\frac{\partial_{x_2}R_1^h}{h}\Big) - \frac{\partial_{x_2}(R_1^h\cdot\operatorname{curl}_h\overline{R}^h)}{h}$$
$$= \partial_{x_1}(R_2^h\cdot\operatorname{curl}_h\overline{R}^h) - \frac{\partial_{x_2}(R_1^h\cdot\operatorname{curl}_h\overline{R}^h)}{h}.$$

Thus, by applying (3.1) we obtain

$$\mathcal{T}_h(\det \widetilde{G}^h_{\rm II}) = \frac{\delta_h^2}{h\varepsilon_h^2}\Big(\partial_{y_1}\mathcal{T}_h(R^h_2\cdot \mathrm{curl}_h\,\overline{R}^h) - \partial_{x_2}\mathcal{T}_h(R^h_1\cdot \mathrm{curl}_h\,\overline{R}^h)\Big).$$

Hence, for $\psi\in C^\infty_0(\Omega\times Y)$ we have

$$\begin{aligned}
&\Big|\int_{\Omega\times Y}\mathcal{T}_h(\det \widetilde{G}^h_{\rm II})\psi\,dxdy_1\Big| \\
&\leq \frac{\delta_h^2}{h\varepsilon_h^2}\int_{\Omega\times Y}\big(|\mathcal{T}_h(R^h_2\cdot \mathrm{curl}_h\,\overline{R}^h)||\partial_{y_1}\psi| + |\mathcal{T}_h(R^h_1\cdot \mathrm{curl}_h\,\overline{R}^h)||\partial_{x_2}\psi|\big)\,dxdy_1 \\
&\leq \frac{\delta_h^2}{h\varepsilon_h^2}\|\mathcal{T}_h(\mathrm{curl}_h\,\overline{R}^h)\|_{L^2(H^{-1})}(\|\mathcal{T}_h(R^h_2)\partial_{y_1}\psi\|_{L^2(H^1_0)} + \|\mathcal{T}_h(R^h_1)\partial_{x_2}\psi\|_{L^2(H^1_0)}) \\
&\leq \frac{C\delta_h^2}{h\varepsilon_h^2}\|\mathcal{T}_h(\mathrm{curl}_h\,\overline{R}^h)\|_{L^2(H^{-1})} \qquad (4.25)
\end{aligned}$$

where we denoted by $L^2(H^1_0)$ the space $L^2(0,\ell;H^1_0((-1/2,1/2)^2\times(0,1);\mathbb{R}^3))$ and by $L^2(H^{-1})$ its dual space. Since

$$\mathrm{curl}_h\,\overline{\nabla}_h f^h = \partial_{x_1}\frac{\partial_{x_2}f^h}{h} - \frac{\partial_{x_2}\partial_{x_1}f^h}{h} = 0$$

we have that

$$\mathcal{T}_h(\mathrm{curl}_h\,\overline{R}^h) = \mathcal{T}_h(\mathrm{curl}_h(\overline{R}^h - \overline{\nabla}_h f^h)) = \frac{1}{h}\,\mathrm{curl}_{(y_1,x_2)}\,\mathcal{T}_h(\overline{R}^h - \overline{\nabla}_h f^h).$$

Thus, by Lemma 4.1 we deduce that

$$\|\mathcal{T}_h(\mathrm{curl}_h\,\overline{R}^h)\|_{L^2(H^{-1})} \leq \frac{1}{h}\|\mathcal{T}_h(\overline{R}^h - \overline{\nabla}_h f^h)\|_{L^2(\Omega\times Y)} \leq C\frac{\varepsilon_h}{h}.$$

Combining this estimate with (4.25), we obtain (4.24). □

In the next two lemmas we further characterise $\widetilde{G}_{\rm II}$ in the two regimes (2.4) and (2.6). We start with the critical case.

Lemma 4.4 (Critical case)**.** *Assume that (2.4) is satisfied and that $(f^h)\subset W^{1,2}(\Omega;\mathbb{R}^3)$ satisfies the energy bound (2.5). Let $(R^h)\subset C^\infty(\overline{\sigma};SO(3))$ be as in the conclusion of Lemma 4.1. Then there exist $(d_1\,|\,d_2\,|\,d_3)\in\mathcal{A}$ and $f\in W^{2,2}(0,\ell;\mathbb{R}^3)$ with $f'=d_1$ a.e. in $(0,\ell)$, such that, up to a subsequence,*

$$\partial_{x_1}f^h\to f' \quad \text{in } L^2(\Omega;\mathbb{R}^3),$$
$$R^h\to R:=(d_1\,|\,d_2\,|\,d_3) \quad \text{in } L^2(\Omega;\mathbb{R}^{3\times3}),$$

and

$$\partial_{x_1}A_1\cdot R_3 = d_1'\cdot d_3, \qquad \partial_{x_1}A_2\cdot R_3 = d_2'\cdot d_3 \quad \text{a.e. in } (0,\ell).$$

Here A is as in the conclusion of Lemma 4.2.

Proof. The first two convergence results follow from [8, Theorem 5.1] and Lemma 4.1. By Lemma 4.2 we have that $A=R-I$ and this implies the last statement. □

Lemma 4.5 (Super-critical case)**.** *Assume that (2.6) is satisfied and that $(f^h)\subset W^{1,2}(\Omega;\mathbb{R}^3)$ satisfies the energy bound (2.5). Then there exist rotations $Q^h\in SO(3)$ and constants $c^h\in\mathbb{R}$ such that, setting $\bar f^h:=(Q^h)^Tf^h-c^h$ and defining u^h and ϑ^h as in (2.7) and (2.8), we have that, up to a subsequence,*

$$u^h\rightharpoonup u \quad \text{weakly in } W^{1,2}(\Omega;\mathbb{R}^3), \qquad \vartheta^h\rightharpoonup\vartheta \quad \text{weakly in } W^{1,2}(0,\ell),$$

for some $u\in\mathcal{A}^\infty$ and some $\vartheta\in W^{1,2}(0,\ell)$. Moreover,

$$\partial_{x_1}A_1\cdot R_3 = u_3'', \qquad \partial_{x_1}A_2\cdot R_3 = \vartheta' \quad \text{a.e. in } (0,\ell),$$

where A and R are as in the conclusion of Lemma 4.2.

