UNIQUENESS OF EQUILIBRIUM WITH SUFFICIENTLY SMALL STRAINS IN FINITE ELASTICITY

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Abstract. The uniqueness of equilibrium for a compressible, hyperelastic body subject to dead-load boundary conditions is considered. It is shown, for both the displacement and mixed problems, that there cannot be two solutions of the equilibrium equations of Finite (Nonlinear) Elasticity whose nonlinear strains are uniformly close to each other. This result is analogous to the result of Fritz John (Comm. Pure Appl. Math. **25**, 617–634, 1972) who proved that, for the displacement problem, there is a most one equilibrium solution with uniformly small strains. The proof in this manuscript utilizes Geometric Rigidity; a new straightforward extension of the Fefferman-Stein inequality to bounded domains; and, an appropriate adaptation, for Elasticity, of a result from the Calculus of Variations. Specifically, it is herein shown that the uniform positivity of the second variation of the energy at an equilibrium solution implies that this mapping is a local minimizer of the energy among deformations whose gradient is sufficiently close, in $\mathrm{BMO} \cap L^1$, to the gradient of the equilibrium solution.

1. Introduction

We herein consider the uniqueness of equilibrium solutions for a compressible, hyperelastic body $\overline{\Omega} \subset \mathbb{R}^n$, subject to dead loads. This problem was previously analyzed by John [34] who showed that for the pure-displacement (Dirichlet) problem there is at most one smooth solution of the equilibrium (Euler-Lagrange) equations among those mappings that have uniformly small *strains*:

$$\mathbf{E}_{\mathbf{u}} := \tfrac{1}{2}\left[(\nabla\mathbf{u})^{\mathrm{T}}\nabla\mathbf{u} - \mathbf{I}\right],$$

where $\nabla\mathbf{u}$ denotes the matrix of partial derivatives of $\mathbf{u} : \overline{\Omega} \to \mathbb{R}^n$ and we write $\mathbf{F}^{\mathrm{T}}$ for the transpose of the n by n matrix $\mathbf{F}$. The main objective of this manuscript is the extension of John's result to the mixed problem. However, our approach also yields the uniqueness of equilibrium in a neighborhood in the space of strains. More precisely we prove that given a smooth solution of the equilibrium equations, $\mathbf{u}_{\mathrm{e}}$, at which the second variation of the energy is uniformly positive, there is no other equilibrium solution, $\mathbf{v}_{\mathrm{e}}$, for which the difference of the two right Cauchy-Green strain tensors:

$$(\nabla\mathbf{u}_{\mathrm{e}})^{\mathrm{T}}\nabla\mathbf{u}_{\mathrm{e}} - (\nabla\mathbf{v}_{\mathrm{e}})^{\mathrm{T}}\nabla\mathbf{v}_{\mathrm{e}} \tag{1.1}$$

is uniformly small.

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In the absence of body forces and surface tractions, the total energy of a deformation $\mathbf{u}:\overline{\Omega}\to\mathbb{R}^n$ of a compressible, hyperelastic body is given by

$$\mathcal{E}(\mathbf{u}) := \int_\Omega W\big(\mathbf{x}, \nabla\mathbf{u}(\mathbf{x})\big)\,\mathrm{d}\mathbf{x},$$

where $W:\overline{\Omega}\times\mathbb{M}^{n\times n}_+\to[0,\infty)$ denotes the stored-energy density and we write $\mathbb{M}^{n\times n}_+$ for the set of n by n matrices with positive determinant. We require that $\mathbf{u}=\mathbf{d}$ on $\mathcal{D}$, where $\mathbf{d}$ is prescribed and $\mathcal{D}\subset\partial\Omega$ is nonempty and relatively open. The pure-displacement problem can then be expressed as the condition $\mathcal{D}=\partial\Omega$, while the genuine-mixed problem is the condition $\mathcal{D}\subsetneq\partial\Omega$. We here consider both problems. With this notation, we call $\mathbf{u}_\mathrm{e}$ an equilibrium solution if it is a weak solution of the corresponding Euler-Lagrange equations:

$$\delta\mathcal{E}(\mathbf{u}_\mathrm{e})[\mathbf{w}] = \int_\Omega \mathbf{S}\big(\mathbf{x}, \nabla\mathbf{u}_\mathrm{e}(\mathbf{x})\big):\nabla\mathbf{w}(\mathbf{x})\,\mathrm{d}\mathbf{x} = 0$$

for all $\mathbf{w}\in W^{1,2}(\Omega;\mathbb{R}^n)$ that satisfy $\mathbf{w}=\mathbf{0}$ on $\mathcal{D}$, while the uniform positivity of the second variation of $\mathcal{E}$ at $\mathbf{u}_\mathrm{e}$ is then the condition that

$$\delta^2\mathcal{E}(\mathbf{u}_\mathrm{e})[\mathbf{w},\mathbf{w}] = \int_\Omega \nabla\mathbf{w}(\mathbf{x}):\mathbb{A}\big(\mathbf{x},\nabla\mathbf{u}_\mathrm{e}(\mathbf{x})\big)\big[\nabla\mathbf{w}(\mathbf{x})\big]\mathrm{d}\mathbf{x} \geq k\int_\Omega |\nabla\mathbf{w}(\mathbf{x})|^2\mathrm{d}\mathbf{x}$$

for some $k>0$ and all $\mathbf{w}\in W^{1,2}(\Omega;\mathbb{R}^n)$ that satisfy $\mathbf{w}=\mathbf{0}$ on $\mathcal{D}$. Here we write $W^{1,2}(\Omega;\mathbb{R}^n)$ for the usual Sobolev space of square-integrable, vector-valued functions whose distributional gradient $\nabla\mathbf{w}$ is square integrable. Also, $\mathbf{H}:\mathbf{K} := \mathrm{trace}(\mathbf{H}\mathbf{K}^\mathrm{T})$ and $\mathbf{S}(\mathbf{x},\mathbf{F})$ and $\mathbb{A}(\mathbf{x},\mathbf{F})$ denote the Piola-Kirchhoff stress and the Elasticity Tensor, respectively:

$$\mathbf{S}(\mathbf{x},\mathbf{F}) := \frac{\partial}{\partial\mathbf{F}}W(\mathbf{x},\mathbf{F}), \qquad \mathbb{A}(\mathbf{x},\mathbf{F}) := \frac{\partial^2}{\partial\mathbf{F}^2}W(\mathbf{x},\mathbf{F}).$$

It is well-known that when the second variation is uniformly positive at an equilibrium solution $\mathbf{u}_\mathrm{e}$, then there is a neighborhood of $\mathbf{u}_\mathrm{e}$ in the Sobolev space $W^{1,\infty}(\Omega;\mathbb{R}^n)$ in which there are no other solutions of the equilibrium equations. In addition, the energy of any other mapping in this neighborhood is strictly greater than the energy of $\mathbf{u}_\mathrm{e}$. These assertions follow readily from a simple analysis of the Taylor expansion of $\mathcal{E}$ that is inherited from the Taylor series for the stored-energy function W:

$$\mathcal{E}(\mathbf{w}+\mathbf{u}_\mathrm{e}) = \mathcal{E}(\mathbf{u}_\mathrm{e}) + \delta\mathcal{E}(\mathbf{u}_\mathrm{e})[\mathbf{w}] + \delta^2\mathcal{E}(\mathbf{u}_\mathrm{e})[\mathbf{w},\mathbf{w}] + \mathcal{R}(\mathbf{u}_\mathrm{e};\mathbf{w})$$

with

$$\big|\mathcal{R}(\mathbf{u}_\mathrm{e};\mathbf{w})\big| \leq C\int_\Omega |\nabla\mathbf{w}(\mathbf{x})|^3\mathrm{d}\mathbf{x}.$$

In particular, the choice $\mathbf{w}=\mathbf{v}-\mathbf{u}_\mathrm{e}$, the fact that $\mathbf{u}_\mathrm{e}$ is an equilibrium solution with uniformly positive second variation, and the standard inequality

$$\int_\Omega |\nabla\mathbf{w}(\mathbf{x})|^3\,\mathrm{d}\mathbf{x} \leq ||\nabla\mathbf{w}||_{L^\infty(\Omega)}\int_\Omega |\nabla\mathbf{w}(\mathbf{x})|^2\,\mathrm{d}\mathbf{x} \tag{1.2}$$

imply that

$$\mathcal{E}(\mathbf{v}) \geq \mathcal{E}(\mathbf{u}_\mathrm{e}) + c\int_\Omega |\nabla\mathbf{w}(\mathbf{x})|^2\,\mathrm{d}\mathbf{x} \tag{1.3}$$

for some $c>0$, provided $||\nabla\mathbf{w}||_{L^\infty(\Omega)}$ is sufficiently small. From this one deduces the claims.

The essential point of John's work is that, while the assumption that $||\mathbf{E_u}||_{L^\infty(\Omega)}$ and $||\mathbf{E_v}||_{L^\infty(\Omega)}$ are small need not imply the same for $||\nabla\mathbf{u} - \nabla\mathbf{v}||_{L^\infty(\Omega)}$, the above argument can be suitably modified to obtain uniqueness for the pure-displacement problem. For the purpose of our work it is convenient for us to separate two key components of his proof. The first is the fact that uniformly small strains $\mathbf{E_u}$ and $\mathbf{E_v}$ imply that $\nabla\mathbf{u} - \nabla\mathbf{v}$ has small norm in the space of functions of Bounded Mean Oscillation, a Geometric-Rigidity result that was obtained by John in various forms [33, 34, 35] and which has been further studied by Friesecke, James, & Müller [24] (see, also, Kohn [38] and Conti & Schweizer [16]). The second is that, while the preceding argument culminating in inequality (1.3) is designed for L^∞ neighborhoods of the gradient, it extends to BMO neighborhoods, although this requires a more sophisticated analysis. Specifically, one requires tools that allow for the replacement of L^∞ by BMO. The canonical example of such a tool is the John-Nirenberg inequality [36], and indeed, this is precisely what John used in his proof of uniqueness.

In this paper we pursue an alternative approach to this replacement through a local analogue of an inequality of Fefferman & Stein [22] for bounded Lipschitz domains. In particular, we make use of results of Iwaniec [31] and Diening, Růžička, & Schumacher [19] to obtain, in Theorem 2.6, an inequality that is valid for any bounded Lipschitz domain Ω: For every $q \in (1, \infty)$ there is a constant $F = F(q) > 0$ such that any $\psi \in L^1(\Omega)$ that satisfies $\psi^\#_\Omega \in L^q(\Omega)$ will also satisfy

$$F^{-1} \int_\Omega |\psi|^q \,\mathrm{d}\mathbf{x} \leq \int_\Omega |\psi^\#_\Omega|^q \,\mathrm{d}\mathbf{x} + \Big| \int_\Omega \psi \,\mathrm{d}\mathbf{x}\Big|^q.$$

Here $\psi^\#_\Omega$ (see $(2.1)_2$) denotes the maximal function of Fefferman & Stein [22]. This inequality implies an interpolation inequality analogous to (1.2) (as well as a more general family of inequalities, see Section 2):

$$\|\nabla\mathbf{w}\|_{L^3(\Omega)} \leq J\Big([\![\nabla\mathbf{w}]\!]_{\mathrm{BMO}(\Omega)} + \Big| \int_\Omega \nabla\mathbf{w}\,\mathrm{d}\mathbf{x}\Big| \Big)^{1/3} \|\nabla\mathbf{w}\|^{2/3}_{L^2(\Omega)}, \tag{1.4}$$

where $[\![\nabla\mathbf{w}]\!]_{\mathrm{BMO}(\Omega)}$ denotes the seminorm of $\nabla\mathbf{w}$ in $\mathrm{BMO}(\Omega)$ (see (2.3)). Therefore, if we replace (1.2) by (1.4), we obtain, in Theorem 3.3, a general uniqueness theorem in the Calculus of Variations for neighborhoods where both $[\![\nabla\mathbf{w}]\!]_{\mathrm{BMO}(\Omega)}$ and $\int_\Omega \nabla\mathbf{w}\,\mathrm{d}\mathbf{x}$ are small. This result is in the spirit of a theorem of Kristensen & Taheri [40] (see, also, Campos Cordero [10]) for the Dirichlet problem under the assumption that the extension of $\nabla\mathbf{w}$ by zero is small as an element of $\mathrm{BMO}(\mathbb{R}^n)$.

With these results established, we can return to the question of uniqueness in Elasticity. In particular, let us observe that for the pure-displacement problem, an integration by parts shows that the integral in (1.4) is zero; thus, the coefficient (with exponent 1/3) in the right-hand side of (1.4) reduces to the $\mathrm{BMO}(\Omega)$-seminorm, whose smallness follows from Geometric Rigidity, and so we obtain John's result. For the mixed problem, with a few elementary calculations we show that for functions which agree on a portion of the boundary one actually has a closeness not just of the seminorms, but of the entire norm of their derivatives in the space $\mathrm{BMO}(\Omega) \cap L^1(\Omega)$. Thus we obtain uniqueness for the mixed problem for small-strain solutions. The general result asserted at the beginning of the introduction then follows by a change of variables to the deformed configuration and an application of the previous analysis.

As noted by Kohn [38, p. 134], there is the question of whether one has an existence theory that produces an equilibrium solution with uniformly small strains. In particular, it is not clear from the existence theory of Ball [4], or any of its many extensions, whether or not $\mathbf{E}_{\mathbf{u}}$ is uniformly small. A few things can be said in this regard. First, a result of Zhang [63] for the displacement problem shows that Ball's minimizer is the equilibrium solution obtained from the implicit function theorem[1] (see, e.g., Valent [59] or Ciarlet [14, Chapter 6]) provided the boundary is smooth and the boundary displacements are sufficiently small. Second, the equilibrium solution obtained from the implicit function theorem will be as smooth as desired when the boundary, the stored energy W, and the boundary displacement $\mathbf{d}$ are all sufficiently smooth. (Though if the boundary is merely Lipschitz then a minimizer need not be smooth.) Finally, unconditional uniqueness of equilibrium solutions is neither desired nor expected in Nonlinear Elasticity. For example, when a thin rod is subjected to uniaxial compression, one expects that the rod will buckle and that there will be more than one buckled equilibrium solution. Thus it may be natural to impose additional restrictions to obtain uniqueness.

Results in the literature have established local uniqueness, uniqueness when the deformation gradient lies in certain subsets of $\mathbb{M}^{n\times n}_+$, uniqueness of the absolute minimizer of the energy when appropriate extra conditions are imposed, and uniqueness of equilibrium solutions for the displacement problem for certain bodies and boundary values. In particular, Knops and Stuart [39] (see, also, Bevan [8] and Taheri [57]) have proven that, for a star-shaped body, the homogeneous deformation $\mathbf{u}_{\rm h}(\mathbf{x}) = \mathbf{F}\mathbf{x} + \mathbf{a}$ is the only smooth equilibrium solution that satisfies a homogeneous displacement boundary condition whenever the energy is globally rank-one convex and strictly quasiconvex at $\mathbf{u}_{\rm h}$. Gurtin and Spector [28] have shown that there is at most one solution of the equilibrium equations that lies in any convex set where the second variation of the energy is strictly positive. Gao, Neff, Roventa, and Thiel [25] have recently established that the convexity of the elastic energy, when considered as a function of the right Cauchy-Green strain tensor, implies that any equilibrium solution $\mathbf{u}_{\rm e}$, at which the Cauchy Stress is positive semi-definite at every point, is an absolute minimizer of the energy. Moreover, if in addition $\mathbf{C}_{\rm e}(\mathbf{x})$ is a point of strict convexity of the energy at every $\mathbf{x} \in \Omega$, then $\mathbf{u}_{\rm e}$ is the unique absolute minimizer of the energy. Sivaloganathan & Spector [53] have demonstrated that, for a large class of polyconvex stored-energy functions, an equilibrium solution that satisfies a certain pointwise inequality is the unique absolute minimizer of the energy. They also gave an elementary proof, for the pure-displacement problem, of John's uniqueness with small strains result that we consider in Section 6.

There is also an extensive literature on nonuniqueness in Nonlinear Elasticity. Post & Sivaloganathan [49] have proven that there are an infinite number of equilibrium solutions for certain displacement problems for an annulus. Antman [3] has shown that, for the pure-traction problem, a thick spherical shell without loads has a second equilibrium solution corresponding

[1] Although the results in [59] are only stated for the pure-displacement and pure-traction problems, it appears that a similar analysis will be valid for the mixed problem *provided* that the parts of the boundary where displacements and tractions are prescribed have disjoint closures, for example, the inside and the outside of a thick spherical shell. See, e.g., Ciarlet [14, Chapter 6].

to an everted deformation. See [53, footnote 3] for additional references that contain examples of nonuniqueness.

Let us mention some related open problems before we proceed to the plan of the paper. Although our technique could, in principle, be applied to the pure-traction problem, we have not considered dead-load tractions applied to the entire boundary since the lack of any displacement boundary condition necessitates an additional mathematical constraint that induces the gradients of two solutions to be close in L^1 (see Proposition 4.8). From a physical point of view the difficulty is a potential axis of equilibrium for the loads that leads to nonuniqueness of equilibrium solutions. For a detailed explanation see, e.g., Valent [59, Chapter 5] or Truesdell & Noll [58, §44] and the references therein. An extension of our results to incompressible elastic bodies is of interest. Difficulties include the constraint that the deformation gradient lie on the manifold $\det \nabla\mathbf{u} = 1$ and the pressure, which appears as a Lagrange multiplier in the equilibrium equations. A uniqueness result for live loads would also be of interest. Here one might want to look at [13, 47, 51, 54], [14, §2.7], or [52, §13.3]. Lastly, our results necessitate that the equilibrium equations have a solution $\mathbf{u}_{\mathrm{e}}$ that is Lipschitz continuous.[2] However, some of our results also require that $\mathbf{u}_{\mathrm{e}}$ be one-to-one on $\overline{\Omega}$, which prohibits self-contact of the boundary of $\mathbf{u}_{\mathrm{e}}(\Omega)$. It would be of interest if this assumption could be excluded. In this regard, see Remark 7.4.

We commence our analysis in Section 2 with a development of the requisite harmonic analysis results. In particular, after we recall some properties of the Hardy-Littlewood and Fefferman-Stein maximal functions, we establish a local analogue of Fefferman and Stein's inequality in Theorem 2.6. We then demonstrate, in Theorem 2.8, how this inequality gives rise to a family of interpolation inequalities that implies (1.4).

In Section 3 we first recall some background material from the Calculus of Variations. We then make use of the interpolation inequality from the previous section to establish two results. The first, Lemma 3.2, shows that whenever two mappings, $\mathbf{u}$ and $\mathbf{v}$, have gradients that are sufficiently close in $\mathrm{BMO} \cap L^1$, the uniform positivity of the second variation of the energy at either mapping implies that the second variation at the other mapping is strictly positive in the direction $\mathbf{w} = \mathbf{v} - \mathbf{u}$, a simple result that we have found to be helpful in establishing uniqueness of equilibrium solutions. Finally, we show, in Theorem 3.3, that any mapping whose gradient is sufficiently close, in $\mathrm{BMO} \cap L^1$, to the gradient of a Lipschitz solution of the Euler-Lagrange equations whose second variation is uniformly positive, will have strictly greater energy than the solution and also cannot satisfy the Euler-Lagrange equations.

In Section 4 we observe that a general version of the relationship between the distance from $\nabla\mathbf{u}$ to the set of rotations and the norm in $\mathrm{BMO}(\Omega)$ of $\nabla\mathbf{u}$ is a consequence of a Geometric-Rigidity result established in [16, 24]: Given a mapping $\mathbf{u} \in W^{1,p}(\Omega;\mathbb{R}^n)$, $1 < p < \infty$, there is a particular rotation $\mathbf{R}_{\mathbf{u}}$ such that the distance in $L^p(\Omega)$ from $\nabla\mathbf{u}$ to $\mathbf{R}_{\mathbf{u}}$ is, up to a multiplicative constant which does not depend on $\mathbf{u}$, a lower bound for the distance in $L^p(\Omega)$ from $\nabla\mathbf{u}$ to the set of rotations (see Proposition 4.3). It follows that, when $\nabla\mathbf{u}$ is uniformly close to the set

[2] The standard existence theory for Nonlinear Elasticity (see, e.g., Ball [4]) yields minimizers in $W^{1,p}(\Omega;\mathbb{R}^n)$ that satisfy only alternative forms of the equilibrium equations. See, e.g., Ball [5, Theorem 2.4].

of rotations, $\nabla \mathbf{u}$ is small in BMO(Ω). We further show in Proposition 4.8 that two mappings in $W^{1,p}(\Omega;\mathbb{R}^n)$, $p > n$, that share the same boundary values on $\mathcal{D}$ will have gradients that are close in $L^1(\Omega)$ whenever the gradients are close to the set of rotations.

In Section 5 we first recall some of the terminology from Continuum Mechanics: bodies, deformations, deformation gradients, strains, and the elastic energy and its first two derivatives: the Piola-Kirchhoff stress tensor and the Elasticity Tensor. We then show, in Lemma 5.6, that the strains $\mathbf{E_u}$ are uniformly small if and only if the gradient of the underlying deformation $\mathbf{u}$ is uniformly close to the set of rotations. Finally, we note, in Theorem 5.8, that when the coefficient (with exponent 1/3) in the right-hand side of (1.4) is small and $\mathbf{u}_\mathrm{e}$ is an equilibrium solution with uniformly positive second variation, then $\mathbf{v} := \mathbf{u}_\mathrm{e} + \mathbf{w}$ cannot be a solution of the equilibrium equations and $\mathbf{v}$ must also have strictly greater energy than $\mathbf{u}_\mathrm{e}$.

In Section 6 we present our uniqueness results for Nonlinear Elasticity when all strains are uniformly small. We first establish that when the reference configuration is stress free and the Elasticity Tensor at the reference configuration is strictly positive definite, then not only is the second variation uniformly positive at the reference configuration, a result that is well-known and which follows from Korn's inequality, but the second variation is uniformly positive at any smooth deformation with sufficiently small strains $\mathbf{E_u}$. We then obtain, in Theorem 6.3, the result mentioned in the first paragraph of this manuscript: There is at most one equilibrium solution $\mathbf{u}_\mathrm{e}$ with sufficiently small strains $\mathbf{E}_\mathrm{e}$ and, moreover, any other deformation with small strains has strictly greater energy than the energy of $\mathbf{u}_\mathrm{e}$.

In Section 7 we extend our results for Elasticity to include one mapping with potentially large strains and a second mapping that is close to it in the space of strains, that is, for which the quantity given in (1.1) is uniformly small. We prove that given an equilibrium solution $\mathbf{u}_\mathrm{e}$ that is a diffeomorphism and for which the second variation of the energy is uniformly positive, any other mapping, $\mathbf{v}$, with right Cauchy-Green tensor, $\mathbf{C_v} = (\nabla \mathbf{v})^{\mathrm{T}} \nabla \mathbf{v}$, uniformly and sufficiently close to $\mathbf{C}_\mathrm{e} = (\nabla \mathbf{u}_\mathrm{e})^{\mathrm{T}} \nabla \mathbf{u}_\mathrm{e}$ cannot be a solution of the equilibrium equations and $\mathbf{v}$ must also have strictly greater energy than the energy of $\mathbf{u}_\mathrm{e}$. Our proof involves a change of variables that replaces the reference configuration Ω by the deformed configuration $\mathbf{u}_\mathrm{e}(\Omega)$. Once this is accomplished, small modifications of our previous analysis then yield the desired result.

Part I: Maximal Functions, the Second Variation, and BMO Local Minimizers

2. Maximal Functions

In this section we first recall some of the properties of the Hardy-Littlewood and Fefferman-Stein maximal functions. We then show that results of Iwaniec [31] and Diening, Růžička, & Schumacher [19] yield a version of the Fefferman-Stein inequality that is valid for many bounded, open regions. This inequality then allows us to give an elementary proof of a result of John [34, p. 632] that bounds the L^q-norm of a function by its L^p-norm, $q > p$, when the function is sufficiently small in $\mathrm{BMO} \cap L^1$ rather than L^∞.

