Quantitative estimates for regular Lagrangian flows with BV vector fields

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Abstract

In this paper, we solve a main open problem mentioned in [7]. Specifically, we prove the well posedness of regular Lagrangian flows to vector fields $\mathbf{B} = (\mathbf{B}^1, ..., \mathbf{B}^d) \in L^1((0,T); L^1 \cap L^\infty(\mathbb{R}^d))$ satisfying $\mathbf{B}^i = \sum_{j=1}^m \mathbf{K}_j^i * b_j$, $b_j \in L^1((0,T), BV(\mathbb{R}^d))$ and $\operatorname{div}(\mathbf{B}) \in L^1((0,T); L^\infty(\mathbb{R}^d))$ for $d \geq 2$, where $(\mathbf{K}_j^i)_{i,j}$ are singular kernels in $\mathbb{R}^d$.

Key words: Ordinary differential with non smooth vector fields; continuity equation; transport equation; regular Lagrangian flow; maximal function; Kakeya maximal function; Riez potential; singular integral operator, maximal singular integral operator; singular integral operator of Kakeya type; BV function; Vlasov-Poisson equation.

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Contents

1 Introduction

In this paper we study the well posedness of flows of ordinary differential equations

$$\begin{cases} \frac{dX(t,x)}{dt} & = \mathbf{B}(t, X(t,x)), \quad t \in [0,T] \\ X(0,x) & = x \end{cases} \tag{1.1}$$

where $\mathbf{B}(t,x) = \mathbf{B}_t(x) : (0,T) \times \mathbb{R}^d \in \mathbb{R}^d$ is a function in $[0,T] \times \mathbb{R}^d$, $d \geq 2$. It is well known that by Peano's Theorem, there exists at least one solution to the problem (1.1) provided that

$\mathbf{B}$ is continuous. Moreover, by the usual Cauchy-Lipshiptz Theorem, one has also uniqueness if $\mathbf{B}$ is a bounded smooth vector field.

The ordinary differential equation (1.1) is related to the continuity equation

$$\begin{cases} \partial_t u(t,x) + \operatorname{div}(\mathbf{B}(t,x)u(t,x)) &= G(t,x)u(t,x) + F(t,x), \\ u(0,x) &= u_0(x), \end{cases} \tag{1.2}$$

for any $(t,x) \in [0,T] \times \mathbb{R}^d$. Indeed, assume that $u_0, \mathbf{B}, G$ and F are smooth and compactly supported. Let $X : [0,T] \times \mathbb{R}^d \to \mathbb{R}^d$ be unique solution of (1.1), it is called the flow of vector field $\mathbf{B}$. We have

$$\det(\nabla_x X(t,x)) = \exp\left(\int_0^t \operatorname{div}(\mathbf{B})(s, X(s,x))ds\right) \neq 0$$

In particular, the map $X(t,.)$ is a diffeomorphism and we denote by $X^{-1}(t,.)$ its inverse. A solution of (1.2) is given in term of the flow X by the following formula

$$\begin{aligned} u(t,x) = u_0(\overline{x}) \exp\left(-\int_0^t (\operatorname{div}(\mathbf{B}) - G)(s, X(s,\overline{x}))ds\right) \\ + \int_0^t F(\tau, X(\tau,\overline{x})) \exp\left(-\int_\tau^t (\operatorname{div}(\mathbf{B}) - G)(s, X(s,\overline{x}))ds\right) d\tau, \end{aligned} \tag{1.3}$$

with $\overline{x} = X^{-1}(t,.)(x)$ (see Appendix for its proof). Therefore, we can say that *the well posedness of* (1.1) *is equivalent to the well posedness of* (1.2).

The continuity equations (often with non-smooth vector fields) are important for describing various quantities in mathematical physics as such mass, energy, momentum, electric charge. Especially, they underlie transport equations as such the convection-diffusion, Boltzmann, Vlasov-Poisson, Euler and Navier-Stokes equations.

Let us start by the seminal work of Diperna and Lions [30], they established existence, uniqueness and stability of distributional solutions of (1.2) for Sobolev $W^{1,1}$ vector fields with bounded divergence. Later some progress was achieved in several papers [38, 17, 19, 33, 26, 27, **?**], finally it was fully extended by Ambrosio [4] to BV vector fields with bounded divergence. The approach by Diperna, Lions and Ambrosio relies on the theory of renormalized solutions of (1.2), roughly speaking renormalized solutions are distributional solutions such that the chain rule holds for u and $\mathbf{B}$ i.e

$$\operatorname{div}(\mathbf{B}h(u)) = \big(h(u) - uh'(u)\big)\operatorname{div}(\mathbf{B}) + h'(u)\operatorname{div}(\mathbf{B}u)$$

for any $h \in C^1(\mathbb{R})$.

In this approach, an important technical tool is the regularization of solutions by a smooth kernel and the analysis of the commutator

$$r_\delta := \rho_\delta * (\operatorname{div}(\mathbf{B}u)) - (\operatorname{div}(\mathbf{B}\rho_\delta * u)) \tag{1.4}$$

exactly the key ingredient is that $r_\delta \to 0$ in L^1_{loc} as $\delta \to 0$ for some ρ. In [30], for $\mathbf{B} \in W^{1,1}$ it is quite simple to take a radial convolution kernel, in the BV case, in [4] Ambrosio chooses a kernel ρ strictly depending on the structure of $\mathbf{B}$. More precisely, he first proves that

$$|r_\delta| \rightharpoonup \sigma, \quad \text{and} \quad \sigma(x) \lesssim \int |\langle M(x)z, \nabla_z \rho(x,z)(z)\rangle|\, dz |D^s\mathbf{B}|(x), \text{with } M = \frac{dD^s\mathbf{B}}{d|D^s\mathbf{B}|},$$

for any smooth kernel ρ, $\int \rho(x,z)dz = 1$ for any $x \in \mathbb{R}^d$, where $D^s\mathbf{B}$ is singular part of $D\mathbf{B}$ with respect to the Lebesgue measure. Then, he takes ρ such that $\int |\langle M(x)z, \nabla\rho(x,z)\rangle|\, dz \lesssim$

$|traceM(x)|$. Using the fact that $\mathrm{div}(\mathbf{B}) << \mathcal{L}^d \iff |traceM(x)||D^s\mathbf{B}|(x) = 0$, then he gets the "defect" measure $\sigma = 0$.

Moreover, Diperna and Lions construct distributional solutions to the continuity equations (1.2) with $\mathbf{B} \in W^{\alpha,1}$ ($\alpha < 1$) and $\mathrm{div}(\mathbf{B}) = 0$ that are not renormalized. A counterexample for non-BV is provided by Depauw [29]. Further results can be found in [6, 28, 10, 11, 13, 14, 34, 23, 8, 9]. For a recent review on the well-posedness theories for the continuity equations (1.2) and ODE (1.1), we refer the reader the lecture notes [7] (and [12]) and video lectures at link: https://www.youtube.com/watch?v=iOD0n2EAMAs

In [24], C. De Lellis and G. Crippa have given an independent proof of the existence and uniqueness of the solutions of (1.1) with Sobolev vector fields, that is without exploiting the connection with the continuity equations (1.2). The basic idea of [24] is to consider the following time dependent quantity

$$\Phi_\delta(t) = \int_{B_R} \log\left(1 + \frac{|X_1(t,x) - X_2(t,x)|}{\delta}\right) dx,$$

where X_1, X_2 are regular Lagrangian flows associated to the same vector field $\mathbf{B}$ and $B_R := B_R(0), R > 0$. We have

$$\Phi_\delta(t) \geq \mathcal{L}^d\left(\left\{x \in B_R : |X_1(t,x) - X_2(t,x)| > \delta^{1/2}\right\}\right) \log\left(1 + \delta^{-1/2}\right). \tag{1.5}$$

However, differentiating in time, one has

$$\begin{aligned}\Phi_\delta(t) = \int_0^t \Phi'_\delta(s) ds &\leq \int_0^t \int_{B_R} \frac{|\mathbf{B}(s, X_1(s,x)) - \mathbf{B}(s, X_2(s,x))|}{\delta + |X_1(s,x) - X_2(s,x)|} dx ds \\ &\leq \int_0^t \int_{B_R} \min\left\{\frac{2||\mathbf{B}||_{L^\infty}}{\delta}, \frac{|\mathbf{B}(s, X_1(s,x)) - \mathbf{B}(s, X_2(s,x))|}{|X_1(s,x) - X_2(s,x)|}\right\} dx. \end{aligned} \tag{1.6}$$

By the standard estimate of the Hardy-Littlewood function $\mathbf{M}$ and changing variable along the flows, we obtain

$$\Phi_\delta(t) \lesssim \int_0^T \int_{B_{R_1}} \min\left\{\delta^{-1}, \mathbf{M}(|\nabla \mathbf{B}_s(.)|)(x)\right\} dx ds \tag{1.7}$$

for some $R_1 > R$, here $\mathbf{B}_s(.) = \mathbf{B}(s,.)$. Using boundedness of $\mathbf{M}$ from L^p to itself for $p > 1$ and (1.5) and (1.7) we deduce that

$$\mathcal{L}^d\left(\left\{x \in B_R : |X_1(t,x) - X_2(t,x)| > \delta^{1/2}\right\}\right) \leq \frac{C}{\log\left(1 + \delta^{-1/2}\right)} \quad \forall \delta > 0$$

provided $\mathbf{B} \in L^1(W^{1,p}), p > 1$. At this point, sending $\delta \to 0$, we get $X_1 = X_2$.

Later, in [35] P.E. Jabin successfully improves this to $\mathbf{B} \in L^1(W^{1,1})$. Besides, also in [35] he uses more information of structure of flows to extend technique to $\mathbf{B} \in L^1(SBV)$ in any dimension and to two-dimension $L^1(BV)$ with local assumption in the direction of flows. Furthermore, in [2] L. Ambrosio, E. Brué and Nguyen show that

$$\int_0^T |D^s\mathbf{B}_t|(B_{R_1}) dt \lesssim \limsup_{\delta \to 0} \frac{1}{|\log(\delta)|} \int_0^T \int_{B_{R_1}} \min\left\{\delta^{-1}, \mathbf{M}(|\nabla \mathbf{B}_s(.)|)\right\} \lesssim \int_0^T |D^s\mathbf{B}_t|(\overline{B}_{R_1}) dt.$$

Therefore, this is reason that De Lellis and Crippa's approach is not able to deal with vector fields $\mathbf{B} \in L^1(BV \backslash W^{1,1})$. Recently, F. Bouchut and G. Crippa in [18] have proven the existence and uniqueness of flows for vector fields with gradient given by singular integrals of L^1 functions i.e $D\mathbf{B} = \mathbf{K} \star g$, $g \in L^1$, where $\mathbf{K}$ is a singular kernel of fundamental type in $\mathbb{R}^d$. Notice that this class is very natural in the study of theory of nonlinear PDEs, such as the Euler equation and the classical Vlasov-Poisson equation, that such class is not contained in BV and neither contains it. To do this, they have used the following maximal singular integral operator:

$$\mathbf{T}(\mu)(x) = \sup_{\varepsilon>0} |(\rho_\varepsilon \star \mathbf{K} \star \mu)(x)|,$$

where $\rho_\varepsilon(.) = \varepsilon^{-d}\rho(./\varepsilon)$, $\rho \in C_c^1$ is such that $\int_{\mathbb{R}^d} \rho dx = 1$. Then, $\Phi_\delta(t) = \circ(|\log(\delta)|)$ is obtained from using the boundedness of such operator from L^1 to weak-L^1 and the fact that

$$\lambda \mathcal{L}^d \left(\{\mathbf{T}(\mu) > \lambda\}\right) \to 0 \ \text{ as } \ \lambda \to \infty, \tag{1.8}$$

for any $\mu \in L^1(\mathbb{R}^d)$, see proof of Lemma 1. Notice that (1.8) is not true for $\mu \in \mathcal{M}_b(\mathbb{R}^d)$; indeed, it is easy to check that if $\mu = \delta_0$ then $\lambda \mathcal{L}^d \left(\{\mathbf{T}(\mu) > \lambda\}\right) \asymp 1, \forall \lambda > 0$ for some ρ and $\mathbf{K}$.

However, later in [15] they have showed such result to case which

$$D\mathbf{B} = \begin{pmatrix} D_{x_1}\mathbf{B}_1 & D_{x_1}\mathbf{B}_2 \\ D_{x_2}\mathbf{B}_1 & D_{x_2}\mathbf{B}_2 \end{pmatrix} = \begin{pmatrix} \mathbf{K}_1 \star f_1 & \mathbf{K}_2 \star f_2 \\ \mathbf{K}_0 \star \mu & \mathbf{K}_3 \star f_3 \end{pmatrix} \quad x = (x_1, x_2), \quad \mathbf{B} = (\mathbf{B}_1, \mathbf{B}_2),$$

where $\mathbf{K}_0, \mathbf{K}_1, \mathbf{K}_2, \mathbf{K}_3$ are singular kernels of fundamental type. This is motivated from the Classical Vlasov-Poisson system associated to $B(x_1, x_2) = (x_2, \mathbf{P} \star \mu(x_1))$, $(x_1, x_2) \in \mathbb{R}^m \times \mathbb{R}^m$, $d = 2m$ and $\mathbf{P}(x_1) = c\frac{x_1}{|x_1|^m}$, $\mu \in \mathcal{M}_b$. In addition, Jabin in [20] has proven the well posedness of this system with $\mathbf{P} \star \mu \in H^{3/4}$ (or $\mu \in H^{-1/4}$). We believe that it is still true with $\mathbf{P} \star \mu \in W^{s,1}$ for any $s > 1/2$. This will be pursued in our forthcoming work. Recently, in [41, 25] Seis has provided a quantitative theory for continuity equation with $W^{1,1}$ vector fields via logarithmic Kantorovich-Rubinstrain distances.

To our knowledge, these results in [15, 35] are the best results for the quantitative ODE estimates at this moment. In this paper, we give quantitative estimates for $\mathbf{K} \star BV$ vector fields with bounded divergence. Namely, we prove the following theorem:

Given a vector field $\mathbf{B} = (\mathbf{B}^1, ..., \mathbf{B}^d) \in L^1([0,T]; L^1_{loc}(\mathbb{R}^d, \mathbb{R}^d))$, we assume that for any $R > 0$, there exist functions $b_{jR} \in L^1([0,T], BV(\mathbb{R}^d))$ for $j = 1, ..., m$; and degree-zero homogeneous functions $(\Omega^i_{jR})_{i,j} \in L^1_{loc}(\mathbb{R}^d)$ $(i = 1, .., d, j = 1, ..., m)$ satisfying $\int_{S^{d-1}} \Omega^i_{jR} = 0$ and $\Omega^i_{jR} \in BV(S^{d-1})$ such that

$$\mathbf{B}^i = \sum_{j=1}^m \left(\frac{\Omega^i_{jR}(.)}{|.|^d}\right) \star b_{jR} \ \text{ in } B_{2R}. \tag{1.9}$$

Main Theorem. *Let* $\mathbf{B}_1, \mathbf{B}_2 \in L^1([0,T]; L^1_{loc}(\mathbb{R}^d, \mathbb{R}^d))$ *satisfy* $||(\frac{|\mathbf{B}_1|}{|x|+1}, \frac{|\mathbf{B}_2|}{|x|+1})||_{L^1((0,T);(L^1+L^\infty)(\mathbb{R}^d))} \leq C_0$ *and let* X_1, X_2 *be regular Lagrangian flows associated to* $\mathbf{B}_1, \mathbf{B}_2$ *resp. with compression constants* $L_1, L_2 \leq L_0$. *Then, if* $\operatorname{div}(\mathbf{B}) \in L^1((0,T), L^1(\mathbb{R}^d))$, *for any* $\kappa \in (0,1), r > 1$ *there exists* $R_0 = R_0(d, T, r, C_0, L_0, \kappa) > 1$, $\delta_0 = \delta_0(d, T, r, C_0, c_{R_0}, L_0, b_{R_0}, \kappa) \in (0,1)$ *such that*

$$\sup_{t\in[0,T]} \mathcal{L}^d \left(\left\{x \in B_r : |X_{1t}(x) - X_{2t}(x)| > \delta^{1/2}\right\}\right) \leq \frac{C(d)L_0}{\delta} || \left(\mathbf{B}_1 - \mathbf{B}, \mathbf{B}_2 - \mathbf{B}\right) ||_{L^1([0,T]\times B_{R_0})} + \kappa \tag{1.10}$$

for any $\delta \in (0, \delta_0)$.

Note that if $\mathbf{B}_1, \mathbf{B}_2 \in L^\infty(\mathbb{R})$, we can take R_0 independent of κ. Moreover, if $\mathbf{B} \in L^1((0,T); BV_{loc}(\mathbb{R}^d, \mathbb{R}^d))$, we can write for any $R > 0$, $\mathbf{B}^i = \sum_{j=1}^d \mathcal{R}_j^2(\chi_R \mathbf{B}^i)$ in $B_R(0)$, where $\chi_R \in C_c^\infty(\mathbb{R}^d)$ satisfies $\chi_R = 1$ in $B_{2R}(0)$ and $\chi_R = 0$ in $B_{4R}(0)^c$, $\mathcal{R}_1, ..., \mathcal{R}_d$ are the Riesz transforms in $\mathbb{R}^d$. Thus, the class of $\mathbf{B}$ in above theorem contains the class of $BV-$ vector fields and hence a main open problem posed by Luigi Ambrosio (see [7]) is solved.

This Theorem is as a consequence of Theorem 2 and Corollary 1 in Section 4. In Section 5, we will use this to deduce the well posedness of regular Lagrangian flows and Transport, Continuity equations. The following is existence and uniqueness result of regular Lagrangian flows.

Proposition. Let B be as above. Assume that $||\frac{|\mathbf{B}|}{|x|+1}||_{L^1((0,T);(L^1+L^\infty)(\mathbb{R}^d))} \leq C_0$ and $\operatorname{div}(\mathbf{B}) \in L^1((0,T), L^1(\mathbb{R}^d))$. Then, there exist a unique regular Lgrangian flows associated to vector field $\mathbf{B}$.