Proof. See [9, Lemma 3.7]. □

Proposition 4.6. *Assume that either* (2.4) *or* (2.6) *holds, and that* (4.11) *is satisfied. Let* $(f^h) \subset W^{1,2}(\Omega;\mathbb{R}^3)$ *satisfy the energy bound* (2.5)*. Then*

$$\liminf_{h\to 0} \frac{1}{\varepsilon_h^2}\int_\Omega W(\nabla_h f^h)\,dx \geq \frac{1}{12}\int_0^\ell \overline{Q}(\partial_{x_1}A_1\cdot R_3, \partial_{x_1}A_2\cdot R_3)\,dx_1,$$

where A and R are as in the conclusion of Lemma 4.2, and $\overline{Q}$ is defined in (2.3)*.*

Proof. Let G^h be the strain tensor defined in (4.8) and let

$$\chi_h(x) := \begin{cases} 1 & \text{if } |G^h(x)| \leq 1/\sqrt{\varepsilon_h},\\ 0 & \text{otherwise.}\end{cases}$$

By applying [8, Lemma 3.2] and by the definition (2.1) of Q we have

$$\liminf_{h\to 0} \frac{1}{\varepsilon_h^2}\int_\Omega W(\nabla_h f^h)\,dx \geq \frac{1}{2}\liminf_{h\to 0}\int_\Omega Q_3(\chi_h G^h)\,dx \geq \liminf_{h\to 0}\int_\Omega Q(\chi_h \widetilde{G}^h)\,dx.$$

Recalling (4.9) and Lemma 4.3, and using the fact that $\chi_h \to 1$ in $L^2(\Omega)$, we deduce that $\chi_h\widetilde{G}^h \xrightharpoonup{2} \widetilde{G}_{\rm I} + x_3\widetilde{G}_{\rm II}$ weakly in $L^2(\Omega\times Y;\mathbb{R}^{2\times 2})$. By the lower semicontinuity of convex integral functionals with respect to weak two-scale convergence (see, e.g., [26, Proposition 1.3]) we obtain

$$\begin{aligned}\liminf_{h\to 0} \frac{1}{\varepsilon_h^2}\int_\Omega W(\nabla_h f^h)\,dx &\geq \int_{\Omega\times Y} Q(\widetilde{G}_{\rm I} + x_3\widetilde{G}_{\rm II})\,dxdy_1\\ &= \int_{\Omega\times Y}\big(Q(\widetilde{G}_{\rm I}) + x_3^2 Q(\widetilde{G}_{\rm II})\big)\,dxdy_1\\ &\geq \int_{\Omega\times Y} x_3^2 Q(\widetilde{G}_{\rm II})\,dxdy_1,\end{aligned}$$

where we used that Q is quadratic and that $\widetilde{G}_{\rm I}$ and $\widetilde{G}_{\rm II}$ do not depend on x_3. Integrating over x_3 yields

$$\liminf_{h\to 0} \frac{1}{\varepsilon_h^2}\int_\Omega W(\nabla_h f^h)\,dx \geq \frac{1}{12}\int_{\sigma\times Y} Q(\widetilde{G}_{\rm II})\,dx_1dx_2dy_1.$$

In view of (4.11), Lemma 4.3 implies that $\det\widetilde{G}_{\rm II} = 0$. Therefore,

$$\frac{1}{12}\int_{\sigma\times Y} Q(\widetilde{G}_{\rm II})\,dx_1dx_2dy_1 = \frac{1}{12}\int_{\sigma\times Y} Q_{\rm ext}(\widetilde{G}_{\rm II})\,dx_1dx_2dy_1 \geq \frac{1}{12}\int_{\sigma\times Y} Q_{\rm ext}^{**}(\widetilde{G}_{\rm II})\,dx_1dx_2dy_1,$$

where $Q_{\rm ext}$ is defined in (2.2). Also by Lemma 4.3 there exist $\gamma \in L^2(0,\ell)$ and $g \in L^2(0,\ell;\mathcal{H}^2)$ such that $\widetilde{G}_{\rm II} = -K_\gamma - \overline{\nabla}^2 g$, where

$$K_\gamma := \begin{pmatrix} \partial_{x_1}A_1\cdot R_3 & \partial_{x_1}A_2\cdot R_3\\ \partial_{x_1}A_2\cdot R_3 & \gamma\end{pmatrix}.$$

By Jensen's inequality and the definition of $\mathcal{H}^2$, we deduce that

$$\begin{aligned}\liminf_{h\to 0}\frac{1}{\varepsilon_h^2}\int_\Omega W(\nabla_h f^h)\,dx &\geq \frac{1}{12}\int_{\sigma\times Y} Q_{\rm ext}^{**}(K_\gamma+\overline{\nabla}^2 g)\,dx_1dx_2dy_1\\ &\geq \frac{1}{12}\int_0^\ell Q_{\rm ext}^{**}\Big(\int_{(-1/2,1/2)\times Y}\big(K_\gamma+\overline{\nabla}^2 g\big)\,dx_2dy_1\Big)\,dx_1\\ &= \frac{1}{12}\int_0^\ell Q_{\rm ext}^{**}(K_\gamma)\,dx_1\\ &\geq \frac{1}{12}\int_0^\ell \overline{Q}(\partial_{x_1}A_1\cdot R_3,\partial_{x_1}A_2\cdot R_3)\,dx_1,\end{aligned}$$

where the last equality follows from the definition of $\overline{Q}$. □

Proposition 4.7 (Critical regime)**.** *Assume that* (2.4) *and* (4.11) *are satisfied, and let* $(f^h)\subset W^{1,2}(\Omega;\mathbb{R}^3)$ *be a sequence of deformations satisfying the energy bound* (2.5)*. Then there exist* $(d_1\,|\,d_2\,|\,d_3)\in\mathcal{A}$ *and* $f\in W^{2,2}(0,\ell;\mathbb{R}^3)$ *with* $f'=d_1$ *a.e. in* $(0,\ell)$*, such that, up to a subsequence,* $f^h\rightharpoonup f$ *weakly in* $W^{1,2}(\Omega;\mathbb{R}^3)$*,* $\nabla_h f^h\rightharpoonup(d_1\,|\,d_2\,|\,d_3)$ *weakly in* $L^2(\Omega;\mathbb{R}^{3\times3})$*, and*

$$\liminf_{h\to0}\frac{1}{\varepsilon_h^2}E^h(f^h)\geq\frac{1}{12}\int_0^\ell\overline{Q}(d_1'\cdot d_3,d_2'\cdot d_3)\,dx_1.$$