2.1. **Preliminaries.** For any domain (nonempty, connected, open set) $\Omega \subset \mathbb{R}^n$, $n \geq 2$, we denote by $L^p(\Omega)$, $p \in [1, \infty)$, the space of real-valued Lebesgue measurable functions, ψ, whose L^p-norm is finite:

$$||\psi||_{p,\Omega}^p := \int_\Omega |\psi(\mathbf{x})|^p \, \mathrm{d}\mathbf{x} < \infty.$$

$L^1_{\mathrm{loc}}(\Omega)$ will consist of those Lebesgue measurable functions that are integrable on every compact subset of Ω. $L^\infty(\Omega)$ will denote those Lebesgue measurable functions whose essential supremum is finite. Given any $\psi \in L^1_{\mathrm{loc}}(V)$, where $V = \mathbb{R}^n$ or V is a bounded domain, the *Hardy-Littlewood* and *Fefferman-Stein maximal functions* of ψ are given by

$$\psi_V^\star(\mathbf{x}) := \sup_{\substack{Q \ni \mathbf{x}, \\ Q \subset V}} \fint_Q |\psi(\mathbf{y})| \, \mathrm{d}\mathbf{y}, \qquad \psi_V^\#(\mathbf{x}) := \sup_{\substack{Q \ni \mathbf{x}, \\ Q \subset V}} \fint_Q \left|\psi(\mathbf{y}) - \langle\psi\rangle_Q\right| \mathrm{d}\mathbf{y}, \tag{2.1}$$

respectively. When $V = \mathbb{R}^n$ we shall omit the subscript V. Here, and in the sequel, *the symbol Q will denote a nonempty, bounded (open) n-dimensional hypercube*[3] *with faces parallel to the coordinate hyperplanes* and

$$\langle\psi\rangle_V := \fint_V \psi(\mathbf{x}) \, \mathrm{d}\mathbf{x} := \frac{1}{|V|} \int_V \psi(\mathbf{x}) \, \mathrm{d}\mathbf{x},$$

with $|V|$ denoting the n-dimensional Lebesgue measure of any bounded domain $V \subset \mathbb{R}^n$. For future reference we note that it is not difficult to show that these functions satisfy the pointwise estimates, for *a.e.* $\mathbf{x} \in V$,

$$\psi(\mathbf{x}) \leq \psi_V^\star(\mathbf{x}), \qquad \psi_V^\#(\mathbf{x}) \leq 2\psi_V^\star(\mathbf{x}). \tag{2.2}$$

The BMO-seminorm is given by

$$|\psi|_{\mathrm{BMO}(V)} := \sup_{Q \subset V} \fint_Q |\psi(\mathbf{x}) - \langle\psi\rangle_Q| \, \mathrm{d}\mathbf{x}, \tag{2.3}$$

while the space BMO(V) (Bounded Mean Oscillation) is defined by

$$\mathrm{BMO}(V) := \{\psi \in L^1_{\mathrm{loc}}(V) : |\psi|_{\mathrm{BMO}(V)} < \infty\}.$$

Here, once again, $V = \mathbb{R}^n$ or V is a bounded domain and we shall omit the V when $V = \mathbb{R}^n$. For future reference we note that

$$|\psi|_{\mathrm{BMO}(V)} = \left\|\psi_V^\#\right\|_{\infty,V} = \operatorname*{ess\,sup}_{\mathbf{x} \in V} \psi_V^\#(\mathbf{x}). \tag{2.4}$$

Suppose now that V is a bounded domain and $\varphi \in L^p(V)$, $p \in [1, \infty]$. Then we define its *extension* $\tilde\varphi : \mathbb{R}^n \to \mathbb{R}$, to all of $\mathbb{R}^n$, by

$$\tilde\varphi(\mathbf{x}) := \begin{cases} \varphi(\mathbf{x}), & \text{if } \mathbf{x} \in V, \\ 0, & \text{if } \mathbf{x} \notin V. \end{cases}$$

Clearly, $\tilde\varphi \in L^p(\mathbb{R}^n)$. Moreover, the Hardy-Littlewood maximal function, $\tilde\varphi^\star$, is given by

$$\tilde\varphi^\star(\mathbf{x}) = (\tilde\varphi)^\star(\mathbf{x}) := \sup_{\substack{Q \ni \mathbf{x}, \\ Q \subset \mathbb{R}^n}} \frac{1}{|Q|} \int_{Q \cap V} |\varphi(\mathbf{y})| \, \mathrm{d}\mathbf{y}, \quad \mathbf{x} \in \mathbb{R}^n. \tag{2.5}$$

In the sequel, we shall make use of a result of Hardy & Littlewood and Wiener.

[3]We shall henceforth refer to a Q as a *cube*, rather than a hypercube.

Proposition 2.1. (See, e.g., Stein [56, p. 5].) *Let $1 < p \leq \infty$. Then there exists a constant*[4] *$H = H(p)$ such that if $\psi \in L^p(\mathbb{R}^n)$, then $\psi^\star \in L^p(\mathbb{R}^n)$ and ψ and $\psi^\star$ satisfy*

$$||\psi^\star||_{p,\mathbb{R}^n} \leq H||\psi||_{p,\mathbb{R}^n}. \tag{2.6}$$

We shall also utilize a more recent result of Diening, Růžička, & Schumacher.

Proposition 2.2. ([19, Theorem 5.23]) *Let $q \in (1, \infty)$ and suppose that $U \subset \mathbb{R}^n$ is a Lipschitz or John domain.*[5] *Then there exists a constant $R = R(q)$ with the following property: If $\psi \in L^1(U)$ and $\psi_U^\# \in L^q(U)$, then $\psi \in L^q(U)$ and*

$$\fint_U \left|\psi - \langle\psi\rangle_U\right|^q \mathrm{d}\mathbf{x} \leq R \fint_U \left|\psi_U^\#\right|^q \mathrm{d}\mathbf{x}. \tag{2.7}$$

Remark 2.3. (1). A key ingredient in the proof of Proposition 2.2 is a result of Iwaniec [31, Lemma 4] that establishes a version of the Fefferman-Stein [22, Theorem 5] inequality when the domain is a cube. (2). As noticed in [19], if $\langle\psi\rangle_U = 0$, then (2.7) together with (2.9) shows that the original Fefferman-Stein inequality is also valid for certain bounded domains. (3). Inequality (2.7) together with Hölder's inequality, (2.4), and the triangle inequality (see (2.14)) yields

$$\fint_U |\psi| \, \mathrm{d}\mathbf{x} \leq R^{1/q} \| \psi \|_{\mathrm{BMO}(U)} + \left| \fint_U \psi \, \mathrm{d}\mathbf{x} \right|. \tag{2.8}$$

This inequality was previously established by Brezis & Nirenberg [9, Lemma A.1] for connected, compact Riemannian manifolds without boundary.

Remark 2.4. (1). By a *Lipschitz domain U* we mean a bounded domain whose boundary ∂U is (strongly) Lipschitz. See, e.g., [20, p. 127], [44, p. 72], or [30, Definition 2.5]. Essentially, a bounded domain is Lipschitz if, in a neighborhood of every boundary point, the boundary is the graph of a Lipschitz function and the domain is on "one side" of this graph. (2). Proposition 2.2 is valid for a class of domains that is larger than Lipschitz domains: *John domains* [33]. Roughly speaking, in a *John domain* there is a particular point that can be connected to every other point by a rectifiable curve; these curves have uniformly bounded length; and the curves do not get too "close to the boundary." See, e.g., [19] or [43] for a precise description.

2.2. **Some Properties of Maximal Functions on Bounded Domains.**

2.2.1. *Extensions of Results Previously Established on Cubes (and on $\mathbb{R}^n$).*

A well-known result is that the Hardy-Littlewood-Wiener inequality, (2.6), is also valid on every bounded domain. We present a proof for the convenience of the reader.

Lemma 2.5. *Let $V \subset \mathbb{R}^n$ be a bounded domain and suppose that $p \in (1, \infty]$. Then there exists a constant $H = H(p) > 0$ such that if $\psi \in L^p(V)$, then $\psi_V^\star \in L^p(V)$ and ψ and $\psi_V^\star$ satisfy*

$$||\psi_V^\star||_{p,V} \leq H||\psi||_{p,V}. \tag{2.9}$$

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[4]Although most of the constants in this manuscript will depend on the dimension n, we shall usually omit this dependence in order to simplify the exposition. However, H does not depend on n.

[5]See Remark 2.4.

Proof for $p \neq \infty$. Fix $p \in (1,\infty)$ and let $\psi \in L^p(V)$. Then, since $\tilde{\psi} = 0$ on $\mathbb{R}^n \setminus V$, $\tilde{\psi} \in L^p(\mathbb{R}^n)$ with

$$\int_{\mathbb{R}^n} |\tilde{\psi}|^p \,\mathrm{d}\mathbf{x} = \int_V |\psi|^p \,\mathrm{d}\mathbf{x}. \tag{2.10}$$

Therefore, we can apply Proposition 2.1 to conclude, with the aid of (2.10), that $\tilde{\psi}^\star \in L^p(\mathbb{R}^n)$ and

$$\int_{\mathbb{R}^n} |\tilde{\psi}^\star|^p \,\mathrm{d}\mathbf{x} \leq H^p \int_V |\psi|^p \,\mathrm{d}\mathbf{x}. \tag{2.11}$$

The definitions of $\psi_V^\star$ and $\tilde{\psi}^\star$, $(2.1)_1$ and (2.5), imply that

$$\psi_V^\star(\mathbf{x}) \leq \tilde{\psi}^\star(\mathbf{x}) \quad \text{for } a.e.\ \mathbf{x} \in V$$

and hence

$$\int_V |\psi_V^\star|^p \,\mathrm{d}\mathbf{x} \leq \int_{\mathbb{R}^n} |\tilde{\psi}^\star|^p \,\mathrm{d}\mathbf{x}. \tag{2.12}$$

The desired result, (2.9), now follows from (2.11) and (2.12). □

We next establish a local version of the Fefferman-Stein inequality that is valid on certain bounded domains.

Theorem 2.6. *Let $q \in (1,\infty)$ and suppose that $U \subset \mathbb{R}^n$ is a Lipschitz (or John) domain. Then there exists a constant $F = F(q) > 0$ such that every $\psi \in L^1(U)$ that satisfies $\psi_U^\# \in L^q(U)$ will also satisfy $\psi \in L^q(U)$ with*

$$⨍_U |\psi|^q \,\mathrm{d}\mathbf{x} \leq F\left(⨍_U |\psi_U^\#|^q \,\mathrm{d}\mathbf{x} + \left| ⨍_U \psi \,\mathrm{d}\mathbf{x}\right|^q \right). \tag{2.13}$$

Before we prove Theorem 2.6, we first note that if we combine it with the Hardy-Littlewood-Wiener inequality on bounded domains, (2.9), we find that a result similar to Iwaniec's [31] version of the Fefferman-Stein inequality for cubes is also valid for John domains (except for the case $q = 1$).

Corollary 2.7. *Let $q \in (1,\infty)$ and suppose that $U \subset \mathbb{R}^n$ is a Lipschitz (or John) domain. Then there exists a constant $S = S(q) > 0$ such that every $\psi \in L^1(U)$ that satisfies $\psi_U^\# \in L^q(U)$ will also satisfy $\psi_U^\star \in L^q(U)$ with*

$$⨍_U |\psi_U^\star|^q \,\mathrm{d}\mathbf{x} \leq S\left(⨍_U |\psi_U^\#|^q \,\mathrm{d}\mathbf{x} + \left| ⨍_U \psi \,\mathrm{d}\mathbf{x}\right|^q \right).$$

Proof of Theorem 2.6. Fix $q \in (1,\infty)$. Then, by the triangle inequality,

$$|\psi(\mathbf{x})| \leq |\psi(\mathbf{x}) - \langle\psi\rangle_U| + |\langle\psi\rangle_U|. \tag{2.14}$$

Thus, if we take (2.14) to the q-th power, use the standard inequality $|a+b|^q \leq 2^{q-1}(|a| + |b|)$, integrate over U, and divide by $|U|$ we find that

$$⨍_U |\psi|^q \,\mathrm{d}\mathbf{x} \leq 2^{q-1} ⨍_U |\psi - \langle\psi\rangle_U|^q \,\mathrm{d}\mathbf{x} + 2^{q-1}\left| ⨍_U \psi \,\mathrm{d}\mathbf{x}\right|^q. \tag{2.15}$$

Finally, Proposition 2.2 yields a constant $R = R(q) > 0$, which does not depend on ψ, such that

$$⨍_U |\psi - \langle\psi\rangle_U|^q \,\mathrm{d}\mathbf{x} \leq R ⨍_U |\psi_U^\#|^q \,\mathrm{d}\mathbf{x}. \tag{2.16}$$

The desired result, (2.13), now follows from (2.15) and (2.16). □

2.2.2. *An Application of the Local Fefferman-Stein Inequality.* We next make use of Theorem 2.6 to establish an interpolation inequality that will be useful when we consider local minimizers of an integral functional in Section 3.

Theorem 2.8. *Let $q > p > 1$ and suppose that $U \subset \mathbb{R}^n$ is a Lipschitz (or John) domain. Then there exists a constant $J = J(p,q) > 0$ such that every $\psi \in \mathrm{BMO}(U) \cap L^1(U)$ satisfies*

$$||\psi||_{q,U} \le J\Big(||\psi||_{\mathrm{BMO}(U)}\Big)^{1-p/q}\Big(||\psi||_{p,U}\Big)^{p/q}, \tag{2.17}$$

where (see (2.3)*)*

$$||\psi||_{\mathrm{BMO}(U)} := [\![\psi]\!]_{\mathrm{BMO}(U)} + \Big|⨍_U \psi \,\mathrm{d}\mathbf{x}\Big|. \tag{2.18}$$

Remark 2.9. (1). Inequality (2.8) shows that (2.18) is an equivalent norm on $\mathrm{BMO}(U) \cap L^1(U)$. (2). Inequality (2.17), for a function that has integral equal to zero and is sufficiently small in $\mathrm{BMO}(Q)$ (Q a cube), was obtained by John [34, p. 632], who showed that it is a consequence of the John-Nirenberg inequality [36].

Remark 2.10. Our proof of Theorem 2.8 makes use of Theorem 2.6, however, since (2.17) is an interpolation inequality, there are other techniques one might use. In particular, there is an analogue of (2.17) for $\mathbb{R}^n$ (see, e.g., Bennett & Sharpley [6, Theorem 8.11]) and so one might try to combine P. Jones' extension theorem [37] with such an inequality. One could also consider an approach that employs complex interpolation theory on metric measure spaces. See Carbonaro, Mauceri & Meda [11, 12].

Proof of Theorem 2.8. Fix $q > p > 1$ and suppose that $\psi \in \mathrm{BMO}(U) \cap L^1(U)$. We first note that (2.4) gives us $\psi_U^\# \in L^\infty(U)$. Consequently, Theorem 2.6 yields $\psi \in L^q(U)$ and a constant $F = F(q) > 0$, which does not depend on ψ, such that

$$F^{-1}⨍_U |\psi|^q \,\mathrm{d}\mathbf{x} \le ⨍_U |\psi_U^\#|^q \,\mathrm{d}\mathbf{x} + \Big|⨍_U \psi \,\mathrm{d}\mathbf{x}\Big|^q. \tag{2.19}$$

Next, in view of Hölder's inequality,

$$\begin{gathered} ⨍_U |\psi_U^\#|^q \,\mathrm{d}\mathbf{x} \le \Big(\|\psi_U^\#\|_{\infty,U}\Big)^{q-p}⨍_U |\psi_U^\#|^p \,\mathrm{d}\mathbf{x}, \\ \Big|⨍_U \psi \,\mathrm{d}\mathbf{x}\Big|^q \le \Big|⨍_U \psi \,\mathrm{d}\mathbf{x}\Big|^{q-p}⨍_U |\psi|^p \,\mathrm{d}\mathbf{x}. \end{gathered} \tag{2.20}$$

Also, Lemma 2.5 together with $(2.2)_2$ yield a constant $H = H(p) > 0$, which does not depend on ψ, such that

$$⨍_U |\psi_U^\#|^p \,\mathrm{d}\mathbf{x} \le (2H)^p⨍_U |\psi|^p \,\mathrm{d}\mathbf{x}. \tag{2.21}$$

If we now combine (2.19), (2.20), and (2.21) we find, with the aid of (2.4) and the standard inequality $|a|^t + |b|^t \le 2(|a| + |b|)^t$ ($t > 0$), that

$$⨍_U |\psi|^q \,\mathrm{d}\mathbf{x} \le 2FK\bigg([\![\psi]\!]_{\mathrm{BMO}(U)} + \Big|⨍_U \psi \,\mathrm{d}\mathbf{x}\Big|\bigg)^{q-p}⨍_U |\psi|^p \,\mathrm{d}\mathbf{x} \tag{2.22}$$

with $K := \max\{1, (2H)^p\}$. The desired result, (2.17), now follows from (2.18) and (2.22). □

3. A Problem from the Calculus of Variations

In this section we consider an energy minimization problem arising in the Calculus of Variations. We use the results in the previous section to determine conditions upon a solution of the corresponding Euler-Lagrange equations which imply that the solution is a local minimizer of the energy in the BMO $\cap L^1$-topology.

3.1. **Further Preliminaries.** We denote the usual inner product of $\mathbf{a}, \mathbf{b} \in \mathbb{R}^d$ ($d = n$ or $d = N$) by $\mathbf{a} \cdot \mathbf{b}$. The norm of $\mathbf{a} \in \mathbb{R}^d$ is then defined by $|\mathbf{a}| := \sqrt{\mathbf{a} \cdot \mathbf{a}}$. We shall write

$$|\mathbf{A}|^2 := \sum_{i=1}^{N} \sum_{j=1}^{n} |A_{ij}|^2, \tag{3.1}$$

for the norm of $\mathbf{A} \in \mathbb{M}^{N \times n}$ (the N by n matrices). Here A_{ij} denotes the component of $\mathbf{A}$ from the i-th row and the j-th column.

We fix a Lipschitz domain $\Omega \subset \mathbb{R}^n$, $n \geq 2$, with boundary $\partial\Omega$. For $1 \leq p \leq \infty$, $W^{1,p}(\Omega; \mathbb{R}^N)$ will denote the usual Sobolev space of (Lebesgue) measurable (vector-valued) functions $\mathbf{u} \in L^p(\Omega; \mathbb{R}^N)$ whose distributional gradient $\nabla \mathbf{u}$ is also contained in L^p. If $\phi \in W^{1,p}(\Omega)$ we shall denote its $W^{1,p}$-norm by

$$||\phi||_{W^{1,p}(\Omega)} := \Big(||\phi||_{p,\Omega}^p + ||\nabla\phi||_{p,\Omega}^p \Big)^{1/p}, \quad 1 \leq p < \infty,$$
$$||\phi||_{W^{1,\infty}(\Omega)} := \max\{ ||\phi||_{\infty,\Omega}, ||\nabla\phi||_{\infty,\Omega} \}, \quad p = \infty.$$

For any $V \subset \mathbb{R}^n$ we denote the closure of V by $\overline{V}$.

3.2. **An Integrand, the Energy, and the Euler-Lagrange Equations.** We take

$$\partial\Omega = \overline{\mathcal{D}} \cup \overline{\mathcal{S}} \quad \text{with } \mathcal{D} \text{ and } \mathcal{S} \text{ relatively open and } \mathcal{D} \cap \mathcal{S} = \varnothing.$$

If $\mathcal{D} \neq \varnothing$ we assume that a Lipschitz-continuous function $\mathbf{d} : \mathcal{D} \to \mathbb{R}^N$ is prescribed. If $\mathcal{S} \neq \varnothing$ we assume that a function $\mathbf{s} \in L^2(\mathcal{S}; \mathbb{R}^N)$ is prescribed. We also suppose that a function $\mathbf{b} \in L^2(\Omega; \mathbb{R}^N)$ is prescribed. In addition, we fix a nonempty, open set $\mathcal{O} \subset \mathbb{M}^{N \times n}$.

Hypothesis 3.1. We suppose that we are given an *integrand* $W : \overline{\Omega} \times \mathcal{O} \to \mathbb{R}$ that satisfies:

1. $\mathbf{F} \mapsto W(\mathbf{x}, \mathbf{F}) \in C^3(\mathcal{O})$, for *a.e.* $\mathbf{x} \in \Omega$;
2. $(\mathbf{x}, \mathbf{F}) \mapsto \mathrm{D}^k W(\mathbf{x}, \mathbf{F})$, $k = 0, 1, 2, 3$, are each (Lebesgue) measurable on their common domain $\Omega \times \mathcal{O}$; and
3. $(\mathbf{x}, \mathbf{F}) \mapsto \mathrm{D}^k W(\mathbf{x}, \mathbf{F})$, $k = 0, 1, 2, 3$, are each bounded on $\overline{\Omega} \times K$ for every compact $K \subset \mathcal{O}$.

Here, and in the sequel,

$$\mathrm{D}^0 W(\mathbf{x}, \mathbf{F}) := W(\mathbf{x}, \mathbf{F}), \qquad \mathrm{D}^k W(\mathbf{x}, \mathbf{F}) := \frac{\partial^k}{\partial \mathbf{F}^k} W(\mathbf{x}, \mathbf{F})$$

denotes k-th derivative of $\mathbf{F} \mapsto W(\cdot, \mathbf{F})$. Note that, for almost every $\mathbf{x} \in \overline{\Omega}$ and every $\mathbf{F} \in \mathcal{O}$,

$$\mathrm{D}W(\mathbf{x}, \mathbf{F}) : \mathbb{M}^{N \times n} \to \mathbb{R}, \qquad \mathrm{D}^2 W(\mathbf{x}, \mathbf{F}) : \mathbb{M}^{N \times n} \times \mathbb{M}^{N \times n} \to \mathbb{R}$$

can be viewed as a linear and a bilinear form, respectively.