Let us describe our idea to prove (1.10). For simplicity, assume that $\mathbf{B}_1(t,x) = \mathbf{B}_2(t,x) = \mathbf{B}(t,x) \equiv \mathbf{B}(x) \in (BV \cap L^\infty)(\mathbb{R}^d, \mathbb{R}^d)$. Thanks to Alberti's rank one Theorem (see section 2), there exist unit vectors $\xi(x) \in \mathbb{R}^d$ and $\eta(x) \in \mathbb{R}^d$ such that $D^s\mathbf{B}(x) = \xi(x) \otimes \eta(x)|D^s\mathbf{B}|(x)$ i.e $D^s_{x_i}\mathbf{B}_j(x) = \xi_j(x)\eta_i(x)|D^s\mathbf{B}|(x)$ for any $i, j = 1, ..., d$. Thus, one gets from $\operatorname{div}(\mathbf{B}) \in L^1([0,T] \times \mathbb{R}^d)$ that $|\langle \xi, \eta \rangle| = 0$ for $|D^s\mathbf{B}|-$a.e in $\mathbb{R}^d$. We first have the following basic inequality: for any $x_1 \neq x_2 \in \mathbb{R}^d$ and $\nu \in S^{d-1}$,

$$|\langle \nu, \mathbf{B}(x_1) - \mathbf{B}(x_2) \rangle| \lesssim \sum_{l=1,2} \int \frac{\mathbf{1}_{|x_l - z| \leq r}}{|x_l - z|^{d-1}} |\langle \nu, \xi(z) \rangle| d\mu(z) + \int \frac{\mathbf{1}_{|x_l - z| \leq r}}{|x_l - z|^{d-1}} d|D^a\mathbf{B}|(z),$$

see Proposition 4, where $\mu = |D^s\mathbf{B}|$ and $r = |x_1 - x_2|$, where $D^a\mathbf{B}$ is regular part of $D\mathbf{B}$ with respect to the Lebesgue measure. We now assume that ξ and η are smooth functions in $\mathbb{R}^d$. Then, choosing $\nu = \eta(x_1)$ and thanks to $|\langle \xi, \eta \rangle| = 0$ for $|\mu|-$a.e in $\mathbb{R}^d$ yields

$$|\langle \nu, \xi(z) \rangle| \leq ||\nabla \eta||_{L^\infty} \left(|x_1 - x_2| + |x_l - z| \right) \text{ for } |\mu| - \text{a.e } \; z \text{ in } \mathbb{R}^d,$$

which implies

$$\frac{|\langle \eta(x_1), \mathbf{B}(x_1) - \mathbf{B}(x_2) \rangle|}{|x_1 - x_2|} \lesssim \sum_{l=1,2} ||\nabla \eta||_{L^\infty} \mathbf{I}_1(\mu)(x_l) + \mathbf{M}(|D^a\mathbf{B}|)(x_l), \tag{1.11}$$

where $\mathbf{I}_1$ is the Riesz potential with the first order in $\mathbb{R}^d$.
Let X_1, X_2 be Regular Lagrangian flows associated to the same vector field $\mathbf{B}$ and $r > 0$. Thus, we derive from (1.11) that

$$\limsup_{\delta \to 0} \frac{1}{|\log(\delta)|} \int_0^T \int_{B_r} \frac{|\langle \eta(X_{1t}), \mathbf{B}(X_{1t}) - \mathbf{B}(X_{2t}) \rangle|}{\delta + |X_{1t} - X_{2t}|} dx dt = 0. \tag{1.12}$$

This suggests us to consider the following new quantity: for $\delta \in (0,1), \gamma > 1$

$$\Phi_\delta^\gamma(t) = \frac{1}{2} \int_{B_r} \log \left(1 + \frac{|X_{1t} - X_{2t}|^2 + \gamma \langle \eta(X_{1t}), X_{1t} - X_{2t}| \rangle^2}{\delta^2} \right) dx. \tag{1.13}$$

We have,

$$\sup_{t\in[0,T]}\Phi_\delta^\gamma(t)=\sup_{t_1\in[0,T]}\int_0^{t_1}\frac{d\Phi_\delta^\gamma(t)}{dt}dt\le\int_0^T\int_{B_r}\frac{\gamma^{1/2}\left|\langle\eta(X_{1t}),\mathbf{B}(X_{1t})-\mathbf{B}(X_{2t})\rangle\right|}{\delta+|X_{1t}-X_{2t}|}dxdt$$
$$+\int_0^T\int_{B_r}\frac{|\mathbf{B}(X_{1t})-\mathbf{B}(X_{2t})|}{\delta+|X_{1t}-X_{2t}|+\gamma^{1/2}|\langle\eta(X_{1t}),X_{1t}-X_{2t}\rangle|}dxdt$$
$$+\int_0^T\int_{B_r}\frac{\gamma^{1/2}\left|\langle\nabla\eta(X_{1t})\mathbf{B}(X_{1t}),X_{1t}-X_{2t}\rangle\right|}{|X_{1t}-X_{2t}|}dxdt.$$

Combining this and (1.12), we get

$$\sup_{t\in[0,T]}\mathcal{L}^d\left(\{x\in B_r:|X_{1t}-X_{2t}|>0\}\right)=\limsup_{\delta\to0}\frac{1}{|\log(\delta)|}\sup_{t\in[0,T]}\Phi_\delta^\gamma(t)$$
$$\le\limsup_{\delta\to0}\frac{1}{|\log(\delta)|}\int_0^T\int_{B_r}\frac{|\mathbf{B}(X_{1t})-\mathbf{B}(X_{2t})|\ dxdt}{\delta+|X_{1t}-X_{2t}|+\gamma^{1/2}|\langle\eta(X_{1t}),X_{1t}-X_{2t}\rangle|}$$
$$:=\limsup_{\delta\to0}A(\delta).$$

Hence, in order to get $X_{1t}=X_{2t}$ for a.e $(x,t)\in B_r\times[0,T]$, we need to show that

$$\limsup_{\delta\to0}A(\delta)=\circ(1)\text{ as }\gamma\to\infty.$$

In fact, we use the following estimate for $\mathbf{B}(x_1)-\mathbf{B}(x_2)$:

$$|\mathbf{B}(x_1)-\mathbf{B}(x_2)|\lesssim\varepsilon^{-d+1}|x_1-x_2|\left(\mathbf{M}(|D^a\mathbf{B}|)(x_1)+\mathbf{M}(|D^a\mathbf{B}|)(x_2)\right)$$
$$+\varepsilon^{-d+1}\sum_{l=1,2}\int\frac{\mathbf{1}_{|x_l-z|\le r}\mathbf{1}_{\left|\frac{x_l-z}{|x_l-z|}-e_l\right|\le\varepsilon}}{|x_l-z|^{d-1}}\frac{|\langle\eta(z),x_1-x_2\rangle|}{|x_1-x_2|}d|\mu|(z)$$
$$+\varepsilon^{-d+2}\sum_{l=1,2}\int\frac{\mathbf{1}_{|x_l-z|\le r}\mathbf{1}_{\left|\frac{x_l-z}{|x_l-z|}-e_l\right|\le\varepsilon}}{|x_l-z|^{d-1}}d|\mu|(z),$$

for any $\varepsilon>0$ where $\mu=|D^s\mathbf{B}|$, $e_1=-e_2=\frac{x_1-x_2}{|x_1-x_2|}$, $r=|x_1-x_2|$ for $l=1,2$ (see Proposition 4 and Lemma 6). Then, using the fact that $|\langle\eta(z),x_1-x_2\rangle|\le|\langle\eta(x_1),x_1-x_2\rangle|+2||\nabla\eta||_{L^\infty}r^2$ for $|z-x_1|\le r$ or $|z-x_2|\le r$ and changing variable along the flows we can estimate

$$A(\delta)\lesssim\frac{\gamma^{-1/2}\varepsilon^{-d+1}}{|\log(\delta)|}\int_{B_{r'}}\min\left\{\frac{\mathbf{I}_1(\mu)}{\delta},\mathbf{M}(\mu)\right\}dx+||\nabla\eta||_{L^\infty}\frac{\varepsilon^{-d+1}}{|\log(\delta)|}\int_{B_{r'}}\mathbf{I}_1(\mu)dx$$
$$+\frac{\varepsilon}{|\log(\delta)|}\int_{B_{r'}}\min\left\{\frac{\mathbf{I}_1(\mu)}{\varepsilon^{d-1}\delta},\mathbf{M}^\varepsilon(\mu)\right\}dx+\frac{\varepsilon^{-d+1}}{|\log(\delta)|}\int_{B_{r'}}\min\left\{\frac{\mathbf{I}_1(|D^aB|)}{\delta},\mathbf{M}(|D^aB|)\right\}dx,$$

for some $r'>r$, where $\mathbf{M}^\varepsilon$ is the Kakeya maximal function in $\mathbb{R}^d$ i.e

$$\mathbf{M}^\varepsilon(\mu)(x)=\sup_{\rho\in(0,2r'),e\in S^{d-1}}⨍_{B_\rho(x)}\varepsilon^{-d+1}\mathbf{1}_{\left|\frac{z-x}{|z-x|}-e\right|\le\varepsilon}d|\mu|(z).$$

We then will deduce that

$$\limsup_{\delta\to0}A(\delta)\lesssim\gamma^{-1/2}\varepsilon^{-d+1}|\mu|(\mathbb{R}^d)+\varepsilon\limsup_{\lambda\to\infty}\lambda\mathcal{L}^d\left(\{\mathbf{M}^\varepsilon(\mu)>\lambda\}\cap B_{r'}\right)$$
$$=\varepsilon\limsup_{\lambda\to\infty}\lambda\mathcal{L}^d\left(\{\mathbf{M}^\varepsilon(\mu)>\lambda\}\cap B_{r'}\right)\text{ as }\gamma\to\infty.$$

So it remains to show that

$$I(\varepsilon) := \varepsilon \limsup_{\lambda\to\infty} \lambda \mathcal{L}^d \left(\{\mathbf{M}^\varepsilon(\mu) > \lambda\} \cap B_{r'}\right) = \circ(1) \quad \text{as} \quad \varepsilon \to 0. \tag{1.14}$$

This estimate is very delicate and its proof is very complicated, hence we will spend Section 3 to establish it. In order to see the key idea for proving the estimate (1.14), we only consider $\mu(x) = |D^s f|(x) \equiv |Df|(x)$ with $f \in BV(\mathbb{R}^d, \mathbb{R})$ such that $\nu = \frac{dD^s f}{d|D^s f|}(x)$ is a constant function in $B_{8r'}$. Set $H_\nu := \left\{x \in \mathbb{R}^d : \langle \nu, x\rangle = 0\right\}$ and $\tilde{H}_\nu := \left\{t\nu \in \mathbb{R}^d : \forall t \in \mathbb{R}\right\}$. We also denote $f^\nu_{y_2}$: $\tilde{H}_\nu \ni y_1 \mapsto f(y_2 + y_1)$ for any $y_2 \in H_\nu$. By assumption one has $d\mu(y) = dDf^\nu_z(y_1)d\mathcal{H}^{d-1}(y_2)$ for any $y_1 = \langle y, \nu\rangle\nu, y_2 = y - \langle y, \nu\rangle\nu$, $y \in B_{8r'}$ and $z \in H_\nu$. We can prove that

$$\mathbf{M}^\varepsilon(\mu)(x) \le C\mathbf{M}^1(|Df^\nu_{x_\nu}|, \tilde{H}_\nu)(\langle x, \nu\rangle\nu), \quad x_\nu := x - \langle x, \nu\rangle\nu, \tag{1.15}$$

where $\mathbf{M}^1(|Df^\nu_{x_\nu}|, \tilde{H}_\nu)$ is the Hardy-Littlewood maximal function of $|Df^\nu_{x_\nu}|$ on $\tilde{H}_\nu$. Indeed, by a standard approximation argument, we only prove for case $|Df^\nu_{x_\nu}| \in L^1(\tilde{H}_\nu, d\mathcal{H}^1)$. By changing of variables, we have for any $\rho \in (0, 2r'), e \in S^{d-1}$, $x \in B_{r'}$

$$\begin{aligned}
&\fint_{B_\rho(x)} \varepsilon^{-d+1}\mathbf{1}_{|\frac{y-x}{|y-x|}-e|\le\varepsilon} d|\mu|(y) \\
&= c_d\rho^{-d}\varepsilon^{-d+1}\int_{S^{d-1}}\int_0^\rho \mathbf{1}_{|\theta-e|\le\varepsilon} Df^\nu_{x_\nu}(\langle x, \nu\rangle\nu - \langle\theta, \nu\rangle\nu s)s^{d-1}dsd\mathcal{H}^{d-1}(\theta) \\
&\le c_d\varepsilon^{-d+1}\int_{S^{d-1}} \mathbf{1}_{|\theta-e|\le\varepsilon} 4\mathbf{M}^1(|Df^\nu_{x_\nu}|, \tilde{H}_\nu)(\langle x, \nu\rangle\nu)d\mathcal{H}^{d-1}(\theta) \\
&\le C\mathbf{M}^1(|Df^\nu_{x_\nu}|, \tilde{H}_\nu)(\langle x, \nu\rangle\nu),
\end{aligned}$$

which implies (1.15). Therefore, we get from (1.15) and weak type (1,1) bound of $\mathbf{M}^1(|Df^\nu_{x_\nu}|, \tilde{H}_\nu)$ that

$$\begin{aligned}
\lambda\mathcal{L}^d\left(\{\mathbf{M}^\varepsilon(\mu) > \lambda\} \cap B_{r'}\right) &\le \lambda\int_{H_\nu} \mathcal{H}^1\left(\left\{x_1 \in \tilde{H}_\nu : C\mathbf{M}^1(|Df^\nu_{x_2}|, \tilde{H}_\nu)(x_1) > \lambda\right\}\right) d\mathcal{H}^{d-1}(x_2) \\
&\le C\int_{H_\nu}\int_{\tilde{H}_\nu} d|Df^\nu_{x_2}|(x_1)d\mathcal{H}^{d-1}(x_2) = C|\mu|(\mathbb{R}^d).
\end{aligned}$$

This gives (1.14). In order to prove (1.14) in general case, we use that $\mu = |D^s\mathbf{B}|$ and the slicing theory of BV functions. And (1.14) is not true for any Radon measure μ, indeed if $\mu = \delta_0$, then $\mathbf{M}^\varepsilon(\mu)(x) = \varepsilon^{-d+1}|x|^{-d}$ and so $I(\varepsilon) \sim \varepsilon^{-d+2}$.

To end this section, let us give an important remark on our result. We deduce from (1.9) that

$$\partial_l\mathbf{B}^i = \sum_{j=1}^m \left(\frac{\Omega^i_{jR}(.)}{|.|^d}\right) \star \mu^l_{jR} \text{ in } \mathcal{D}'(B_{2R}) \tag{1.16}$$

where $\mu^l_{jR} = \partial_l b_{jR}$, $l, i = 1..., d, j = 1, ..., m$ are bounded Radon measures in $\mathbb{R}^d$. Thus,

A natural question is that whether above Proposition is still true for a class of vector fields $\mathbf{B}$ *satisfying* (1.16) *with arbitrary Radon measures* μ^l_{jR} *in* $\mathbb{R}^d$.

The following proposition is to give a negative answer.

Proposition 1. *There exist a vector field $B : \mathbb{R}^2 \to \mathbb{R}^2$ and degree-zero homogeneous functions $\Omega^i_1, ..., \Omega^i_m \in (L^\infty \cap BV)(S^1)$, $i = 1, 2$ with $\frac{|B(x)|}{|x|+1} \in L^1(\mathbb{R}^2) + L^\infty(\mathbb{R}^2)$, $\operatorname{div}(B) = 0$, $\int_{S^1} \Omega_l = 0$ such that for any $R > 1$ we have*

$$\partial_l \mathbf{B}^i = \sum_{j=1}^m \left(\frac{\Omega^i_j(.)}{|.|^2} \right) \star \mu^l_{jR} \quad in \quad \mathcal{D}'(B_R) \tag{1.17}$$

for some $\mu^l_{jR} \in \mathcal{M}_b(\mathbb{R}^2)$ $i, l = 1, 2$ and $j = 1, ..., m$ and problem (1.1) *is ill-posed with this vector field, i.e there exist two different regular Lagrangian flows X_1, X_2 associated to $\mathbf{B}$.*

We will prove proposition 1 in Appendix section.

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2 Main Notation and preliminary results

We begin with some notations which will be used in this paper.

• $x.y$, $\langle x, y \rangle$ denote the usual scalar product of $x, y \in \mathbb{R}^d$;
• $a \wedge b$ denotes $\min\{a, b\}$;
• S^{d-1} denotes the $(d-1)-$dimensional unit sphere in $\mathbb{R}^d$;
• $\mathbf{1}_E$ is the characteristic function of the set E, defined as $\mathbf{1}_E(x) = 1$ if $x \in E$ and $\mathbf{1}_E(x) = 0$ otherwise;
• $B_r(x)$ is the open ball in $\mathbb{R}^d$ with radius r and center x; B_r is the open ball in $\mathbb{R}^d$ with radius and center 0; if X is a vector subspace of $\mathbb{R}^d$, for any $x \in X$, $B_r(x, X)$ is the open ball in X with radius r and center x i.e $B_r(x, X) = B_r(x) \cap X$.
• $\mathcal{M}_b(X)$ is a set of bounded Radon measure in a metric space X; $\mathcal{M}^+_b(X)$ is a set of positive bounded Radon measure in X;
• $|\mu|$ is the total variation of a measure μ; μ^s, μ^a are the singular component and regular component of μ with respect to the Lebesgue measure, respectively;
• $\mathcal{L}^d$ is the Lebesgue measure on $\mathbb{R}^d$ and $\mathcal{H}^k$ is the $k-$dimensional Hausdorff measure;
• $BV(\mathbb{R}^d, \mathbb{R}^m)$ is a set of $\mathbb{R}^m-$valued functions with bounded variation in $\mathbb{R}^d$;
• $f \star g$ is the convolution of f and g, in particular if $f, g \in \mathbb{R}^l$, then $f \star g := \sum_{j=1}^l f_j \star g_j$; if $f \in \mathbb{R}^l, g \in \mathbb{R}$, then $f \star g = g \star f := (f_1 \star g, f_2 \star g, ..., f_l \star g)$
• $f_\# \mu$ is the push-forward of μ via a Borel map f, more specifically, a Borel map $f : \mathbb{R}^l \to \mathbb{R}^m$, and a measure μ in $\mathbb{R}^l$ then $f_\# \mu$ is a measure in $\mathbb{R}^m$ given by $f_\# \mu(B) = \mu(f^{-1}(B))$ for any Borel set $B \subset \mathbb{R}^m$; this is equivalent to $\int_{\mathbb{R}^m} \phi df_\# \mu = \int_{\mathbb{R}^l} \phi \circ f d\mu$ for any $\phi : \mathbb{R}^m \to [0, +\infty]$ Borel.
• $⨍_E f d\omega$ denotes the average of the function f over the set E with respect to the positive measure ω i.e $⨍_E f d\omega := \frac{1}{\omega(E)} \int_E f d\omega$;
• $\{f > \lambda\}$, $\{f < \lambda\}$ stand for $\{x : f(x) > \lambda\}$, $\{x : f(x) < \lambda\}$ respectively;
• E^c is the complement of set E;
• C is a common constant whose value may change from line to line. In particular cases, we want to clarify the dependence of the constant on relevant parameters, we will use $C(\varepsilon, \kappa, ...)$.