Proof. This follows immediately from Lemma 4.4 and Proposition 4.6. □

Proposition 4.8 (Super-critical regime)**.** *Assume that* (2.6) *and* (4.11) *are satisfied, and let* $(f^h)\subset W^{1,2}(\Omega;\mathbb{R}^3)$ *be a sequence of deformations satisfying the energy bound* (2.5)*. Then there exist rotations* $Q^h\in SO(3)$ *and constants* $c^h\in\mathbb{R}$ *such that, setting* $\bar f^h:=(Q^h)^Tf^h-c^h$*, up to a subsequence,*

$$\bar f^h\to x_1e_1\quad \text{in } W^{1,2}(\Omega;\mathbb{R}^3)\qquad\text{and}\qquad \nabla_h\bar f^h\to I\quad\text{in } L^2(\Omega;\mathbb{R}^{3\times3}).$$

Moreover, defining u^h *and* ϑ^h *as in* (2.7) *and* (2.8)*, we have that there exist* $u\in\mathcal{A}^\infty$ *and* $\vartheta\in W^{1,2}(0,\ell)$ *such that, up to a subsequence,* $u^h\rightharpoonup u$ *weakly in* $W^{1,2}(\Omega;\mathbb{R}^3)$ *and* $\vartheta^h\rightharpoonup\vartheta$ *weakly in* $W^{1,2}(0,\ell)$*. Moreover,*

$$\liminf_{h\to0}\frac{1}{\varepsilon_h^2}\int_\Omega W(\nabla_h y^h)\,dx\geq\frac{1}{12}\int_0^\ell\overline{Q}(u_3'',\vartheta')\,dx_1.$$

Proof. This follows immediately from Lemma 4.5 and Proposition 4.6. □

5. Construction of the recovery sequence

The purpose of this section is to prove sharpness of the lower bound. The critical case (2.4) differs from the super-critical case (2.6), but the basic construction is the same.

5.1. **Basic construction.** The basic construction uses the following result.

Lemma 5.1. *Let* $C_0>0$ *and let* τ*,* $\mu\in C^0([0,\ell])$ *be such that there exists* $q\in C^1([0,\ell];\mathbb{R}^2)$ *with* $|q|=1$*,* $|q'|\leq C_0$*,* $q\cdot e_2\geq C_0^{-1}$*, and such that* $(-\tau,\mu)$ *is parallel to* q *everywhere on* $(0,\ell)$*. Then there exists an open rectangle* $\widetilde U\subset\mathbb{R}^2$*, depending only on* C_0 *and* ℓ*, that contains* $[0,\ell]\times\{0\}$ *and there exist a* C^1 *extension of* q *and continuous extensions of* μ*,* τ *(not renamed) to the interval* $\{x_1\in\mathbb{R}:(x_1,0)\in\widetilde U\}$ *such that the map* $\Phi:(x_1,s)\mapsto x_1e_1+sq(x_1)$ *is a* C^1 *diffeomorphism from* $\widetilde U$ *onto* $U=\Phi(\widetilde U)$*.*

Moreover, there exists an isometric immersion $v \in C^2(\overline{U};\mathbb{R}^3)$, unique up to rigid motions, such that the second fundamental form A_v of v satisfies

$$A_v = \begin{pmatrix} \mu & \tau \\ \tau & \frac{\tau^2}{\mu} \end{pmatrix} \quad \text{on } (0,\ell)\times\{0\},$$

where the right-hand side is defined to be zero at zeros of μ. More precisely, v is given by

$$v(x_1e_1 + sq(x_1)) = v_0 + \int_0^{x_1} r^T(t)e_1\,dt + sr^T(x_1)\begin{pmatrix} q_1(x_1) \\ q_2(x_1) \\ 0 \end{pmatrix},$$

where $v_0 \in \mathbb{R}^3$ and $r : [0,\ell] \to SO(3)$ is a solution of

$$r' = \begin{pmatrix} 0 & 0 & \mu \\ 0 & 0 & \tau \\ -\mu & -\tau & 0 \end{pmatrix} r. \tag{5.1}$$

In particular,

$$\nabla v(x_1e_1 + sq(x_1)) = r^T(x_1)(e_1\otimes e_1 + e_2\otimes e_2) \tag{5.2}$$

and

$$A_v(x_1e_1 + sq(x_1)) = \frac{1}{\mu + sq'\cdot(\mu,\tau)}\begin{pmatrix} \mu \\ \tau \end{pmatrix}\otimes\begin{pmatrix} \mu \\ \tau \end{pmatrix}, \tag{5.3}$$

where the right-hand side is defined to be zero if $\mu(x_1) = 0$.

Proof. Compare [6, Lemma 12 and Proposition 13]. Regarding (5.3), observe that the hypotheses imply that $|\mu| \geq C_0^{-1}|\tau|$, so that, if $\mu(x_1) = 0$, then $\tau(x_1) = 0$.

All assertions about v can be verified directly from the explicit formula. In fact, (5.2) follows at once. Moreover, (5.2) implies that the normal vector to v satisfies $\nu(x_1e_1 + sq(x_1)) = r^T(x_1)e_3$. From this one readily deduces that

$$A_v\left(x_1e_1 + sq(x_1)\right)\left(e_1 + sq'(x_1)\right) = \begin{pmatrix} \mu \\ \tau \end{pmatrix}. \tag{5.4}$$

Since v is an isometric immersion, there exists $\lambda : \widetilde{U} \to \mathbb{R}$ such that

$$A_v\left(x_1e_1 + sq(x_1)\right) = \lambda(x_1,s)\begin{pmatrix} \mu \\ \tau \end{pmatrix}\otimes\begin{pmatrix} \mu \\ \tau \end{pmatrix}.$$

Together with (5.4) this implies (5.3). □

Remark 5.2. (i) Solutions r of (5.1) are naturally associated with geodesics on surfaces. In fact, let $\gamma : [0,\ell] \to \Sigma$ be a geodesic on an immersed surface $\Sigma \subset \mathbb{R}^3$ and denote by $n(t)$ the normal vector to Σ at $\gamma(t)$. Then $r = (\gamma' \mid n\wedge\gamma' \mid n)^T$ satisfies (5.1), where μ is the normal curvature of γ and τ is its geodesic torsion. The corresponding isometric immersion v constructed in Lemma 5.1 is a tangent developable to Σ along γ. Observe that the normal curvature μ of γ is allowed to become zero and that v is an isometric parametrization of the tangent developable.

(ii) Conversely, if r solves (5.1) and we define v and U as in Lemma 5.1, then $\gamma(t) = \int_0^t r^T(x_1)e_1\,dx_1$ is a geodesic on $v(U)$ with normal curvature μ and geodesic torsion τ.