We denote the set of *Admissible Mappings* by[6]

$$\mathrm{AM} := \{\mathbf{u} \in W^{1,\infty}(\Omega;\mathbb{R}^n) : \nabla\mathbf{u} \in \mathcal{O} \text{ and } \ \mathbf{u} = \mathbf{d} \text{ on } \mathcal{D} \text{ or } \langle\mathbf{u}\rangle_\Omega = \mathbf{0} \text{ if } \mathcal{D} = \varnothing\},$$

where $\nabla\mathbf{u} \in \mathcal{O}$ signifies that $\nabla\mathbf{u}(\mathbf{x}) \in \mathcal{O}$ for *a.e.* $\mathbf{x} \in \Omega$. The *energy* of $\mathbf{u} \in \mathrm{AM}$ is defined by

$$\mathcal{E}(\mathbf{u}) := \int_\Omega \big[W\big(\mathbf{x}, \nabla\mathbf{u}(\mathbf{x})\big) - \mathbf{b}(\mathbf{x}) \cdot \mathbf{u}(\mathbf{x})\big]\,\mathrm{d}\mathbf{x} - \int_{\mathcal{S}} \mathbf{s}(\mathbf{x}) \cdot \mathbf{u}(\mathbf{x})\,\mathrm{d}\mathcal{H}^{n-1}_{\mathbf{x}}, \tag{3.2}$$

where $\mathcal{H}^k$ denotes k-dimensional Hausdorff measure. We shall assume that we are given a $\mathbf{u}_\mathrm{e} \in \mathrm{AM}$ that is a weak solution of the *Euler-Lagrange equations* corresponding to (3.2), i.e.,

$$0 = \int_\Omega \big[\mathrm{D}W\big(\mathbf{x}, \nabla\mathbf{u}_\mathrm{e}(\mathbf{x})\big)[\nabla\mathbf{w}(\mathbf{x})] - \mathbf{b}(\mathbf{x}) \cdot \mathbf{w}(\mathbf{x})\big]\,\mathrm{d}\mathbf{x} - \int_{\mathcal{S}} \mathbf{s}(\mathbf{x}) \cdot \mathbf{w}(\mathbf{x})\,\mathrm{d}\mathcal{H}^{n-1}_{\mathbf{x}} \tag{3.3}$$

for all *variations* $\mathbf{w} \in \mathrm{Var}$, where

$$\mathrm{Var} := \{\mathbf{w} \in W^{1,2}(\Omega;\mathbb{R}^N) : \mathbf{w} = \mathbf{0} \text{ on } \mathcal{D} \ \text{ or } \ \langle\mathbf{w}\rangle_\Omega = \mathbf{0} \text{ if } \mathcal{D} = \varnothing\}.$$

If $\mathcal{S} = \varnothing$ then $\mathbf{u}_\mathrm{e}$ is a solution of the *Dirichlet* problem. If $\mathcal{D} = \varnothing$ then $\mathbf{u}_\mathrm{e}$ is a solution of the *Neumann* problem. Otherwise, $\mathbf{u}_\mathrm{e}$ is a solution of the *mixed* problem. For future reference we note that, for the Dirichlet problem, the divergence theorem implies that, for all $\mathbf{w} \in \mathrm{Var}$,

$$\int_\Omega \nabla\mathbf{w}(\mathbf{x})\,\mathrm{d}\mathbf{x} = \mathbf{0}. \tag{3.4}$$

We are interested in the local minimality (in an appropriate topology) of solutions of (3.3). For future use we note that, for every $\mathbf{u}, \mathbf{v} \in \mathrm{AM}$, (3.2) gives us

$$\mathcal{E}(\mathbf{v}) - \mathcal{E}(\mathbf{u}) = \int_\Omega \big[W\big(\nabla\mathbf{v}\big) - W\big(\nabla\mathbf{u}\big) - \mathbf{b} \cdot \mathbf{w}\big]\mathrm{d}\mathbf{x} - \int_{\mathcal{S}} \mathbf{s} \cdot \mathbf{w}\,\mathrm{d}\mathcal{H}^{n-1}_{\mathbf{x}},$$

where $\mathbf{w} := \mathbf{v} - \mathbf{u} \in W^{1,\infty}(\Omega;\mathbb{R}^N) \cap \mathrm{Var}$. It follows that, when $\mathbf{u}_\mathrm{e} \in \mathrm{AM}$ is a solution of the Euler-Lagrange equations, (3.3), we have the identity, for every $\mathbf{v} \in \mathrm{AM}$,

$$\mathcal{E}(\mathbf{v}) - \mathcal{E}(\mathbf{u}_\mathrm{e}) = \int_\Omega \Big(W\big(\mathbf{x}, \nabla\mathbf{v}(\mathbf{x})\big) - W\big(\mathbf{x}, \nabla\mathbf{u}_\mathrm{e}(\mathbf{x})\big) - \mathrm{D}W\big(\mathbf{x}, \nabla\mathbf{u}_\mathrm{e}(\mathbf{x})\big)[\nabla\mathbf{w}(\mathbf{x})]\Big)\mathrm{d}\mathbf{x}. \tag{3.5}$$

For future reference we note that the second variation of the energy is continuous in a certain "direction" in the $\mathrm{BMO} \cap L^1$-topology.

Lemma 3.2. *Let W satisfy (1)–(3) of Hypothesis 3.1. Suppose that $\mathbf{u} \in \mathrm{AM}$ satisfies, for some $\hat{k} > 0$ and all $\mathbf{z} \in \mathrm{Var}$,*

$$\begin{gathered}\int_\Omega \mathrm{D}^2W\big(\mathbf{x}, \nabla\mathbf{u}(\mathbf{x})\big)\big[\nabla\mathbf{z}(\mathbf{x}), \nabla\mathbf{z}(\mathbf{x})\big]\,\mathrm{d}\mathbf{x} \geq 8\hat{k}\int_\Omega |\nabla\mathbf{z}(\mathbf{x})|^2\mathrm{d}\mathbf{x}, \\ \nabla\mathbf{u}(\mathbf{x}) \in \mathcal{B} \textit{ for a.e. } \mathbf{x} \in \Omega,\end{gathered} \tag{3.6}$$

where $\mathcal{B}$ is a nonempty, bounded, open set with $\overline{\mathcal{B}} \subset \mathcal{O} \subset \mathbb{M}^{N\times n}$. Then there exists an $\varepsilon > 0$ such that any $\mathbf{v} \in \mathrm{AM}$ that satisfies, for a.e. $\mathbf{x} \in \Omega$,

$$\nabla\mathbf{v}(\mathbf{x}) \in \mathcal{B}, \qquad [\![\nabla\mathbf{v} - \nabla\mathbf{u}]\!]_{\mathrm{BMO}(\Omega)} < \varepsilon, \qquad \Big|\fint_\Omega (\nabla\mathbf{v} - \nabla\mathbf{u})\,\mathrm{d}\mathbf{x}\Big| < \varepsilon \tag{3.7}$$

will also satisfy

$$\int_\Omega \mathrm{D}^2W(\mathbf{x}, \nabla\mathbf{v}(\mathbf{x}))\big[\nabla\mathbf{w}(\mathbf{x}), \nabla\mathbf{w}(\mathbf{x})\big]\,\mathrm{d}\mathbf{x} \geq 4\hat{k}\int_\Omega \big|\nabla\mathbf{w}(\mathbf{x})\big|^2\,\mathrm{d}\mathbf{x}, \quad \mathbf{w} := \mathbf{v} - \mathbf{u}. \tag{3.8}$$

[6]Since Ω is a Lipschitz domain, each $\mathbf{u} \in \mathrm{AM}$ has a representative that is Lipschitz continuous.

Proof. For clarity of exposition, we suppress the variable $\mathbf{x}$. Let $\mathbf{u} \in \mathrm{AM}$ satisfy (3.6) for all $\mathbf{z} \in \mathrm{Var}$. Suppose that $\mathbf{v} \in \mathrm{AM}$ satisfies (3.7) for some $\varepsilon > 0$ to be determined later and define $\mathbf{w} := \mathbf{v} - \mathbf{u}$. Then, Lemma A.3 with $\mathbf{G} = \nabla\mathbf{v}$, $\mathbf{F} = \nabla\mathbf{u}$, and $\mathbf{L} = \mathbf{G} - \mathbf{F} = \nabla\mathbf{w}$ yields a constant $\hat{c} = \hat{c}(\mathcal{B}) > 0$ such that, for *a.e.* $\mathbf{x} \in \Omega$,

$$\mathrm{D}^2 W(\nabla\mathbf{v})[\nabla\mathbf{w}, \nabla\mathbf{w}] \geq \mathrm{D}^2 W(\nabla\mathbf{u})[\nabla\mathbf{w}, \nabla\mathbf{w}] - \hat{c}|\nabla\mathbf{w}|^3. \tag{3.9}$$

If we now integrate (3.9) over Ω and make use of the uniform positivity of the second variation, $(3.6)_1$, we find that

$$\int_\Omega \mathrm{D}^2 W(\nabla\mathbf{v})[\nabla\mathbf{w}, \nabla\mathbf{w}]\,\mathrm{d}\mathbf{x} \geq 8\hat{k}\int_\Omega |\nabla\mathbf{w}|^2\,\mathrm{d}\mathbf{x} - \hat{c}\int_\Omega |\nabla\mathbf{w}|^3\,\mathrm{d}\mathbf{x}. \tag{3.10}$$

We next note that Theorem 2.8 yields a $J > 0$ such that, for the given $\mathbf{u}$ and $\mathbf{v}$ that satisfy $(3.7)_{2,3}$,

$$2\varepsilon J\int_\Omega |\nabla\mathbf{w}|^2\,\mathrm{d}\mathbf{x} \geq \int_\Omega |\nabla\mathbf{w}|^3\,\mathrm{d}\mathbf{x}, \qquad \mathbf{w} := \mathbf{v} - \mathbf{u}. \tag{3.11}$$

It is now clear that (3.8) follows from (3.10) and (3.11) when ε is sufficiently small. □

3.3. **Implications of the Positivity of the Second Variation.** In this subsection we show that any admissible mapping $\mathbf{v}$ with gradient sufficiently close, in $\mathrm{BMO} \cap L^1$, to the gradient of a Lipschitz solution of the Euler-Lagrange equations whose second variation is uniformly positive, will have strictly greater energy than the solution. In addition, it will follow that such a $\mathbf{v}$ cannot itself satisfy the Euler-Lagrange equations.

Theorem 3.3. *Let W satisfy (1)–(3) of Hypothesis 3.1. Suppose that $\mathbf{u}_\mathrm{e} \in \mathrm{AM}$ is a weak solution of the Dirichlet, Neumann, or mixed problem, i.e.,* (3.3)*, that satisfies, for some $\hat{k} > 0$ and all $\mathbf{z} \in \mathrm{Var}$,*

$$\begin{gathered}\int_\Omega \mathrm{D}^2 W\big(\mathbf{x}, \nabla\mathbf{u}_\mathrm{e}(\mathbf{x})\big)\big[\nabla\mathbf{z}(\mathbf{x}), \nabla\mathbf{z}(\mathbf{x})\big]\,\mathrm{d}\mathbf{x} \geq 8\hat{k}\int_\Omega |\nabla\mathbf{z}(\mathbf{x})|^2\mathrm{d}\mathbf{x}, \\ \nabla\mathbf{u}_\mathrm{e}(\mathbf{x}) \in \mathcal{B} \text{ for a.e. } \mathbf{x} \in \Omega,\end{gathered} \tag{3.12}$$

where $\mathcal{B}$ is a nonempty, bounded, open set with $\overline{\mathcal{B}} \subset \mathcal{O} \subset \mathbb{M}^{N\times n}$. Then there exists a $\delta > 0$ such that any $\mathbf{v} \in \mathrm{AM}$ that satisfies, for a.e. $\mathbf{x} \in \Omega$,

$$\nabla\mathbf{v}(\mathbf{x}) \in \mathcal{B}, \qquad [\![\nabla\mathbf{v} - \nabla\mathbf{u}_\mathrm{e}]\!]_{\mathrm{BMO}(\Omega)} < \delta, \qquad \left|\fint_\Omega (\nabla\mathbf{v} - \nabla\mathbf{u}_\mathrm{e})\,\mathrm{d}\mathbf{x}\right| < \delta \tag{3.13}$$

will also satisfy

$$\mathcal{E}(\mathbf{v}) \geq \mathcal{E}(\mathbf{u}_\mathrm{e}) + \hat{k}\int_\Omega |\nabla\mathbf{v} - \nabla\mathbf{u}_\mathrm{e}|^2\mathrm{d}\mathbf{x}. \tag{3.14}$$

In particular, $\mathbf{v} \not\equiv \mathbf{u}_\mathrm{e}$ will have strictly greater energy than $\mathbf{u}_\mathrm{e}$. Moreover, $\mathbf{v}$ cannot be a solution of the Euler-Lagrange equations, (3.3)*.*

Remark 3.4. (1). For the Dirichlet problem, (3.4) shows that the integral in $(3.13)_3$ is equal to zero; consequently, $(3.13)_3$ is trivially satisfied for any $\delta > 0$. (2). Since we have assumed that $\mathbf{u}_\mathrm{e} \in W^{1,\infty}(\Omega; \mathbb{R}^n)$, sets $\mathcal{B} \subset \mathbb{M}^{N\times n}$ that satisfy $(3.12)_2$ do exist, e.g.,

$$\mathcal{B} := \mathcal{B}(||\nabla\mathbf{u}_\mathrm{e}||_\infty) = \{\mathbf{F} \in \mathbb{M}^{N\times n} : |\mathbf{F}| < 1 + ||\nabla\mathbf{u}_\mathrm{e}||_{\infty,\Omega}\}.$$

However, the integrand $W : \overline{\Omega} \times \mathcal{O} \to \mathbb{R}$ need not be defined on all of $\overline{\Omega} \times \mathcal{B}(||\nabla \mathbf{u}_{\rm e}||_\infty)$. For example, in Nonlinear Elasticity (see Section 5.2) one usually assumes that[7]

$$\mathcal{O} = \{\mathbf{F} \in \mathbb{M}^{n\times n} : \det \mathbf{F} > 0\}$$

in which case $\mathbf{0} \notin \mathcal{O}$ and hence $\mathcal{B}(||\nabla \mathbf{u}_{\rm e}||_\infty) \not\subset \mathcal{O}$.

Remark 3.5. Kristensen & Taheri [40, Section 6] and Campos Cordero [10, Section 4] have each obtained a result that is analogous to Theorem 3.3 for Dirichlet boundary data. In particular, they show that, under weaker smoothness hypotheses than used here ($\mathbf{F} \mapsto W(\mathbf{F}) \in C^2(\mathbb{M}^{N\times n})$ and $(\mathbf{x}, \mathbf{F}) \mapsto W(\mathbf{x}, \mathbf{F}) \in C^2(\overline{\Omega} \times \mathbb{M}^{N\times n})$, respectively), one has uniqueness in the regime where the extension by zero of $\mathbf{H}(\mathbf{x}) := \nabla \mathbf{v}(\mathbf{x}) - \nabla \mathbf{u}_{\rm e}(\mathbf{x})$ is sufficiently small as an element of $\mathrm{BMO}(\mathbb{R}^n)$. The extension of our result to C^2 integrands appears to depend on a particular generalization of the Fefferman-Stein inequality to bounded domains: more precisely, a version of Theorem 2.6 for certain Orlicz spaces. The proofs of Lemma 6.2 in [40] and Lemmas 4.6 and 4.7 in [10] modify the Fefferman-Stein inequality on all of $\mathbb{R}^n$ by introducing the modulus of continuity, ω, of $\mathrm{D}^2 W$ in Taylor's theorem and then making use of $t \mapsto t^2\omega(t)$ as an N-function (see, e.g., [1]). Such an extension for *cubes* has been obtained by Verde & Zecca [60, Theorem 2.1], however, we are not aware of any corresponding proof for Lipschitz (or John) domains.

Proof of Theorem 3.3. For clarity of exposition, we suppress the variable $\mathbf{x}$. Let $\mathbf{u}_{\rm e} \in \mathrm{AM}$ be a solution of the Euler-Lagrange equations, (3.3), that satisfies (3.12) for all $\mathbf{z} \in \mathrm{Var}$. Suppose that $\mathbf{v} \in \mathrm{AM}$ satisfies (3.13) for some $\delta > 0$ to be determined later and define $\mathbf{w} := \mathbf{v} - \mathbf{u}_{\rm e} \in \mathrm{Var}$. Then, Lemma A.1 with $\mathbf{G} = \nabla \mathbf{v}$, $\mathbf{F} = \nabla \mathbf{u}_{\rm e}$, and $\mathbf{H} = \mathbf{G} - \mathbf{F} = \nabla \mathbf{w}$, yields a constant $c = c(\mathcal{B}) > 0$ such that, for *a.e.* $\mathbf{x} \in \Omega$,

$$W(\nabla \mathbf{v}) \geq W(\nabla \mathbf{u}_{\rm e}) + \mathrm{D}W(\nabla \mathbf{u}_{\rm e})[\nabla \mathbf{w}] + \tfrac{1}{2}\mathrm{D}^2 W(\nabla \mathbf{u}_{\rm e})[\nabla \mathbf{w}, \nabla \mathbf{w}] - c|\nabla \mathbf{w}|^3. \tag{3.15}$$

If we now integrate (3.15) over Ω and make use of the uniform positivity of the second variation, $(3.12)_1$, we find, with the aid of (3.5) (which is a consequence of the fact that $\mathbf{u}_{\rm e}$ satisfies the Euler-Lagrange equations (3.3)), that

$$\mathcal{E}(\mathbf{v}) \geq \mathcal{E}(\mathbf{u}_{\rm e}) + 2\hat{k} \int_\Omega |\nabla \mathbf{w}|^2 \,\mathrm{d}\mathbf{x} - c \int_\Omega |\nabla \mathbf{w}|^3 \,\mathrm{d}\mathbf{x}. \tag{3.16}$$

We next note that Theorem 2.8 yields a $J > 0$ such that, for the given $\mathbf{u}_{\rm e}$ and $\mathbf{v}$ that satisfy $(3.13)_{2,3}$,

$$2\delta J \int_\Omega |\nabla \mathbf{w}|^2 \,\mathrm{d}\mathbf{x} \geq \int_\Omega |\nabla \mathbf{w}|^3 \,\mathrm{d}\mathbf{x}, \qquad \mathbf{w} := \mathbf{v} - \mathbf{u}. \tag{3.17}$$

It is now clear that (3.14) follows from (3.16) and (3.17) when δ is sufficiently small.

Now, suppose that $\mathcal{E}(\mathbf{v}) = \mathcal{E}(\mathbf{u}_{\rm e})$. Then (3.14) yields $\nabla \mathbf{v} = \nabla \mathbf{u}_{\rm e}$ in Ω and hence, since Ω is open and connected, $\mathbf{v} = \mathbf{u}_{\rm e} + \mathbf{a}$ for some $\mathbf{a} \in \mathbb{R}^N$. However, $\mathbf{w} = \mathbf{v} - \mathbf{u}_{\rm e} \in \mathrm{Var}$ and so either $\mathbf{v} = \mathbf{u}_{\rm e}$ on $\mathcal{D}$ or $\langle \mathbf{w} \rangle_\Omega = \mathbf{0}$, both of which force $\mathbf{a} = \mathbf{0}$. Thus, $\mathcal{E}(\mathbf{v}) = \mathcal{E}(\mathbf{u}_{\rm e})$ implies $\mathbf{v} \equiv \mathbf{u}_{\rm e}$.

Finally, we note that Lemma 3.2 shows that, if $\delta \in (0, \varepsilon)$, then the second variation of the energy is uniformly positive in the direction $\mathbf{v} - \mathbf{u}_{\rm e}$ at $\mathbf{v}$, that is, $\mathbf{v}$ satisfies (3.12) with $\mathbf{u}_{\rm e}$ replaced by $\mathbf{v}$ and $\mathbf{z} = \mathbf{v} - \mathbf{u}_{\rm e}$. Now, suppose for the sake of contradiction that $\mathbf{v} \not\equiv \mathbf{u}_{\rm e}$ is

[7]Here $\det \mathbf{F}$ denotes the determinant of $\mathbf{F} \in \mathbb{M}^{n\times n}$.

also a solution of (3.3). Then, the above argument, with $\mathbf{u}_e$ replaced by $\mathbf{v}$ and $\mathbf{v}$ replaced by $\mathbf{u}_e$, shows that $\mathcal{E}(\mathbf{u}_e) > \mathcal{E}(\mathbf{v})$, which contradicts $\mathcal{E}(\mathbf{v}) > \mathcal{E}(\mathbf{u}_e)$. Thus, two distinct solutions of (3.3), both of which satisfy (3.13), is not possible. □

Part II: Rotations, Sobolev Mappings, and Nonlinear Elasticity

4. Rotations, Geometric Rigidity, and Sobolev Mappings

In this section we consider the set of n-dimensional rotations with an interest in a comparison of the distance of a Sobolev mapping from this set to the distance the mapping has from a single rotation.

4.1. **Additional Preliminaries.** We shall write $\mathbf{H}:\mathbf{K} := \mathrm{tr}(\mathbf{H}\mathbf{K}^{\mathrm{T}})$ for the inner product of $\mathbf{H}, \mathbf{K} \in \mathbb{M}^{n\times n}$, where tr denotes the trace and $\mathbf{K}^{\mathrm{T}}$ denotes the transpose of $\mathbf{K}$. The norm of $\mathbf{H} \in \mathbb{M}^{n\times n}$, which is defined by (3.1), is then equal to $\sqrt{\mathbf{H}:\mathbf{H}}$. We shall denote the set of n-dimensional *rotations* by $\mathrm{SO}(n)$; thus, every $\mathbf{R} \in \mathrm{SO}(n)$ satisfies $\mathbf{R}^{\mathrm{T}}\mathbf{R} = \mathbf{R}\mathbf{R}^{\mathrm{T}} = \mathbf{I}$ and $\det \mathbf{R} = 1$, where $\mathbf{I} \in \mathbb{M}^{n\times n}$ denotes the identity matrix. If $\mathbf{V} \in \mathbb{M}^{n\times n}$ is invertible, we use the notation $\mathbf{V}^{-1}$ to denote its inverse, viz., $\mathbf{V}\mathbf{V}^{-1} = \mathbf{V}^{-1}\mathbf{V} = \mathbf{I}$.

We use the notation $\wedge$ to denote the exterior ("wedge") product (see, e.g., [21, Chapter 1], [32, Chapter 9], or [55, Chapter 4]). For $n \geq 3$ we shall identify the space $\Lambda_{n-1}\mathbb{R}^{n-1}$, of alternating $n-1$ tensors on $\mathbb{R}^n$, with $\mathbb{R}^n$ itself by means of the mapping

$$(\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_{n-1}) \mapsto \mathbf{a}_1 \wedge \mathbf{a}_2 \wedge \ldots \wedge \mathbf{a}_{n-1}.$$

We note that this mapping is multilinear, alternating, and satisfies

$$\mathbf{e}_1 \wedge \mathbf{e}_2 \wedge \ldots \wedge \mathbf{e}_{n-1} = \mathbf{e}_n \tag{4.1}$$

when $\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n$ is *any orthonormal basis with the standard orientation for* $\mathbb{R}^n$. We shall also make use of the identities, for all rotations $\mathbf{Q} \in \mathrm{SO}(n)$,

$$\begin{gathered}\mathbf{Q}\mathbf{e}_1 \wedge \mathbf{Q}\mathbf{e}_2 \wedge \ldots \wedge \mathbf{Q}\mathbf{e}_{n-1} = \mathbf{Q}(\mathbf{e}_1 \wedge \mathbf{e}_2 \wedge \ldots \wedge \mathbf{e}_{n-1}) = \mathbf{Q}\mathbf{e}_n, \\ |\mathbf{a}_1 \wedge \mathbf{a}_2 \wedge \ldots \wedge \mathbf{a}_{n-1}| \leq X \prod_{k=1}^{n-1} |\mathbf{a}_k|,\end{gathered} \tag{4.2}$$

for all $\mathbf{a}_k \in \mathbb{R}^n$, where $X = X(n) > 0$ is a constant that depends only on the dimension n.

Remark 4.1. (1). When $n = 3$ the usual cross product can be substituted for the wedge product; also $X(3) = 1$. (2). Equation $(4.2)_1$ follows from (4.1); the exterior product of the first $n-1$ vectors in any standardly oriented orthonormal basis yields the unique unit vector, with the proper orientation, that is perpendicular to each of the other vectors. For $(4.2)_2$ see, e.g., [32, p. 220].

4.2. **The Geometric-Rigidity Theory of Friesecke, James, & Müller.** In Theorem 3.1 in [24] the authors have shown that, given a Sobolev mapping $\mathbf{u}$, there exists a rotation $\mathbf{R}_{\mathbf{u}}$ such that the distance from $\nabla\mathbf{u}$ to $\mathbf{R}_{\mathbf{u}}$ is, up to a multiplicative constant which does not depend on

u, a lower bound for the distance from $\nabla\mathbf{u}$ to the set of n-dimensional rotations. Their measure of distance from the set of rotations is the L^2-norm of the functional

$$\operatorname{dist}\big(\nabla\mathbf{v}(\mathbf{x}),\mathrm{SO}(n)\big) := \min_{\mathbf{Q}\in\mathrm{SO}(n)} |\nabla\mathbf{v}(\mathbf{x}) - \mathbf{Q}|.$$

However, as noted by Conti & Schweizer [16, p. 854], L^2 can be replaced by L^p for any $p \in (1,\infty)$.

Before we state the Geometric-Rigidity result of interest in this manuscript, we first note that, when the Jacobian of a mapping is strictly positive, the distance to the set of rotations can be expressed in an alternative form. We give a proof for the convenience of the reader.