2.1 *BV functions.* Given $b \in BV(\mathbb{R}^d, \mathbb{R}^m)$, we have the canonical decomposition of Db as

$D^a b + D^s b$, with $|D^a b| << \mathcal{L}^d$ and $|D^s b| \perp \mathcal{L}^d$. The following deep result of Alberti will be used in the proof of the main Theorem 2. Its proof can be found in [1], see also [36].

Proposition 2 (Alberti's rank one Theorem)**.** *There exist unit vectors $\xi(x) \in \mathbb{R}^m, \eta(x) \in \mathbb{R}^d$ such that $D^s b(x) = \xi(x) \otimes \eta(x) |D^s b|(x)$ i.e $D^s_{x_i} b_j(x) = \xi_j(x)\eta_i(x)|D^s b|(x)$ for any $i = 1, ..., d, j = 1, ..., m$.*

Notice that the pair of unit vector (ξ, η) is uniquely determined $|D^s b|-$a.e up to a change of sign. Case $m = d$, we can write the distributional divergence $\text{div}(b)$ as $\text{div}(b) = \text{trace}(D^a b)\mathcal{L}^d + \langle \xi, \eta \rangle |D^s b|$, thus, $\text{div}(b) << \mathcal{L}^d$ if and only if $\xi \perp \eta \quad |D^s b| -$ a.e. in $\mathbb{R}^d$.

For $e \in S^{d-1}$, let us introduce the hyperplane orthogonal to e: $H_e := \left\{x \in \mathbb{R}^d : \langle e, x \rangle = 0\right\}$ and the line of e: $\tilde{H}_e := \left\{te \in \mathbb{R}^d : \forall t \in \mathbb{R}\right\}$. Given a Borel function f in $\mathbb{R}^d$, we denote $f^e_{y_1} : \tilde{H}_e \ni z_1 \mapsto f(y_1 + z_1)$ for $y_1 \in H_e$. The following characterization of BV by hyperplanes will be used in proof of main Theorem 1.

Proposition 3. *([3, Section 3.11]) Let $f : \mathbb{R}^d \to \mathbb{R}$ be a Borel and $e \in S^{d-1}$. Then, $f \in BV(\mathbb{R}^d)$ to equivalent to $f^e_{y_1} \in BV(\tilde{H}_e)$, $\mathcal{H}^{d-1}$-a.e y_1 in H_e and $\int_{H_e} ||f^e_{y_1}||_{BV(\tilde{H}_e)} d\mathcal{H}^{d-1}(y_1) < \infty$. Moreover, for any rotation $\mathbf{R}$ in $\mathbb{R}^d$ with $e = \mathbf{R}e_1$, $e_1 = (1, ..., 0)$*

$$dD^s f^e_{y_1}(t) d\mathcal{H}^{d-1}(y_1) = \langle e, \eta(t + y_1) \rangle d(\mathbf{R}_\# |D^s f|)(t, y_1) \ \forall (t, y_1) \in \tilde{H}_e \otimes H_e$$

where $\eta(x) = \frac{dD^s f(x)}{d|D^s f|(x)}$. In particular, for any Borel function $\phi : \mathbb{R}^d \to \mathbb{R}^+$ there holds

$$\int_{H_e} \int_{\tilde{H}_e} \phi(t + y_1) d|D^s f^e_{y_1}|(t) d\mathcal{H}^{d-1}(y_1) = \int_{\mathbb{R}^d} \phi(x) \left|\langle e, \eta(x) \rangle\right| d|\mu|^s(x). \tag{2.1}$$

Remark 1. Proposition 3 gives that if $f \in BV(\mathbb{R}^d)$ then $D_{x_1} f \in L^1(\mathbb{R}^{d-1}, \mathcal{M}_{b,x_1}(\mathbb{R}))$ i.e the map $(x_2, ..., x_d) \mapsto ||D_{x_1} f(., x_2, ..., x_d)||_{\mathcal{M}_b(\mathbb{R})}$ is $L^1(\mathbb{R}^{d-1}, d\mathcal{H}^{d-1})$. It is quite surprising that in [39], we construct a measure $\mu \in L^1(\mathbb{R}^{d-1}, \mathcal{M}_{b,x_1}(\mathbb{R}))$ such that $||\mu - D_{x_1} f||_{\mathcal{M}_b(\mathbb{R}^d)} \geq 1$ for any $f \in BV(\mathbb{R}^d)$. In other words, $\{D_{x_1} f \in \mathcal{M}_b(\mathbb{R}^d) : f \in BV(\mathbb{R}^d)\}$ is not dense in $L^1(\mathbb{R}^{d-1}, \mathcal{M}_{b,x_1}(\mathbb{R}))$.

We next have an extension of [20, Proposition 4.2]. It is one of main tools to be used in proof of main theorem 2.

Proposition 4. *Let $\varepsilon \in (0, 1/100)$, $f \in BV_{loc}(\mathbb{R}^d)$. Then, for every $x, y \in \mathbb{R}^d$, $x \neq y$,*

$$f(x) - f(y) = \int_{\mathbb{R}^d} \frac{\varepsilon^{-d+1}}{|x-z|^{d-1}} \Theta_1^{\varepsilon, \mathbf{e}_1}\left(\frac{x-z}{|x-y|}\right) \mathbf{e}_1 . dDf(z) + \int_{\mathbb{R}^d} \frac{\varepsilon^{-d+2}}{|x-z|^{d-1}} \Theta_2^{\varepsilon, \mathbf{e}_1}\left(\frac{x-z}{|x-y|}\right) dDf(z)$$
$$- \int_{\mathbb{R}^d} \frac{\varepsilon^{-d+1}}{|y-z|^{d-1}} \Theta_1^{\varepsilon, \mathbf{e}_2}\left(\frac{y-z}{|x-y|}\right) \mathbf{e}_2 . dDf(z) - \int_{\mathbb{R}^d} \frac{\varepsilon^{-d+2}}{|y-z|^{d-1}} \Theta_2^{\varepsilon, \mathbf{e}_2}\left(\frac{y-z}{|x-y|}\right) dDf(z) \tag{2.2}$$

where $\mathbf{e}_1 = -\mathbf{e}_2 = \frac{x-y}{|x-y|}$ and for $e \in S^{d-1}$, $\varepsilon \in (0, 1/100)$, $\Theta_1^{\varepsilon, e} : \mathbb{R}^d \to \mathbb{R}_+$ and $\Theta_2^{\varepsilon, e} : \mathbb{R}^d \to \mathbb{R}^d$ are bounded functions such that $\Theta_1^{\varepsilon, e}, \Theta_2^{\varepsilon, e} \in C^\infty(\mathbb{R}^d \backslash \{0\}) \cap L^\infty_c(\mathbb{R}^d)$,

$$\text{supp}(\Theta_1^{\varepsilon, e}), \text{supp}(\Theta_2^{\varepsilon, e}) \subset B_{3/4}(0) \cap \left\{x : \left|e - \frac{x}{|x|}\right| \leq \varepsilon\right\},$$

and

$$|\Theta_l^{\varepsilon, e}(x)| + \varepsilon |x| |\nabla \Theta_l^{\varepsilon, e}(x)| \leq C(d) \ \forall \ x \in \mathbb{R}^d, l = 1, 2;$$
$$\varepsilon^{-d+1} \int_{\mathbb{R}^d} \Theta_1^{\varepsilon, e} dx + \varepsilon^{-d+1} \int_{\mathbb{R}^d} |\Theta_2^{\varepsilon, e}| dx \leq C(d).$$

Proof of Proposition 4. Let $\rho : \mathbb{R} \to [0, \infty)$ be a C_c function such that $\rho \in C^\infty([0, 1])$, $\rho(t) = 1$ for $0 \le t \le 1/4$, $\rho(t) = 0$ for $t \ge \frac{3}{4}$ and $t < 0$, $\rho(t) + \rho(1 - t) = 1$ for $0 \le t \le 1$. Let $\psi : (0, \infty) \to (0, \infty)$ be a C_b^∞ function such that $\psi(t) = 0$ for $t > 1$, $\psi(t) = 1$ in $(0, \varepsilon_0)$ for some $\varepsilon_0 \in (0, 1)$ and $\int_{\mathbb{R}^{d-1}} \psi(|h|)dh = 1$. We define for $(a, b, c) \in S^{d-1} \times S^{d-1} \times (0, \infty)$

$$\Psi_1(a, b, c) = \frac{\rho((a.b)c)\psi\left(\frac{|a-(a.b)b|}{4(a.b)(1-(a.b)c)}\right)}{4^{d-1}(a.b)^d(1 - (a.b)c)^{d-1}}, \quad \Psi_2(a, b, c) = \frac{\rho((a.b)c)\psi\left(\frac{|a-(a.b)b|}{4(a.b)(1-(a.b)c)}\right)}{4^{d-1}(a.b)^{d-1}(1 - (a.b)c)^d} c(a - (a.b)b).$$

Since $\Psi^1(a, b, c) = \Psi^1(-a, -b, c), \Psi^2(a, b, c) = -\Psi^2(-a, -b, c)$, thus it is not hard to obtain from the proof of [20, Proposition 4.2] that

$$\begin{aligned} f(x) - f(y) = &\int_{\mathbb{R}^d} \frac{1}{|x - z|^{d-1}} \Psi_1\left(\frac{x - z}{|x - z|}, \frac{x - y}{|x - y|}, \frac{|x - z|}{|x - y|}\right) \frac{x - z}{|x - z|} dDf(z) \\ &- \int_{\mathbb{R}^d} \frac{1}{|x - z|^{d-1}} \Psi_2\left(\frac{x - z}{|x - z|}, \frac{x - y}{|x - y|}, \frac{|x - z|}{|x - y|}\right) dDf(z) \\ &- \int_{\mathbb{R}^d} \frac{1}{|y - z|^{d-1}} \Psi_1\left(\frac{y - z}{|y - z|}, \frac{y - x}{|y - x|}, \frac{|y - z|}{|x - y|}\right) \frac{y - z}{|z - y|} dDf(z) \\ &+ \int_{\mathbb{R}^d} \frac{1}{|y - z|^{d-1}} \Psi_2\left(\frac{y - z}{|y - z|}, \frac{y - x}{|y - x|}, \frac{|y - z|}{|x - y|}\right) dDf(z). \end{aligned}$$

Replacing ψ by $\frac{8^{d-1}}{\varepsilon^{d-1}}\psi(8\frac{\cdot}{\varepsilon})$, we obtain (2.2) where $\Theta_l^{\varepsilon,e}(z) = \phi_l^\varepsilon(z/|z|, e, |z|)$ for $(e, z) \in S^{d-1} \times \mathbb{R}^d$; and

$$\phi_1^\varepsilon(a, b, c) = 2^{d-1} \frac{\rho((a.b)c)\psi\left(\frac{2|a-(a.b)b|}{\varepsilon(a.b)(1-(a.b)c)}\right)}{(a.b)^d(1 - (a.b)c)^{d-1}},$$

$$\phi_2^\varepsilon(a, b, c) = 2^{d-1} \frac{\rho((a.b)c)\psi\left(\frac{2|a-(a.b)b|}{\varepsilon(a.b)(1-(a.b)c)}\right)}{(a.b)^d(1 - (a.b)c)^{d-1}} \frac{a - b}{\varepsilon} - 2^{d-1} \frac{\rho((a.b)c)\psi\left(\frac{2|a-(a.b)b|}{\varepsilon(a.b)(1-(a.b)c)}\right)}{(a.b)^{d-1}(1 - (a.b)c)^d} c \frac{(a - (a.b)b)}{\varepsilon}.$$

Note that $\rho((a.b)c)\psi\left(\frac{2|a-(a.b)b|}{\varepsilon(a.b)(1-(a.b)c)}\right) \ne 0$ implies $|a - (a.b)b| \le \frac{\varepsilon}{2}$ and $(a.b)c \le 3/4$; so,

$$|a - b| = \sqrt{2(1 - (a.b))} \le \sqrt{2(1 - (a.b)^2)} = \sqrt{2|a - (a.b)b|^2} \le \varepsilon/\sqrt{2},$$

and $a.b \ge 1 - \varepsilon/2 \ge 1/2$, $c \le 3/4$. Hence, it is easy to check that $\Theta_1^{\varepsilon,e}, \Theta_2^{\varepsilon,e} \in C^\infty(\mathbb{R}^d\backslash\{0\}) \cap L_c^\infty(\mathbb{R}^d)$, $\text{supp}(\Theta_1^{\varepsilon,e}), \text{supp}(\Theta_2^{\varepsilon,e}) \subset B_{3/4}(0) \cap \left\{x : \left|e - \frac{x}{|x|}\right| \le \varepsilon\right\}$ and $|\Theta_l^{\varepsilon_1,e}(x)| + \varepsilon_1|x||\nabla\Theta_l^{\varepsilon_1,e}(x)| \le C(d)$ $\forall\ x \in \mathbb{R}^d, l = 1, 2$ and $\varepsilon^{-d+1}\int_{\mathbb{R}^d} \Theta_1^{\varepsilon,e}dx + \varepsilon^{-d+1}\int_{\mathbb{R}^d} |\Theta_2^{\varepsilon,e}|dx \le C(d)$. The proof is complete. □

2.2 *The Hardy-Littlewood maximal function and Riesz potential.* We recall some basic properties of the Hardy-Littlewood maximal function and Riesz potential. Given a positive Radon measure μ in a vector subspace X of $\mathbb{R}^d$ with $\dim(X) = k$, $k = 1, ..., d$. The Hardy-Littlewood maximal function of μ on X is defined by

$$\mathbf{M}^k(\mu, X)(x) = \sup_{r>0} \frac{1}{\mathcal{H}^k(B_r(x, X))} \int_{B_r(x,X)} d|\mu| \quad \forall\ x \in X.$$

If $X = \mathbb{R}^d$, we write $\mathbf{M}(\mu)$ instead of $\mathbf{M}^k(\mu, X)$. It is well known that $\mathbf{M}^k(., X)$ is bounded from $L^p(X, d\mathcal{H}^k)$ to $L^p(X, d\mathcal{H}^k)$ and $\mathcal{M}_b^+(X)$ to $L^{1,\infty}(X, d\mathcal{H}^k)$ for $1 < p \le \infty$ i.e

$$||\mathbf{M}^k(\mu, X)||_{L^p(X,d\mathcal{H}^k)} \le C(k)||\mu||_{L^p(X,d\mathcal{H}^k)} \quad \text{for any } \mu \in L^p(X, d\mathcal{H}^k); \tag{2.3}$$

$$\sup_{\lambda>0} \lambda\mathcal{H}^k\left(\left\{\mathbf{M}^k(\mu, X) > \lambda\right\} \cap X\right) \le C(k)|\mu|(X) \quad \text{for any } \mu \in \mathcal{M}_b^+(X); \tag{2.4}$$

(see [42], [43],[5]).

The Riesz potential of μ on X is defined by

$$\mathbf{I}_\alpha^k(\mu, X)(x) = \int_X \frac{1}{|x-z|^{k-\alpha}} d\mu(z) \ \ \forall\, x \in X, 0 < \alpha < k.$$

If $X = \mathbb{R}^d$, we write $\mathbf{I}_\alpha(\mu)$ instead of $\mathbf{I}_\alpha^k(\mu, X)$. We have that $\mathbf{I}_\alpha^k(., X)$ is bounded from $L^p(X, d\mathcal{H}^k)$ to $L^{\frac{kp}{k-\alpha p}}(X, d\mathcal{H}^k)$ for $p > 1, 0 < \alpha p < k$; and bounded from $\mathcal{M}_b^+(X)$ to $L^{\frac{k}{k-\alpha},\infty}(X, d\mathcal{H}^k)$ for $0 < \alpha < k$, see [42].
It is easy to see that for $\alpha > 0$

$$\sup_{r>0} r^{-\alpha} \int_X \frac{\mathbf{1}_{|x-z|\leq r}}{|x-z|^{k-\alpha}} d\mu(z) \leq C(k,\alpha) \mathbf{M}^k(\mu, X)(x) \ \ \forall\, x \in X. \tag{2.5}$$

Thanks to (2.4), one gets

$$\begin{aligned}
&\lambda \mathcal{H}^k \left(\left\{ \mathbf{M}^k(\mu, X) > \lambda \right\} \cap X \right) \\
&\leq \lambda \mathcal{H}^k \left(\left\{ \mathbf{M}^k(\mu^s, X) > \lambda/2 \right\} \cap X \right) + \lambda \mathcal{H}^k \left(\left\{ \mathbf{M}^k(\mu^a \mathbf{1}_{|\mu^a| \geq \lambda/4}, X) > \lambda/2 \right\} \cap X \right) \\
&\leq C(k)|\mu|^s(X) + C(k) \int_X \mathbf{1}_{|\mu|^a \geq \lambda/4} |\mu|^a dx.
\end{aligned}$$

provided $|\mu|(X) < \infty$. Thus,

$$\limsup_{\lambda\to\infty} \lambda \mathcal{H}^k \left(\left\{ \mathbf{M}^k(\mu, X) > \lambda \right\} \cap X \right) \leq C(k)|\mu|^s(X). \tag{2.6}$$

Moreover, in [2] we showed that for any $\lambda > 0$,

$$\lambda \mathcal{H}^k \left(\left\{ \mathbf{M}^k(\mu, X) > \lambda \right\} \cap X \right) \geq C(k)|\mu|^s(X). \tag{2.7}$$

Therefore, it is not hard to see from (2.6) and (2.7) that for any $B_R := B_R(0, X) \subset X$,

$$C_1(k)|\mu|^s(B_R) \leq \limsup_{\lambda\to\infty} \lambda \mathcal{H}^k \left(\left\{ \mathbf{M}^k(\mu, X) > \lambda \right\} \cap B_R \right) \leq C_2(k)|\mu|^s(\overline{B_R}). \tag{2.8}$$

Again, (2.6) and (2.7) imply that $\mu << \mathcal{H}^k$ in X if any only if

$$\limsup_{\lambda\to\infty} \lambda \mathcal{H}^k \left(\left\{ \mathbf{M}^k(\mu, X) > \lambda \right\} \cap X \right) = 0. \tag{2.9}$$

Next is a basic estimate of the Hardy-Littlewood maximal function, it will be used several times in this paper.