(iii) In order to match the compactness result, the recovery sequence constructed below must be able to produce limiting second fundamental forms which violate the zero Gauss curvature constraint satisfied by isometric immersions. Indeed, the weak limit of the second fundamental forms of a sequence of isometric immersions v_k with uniformly L^2-bounded curvature does not in general have zero Gauss curvature. One can construct examples by choosing τ_k and

μ_k suitably and then defining r_k implicitly via (5.1). Below we give an example where r_k is given more explicitly.

(iv) Conversely, a trivial example of a sequence of isometric immersions whose second fundamental forms are nontrivial but converge weakly to zero, is obtained by choosing $\mu_k(x_1) = \sin(kx_1)$ for each $k \in \mathbb{N}$ and setting

$$v_k(x_1, x_2) = \int_0^{x_1} \cos\Big(\int_0^t \mu_k(s)\,ds\Big)dt\,e_1 + x_2 e_2 + \int_0^{x_1} \sin\Big(\int_0^t \mu_k(s)\,ds\Big)dt\,e_3.$$

In this case $\tau_k = 0$ and the weak limit of the second fundamental forms is zero.

Example 5.3. Let $\ell = 1$ and for each $k \in \mathbb{N}$ define the following functions on $[0, 1]$:

$$\rho_k(t) = \frac{\pi}{8}\sin(kt), \qquad \zeta_k(t) = \frac{\pi}{2} + \int_0^t \cos(\rho_k(s))\,ds,$$

$$\Psi_k(t) = \int_0^t \frac{\sin(\rho_k(s))}{\sin(\zeta_k(s))}\,ds, \qquad \alpha_k(t) = -\int_0^t \sin(\rho_k(s))\cot(\zeta_k(s))\,ds.$$

Observe that ζ_k takes values in $[\frac{\pi}{2} - 1, \frac{\pi}{2} + 1]$, so that $|\sin\zeta_k| \geq 1/2$. Hence the above expressions are well-defined. Now define $r_k : [0, 1] \to SO(3)$ by setting

$$r_k^T = \begin{pmatrix} \cos\Psi_k\cos\zeta_k & -\sin\Psi_k & \cos\Psi_k\sin\zeta_k \\ \sin\Psi_k\cos\zeta_k & \cos\Psi_k & \sin\Psi_k\sin\zeta_k \\ -\sin\zeta_k & 0 & \cos\zeta_k \end{pmatrix} \begin{pmatrix} \cos\alpha_k & -\sin\alpha_k & 0 \\ \sin\alpha_k & \cos\alpha_k & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

A direct computation shows that r_k then satisfies (5.1) with

$$\mu_k = -\cos(\rho_k - \alpha_k) \qquad \text{and} \qquad \tau_k = -\sin(\rho_k - \alpha_k).$$

Therefore, we can take

$$q_k = (\tau_k, -\mu_k) = \big(-\sin(\rho_k - \alpha_k), \cos(\rho_k - \alpha_k)\big).$$

In particular, $|\mu_k| = q_k \cdot e_2 \geq \cos(3\pi/8)$, because $|\sin\rho_k| \leq |\rho_k| \leq \frac{\pi}{8}$ and $|\alpha_k| \leq \frac{\pi}{4}$, so that

$$|\rho_k - \alpha_k| \leq \frac{\pi}{8} + \frac{\pi}{4} = \frac{3\pi}{8}.$$

Each q_k' is uniformly bounded on $[0, 1]$, although the constant depends on k. Thus, we can define an isometry $v_k : U_k \to \mathbb{R}^3$ as in Lemma 5.1, with the above r_k and q_k and suitable domains U_k. For its second fundamental form on $[0, 1] \times \{0\}$ we obtain

$$A_{v_k} = \begin{pmatrix} \mu_k & \tau_k \\ \tau_k & \frac{\tau_k^2}{\mu_k} \end{pmatrix} = \begin{pmatrix} \mu_k & \tau_k \\ \tau_k & \frac{1}{\mu_k} - \mu_k \end{pmatrix}.$$

Observe that $\sin\rho_k \rightharpoonup 0$ weakly* in $L^\infty(0, 1)$ because the sine function is odd. Since ζ_k converges uniformly, we conclude that $\alpha_k \to 0$ uniformly. Thus, the weak* limits of μ_k and of τ_k agree with those of $-\cos\rho_k$ and of $-\sin\rho_k$, respectively. The latter is zero, while the former is given by

$$\overline{\mu} := -\frac{1}{2\pi}\int_0^{2\pi} \cos\Big(\frac{\pi}{8}\sin y\Big)\,dy,$$

which is a constant strictly between -1 and zero. Since the constant

$$\frac{1}{\mu^*} := -\frac{1}{2\pi}\int_0^{2\pi} \frac{1}{\cos\big(\frac{\pi}{8}\sin y\big)}\,dy$$

is less than -1, it differs from $\overline{\mu}$. It is easy to see that

$$A_{v_k} \rightharpoonup \begin{pmatrix} \overline{\mu} & 0 \\ 0 & \frac{1}{\mu^*} - \overline{\mu} \end{pmatrix} \qquad \text{weakly* in } L^\infty(0, 1).$$

Therefore the determinant of the weak* limit of (A_{v_k}) is not identically zero. This concludes the example.

We will now provide the basic construction. Set $\eta_h := \varepsilon_h/\delta_h$. Let $\Xi,\ \Theta \in C^2([0,\ell])$. Following [9] we define $S : [0,\ell] \to \mathbb{R}^{3\times 3}$ by

$$S = \begin{pmatrix} 0 & 0 & -\Xi \\ 0 & 0 & -\Theta \\ \Xi & \Theta & 0 \end{pmatrix} \tag{5.5}$$

and $R^h : [0,\ell] \to SO(3)$ as the solution of $(R^h)' = \eta_h R^h S'$ with initial condition $R^h(0) = R^h_0$, where $R^h_0 \in SO(3)$ will be chosen later.