Lemma 4.2. *Let* $\mathbf{F}\in\mathbb{M}^{n\times n}$ *with polar decomposition* $\mathbf{F}=\mathbf{R}\mathbf{U}$ *satisfy* $\det\mathbf{F}>0$*. Then*

$$\operatorname{dist}\big(\mathbf{F},\mathrm{SO}(n)\big) = \big|\sqrt{\mathbf{F}^{\mathrm{T}}\mathbf{F}} - \mathbf{I}\big| = |\mathbf{U}-\mathbf{I}|.$$

Proof. Recall that (see, e.g., [27, Chapter I] or [14, Section 3.2]) $\mathbf{F}\in\mathbb{M}^{n\times n}$ with $\det\mathbf{F}>0$ has a unique polar decomposition $\mathbf{F}=\mathbf{R}\mathbf{U}$, where $\mathbf{U}:=\sqrt{\mathbf{F}^{\mathrm{T}}\mathbf{F}}$ is symmetric and strictly positive definite and $\mathbf{R}:=\mathbf{F}\mathbf{U}^{-1}\in\mathrm{SO}(n)$. Then, for any $\mathbf{Q}\in\mathrm{SO}(n)$,

$$|\mathbf{F}-\mathbf{Q}|^2 = |\mathbf{F}|^2 - 2\mathbf{F}:\mathbf{Q} + n = |\mathbf{U}|^2 - 2\mathbf{U}:\mathbf{R}^{\mathrm{T}}\mathbf{Q} + n. \tag{4.3}$$

Next, by the spectral theorem,

$$\mathbf{U}:\mathbf{R}^{\mathrm{T}}\mathbf{Q} = \sum_{k=1}^{n}\lambda_k\big[\mathbf{f}_k\otimes\mathbf{f}_k\big]:\mathbf{R}^{\mathrm{T}}\mathbf{Q} = \sum_{k=1}^{n}\lambda_k\mathbf{f}_k\cdot\mathbf{R}^{\mathrm{T}}\mathbf{Q}\mathbf{f}_k, \tag{4.4}$$

where $\lambda_k>0$ and $\{\mathbf{f}_k : k=1,2,\dots,n\}$ is an orthonormal basis for $\mathbb{R}^n$. Consequently, in view of (4.3) and (4.4) the minimum of $|\mathbf{F}-\mathbf{Q}|$ will occur when each of the quantities $\mathbf{f}_k\cdot\mathbf{R}^{\mathrm{T}}\mathbf{Q}\mathbf{f}_k$ is maximized, that is, when $\mathbf{R}^{\mathrm{T}}\mathbf{Q}=\mathbf{I}$. Therefore,

$$\operatorname{dist}\big(\mathbf{F},\mathrm{SO}(n)\big) := \min_{\mathbf{Q}\in\mathrm{SO}(n)}|\mathbf{F}-\mathbf{Q}| = |\mathbf{R}\mathbf{U}-\mathbf{R}| = |\mathbf{U}-\mathbf{I}|,$$

as claimed. □

We now state the result that we shall utilize.

Proposition 4.3. ([24, Section 3]) and [16, Section 2.4]) *Let* $1<p<\infty$*. Suppose that* $\Omega\subset\mathbb{R}^n$*,* $n\geq 2$*, is a bounded Lipschitz domain. Then there exists a constant* $C=C(p,\Omega)$ *with the following property: For each* $\mathbf{v}\in W^{1,p}(\Omega;\mathbb{R}^n)$ *there is an associated rotation* $\mathbf{R}=\mathbf{R}(p,\mathbf{v},\Omega)\in\mathrm{SO}(n)$ *such that*

$$⨍_\Omega|\nabla\mathbf{v}(\mathbf{x})-\mathbf{R}|^p\,\mathrm{d}\mathbf{x} \leq C^p ⨍_\Omega\Big[\operatorname{dist}\big(\nabla\mathbf{v}(\mathbf{x}),\mathrm{SO}(n)\big)\Big]^p\mathrm{d}\mathbf{x}. \tag{4.5}$$

Moreover, (4.5) *is scale invariant, i.e.,* $C(p,\lambda\Omega+\mathbf{a})=C(p,\Omega)$ *for all* $\lambda>0$ *and* $\mathbf{a}\in\mathbb{R}^n$*. In addition, there exists a constant* $M=M(n)>0$ *such that, for all* $\mathbf{v}\in W^{1,\infty}(\Omega;\mathbb{R}^n)$*,*

$$\big|\nabla\mathbf{v}\big|_{\mathrm{BMO}(\Omega)} \leq M\big\|\operatorname{dist}\big(\nabla\mathbf{v},\mathrm{SO}(n)\big)\big\|_{\infty,\Omega}. \tag{4.6}$$

Remark 4.4. (1). When $p=1$ or $p=\infty$ the estimate corresponding to (4.5) is *not* valid. See John [33, pp. 393–394] for a counterexample when $p=\infty$. (2). When $p=1$ Conti & Schweizer [16, p. 853] obtained a so-called *weak-type* estimate as well as an estimate where the integral on the right-hand side of (4.5), which we here denote by ρ, is replaced by $\rho\max\{-\ln\rho,1\}$. (3). The result in [24] corresponding to (4.6) differs slightly. However, the above version is a direct consequence of (4.5), Hölders inequality, the scale invariance of C, and the definition of $\mathrm{BMO}(\Omega)$. (4). Inequalities (4.5) and (4.6) were first obtained by John [33, 35] when Ω is a cube, $\mathbf{v}$ is C^1, and the norm on the right-hand side of (4.6) is sufficiently small. (5). Conti, Dolzmann, & Müller [17, Section 4] have obtained a version of (4.5) for the Lorentz spaces $L^{p,q}(\Omega)$, $p \in (1,\infty)$, $q \in [1,\infty]$. (6). Ciarlet & Mardare [15] have obtained a version of (4.5) (but not (4.6)) that involves two mappings. See Remark 7.3 in this manuscript for a brief description of one of their results. (7). See, also, Rešetnjak [50] and Benyamini & Lindenstrauss [7, Chapter 14].

Remark 4.5. The distance of the mapping $\mathbf{v}$ to the closest rigid mapping, $\mathbf{r}(\mathbf{x}) = \mathbf{R}\mathbf{x} + \mathbf{a}$, is also of interest. Such estimates follow from (4.5) upon application of a standard embedding theorem or the Poincaré inequality. John [33, 35] obtained such a result for cubes when the L^∞-norm on the right-hand side of (4.6) is sufficiently small. Kohn [38] proved a similar result for Lipschitz domains when the mappings were bi-Lipschitz, but without the need for an L^∞ bound. He also obtained a bound similar to (4.5) for bi-Lipschitz mappings.

Remark 4.6. If $\mathbf{G} := \langle \nabla \mathbf{v} \rangle_\Omega$ satisfies $\det \mathbf{G} > 0$, a short computation (see the proof of Lemma 4.2) shows that

$$\min_{\mathbf{Q}\in \mathrm{SO}(n)} \int_\Omega |\nabla \mathbf{v}(\mathbf{x}) - \mathbf{Q}|^2 \, \mathrm{d}\mathbf{x}$$

is achieved when $\mathbf{Q} := \mathbf{G}\mathbf{V}^{-1}$, where $\mathbf{V} = \sqrt{\mathbf{G}^{\mathrm{T}}\mathbf{G}}$, i.e., $\mathbf{G}$ has polar decomposition $\mathbf{G} = \mathbf{Q}\mathbf{V}$. This was first noticed by John [33, 35].

4.3. **Sobolev Mappings and Rotations.** In this subsection we show that the imposition of a Dirichlet boundary condition on a nonempty, relatively open subset of the boundary yields a relationship between Sobolev mappings and rotations. Recall that $\Omega \subset \mathbb{R}^n$ is a fixed Lipschitz domain and suppose that $\mathcal{D} \subset \partial\Omega$ is a nonempty, relatively open set.

Lemma 4.7. *Fix $p \in (n,\infty)$. Then there exists a constant $A = A(p,\Omega,\mathcal{D}) > 0$ such that every pair of mappings $\mathbf{u}^{(i)} \in W^{1,p}(\Omega;\mathbb{R}^n)$, $i = 1,2$, that satisfies $\mathbf{u}^{(1)}(\mathbf{x}) = \mathbf{u}^{(2)}(\mathbf{x})$ for $\mathbf{x} \in \mathcal{D}$, will also satisfy*

$$\big|\mathbf{R}^{(1)} - \mathbf{R}^{(2)}\big| < A\Big(\big\|\nabla\mathbf{u}^{(1)} - \mathbf{R}^{(1)}\big\|_{p,\Omega} + \big\|\nabla\mathbf{u}^{(2)} - \mathbf{R}^{(2)}\big\|_{p,\Omega}\Big) \tag{4.7}$$

for every pair of rotations $\mathbf{R}^{(i)} \in \mathrm{SO}(n)$, $i = 1,2$.

Before we prove Lemma 4.7, we first present an interesting consequence.

Proposition 4.8. *Fix $p \in (n,\infty)$. Then there exists a constant $A^* = A^*(p,\Omega,\mathcal{D}) > 0$ such that every pair of mappings $\mathbf{u}^{(i)} \in W^{1,p}(\Omega;\mathbb{R}^n)$, $i = 1,2$, that satisfies $\mathbf{u}^{(1)}(\mathbf{x}) = \mathbf{u}^{(2)}(\mathbf{x})$ for $\mathbf{x} \in \mathcal{D}$, will also satisfy*

$$\big\|\nabla\mathbf{u}^{(1)} - \nabla\mathbf{u}^{(2)}\big\|_{1,\Omega} \le A^*\Big(\big\|\operatorname{dist}\big(\nabla\mathbf{u}^{(1)},\mathrm{SO}(n)\big)\big\|_{p,\Omega} + \big\|\operatorname{dist}\big(\nabla\mathbf{u}^{(2)},\mathrm{SO}(n)\big)\big\|_{p,\Omega}\Big). \tag{4.8}$$

Proof. Fix $p > n$ and suppose that $\mathbf{u}^{(i)} \in W^{1,p}(\Omega;\mathbb{R}^n)$, $i = 1,2$. Then, in view of Proposition 4.3, there exist rotations $\mathbf{R}^{(i)} \in \mathrm{SO}(n)$ that satisfy

$$\big\|\nabla\mathbf{u}^{(i)} - \mathbf{R}^{(i)}\big\|_{p,\Omega} \le C\big\|\operatorname{dist}\big(\nabla\mathbf{u}^{(i)},\mathrm{SO}(n)\big)\big\|_{p,\Omega} \tag{4.9}$$

for some constant $C = C(p,\Omega)$. If we now add and subtract $\mathbf{R}^{(1)}$ and $\mathbf{R}^{(2)}$ from $\nabla\mathbf{u}^{(1)} - \nabla\mathbf{u}^{(2)}$ and take the L^1-norm of the result we find, with the aid of the triangle inequality, that

$$\left\|\nabla\mathbf{u}^{(1)} - \nabla\mathbf{u}^{(2)}\right\|_{1,\Omega} \leq |\Omega|\left|\mathbf{R}^{(1)} - \mathbf{R}^{(2)}\right| + \sum_{i=1}^{2} \left\|\nabla\mathbf{u}^{(i)} - \mathbf{R}^{(i)}\right\|_{1,\Omega}. \tag{4.10}$$

The desired result, (4.8), now follows from (4.10), (4.9), Lemma 4.7, and Hölder's inequality. □

Proof of Lemma 4.7. Given $\mathbf{u}^{(i)} \in W^{1,p}(\Omega;\mathbb{R}^n)$ and $\mathbf{R}^{(i)} \in \mathrm{SO}(n)$ define, for $i = 1, 2$,

$$d_i := ||\nabla\mathbf{u}^{(i)} - \mathbf{R}^{(i)}||_{p,\Omega}, \qquad \mathbf{a}^{(i)} := \langle \mathbf{u}^{(i)} - \mathbf{R}^{(i)}\mathbf{x}\rangle_\Omega.$$

Then, the Poincaré inequality (see, e.g., [41, p. 361] or [42, p. 218]) yields a constant $P > 0$, which is independent of $\mathbf{u}^{(i)}$, $\mathbf{R}^{(i)}$, and $\mathbf{a}^{(i)}$, such that

$$||\mathbf{u}^{(i)} - \mathbf{R}^{(i)}\mathbf{x} - \mathbf{a}^{(i)}||_{W^{1,p}(\Omega)} \leq P d_i. \tag{4.11}$$

Next, since $p > n$ we have the imbedding (see, e.g., [1, Section 4.27]) $W^{1,p}(\Omega) \to C^{0,\lambda}(\overline{\Omega})$, i.e., there is a constant $M > 0$ such that, for every $\mathbf{x}, \mathbf{y} \in \overline{\Omega}$ with $\mathbf{x} \neq \mathbf{y}$,

$$||\mathbf{v}^{(i)}||_{\infty,\Omega} + \frac{|\mathbf{v}^{(i)}(\mathbf{x}) - \mathbf{v}^{(i)}(\mathbf{y})|}{|\mathbf{x}-\mathbf{y}|^\lambda} \leq M||\mathbf{v}^{(i)}||_{W^{1,p}(\Omega)}, \tag{4.12}$$

where $\mathbf{v}^{(i)}(\mathbf{x}) := \mathbf{u}^{(i)}(\mathbf{x}) - \mathbf{R}^{(i)}\mathbf{x} - \mathbf{a}^{(i)}$. Here $\lambda := 1 - n/p$. We now note that (4.11) together with (4.12) implies that, for all $\mathbf{x}, \mathbf{y} \in \overline{\Omega}$,

$$|\mathbf{v}^{(i)}(\mathbf{x}) - \mathbf{v}^{(i)}(\mathbf{y})| \leq MPd_i|\mathbf{x}-\mathbf{y}|^\lambda. \tag{4.13}$$

Now, suppose that $\mathbf{x}, \mathbf{y} \in \mathcal{D}$; then $\mathbf{u}^{(1)}(\mathbf{x}) = \mathbf{u}^{(2)}(\mathbf{x})$ and $\mathbf{u}^{(1)}(\mathbf{y}) = \mathbf{u}^{(2)}(\mathbf{y})$. Thus,

$$\begin{aligned}\big(\mathbf{R}^{(1)} - \mathbf{R}^{(2)}\big)[\mathbf{y}-\mathbf{x}] &= \mathbf{R}^{(1)}[\mathbf{y}-\mathbf{x}] - \mathbf{R}^{(2)}[\mathbf{y}-\mathbf{x}] \\ &\quad + \big(\mathbf{u}^{(1)}(\mathbf{x}) - \mathbf{u}^{(1)}(\mathbf{y})\big) - \big(\mathbf{u}^{(2)}(\mathbf{x}) - \mathbf{u}^{(2)}(\mathbf{y})\big).\end{aligned} \tag{4.14}$$

Define $\mathbf{R} := [\mathbf{R}^{(1)}]^{\mathrm{T}}\mathbf{R}^{(2)} \in \mathrm{SO}(n)$ and note that, for all $\mathbf{b} \in \mathbb{R}^n$,

$$\left|\big(\mathbf{R}^{(1)} - \mathbf{R}^{(2)}\big)\mathbf{b}\right| = \left|\mathbf{R}^{(1)}\big(\mathbf{I} - \big[\mathbf{R}^{(1)}\big]^{\mathrm{T}}\mathbf{R}^{(2)}\big)\mathbf{b}\right| = |(\mathbf{I}-\mathbf{R})\mathbf{b}|. \tag{4.15}$$

Therefore, if we take the norm of (4.14), the triangle inequality together with (4.13), the definition of the $\mathbf{v}^{(i)}$, and (4.15) yield

$$\left|(\mathbf{I}-\mathbf{R})[\mathbf{y}-\mathbf{x}]\right| \leq MPd|\mathbf{y}-\mathbf{x}|^\lambda \text{ for all } \mathbf{x}, \mathbf{y} \in \mathcal{D}, \tag{4.16}$$

where $d := d_1 + d_2$.

Next, $\partial\Omega$ is Lipschitz; thus, we can fix an $\mathbf{x}_\mathrm{o} \in \mathcal{D}$ where $\partial\Omega$ has a unique outward unit normal vector and tangent hyperplane. Then, *with a change in coordinates,* let $\mathbf{x}_\mathrm{o} = \mathbf{0}$ and suppose that $\{\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n\}$ is a basis for $\mathbb{R}^n$ (with the standard orientation) with $\mathbf{e}_n$ the outward unit normal at $\mathbf{0}$ and the tangent hyperplane, $\mathcal{T} \subset \mathbb{R}^n$, at $\mathbf{0}$ given as the span of $\{\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_{n-1}\}$. Moreover, since $\mathcal{D}$ is relatively open and $\partial\Omega$ is Lipschitz, there exists an open ball $B = B(\mathbf{0}, 2r) \subset \mathbb{R}^{n-1}$ and a Lipschitz function $\gamma : B \to \mathbb{R}$ such that $(\mathbf{z}, \gamma(\mathbf{z}))$ with $\mathbf{z} \in B$ is a relatively open subset of $\mathcal{D}$ and $\gamma(\mathbf{0}) = 0$.

For any $\mathbf{z} \in \mathbb{R}^{n-1}$ that satisfies $|\mathbf{z}| \leq r$, inequality (4.16) implies that

$$|(\mathbf{I}-\mathbf{R})\mathbf{t}| \leq |(\mathbf{I}-\mathbf{R})\mathbf{y}_\gamma| \leq MPd|\mathbf{y}_\gamma|^\lambda, \qquad \mathbf{y}_\gamma := (\mathbf{z}, \gamma(\mathbf{z})), \qquad \mathbf{t} = (\mathbf{z}, 0) \in \mathcal{T}. \tag{4.17}$$

Also, γ is Lipschitz continuous; consequently, there exists a $L > 0$ such that (recall that $\gamma(\mathbf{0}) = 0$)

$$|\gamma(\mathbf{z})| \leq L|\mathbf{z}| \quad \text{and hence} \quad |\mathbf{y}_\gamma| \leq \sqrt{1+L^2}\,|\mathbf{z}|. \tag{4.18}$$

If we now combine (4.17) and (4.18) we find that, for all $\mathbf{t} \in \mathcal{T}$ with $|\mathbf{t}| \leq r$,

$$|(\mathbf{I}-\mathbf{R})\mathbf{t}| \leq Gd|\mathbf{z}|^\lambda = Gd|\mathbf{t}|^\lambda, \qquad \mathbf{t} = (\mathbf{z}, 0), \tag{4.19}$$

where $G = G(p, n, \Omega) := MP(1+L^2)^{\lambda/2}$. In particular, the choice $\mathbf{t} = r\mathbf{e}_k$, $k = 1, 2, 3, \ldots, n-1$ in (4.19) yields

$$|(\mathbf{I}-\mathbf{R})\mathbf{e}_k| \leq Gdr^{\lambda-1} \ \text{ for } 1 \leq k \leq n-1. \tag{4.20}$$

Finally, we shall show that[8], if $n \geq 3$, then (4.20) is also satisfied when $k = n$ and G is replaced by $(n-1)GX$, where X is the constant from $(4.2)_2$. This will imply that (see (4.15) and $(4.17)_2$)

$$\left|\mathbf{R}^{(1)} - \mathbf{R}^{(2)}\right| \leq \sqrt{n} \sup_{|\mathbf{e}|=1} |(\mathbf{I}-\mathbf{R})\mathbf{e}| \leq \sqrt{n}(n-1)XG(d_1+d_2)r^{\lambda-1},$$

which is (4.7) with $A = \sqrt{n}(n-1)MPXr^{\lambda-1}(1+L^2)^{\lambda/2}$.

In order to estimate $|\mathbf{R}\mathbf{e}_n - \mathbf{e}_n|$ we first make use of (4.1) and $(4.2)_1$ to write

$$\mathbf{R}\mathbf{e}_n - \mathbf{e}_n = \left[\mathbf{R}\mathbf{e}_1 \wedge \mathbf{R}\mathbf{e}_2 \wedge \ldots \wedge \mathbf{R}\mathbf{e}_{n-1}\right] - \left[\mathbf{e}_1 \wedge \mathbf{e}_2 \wedge \ldots \wedge \mathbf{e}_{n-1}\right]. \tag{4.21}$$

Then, if we subtract and then add terms of the form

$$\mathbf{e}_1 \wedge \ldots \wedge \mathbf{e}_{k-1} \wedge \mathbf{R}\mathbf{e}_k \wedge \ldots \wedge \mathbf{R}\mathbf{e}_{n-1}$$

to the right-hand side of (4.21), we find that

$$\mathbf{R}\mathbf{e}_n - \mathbf{e}_n = \sum_{k=1}^{n-1} \left[\mathbf{e}_1 \wedge \ldots \wedge \mathbf{e}_{k-1} \wedge (\mathbf{R}\mathbf{e}_k - \mathbf{e}_k) \wedge \mathbf{R}\mathbf{e}_{k+1} \wedge \ldots \wedge \mathbf{R}\mathbf{e}_{n-1}\right]. \tag{4.22}$$

Taking the norm of (4.22) and making use of the triangle inequality together with $(4.2)_2$ and the fact that, for all k, $|\mathbf{R}\mathbf{e}_k| = |\mathbf{e}_k| = 1$ yields, with the aid of (4.20),

$$|\mathbf{R}\mathbf{e}_n - \mathbf{e}_n| \leq \sum_{k=1}^{n-1} X|\mathbf{R}\mathbf{e}_k - \mathbf{e}_k| \leq (n-1)GXdr^{\lambda-1},$$

as claimed, which completes the proof. □

5. Nonlinear Elasticity

In the remainder of this manuscript we shall focus on the minimization problem that arises when one considers the theory of Nonlinear Elasticity.

[8]Recall that in 2-dimensions all rotations commute. Consider the rotation, $\mathbf{Q}_{12}$, that satisfies $\mathbf{Q}_{12}\mathbf{e}_1 = \mathbf{e}_2$. It follows that $(\mathbf{I}-\mathbf{R})\mathbf{e}_2 = (\mathbf{I}-\mathbf{R})\mathbf{Q}_{12}\mathbf{e}_1 = \mathbf{Q}_{12}(\mathbf{I}-\mathbf{R})\mathbf{e}_1$ and hence $|(\mathbf{I}-\mathbf{R})\mathbf{e}_2| = |\mathbf{Q}_{12}(\mathbf{I}-\mathbf{R})\mathbf{e}_1| = |(\mathbf{I}-\mathbf{R})\mathbf{e}_1|$, which, by (4.20), is bounded above by $Gdr^{\lambda-1}$.

5.1. **More Preliminaries.** Sym_n will denote the space of *symmetric* $\mathbf{B} \in \mathbb{M}^{n\times n}$, i.e., $\mathbf{B} = \mathbf{B}^{\mathrm{T}}$, while Psym_n will denote those $\mathbf{C} \in \mathrm{Sym}_n$ that are *strictly positive definite*, that is, $\mathbf{a} \cdot \mathbf{C}\mathbf{a} > 0$ for all nonzero $\mathbf{a} \in \mathbb{R}^n$. In the sequel we shall have occasion to consider a function defined on $\overline{\Omega} \times \mathcal{O}$, where $\Omega \subset \mathbb{R}^n$ is a bounded Lipschitz domain and $\mathcal{O} \subset \mathbb{M}^{n\times n}$ is a nonempty, open set.

Definition 5.1. Let $\Phi : \overline{\Omega} \times \mathcal{O} \to \mathbb{R}$. We say that $\mathbf{F} \mapsto \Phi(\mathbf{x}, \mathbf{F})$ is *continuous, almost uniformly in* $\mathbf{x} \in \Omega$, *at* $\mathbf{F}_{\mathrm{o}} \in \mathcal{O}$, provided that, for every $\varepsilon > 0$, there exists a $\delta > 0$ such that, for *a.e.* $\mathbf{x} \in \Omega$,

$$|\Phi(\mathbf{x}, \mathbf{F}) - \Phi(\mathbf{x}, \mathbf{F}_{\mathrm{o}})| < \varepsilon \quad \text{whenever} \quad |\mathbf{F} - \mathbf{F}_{\mathrm{o}}| < \delta.$$

More generally, we say that $\mathbf{F} \mapsto \Phi(\mathbf{x}, \mathbf{F})$ is C^2, *almost uniformly in* $\mathbf{x}$, *on* $\mathcal{O}$, provided $\mathbf{F} \mapsto \Phi(\mathbf{x}, \mathbf{F})$ and its first two derivatives are each continuous, almost uniformly in *a.e.* $\mathbf{x} \in \Omega$, at every $\mathbf{F} \in \mathcal{O}$.