Lemma 1. *let X be a vector subspace of $\mathbb{R}^d$ with $\dim(X) = k$ and $q > 1$. Then, for any $\mu \in \mathcal{M}_b^+(X)$ and ball $B_R := B_R(0, X) \subset X$ and $f \in L^q(B_R)$ there holds,*

$$\limsup_{\delta\to 0} \frac{1}{|\log(\delta)|} \int_{B_R} (\delta^{-1}|f|) \wedge \mathbf{M}^k(\mu, X) d\mathcal{H}^k \leq C(k,q) \mu^s(\overline{B_R}). \tag{2.10}$$

Moreover, for any $0 < \delta << 1$,

$$\frac{1}{|\log(\delta)|} \int_{B_R} (\delta^{-1}|f|) \wedge \mathbf{M}^k(\mu, X) d\mathcal{H}^k \leq C(k,q) \left(R^k + \mu(X) + ||f||_{L^q(B_R)} \right). \tag{2.11}$$

Proof. Set $A(\lambda) = \sup_{\lambda'>\lambda} \lambda' \mathcal{H}^k\left(\{\mathbf{M}^k(\mu,X) > \lambda'\} \cap B_R(0,X)\right) \le C(k)\mu(X)$. One has for any $0<\delta<<1$ and $0<\lambda_1<\lambda_2<\infty$,

$$\frac{1}{|\log(\delta)|}\int_{B_R}(\delta^{-1}|f|)\wedge \mathbf{M}^k(\mu,X)d\mathcal{H}^k = \frac{1}{|\log(\delta)|}\int_0^\infty \mathcal{H}^k\left(\left\{(\delta^{-1}|f|)\wedge \mathbf{M}^k(\mu,X)>\lambda\right\}\cap B_R\right)d\lambda$$
$$\le \frac{1}{|\log(\delta)|}\int_0^{\lambda_1}\mathcal{H}^k(B_R)d\lambda + \frac{1}{|\log(\delta)|}\int_{\lambda_1}^{\lambda_2}A(\lambda_1)\frac{d\lambda}{\lambda} + \frac{1}{|\log(\delta)|}\int_{\lambda_2}^\infty \mathcal{H}^k\left(\{|f|>\delta\lambda\}\cap B_R\right)d\lambda$$
$$\le \frac{\lambda_1}{|\log(\delta)|}\mathcal{H}^k(B_R) + \frac{\log(\lambda_2/\lambda_1)}{|\log(\delta)|}A(\lambda_1) + \frac{1}{q|\log(\delta)|\lambda_1^{q-1}\delta^q}||f||_{L^q(B_R)}.$$

Choosing $\lambda_1 = |\log(\delta)|^{1/2}, \lambda_2 = \delta^{-\frac{q}{q-1}}$ and thanks to (2.4) and (2.8) we obtain (2.10) and (2.11). The proof is complete. □

2.3 *Singular integral operators with rough kernels.* In this subsection, we provide some basic properties of singular integral operators with rough convolution kernels. In this paper, we consider the following general kernel in $\mathbb{R}^d$:

$$\mathbf{K}(x) = \Omega(x)K(x) \quad \forall\, x\in\mathbb{R}^d\backslash\{0\} \tag{2.12}$$

where

i.) $K\in C^1(\mathbb{R}^d\backslash\{0\})$,

$$|K(x)| + |x||\nabla K(x)| + |x||D^2K(x)| \le \frac{1}{|x|^d} \quad \forall\, x\in\mathbb{R}^d, \tag{2.13}$$

ii.) $\Omega(\theta) = \Omega(r\theta)$ for any $r>0, \theta\in S^{d-1}$ and

$$||\Omega||_{W^{\alpha_0,1}(B_2\backslash B_1)} := \int_{B_2\backslash B_1}|\Omega| + \int_{B_2\backslash B_1}\int_{B_2\backslash B_1}\frac{|\Omega(x)-\Omega(y)|}{|x-y|^{d+\alpha_0}}dxdy \le c_1, \tag{2.14}$$

for some $\alpha_0\in(0,1)$ and $c_1>0$.

iii.) (the "cancellation" condition)

$$\sup_{0<R_1<R_2<\infty}\left|\int_{R_1<|x|<R_2}\mathbf{K}(x)dx\right| \le c_2 \tag{2.15}$$

for some $c_2>0$.

We say that the kernel $\mathbf{K}$ is a singular Kernel of fundamental type in $\mathbb{R}^d$ if $\Omega\in C^1(S^{d-1})$.

Remark 2. From (2.13) one has,

$$|K(x-y) - K(x)| \le \frac{2^{d+1}|y|}{|x|^{d+1}} \quad \forall |y| < |x|/2. \tag{2.16}$$

Remark 3. If $K(x) = |x|^{-d}$ for any $x\in\mathbb{R}^d\backslash\{0\}$, then (2.15) implies $\int_{S^{d-1}}\Omega(\theta)d\mathcal{H}^{d-1}(\theta) = 0$. Moreover, if we set

$$\Omega_n(x) := \int_0^\infty \tilde{\Omega}\star\varrho_n\left(\frac{x}{|x|}r\right)r^{n-1}dr \tag{2.17}$$

where ϱ_n is a standard sequence of mollifiers in $\mathbb{R}^n$ and $\tilde{\Omega}(x) := \frac{1}{\log(2)}\frac{\Omega(x)}{|x|^d}\mathbf{1}_{1\le|x|\le 2}$, then $\int_{S^{d-1}}\Omega_n(\theta)d\mathcal{H}^{d-1}(\theta) = 0$ for any n, $\Omega_n\in C_b^\infty(S^{d-1})$, $\Omega_n(\theta) = \Omega_n(r\theta)$ for any $r>0, \theta\in S^{d-1}$ and

$$||\Omega_n - \Omega||_{W^{\alpha_0/2,1}(B_2\backslash B_1)} \le Cc_1 n^{-\alpha_0/2}\ \forall\, n. \tag{2.18}$$

Remark 4. Since $\Omega(\theta)=\Omega(t\theta)$ for any $t>0,\theta\in S^{d-1}$, so by Sobolev inequality one gets

$$||\Omega||_{L^q(S^{d-1})}+\sup_{|h|\leq 1/2}|h|^{-\alpha_0/2}\left(||\Omega(.-h)-\Omega(.)||_{L^q(B_2\backslash B_1)}+||\Omega(.-h)-\Omega(.)||_{L^q(S^{d-1})}\right)$$
$$+\left(\int_{1<|x|<2}\sup_{0<\rho<1/2}\fint_{B_\rho(0)}\frac{|\Omega(x-h)-\Omega(x)|^q}{|h|^{\frac{\alpha_0 q}{2}}}dhdx\right)^{1/q}\leq C(d,\alpha_0)||\Omega||_{W^{\alpha_0,1}(B_2\backslash B_1)}\leq C(d,\alpha_0)c_1. \tag{2.19}$$

for any $1\leq q\leq q_0=\frac{d}{d-\alpha_0/2}$ and

$$||\Omega||_{W^{\alpha_0,1}(B_2\backslash B_1)}\leq C(d,\alpha_0)||\Omega||_{BV(S^{d-1})}.$$

Remark 5. Thanks to (2.13) and Minkowski's inequality, one has

$$||(\mathbf{1}_{|.|>\varepsilon}\mathbf{K}(.))\star\mu||_{L^{q_0}(\mathbb{R}^d)}\leq C(d,\alpha_0)\varepsilon^{-\frac{(q_0-1)d}{q_0}}||\Omega||_{L^{q_0}(S^{d-1})}|\mu|(\mathbb{R}^d). \tag{2.20}$$

for $\varepsilon>0$, $q_0=\frac{d}{d-\alpha_0/2}$ and $\mu\in\mathcal{M}_b(\mathbb{R}^d)$.

The following is L^p and weak type $(1,1)$ boundedness of singular integral operators associated to the kernel $\mathbf{K}$.

Proposition 5. *Let* $\mathbf{K}$ *be as in* (2.12) *with constants* $c_1,c_2>0$, $\alpha_0\in(0,1)$. *Let* $\chi\in C_c(\mathbb{R}^d,[0,1])$ *be such that* $\chi=1$ *in* $|x|>3$ *and* $\chi=0$ *in* $|x|<2$. *For* $f\in C_c^\infty(\mathbb{R}^d)$, *we define*

$$\mathbf{T}^1(f)(x)=\mathbf{K}\star f(x),\ \ \mathbf{T}^2(f)(x)=\sup_{\varepsilon>0}\left|\left(\chi(\frac{\cdot}{\varepsilon})\mathbf{K}\right)\star f(x)\right|,\ \ \mathbf{T}^3(f)(x)=\sup_{\varepsilon>0}\left|\left(\mathbf{1}_{|.|>\varepsilon}\mathbf{K}\right)\star f(x)\right|.$$

Then, $\mathbf{T}^1$ *and* $\mathbf{T}^2,\mathbf{T}^3$ *extend to bounded operator from* $L^p\to L^p(p>1)$ *and* $L^1\to L^{1,\infty}$ *with norms*

$$\sum_{j=1,2,3}||\mathbf{T}^j||_{L^p\to L^p}+||\mathbf{T}^j||_{L^1\to L^{1,\infty}}\leq C(d,p,\alpha_0)(c_1+c_2) \tag{2.21}$$

Moreover, we also get

$$\sum_{j=1,2,3}||\mathbf{T}^j||_{\mathcal{M}_b\to L^{1,\infty}}\leq C(d,\alpha_0)(c_1+c_2). \tag{2.22}$$

and for any $\mu\in\mathcal{M}_b(\mathbb{R}^d)$, *there holds*

$$\sum_{j=1,2,3}\limsup_{\lambda\to\infty}\lambda\mathcal{L}^d\left(\{|\mathbf{T}^j(\mu)|>\lambda\}\right)\leq C(d,\alpha_0)(c_1+c_2)|\mu|^s(\mathbb{R}^d). \tag{2.23}$$

Proof. First, we need to check that

$$\sup_{R>0}\int_{R<|x|<2R}|\mathbf{K}(x)|dx\leq Cc_1, \tag{2.24}$$
$$\sup_{y\neq 0}\int_{|x|\geq 2|y|}|\mathbf{K}(x-y)-\mathbf{K}(x)|\,dx\leq C(d,\alpha_0)c_1. \tag{2.25}$$

Indeed, by (2.13) one has

$$\sup_{R>0}\int_{R<|x|<2R}|\mathbf{K}(x)|dx\leq\sup_{R>0}\int_{R<|x|<2R}\frac{|\Omega(x)|}{|x|^d}dx=\log(2)\int_{S^{d-1}}|\Omega|\leq Cc_1,$$

which implies (2.24). Moreover, for any $y\neq 0$,

$$\begin{aligned}&\int_{|x|\geq 2|y|}|\mathbf{K}(x-y)-\mathbf{K}(x)|\,dx\overset{(2.16)}{\leq}C\int_{|x|\geq 2|y|}\frac{|\Omega(x)||y|}{|x|^{d+1}}dx+C\int_{|x|\geq 2|y|}\frac{1}{|x|^d}|\Omega(x-y)-\Omega(x)|\,dx\\&\leq C\int_{2|y|}^{\infty}\int_{S^{d-1}}|\Omega(\theta)|\frac{|y|}{r^2}d\mathcal{H}^{d-1}(\theta)dr+C\sum_{j=1}^{\infty}\frac{1}{(2^j|y|)^d}\int_{2^j|y|<|x|<2^{j+1}|y|}|\Omega(x-y)-\Omega(x)|\,dx\\&\leq C\int_{S^{d-1}}|\Omega(\theta)|d\mathcal{H}^{d-1}(\theta)+C\int_0^{3/4}\sup_{|h|\leq\rho}\int_{1<|x|<2}|\Omega(x-h)-\Omega(x)|dx\frac{d\rho}{\rho}\\&\overset{(2.19)}{\leq}Cc_1,\end{aligned}$$

which implies (2.25). Therefore, $\mathbf{K}$ satisfies (2.24), (2.25) and (2.15), so by [31, Theorem 5.4.1,5.4.5 and 5.3.5] we obtain (2.21).
We now prove (2.22), let $\mu\in\mathcal{M}_b(\mathbb{R}^d)$. Thanks to (2.20), one has $(\chi(./\varepsilon)\mathbf{K})\star\mu\in L^1_{loc}(\mathbb{R}^d)$ for any $\varepsilon>0$. Thus, for any $\varepsilon>0$, $\lim_{n\to\infty}(\chi(./\varepsilon)\mathbf{K})\star(\varrho_n\star\mu)=(\chi(./\varepsilon)\mathbf{K})\star\mu$ a.e in $\mathbb{R}^d$ where ϱ_n is a standard sequence of mollifiers in $\mathbb{R}^d$. This implies that $\mathbf{1}_{|\mathbf{T}^2(\mu)|>\lambda}\leq\liminf_{n\to\infty}\mathbf{1}_{|\mathbf{T}^2(\varrho_n\star\mu)|>\lambda}$ a.e in $\mathbb{R}^d$, for any $\lambda>0$. On the other hand, by (2.21),

$$\sup_{\lambda>0}\lambda\mathcal{L}^d\left(\{|\mathbf{T}^2(\varrho_n\star\mu)|>\lambda\}\right)\leq C(d,\alpha_0)(c_1+c_2)||\varrho_n\star\mu||_{L^1(\mathbb{R}^d)}\leq C(d,\alpha_0)(c_1+c_2)|\mu|(\mathbb{R}^d).$$

By applying Fatou's lemma, we find $\sup_{\lambda>0}\lambda\mathcal{L}^d\left(\{|\mathbf{T}^2(\mu)|>\lambda\}\right)\leq C(d,\alpha_0)(c_1+c_2)|\mu|(\mathbb{R}^d)$. Similarly, we also get $\sup_{\lambda>0}\lambda\mathcal{L}^d\left(\{|\mathbf{T}^3(\mu)|>\lambda\}\right)\leq C(d,\alpha_0)(c_1+c_2)|\mu|(\mathbb{R}^d)$. Hence, we conclude (2.22) since $|\mathbf{T}^1(\mu)|\leq|\mathbf{T}^3(\mu)|$. To get (2.23), one has for $R>1$ and $\gamma>1$

$$\begin{aligned}\lambda\mathcal{L}^d\left(\{|\mathbf{T}^j(\mu)|>\lambda\}\right)&\leq\lambda\mathcal{L}^d\left(\{|\mathbf{T}^j(\mu^s)|>\lambda/4\}\right)+\lambda\mathcal{L}^d\left(\{|\mathbf{T}^j(\mu^{\mathrm{a}}\mathbf{1}_{B_R^c})|>\lambda/4\}\right)\\&+\lambda\mathcal{L}^d\left(\{|\mathbf{T}^j(\mu^{\mathrm{a}}\mathbf{1}_{|\mu|^a>\gamma}\mathbf{1}_{B_R})|>\lambda/4\}\right)+\lambda\mathcal{L}^d\left(\{|\mathbf{T}(\mu^{\mathrm{a}}1_{|\mu|^a\leq\gamma}\mathbf{1}_{B_R})|>\lambda/4\}\right).\end{aligned}$$

Using the boundedness of $\mathbf{T}$ from $\mathcal{M}_b(\mathbb{R}^d)$ to $L^{1,\infty}(\mathbb{R}^d)$ for first three terms and the boundedness of $\mathbf{T}$ from $L^2(\mathbb{R}^d)$ to itself for last term yields

$$\begin{aligned}&\lambda\mathcal{L}^d\left(\{|\mathbf{T}^j(\mu)|>\lambda\}\right)\\&\quad\leq C(d,\alpha_0)(c_1+c_2)\left(|\mu|^s(\mathbb{R}^d)+\int_{B_R^c}|\mu|^a+\int_{B_R}\mathbf{1}_{|\mu|^a>\gamma}|\mu|^a+\lambda^{-1}\int_{B_R}\mathbf{1}_{|\mu|^a\leq\gamma}(|\mu|^a)^2\right).\end{aligned}$$

This implies (2.23) by letting $\lambda\to\infty$ and then $\gamma\to\infty$, $R\to\infty$. The proof is complete. □

Remark 6. Since $\mathbf{T}^j(\mathbf{1}_{B^c_{R+\varepsilon}}\mu)\in L^1(B_R)$ for any $B_R\subset\mathbb{R}^d$ and $\varepsilon>0$, so

$$\limsup_{\lambda\to\infty}\lambda\mathcal{L}^d\left(\{|\mathbf{T}^j(\mu)|>\lambda\}\cap B_R\right)\leq\limsup_{\lambda\to\infty}\lambda\mathcal{L}^d\left(\{|\mathbf{T}^j(\mathbf{1}_{B_{R+\varepsilon}}\mu)|>\lambda/2\}\right)\ \forall\,\varepsilon>0.$$

Applying (2.23) to $\mathbf{1}_{B_{R+\varepsilon}}\mu$ and then letting $\varepsilon\to 0$, we find that

$$\sum_{j=1,2,3}\limsup_{\lambda\to\infty}\lambda\mathcal{L}^d\left(\{|\mathbf{T}^j(\mu)|>\lambda\}\cap B_R\right)\leq C(d,\alpha_0)(c_1+c_2)|\mu|^s(\overline{B}_R).$$

Remark 7. It is unknown whether $\mathbf{T}^3$ is bounded from $L^1(\mathbb{R}^d)$ to $L^{1,\infty}(\mathbb{R}^d)$ where $\Omega \in W^{\alpha_0,1}(B_2\backslash B_1)$ is replaced by $\Omega \in L^q(S^{d-1})$ for $q>1$. This is an interesting open problem posed by A. Seeger.

We also have L^p and weak type $(1,1)$ boundedness of singular maximal operator:

$$\mathbf{M}^{\tilde{\Omega}}f(x)=\sup_{\rho>0}\rho^{-n}\int_{B_\rho(x)}\left|\tilde{\Omega}\left(\frac{x-y}{|x-y|}\right)\right||f(y)|dy \quad \text{with } \tilde{\Omega}\in L^1(S^{d-1}). \tag{2.26}$$

Proposition 6 ([43],[21],[22])**.** *We have for any $p>1,q>1$,*

$$||\mathbf{M}^{\tilde{\Omega}}||_{L^p\to L^p}\leq C(d,p)||\tilde{\Omega}||_{L^1(S^{d-1})},\quad ||\mathbf{M}^{\tilde{\Omega}}||_{L^1\to L^{1,\infty}}\leq C(d,q)||\tilde{\Omega}||_{L^q(S^{d-1})} \tag{2.27}$$

By a standard approximation, we obtain from (2.27) that

$$||\mathbf{M}^{\tilde{\Omega}}||_{\mathcal{M}_b\to L^{1,\infty}}\leq C(d,q)||\tilde{\Omega}||_{L^q(S^{d-1})}\quad \forall\, q>1. \tag{2.28}$$

Proposition 7. *Let $\mathbf{K}$ be as in (2.12) with constants $c_1,c_2>0$, $\alpha_0\in(0,1)$. Let $\{\phi^e\}_e\subset C^1(\mathbb{R}^d\backslash\{0\})\cap L^\infty_c(\mathbb{R}^d)$ be a family of kernels such that $\operatorname{supp}(\phi^e)\subset B_1$, $\sup_{x\in\mathbb{R}^d,e}|\phi^e(x)|+|x||\nabla\phi^e(x)|\leq c_0$. For $\alpha\in(0,d)$ and $f\in C^\infty_c(\mathbb{R}^d)$ we define*

$$\mathbf{T}(f)(x)=\sup_e\sup_{\rho>0}\left|\left(\frac{\rho^{-\alpha}}{|.|^{d-\alpha}}\phi^e(\frac{\cdot}{\rho})\right)\star\mathbf{K}\star f(x)\right|\ \forall\ x\in\mathbb{R}^d.$$

Then, $\mathbf{T}$ extends to bounded operator from $L^p(\mathbb{R}^d)\to L^p(\mathbb{R}^d)(p>1)$ and $L^1(\mathbb{R}^d)\to L^{1,\infty}(\mathbb{R}^d)$ with norms

$$||\mathbf{T}||_{L^p\to L^p}+||\mathbf{T}||_{\mathcal{M}_b\to L^{1,\infty}}\leq C(d,p,\alpha,\alpha_0)c_0(c_1+c_2) \tag{2.29}$$

Moreover, for any $\mu\in\mathcal{M}_b(\mathbb{R}^d)$

$$\limsup_{\lambda\to\infty}\lambda\mathcal{L}^d\left(\{\mathbf{T}(\mu)>\lambda\}\right)\leq C(d,\alpha,\alpha_0)c_0(c_1+c_2)|\mu|^s(\mathbb{R}^d), \tag{2.30}$$

In particular, for any $B_R\subset\mathbb{R}^d$ and $f\in L^q(B_R)$ for $q>1$,

$$\limsup_{\delta\to0}\frac{1}{|\log(\delta)|}\int_{B_R}\min\left\{\delta^{-1}|f|,\mathbf{T}(\mu)\right\}\leq C(d,\alpha,\alpha_0)c_0(c_1+c_2)|\mu|^s(\overline{B}_R). \tag{2.31}$$

Proposition 7 is still true for any $\alpha\geq d$. This was proven in [18] for smooth kernel case (i.e $\Omega\in C^1_b(S^{d-1})$).