Assume that there exist $C_0 > 0$ and $q \in C^1([0,\ell];\mathbb{R}^2)$ such that the hypotheses of Lemma 5.1 are satisfied with $\tau = \Theta'$ and $\mu = \Xi'$. Note that then they are satisfied, with the same $q = (q_1, q_2)$ and C_0, for $\tau = \eta_h\Theta'$ and $\mu = \eta_h\Xi'$. Hence by Lemma 5.1 there exist neighbourhoods $\widetilde{U}$, $U \subset \mathbb{R}^2$ of $[0,\ell]\times\{0\}$ such that the map $\Phi : \widetilde{U} \to U$ given by $\Phi(x_1, s) = x_1e_1 + sq(x_1)$ is a C^1 diffeomorphism. Moreover, for every $v^h_0 \in \mathbb{R}^3$ the map $v^h : U \to \mathbb{R}^3$ defined by

$$v^h\big(x_1e_1 + sq(x_1)\big) = v^h_0 + \int_0^{x_1} R^h(t)e_1\,dt + sR^h(x_1)\begin{pmatrix} q_1(x_1) \\ q_2(x_1) \\ 0 \end{pmatrix} \tag{5.6}$$

is an isometric immersion such that, denoting the normal vector by $\nu^h = \partial_1 v^h \wedge \partial_2 v^h$,

$$\big(\partial_1 v^h \,|\, \partial_2 v^h \,|\, \nu^h\big) = R^h \circ \rho. \tag{5.7}$$

Here $\rho : U \to \mathbb{R}$ denotes the first component of the inverse of Φ. According to (5.3) (with $\mu = \eta_h\Xi'$ and $\tau = \eta_h\Theta'$), the second fundamental form of v^h satisfies

$$\frac{1}{\eta_h} A_{v^h}(x_1e_1 + sq(x_1)) = \frac{\chi_{\{\Xi'\neq 0\}}}{\Xi' + sq'\cdot(\Xi',\Theta')}\begin{pmatrix}\Xi' \\ \Theta'\end{pmatrix}\otimes\begin{pmatrix}\Xi' \\ \Theta'\end{pmatrix}. \tag{5.8}$$

We claim that there exists a (unique) Lipschitz continuous function $p : [0,\ell] \to \mathbb{R}^3$ such that

$$\frac{1}{2}Q_3\begin{pmatrix} \chi_{\{\Xi'\neq 0\}}\Xi' & \chi_{\{\Xi'\neq 0\}}\Theta' & \frac{1}{2}p_1 \\ \chi_{\{\Xi'\neq 0\}}\Theta' & \chi_{\{\Xi'\neq 0\}}\frac{(\Theta')^2}{\Xi'} & \frac{1}{2}p_2 \\ \frac{1}{2}p_1 & \frac{1}{2}p_2 & p_3 \end{pmatrix} = \chi_{\{\Xi'\neq 0\}} Q\begin{pmatrix} \Xi' & \Theta' \\ \Theta' & \frac{(\Theta')^2}{\Xi'}\end{pmatrix}. \tag{5.9}$$

In order to write down the explicit formula for p, let $\mathcal{L} : \mathbb{R}^{3\times 3}_{\mathrm{sym}} \to \mathbb{R}^{3\times 3}_{\mathrm{sym}}$ be the linear symmetric and positive definite operator satisfying $Q_3(F) = F : \mathcal{L}F$ for all $F \in \mathbb{R}^{3\times 3}_{\mathrm{sym}}$. Since $\mathcal{L}$ is symmetric and positive definite, the matrix $B \in \mathbb{R}^{3\times 3}$ with entries

$$B_{ij} = \mathrm{sym}(e_i \otimes e_3) : \mathcal{L}(\mathrm{sym}(e_j \otimes e_3))$$

is symmetric and positive definite as well. Hence B is invertible.

Since Q_3 is quadratic, for every $F \in \mathbb{R}^{3\times 3}_{\mathrm{sym}}$ there exists a unique minimiser $p \in \mathbb{R}^3$ of $Q_3(F + \mathrm{sym}(p \otimes e_3))$. It satisfies

$$e_i \otimes e_3 : \mathcal{L}\left(F + \mathrm{sym}(p\otimes e_3)\right) = 0 \quad \text{for } i = 1,2,3.$$

This can be rewritten as the explicit formula $p = -B^{-1}(\mathcal{L}F)e_3$.

The hypotheses on Θ and Ξ, together with the bound $|\Theta'| \le C_0|\Xi'|$, ensure that $\chi_{\{\Xi'\neq 0\}}\Xi'$, $\chi_{\{\Xi'\neq 0\}}\Theta'$ and $\chi_{\{\Xi'\neq 0\}}(\Theta')^2/\Xi'$ are Lipschitz continuous. Hence the solution

$$p = -\chi_{\{\Xi'\neq 0\}}B^{-1}\mathcal{L}\begin{pmatrix} \Xi' & \Theta' & 0 \\ \Theta' & \frac{(\Theta')^2}{\Xi'} & 0 \\ 0 & 0 & 0\end{pmatrix}$$

of (5.9) is Lipschitz, too.

We now define the basic recovery sequence. Let $\beta^h:\Omega\to\mathbb{R}^3$ and $f^h:\Omega\to\mathbb{R}^3$ be defined by $\beta^h(x):=R^h(\rho(x_1,hx_2))p(x_1)$ and

$$f^h(x):=v^h(x_1,hx_2)+\delta_h x_3\nu^h(x_1,hx_2)+\varepsilon_h\delta_h\frac{x_3^2}{2}\beta^h(x). \tag{5.10}$$

Observe that $\sigma_h=(0,\ell)\times(-h/2,h/2)\subset U$ for every h small enough. We compute

$$\begin{aligned}\nabla_h f^h(x) &= (\partial_1 v^h\,|\,\partial_2 v^h\,|\,\nu^h)(x_1,hx_2)+\delta_h x_3(\partial_1\nu^h\,|\,\partial_2\nu^h\,|\,0)(x_1,hx_2)\\ &\quad+\varepsilon_h x_3\beta^h(x)\otimes e_3+\frac{\delta_h\varepsilon_h x_3^2}{2}\nabla_h\beta^h(x).\end{aligned}$$

By (5.7) the first term on the right-hand side equals $R^h(\rho(x_1,hx_2))$. Hence,

$$\begin{aligned}\left(R^h(\rho(x_1,hx_2))\right)^T\nabla_h f^h(x) &= I-\delta_h x_3 A_{v^h}(x_1,hx_2)+\varepsilon_h x_3 p(x_1)\otimes e_3\\ &\quad+\frac{\delta_h\varepsilon_h x_3^2}{2}\left(R^h(\rho(x_1,hx_2))\right)^T\nabla_h\beta^h(x),\end{aligned}\tag{5.11}$$

where we regard A_{v^h} as the 3×3 matrix field with zeros in the last row and column. By (5.8) we deduce that

$$\frac{1}{\eta_h}A_{v^h}(x_1,hx_2)\to\chi_{\{\Xi'\neq0\}}\begin{pmatrix}\Xi' & \Theta'\\ \Theta' & \frac{(\Theta')^2}{\Xi'}\end{pmatrix} \tag{5.12}$$

uniformly for $(x_1,x_2)\in\sigma$. By frame indifference and Taylor expansion we conclude that

$$\begin{aligned}\lim_{h\to0}\frac{1}{\varepsilon_h^2}W(\nabla_h f^h(x)) &= \frac{x_3^2}{2}\lim_{h\to0}Q_3\left(\frac{1}{\eta_h}A_{v^h}(x_1,hx_2)+p(x_1)\otimes e_3\right)\\ &= \chi_{\{\Xi'\neq0\}}x_3^2 Q\begin{pmatrix}\Xi' & \Theta'\\ \Theta' & \frac{(\Theta')^2}{\Xi'}\end{pmatrix}\end{aligned}\tag{5.13}$$

strongly in $L^\infty(\Omega)$. In the last step we used (5.12) and (5.9).