5.2. **The Constitutive Relation.** We consider a *body* that for convenience we identify with the closure of a *bounded Lipschitz domain* $\Omega \subset \mathbb{R}^n$, $n = 2$ or $n = 3$, which it occupies in a fixed reference configuration. A *deformation* of $\overline{\Omega}$ is a mapping that lies in the space

$$\mathrm{Def} := \{\mathbf{u} \in W^{1,1}(\Omega; \mathbb{R}^n) : \det \nabla \mathbf{u} > 0 \ a.e.\},$$

where $\det \mathbf{F}$ denotes the determinant of $\mathbf{F} \in \mathbb{M}^{n\times n}$. We define $\mathcal{O} \subset \mathbb{M}^{n\times n}$ by

$$\mathcal{O} := \mathbb{M}^{n\times n}_+ = \{\mathbf{F} \in \mathbb{M}^{n\times n} : \det \mathbf{F} > 0\}.$$

We assume that the body is composed of a hyperelastic material with *stored-energy density* $W : \overline{\Omega} \times \mathbb{M}^{n\times n}_+ \to [0, \infty)$. $W(\mathbf{x}, \nabla \mathbf{u}(\mathbf{x}))$ gives the elastic energy stored at almost every point $\mathbf{x} \in \Omega$ of the body when it undergoes the deformation $\mathbf{u} \in \mathrm{Def}$. We assume that the response of the material is *Invariant under a Change in Observer* and hence that[9]

$$W(\mathbf{x}, \mathbf{Q}\mathbf{F}) = W(\mathbf{x}, \mathbf{F}) \quad \text{for every } \mathbf{F} \in \mathbb{M}^{n\times n}_+ \text{ and } \mathbf{Q} \in \mathrm{SO}(n). \tag{5.1}$$

In the sequel we shall have occasion to assume that W also satisfies (1)–(3) in Hypothesis 3.1. For the moment we suppose that $\mathbf{F} \mapsto W(\mathbf{x}, \mathbf{F})$ is C^2.

Rather than view the derivatives of W as multilinear forms, as we did in Section 3.2, we shall instead follow the usual convention in Continuum Mechanics (see, e.g., [14, 27]); the (Piola-Kirchhoff) *stress* is the derivative

$$\mathbf{S}(\mathbf{x}, \mathbf{F}) := \frac{\partial}{\partial \mathbf{F}} W(\mathbf{x}, \mathbf{F}), \qquad \mathbf{S} : \overline{\Omega} \times \mathbb{M}^{n\times n}_+ \to \mathbb{M}^{n\times n}.$$

The *Elasticity Tensor* is the second derivative of $\mathbf{F} \mapsto W(\mathbf{x}, \mathbf{F})$, that is,

$$\mathbb{A}(\mathbf{x}, \mathbf{F}) := \frac{\partial^2}{\partial \mathbf{F}^2} W(\mathbf{x}, \mathbf{F}), \qquad \mathbb{A} : \overline{\Omega} \times \mathbb{M}^{n\times n}_+ \to \mathrm{Lin}(\mathbb{M}^{n\times n}; \mathbb{M}^{n\times n}),$$

where $\mathrm{Lin}(\mathcal{U}; \mathcal{V})$ denotes the set of linear maps from the vector space $\mathcal{U}$ to the vector space $\mathcal{V}$.

Remark 5.2. In the notation of Section 3.2 and in view of the symmetry of the second gradient

$$\mathbf{S}(\mathbf{x}, \mathbf{F}) : \mathbf{H} = \mathrm{D}W(\mathbf{x}, \mathbf{F})[\mathbf{H}],$$

$$\mathbf{H} : \mathbb{A}(\mathbf{x}, \mathbf{F})[\mathbf{K}] = \mathbf{K} : \mathbb{A}(\mathbf{x}, \mathbf{F})[\mathbf{H}] = \mathrm{D}^2 W(\mathbf{x}, \mathbf{F})[\mathbf{H}, \mathbf{K}],$$

[9]All of the equations (and inequalities) in this section are valid only for almost every $\mathbf{x} \in \Omega$. For clarity of exposition we have sometimes suppressed this dependence on $\mathbf{x}$.

for all $\mathbf{F} \in \mathbb{M}^{n\times n}_+$ and all $\mathbf{H}, \mathbf{K} \in \mathbb{M}^{n\times n}$.

Definition 5.3. The reference configuration is said to be *stress free* provided that,

$$\mathbf{S}(\mathbf{x}, \mathbf{I}) = \mathbf{0} \quad \text{for } a.e.\ \mathbf{x} \in \Omega. \tag{5.2}$$

If the reference configuration is stress free, then Elasticity Tensor at the reference configuration is said to be *uniformly positive definite*[10], provided that there exists a constant $c > 0$ such that, for every $\mathbf{H} \in \mathbb{M}^{n\times n}$ and $a.e.$ $\mathbf{x} \in \Omega$,

$$\mathbf{H} : \mathbb{A}(\mathbf{x}, \mathbf{I})[\mathbf{H}] \geq c|\mathbf{H} + \mathbf{H}^{\mathrm{T}}|^2.$$

We next note, once again, that every $\mathbf{F} \in \mathbb{M}^{n\times n}_+$ has a unique polar decomposition $\mathbf{F} = \mathbf{R}\mathbf{U}$, where $\mathbf{U} := \sqrt{\mathbf{F}^{\mathrm{T}}\mathbf{F}} \in \mathrm{Psym}_n$ and $\mathbf{R} := \mathbf{F}\mathbf{U}^{-1} \in \mathrm{SO}(n)$. Equation (5.1) then implies that $W(\mathbf{x}, \mathbf{F}) = W(\mathbf{x}, \mathbf{U})$. With this in mind we define $\sigma : \overline{\Omega} \times \mathrm{Psym}_n \to \mathbb{R}$ by

$$\sigma(\mathbf{x}, \mathbf{C}) := W(\mathbf{x}, \sqrt{\mathbf{C}}). \tag{5.3}$$

Since $\mathbf{C} \mapsto \sqrt{\mathbf{C}}$ is C^∞ on Psym_n our assumptions (1)–(3) in Hypothesis 3.1 yield the same properties for σ. In particular, we can differentiate the identity

$$W(\mathbf{x}, \mathbf{F}) = W(\mathbf{x}, \mathbf{U}) = \sigma(\mathbf{x}, \mathbf{U}^2) = \sigma(\mathbf{x}, \mathbf{F}^{\mathrm{T}}\mathbf{F}). \tag{5.4}$$

However, we shall need additional smoothness assumptions on W in order to show that the second variation is uniformly positive near the set of rotations. In the sequel we shall therefore sometimes assume that (see Definition 5.1)

$$\mathbf{C} \mapsto \sigma(\mathbf{x}, \mathbf{C}) \text{ is } C^2, \text{ almost uniformly in } \mathbf{x}, \text{ on } \mathrm{Psym}_n, \tag{5.5}$$

and hence, in view of (5.3)–(5.4), that $\mathbf{F} \mapsto W(\mathbf{x}, \mathbf{F})$ is C^2, almost uniformly in $\mathbf{x}$, on $\mathbb{M}^{n\times n}_+$.

Remark 5.4. Note that (5.4) implies that W satisfies (5.1).

The next well-known result shows that our assumptions on W yield similar properties for σ.

Lemma 5.5. *Let σ satisfy (5.3)–(5.5). Then, for all $\mathbf{F} \in \mathbb{M}^{n\times n}_+$, all $\mathbf{H} \in \mathbb{M}^{n\times n}$, and a.e. $\mathbf{x} \in \Omega$,*

$$\begin{aligned} \mathbf{S}(\mathbf{x}, \mathbf{F}) &= 2\mathbf{F}\, \mathrm{D}\sigma(\mathbf{x}, \mathbf{F}^{\mathrm{T}}\mathbf{F}), \\ \mathbf{H} : \mathbb{A}(\mathbf{x}, \mathbf{F})[\mathbf{H}] &= \big(\mathbf{H}^{\mathrm{T}}\mathbf{F} + \mathbf{F}^{\mathrm{T}}\mathbf{H}\big) : \mathrm{D}^2\sigma(\mathbf{x}, \mathbf{F}^{\mathrm{T}}\mathbf{F})[\mathbf{H}^{\mathrm{T}}\mathbf{F} + \mathbf{F}^{\mathrm{T}}\mathbf{H}] \\ &\quad + 2\mathrm{D}\sigma(\mathbf{x}, \mathbf{F}^{\mathrm{T}}\mathbf{F}) : [\mathbf{H}^{\mathrm{T}}\mathbf{H}]. \end{aligned} \tag{5.6}$$

Moreover, suppose that, for a.e. $\mathbf{x} \in \Omega$, $\mathbf{S}(\mathbf{x}, \mathbf{I}) = \mathbf{0}$ and

$$\mathbf{H} : \mathbb{A}(\mathbf{x}, \mathbf{I})[\mathbf{H}] \geq c|\mathbf{H} + \mathbf{H}^{\mathrm{T}}|^2 \quad \textit{for all } \mathbf{H} \in \mathbb{M}^{n\times n}. \tag{5.7}$$

Then, for a.e. $\mathbf{x} \in \Omega$, $\mathrm{D}\sigma(\mathbf{x}, \mathbf{I}) = \mathbf{0}$ and

$$\mathbf{B} : \mathrm{D}^2\sigma(\mathbf{x}, \mathbf{I})[\mathbf{B}] \geq c|\mathbf{B}|^2 \quad \textit{for all } \mathbf{B} \in \mathrm{Sym}_n. \tag{5.8}$$

Here $\mathrm{D}^k\sigma(\mathbf{x}, \mathbf{C})$ denotes the k-th derivative of the function $\mathbf{C} \mapsto \sigma(\mathbf{x}, \mathbf{C})$.

[10] One consequence of (5.1) and (5.2) is that $\mathbb{A}(\mathbf{x}, \mathbf{I})[\mathbf{K}] = \mathbf{0}$ for all $\mathbf{K} \in \mathbb{M}^{n\times n}$ that satisfy $\mathbf{K}^{\mathrm{T}} = -\mathbf{K}$.

Proof. If we differentiate (5.4) with respect to $\mathbf{F}$, we find that, for all $\mathbf{F} \in \mathbb{M}^{n\times n}_+$ and $\mathbf{H} \in \mathbb{M}^{n\times n}$,

$$\mathbf{S}(\mathbf{x},\mathbf{F}) : \mathbf{H} = \mathrm{D}\sigma(\mathbf{x},\mathbf{F}^{\mathrm{T}}\mathbf{F}) : [\mathbf{H}^{\mathrm{T}}\mathbf{F} + \mathbf{F}^{\mathrm{T}}\mathbf{H}], \tag{5.9}$$

which implies $(5.6)_1$. If we then differentiate (5.9) with respect to $\mathbf{F}$ we deduce $(5.6)_2$. Next, let $\mathbf{F} = \mathbf{I}$ in $(5.6)_1$, to conclude, with the aid of $\mathbf{S}(\mathbf{x},\mathbf{I}) = \mathbf{0}$, that $\mathrm{D}\sigma(\mathbf{x},\mathbf{I}) = \mathbf{0}$.

If we take $\mathbf{F} = \mathbf{I}$ in $(5.6)_2$ we find that

$$\mathbf{H} : \mathbb{A}(\mathbf{x},\mathbf{I})[\mathbf{H}] = \big(\mathbf{H}^{\mathrm{T}} + \mathbf{H}\big) : \mathrm{D}^2\sigma(\mathbf{x},\mathbf{I})[\mathbf{H}^{\mathrm{T}} + \mathbf{H}],$$

which together with (5.7) yields

$$\big(\mathbf{H}^{\mathrm{T}} + \mathbf{H}\big) : \mathrm{D}^2\sigma(\mathbf{x},\mathbf{I})[\mathbf{H}^{\mathrm{T}} + \mathbf{H}] \geq c|\mathbf{H}^{\mathrm{T}} + \mathbf{H}|^2. \tag{5.10}$$

Finally, inequality (5.10) yields (5.8) for all symmetric $\mathbf{B}$. □

The matrix $\mathbf{C} := \mathbf{F}^{\mathrm{T}}\mathbf{F}$ is known as the *right Cauchy-Green strain tensor.* It can be used to measure the change in the length of a curve in the reference configuration after it is deformed by $\mathbf{u}$. The matrix

$$\mathbf{E} := \tfrac{1}{2}(\mathbf{C} - \mathbf{I}) = \tfrac{1}{2}(\mathbf{F}^{\mathrm{T}}\mathbf{F} - \mathbf{I}) \tag{5.11}$$

is sometimes referred to as the (nonlinear) *strain.*[11] The linearization of $\mathbf{E}$ at $\mathbf{F} = \mathbf{I}$ yields the strain tensor used in the classical theory of Linear Elasticity. The advantage of using $\mathbf{E}$, rather than $\mathbf{C}$, is that $\mathbf{E} = \mathbf{0}$ corresponds to an undeformed body. We next note that *a deformation has uniformly small strains if and only if it is uniformly close to the set of rotations.*

Lemma 5.6. *Let* $\mathbf{F} \in \mathbb{M}^{n\times n}_+$. *Then*

$$\big[\operatorname{dist}\big(\mathbf{F},\mathrm{SO}(n)\big)\big]^2 \leq 2\sqrt{n}\,|\mathbf{E}| \leq \sqrt{n}\operatorname{dist}\big(\mathbf{F},\mathrm{SO}(n)\big)\big[\operatorname{dist}\big(\mathbf{F},\mathrm{SO}(n)\big) + 2\sqrt{n}\,\big]. \tag{5.12}$$

Proof. Define $\mathbf{A} \in \mathbb{M}^{n\times n}$ by $\mathbf{A} := \operatorname{diag}\{|a_1|, |a_2|, \ldots, |a_n|\}$, where $a_k \in \mathbb{R}$. Then, by the Cauchy-Schwarz inequality,

$$\left(\sum_{k=1}^{n} |a_k|\right)^2 = |\mathbf{A} : \mathbf{I}|^2 \leq |\mathbf{A}|^2|\mathbf{I}|^2 = n\sum_{k=1}^{n} |a_k|^2. \tag{5.13}$$

Next, by the spectral theorem, $\mathbf{U} = \sqrt{\mathbf{C}}$ has eigenvalues $0 < \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n$. Since $|\lambda_k - 1|^2 \leq |\lambda_k^2 - 1|$ the choice $a_k = \lambda_k^2 - 1$ in (5.13) yields, with the aid of (5.11),

$$|\mathbf{U} - \mathbf{I}|^4 = \left(\sum_{k=1}^{n} |\lambda_k - 1|^2\right)^2 \leq n\sum_{k=1}^{n} |\lambda_k^2 - 1|^2 = 4n|\mathbf{E}|^2,$$

which together with Lemma 4.2 establishes the first inequality in (5.12).

The identity $\mathbf{C} = \mathbf{U}^2$ together with (5.11), Lemma 4.2, and the triangle inequality gives us

$$\begin{aligned} 2|\mathbf{E}| &= |(\mathbf{U} - \mathbf{I})(\mathbf{U} + \mathbf{I})| \leq \operatorname{dist}\big(\mathbf{F},\mathrm{SO}(n)\big)\big(|\mathbf{U}| + \sqrt{n}\,\big), \\ |\mathbf{U}| &= |\mathbf{U} - \mathbf{I} + \mathbf{I}| \leq \operatorname{dist}\big(\mathbf{F},\mathrm{SO}(n)\big) + \sqrt{n}, \end{aligned}$$

which together yield the second inequality in (5.12). □

[11] See, e.g., [46, Section 2.2.7] for a discussion of various measures of strain.

Remark 5.7. We note for future reference that $|\lambda_k - 1| \leq |\lambda_k - 1||\lambda_k + 1| = |\lambda_k^2 - 1|$ and hence, in view of Lemma 4.2 and (5.11),

$$\big[\operatorname{dist}\big(\mathbf{F}, \mathrm{SO}(n)\big)\big]^2 = |\mathbf{U} - \mathbf{I}|^2 = \sum_{k=1}^{n} |\lambda_k - 1|^2 \leq \sum_{k=1}^{n} |\lambda_k^2 - 1|^2 = 4|\mathbf{E}|^2. \tag{5.14}$$

Although (5.14) does not scale properly for large strains, its use will simplify the small strain computation in one of our proofs.

5.3. **Equilibrium Solutions and Energy Minimizers in Nonlinear Elasticity.** We assume the body is subject to dead loads. As in Section 3.2 we shall let

$$\partial\Omega = \overline{\mathcal{D}} \cup \overline{\mathcal{S}} \quad \text{with } \mathcal{D} \text{ and } \mathcal{S} \text{ relatively open and } \mathcal{D} \cap \mathcal{S} = \varnothing.$$

In addition, we shall suppose that $\mathcal{D} \neq \varnothing$. We assume that a Lipschitz-continuous function $\mathbf{d} : \mathcal{D} \to \mathbb{R}^n$ is prescribed; $\mathbf{d}$ will give the deformation of $\mathcal{D}$. If $\mathcal{S} \neq \varnothing$ we assume that a function $\mathbf{s} \in L^2(\mathcal{S}; \mathbb{R}^n)$ is prescribed; for $\mathcal{H}^{n-1}$-*a.e.* $\mathbf{x} \in \mathcal{S}$, $\mathbf{s}(\mathbf{x})$ will give the surface force exerted on the body, at the point $\mathbf{x}$, by its environment. Finally, we suppose that a function $\mathbf{b} \in L^2(\Omega; \mathbb{R}^n)$ is prescribed; for *a.e.* $\mathbf{x} \in \Omega$, $\mathbf{b}(\mathbf{x})$ will give the body force exerted on the body, at the point $\mathbf{x}$, by its environment. The set of *Admissible Deformations* will be denoted by

$$\mathrm{AD} := \{\mathbf{u} \in \mathrm{Def} \cap W^{1,\infty}(\Omega; \mathbb{R}^n) : \mathbf{u} = \mathbf{d} \text{ on } \mathcal{D}\}.$$

The *total energy* of an admissible deformation $\mathbf{u} \in \mathrm{AD}$ is defined to be

$$\mathcal{E}(\mathbf{u}) := \int_\Omega \big[W\big(\mathbf{x}, \nabla\mathbf{u}(\mathbf{x})\big) - \mathbf{b}(\mathbf{x}) \cdot \mathbf{u}(\mathbf{x})\big] \mathrm{d}\mathbf{x} - \int_{\mathcal{S}} \mathbf{s}(\mathbf{x}) \cdot \mathbf{u}(\mathbf{x}) \, \mathrm{d}\mathcal{H}^{n-1}_{\mathbf{x}}. \tag{5.15}$$

We shall assume that we are given a deformation, $\mathbf{u}_\mathrm{e} \in \mathrm{AD}$, that is a weak solution of the *Equilibrium Equations* corresponding to (5.15), i.e.,

$$0 = \int_\Omega \big[\mathbf{S}\big(\mathbf{x}, \nabla\mathbf{u}_\mathrm{e}(\mathbf{x})\big) : \nabla\mathbf{w} - \mathbf{b}(\mathbf{x}) \cdot \mathbf{w}(\mathbf{x})\big] \mathrm{d}\mathbf{x} - \int_{\mathcal{S}} \mathbf{s}(\mathbf{x}) \cdot \mathbf{w}(\mathbf{x}) \, \mathrm{d}\mathcal{H}^{n-1}_{\mathbf{x}} \tag{5.16}$$

for all *variations* $\mathbf{w} \in \mathrm{Var}$, where

$$\mathrm{Var} := \{\mathbf{w} \in W^{1,2}(\Omega; \mathbb{R}^n) : \mathbf{w} = \mathbf{0} \text{ on } \mathcal{D}\}.$$

If $\mathcal{D} = \partial\Omega$ we shall call $\mathbf{u}_\mathrm{e}$ a weak solution of the *(pure) displacement problem.* Otherwise, we shall refer to such a $\mathbf{u}_\mathrm{e}$ as a weak solution of the *(genuine) mixed problem.* If, in addition, $W \in C^2(\Omega \times \mathbb{M}^{n\times n}_+)$ and $\mathbf{u}_\mathrm{e} \in C^2(\Omega; \mathbb{R}^n) \cap C^1(\overline{\Omega}; \mathbb{R}^n)$, then $\mathbf{u}_\mathrm{e}$ will be a *classical solution of the equations of equilibrium*, i.e.,

$$\operatorname{Div} \mathbf{S}(\nabla\mathbf{u}_\mathrm{e}) + \mathbf{b} = \mathbf{0} \;\text{ in } \Omega,$$
$$\mathbf{S}(\nabla\mathbf{u}_\mathrm{e})\mathbf{n} = \mathbf{s} \;\text{ on } \mathcal{S}, \qquad \mathbf{u}_\mathrm{e} = \mathbf{d} \;\text{ on } \mathcal{D},$$

where $\mathbf{n}(\mathbf{x})$ is the outward unit normal to Ω at $\mathcal{H}^{n-1}$-*a.e.* $\mathbf{x} \in \mathcal{S}$ and $\operatorname{Div} \mathbf{S} \in \mathbb{R}^n$ is given by $(\operatorname{Div} \mathbf{S})_i = \sum_j \frac{\partial}{\partial \mathbf{x}_j} \mathbf{S}_{ij}$. We are interested in conditions under which a weak solution of the equilibrium equations, $\mathbf{u}_\mathrm{e} \in \mathrm{AD}$, is a local minimizer of the total energy $\mathcal{E}$. We are also interested in conditions under which $\mathbf{u}_\mathrm{e}$ is the unique weak solution of the equilibrium equations that lies in a neighborhood of $\mathbf{u}_\mathrm{e}$.

5.4. **Uniqueness in** $\mathrm{BMO} \cap L^1$ **Neighborhoods in Elasticity.** We next make note of a direct implication of Theorem 3.3 for Elasticity.

Theorem 5.8. *Let W satisfy (1)–(3) of Hypothesis 3.1. Suppose that $\mathbf{u}_{\mathrm{e}} \in \mathrm{AD}$ is a weak solution of the pure-displacement or mixed problem that satisfies, for some $\varepsilon > 0$ and $k > 0$,*

$$\det \nabla \mathbf{u}_{\mathrm{e}} > \varepsilon \quad a.e., \qquad \int_\Omega \nabla \mathbf{w} : \mathbb{A}(\nabla \mathbf{u}_{\mathrm{e}})[\nabla \mathbf{w}] \, \mathrm{d}\mathbf{x} \geq 4k \int_\Omega |\nabla \mathbf{w}|^2 \mathrm{d}\mathbf{x},$$

for all $\mathbf{w} \in \mathrm{Var}$. Let $\tau \in \mathbb{R}$ satisfy $\tau > ||\nabla \mathbf{u}_{\mathrm{e}}||_{\infty,\Omega}$ and $\tau^{-1} < \varepsilon$. Then there exists a $\delta = \delta(\tau) > 0$ such that any $\mathbf{v} \in \mathrm{AD}$ that satisfies $\det \nabla \mathbf{v} > \tau^{-1}$ a.e.,

$$||\nabla \mathbf{v}||_{\infty,\Omega} < \tau, \qquad \left| \nabla \mathbf{v} - \nabla \mathbf{u}_{\mathrm{e}} \right|_{\mathrm{BMO}(\Omega)} < \delta, \qquad \left| \fint_\Omega (\nabla \mathbf{v} - \nabla \mathbf{u}_{\mathrm{e}}) \, \mathrm{d}\mathbf{x} \right| < \delta, \tag{5.17}$$

cannot be a weak solution of the equations of equilibrium. Moreover,

$$\mathcal{E}(\mathbf{v}) \geq \mathcal{E}(\mathbf{u}_{\mathrm{e}}) + k \int_\Omega |\nabla \mathbf{v} - \nabla \mathbf{u}_{\mathrm{e}}|^2 \mathrm{d}\mathbf{x}$$

and hence $\mathbf{v}$ will have strictly greater energy than $\mathbf{u}_{\mathrm{e}}$.

A physical interpretation of hypothesis $(5.17)_2$ is of interest. In the remainder of the paper we will show that, in certain situations, sufficiently small strains or small strain differences imply that $(5.17)_2$ is satisfied.