Proof of Proposition 7. Set

$$\mathbf{T}^1(f)(x)=\sup_{\rho>0}\left|\int_{|x-z|>2\rho}\mathbf{K}(x-z)f(z)dz\right|,\quad \mathbf{T}^2_{e,\rho}(f)(x)=\sup_e\sup_{\rho>0}\left|\int_{\mathbb{R}^d}\mathbf{K}_{e,\rho}(x-z)f(z)dz\right|$$

for $f\in C^\infty_c(\mathbb{R}^d)$, where

$$\mathbf{K}_{e,\rho}(x)=\int_{\mathbb{R}^d}\frac{\rho^{-\alpha}}{|y|^{d-\alpha}}\phi^e(\frac{y}{\rho})\mathbf{K}(x-y)dy-\int_{\mathbb{R}^d}\frac{1}{|y|^{d-\alpha}}\phi^e(y)dy\mathbf{1}_{|x|>2\rho}\mathbf{K}(x).$$

Clearly,

$$|\mathbf{T}(f)|\leq C(d,\alpha)c_0\mathbf{T}^1(f)+\mathbf{T}^2(f). \tag{2.32}$$

We show that for any $x \in \mathbb{R}^d$, one has

$$|\mathbf{K}_{e,\rho}(x)| \le \frac{C(d,\alpha,\alpha_0)c_0}{|x|^{d-\alpha}} \min\left\{\frac{1}{\rho^\alpha}, \frac{\rho^{\frac{\alpha_0}{2}}}{|x|^{\frac{\alpha_0}{2}+\alpha}}\right\} \Omega_1(x/|x|), \tag{2.33}$$

where

$$\Omega_1(\theta) = c_1 + c_2 + |\Omega(\theta)| + \sup_{r\in(0,1/2)} r^{-\alpha_0/2} \fint_{B_r(0)} |\Omega(\theta - z) - \Omega(\theta)|\, dz.$$

Note that (2.19) gives $||\Omega_1||_{L^q(S^{d-1})} \le C(d,\alpha_0)(c_1+c_2)$ with $q = \frac{d}{d-\alpha_0/2}$. Then, we find that

$$\mathbf{T}^2(f) \le C(d,\alpha,\alpha_0)c_0 \mathbf{M}^{\Omega_1}(f). \tag{2.34}$$

Indeed, we need to check that for any $\rho > 0$

$$\left|\int_{\mathbb{R}^d} \mathbf{K}(x-y)\frac{1}{|y|^{d-\alpha}}\phi^e(\frac{y}{\rho})dy\right| \le \frac{C(d,\alpha)c_0\Omega_1(x/|x|)}{|x|^{d-\alpha}}. \tag{2.35}$$

Let χ be a smooth function in $\mathbb{R}^d$ such that $\chi(y) = 1$ if $|y| \le 1$ and $\chi(y) = 0$ if $|y| > 2$. One has $\sum_{j=-4}^{\infty}\left(\chi(2^j\rho^{-1}y) - \chi(2^{j+1}\rho^{-1}y)\right) = 1$ for any $y \in B_\rho$. So,

$$\left|\int_{\mathbb{R}^d} \mathbf{K}(x-y)\frac{1}{|y|^{d-\alpha}}\phi^e(\frac{y}{\rho})dy\right| \le \sum_{j=-4}^{\infty} |\mathbf{K}_j(x)|, \tag{2.36}$$

with

$$\mathbf{K}_j(x) = \int_{\mathbb{R}^d} \mathbf{K}(x-y)\frac{1}{|y|^{d-\alpha}}\phi^e(\frac{y}{\rho})\left(\chi(2^j\rho^{-1}y) - \chi(2^{j+1}\rho^{-1}y)\right)dy.$$

Let us fix $x \in \mathbb{R}^d$. Assume that $2^{-j_0}\rho < |x| \le 2^{-j_0+1}\rho$ for $j_0 \in \mathbb{Z}$. One has

$$\begin{aligned}
\sum_{j=-4}^{j_0-3} |\mathbf{K}_j(x)| &\le Cc_0 \sum_{j=-4}^{j_0-3} \int_{\mathbb{R}^d} |\Omega(x-y)| \frac{\mathbf{1}_{2^{-j-1}\rho<|y|\le 2^{-j+1}\rho}}{|x-y|^d|y|^{d-\alpha}}dy \\
&\le Cc_0 \sum_{j=-4}^{j_0-3} \int_{\mathbb{R}^d} |\Omega(z)| \frac{\mathbf{1}_{2^{-j-2}\rho<|z|\le 2^{-j+2}\rho}}{|z|^d(2^{-j}\rho)^{d-\alpha}}dz \\
&\le Cc_0||\Omega||_{L^1(S^{d-1})} \sum_{j=-4}^{j_0-3} \frac{1}{(2^{-j}\rho)^{d-\alpha}} \overset{(2.19)}{\le} \frac{Cc_0c_1}{|x|^{d-\alpha}},
\end{aligned} \tag{2.37}$$

and

$$\begin{aligned}
&\sum_{j=j_0+3}^{\infty} |\mathbf{K}_j(x)| \le Cc_0 \sum_{j=j_0+3}^{\infty} \int_{\mathbb{R}^d} |\Omega(x-y)| \frac{\mathbf{1}_{2^{-j-1}\rho<|y|\le 2^{-j+1}\rho}}{|x|^d|y|^{d-\alpha}}dy \\
&\le Cc_0 \sum_{j=j_0+3}^{\infty} \int_{\mathbb{R}^d} |\Omega(x-y) - \Omega(x)| \frac{\mathbf{1}_{2^{-j-1}\rho<|y|\le 2^{-j+1}\rho}}{|x|^d|y|^{d-\alpha}}dy + Cc_0 \sum_{j=j_0+3}^{\infty} \frac{|\Omega(x)|}{|x|^d}(2^{-j}\rho)^\alpha \\
&\le \frac{Cc_0}{|x|^{d-\alpha}} \sum_{j=j_0+3}^{\infty} \int_{\mathbb{R}^d} \left|\Omega(\frac{x}{|x|} - z) - \Omega(\frac{x}{|x|})\right| \frac{\mathbf{1}_{2^{j_0-j-2}<|z|\le 2^{j_0-j+2}}}{|z|^{d-\alpha}}dz + \frac{Cc_0|\Omega(x)|}{|x|^{d-\alpha}}s \\
&\le \frac{Cc_0}{|x|^{d-\alpha}} \sum_{j=j_0+3}^{\infty} 2^{(j_0-j)\alpha} \sup_{r\in(0,1/2)} \fint_{B_r(0)} \left|\Omega(\frac{x}{|x|} - z) - \Omega(\frac{x}{|x|})\right| dz + \frac{Cc_0|\Omega(x)|}{|x|^{d-\alpha}} \\
&\le \frac{Cc_0\Omega_1(x/|x|)}{|x|^{d-\alpha}}.
\end{aligned} \tag{2.38}$$

Next, let us fix $j=j_0-2,...,j_0+2$, thanks to Gagliardo-Nirenberg interpolation inequality, we find

$$\sup_{2^{-j_0}\rho<|y|\leq 2^{-j_0+1}\rho}|\mathbf{K}_j(y)|\leq C|x|^{1/2}||\nabla\mathbf{K}_j||_{L^{2d}(\mathbb{R}^d)}+C\frac{1}{|x|^{d/2}}\left(\int_{\mathbb{R}^d}|\mathbf{K}_j|^2dy\right)^{1/2}.$$

By Proposition 5, one obtains

$$\sup_{2^{-j_0}\rho<|y|\leq 2^{-j_0+1}\rho}|\mathbf{K}_j(y)|\leq C(c_1+c_2)|x|^{1/2}\left(\int_{\mathbb{R}^d}\left|\nabla\left(\frac{\phi^e(\frac{y}{\rho})}{|y|^{d-\alpha}}\left(\chi(2^j\rho^{-1}y)-\chi(2^{j+1}\rho^{-1}y)\right)\right)\right|^{2d}\right)^{\frac{1}{2d}}$$

$$+C(c_1+c_2)\frac{1}{|x|^{d/2}}\left(\int_{\mathbb{R}^d}\left|\frac{\phi^e(\frac{y}{\rho})}{|y|^{d-\alpha}}\left(\chi(2^j\rho^{-1}y)-\chi(2^{j+1}\rho^{-1}y)\right)\right|^2dy\right)^{1/2}$$

$$\leq Cc_0(c_1+c_2)\left(|x|^{1/2}(2^{-j_0}\rho)^{-d+\alpha-\frac{1}{2}}+\frac{1}{|x|^{d/2}}(2^{-j_0}\rho)^{-d/2+\alpha}\right)\leq\frac{Cc_0(c_1+c_2)}{|x|^{d-\alpha}}. \tag{2.39}$$

Thus, we get (2.35) from (2.36), (2.37), (2.38) and (2.39). On the other hand, for $|x|>2\rho$

$$|\mathbf{K}_{e,\rho}(x)|\leq|\int_{\mathbb{R}^d}\frac{\rho^{-\alpha}}{|y|^{d-\alpha}}\phi^e(\frac{y}{\rho})\left(\mathbf{K}(x-y)-\mathbf{K}(x)\right)dy|$$

$$\overset{(2.13),(2.16)}{\leq}Cc_0|\Omega(x)|\int_{|y|<\rho}\frac{\rho^{-\alpha}}{|y|^{d-\alpha}}\frac{|y|}{|x|^{d+1}}dy+\frac{Cc_0}{|x|^d}\int_{|y|<\rho}\frac{\rho^{-\alpha}}{|y|^{d-\alpha}}|\Omega(x-y)-\Omega(x)|dy$$

$$\leq Cc_0\left(\frac{|\Omega(x)|\rho}{|x|^{d+1}}+\frac{1}{|x|^d}\int_{|z|<\rho/|x|}\frac{(\rho/|x|)^{-\alpha}}{|z|^{d-\alpha}}\left|\Omega(\frac{x}{|x|}-z)-\Omega(\frac{x}{|x|})\right|dz\right)$$

$$\leq Cc_0\left(\frac{|\Omega(x)|\rho}{|x|^{d+1}}+\sum_{j=0}^{\infty}\frac{1}{|x|^d}\frac{(\rho/|x|)^{-\alpha}}{(2^{-j}\rho/|x|)^{d-\alpha}}\int_{2^{-j-1}\rho/|x|\leq|z|<2^{-j}\rho/|x|}\left|\Omega(\frac{x}{|x|}-z)-\Omega(\frac{x}{|x|})\right|dz\right)$$

$$\leq Cc_0\left(\frac{|\Omega(x)|\rho}{|x|^{d+1}}+\sum_{j=0}^{\infty}\frac{1}{|x|^d}\frac{(\rho/|x|)^{-\alpha}(2^{-j}\rho/|x|)^{d+\frac{\alpha_0}{2}}}{(2^{-j}\rho/|x|)^{d-\alpha}}\sup_{0<r<\rho/|x|}r^{-\frac{\alpha_0}{2}}\fint_{B_r(0)}\left|\Omega(\frac{x}{|x|}-z)-\Omega(\frac{x}{|x|})\right|dz\right)$$

$$\leq Cc_0\frac{\Omega_1(x/|x|)\rho^{\frac{\alpha_0}{2}}}{|x|^{d+\frac{\alpha_0}{2}}}. \tag{2.40}$$

Thus, from this and (2.35) we find (2.33).
Finally, it follows from (2.32) and (2.34) that $|\mathbf{T}(f)|\leq C(d,\alpha,\alpha_0)c_0\left(\mathbf{T}^1(f)+\mathbf{M}^{\Omega_1}(f)\right)$. Thanks to Proposition 5 and 6, we get $||\left(\mathbf{T}^1,\mathbf{M}^{\Omega_1}\right)||_{L^p\to L^p}+||\left(\mathbf{T}^1,\mathbf{M}^{\Omega_1}\right)||_{L^1\to L^{1,\infty}}\leq C(d,p,\alpha_0)c_0(c_1+c_2)$. This gives $||\mathbf{T}||_{L^p\to L^p}+||\mathbf{T}||_{L^1\to L^{1,\infty}}\leq C(d,p,\alpha,\alpha_0)c_0(c_1+c_2)$. By a standard approximation (see proof of proposition 5), we also obtain that $||\mathbf{T}||_{\mathcal{M}_b\to L^{1,\infty}}\leq C(d,p,\alpha,\alpha_0)c_0(c_1+c_2)$. So, we find (2.29). Then, similar to proof of (2.23) and (2.10), we obtain (2.30) and (2.31) from (2.29). The proof is complete. □

Remark 8. We denote for $\rho_0>0$, and $\alpha_1\in(0,\alpha]$ and $\mu\in\mathcal{M}_b(\mathbb{R}^d)$

$$\mathbf{T}^{\alpha_1}(\mu)(x)=\sup_e\sup_{\rho\in(0,\rho_0)}\left|\left(\frac{\rho^{\alpha_1-\alpha}}{|.|^{d-\alpha}}\phi^e(\frac{.}{\rho})\right)\star\mathbf{K}\star\mu(x)\right|\ \forall\ x\in\mathbb{R}^d.$$

Then,

$$||\mathbf{T}^{\alpha_1}(\mu)||_{L^{q_0}(\mathbb{R}^d)}\leq C(\rho_0)(c_1+c_2)|\mu|(\mathbb{R}^d),\quad q_0=\frac{d}{d-\frac{1}{4}\min\{\alpha,\alpha_0,\alpha_1\}}>1. \tag{2.41}$$

Indeed, we deduce from (2.40) and (2.35) that $|\left(\frac{\rho^{-\alpha}}{|.|^{d-\alpha}}\phi^e(\frac{\cdot}{\rho})\right)\star\mathbf{K}(x)| \leq C\left(\frac{\rho^{-\alpha}}{|x|^{d-\alpha}}\wedge\frac{1}{|x|^d}\right)\Omega_1(\frac{x}{|x|})$ for any $x\in\mathbb{R}^d$. Thus, $\mathbf{T}^{\alpha_1}(\mu)(x) \leq C(\rho_0)P\star|\mu|(x)$, $P(x) = \left(\frac{1}{|x|^{d-\beta_0}}\wedge\frac{1}{|x|^d}\right)\Omega_1(\frac{x}{|x|})$. Then, by Minkowski's inequality, one has $||\mathbf{T}^{\alpha_1}(\mu)||_{L^{q_0}(\mathbb{R}^d)} \leq C(\rho_0)||\Omega_1||_{L^{q_0}(S^{d-1})}|\mu|(\mathbb{R}^d)$ which implies (2.41).

Remark 9. As Remark (8), we also show that for $\rho_0>0$,

$$\mathbf{P}(\mu)(x) = \sup_e \sup_{\rho\in(0,\rho_0)} |\left(\frac{\rho^{-\alpha}}{|.|^{d-\alpha}}\phi^e(\frac{\cdot}{\rho})\right)\star\mathbf{K}\star((\psi(.)-\psi(x))\mu)(x)| \in L^{q_0}_{loc}(\mathbb{R}^d) \tag{2.42}$$

for some $q_0>1$, with $\psi\in W^{1,\infty}(\mathbb{R}^d)$, exactly

$$||\mathbf{P}(\mu)||_{L^{q_0}(B_R(0))} \leq C(R)||\psi||_{W^{1,\infty}(\mathbb{R}^d)}|\mu|(\mathbb{R}^d) \quad \forall\, R>0. \tag{2.43}$$

Furthermore, if $\Omega\in C^1_b(S^{d-1})$ then $\mathbf{P}(\mu)(x)\leq C\mathbf{I}_1(\mu)(x)$.

Remark 10. If $\mu_t(x)=\mu(t,x)\in L^1([0,T],\mathcal{M}(\mathbb{R}^d))$ and $f\in L^1((0,T),L^q(B_R))$ for $q>1$, it follows from (2.31) and Dominated convergence theorem that

$$\limsup_{\delta\to0}\frac{1}{|\log(\delta)|}\int_0^T\int_{B_R}\min\left\{\delta^{-1}|f(x,t)|,\mathbf{T}(\mu_t)(x)\right\}dxdt \leq C(d,\alpha,\alpha_0,c_0)\int_0^T|\mu_t|^s(\overline{B}_R)dt. \tag{2.44}$$

Remark 11. We do not know how to prove Theorem 7 when $\alpha_0=0$.

3 Kakeya singular integral operators

This section we introduce the Kakeya singular integral operators integral operators and establish a strong version of (2.30) for this operator. It is a main tool of this paper.