5.2. **Recovery sequence in the critical case.** The following lemma will be used in the construction of the recovery sequence. It summarises [6, Proposition 9 and Lemma 16], see also the argument in the proof of [6, Theorem 5(ii)].

Lemma 5.4. *Let $M\in L^2(0,\ell;\mathbb{R}^{2\times2}_{\rm sym})$. Then there exist β_n, $\lambda_n\in C^\infty([0,\ell])$ with $|\beta_n|<\frac{\pi}{2}$ on $[0,\ell]$ such that*

$$M_n=\lambda_n\begin{pmatrix}\cos\beta_n\\ \sin\beta_n\end{pmatrix}\otimes\begin{pmatrix}\cos\beta_n\\ \sin\beta_n\end{pmatrix}$$

converge to M weakly in $L^2(0,\ell;\mathbb{R}^{2\times2}_{\rm sym})$ and

$$\lim_{n\to\infty}\int_0^\ell Q(M_n)\,dx_1=\int_0^\ell Q^{**}_{\rm ext}(M)\,dx_1.$$

We will now prove the following upper bound.

Proposition 5.5 (Critical case)**.** *Assume* (2.4)*. For every $(d_1\,|\,d_2\,|\,d_3)\in\mathcal{A}$ and every $f\in W^{2,2}(0,\ell;\mathbb{R}^3)$ with $f'=d_1$ a.e. in $(0,\ell)$, there exists a sequence $(f^h)\subset W^{1,2}(\Omega;\mathbb{R}^3)$ such that $f^h\rightharpoonup f$ weakly in $W^{1,2}(\Omega;\mathbb{R}^3)$, $\nabla_h f^h\rightharpoonup(d_1\,|\,d_2\,|\,d_3)$ weakly in $L^2(\Omega;\mathbb{R}^{3\times3})$, and*

$$\limsup_{h\to0}\frac{1}{\varepsilon_h^2}E^h(f^h)\le\frac{1}{12}\int_0^\ell\overline{Q}(d_1'\cdot d_3,d_2'\cdot d_3)\,dx_1.$$

Proof. Let $(d_1\,|\,d_2\,|\,d_3)\in\mathcal{A}$ and let $f\in W^{2,2}(0,\ell;\mathbb{R}^3)$ be such that $f'=d_1$ a.e. in $(0,\ell)$. We set

$$K_\gamma:=\begin{pmatrix} d_1'\cdot d_3 & d_2'\cdot d_3\\ d_2'\cdot d_3 & \gamma\end{pmatrix},$$

where $\gamma\in L^2(0,\ell)$ is chosen so that

$$\overline{Q}(d_1'\cdot d_3,d_2'\cdot d_3)=Q^{**}_{\rm ext}(K_\gamma). \tag{5.14}$$

By Lemma 5.4 there exists a sequence $(M_n)\subset C^\infty([0,\ell];\mathbb{R}^{2\times2}_{\rm sym})$ such that $\det M_n=0$, $M_n\rightharpoonup K_\gamma$ weakly in $L^2(0,\ell;\mathbb{R}^{2\times2}_{\rm sym})$, and

$$\lim_{n\to\infty}\int_0^\ell Q(M_n)\,dx_1=\int_0^\ell Q^{**}_{\rm ext}(K_\gamma)\,dx_1=\int_0^\ell\overline{Q}(d_1'\cdot d_3,d_2'\cdot d_3)\,dx_1. \tag{5.15}$$

The last equality follows from (5.14). Moreover, since $|\beta_n|<\pi/2$ in the conclusion of Lemma 5.4, we see that for each n the hypotheses of Lemma 5.1 are satisfied with $q=(-\sin\beta_n,\cos\beta_n)$ and $\mu=(M_n)_{11}$ and $\tau=(M_n)_{12}$.

For each n we can apply the 'basic construction' given in Section 5.1 with $\Xi_n(t)=\int_0^t(M_n)_{11}\,dx_1$ and $\Theta_n(t)=\int_0^t(M_n)_{12}\,dx_1$, as well as $R_0^h=R_0:=(d_1\,|\,d_2\,|\,d_3)^T(0)$ and $v_0^h=f(0)$. Define S_n as in (5.5) with Ξ and Θ replaced by Ξ_n and Θ_n. Let $R_n^h:[0,\ell]\to SO(3)$ be the solution of $(R_n^h)'=\eta_hR_n^hS_n'$ with initial condition $R_n^h(0)=R_0$. Let $R_n:[0,\ell]\to\mathbb{R}^{3\times3}$ solve $R_n'=R_nS_n'$, with $R_n(0)=R_0$. Let (f_n^h) be the corresponding sequence of deformations defined accordingly to (5.10). Since $\eta_h\to1$, we have $R_n^h\to R_n$ uniformly, as $h\to0$. Hence, (5.11) implies that $\nabla_hf_n^h\to R_n$ in $L^\infty(\Omega;\mathbb{R}^3)$, as $h\to0$. By (5.13) we obtain

$$\lim_{h\to0}\frac{1}{\varepsilon_h^2}\int_\Omega W(\nabla_hf_n^h)\,dx=\frac{1}{12}\int_0^\ell Q(M_n)\,dx_1$$

for every n. Using (5.15) and the weak convergence of R_n to $(d_1\,|\,d_2\,|\,d_3)^T$ in $W^{1,2}(0,\ell;\mathbb{R}^{3\times3})$, we reach the conclusion by a diagonal argument. □