6. Deformations with Small Strain

In this section we focus on deformations $\mathbf{u}$ whose nonlinear strains $\mathbf{E}_{\mathbf{u}}$ are sufficiently small. We show, in particular, that uniformly small strains implies that the deformation gradient is small in BMO.

6.1. **The Positivity of the Second Variation for Deformations with Small Strain.** We now consider the sign of the second variation for deformations that have sufficiently small strains. The next result shows that a stress-free reference configuration together with the uniform positivity of the Elasticity Tensor at this reference configuration yields the uniform positivity of the second variation of the total energy at any admissible deformation, $\mathbf{u} \in \mathrm{AD}$, that either is C^1 and has sufficiently small strains, or is sufficiently close to a single rotation.

Proposition 6.1. *Let $\mathbf{F} \mapsto W(\mathbf{x}, \mathbf{F})$ be C^2, almost uniformly in $\mathbf{x}$, on $\mathbb{M}^{n\times n}_+$ and satisfy* (5.1)*. Suppose that, for a.e. $\mathbf{x} \in \Omega$, $\mathbf{S}(\mathbf{x}, \mathbf{I}) = \mathbf{0}$ and*

$$\mathbf{H} : \mathbb{A}(\mathbf{x}, \mathbf{I})[\mathbf{H}] \geq c |\mathbf{H} + \mathbf{H}^{\mathrm{T}}|^2 \tag{6.1}$$

for some constant $c > 0$ and every $\mathbf{H} \in \mathbb{M}^{n\times n}$. Then there exists a $\delta_{\mathrm{o}} \in (0, 1)$ such that any admissible deformation $\mathbf{u} \in \mathrm{AD}$ that satisfies both

$$\mathbf{u} \in C^1(\overline{\Omega}; \mathbb{R}^n) \quad and \quad ||(\nabla \mathbf{u})^{\mathrm{T}} \nabla \mathbf{u} - \mathbf{I}||_{\infty,\Omega} < \delta_{\mathrm{o}} \tag{6.2}$$

or, merely,

$$||\nabla \mathbf{u} - \mathbf{Q}||_{\infty,\Omega} < \delta_{\mathrm{o}} \tag{6.3}$$

for some $\mathbf{Q} \in \mathrm{SO}(n)$*, will also satisfy*

$$\int_\Omega \nabla\mathbf{w}(\mathbf{x}) : \mathbb{A}\big(\mathbf{x}, \nabla\mathbf{u}(\mathbf{x})\big)\big[\nabla\mathbf{w}(\mathbf{x})\big]\,\mathrm{d}\mathbf{x} \geq 4k\int_\Omega |\nabla\mathbf{w}(\mathbf{x})|^2\mathrm{d}\mathbf{x}, \tag{6.4}$$

for some $k > 0$ *and all* $\mathbf{w} \in \mathrm{Var}$.

Remark 6.2. (1). Note that Lemma 5.6 and $(6.2)_2$ imply that the distance from $\nabla\mathbf{u}$ to the set of rotations is small. (2). The additional smoothness of $\mathbf{u}$, $(6.2)_1$, is necessitated by our use of a version of Korn's inequality with nonconstant coefficients. See Appendix B.

Proof of Proposition 6.1. We first note that the result is well-known when $\nabla\mathbf{u}$ satisfies (6.3) (see, e.g., [28, Theorem 5]). We shall therefore assume that $\mathbf{u} \in \mathrm{AD}$ satisfies (6.2) for some $\delta_\mathrm{o} \in (0,1)$ to be determined. Suppose that $\varepsilon > 0$ is an additional small parameter to be determined. Then, by hypothesis and Lemma 5.5, $\mathbf{S}(\mathbf{x},\mathbf{I}) = \mathrm{D}\sigma(\mathbf{x},\mathbf{I}) = \mathbf{0}$. The continuity of $\mathrm{D}\sigma$ (almost uniformly in $\mathbf{x}$) then yields an $\eta > 0$ such that, for *a.e.* $\mathbf{x} \in \Omega$,

$$|\mathrm{D}\sigma(\mathbf{x}, \mathbf{F}^\mathrm{T}\mathbf{F})| < \varepsilon \quad \text{whenever} \quad |\mathbf{F}^\mathrm{T}\mathbf{F} - \mathbf{I}| < \eta. \tag{6.5}$$

Thus, in view of $(6.2)_2$, if we choose $\delta_\mathrm{o} < \eta$, it follows that, for *a.e.* $\mathbf{x} \in \Omega$,

$$2|\mathrm{D}\sigma(\mathbf{x}, \mathbf{F}^\mathrm{T}\mathbf{F}) : (\mathbf{H}^\mathrm{T}\mathbf{H})| < 2\varepsilon|\mathbf{H}|^2. \tag{6.6}$$

We next consider

$$\big(\mathbf{H}^\mathrm{T}\mathbf{F} + \mathbf{F}^\mathrm{T}\mathbf{H}\big) : \mathrm{D}^2\sigma(\mathbf{x}, \mathbf{F}^\mathrm{T}\mathbf{F})[\mathbf{H}^\mathrm{T}\mathbf{F} + \mathbf{F}^\mathrm{T}\mathbf{H}].$$

Define $\mathbf{B} := \mathbf{H}^\mathrm{T}\mathbf{F} + \mathbf{F}^\mathrm{T}\mathbf{H} \in \mathrm{Sym}_n$ and rewrite this quadratic form (in $\mathbf{B}$) as

$$\mathbf{B} : \mathrm{D}^2\sigma(\mathbf{x}, \mathbf{F}^\mathrm{T}\mathbf{F})[\mathbf{B}] = \mathbf{B} : \mathrm{D}^2\sigma(\mathbf{x}, \mathbf{I})[\mathbf{B}] + \mathbf{B} : \big(\mathrm{D}^2\sigma(\mathbf{x}, \mathbf{F}^\mathrm{T}\mathbf{F}) - \mathrm{D}^2\sigma(\mathbf{x}, \mathbf{I})\big)[\mathbf{B}]. \tag{6.7}$$

Then, given $\varepsilon > 0$, the continuity of $\mathrm{D}^2\sigma$ (almost uniformly in $\mathbf{x}$) yields a $\beta > 0$ such that, for *a.e.* $\mathbf{x} \in \Omega$,

$$\big|\mathrm{D}^2\sigma(\mathbf{x}, \mathbf{F}^\mathrm{T}\mathbf{F}) - \mathrm{D}^2\sigma(\mathbf{x}, \mathbf{I})\big| < \varepsilon \quad \text{whenever} \quad |\mathbf{F}^\mathrm{T}\mathbf{F} - \mathbf{I}| < \beta. \tag{6.8}$$

In view of $(6.2)_2$, a choice of $\delta_\mathrm{o} < \beta$ yields

$$\big|\mathbf{B} : \big(\mathrm{D}^2\sigma(\mathbf{x}, \mathbf{F}^\mathrm{T}\mathbf{F}) - \mathrm{D}^2\sigma(\mathbf{x}, \mathbf{I})\big)\big[\mathbf{B}\big]\big| \leq \varepsilon|\mathbf{B}|^2, \tag{6.9}$$

for *a.e.* $\mathbf{x} \in \Omega$. Lastly, in view of (6.1) and Lemma 5.5, the remaining term in (6.7) satisfies

$$\mathbf{B} : \mathrm{D}^2\sigma(\mathbf{x}, \mathbf{I})[\mathbf{B}] \geq c|\mathbf{B}|^2. \tag{6.10}$$

If we let $\mathbf{F} = \nabla\mathbf{u}(\mathbf{x})$ and $\mathbf{H} = \nabla\mathbf{w}(\mathbf{x})$ in $(5.6)_2$, integrate over Ω, and make use of (6.6), (6.7), (6.9), and (6.10), we conclude that

$$\begin{aligned}\int_\Omega \nabla\mathbf{w} : \mathbb{A}(\nabla\mathbf{u})[\nabla\mathbf{w}]\,\mathrm{d}\mathbf{x} \geq (c-\varepsilon)\int_\Omega \big|(\nabla\mathbf{w})^\mathrm{T}\nabla\mathbf{u} + (\nabla\mathbf{u})^\mathrm{T}\nabla\mathbf{w}\big|^2\,\mathrm{d}\mathbf{x} \\ -2\varepsilon\int_\Omega |\nabla\mathbf{w}|^2\,\mathrm{d}\mathbf{x}.\end{aligned} \tag{6.11}$$

We now assume that $\mathbf{u} \in C^1(\overline{\Omega}; \mathbb{R}^n)$. A generalized Korn's inequality, Proposition B.1, then yields the existence of a constant $K > 0$ such that

$$\int_\Omega \big|(\nabla\mathbf{w})^\mathrm{T}\nabla\mathbf{u} + (\nabla\mathbf{u})^\mathrm{T}\nabla\mathbf{w}\big|^2\,\mathrm{d}\mathbf{x} \geq K\int_\Omega |\nabla\mathbf{w}|^2\,\mathrm{d}\mathbf{x},$$

which together with (6.11) gives us

$$\int_\Omega \nabla\mathbf{w} : \mathbb{A}(\nabla\mathbf{u})[\nabla\mathbf{w}]\,\mathrm{d}\mathbf{x} \ge \big[K(c-\varepsilon) - 2\varepsilon\big]\int_\Omega |\nabla\mathbf{w}|^2\,\mathrm{d}\mathbf{x}. \tag{6.12}$$

Finally, we return to ε and $\delta_{\rm o}$. Choose $\varepsilon > 0$ that satisfies $\varepsilon < \min\{c, Kc/(K+2)\}$ so that (6.12) will yield (6.4). Then choose $\delta_{\rm o} > 0$ so that $\delta_{\rm o} < \min\{\eta, \beta, 1\}$, where η and β are determined by ε in (6.5) and (6.8), respectively. That concludes the proof. □

6.2. **Uniqueness of Equilibrium that have Sufficiently Small Strains.** We are now ready to apply the results obtained for general integrands in the Calculus of Variations to elastic deformations with small strains.

Theorem 6.3. *Let $\mathbf{F} \mapsto W(\mathbf{x}, \mathbf{F})$ be C^2, almost uniformly in $\mathbf{x}$, on $\mathbb{M}^{n\times n}_+$. Suppose that W satisfies* (5.1) *and (1)–(3) of Hypothesis 3.1. Assume, in addition, that, for a.e. $\mathbf{x} \in \Omega$, $\mathbf{S}(\mathbf{x}, \mathbf{I}) = \mathbf{0}$ and*

$$\mathbf{H} : \mathbb{A}(\mathbf{x}, \mathbf{I})[\mathbf{H}] \ge c|\mathbf{H} + \mathbf{H}^{\rm T}|^2$$

for some constant $c > 0$ and every $\mathbf{H} \in \mathbb{M}^{n\times n}$. Then there exists a $\delta \in (0, 1)$ such that any solution, $\mathbf{u}_{\rm e} \in \mathrm{AD}$, of the equilibrium equations (5.16)*, for either the pure-displacement problem or the mixed problem, that satisfies both*

$$\mathbf{u}_{\rm e} \in C^1(\overline{\Omega}; \mathbb{R}^n) \quad \textit{and} \quad ||(\nabla\mathbf{u}_{\rm e})^{\rm T}\nabla\mathbf{u}_{\rm e} - \mathbf{I}||_{\infty,\Omega} < \delta \tag{6.13}$$

or, merely,

$$||\nabla\mathbf{u}_{\rm e} - \mathbf{Q}||_{\infty,\Omega} < \delta \tag{6.14}$$

for some $\mathbf{Q} \in \mathrm{SO}(n)$, is the unique absolute minimizer of the energy among $\mathbf{v} \in \mathrm{AD}$ that satisfy

$$||(\nabla\mathbf{v})^{\rm T}\nabla\mathbf{v} - \mathbf{I}||_{\infty,\Omega} < \delta. \tag{6.15}$$

Moreover, there are no other equilibrium solutions, $\widehat{\mathbf{u}}_{\rm e} \in \mathrm{AD}$, that satisfy (6.15) *with $\mathbf{v} = \widehat{\mathbf{u}}_{\rm e}$.*

Theorem 6.3 establishes that there is at most one solution with (sufficiently) small strains for both the pure-displacement and the mixed problem in Nonlinear Elasticity. For the pure-displacement problem, essentially the same result (with a similar proof) was first established by John [34]. A more recent elementary proof, under different hypotheses, can be found in [53].

Remark 6.4. Theorem 6.3 does not yield the *existence* of any solutions of the equilibrium equations that satisfy (6.13). However, suppose that the stored-energy density, the boundary, and the data: $(\mathbf{d}, \mathbf{s}, \mathbf{b})$ are sufficiently smooth and either $\mathcal{D} = \partial\Omega$ (the displacement problem) or both $\partial\mathcal{S} = \varnothing$ and $\partial\mathcal{D} = \varnothing$, e.g., a thick spherical shell with $\mathcal{S}$ and $\mathcal{D}$ the inner and outer boundaries. Then results of Valent [59], which make use of estimates for systems of linear elliptic equations and the implicit function theorem, yield the existence of a solution that satisfies (6.13) whenever $\mathbf{s}$ and $\mathbf{b}$ are sufficiently small and $\mathbf{d}$ is sufficiently close to the identity.

Remark 6.5. In Theorem 6.3 it is irrelevant whether or not the equilibrium solution is injective. This may engender curious consequences. For example, suppose that one can show that a non-injective equilibrium solution with (sufficiently) small strains exists. Then Theorem 6.3 implies, in particular, that there are *no injective equilibrium solutions with small strains.*

Proof of Theorem 6.3. We shall assume that $\mathbf{u}_\mathrm{e}$ satisfies (6.13). The proof when $\mathbf{u}_\mathrm{e}$ satisfies (6.14) is similar. Let $\mathbf{u}_\mathrm{e} \in \mathrm{AD}$ be a solution of (5.16) that satisfies (6.13) for some $\delta \in (0,1)$ to be determined. Then, in view of Proposition 6.1, there exists a $\delta_\mathrm{o} \in (0,1)$ and a $k > 0$ such that, for all $\mathbf{w} \in \mathrm{Var}$,

$$\int_\Omega \nabla\mathbf{w} : \mathbb{A}(\nabla\mathbf{u}_\mathrm{e})[\nabla\mathbf{w}]\,\mathrm{d}\mathbf{x} \geq 4k\int_\Omega |\nabla\mathbf{w}|^2\mathrm{d}\mathbf{x}, \tag{6.16}$$

provided $\delta < \delta_\mathrm{o}$. Now, let $\mathbf{v} \in \mathrm{AD}$ satisfy (6.15) for some $\delta \in (0,\delta_\mathrm{o})$ to be determined.

Next, fix $p > n$. Then Proposition 4.8 together with (5.11), (5.14), $(6.13)_2$, and (6.15) yield a constant $A^* > 0$ such that

$$||\nabla\mathbf{u}_\mathrm{e} - \nabla\mathbf{v}||_{1,\Omega} < 2A^*|\Omega|^{1/p}\delta. \tag{6.17}$$

Also, in view of Proposition 4.3 (Geometric Rigidity), there exists a constant $M > 0$ such that

$$\boldsymbol{|}\nabla\mathbf{u}_\mathrm{e}\boldsymbol{|}_{\mathrm{BMO}(\Omega)} < M\delta, \qquad \boldsymbol{|}\nabla\mathbf{v}\boldsymbol{|}_{\mathrm{BMO}(\Omega)} < M\delta,$$

and hence, by the triangle inequality,

$$\boldsymbol{|}\nabla\mathbf{u}_\mathrm{e} - \nabla\mathbf{v}\boldsymbol{|}_{\mathrm{BMO}(\Omega)} < 2M\delta. \tag{6.18}$$

Finally, if we define

$$\mathcal{B} := \{\mathbf{F} \in \mathbb{M}^{n\times n} : \mathrm{dist}(\mathbf{F}, \mathrm{SO}(n)) < \delta < 1\} \subset \mathbb{M}^{n\times n}_+,$$

we find that, for almost every $\mathbf{x} \in \Omega$,

$$\nabla\mathbf{u}_\mathrm{e}(\mathbf{x}) \in \mathcal{B}, \qquad \nabla\mathbf{v}(\mathbf{x}) \in \mathcal{B}. \tag{6.19}$$

We now take note of (6.16), (6.17), (6.18), and (6.19) and choose $\delta \in (0,\delta_\mathrm{o})$ sufficiently small so that $\mathbf{u}_\mathrm{e}$ and $\mathbf{v}$ satisfy the hypotheses of Theorem 3.3. We then find that $\mathbf{u}_\mathrm{e}$ and $\mathbf{v}$ satisfy the conclusions of that theorem, i.e.,

$$\mathcal{E}(\mathbf{v}) \geq \mathcal{E}(\mathbf{u}_\mathrm{e}) + k\int_\Omega |\nabla\mathbf{v} - \nabla\mathbf{u}_\mathrm{e}|^2\mathrm{d}\mathbf{x};$$

$\mathbf{v} \not\equiv \mathbf{u}_\mathrm{e}$ has strictly greater energy than $\mathbf{u}_\mathrm{e}$; and $\mathbf{v} \not\equiv \mathbf{u}_\mathrm{e}$ cannot be an equilibrium solution. □

7. Uniqueness of Equilibrium with Sufficiently Small Strain Differences; Change of Reference Configuration

In this section we extend the uniqueness results obtained in Section 6.2. In particular, we show that the positivity of the second variation at a weak solution of the equilibrium equations, $\mathbf{u}_\mathrm{e}$, that is a diffeomorphism, implies that $\mathbf{u}_\mathrm{e}$ is a strict minimizer of the energy among those admissible deformations $\mathbf{v}$ whose right Cauchy-Green strain tensor $\mathbf{C}_\mathbf{v} := (\nabla\mathbf{v})^\mathrm{T}\nabla\mathbf{v}$ is uniformly and sufficiently close to $\mathbf{C}_\mathrm{e} := (\nabla\mathbf{u}_\mathrm{e})^\mathrm{T}\nabla\mathbf{u}_\mathrm{e}$. We also show that such a $\mathbf{v}$ cannot be a weak solution of the equilibrium equations. We begin with some additional notations.

Recall that we consider a body that we identify with the closure of a bounded, Lipschitz domain $\Omega \subset \mathbb{R}^n$, $n = 2$ or $n = 3$, that it occupies in a fixed reference configuration. We let $C^0(\overline{\Omega};\mathbb{R}^n)$ denote those maps $\mathbf{u} : \overline{\Omega} \to \mathbb{R}^n$ that are bounded and uniformly continuous on the closure of Ω. We shall write $\mathbf{u} \in C^1(\overline{\Omega};\mathbb{R}^n)$ provided that both $\mathbf{u}$ and its classical gradient

$\nabla\mathbf{u}$ are bounded and uniformly continuous on the closure of Ω. Note that, for each $\mathbf{x}\in\Omega$, $\nabla\mathbf{u}(\mathbf{x})\in\mathbb{M}^{n\times n}$ with components $[\nabla\mathbf{u}]_{ij}=\partial u_i/\partial x_j$. As in Section 5.3, we shall let

$$\partial\Omega=\overline{\mathcal{D}}\cup\overline{\mathcal{S}}\quad\text{with } \mathcal{D} \text{ and } \mathcal{S} \text{ relatively open},\ \ \mathcal{D}\cap\mathcal{S}=\varnothing,$$

and $\mathcal{D}\neq\varnothing$. In addition, we suppose that functions $\mathbf{d}\in C^1(\overline{\mathcal{D}};\mathbb{R}^n)$, $\mathbf{b}\in L^2(\Omega;\mathbb{R}^n)$, and, if $\mathcal{S}\neq\varnothing$, $\mathbf{s}\in L^2(\mathcal{S};\mathbb{R}^n)$ are prescribed. We assume that $\mathbf{d}$ is one-to-one.

We next define what we mean by a diffeomorphism and we also recall our definition of admissible deformations and variations from Section 5.

Definition 7.1. Let $\mathbf{u}:\overline{\Omega}\to\mathbb{R}^n$ be an injective mapping with inverse $\mathbf{u}^{\text{-}1}:\mathbf{u}(\overline{\Omega})\to\overline{\Omega}$. We call $\mathbf{u}$ an (orientation preserving) *diffeomorphism* provided that

(1) $\mathbf{u}\in C^1(\overline{\Omega};\mathbb{R}^n)$;
(2) $\mathbf{u}^{\text{-}1}\in C^1(\mathbf{u}(\overline{\Omega});\mathbb{R}^n)$; and
(3) $\det\nabla\mathbf{u}>0$ on the compact set $\overline{\Omega}$.

Next, recall that

$$\begin{aligned}\mathrm{AD}&:=\{\mathbf{u}\in W^{1,\infty}(\Omega;\mathbb{R}^n):\det\nabla\mathbf{u}>0\ a.e.,\ \mathbf{u}=\mathbf{d}\text{ on }\mathcal{D}\},\\ \mathrm{Var}&:=\{\mathbf{w}\in W^{1,2}(\Omega;\mathbb{R}^n):\mathbf{w}=\mathbf{0}\text{ on }\mathcal{D}\}.\end{aligned}\tag{7.1}$$

The main result of this section is the following theorem.

Theorem 7.2. *Let W satisfy (1)–(3) of Hypothesis 3.1. Suppose that*

(A) $\mathbf{u}_\mathrm{e}\in\mathrm{AD}$ *is a diffeomorphism;*
(B) $\mathbf{u}_\mathrm{e}$ *is a weak solution of the equilibrium equations; and*
(C) $\mathbf{u}_\mathrm{e}$ *satisfies*

$$\int_\Omega\nabla\mathbf{w}(\mathbf{x}):\mathbb{A}\big(\mathbf{x},\nabla\mathbf{u}_\mathrm{e}(\mathbf{x})\big)\big[\nabla\mathbf{w}(\mathbf{x})\big]\,\mathrm{d}\mathbf{x}\geq 4k\int_\Omega|\nabla\mathbf{w}(\mathbf{x})|^2\mathrm{d}\mathbf{x},$$

for some $k>0$ and all $\mathbf{w}\in\mathrm{Var}$.

Then there exists an $\varepsilon>0$ such that any $\mathbf{v}\in\mathrm{AD}$ that satisfies

$$0<\big\|(\nabla\mathbf{v})^{\mathrm{T}}\nabla\mathbf{v}-(\nabla\mathbf{u}_\mathrm{e})^{\mathrm{T}}\nabla\mathbf{u}_\mathrm{e}\big\|_{\infty,\Omega}<\varepsilon\tag{7.2}$$

has strictly greater energy than $\mathbf{u}_\mathrm{e}$. Moreover, there are no other weak solutions of the equilibrium equations, $\mathbf{v}_\mathrm{e}\in\mathrm{AD}$, that satisfy (7.2) with $\mathbf{v}=\mathbf{v}_\mathrm{e}$.

Remark 7.3. Our proof of Theorem 7.2 requires that we show that all of the hypotheses of Theorem 3.3 are satisfied. A direct application of Theorem 3.3 would necessitate us to make use of (7.2) to demonstrate that $\mathbf{u}_\mathrm{e}$ and $\mathbf{v}$ satisfy $(3.13)_{2,3}$. In this regard, Ciarlet & Mardare [15] have obtained extensions of the Geometric-Rigidity results of [24] and [16] (Proposition 4.3 in this manuscript) that include a second mapping $\mathbf{u}_\mathrm{e}\in C^1(\overline{\Omega};\mathbb{R}^n)$ with $\det\nabla\mathbf{u}_\mathrm{e}>0$ on $\overline{\Omega}$, but which need not be injective. Their results imply that there exists a constant $K=K(p,\mathbf{u}_\mathrm{e},\Omega)$ such that any $\mathbf{v}\in W^{1,p}(\Omega;\mathbb{R}^n)$, $2\leq p<\infty$, that satisfies $\det\nabla\mathbf{v}>0$ *a.e.* and $\mathbf{v}=\mathbf{u}_\mathrm{e}$ on $\mathcal{D}$ will also satisfy

$$\|\nabla\mathbf{v}-\nabla\mathbf{u}_\mathrm{e}\|_{p,\Omega}^2\leq K\big\|(\nabla\mathbf{v})^{\mathrm{T}}\nabla\mathbf{v}-(\nabla\mathbf{u}_\mathrm{e})^{\mathrm{T}}\nabla\mathbf{u}_\mathrm{e}\big\|_{p/2,\Omega}.\tag{7.3}$$

This result together with (7.2) yields the integral estimate $(3.13)_3$. However, Theorem 3.3 also requires the BMO-estimate $(3.13)_2$. Unfortunately, a BMO-estimate such as (4.6) does not follow from (7.3) due to the dependence of the constant K upon the mapping $\mathbf{u}_e$. To obtain (4.6) from (4.5) one must make use of the fact that the constant C in (4.5) is the same for all cubes contained in the region.