Assume that $\{\phi^{e,\varepsilon}\}_{\varepsilon,e}\subset C^1(\mathbb{R}^d\backslash\{0\},\mathbb{R}^d)\cap L^\infty_c(\mathbb{R}^d,\mathbb{R}^d)$ is a family of kernels such that

$$\operatorname{supp}(\phi^{e,\varepsilon})\subset B_1(0)\cap\left\{x:\left|e-\frac{x}{|x|}\right|\leq\varepsilon\right\},\quad \textit{and}\quad |\phi^{e,\varepsilon}(x)|+\varepsilon|x||\nabla\phi^{e,\varepsilon}(x)|\leq c_0 \tag{3.1}$$

for any $x\in\mathbb{R}^d,\varepsilon\in(0,1/10)$, $e\in S^{d-1}$. Let $\mathbf{K}$ be as in (2.12) with constants $c_1,c_2>0$, $\alpha_0\in(0,1)$. Assume that there exist a sequence of $\Omega_n\in C^2_b(S^{d-1})$ such that $\Omega_n(\theta)=\Omega_n(r\theta)$ for any $r>0,\theta\in S^{d-1}$ and

$$||\Omega_n||_{W^{\alpha_0,1}(B_2\backslash B_1)}\leq 2c_1,\quad \lim_{n\to\infty}||\Omega_n-\Omega||_{W^{\alpha_0,1}(B_2\backslash B_1)}=0 \tag{3.2}$$

and $\mathbf{K}_n(x):=\Omega_n(x)K(x)$ satisfies (2.15) i.e

$$\sup_{0<R_1<R_2<0}\left|\int_{R_1<|x|<R_2}\mathbf{K}_n(x)dx\right|\leq c_3 \tag{3.3}$$

for some $c_3>0$, moreover,

$$\lim_{n\to\infty}\sup_{0<R_1<R_2<0}\left|\int_{R_1<|x|<R_2}(\mathbf{K}_n(x)-\mathbf{K}(x))\,dx\right|=0. \tag{3.4}$$

For any $\mu \in C_c^\infty(\mathbb{R}^d,\mathbb{R}^d)$ and $\rho_0>0$, the Kakeya singular integral operator $\mathbf{T}_\varepsilon$ is given by

$$\mathbf{T}_\varepsilon(\mu)(x) := \mathbf{T}_\varepsilon^{\mathbf{K}}(\mu)(x) = \sup_{\rho\in(0,\rho_0),e\in S^{d-1}} \frac{\varepsilon^{-d+1}}{\rho^\alpha}\left|\left(\frac{1}{|.|^{d-\alpha}}\phi_\rho^{e,\varepsilon}(.)\right)\star\mathbf{K}\star\mu(x)\right| \quad \forall \ x\in\mathbb{R}^d, \quad (3.5)$$

for some $\alpha\in(0,d)$, where $\phi_\rho^{e,\varepsilon}(.) = \phi^{e,\varepsilon}(\frac{.}{\rho})$. Set

$$\mathbf{T}_\varepsilon^{1,n} := \mathbf{T}_\varepsilon^{\mathbf{K}_n}, \quad \mathbf{T}_\varepsilon^{2,n} := \mathbf{T}_\varepsilon^{\mathbf{K}_n-\mathbf{K}}.$$

Thanks to Proposition 7 and conditions (3.2), (3.3), (3.4), there exists a constant C depending on $d,\alpha,\alpha_0,p,\rho_0,c_0$ such that for any $\mu\in\mathcal{M}_b(\mathbb{R}^d)$

$$\limsup_{\lambda\to\infty}\lambda\mathcal{L}^d\left(\{\mathbf{T}_\varepsilon(\mu)>\lambda\}\right)+\lambda\mathcal{L}^d\left(\{\mathbf{T}_\varepsilon^{1,n}(\mu)>\lambda\}\right)\le C\varepsilon^{-d+1}(c_1+c_2)|\mu|^s(\mathbb{R}^d), \quad (3.6)$$

$$\limsup_{\lambda\to\infty}\lambda\mathcal{L}^d\left(\{\mathbf{T}_\varepsilon^{2,n}(\mu)>\lambda\}\right)\le C\varepsilon^{-d+1}c_n|\mu|^s(\mathbb{R}^d), \quad (3.7)$$

for any $\varepsilon\in(0,1/10),\forall n$, where $\lim_{n\to\infty}c_n=0$.

Remark 12. Remark that if $K(x)=|x|^{-d}$, Ω_n is given by (2.17) in Remark 3 satisfies (3.2), (3.3) and (3.4).

Remark 13. if $\mathbf{K}=\sum_{i=1}^d\mathcal{R}_j^2=\delta_0$ where $\mathcal{R}_1,...,\mathcal{R}_d$ are the Riesz transforms in $\mathbb{R}^d$, we thus get $\mathbf{T}_\varepsilon(\mu)\le C(d,c_0)\mathbf{M}^\varepsilon(\mu)$, where $\mathbf{M}^\varepsilon$ is the Kakeya maximal function in $\mathbb{R}^d$ i.e

$$\mathbf{M}^\varepsilon(\mu)(x) = \sup_{\rho>0,e\in S^{d-1}} \fint_{B_\rho(x)}\varepsilon^{-d+1}\mathbf{1}_{|\frac{z-x}{|z-x|}-e|\le\varepsilon}d|\mu|(z), \quad \forall \ x\in\mathbb{R}^d.$$

Our main result is the following:

Theorem 1. *Assume that $\mu=Df$, $f\in BV(\mathbb{R}^d)$. Then, we have*

$$\limsup_{\lambda\to\infty}\lambda\mathcal{L}^d\left(\{\mathbf{T}_\varepsilon(\mu)>\lambda\}\right)\le C|\log(\varepsilon)||\mu|^s(\mathbb{R}^d) \quad (3.8)$$

for any $\varepsilon\in(0,1/10)$, where C depends on $d,\alpha,\alpha_0,c_0,c_1,c_2,c_3$. In particular, for any $B_R\subset\mathbb{R}^d$ and $f\in L^q(B_R)$ for $q>1$,

$$\limsup_{\delta\to0}\frac{1}{|\log(\delta)|}\int_{B_R}\min\left\{\delta^{-1}|f|,\mathbf{T}_\varepsilon(\mu)\right\}dx\le C|\log(\varepsilon)||\mu|^s(\overline{B}_R) \quad (3.9)$$

for any $\varepsilon\in(0,1/10)$.

Remark 14. Estimate (3.8) is not true for all $\mu\in\mathcal{M}_b(\mathbb{R}^d,\mathbb{R}^d)$. Indeed, let $d\mu=d\delta_{\{0\}}$ and $|\phi^{e,\varepsilon}(e)|\ge1$ for any $e\in S^{d-1}$ and $\varepsilon>0$, let $\mathbf{T}_{j,\varepsilon}$ be $\mathbf{T}_\varepsilon$ associated to $\mathbf{K}(x)=\mathbf{K}_j(x)=\frac{|x|^2-x_j^2d}{|x|^{d+2}}$. One has $\sum_{j=1}^d\mathbf{T}_{j,\varepsilon}(\mu)(x)\ge C\frac{\varepsilon^{-d+1}}{|x|^d}|\phi^{x/|x|,\varepsilon}(x/|x|)|\ge C\frac{\varepsilon^{-d+1}}{|x|^d}$. Thus, for any $\lambda>1$

$$\lambda\mathcal{L}^d\left(\{\mathbf{T}_{1,\varepsilon}(\mu)>\lambda\}\right)\ge d^{-1}\mathcal{L}^d(\{\sum_{j=1}^d\mathbf{T}_{j,\varepsilon}(\mu)>d\lambda\})\ge C\varepsilon^{-d+1}.$$

As we discussed in Remark 9, $\{D_{x_1}f\in\mathcal{M}_b(\mathbb{R}^d):f\in BV(\mathbb{R}^d)\}$ is not dense in $L^1(\mathbb{R}^{d-1},\mathcal{M}_{b,x_1}(\mathbb{R}))$; so, a natural question is that *whether* (3.8) holds for any $\mu\in L^1(\mathbb{R}^{d-1},\mathcal{M}_b(\mathbb{R}))$.

To prove Theorem 1, we need the following lemmas:

Lemma 2. *Let $\omega \in \mathcal{M}_b^+(\mathbb{R})$ and $a : \mathbb{R}^d \to \mathbb{R}^+$ be a Borel function. Then, for any $\rho > 0$,*

$$\int_{\mathbb{R}^{d-1}} \int_{\mathbb{R}} 1_{\rho<|y_1+y_2|\leq 2\rho} a(y_1+y_2) d\omega(y_1) d\mathcal{H}^{d-1}(y_2) \leq 4(2\rho)^d \left[\int_{S^{d-1}} \sup_{r\in[\rho,2\rho]} a(r\theta) d\mathcal{H}^{d-1}(\theta)\right] \mathbf{M}^1(\omega,\mathbb{R})(0). \tag{3.10}$$

Proof of Lemma 2. Let $d\omega_\kappa(y) = \mathbf{1}_{|y|>\kappa} d\omega(y)$ for $\kappa \in (0, \rho/100)$. Let ϱ_m be a standard sequence of mollifiers in $\mathbb{R}$. For any $m > 4/\kappa$, we have $\operatorname{supp}(\varrho_m \star \omega_\kappa) \subset \{z : |z| > \kappa/2\}$ and

$$\begin{aligned}
&\int_{\mathbb{R}^{d-1}} \int_{\mathbb{R}} \mathbf{1}_{\rho<|y_1+y_2|\leq 2\rho} a(y_1+y_2)(\varrho_m \star \omega_\kappa)(y_1) d\mathcal{H}^1(y_1) d\mathcal{H}^{d-1}(y_2) \\
&= \int_{S^{d-1}} \int_\rho^{2\rho} r^{d-1} a(r\theta) \mathbf{1}_{|r\theta_1|>\kappa/2} (\varrho_m \star \omega_\kappa)(r\theta_1) dr d\mathcal{H}^{d-1}(\theta) \\
&\leq (2\rho)^{d-1} \int_{S^{d-1}} \left(\sup_{r'\in[\rho,2\rho]} a(r'\theta)\right) \int_\rho^{2\rho} \mathbf{1}_{|r\theta_1|>\kappa/2} (\varrho_m \star \omega_\kappa)(r\theta_1) dr d\mathcal{H}^{d-1}(\theta).
\end{aligned}$$

On the other hand,

$$\begin{aligned}
&\int_\rho^{2\rho} \mathbf{1}_{|r\theta_1|>\kappa/2} (\varrho_m \star \omega_\kappa)(r\theta_1) dr \leq \int_{\mathbb{R}} \int_\rho^{2\rho} \mathbf{1}_{|r\theta_1|>\kappa/2} \varrho_m(r\theta_1 - z) dr d\omega(z) \\
&\leq \frac{\mathbf{1}_{|\theta_1|>\frac{\kappa}{4\rho}}}{|\theta_1|} \int_{\mathbb{R}} \int_{-2|\theta_1|\rho}^{2|\theta_1|\rho} \varrho_m(r-z) dr d\omega(z) \leq \frac{\mathbf{1}_{|\theta_1|>\frac{\kappa}{4\rho}}}{|\theta_1|} \int_{\mathbb{R}} \mathbf{1}_{|z|<2|\theta_1|\rho+\frac{2}{m}} d\omega(z) \\
&\leq \frac{\mathbf{1}_{|\theta_1|>\frac{\kappa}{4\rho}}}{|\theta_1|} \int_{-4|\theta_1|\rho}^{4|\theta_1|\rho} d\omega(z) \leq 8\rho \mathbf{M}^1(\omega,\mathbb{R})(0).
\end{aligned}$$

Thus, by Fatou's Lemma, letting $m \to \infty$ and then $\kappa \to \infty$ we get (3.10). The proof is complete. □

Remark 15. From proof of Lemma 2 we can see that for any $e_0 \in S^{d-1}$ and $\mu \in \mathcal{M}_b^+(\mathbb{R}^d)$ and $\omega \in \mathcal{M}_b^+(\tilde{H}_{e_0})$ if $\mu \leq \omega \otimes \mathcal{H}^{d-1}$ then

$$\mathbf{M}^\varepsilon(\mu)(x) \leq C(d) \mathbf{M}^1(\omega, \tilde{H}_{e_0})(\langle e_0, x\rangle e_0) \quad \forall\, x \in \mathbb{R}^d, \varepsilon > 0. \tag{3.11}$$

Lemma 3. *Let $\{e_1, ..., e_d\}$ be an orthonormal basis in $\mathbb{R}^d$. For any $x_i \in \tilde{H}_{e_i}$, $i = 1, ..., d$, we denote $\nu^1_{k,\sum_{i=d-k+1}^d x_i}, \nu^2_{k,\sum_{i=d-k+1}^d x_i} \in \mathcal{M}^+(\bigotimes_{i=1}^{d-k} \tilde{H}_{e_i})$ by*

$$\begin{aligned}
d\nu^1_{k,\sum_{i=d-k+1}^d x_i}(y_{d-k}, ..., y_1) &= d|Df^{e_{d-k}}_{\sum_{i=1}^{d-k-1} y_i + \sum_{i=d-k+1}^d x_i}|(y_{d-k}) d\mathcal{H}^1(y_{d-k-1})...d\mathcal{H}^1(y_1), \\
d\nu^2_{k,\sum_{i=d-k+1}^d x_i}(y_{d-k}, ..., y_1) &= \mathbf{1}_{|\sum_{i=1}^{d-k} y_{0i} - \sum_{i=1}^{d-k} y_i| \leq 2\varepsilon} d\nu^1_{k,\sum_{i=d-k+1}^d x_i}(y_{d-k}, ..., y_1).
\end{aligned}$$

Then, for any $y_{0i}, x_i \in \tilde{H}_{e_i}$ $i = 1, ..., d$

$$
\begin{aligned}
M :=& \int_{\tilde{H}_{e_1}} ... \int_{\tilde{H}_{e_d}} 1 \wedge \left(\frac{\rho}{|\sum_{i=1}^d (x_i - y_i)|} \right)^{d+2} \mathbf{1}_{|\sum_{i=1}^d (y_{0i} - y_i)| \leq \varepsilon} \\
& \times \left| f(\sum_{i=1}^d y_i) - f(y_1 + \sum_{i=2}^d x_i) \right| d\mathcal{H}^1(y_d) ... d\mathcal{H}^1(y_1) \\
\leq& \sum_{k=0}^{d-2} \frac{C(d)\rho^{d+\frac{5}{4}}}{\varepsilon} \mathbf{I}_{\frac{3}{4}}^{d-k} (\nu^1_{k, \sum_{i=d-k+1}^d x_i}, \bigotimes_{i=1}^{d-k} \tilde{H}_{e_i})(\sum_{i=1}^{d-k} x_i) \\
&+ \sum_{k=0}^{d-2} C(d) \rho^{d+1} \mathbf{1}_{|\sum_{i=1}^d (y_{0i} - x_i)| \leq 2\varepsilon} \mathbf{M}^{d-k} (\nu^2_{k, \sum_{i=d-k+1}^d x_i}, \bigotimes_{i=1}^{d-k} \tilde{H}_{e_i})(\sum_{i=1}^{d-k} x_i),
\end{aligned}
$$

and

$$
\begin{aligned}
& \int_{\tilde{H}_{e_1}} ... \int_{\tilde{H}_{e_d}} 1 \wedge \left(\frac{\rho}{|\sum_{i=1}^d (x_i - y_i)|} \right)^{d+1} \left| f(\sum_{i=1}^d y_i) - f(y_1 + \sum_{i=2}^d x_i) \right| d\mathcal{H}^1(y_d) ... d\mathcal{H}^1(y_1) \\
& \leq \sum_{k=0}^{d-2} C(d) \rho^{d+\frac{1}{4}} \mathbf{I}_{\frac{3}{4}}^{d-k} (\nu^1_{k, \sum_{i=d-k+1}^d x_i}, \bigotimes_{i=1}^{d-k} \tilde{H}_{e_i})(\sum_{i=1}^{d-k} x_i).
\end{aligned}
$$

We will prove Lemma (3) in Appendix section. Now, we are ready to prove Theorem 1.

Proof of Theorem 1. **Step 1:** It is enough to show that

$$
\limsup_{\lambda \to \infty} \lambda \mathcal{L}^d \left(\left\{ \mathbf{T}_\varepsilon^{1,n}(\mu) > \lambda \right\} \cap B_R \right) \leq C |\log(\varepsilon)| |\mu|^s(\mathbb{R}^d), \tag{3.12}
$$

for any $B_R \subset \mathbb{R}^d$, $\varepsilon \in (0, 1/10)$ and $n \in \mathbb{N}$. We now assume that (3.12) is proven. Let $\chi \in C_c^\infty(\mathbb{R}^d)$ be such that $\chi = 1$ in $B_{R/4}$ and $\chi = 0$ in $B^c_{R/2}$. Thanks to (3.12) and using the fact that $\mathbf{T}_\varepsilon^{1,n}(D(\chi f)) \in L^\infty(B_R^c)$, one gets

$$
\begin{aligned}
& \limsup_{\lambda \to \infty} \lambda \mathcal{L}^d \left(\left\{ \mathbf{T}_\varepsilon^{1,n}(\mu) > \lambda \right\} \right) \\
& \quad \leq \limsup_{\lambda \to \infty} \lambda \mathcal{L}^d \left(\left\{ \mathbf{T}_\varepsilon^{1,n}(\mu) > \lambda \right\} \cap B_R \right) + \limsup_{\lambda \to \infty} \lambda \mathcal{L}^d \left(\left\{ \mathbf{T}_\varepsilon^{1,n}(\mu) > \lambda \right\} \cap B_R^c \right) \\
& \quad \leq C |\log(\varepsilon)| |\mu|^s(\mathbb{R}^d) + \limsup_{\lambda \to \infty} \lambda \mathcal{L}^d \left(\left\{ \mathbf{T}_\varepsilon^{1,n}(D((1-\chi)f)) > \lambda/2 \right\} \right).
\end{aligned}
$$

So, by (3.6), (3.7) and using the fact that $\mathbf{T}_\varepsilon(\mu) \leq \mathbf{T}_\varepsilon^{1,n}(\mu) + \mathbf{T}_\varepsilon^{2,n}(\mu)$, we have

$$
\begin{aligned}
\limsup_{\lambda \to \infty} \lambda \mathcal{L}^d \left(\{ \mathbf{T}_\varepsilon(\mu) > \lambda \} \right) &\leq C |\log(\varepsilon)| |\mu|^s(\mathbb{R}^d) + C(\varepsilon) |D^s((1-\chi)f)|(\mathbb{R}^d) + C(\varepsilon) c_n |\mu|^s(\mathbb{R}^d) \\
&\leq C |\log(\varepsilon)| |\mu|^s(\mathbb{R}^d) + C(\varepsilon) |\mu|^s(B^c_{R/4}) + C(\varepsilon) c_n |\mu|^s(\mathbb{R}^d).
\end{aligned}
$$

This implies (3.8) by letting $R \to \infty$, $n \to \infty$. Moreover, as proof of Lemma 1 we also get (3.9). We are going to prove (3.12) in several steps.

Step 2. Let $\eta : \mathbb{R}^d \to S^{d-1}$ be such that $\eta(x) = (1, ..., 0) \in S^{d-1}$ if $x \notin \mathrm{supp}(\mu^s)$ and $\eta(x) = \frac{d\mu^s(x)}{d|\mu|^s(x)}$ if $x \in \mathrm{supp}(\mu^s)$. Let $\eta^\kappa : \mathbb{R}^d \to S^{d-1}$ be smooth functions such that $\eta^\kappa \to \eta$ $|\mu|^s-$a.e in $\mathbb{R}^d$ and $\lim_{\kappa \to 0} \int_{\mathbb{R}^d} |\eta^\kappa - \eta| d|\mu|^s = 0$. Let $\varphi_r \in C_b^\infty(\mathbb{R}^d)$ be such that $\varphi_r(z) = 1$ if $|z| > 2r$ and $\varphi_r(z) = 0$ if $|z| \leq r$ and $||\nabla \varphi_r||_{L^\infty(\mathbb{R}^d)} \leq C r^{-1}$.