5.3. **Recovery sequence in the super-critical case.**

Proposition 5.6 (Super-critical case)**.** *Assume* (2.6)*. For every $u\in\mathcal{A}^\infty$ and every $\vartheta\in W^{1,2}(0,\ell)$, there exists a sequence $(f^h)\subset W^{1,2}(\Omega;\mathbb{R}^3)$ such that $f^h\to x_1e_1$ in $W^{1,2}(\Omega;\mathbb{R}^3)$ and $\nabla_hf^h\to I$ in $L^2(\Omega;\mathbb{R}^{3\times3})$. Moreover, defining*

$$u_1^h:=\frac{f_1^h-x_1}{(\varepsilon_h/\delta_h)^2},\qquad u_2^h:=\frac{f_2^h-hx_2}{\varepsilon_h/\delta_h},\qquad u_3^h:=\frac{f_3^h-\delta_hx_3}{\varepsilon_h/\delta_h},$$

$$\vartheta^h:=\frac{6}{h\varepsilon_h}\int_\omega(\delta_hx_2f_3^h-hx_3f_2^h)\,dx_2dx_3,$$

we have that $u^h\rightharpoonup u$ weakly in $W^{1,2}(\Omega;\mathbb{R}^3)$, $\vartheta^h\rightharpoonup\vartheta$ weakly in $W^{1,2}(0,\ell)$, and

$$\limsup_{h\to0}\frac{1}{\varepsilon_h^2}E^h(f^h)\le\frac{1}{12}\int_0^\ell\overline{Q}(u_3'',\vartheta')\,dx_1. \tag{5.16}$$

Proof. Let $u\in\mathcal{A}^\infty$ and let $\vartheta\in W^{1,2}(0,\ell)$. We set

$$K_\gamma:=\begin{pmatrix}u_3'' & \vartheta'\\ \vartheta' & \gamma\end{pmatrix},$$

where $\gamma\in L^2(0,\ell)$ is chosen so that

$$\overline{Q}(u_3'',\vartheta')=Q^{**}_{\rm ext}(K_\gamma).$$

By Lemma 5.4 there exists a sequence $(M_n) \subset C^\infty([0,\ell];\mathbb{R}^{2\times 2}_{\rm sym})$ such that $\det M_n = 0$, $M_n \rightharpoonup K_\gamma$ weakly in $L^2(0,\ell;\mathbb{R}^{2\times 2}_{\rm sym})$, and

$$\lim_{n\to\infty}\int_0^\ell Q(M_n)\,dx_1 = \int_0^\ell Q^{**}_{\rm ext}(K_\gamma)\,dx_1 = \int_0^\ell \overline{Q}(u_3'',\vartheta')\,dx_1.$$

For each n we apply the 'basic construction' with $\Xi_n(t) = u_3'(0) + \int_0^t (M_n)_{11}\,dx_1$ and $\Theta_n(t) = \vartheta(0) + \int_0^t (M_n)_{12}\,dx_1$. We choose R_0^h as the nearest point projection of $I + \eta_h S(0)$ onto $SO(3)$ and we take $v_0^h = (\eta_h^2 u_1(0), 0, \eta_h u_3(0))$. We recall that $\eta_h = \varepsilon_h/\delta_h$ and that S is defined in (5.5). By (5.13) we deduce

$$\lim_{h\to 0}\frac{1}{\varepsilon_h^2}\int_\Omega W(\nabla_h f_n^h)\,dx = \frac{1}{12}\int_0^\ell Q(M_n)\,dx_1.$$

This yields (5.16) after taking a suitable diagonal sequence. Notice, moreover, that $\Xi_n \rightharpoonup u_3'$ and $\Theta_n \rightharpoonup \vartheta$ weakly in $W^{1,2}(0,\ell)$. The remaining assertions will therefore follow by a diagonal argument once we prove them (with strong convergence instead of weak) for each fixed n, with Θ_n instead of ϑ and Ξ_n instead of u_3'.

Hereafter the index n is fixed and will be omitted in the notation. Define $L^h : [0,\ell] \to \mathbb{R}^{3\times 3}$ via

$$R^h = I + \eta_h S + \eta_h^2 L^h. \tag{5.17}$$

Since $(R^h)' = \eta_h R^h S'$, we find that L^h satisfies

$$(L^h)' = \eta_h L^h S' + SS'. \tag{5.18}$$

Hence by the variation of constants formula

$$L^h(t) = \left(L^h(0)(R_0^h)^T + \int_0^t (SS')(s)(R^h)^T(s)\,ds\right) R^h(t). \tag{5.19}$$

Observe that by definition of R_0^h we have

$$\limsup_{h\to\infty}|L^h(0)| < \infty,$$

since $S(0)$ is in the tangent space to $SO(3)$ at I. Thus, (5.19) implies that L^h remains uniformly bounded in $L^\infty(0,\ell)$. Hence by (5.18) so is $(L^h)'$, and after taking derivatives in (5.18) we conclude that so is $(L^h)''$ and so on. In other words,

$$\limsup_{h\to 0}\|L^h\|_{W^{k,\infty}(0,\ell)} < \infty \quad \text{for all } k\in\mathbb{N}. \tag{5.20}$$

We now define the functions $V : U \to \mathbb{R}$, $V^h : U \to \mathbb{R}$, and $Y^h : U \to \mathbb{R}^2$ as follows:

$$V(\Phi(x_1,s)) := u_3(0) + \int_0^{x_1}\Xi(t)\,dt + sq(x_1)\cdot\begin{pmatrix}\Xi(x_1)\\ \Theta(x_1)\end{pmatrix},$$

$$V^h(\Phi(x_1,s)) := V(\Phi(x_1,s)) + \eta_h\int_0^{x_1} L^h_{31}(t)\,dt + \eta_h s e_3\cdot L^h(x_1)\begin{pmatrix}q_1(x_1)\\ q_2(x_1)\\ 0\end{pmatrix},$$

$$Y^h(\Phi(x_1,s)) := \begin{pmatrix}u_1(0)\\ 0\end{pmatrix} + \int_0^{x_1}\widetilde{L}^h(t)e_1\,dt + s\widetilde{L}^h(x_1)q(x_1),$$

where we recall that $\widetilde{L}^h$ denotes the upper left 2×2 submatrix of L^h.

From the definitions of Φ^h and L^h, we have

$$\nabla Y^h = \widetilde{L}^h\circ\rho \tag{5.21}$$

and

$$\nabla V^h == \nabla V + \eta_h(\overline{L}^h\circ\rho)^T e_3 = (\overline{S} + \eta_h\overline{L}^h)^T\circ\rho\; e_3,$$

where

$$\nabla V = \begin{pmatrix} \Xi \circ \rho \\ \Theta \circ \rho \end{pmatrix}$$

and $\overline{S}$, $\overline{L}^h$ denote the submatrices given by the first two columns of S, L^h, respectively. Notice that, in view of (5.20),

$$\nabla V^h \to \begin{pmatrix} \Xi \circ \rho \\ \Theta \circ \rho \end{pmatrix} \quad \text{in } W^{2,\infty}(U;\mathbb{R}^2) \tag{5.22}$$

and $V^h \to V$ in $W^{3,\infty}(U)$.