Remark 7.4. At the end of the introduction we noted that it would be of interest to prove some of our results, e.g., Theorem 7.2, without the assumption that $\mathbf{u}_e$ is one-to-one on $\overline{\Omega}$. This is of particular interest when the restriction of $\mathbf{u}_e$ to $\mathcal{S}$ is not one-to-one and the deformed body then exhibits self-contact (see, e.g., Ciarlet [14, Section 5.6]). The main difficulty is that the boundary of $\mathbf{u}_e(\Omega)$ may then fail to be Lipschitz since the deformed region may be on both "sides" of its boundary. Here one might want to attempt to follow the approach in [15] that partitions Ω into subdomains upon which $\mathbf{u}_e$ is injective. We also note that much of our proof is valid if $\mathbf{u}_e$ is bi-Lipschitz, rather than a diffeomorphism. However, once again, $\mathbf{u}_e(\Omega)$ may then fail to be Lipschitz. See the counterexample in [26, Section 1.2].

Our proof of Theorem 7.2 involves a change in reference configuration.[12] This change of variables will show that our assumption that two strain tensors are close to each other yields a new deformation whose gradient is close to the set of rotations. We shall then make use of Geometric Rigidity and Theorem 3.3. We postpone the proof of Theorem 7.2 to the end of this section.

7.1. **Change of Reference Configuration.** In this subsection we present the required change of variables that makes the deformed configuration into a new reference configuration. Those readers already familiar with this procedure may prefer to skip to Section 7.2.

7.1.1. *The Body and its Deformed Image.* We first recall some properties of domains and their image under injective mappings. Let $U \subset \mathbb{R}^n$, $n \geq 2$, be a bounded domain. Suppose that $\mathbf{u} \in C^0(U; \mathbb{R}^n)$ is injective. Then standard results in topology and degree theory (see, e.g., [23, Theorem 3.30]) imply that $\mathbf{u}(U)$ is also a bounded domain. Since $\Omega \subset \mathbb{R}^n$, $n \geq 2$, denotes a bounded Lipschitz domain, when $\mathbf{u} \in C^0(\overline{\Omega}; \mathbb{R}^n)$ is injective it then follows that $\mathbf{u}(\partial\Omega) = \partial\mathbf{u}(\Omega)$. The next result is well known. We sketch the proof for the interested reader.

Proposition 7.5. *Suppose that* $\mathbf{u} \in C^1(\overline{\Omega}; \mathbb{R}^n)$ *is a diffeomorphism. Then* $\mathbf{u}(\Omega)$ *is a bounded Lipschitz domain;* $\mathbf{u}$ *and* $\mathbf{u}^{-1}$ *satisfy*

$$\big[\nabla_{\mathbf{x}}\mathbf{u}(\mathbf{x})\big]^{-1} = \nabla_{\mathbf{y}}\mathbf{u}^{-1}(\mathbf{y}) \quad \textit{with } \mathbf{y} = \mathbf{u}(\mathbf{x}). \tag{7.4}$$

Moreover, if $\widehat{\mathbf{z}} \in W^{1,p}(\mathbf{u}(\Omega); \mathbb{R}^n)$ *and* $\mathbf{w} \in W^{1,p}(\Omega; \mathbb{R}^n)$, $p \in [1, \infty]$, *then* $\widehat{\mathbf{z}} \circ \mathbf{u} \in W^{1,p}(\Omega; \mathbb{R}^n)$, $\mathbf{w} \circ \mathbf{u}^{-1} \in W^{1,p}(\mathbf{u}(\Omega); \mathbb{R}^n)$, *and*

$$\begin{aligned} \nabla_{\mathbf{x}}(\widehat{\mathbf{z}} \circ \mathbf{u})(\mathbf{x}) &= \big[\nabla_{\mathbf{y}}\widehat{\mathbf{z}}\big(\mathbf{u}(\mathbf{x})\big)\big]\nabla\mathbf{u}(\mathbf{x}) \quad \textit{for a.e. } \mathbf{x} \in \Omega, \\ \nabla_{\mathbf{y}}(\mathbf{w} \circ \mathbf{u}^{-1})(\mathbf{y}) &= \big[\nabla_{\mathbf{x}}\mathbf{w}\big(\mathbf{u}^{-1}(\mathbf{y})\big)\big]\nabla\mathbf{u}^{-1}(\mathbf{y}) \quad \textit{for a.e. } \mathbf{y} \in \mathbf{u}(\Omega). \end{aligned} \tag{7.5}$$

[12] A change in reference configuration is a standard procedure in Continuum Mechanics. See, e.g., Ciarlet [14, Chapter 1].

Remark 7.6. We note that the change of variables formula also shows that diffeomorphisms map sets of measure zero to sets of measure zero, e.g., if $\det \nabla \mathbf{v}(\mathbf{x}) > 0$ for *a.e.* $\mathbf{x} \in \Omega$ then $\det \nabla(\mathbf{v} \circ \mathbf{u}^{-1})(\mathbf{y}) > 0$ for *a.e.* $\mathbf{y} \in \mathbf{u}(\Omega)$.

Sketch of the proof of Proposition 7.5. The set $\mathbf{u}(\overline{\Omega})$ is compact and hence bounded. Equation (7.4) follows from the chain rule for diffeomorphisms. We next show that $\mathbf{u}(\Omega)$ is a Lipschitz domain. We note that a result of Whitney [61] implies that the Whitney extension theorem (see, e.g., [20, Section 6.5]) applies to Lipschitz domains and hence that $\mathbf{u}$ has a C^1 extension to $\mathbb{R}^n$. Fix a point $\mathbf{x}_\mathrm{o} \in \partial\Omega$. Then, since $\det \nabla \mathbf{u}(\mathbf{x}_\mathrm{o}) > 0$, $\nabla \mathbf{u}(\mathbf{x}_\mathrm{o})$ is invertible. The inverse function theorem states that (the extension of) $\mathbf{u}$ is a diffeomorphism on $B(\mathbf{x}_\mathrm{o}, r)$ for some $r > 0$. A result of Hofmann, Mitrea, & Taylor [30, Section 4.1] then shows that $\mathbf{u}(\Omega)$ is Lipschitz at the point $\mathbf{u}(\mathbf{x}_\mathrm{o})$. Thus, $\mathbf{u}(\Omega)$ is a bounded Lipschitz domain. Finally, we note that, for $1 \le p \le \infty$, $(7.5)_{1,2}$ are each a consequence of the chain rule for the composition of a Sobolev function with a diffeomorphism (see, e.g., [2, Section 4.26]). □

7.1.2. *Body and Surface Forces, the Energy, the Stress, and the Elasticity Tensor.* We now consider $\mathbf{d}$, $\mathbf{s}$, the stored-energy density W and its first and second derivatives, the Piola-Kirchhoff stress $\mathbf{S}$ and the Elasticity Tensor $\mathbb{A}$. We show how each transforms from the reference configuration $\overline{\Omega}$ to the deformed configuration $\mathbf{u}(\overline{\Omega})$.

Definition 7.7. Given a stored-energy density $W : \overline{\Omega} \times \mathbb{M}^{n\times n}_+ \to [0, \infty)$ and a diffeomorphism $\mathbf{u} \in C^1(\overline{\Omega}; \mathbb{R}^n)$, we define $W_u : \mathbf{u}(\overline{\Omega}) \times \mathbb{M}^{n\times n}_+ \to [0, \infty)$, *the stored-energy density with respect to the deformed configuration* $\mathbf{u}(\overline{\Omega})$, by

$$W_u(\mathbf{y}, \mathbf{G}) := W\big(\mathbf{x}, \mathbf{G}\mathbf{F}\big)(\det \mathbf{F})^{-1}, \tag{7.6}$$

where $\mathbf{y} = \mathbf{u}(\mathbf{x})$ and $\mathbf{F} = \mathbf{F}(\mathbf{x}) := \nabla \mathbf{u}(\mathbf{x})$. Given a body force field $\mathbf{b} \in L^2(\Omega; \mathbb{R}^n)$ and, if $\mathcal{S} \neq \varnothing$, a surface traction field $\mathbf{s} \in L^2(\mathcal{S}; \mathbb{R}^n)$ we define $\mathbf{b}_u : \mathbf{u}(\Omega) \to \mathbb{R}^n$ and $\mathbf{s}_u : \mathbf{u}(\mathcal{S}) \to \mathbb{R}^n$, the body force and surface tractions in the deformed configuration, by, for *a.e.* $\mathbf{y} \in \mathbf{u}(\Omega)$,

$$\mathbf{b}_u(\mathbf{y}) := \mathbf{b}(\mathbf{x})(\det \mathbf{F})^{-1}, \qquad \mathbf{s}_u(\mathbf{y}) := \mathbf{s}(\mathbf{x})|\mathbf{F}^{-\mathrm{T}}\mathbf{n}(\mathbf{x})|^{-1}(\det \mathbf{F})^{-1},$$

for $\mathcal{H}^{n-1}$-*a.e.* $\mathbf{y} \in \mathbf{u}(\mathcal{S})$, where $\mathbf{n}(\mathbf{x})$ denotes the outward unit normal to Ω (which exists at $\mathcal{H}^{n-1}$-*a.e.* $\mathbf{x} \in \partial\Omega$, since $\partial\Omega$ is Lipschitz.)

The next result is a simple consequence of the standard chain rule for C^1 functions.

Lemma 7.8. *Let* $\mathbf{u} \in C^1(\overline{\Omega}; \mathbb{R}^n)$ *be a diffeomorphism. Suppose that* $W : \overline{\Omega} \times \mathbb{M}^{n\times n}_+ \to [0, \infty)$ *is such that* W *satisfies (1)–(3) of Hypothesis 3.1. Then* W_u*, defined by* (7.6)*, also satisfies (1)–(3) of Hypothesis 3.1. Moreover, for a.e.* $\mathbf{x} \in \Omega$*, every* $\mathbf{G} \in \mathbb{M}^{n\times n}_+$*, and every* $\mathbf{H} \in \mathbb{M}^{n\times n}$*,*

$$\begin{aligned} \mathbf{S}_u(\mathbf{u}(\mathbf{x}), \mathbf{G}) : \mathbf{H} &:= \Big[\frac{\partial}{\partial \mathbf{G}} W_u(\mathbf{u}(\mathbf{x}), \mathbf{G})\Big] : \mathbf{H} = \mathbf{S}(\mathbf{x}, \mathbf{G}\mathbf{F}) : [\mathbf{H}\mathbf{F}](\det \mathbf{F})^{-1}, \\ \mathbf{H} : \mathbb{A}_u(\mathbf{u}(\mathbf{x}), \mathbf{G})[\mathbf{H}] &:= \frac{\partial}{\partial \mathbf{G}}\Big(\mathbf{S}_u\big(\mathbf{u}(\mathbf{x}), \mathbf{G}\big) : \mathbf{H}\Big)[\mathbf{H}] = [\mathbf{H}\mathbf{F}] : \mathbb{A}(\mathbf{x}, \mathbf{G}\mathbf{F})[\mathbf{H}\mathbf{F}](\det \mathbf{F})^{-1}, \end{aligned} \tag{7.7}$$

where $\mathbf{F} = \mathbf{F}(\mathbf{x}) := \nabla \mathbf{u}(\mathbf{x})$.

If we combine Proposition 7.5, Lemma 7.8, and the change of variables formula for injective Lipschitz mappings we conclude the following.

Proposition 7.9. *Let* $\mathbf{u} \in C^1(\overline{\Omega};\mathbb{R}^n)$ *be a diffeomorphism. Suppose that* W *satisfies (1)–(3) of Hypothesis 3.1. Assume further that* $\widehat{\mathbf{v}} \in W^{1,\infty}(\mathbf{u}(\Omega);\mathbb{R}^n)$ *and* $\widehat{\mathbf{w}} \in W^{1,2}(\mathbf{u}(\Omega);\mathbb{R}^n)$*. Define* $\mathbf{v} := \widehat{\mathbf{v}} \circ \mathbf{u} : \Omega \to \mathbb{R}^n$ *and* $\mathbf{w} := \widehat{\mathbf{w}} \circ \mathbf{u} : \Omega \to \mathbb{R}^n$*. Then* $\mathbf{v} \in W^{1,\infty}(\Omega;\mathbb{R}^n)$, $\mathbf{w} \in W^{1,2}(\Omega;\mathbb{R}^n)$, *and*

$$
\begin{aligned}
\int_\Omega W\big(\mathbf{x}, \nabla\mathbf{v}(\mathbf{x})\big)\,\mathrm{d}\mathbf{x} &= \int_{\mathbf{u}(\Omega)} W_u\big(\mathbf{y}, \nabla\widehat{\mathbf{v}}(\mathbf{y})\big)\,\mathrm{d}\mathbf{y},\\
\int_\Omega \mathbf{S}\big(\mathbf{x}, \nabla\mathbf{v}(\mathbf{x})\big) : \nabla\mathbf{w}(\mathbf{x})\,\mathrm{d}\mathbf{x} &= \int_{\mathbf{u}(\Omega)} \mathbf{S}_u\big(\mathbf{y}, \nabla\widehat{\mathbf{v}}(\mathbf{y})\big) : \nabla\widehat{\mathbf{w}}(\mathbf{y})\,\mathrm{d}\mathbf{y},\\
\int_\Omega \nabla\mathbf{w}(\mathbf{x}) : \mathbb{A}\big(\mathbf{x}, \nabla\mathbf{v}(\mathbf{x})\big)[\nabla\mathbf{w}(\mathbf{x})]\,\mathrm{d}\mathbf{x} &= \int_{\mathbf{u}(\Omega)} \nabla\widehat{\mathbf{w}}(\mathbf{y}) : \mathbb{A}_u\big(\mathbf{y}, \nabla\widehat{\mathbf{v}}(\mathbf{y})\big)[\nabla\widehat{\mathbf{w}}(\mathbf{y})]\,\mathrm{d}\mathbf{y},\\
\int_\Omega \mathbf{b}(\mathbf{x}) \cdot \mathbf{w}(\mathbf{x})\,\mathrm{d}\mathbf{x} &= \int_{\mathbf{u}(\Omega)} \mathbf{b}_u(\mathbf{y}) \cdot \widehat{\mathbf{w}}(\mathbf{y})\,\mathrm{d}\mathbf{y},\\
\int_{\mathcal{S}} \mathbf{s}(\mathbf{x}) \cdot \mathbf{w}(\mathbf{x})\,\mathrm{d}\mathcal{H}^{n-1}_{\mathbf{x}} &= \int_{\mathbf{u}(\mathcal{S})} \mathbf{s}_u(\mathbf{y}) \cdot \widehat{\mathbf{w}}(\mathbf{y})\,\mathrm{d}\mathcal{H}^{n-1}_{\mathbf{y}}.
\end{aligned}
\tag{7.8}
$$

Remark 7.10. Equations (7.8) remain valid if $\mathbf{v} \in W^{1,\infty}(\Omega;\mathbb{R}^n)$ and $\mathbf{w} \in W^{1,2}(\Omega;\mathbb{R}^n)$ are prescribed and $\widehat{\mathbf{v}} := \mathbf{v} \circ \mathbf{u}^{-1} \in W^{1,\infty}(\mathbf{u}(\Omega);\mathbb{R}^n)$ and $\widehat{\mathbf{w}} := \mathbf{w} \circ \mathbf{u}^{-1} \in W^{1,2}(\mathbf{u}(\Omega);\mathbb{R}^n)$ are defined.

Proof of Proposition 7.9. We shall prove $(7.8)_2$. The proofs of the other equations are similar.[13] Let W satisfy (1)–(3) of Hypothesis 3.1 and suppose that $\mathbf{u}$, $\mathbf{v}$, $\widehat{\mathbf{v}}$, $\mathbf{w}$, and $\widehat{\mathbf{w}}$ are as given in the statement of the proposition. Then, by Proposition 7.5, $\mathbf{v} \in W^{1,\infty}(\Omega;\mathbb{R}^n)$, $\mathbf{w} \in W^{1,2}(\Omega;\mathbb{R}^n)$, and

$$
\nabla_{\mathbf{x}}\mathbf{v}(\mathbf{x}) = \nabla_{\mathbf{y}}\widehat{\mathbf{v}}\big(\mathbf{u}(\mathbf{x})\big)\nabla\mathbf{u}(\mathbf{x}), \qquad \nabla_{\mathbf{x}}\mathbf{w}(\mathbf{x}) = \nabla_{\mathbf{y}}\widehat{\mathbf{w}}\big(\mathbf{u}(\mathbf{x})\big)\nabla\mathbf{u}(\mathbf{x}), \tag{7.9}
$$

for *a.e.* $\mathbf{x} \in \Omega$. Therefore, in view of (7.9) and $(7.7)_1$ with $\mathbf{G} = \nabla_{\mathbf{y}}\widehat{\mathbf{v}}(\mathbf{u}(\mathbf{x}))$, $\mathbf{H} = \nabla_{\mathbf{y}}\widehat{\mathbf{w}}(\mathbf{u}(\mathbf{x}))$, and $\mathbf{F} = \nabla\mathbf{u}(\mathbf{x})$,

$$
\mathbf{S}\big(\mathbf{x}, \nabla\mathbf{v}(\mathbf{x})\big) : \nabla\mathbf{w}(\mathbf{x}) = \mathbf{S}_u\big(\mathbf{y}, \nabla\widehat{\mathbf{v}}(\mathbf{y})\big) : \nabla\widehat{\mathbf{w}}(\mathbf{y})\big[\det\nabla\mathbf{u}(\mathbf{x})\big], \quad \mathbf{y} := \mathbf{u}(\mathbf{x}). \tag{7.10}
$$

Finally, we integrate (7.10) over Ω and then apply the change of variables formula for injective Lipschitz mappings (see, e.g., [21, Theorem 3.2.5]) to deduce the desired result, $(7.8)_2$. ☐

We now fix a diffeomorphism $\mathbf{u} \in \mathrm{AD}$ (see $(7.1)_1$) and consider $\mathbf{u}(\Omega)$ as a new reference configuration. We first define the admissible deformations and the corresponding variations that originate at this reference configuration.

Definition 7.11. Fix a *diffeomorphism* $\mathbf{u} \in C^1(\overline{\Omega};\mathbb{R}^n)$ that satisfies $\mathbf{u} \in \mathrm{AD}$ and define

$$
\begin{aligned}
\mathrm{AD}_u &:= \{\widehat{\mathbf{v}} \in W^{1,\infty}(\mathbf{u}(\Omega);\mathbb{R}^n) : \det\nabla\widehat{\mathbf{v}} > 0 \textit{ a.e.},\ \widehat{\mathbf{v}} = \mathbf{i} \ \text{ on } \mathbf{u}(\mathcal{D})\},\\
\mathrm{Var}_u &:= \{\widehat{\mathbf{w}} \in W^{1,2}(\mathbf{u}(\Omega);\mathbb{R}^n) : \widehat{\mathbf{w}} = \mathbf{0} \text{ on } \mathbf{u}(\mathcal{D})\}.
\end{aligned}
$$

Recall that the total energy $\mathcal{E}$ of $\mathbf{v} \in \mathrm{AD}$ is defined by

$$
\mathcal{E}(\mathbf{v}) := \int_\Omega \big[W\big(\mathbf{x}, \nabla\mathbf{v}(\mathbf{x})\big) - \mathbf{b}(\mathbf{x}) \cdot \mathbf{v}(\mathbf{x})\big]\mathrm{d}\mathbf{x} - \int_{\mathcal{S}} \mathbf{s}(\mathbf{x}) \cdot \mathbf{v}(\mathbf{x})\,\mathrm{d}\mathcal{H}^{n-1}_{\mathbf{x}} \tag{7.11}
$$

[13] Equation $(7.8)_5$ is based upon the identities $\mathbf{s} = \mathbf{S}\mathbf{n}$, $\mathbf{s}_u = \mathbf{S}_u\mathbf{m}$, $\mathbf{m} = (\mathbf{F}^{-\mathrm{T}}\mathbf{n})/|\mathbf{F}^{-\mathrm{T}}\mathbf{n}|$, and (cf. $(7.7)_1$) $\mathbf{S}\mathbf{F}^{\mathrm{T}} = (\det\mathbf{F})\mathbf{S}_u$, where $\mathbf{m}$ denotes the outward unit normal to $\mathbf{u}(\Omega)$. See, e.g., [14, Section 1.7].

and $\mathbf{v}_e \in \mathrm{AD}$ is a weak solution of the equilibrium equations corresponding to (7.11) if

$$0 = \int_\Omega \big[\mathbf{S}\big(\mathbf{x}, \nabla\mathbf{v}_e(\mathbf{x})\big) : \nabla\mathbf{w}(\mathbf{x}) - \mathbf{b}(\mathbf{x})\cdot\mathbf{w}(\mathbf{x})\big]\mathrm{d}\mathbf{x} - \int_{\mathcal{S}} \mathbf{s}(\mathbf{x})\cdot\mathbf{w}(\mathbf{x})\,\mathrm{d}\mathcal{H}^{n-1}_{\mathbf{x}} \tag{7.12}$$

for all variations $\mathbf{w} \in \mathrm{Var}$.

Lemma 7.12. *Let $\mathbf{u} \in \mathrm{AD}$ be a diffeomorphism and suppose that $\mathbf{v} \in \mathrm{AD}$. Then $\mathbf{v}$ is a weak solution of the equilibrium equations (7.12) if and only if $\widehat{\mathbf{v}} := \mathbf{v}\circ\mathbf{u}^{-1}$ is a weak solution of the equilibrium equations corresponding to the energy*

$$\mathcal{E}_u(\widehat{\mathbf{z}}) := \int_{\mathbf{u}(\Omega)} \big[W_u\big(\mathbf{y}, \nabla\widehat{\mathbf{z}}(\mathbf{y})\big) - \mathbf{b}_u(\mathbf{y})\cdot\widehat{\mathbf{z}}(\mathbf{y})\big]\mathrm{d}\mathbf{y} - \int_{\mathbf{u}(\mathcal{S})} \mathbf{s}_u(\mathbf{y})\cdot\widehat{\mathbf{z}}(\mathbf{y})\,\mathrm{d}\mathcal{H}^{n-1}_{\mathbf{y}}. \tag{7.13}$$

Moreover, necessary and sufficient conditions for the uniform positivity of the second variation of $\mathcal{E}$ at $\mathbf{v}$ is that the second variation of $\mathcal{E}_u$ be uniformly positive at $\widehat{\mathbf{v}}$.

Proof. The first assertion follows from $(7.8)_{2,4,5}$, (7.12), and an argument similar to the following one. To prove sufficiency, suppose that the second variation of $\mathcal{E}_u$ is uniformly positive at $\widehat{\mathbf{v}}$ with constant k. Fix $\mathbf{w} \in \mathrm{Var}$ and define $\widehat{\mathbf{w}} := \mathbf{w}\circ\mathbf{u}^{-1}$. Then, by Proposition 7.5, $\widehat{\mathbf{w}} \in W^{1,2}(\mathbf{u}(\Omega);\mathbb{R}^n)$ with

$$\nabla\widehat{\mathbf{w}}(\mathbf{y}) = \nabla_{\mathbf{x}}\mathbf{w}\big(\mathbf{u}^{-1}(\mathbf{y})\big)\nabla\mathbf{u}^{-1}(\mathbf{y}) \quad \text{for } a.e.\ \mathbf{y} \in \mathbf{u}(\Omega). \tag{7.14}$$

Moreover, since $\mathbf{w} = \mathbf{0}$ on $\mathcal{D}$ it follows that $\widehat{\mathbf{w}} = \mathbf{0}$ on $\mathbf{d}(\mathcal{D})$ and hence that $\widehat{\mathbf{w}} \in \mathrm{Var}_u$.