Let us define $S_\tau = \{y \in 2\tau\mathbb{Z}^d : y \in B_{R+4\rho_0}\}$, for $\tau \in (0, \rho_0/100)$. There exists a sequence of smooth functions $\{\chi^\tau_{y_\tau}\}_{y_\tau \in S_\tau}$ such that $0 \le \chi^\tau_{y_\tau}(y) \le 1$, $\sum_{y_\tau \in S_\tau} \chi^\tau_{y_\tau}(y) = 1 \ \ \forall y \in B_{R+4\rho_0}$ and $\chi^\tau_{y_\tau} = 1$ in $B_\tau(y_\tau)$, $\mathrm{supp}(\chi^\tau_{y_\tau}) \subset B_{2\tau}(y_\tau)$, $|\nabla \chi^\tau_{y_\tau}(y)| \le C\tau^{-1} \ \forall \ y \in \mathbb{R}^d$.
Note that $\mathrm{Card}(S_\tau) \sim \left(\frac{R+\rho_0}{\tau}\right)^d$, $B_\tau(y_\tau) \cap B_\tau(y'_\tau) = \emptyset$ for $y_\tau, y'_\tau \in S_\tau$, $y_\tau \ne y'_\tau$; and

$$\sum_{y_\tau \in S_\tau} \mathbf{1}_{B_{100\tau}(y_\tau)}(y) \le C(d) 1_{B_{R+6\rho_0}}(y) \ \ \forall \ \ y \in \mathbb{R}^d. \tag{3.13}$$

Set $\chi_0 = \sum_{y_\tau \in S_\tau} \chi^\tau_{y_\tau}$. For any $y_\tau \in S_\tau$, we denote

$$\eta^\kappa_{y_\tau} = \eta^\kappa(y_\tau).$$

Because of $\mu^s = \eta\langle \eta, \mu^s\rangle$, one has

$$\mu = (1-\chi_0)\mu + \chi_0\mu^a + \chi_0(\eta - \eta^\kappa)\langle \eta, \mu^s\rangle + \chi_0\eta^\kappa\langle (\eta - \eta^\kappa), \mu^s\rangle + \chi_0\eta^\kappa\langle \eta^\kappa, \mu^s\rangle;$$

and

$$\chi_0\eta^\kappa\langle \eta^\kappa, \mu^s\rangle = (\sum_{y_\tau \in S_\tau} \chi^\tau_{y_\tau})\eta^\kappa\langle \eta^\kappa, \mu^s\rangle = \sum_{y_\tau \in S_\tau} \chi^\tau_{y_\tau}\eta^\kappa\langle (\eta^\kappa - \eta^\kappa_{y_\tau}), \mu^s\rangle$$
$$+ \sum_{y_\tau \in S_\tau} \chi^\tau_{y_\tau}(\eta^\kappa - \eta^\kappa_{y_\tau})\langle \eta^\kappa_{y_\tau}, \mu^s\rangle + \sum_{y_\tau \in S_\tau} \chi^\tau_{y_\tau}\eta^\kappa_{y_\tau}\langle \eta^\kappa_{y_\tau}, \mu\rangle - \sum_{y_\tau \in S_\tau} \chi^\tau_{y_\tau}\eta^\kappa_{y_\tau}\langle \eta^\kappa_{y_\tau}, \mu^a\rangle.$$

Hence, with $\tilde{\mathbf{K}}^n = \frac{\varepsilon^{-d+1}}{\rho^\alpha}\left(\frac{1}{|.|^{d-\alpha}}\phi^{e,\varepsilon}_\rho(.)\right) \star \mathbf{K}_n$ and $\zeta \in (0, 1/10)$ we write

$$\begin{aligned}
\tilde{\mathbf{K}}^n \star \mu &= \tilde{\mathbf{K}}^n \star ((1-\chi_0)\mu) + \tilde{\mathbf{K}}^n \star (\chi_0\mu^a) + \tilde{\mathbf{K}}^n \star (\chi_0(\eta-\eta^\kappa)\langle \eta, \mu^s\rangle) + \tilde{\mathbf{K}}^n \star (\chi_0\eta^\kappa\langle (\eta-\eta^\kappa), \mu^s\rangle) \\
&+ \sum_{y_\tau \in S_\tau} \tilde{\mathbf{K}}^n \star (\chi^\tau_{y_\tau}\eta^\kappa\langle (\eta^\kappa - \eta^\kappa_{y_\tau}), \mu^s\rangle) + \sum_{y_\tau \in S_\tau} \tilde{\mathbf{K}}^n \star (\chi^\tau_{y_\tau}(\eta^\kappa - \eta^\kappa_{y_\tau})\langle \eta^\kappa_{y_\tau}, \mu^s\rangle) \\
&- \sum_{y_\tau \in S_\tau} \tilde{\mathbf{K}}^n \star (\chi^\tau_{y_\tau}\eta^\kappa_{y_\tau}\langle \eta^\kappa_{y_\tau}, \mu^a\rangle) + \sum_{y_\tau \in S_\tau} \frac{\varepsilon^{-d+1}}{\rho^\alpha}\left(\frac{(1-\varphi_{\zeta\rho})}{|.|^{d-\alpha}}\phi^{e,\varepsilon}_\rho(.)\right) \star \mathbf{K}_n \star (\chi^\tau_{y_\tau}\eta^\kappa_{y_\tau}\langle \eta^\kappa_{y_\tau}, \mu\rangle) \\
&+ \sum_{y_\tau \in S_\tau} \frac{\varepsilon^{-d+1}}{\rho^\alpha}\left(\frac{\varphi_{\zeta\rho}}{|.|^{d-\alpha}}\phi^{e,\varepsilon}_\rho(.)\right) \star \mathbf{K}_n \star (\chi^\tau_{y_\tau}\eta^\kappa_{y_\tau}\langle \eta^\kappa_{y_\tau}, \mu\rangle) := \sum_{i=1}^9 I^{e,\rho}_{i,\varepsilon}.
\end{aligned}$$

Step 3: In this prove, we denote $A_i(\lambda, \varepsilon) = \lambda\mathcal{L}^d\left(\left\{\sup_{\rho \in (0,\rho_0), e \in S^{d-1}} |I^{e,\rho}_{i,\varepsilon}| > \lambda\right\} \cap B_R\right)$.
Thus, for $\lambda > 1$,

$$\lambda\mathcal{L}^d\left(\left\{\mathbf{T}^{1,n}_\varepsilon(\mu) > \lambda\right\} \cap B_R\right) \le 9\sum_{i=1}^9 A_i(\lambda/9, \varepsilon). \tag{3.14}$$

Thanks to (3.6) we have

$$\limsup_{\lambda\to\infty} \sum_{i=2,7} A_i(\lambda, \varepsilon) = 0, \tag{3.15}$$

$$\limsup_{\lambda\to\infty} \sum_{i=3,4} A_i(\lambda, \varepsilon) \le C(\varepsilon)|||\eta - \eta^\kappa||\mu^s|||_{\mathcal{M}(\mathbb{R}^d)}, \tag{3.16}$$

$$\limsup_{\lambda\to\infty} \sum_{i=5,6} A_i(\lambda, \varepsilon) \le C(\varepsilon)||\sum_{y_\tau \in S_\tau} \chi^\tau_{y_\tau}|\eta^\kappa - \eta^\kappa_{y_\tau}||\mu^s|||_{\mathcal{M}(\mathbb{R}^d)} \le C(\varepsilon, \kappa)\tau|\mu|^s(\mathbb{R}^d). \tag{3.17}$$

Here in the last inequality we have used the fact that

$$\sum_{y_\tau \in S_\tau} \chi^\tau_{y_\tau}(x)|\eta^\kappa(x) - \eta^\kappa_{y_\tau}| \leq C(d)||\nabla \eta^\kappa||_{L^\infty(\mathbb{R}^d)} \sum_{y_\tau \in S_\tau} \mathbf{1}_{B_{2\tau}(y_\tau)}(x)|x - y_\tau|$$
$$\overset{(3.13)}{\leq} C(d)||\nabla \eta^\kappa||_{L^\infty(\mathbb{R}^d)} \tau \mathbf{1}_{B_{R+6\rho_0}}(x) \quad \forall x \in \mathbb{R}^d.$$

Again, applying (3.6) (where ρ is replaced by $\zeta\rho$) yields

$$\limsup_{\lambda\to\infty} A_8(\lambda, \varepsilon) \leq C(\varepsilon)\zeta^\alpha || \sum_{y_\tau \in S_\tau} \chi^\tau_{y_\tau} |\mu^s| |||_{\mathcal{M}(\mathbb{R}^d)} \leq C(\varepsilon)\zeta^\alpha |\mu|^s(\mathbb{R}^d). \tag{3.18}$$

On the other hand, it is easy to see that $\sup_{\rho\in(0,\rho_0), e\in S^{d-1}} |I^{e,\rho}_{1,\varepsilon}(.)| \in L^\infty(B_R)$, so

$$\limsup_{\lambda\to\infty} A_1(\lambda, \varepsilon) = 0. \tag{3.19}$$

Therefore, we deduce from (3.14) and (3.15)-(3.19) that

$$\limsup_{\lambda\to\infty} \lambda \mathcal{L}^d \left(\left\{\mathbf{T}^{1,n}_\varepsilon(\mu) > \lambda\right\} \cap B_R\right) \leq C(n, \varepsilon)|||\eta - \eta^\kappa||\mu|^s||_{\mathcal{M}(\mathbb{R}^d)}$$
$$+ C(n, \varepsilon, \kappa)\tau|\mu|^s(\mathbb{R}^d) + C(n, \varepsilon)\zeta^\alpha |\mu|^s(\mathbb{R}^d) + 9 \limsup_{\lambda\to\infty} A_9(\lambda, \varepsilon). \tag{3.20}$$

In next steps, we will deal with $A_9(\lambda, \varepsilon)$.

Step 4: One has

$$I^{e,\rho}_{9,\varepsilon}(x) = \sum_{y_\tau \in S_\tau} \frac{\varepsilon^{-d+1}}{\rho^\alpha} \int_{\mathbb{R}^d} \left[\int_{\mathbb{R}^d} \mathbf{K}_n(z)\varphi_{\zeta\rho}(x - y - z) \frac{\langle \phi^{e,\varepsilon}_\rho(x - y - z), \eta^\kappa_{y_\tau}\rangle}{|x - y - z|^{d-\alpha}} dz\right] \chi^\tau_{y_\tau}(y) d\langle \eta^\kappa_{y_\tau}, \mu(y)\rangle$$
$$= \sum_{y_\tau \in S_\tau} \int_{\mathbb{R}^d} \mathbf{K}^{e,\rho}_{\varepsilon,n}(x - y)\chi^\tau_{y_\tau}(y) d\langle \eta^\kappa_{y_\tau}, \mu(y)\rangle + \mathbf{c}(\varepsilon, \kappa, \tau, \zeta) \sum_{y_\tau \in S_\tau} (\varphi_\rho \mathbf{K}_n) \star (\chi^\tau_{y_\tau}\langle \eta^\kappa_{y_\tau}, \mu\rangle)(x)$$
$$= I^{e,\rho}_{10,\varepsilon}(x) + I^{e,\rho}_{11,\varepsilon}(x),$$

where

$$\mathbf{K}^{e,\rho}_{\varepsilon,n}(z') = \frac{\varepsilon^{-d+1}}{\rho^\alpha} \int_{\mathbb{R}^d} \mathbf{K}_n(z)\varphi_{\zeta\rho}(z' - z) \frac{\langle \phi^{e,\varepsilon}_\rho(z' - z), \eta^\kappa_{y_\tau}\rangle}{|z' - z|^{d-\alpha}} dz - \mathbf{c}(\varepsilon, \kappa, \tau, \zeta)\varphi_\rho(z')\mathbf{K}_n(z') \quad \forall \ z' \in \mathbb{R}^d, \tag{3.21}$$

and

$$\mathbf{c}(\varepsilon, \kappa, \tau, \zeta) = \frac{\varepsilon^{-d+1}}{\rho^\alpha} \int_{\mathbb{R}^d} \varphi_{\zeta\rho}(z' - z) \frac{\langle \phi^{e,\varepsilon}((z' - z)/\rho), \eta^\kappa_{y_\tau}\rangle}{|z' - z|^{d-\alpha}} dz = \varepsilon^{-d+1} \int_{\mathbb{R}^d} \varphi_\zeta(z) \frac{\langle \phi^{e,\varepsilon}(z), \eta^\kappa_{y_\tau}\rangle}{|z|^{d-\alpha}} dz. \tag{3.22}$$

Note that $|\mathbf{c}(\varepsilon, \kappa, \tau, \zeta)| \leq C$ for all $\kappa, \varepsilon, \zeta > 0, e \in S^{d-1}$ and by (2.33) in the Proof of Proposition 7, we have $|\mathbf{K}^{e,\rho}_{\varepsilon,n}(x)| \leq C(n, \varepsilon, \zeta)\frac{1}{|x|^{d-\alpha}} \min\left\{\frac{1}{\rho^\alpha}, \frac{\rho}{|x|^{1+\alpha}}\right\}$ for any $x \in \mathbb{R}^d\backslash\{0\}$. Similarly, we also have $|\nabla\mathbf{K}^{e,\rho}_{\varepsilon,n}(x)| \leq C(n, \varepsilon, \zeta)\frac{1}{|x|^{d-\alpha+1}} \min\left\{\frac{1}{\rho^\alpha}, \frac{\rho}{|x|^{1+\alpha}}\right\}$ for any $x \in \mathbb{R}^d\backslash\{0\}$. Moreover, since $|\varphi_{\zeta\rho}(z)| \leq C1_{|z|>\zeta\rho}$, so we have for any $|x| \leq \zeta\rho/4$ that $|\mathbf{K}^{e,\rho}_{\varepsilon,n}(x)| + \rho|\nabla\mathbf{K}^{e,\rho}_{\varepsilon,n}(x)| \leq C(n, \varepsilon, \zeta)\frac{1}{\rho^d}$. Thus,

$$|\mathbf{K}^{e,\rho}_{\varepsilon,n}(x)| \leq C(n, \varepsilon, \zeta) \min\left\{\frac{1}{\rho^d}, \frac{\rho}{|x|^{d+1}}\right\}, \quad |\nabla\mathbf{K}^{e,\rho}_{\varepsilon,n}(x)| \leq C(n, \varepsilon, \zeta) \min\left\{\frac{1}{\rho^{d+1}}, \frac{\rho}{|x|^{d+2}}\right\}. \tag{3.23}$$

Thanks to proposition 5, we get

$$\limsup_{\lambda\to\infty} A_{11}(\lambda,\varepsilon) \le C|| \sum_{y_\tau\in S_\tau} \chi^\tau_{y_\tau}|\mu|^s||_{\mathcal{M}(\mathbb{R}^d)} \le C|\mu|^s(\mathbb{R}^d). \tag{3.24}$$

Using integration by parts, we have

$$\begin{aligned}
&\int_{\mathbb{R}^d} \mathbf{K}^{e,\rho}_{\varepsilon,n}(x-y)\chi^\tau_{y_\tau}(y)d\langle \eta^\kappa_{y_\tau},\mu(y)\rangle = -\int_{\mathbb{R}^d} \eta^\kappa_{y_\tau}.\nabla_y\left[\mathbf{K}^{e,\rho}_{\varepsilon,n}(x-y)\chi^\tau_{y_\tau}(y)\right] f(y)dy,\\
&\int_{H_{\eta^\kappa_{y_\tau}}}\int_{\tilde{H}_{\eta^\kappa_{y_\tau}}} \mathbf{K}^{e,\rho}_{\varepsilon,n}(x-y_1-y_2)\chi^\tau_{y_\tau}(y_1+y_2)dDf^{\eta^\kappa_{y_\tau}}_{\tilde{x}_{\kappa,y_\tau}}(y_1)d\mathcal{H}^{d-1}(y_2)\\
&\qquad = -\int_{\mathbb{R}^d} \eta^\kappa_{y_\tau}.\nabla_y\left[\mathbf{K}^{e,\rho}_{\varepsilon,n}(x-y)\chi^\tau_{y_\tau}(y)\right] f(\langle y,\eta^\kappa_{y_\tau}\rangle\eta^\kappa_{y_\tau}+\tilde{x}_{\kappa,y_\tau})dy.
\end{aligned}$$

So,

$$\begin{aligned}
I^{e,\rho}_{10,\varepsilon}(x) &= -\sum_{y_\tau\in S_\tau}\int_{\mathbb{R}^d} \eta^\kappa_{y_\tau}.\nabla_y\left[\mathbf{K}^{e,\rho}_{\varepsilon,n}(x-y)\chi^\tau_{y_\tau}(y)\right]\left[f(y)-f(\langle y,\eta^\kappa_{y_\tau}\rangle\eta^\kappa_{y_\tau}+\tilde{x}_{\kappa,y_\tau})\right]dy\\
&+\sum_{y_\tau\in S_\tau}\int_{H_{\eta^\kappa_{y_\tau}}}\int_{\tilde{H}_{\eta^\kappa_{y_\tau}}} \mathbf{K}^{e,\rho}_{\varepsilon,n}(x-y_1-y_2)\chi^\tau_{y_\tau}(y_1+y_2)dD^sf^{\eta^\kappa_{y_\tau}}_{\tilde{x}_{\kappa,y_\tau}}(y_1)d\mathcal{H}^{d-1}(y_2)\\
&+\sum_{y_\tau\in S_\tau}\int_{H_{\eta^\kappa_{y_\tau}}}\int_{\tilde{H}_{\eta^\kappa_{y_\tau}}} \mathbf{K}^{e,\rho}_{\varepsilon,n}(x-y_1-y_2)\chi^\tau_{y_\tau}(y_1+y_2)\langle\eta^\kappa_{y_\tau},D^af(y_1+\tilde{x}_{\kappa,y_\tau})\rangle d\mathcal{H}^1(y_1)d\mathcal{H}^{d-1}(y_2)\\
&=\sum_{i=12}^{14} I^{e,\rho}_{i,\varepsilon}(x),
\end{aligned}$$

where throughout this proof we denote

$$\tilde{x}_{\kappa,y_\tau} = x-\langle x,\eta^\kappa_{y_\tau}\rangle\eta^\kappa_{y_\tau}.$$

Thus,

$$A_9(\lambda,\varepsilon) \le \sum_{i=11}^{14}\lambda\mathcal{L}^d\left(\left\{\sup_{\rho\in(0,\rho_0),e\in S^{d-1}}|I^{e,\rho}_{i,\varepsilon}|>\lambda/4\right\}\cap B_R\right) = 4\sum_{i=11}^{14}A_i(\lambda/4,\varepsilon). \tag{3.25}$$