By (5.20) and (5.21) the sequence (Y^h) is uniformly bounded in $W^{2,\infty}(U;\mathbb{R}^2)$. In particular, after taking subsequences, there exists $Y \in W^{1,\infty}(U;\mathbb{R}^2)$ such that $Y^h \to Y$ in $W^{1,\infty}(U;\mathbb{R}^2)$.

Inserting (5.17) into (5.6) and applying the inverse of Φ, we have

$$v^h(x) = \begin{pmatrix} x_1 + \eta_h^2 Y_1^h(x) \\ x_2 + \eta_h^2 Y_2^h(x) \\ \eta_h V^h(x) \end{pmatrix}. \tag{5.23}$$

Passing to the limit in

$$0 = \frac{1}{\eta_h^2}\left((\nabla v^h)^T(\nabla v^h) - I\right) = 2\,\mathrm{sym}\,\nabla Y^h + \nabla V^h \otimes \nabla V^h + \eta_h^2 (\nabla Y^h)^T(\nabla Y^h),$$

we find

$$2\,\mathrm{sym}\,\nabla Y + \nabla V \otimes \nabla V = 0.$$

In particular,

$$\partial_1 Y_1 = -\frac{1}{2}\left(\partial_1 V\right)^2 = -\frac{1}{2}(\Xi \circ \rho)^2. \tag{5.24}$$

Inserting (5.23) into (5.10) and using $\nu_3^h = R_{33}^h \circ \rho$ and (5.17), we have

$$f^h(x) = \begin{pmatrix} x_1 \\ hx_2 \\ \delta_h x_3 \end{pmatrix} + \begin{pmatrix} \eta_h^2 Y_1^h(x_1, hx_2) \\ \eta_h^2 Y_2^h(x_1, hx_2) \\ \eta_h V^h(x_1, hx_2) \end{pmatrix} + \delta_h x_3 \begin{pmatrix} \nu_1^h(x_1, hx_2) \\ \nu_2^h(x_1, hx_2) \\ \eta_h^2 L_{33}^h\left(\rho(x_1, hx_2)\right) \end{pmatrix} + \varepsilon_h \delta_h \frac{x_3^2}{2} \beta^h(x). \tag{5.25}$$

Since $\nu^h = R^h(\rho)e_3$ by (5.7), we see that, for $\alpha = 1, 2$,

$$\nu_\alpha^h = \eta_h\left(S_{\alpha 3} + \eta_h L_{\alpha 3}^h\right) \circ \rho, \tag{5.26}$$

so that $\|\nu_\alpha^h\|_{W^{1,\infty}(U)} \le C\eta_h$ by (5.20). Moreover, $\|L_{33}^h \circ \rho\|_{W^{1,\infty}(U)} \le C$. Hence,

$$\left\| \begin{pmatrix} f_1^h(x) - x_1 \\ f_2^h(x) - hx_2 \\ f_3^h(x) - \delta_h x_3 \end{pmatrix} - \begin{pmatrix} \eta_h^2 Y_1^h(x_1, hx_2) \\ \eta_h^2 Y_2^h(x_1, hx_2) \\ \eta_h V^h(x_1, hx_2) \end{pmatrix} \right\|_{W^{1,\infty}(\Omega)} \le C\varepsilon_h, \tag{5.27}$$

because β^h is uniformly bounded in $W^{1,\infty}(\Omega)$.

From (5.27) we immediately deduce that

$$\frac{f_2^h - hx_2}{\eta_h} \to 0$$

strongly in $W^{1,\infty}(\Omega)$. From this we eventually deduce that $u_2^h \rightharpoonup 0$ weakly in $W^{1,2}(\Omega)$.

Since $V^h \to V$ uniformly, we have $V^h(x_1, hx_2) \to V(x_1, 0)$. Similarly, $Y^h(x_1, hx_2) \to Y(x_1, 0)$. Hence (5.27), together with (2.6), implies that

$$\frac{f_1^h(x) - x_1}{\eta_h^2} \to Y_1(x_1, 0) \tag{5.28}$$

$$\frac{f_3^h(x) - \delta_h x_3}{\eta_h} \to V(x_1, 0) \tag{5.29}$$

strongly in $W^{1,\infty}(\Omega)$. Since by definition $V(x_1,0) = u_3(0) + \int_0^{x_1} \Xi(t)\,dt$, the convergence of (u_3^h) follows from (5.29). By (5.24) and the fact that $\rho(x_1,0) = x_1$ we have

$$Y_1(x_1,0) = u_1(0) - \frac{1}{2}\int_0^{x_1} \Xi^2(t)\,dt,$$

so that the convergence of (u_1^h) follows from (5.28) and the fact that $u_1' = -\frac{1}{2}(u_3')^2$.

From (5.25) and the x_3-independence of Y_2^h we deduce that

$$\vartheta^h(x_1) = 6\int_{-\frac{1}{2}}^{\frac{1}{2}} \frac{x_2}{h} V^h(x_1,hx_2)\,dx_2 - 6\int_\omega \frac{x_3^2}{\eta_h}\nu_2^h(x_1,hx_2)\;dx_2\,dx_3 + \zeta^h(x_1), \tag{5.30}$$

where $\zeta^h \to 0$ in $W^{1,\infty}(0,\ell)$. By (5.26) and since $S_{23} = -\Theta$, the second term on the right-hand side of (5.30) converges to $\frac{1}{2}\Theta$ strongly in $W^{1,\infty}(0,\ell)$. The first term on the right-hand side of (5.30) equals

$$6\int_{-\frac{1}{2}}^{\frac{1}{2}} \frac{x_2}{h}\big(V^h(x_1,hx_2) - V^h(x_1,0)\big)\,dx_2 = 6\int_{-\frac{1}{2}}^{\frac{1}{2}} \frac{x_2}{h}\Big(\int_0^{hx_2} \partial_2 V^h(x_1,t)\,dt\Big)\,dx_2.$$

From (5.22) we therefore deduce that the first term on the right-hand side of (5.30) converges to $\frac{1}{2}\Theta$ strongly in $W^{1,\infty}(0,\ell)$. We conclude that ϑ^h converges to Θ strongly in $W^{1,\infty}(0,\ell)$. □

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Conflict of Interest

The authors declare that they have no conflict of interest.

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