Next, the assumed uniform positivity together with $(7.8)_3$ shows that the second variation of $\mathcal{E}$ at $\mathbf{v}$ in the direction $\mathbf{w}$ is bounded below by

$$k\int_{\mathbf{u}(\Omega)} |\nabla\widehat{\mathbf{w}}(\mathbf{y})|^2\,\mathrm{d}\mathbf{y} = k\int_\Omega |\nabla\mathbf{w}(\mathbf{x})|^2 \det\nabla\mathbf{u}(\mathbf{x})\,\mathrm{d}\mathbf{x},$$

where the last equality follows from (7.14) and the change of variables formula. The desired result now follows since $\det\nabla\mathbf{u}$ is bounded away from zero on the compact set $\overline{\Omega}$. The necessity argument is similar. □

7.2. **Proof of Theorem 7.2.** Our proof of Theorem 7.2 will require us to show that (7.2) implies that the gradient of some mapping is sufficiently close to the set of rotations. We first define this mapping and show that the distance of its gradient from the rotations is bounded above by a constant times the strain difference given in (7.2).

Lemma 7.13. *Let $\mathbf{u}_e, \mathbf{v} \in \mathrm{AD}$ with $\mathbf{u}_e$ a diffeomorphism. Define $\mathbf{F}_e := \nabla\mathbf{u}_e$, $\mathbf{G} := \nabla\mathbf{v}$,*

$$\Upsilon_e := \sup_{\mathbf{x}\in\overline{\Omega}} |\mathbf{F}_e(\mathbf{x})|, \qquad \upsilon_e := \inf_{\mathbf{x}\in\overline{\Omega}} \big|[\mathbf{F}_e(\mathbf{x})]^{-1}\big|^{-1}, \qquad d(\mathbf{x}) := \mathrm{dist}\big(\mathbf{G}\mathbf{F}_e^{-1}, \mathrm{SO}(n)\big). \tag{7.15}$$

Then

$$\upsilon_e^2 d^2 \le \sqrt{n}\,|\mathbf{G}^{\mathrm{T}}\mathbf{G} - \mathbf{F}_e^{\mathrm{T}}\mathbf{F}_e| \le \Upsilon_e^2 d\sqrt{n}\,\big(d + 2\sqrt{n}\,\big). \tag{7.16}$$

Proof. We first note that $\nabla\mathbf{u}_e \in C^0(\overline{\Omega};\mathbb{R}^n)$ with $\det\nabla\mathbf{u}_e > 0$ on the compact set $\overline{\Omega}$ and hence Υ_e and υ_e are strictly positive and finite. Define

$$\mathbf{F} = \mathbf{F}(\mathbf{x}) := \mathbf{G}\mathbf{F}_e^{-1}, \qquad \mathbf{E} = \mathbf{E}(\mathbf{x}) := \tfrac{1}{2}\big(\mathbf{F}^{\mathrm{T}}\mathbf{F} - \mathbf{I}\big).$$

Then Lemma 5.6 shows that $\mathbf{E}$ and d, given by $(7.15)_3$, satisfy

$$d^2 \le 2\sqrt{n}\,|\mathbf{E}| \le d\sqrt{n}\,\big(d + 2\sqrt{n}\,\big). \tag{7.17}$$

Next, consider

$$\mathbf{G}^{\mathrm{T}}\mathbf{G} - \mathbf{F}_{\mathrm{e}}^{\mathrm{T}}\mathbf{F}_{\mathrm{e}} = \mathbf{F}_{\mathrm{e}}^{\mathrm{T}}\big[(\mathbf{G}\mathbf{F}_{\mathrm{e}}^{-1})^{\mathrm{T}}\mathbf{G}\mathbf{F}_{\mathrm{e}}^{-1} - \mathbf{I}\big]\mathbf{F}_{\mathrm{e}} = 2\mathbf{F}_{\mathrm{e}}^{\mathrm{T}}\mathbf{E}\mathbf{F}_{\mathrm{e}}, \tag{7.18}$$

$$|\mathbf{E}| = \big|\mathbf{F}_{\mathrm{e}}^{-\mathrm{T}}\big(\mathbf{F}_{\mathrm{e}}^{\mathrm{T}}\mathbf{E}\mathbf{F}_{\mathrm{e}}\big)\mathbf{F}_{\mathrm{e}}^{-1}\big| \leq \big|\mathbf{F}_{\mathrm{e}}^{-1}\big|^2\big|\mathbf{F}_{\mathrm{e}}^{\mathrm{T}}\mathbf{E}\mathbf{F}_{\mathrm{e}}\big|, \qquad \big|\mathbf{F}_{\mathrm{e}}^{\mathrm{T}}\mathbf{E}\mathbf{F}_{\mathrm{e}}\big| \leq |\mathbf{E}||\mathbf{F}_{\mathrm{e}}|^2. \tag{7.19}$$

Thus, if we now combine (7.17) and (7.19) we find that

$$\tfrac{1}{2}\tfrac{d^2}{\sqrt{n}}|\mathbf{F}_{\mathrm{e}}^{-1}|^{-2} \leq |\mathbf{F}_{\mathrm{e}}^{-1}|^{-2}|\mathbf{E}| \leq \big|\mathbf{F}_{\mathrm{e}}^{\mathrm{T}}\mathbf{E}\mathbf{F}_{\mathrm{e}}\big| \leq |\mathbf{E}||\mathbf{F}_{\mathrm{e}}|^2 \leq \tfrac{1}{2}d\big(d + 2\sqrt{n}\,\big)|\mathbf{F}_{\mathrm{e}}|^2. \tag{7.20}$$

Finally, $(7.15)_{1,2}$, (7.18), and (7.20) yield the desired result, (7.16). □

Remark 7.14. (1). The mapping whose gradient is close to the set of rotations is $\mathbf{v}\circ\mathbf{u}_{\mathrm{e}}^{-1}$. (2). If we make use of (5.14), in place of the first inequality in Lemma 5.6, we find that

$$\upsilon_{\mathrm{e}}^2 d \leq |\mathbf{G}^{\mathrm{T}}\mathbf{G} - \mathbf{F}_{\mathrm{e}}^{\mathrm{T}}\mathbf{F}_{\mathrm{e}}|. \tag{7.21}$$

Once again, although (7.21) does not scale properly for large d, its use will simplify the computation, which involves small-strain differences, in the next proof.

Proof of Theorem 7.2. Let $\mathbf{u}_{\mathrm{e}} \in \mathrm{AD}$ satisfy hypotheses (A)–(C) of the theorem. Define $\Omega_e := \mathbf{u}_{\mathrm{e}}(\Omega)$. Then, by Proposition 7.5, Ω_e is a bounded Lipschitz domain. Suppose that $\varepsilon > 0$ is a small parameter to be determined and let $\mathbf{v} \in \mathrm{AD}$ satisfy (7.2). Define, $\widehat{\mathbf{u}}_{\mathrm{e}}, \widehat{\mathbf{v}} \in \mathrm{AD}_{u_e}$ by

$$\widehat{\mathbf{u}}_{\mathrm{e}} := \mathbf{u}_{\mathrm{e}} \circ \mathbf{u}_{\mathrm{e}}^{-1} = \mathbf{i}, \qquad \widehat{\mathbf{v}} := \mathbf{v} \circ \mathbf{u}_{\mathrm{e}}^{-1}. \tag{7.22}$$

Let $\delta \in (0,1)$ be given as in Theorem 3.3. We shall determine ε such that $\widehat{\mathbf{u}}_{\mathrm{e}}$ and $\widehat{\mathbf{v}}$ satisfy the hypotheses of Theorem 3.3 (with $\mathbf{u}_{\mathrm{e}}, \mathbf{v}, \Omega$, and $\mathcal{E}$ replaced by $\widehat{\mathbf{u}}_{\mathrm{e}}, \widehat{\mathbf{v}}, \Omega_e$, and $\mathcal{E}_{u_e}$). In view of Lemma 7.12 and assumptions (A)–(C), $\widehat{\mathbf{u}}_{\mathrm{e}} = \mathbf{i}$ is a weak equilibrium solution for $\mathcal{E}_{u_e}$, given by (7.13) with $u = u_e$, at which the second variation of $\mathcal{E}_{u_e}$ is uniformly positive. Thus, $\widehat{\mathbf{u}}_{\mathrm{e}}$ satisfies $(3.12)_1$. Trivially, $\mathrm{dist}(\mathbf{I}, \mathrm{SO}(n)) = 0$ and the rotation associated with $\mathbf{i}$ in Proposition 4.3 is $\mathbf{I}$. Next, if we combine (7.2) and (7.21) we find, with the aid of (7.4), $(7.22)_2$, Remark 7.6, and the chain rule, that

$$\mathrm{dist}\big(\nabla\widehat{\mathbf{v}}(\mathbf{y}), \mathrm{SO}(n)\big) < \upsilon_{\mathrm{e}}^{-2}\varepsilon \ \text{ for } a.e.\ \mathbf{y} \in \Omega_e. \tag{7.23}$$

Next, fix $p > n$. Then (7.23), Proposition 4.8, and Proposition 4.3 yield

$$||\nabla\widehat{\mathbf{v}} - \mathbf{I}||_{1,\Omega_e} < \upsilon_{\mathrm{e}}^{-2}A^*|\Omega_e|^{1/p}\varepsilon. \qquad \lVert\nabla\widehat{\mathbf{v}} - \mathbf{I}\rVert_{\mathrm{BMO}(\Omega_e)} = \lVert\nabla\widehat{\mathbf{v}}\rVert_{\mathrm{BMO}(\Omega_e)} < M\upsilon_{\mathrm{e}}^{-2}\varepsilon$$

for some constants $A^* > 0$ and $M > 0$.

Now, let $\varepsilon > 0$ be sufficiently small so that

$$\max\{M\varepsilon,\ \varepsilon,\ A^*|\Omega_e|^{1/p}\varepsilon\} < \upsilon_{\mathrm{e}}^2\delta.$$

In addition, define

$$\mathcal{B} := \{\mathbf{G} \in \mathbb{M}_+^{n\times n} : \mathrm{dist}(\mathbf{G}, \mathrm{SO}(n)) < \delta < 1\}.$$

Then the hypotheses of Theorem 3.3 have been satisfied; consequently that result yields

$$\mathcal{E}_{u_e}(\widehat{\mathbf{v}}) \geq \mathcal{E}_{u_e}(\mathbf{i}) + k\int_{\Omega_e} |\nabla\widehat{\mathbf{v}} - \mathbf{I}|^2 \mathrm{d}\mathbf{y}. \tag{7.24}$$

Moreover, $\widehat{\mathbf{v}}$ *cannot be a weak solution of the equilibrium equations corresponding to* $\mathcal{E}_{u_e}$.

Next, Proposition 7.9 together with (7.11) and (7.13) shows that $\mathcal{E}_{u_e}(\widehat{\mathbf{v}}) = \mathcal{E}(\mathbf{v})$ and $\mathcal{E}_{u_e}(\mathbf{i}) = \mathcal{E}(\mathbf{u}_e)$; thus, by (7.24),

$$\mathcal{E}(\mathbf{v}) \geq \mathcal{E}(\mathbf{u}_\mathrm{e}) + k\int_{\Omega_e} |\nabla\widehat{\mathbf{v}} - \mathbf{I}|^2 \mathrm{d}\mathbf{y}.$$

Consequently, $\mathcal{E}(\mathbf{v}) > \mathcal{E}(\mathbf{u}_\mathrm{e})$ unless $\nabla\widehat{\mathbf{v}} \equiv \mathbf{I}$. However, $\nabla\widehat{\mathbf{v}} = \mathbf{I}$ on the connected open set Ω_e together with $\widehat{\mathbf{v}} = \mathbf{i}$ on $\mathcal{D}$ yields $\widehat{\mathbf{v}} \equiv \mathbf{i}$. Equivalently, (cf. $(7.22)_2$) $\mathbf{v} \circ \mathbf{u}_\mathrm{e}^{-1} = \mathbf{i}$ and so $\mathbf{v} = \mathbf{u}_\mathrm{e}$. Therefore, $\mathbf{v} \not\equiv \mathbf{u}_\mathrm{e}$ will have strictly greater energy than $\mathbf{u}_\mathrm{e}$.

Finally, if $\mathbf{v}$ were to satisfy (7.12), then Lemma 7.12 would imply that $\widehat{\mathbf{v}} = \mathbf{v} \circ \mathbf{u}_\mathrm{e}^{-1}$ is a weak solution of the equilibrium equations corresponding to $\mathcal{E}_e$. However, this is not possible (see the sentence in italics following (7.24)). □

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Appendix A. Versions of Taylor's Theorem for Non-convex Sets

Recall that $\Omega \subset \mathbb{R}^n$, $n \geq 2$, is a Lipschitz domain and $\mathcal{O} \subset \mathbb{M}^{N\times n}$ is a nonempty, open set. If $\mathcal{B} \subset \mathbb{M}^{N\times n}$ is a nonempty, bounded, open set that satisfies $\overline{\mathcal{B}} \subset \mathcal{O}$, then, for $\varepsilon > 0$ and sufficiently small, the set

$$\mathcal{B}_\varepsilon := \{\mathbf{K} \in \mathbb{M}^{N\times n} : |\mathbf{K} - \mathbf{F}| < \varepsilon \ \text{ for some } \mathbf{F} \in \mathcal{B}\} \tag{A.1}$$

is a nonempty, bounded, open set that satisfies $\overline{\mathcal{B}}_\varepsilon \subset \mathcal{O}$.

Lemma A.1. *Let Ω, $\mathcal{O}$, and $W : \overline{\Omega} \times \mathcal{O} \to \mathbb{R}$ be as given in (1)–(3) of Hypothesis 3.1. Suppose that $\mathcal{B} \subset \mathbb{M}^{N\times n}$ is a nonempty, bounded, open set that satisfies $\overline{\mathcal{B}} \subset \mathcal{O}$. Then there exists a constant $c = c(\mathcal{B}) > 0$ such that, for every $\mathbf{F}, \mathbf{G} \in \overline{\mathcal{B}}$ and almost every $\mathbf{x} \in \Omega$,*

$$W(\mathbf{x}, \mathbf{G}) \geq W(\mathbf{x}, \mathbf{F}) + \mathrm{D}W(\mathbf{x}, \mathbf{F})[\mathbf{H}] + \tfrac{1}{2}\mathrm{D}^2W(\mathbf{x}, \mathbf{F})[\mathbf{H}, \mathbf{H}] - c|\mathbf{H}|^3, \tag{A.2}$$

where $\mathbf{H} := \mathbf{G} - \mathbf{F}$.

Remark A.2. If $\mathcal{B}$ is convex, then Lemma A.1 follows from Taylor's theorem.

Proof of Lemma A.1. Given $\mathcal{B} \subset \overline{\mathcal{B}} \subset \mathcal{O}$, let $\mathcal{B}_\varepsilon$ (defined by (A.1)) satisfy $\overline{\mathcal{B}}_\varepsilon \subset \mathcal{O}$. Define

$$c := \sup_{\substack{\mathbf{F}\in\overline{\mathcal{B}},\, \mathbf{x}\in\overline{\Omega} \\ \mathbf{G}\in\overline{\mathcal{B}}_\varepsilon}} \frac{W(\mathbf{x}, \mathbf{F}) - W(\mathbf{x}, \mathbf{G}) + \mathrm{D}W(\mathbf{x}, \mathbf{F})[\mathbf{H}] + \frac{1}{2}\mathrm{D}^2W(\mathbf{x}, \mathbf{F})[\mathbf{H}, \mathbf{H}]}{|\mathbf{H}|^3}, \tag{A.3}$$

where $\mathbf{H} := \mathbf{G} - \mathbf{F}$. We need only show that the supremum is finite in order to conclude that (A.2) is satisfied for all $\mathbf{F}, \mathbf{G} \in \overline{\mathcal{B}}$ and *a.e.* $\mathbf{x} \in \Omega$. Suppose that the right-hand side of (A.3) is not bounded. In view of (3) of Hypothesis 3.1, the numerator in (A.3) is bounded on the compact set $\overline{\Omega} \times \overline{\mathcal{B}}_\varepsilon \times \overline{\mathcal{B}}$; thus, there must exist sequences $\mathbf{x}_k \in \overline{\Omega}$, $\mathbf{F}_k \in \overline{\mathcal{B}}$, and $\mathbf{G}_k \in \overline{\mathcal{B}}_\varepsilon$ such that $\mathbf{H}_k := \mathbf{G}_k - \mathbf{F}_k \to \mathbf{0}$. It follows that there exists $\mathbf{P} \in \overline{\mathcal{B}}$ such that, for a subsequence (not relabeled) $\mathbf{F}_k, \mathbf{G}_k \to \mathbf{P}$.

We note that $\mathbf{P} \in \mathcal{B}_\varepsilon$, an open set; thus exists a $\delta > 0$ such that the open ball of radius 2δ centered at $\mathbf{P}$, $B(\mathbf{P}, 2\delta) \subset \mathcal{B}_\varepsilon$. Then, for k sufficiently large, $\mathbf{F}_k, \mathbf{G}_k \in B(\mathbf{P}, \delta)$. In addition, since $\mathbf{F} \mapsto W(\cdot, \mathbf{F})$ is C^3, $\overline{\Omega} \times \overline{B}(\mathbf{P}, \delta)$ is compact, and the unit ball in $\mathbb{M}^{N\times n}$ is compact, it follows from (3) of Hypothesis 3.1 that

$$c^* := \sup_{\substack{\mathbf{x}\in\overline{\Omega} \\ \mathbf{N}\in\overline{B}(\mathbf{P},\delta)}} |\mathrm{D}^3 W(\mathbf{x}, \mathbf{N})| < \infty, \quad \text{where}$$
$$|\mathrm{D}^3 W(\mathbf{x}, \mathbf{N})| := \sup_{\substack{|\mathbf{K}|\le 1 \\ |\mathbf{L}|\le 1,\, |\mathbf{R}|\le 1}} \left|\mathrm{D}^3 W(\mathbf{x}, \mathbf{N})[\mathbf{K}, \mathbf{L}, \mathbf{R}]\right|.$$

Next, choose k_{o} such that $\mathbf{F}_k, \mathbf{G}_k \in B(\mathbf{P}, \delta)$, for all $k \ge k_{\mathrm{o}}$, and apply Taylor's theorem (see, e.g., [62, Section 4.6]) to the function $\mathbf{F} \mapsto W(\mathbf{x}_k, \mathbf{F})$ at $\mathbf{F}_k$ and $\mathbf{G}_k$ to conclude that, for all $k \ge k_{\mathrm{o}}$,

$$\begin{aligned} W(\mathbf{x}_k, \mathbf{G}_k) = W(\mathbf{x}_k, \mathbf{F}_k) &+ \mathrm{D}W(\mathbf{x}_k, \mathbf{F}_k)[\mathbf{H}_k] \\ &+ \tfrac{1}{2}\mathrm{D}^2 W(\mathbf{x}_k, \mathbf{F}_k)[\mathbf{H}_k, \mathbf{H}_k] + \tfrac{1}{6} R(\mathbf{x}_k, \mathbf{F}_k, \mathbf{H}_k), \end{aligned} \tag{A.4}$$

where $\mathbf{H}_k := \mathbf{G}_k - \mathbf{F}_k$ and

$$|R(\mathbf{x}_k, \mathbf{F}_k, \mathbf{H}_k)| \le |\mathbf{H}_k|^3 \sup_{t\in[0,1]} |\mathrm{D}^3 W(\mathbf{x}_k, \mathbf{F}_k + t\mathbf{H}_k)| \le c^* |\mathbf{H}_k|^3. \tag{A.5}$$

Then, in view of (A.4) and (A.5),

$$W(\mathbf{x}_k, \mathbf{F}_k) - W(\mathbf{x}_k, \mathbf{G}_k) + \mathrm{D}W(\mathbf{x}_k, \mathbf{F}_k)[\mathbf{H}_k] + \tfrac{1}{2}\mathrm{D}^2 W(\mathbf{x}_k, \mathbf{F}_k)[\mathbf{H}_k, \mathbf{H}_k] \le \tfrac{1}{6} c^* |\mathbf{H}_k|^3.$$

This contradicts our assumption that the right-hand side of (A.3) becomes arbitrarily large when $\mathbf{F} = \mathbf{F}_k$, $\mathbf{G} = \mathbf{G}_k$, $\mathbf{x} = \mathbf{x}_k$, and $k \to \infty$. □

Lemma A.3. *Let Ω, $\mathcal{O}$, $W : \overline{\Omega} \times \mathcal{O} \to \mathbb{R}$, and $\mathcal{B} \subset \mathbb{M}^{N\times n}$ be as given in the statement of Lemma A.1. Then there exists a constant $\hat{c} = \hat{c}(\mathcal{B}) > 0$ such that, for every $\mathbf{F}, \mathbf{G} \in \overline{\mathcal{B}}$, every $\mathbf{L} \in \mathbb{M}^{N\times n}$, and almost every $\mathbf{x} \in \Omega$,*

$$\mathrm{D}^2 W(\mathbf{x}, \mathbf{G})[\mathbf{L}, \mathbf{L}] \ge \mathrm{D}^2 W(\mathbf{x}, \mathbf{F})[\mathbf{L}, \mathbf{L}] - \hat{c}|\mathbf{G} - \mathbf{F}||\mathbf{L}|^2.$$

The proof of the above result is similar to to the proof of Lemma A.1 with the constant $\hat{c}$ now given by

$$\hat{c} := \sup_{\substack{\mathbf{F}\in\overline{\mathcal{B}},\, \mathbf{x}\in\overline{\Omega} \\ \mathbf{G}\in\overline{\mathcal{B}}_\varepsilon,\, |\mathbf{K}|=1}} \frac{\mathrm{D}^2 W(\mathbf{x}, \mathbf{F})[\mathbf{K}, \mathbf{K}] - \mathrm{D}^2 W(\mathbf{x}, \mathbf{G})[\mathbf{K}, \mathbf{K}]}{|\mathbf{G} - \mathbf{F}|}.$$

Appendix B. A Generalized Korn Inequality

Our first result in Section 6.1 required a more general version of Korn's inequality than is usually needed in Nonlinear Elasticity. The precise version we used can be found in a paper of Pompe [48, Corollary 4.1].

Proposition B.1. (Korn's Inequality with Variable Coefficients) *Let $\mathbf{F} \in C(\overline{\Omega}; \mathbb{M}^{n\times n})$ satisfy $\det \mathbf{F}(\mathbf{x}) \ge \mu > 0$ for all $\mathbf{x} \in \overline{\Omega}$. Then there exists a constant $K > 0$ such that*

$$\int_\Omega \left| \left[\mathbf{F}(\mathbf{x})\right]^{\mathrm{T}} \nabla \mathbf{w}(\mathbf{x}) + \left[\nabla \mathbf{w}(\mathbf{x})\right]^{\mathrm{T}} \mathbf{F}(\mathbf{x}) \right|^2 \mathrm{d}\mathbf{x} \ge K \int_\Omega \left|\nabla \mathbf{w}(\mathbf{x})\right|^2 \mathrm{d}\mathbf{x}, \tag{B.1}$$

for every $\mathbf{w} \in W^{1,2}(\Omega;\mathbb{R}^n)$ *that satisfies* $\mathbf{w} = \mathbf{0}$ *on* $\mathcal{D}$*.*

Remark B.2. The standard version of Korn's inequality occurs when $\mathbf{F}(\mathbf{x}) \equiv \mathbf{I}$ in (B.1). Proposition B.1 is *not* generally valid if one assumes only that $\mathbf{F} \in L^\infty(\Omega;\mathbb{M}^{n\times n})$. Counterexamples can be found in Neff & Pompe [45] and the references therein. Proposition B.1 can also be obtained[14] from results of Hlaváček & Nečas [29] that address the problem of coercivity for formally positive quadratic forms of vector-valued functions (e.g., the left-hand side of (B.1)). However, [29] does not establish precisely (B.1).

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