Step 5:. To treat $A_{13}(\lambda,\varepsilon)$ and $A_{14}(\lambda,\varepsilon)$, we need to show the following inequality:

$$\begin{aligned}
&\left|\int_{H_{\eta^\kappa_{y_\tau}}}\int_{\tilde{H}_{\eta^\kappa_{y_\tau}}} \mathbf{K}^{e,\rho}_{\varepsilon,n}(x-y_1-y_2)\chi^\tau_{y_\tau}(y_1+y_2)d\nu(y_1)d\mathcal{H}^{d-1}(y_2)\right|\\
&\quad\le C|\log(\varepsilon)|\mathbf{1}_{B_{4\tau}(y_\tau)}(x)\mathbf{M}^1(1_{B_{2\tau}(y_{\tau,1})}\nu,\tilde{H}_{\eta^\kappa_{y_\tau}})(x_1)\\
&\quad+C(n,\varepsilon,\zeta,\tau)\rho\mathbf{1}_{B_{4\tau}(y_\tau)^c}(x)\int_{H_{\eta^\kappa_{y_\tau}}}\int_{\tilde{H}_{\eta^\kappa_{y_\tau}}}\mathbf{1}_{B_{2\tau}(y_\tau)}(y_1+y_2)d\nu(y_1)d\mathcal{H}^{d-1}(y_2)
\end{aligned}\tag{3.26}$$

for any $\nu\in\mathcal{M}_b(\tilde{H}_{\eta^\kappa_{y_\tau}})$ and $x\in\mathbb{R}^d$ where

$$x_1 = \langle x,\eta^\kappa_{y_\tau}\rangle\eta^\kappa_{y_\tau},\quad y_{\tau,1} = \langle y_\tau,\eta^\kappa_{y_\tau}\rangle\eta^\kappa_{y_\tau}.$$

Indeed, let $\tau_{x_1}(z)=x_1-z$ for any $z\in\tilde{H}_{\eta^\kappa_{y_\tau}}$, by Lemma 2 (with $a=|\mathbf{K}^{e,\rho}_\varepsilon(.)|, \omega=(\tau_{x_1})_\#(\mathbf{1}_{B_{2\tau}(y_{\tau,1},\tilde{H}_{\eta^\kappa_{y_\tau}})}|\nu|)$) we have

$$
\begin{aligned}
&\sum_{j=-\infty}^{\infty}\int_{H_{\eta^\kappa_{y_\tau}}}\int_{\tilde{H}_{\eta^\kappa_{y_\tau}}}\mathbf{1}_{2^j\rho<|x-y_1-y_2|\le 2^{j+1}\rho}|\mathbf{K}^{e,\rho}_{\varepsilon,n}(x-y_1-y_2)|\chi^\tau_{y_\tau}(y_1+y_2)d|\nu|(y_1)d\mathcal{H}^{d-1}(y_2)\\
&\le C\sum_{j=-\infty}^{\infty}\int_{H_{\eta^\kappa_{y_\tau}}}\int_{\tilde{H}_{\eta^\kappa_{y_\tau}}}\mathbf{1}_{2^j\rho<|y_1+y_2|\le 2^{j+1}\rho}|\mathbf{K}^{e,\rho}_{\varepsilon,n}(y_1+y_2)|d(\tau_{x_1})_\#(\mathbf{1}_{B_{2\tau}(y_{\tau,1},\tilde{H}_{\eta^\kappa_{y_\tau}})}|\nu|)(y_1)d\mathcal{H}^{d-1}(y_2)\\
&\le \sum_{j=-\infty}^{\infty}C\left[(2^j\rho)^d\int_{S^{d-1}}\sup_{r\in[2^j\rho,2^{j+1}\rho]}|\mathbf{K}^{e,\rho}_{\varepsilon,n}(r\theta)|d\mathcal{H}^{d-1}(\theta)\right]\mathbf{M}^1((\tau_{x_1})_\#(\mathbf{1}_{B_{2\tau}(y_{\tau,1},\tilde{H}_{\eta^\kappa_{y_\tau}})}|\nu|),\tilde{H}_{\eta^\kappa_{y_\tau}})(0)\\
&= C\left[\sum_{j=-\infty}^{\infty}(2^j\rho)^d\int_{S^{d-1}}\sup_{r\in[2^j\rho,2^{j+1}\rho]}|\mathbf{K}^{e,\rho}_{\varepsilon,n}(r\theta)|d\mathcal{H}^{d-1}(\theta)\right]\mathbf{M}^1(\mathbf{1}_{B_{2\tau}(y_{\tau,1},\tilde{H}_{\eta^\kappa_{y_\tau}})}|\nu|,\tilde{H}_{\eta^\kappa_{y_\tau}})(x_1)
\end{aligned}
$$

So, by (3.32) in Lemma 4 below, we obtain that

$$
\left|\int_{H_{\eta^\kappa_{y_\tau}}}\int_{\tilde{H}_{\eta^\kappa_{y_\tau}}}\mathbf{K}^{e,\rho}_{\varepsilon,n}(x-y_1-y_2)\chi^\tau_{y_\tau}(y_1+y_2)d\nu(y_1)d\mathcal{H}^{d-1}(y_2)\right|\le C|\log(\varepsilon)|\mathbf{M}^1(1_{B_{2\tau}(y_{\tau,1})}\nu,\tilde{H}_{\eta^\kappa_{y_\tau}})(x_1). \tag{3.27}
$$

On the other hand, for any $x\notin B_{4\tau}(y_\tau)$,

$$
\begin{aligned}
&\left|\int_{H_{\eta^\kappa_{y_\tau}}}\int_{\tilde{H}_{\eta^\kappa_{y_\tau}}}\mathbf{K}^{e,\rho}_{\varepsilon,n}(x-y_1-y_2)\chi^\tau_{y_\tau}(y_1+y_2)d\nu(y_1)d\mathcal{H}^{d-1}(y_2)\right|\\
&\quad\le C\int_{H_{\eta^\kappa_{y_\tau}}}\int_{\tilde{H}_{\eta^\kappa_{y_\tau}}}\mathbf{1}_{|x-y_1-y_2|>2\tau}\mathbf{1}_{|y_1+y_2-y_\tau|<2\tau}|\mathbf{K}^{e,\rho}_{\varepsilon,n}(x-y_1-y_2)|d|\nu|(y_1)d\mathcal{H}^{d-1}(y_2)\\
&\quad\overset{(3.23)}{\le} C(n,\varepsilon,\zeta,\tau)\rho\int_{H_{\eta^\kappa_{y_\tau}}}\int_{\tilde{H}_{\eta^\kappa_{y_\tau}}}\mathbf{1}_{|y_1+y_2-y_\tau|<2\tau}d|\nu|(y_1)d\mathcal{H}^{d-1}(y_2).
\end{aligned}
$$

From this and (3.27), we find (3.26).

Step 6: Estimate $A_{13}(\lambda,\varepsilon)$ and $A_{14}(\lambda,\varepsilon)$.
We set

$$
\omega^\tau_{y_\tau,z_2}:=\mathbf{1}_{B_{2\tau}(y_{\tau,1},\tilde{H}_{\eta^\kappa_{y_\tau}})}|D^sf^{\eta^\kappa_{y_\tau}}_{z_2}|\quad\forall\ z_2\in H_{\eta^\kappa_{y_\tau}}.
$$

We then apply (3.26) for $\nu(y_1)=D^sf^{\eta^\kappa_{y_\tau}}_{\tilde{x}_{\kappa,y_\tau}}(y_1)$ to get that

$$
\begin{aligned}
I^{e,\rho}_{13,\varepsilon}(x)\le C|\log(\varepsilon)|&\sum_{y_\tau\in S_\tau}\mathbf{1}_{B_{4\tau}(y_\tau)}(x)\mathbf{M}^1(\omega^\tau_{y_\tau,\tilde{x}_{\kappa,y_\tau}},\tilde{H}_{\eta^\kappa_{y_\tau}})(\langle x,\eta^\kappa_{y_\tau}\rangle\eta^\kappa_{y_\tau})\\
&+\sum_{y_\tau\in S_\tau}C(\varepsilon,\zeta,\tau)\rho\int_{H_{\eta^\kappa_{y_\tau}}}\int_{\tilde{H}_{\eta^\kappa_{y_\tau}}}\mathbf{1}_{B_{2\tau}(y_\tau)}(y_1+y_2)d|D^sf^{\eta^\kappa_{y_\tau}}_{\tilde{x}_{\kappa,y_\tau}}|(y_1)d\mathcal{H}^{d-1}(y_2).
\end{aligned}
$$

By (2.1) in Proposition 3 and (3.13), we have

$$I^{e,\rho}_{13,\varepsilon}(x) \le C|\log(\varepsilon)| \sum_{y_\tau \in S_\tau} \mathbf{1}_{B_{4\tau}(y_\tau)}(x)\mathbf{M}^1(\omega^\tau_{y_\tau,\tilde{x}_{\kappa,y_\tau}}, \tilde{H}_{\eta^\kappa_{y_\tau}})(\langle x, \eta^\kappa_{y_\tau}\rangle \eta^\kappa_{y_\tau})$$
$$+ \sum_{y_\tau \in S_\tau} C(\varepsilon,\zeta,\tau)\rho \int_{\mathbb{R}^d} \mathbf{1}_{B_{2\tau}(y_\tau)}(y) d|\mu|^s(y)$$
$$\le C|\log(\varepsilon)| \sum_{y_\tau \in S_\tau} \mathbf{1}_{B_{4\tau}(y_\tau)}(x)\mathbf{M}^1(\omega^\tau_{y_\tau,\tilde{x}_{\kappa,y_\tau}}, \tilde{H}_{\eta^\kappa_{y_\tau}})(\langle x, \eta^\kappa_{y_\tau}\rangle \eta^\kappa_{y_\tau}) + C(\varepsilon,\zeta,\tau)\rho|\mu|^s(\mathbb{R}^d).$$

Thus, for $\lambda >> 1$

$$A_{13}(\lambda,\varepsilon) \le \lambda \mathcal{L}^d \left(\left\{ x \in B_R : C|\log(\varepsilon)| \sum_{y_\tau \in S_\tau} \mathbf{1}_{B_{4\tau}(y_\tau)}(x)\mathbf{M}^1(\omega^\tau_{y_\tau,\tilde{x}_{\kappa,y_\tau}}, \tilde{H}_{\eta^\kappa_{y_\tau}})(\langle x, \eta^\kappa_{y_\tau}\rangle \eta^\kappa_{y_\tau}) > \lambda \right\} \right)$$
$$\le \sum_{y'_\tau \in S_\tau} \lambda \mathcal{L}^d \left(\left\{ x \in B_{2\tau}(y'_\tau) : C|\log(\varepsilon)| \sum_{y_\tau \in S_\tau} \mathbf{1}_{B_{4\tau}(y_\tau)}(x)\mathbf{M}^1(\omega^\tau_{y_\tau,\tilde{x}_{\kappa,y_\tau}}, \tilde{H}_{\eta^\kappa_{y_\tau}})(\langle x, \eta^\kappa_{y_\tau}\rangle \eta^\kappa_{y_\tau}) > \lambda \right\} \right)$$
$$\overset{(3.13)}{\le} C \sum_{y_\tau \in S_\tau} \lambda \mathcal{L}^d \left(\left\{ x \in B_{8\tau}(y_\tau) : C|\log(\varepsilon)|\mathbf{M}^1(\omega^\tau_{y_\tau,\tilde{x}_{\kappa,y_\tau}}, \tilde{H}_{\eta^\kappa_{y_\tau}})(\langle x, \eta^\kappa_{y_\tau}\rangle \eta^\kappa_{y_\tau}) > \lambda \right\} \right).$$

Thanks to the boundedness of $\mathbf{M}^1(., \tilde{H}_{\eta^\kappa_{y_\tau}})$ from $\mathcal{M}(\tilde{H}_{\eta^\kappa_{y_\tau}})$ to $L^{1,\infty}(\tilde{H}_{\eta^\kappa_{y_\tau}})$ yields for $\lambda >> 1$

$$A_{13}(\lambda,\varepsilon) \le C \sum_{y_\tau \in S_\tau} \lambda \int_{H_{\eta^\kappa_{y_\tau}}} 1_{|z_2-(y_\tau-y_{\tau,1})|\le 8\tau} \mathcal{H}^1 \left(\left\{ C|\log(\varepsilon)|\mathbf{M}^1\left(\omega^\tau_{y_\tau,z_2}, \tilde{H}_{\eta^\kappa_{y_\tau}}\right) > \lambda \right\} \right) d\mathcal{H}^{d-1}(z_2)$$
$$\le C|\log(\varepsilon)| \sum_{y_\tau \in S_\tau} \int_{H_{\eta^\kappa_{y_\tau}}} 1_{|z_2-(y_\tau-y_{\tau,1})|\le 8\tau} \int_{\tilde{H}_{\eta^\kappa_{y_\tau}}} \mathbf{1}_{B_{2\tau}(y_{1,\tau},\tilde{H}_{\eta^\kappa_{y_\tau}})}(z_1) d|D^s f^{\eta^\kappa_{y_\tau}}_{z_2}|(z_1) d\mathcal{H}^{d-1}(z_2)$$
$$\le C|\log(\varepsilon)| \sum_{y_\tau \in S_\tau} \int_{B_{10\tau}(y_\tau)} d|D^s f|(z)$$
$$\le C|\log(\varepsilon)||\mu|^s(\mathbb{R}^d).$$

Here we have used (2.1) in Proposition 3 for the third inequality and (3.13) for the last one. Thus,

$$\limsup_{\lambda\to\infty} A_{13}(\lambda,\varepsilon) \le C|\log(\varepsilon)||\mu|^s(\mathbb{R}^d). \tag{3.28}$$

Similarly, we also have

$$A_{14}(\lambda,\varepsilon) \le C|\log(\varepsilon)|||\mu^a||_{L^1(B_{R+6\rho_0})} \quad \forall \quad \lambda >> 1.$$

Since, $\limsup_{\lambda\to\infty} \lambda \mathcal{H}^1 \left(\left\{ \mathbf{M}^1(1_{B_{2\tau}(y_{1,\tau},\tilde{H}_{\eta^\kappa_{y_\tau}})}|D^a f(.+z_2)|, \tilde{H}_{\eta^\kappa_{y_\tau}}) > \lambda \right\} \right) = 0$ for $\mathcal{H}^{d-1}$-a.e z_2 in $H_{\eta^\kappa_{y_\tau}}$, so by dominated convergence theorem we get

$$\limsup_{\lambda\to\infty} A_{14}(\lambda,\varepsilon) = 0. \tag{3.29}$$

Step 7:. We will prove that

$$\limsup_{\lambda\to\infty} A_{12}(\lambda,\varepsilon) \le C(n,\varepsilon,\zeta)||(\eta-\eta^\kappa)|\mu|^s||_{\mathcal{M}(\mathbb{R}^d)} + C(n,\varepsilon,\zeta,\kappa)\tau|\mu|^s(\mathbb{R}^d). \tag{3.30}$$

Let $\{\eta_1^\kappa(y_\tau),\eta_2^\kappa(y_\tau),...,\eta_d^\kappa(y_\tau)\}$ be an orthonormal basis in $\mathbb{R}^d$ such that $\eta_1^\kappa(y_\tau)=\eta_{y_\tau}^\kappa$. So, for any $x\in\mathbb{R}^d$, throughout this proof we denote

$$x_{\eta_i^\kappa(y_\tau)}=\langle x,\eta_i^\kappa(y_\tau)\rangle\eta_i^\kappa(y_\tau),\quad x^{1,j}_{\eta_i^\kappa(y_\tau)}=\sum_{i=1}^{j}x_{\eta_i^\kappa(y_\tau)},\quad x^{2,j}_{\eta_i^\kappa(y_\tau)}=\sum_{i=j+1}^{d}x_{\eta_i^\kappa(y_\tau)}.$$

By (3.23), we have

$$\begin{aligned}|I^{e,\rho}_{12,\varepsilon}(x)|&\leq C(n,\varepsilon,\zeta)\frac{1}{\rho^{d+1}}\sum_{y_\tau\in S_\tau}\int_{\mathbb{R}^d}\left(1\wedge\left(\frac{\rho}{|x-y|}\right)^{d+2}\right)\mathbf{1}_{|y_\tau-y|\leq 2\tau}\left|f(y)-f(y_{\eta^\kappa_{y_\tau}}+\sum_{i=2}^{d}x_{\eta_i^\kappa(y_\tau)})\right|dy\\&\quad+C(n,\varepsilon,\tau,\zeta)\frac{1}{\rho^{d}}\sum_{y_\tau\in S_\tau}\int_{\mathbb{R}^d}\left(1\wedge\left(\frac{\rho}{|x-y|}\right)^{d+1}\right)\left|f(y)-f(y_{\eta^\kappa_{y_\tau}}+\sum_{i=2}^{d}x_{\eta_i^\kappa(y_\tau)})\right|dy.\end{aligned}$$

Applying Lemma 3 to $\{e_1,...,e_d\}=\{\eta_1^\kappa(y_\tau),\eta_2^\kappa(y_\tau),...,\eta_d^\kappa(y_\tau)\}$ and $x_i=x_{\eta_i^\kappa(y_\tau)}$ for $i=1,...,d$ and $\varepsilon=2\tau$, we find that

$$\begin{aligned}I^{e,\rho}_{12,\varepsilon}(x)&\leq C(n,\varepsilon,\tau,\zeta)\rho^{\frac{1}{4}}\sum_{k=0}^{d-2}\sum_{y_\tau\in S_\tau}\mathbf{I}^{d-k}_{\frac{3}{4}}(\nu^1_{k,x^{2,d-k}_{\eta_i^\kappa(y_\tau)}},\bigotimes_{i=1}^{d-k}\tilde{H}_{\eta_i^\kappa(y_\tau)})(x^{1,d-k}_{\eta_i^\kappa(y_\tau)})\\&\quad+C(n,\varepsilon,\zeta)\sum_{k=0}^{d-2}\sum_{y_\tau\in S_\tau}1_{B_{4\tau}(y_\tau)}(x)\mathbf{M}^{d-k}(\nu^2_{k,x^{2,d-k}_{\eta_i^\kappa(y_\tau)}},\bigotimes_{i=1}^{d-k}\tilde{H}_{\eta_i^\kappa(y_\tau)})(x^{1,d-k}_{\eta_i^\kappa(y_\tau)}),\end{aligned}$$

where

$$\begin{aligned}&d\nu^1_{k,z}(y_{d-k},...,y_1)=d|Df^{\eta^\kappa_{d-k}(y_\tau)}_{\sum_{i=1}^{d-k-1}y_i+z}|(y_{d-k})d\mathcal{H}^1(y_{d-k-1})...d\mathcal{H}^1(y_1),\\&d\nu^2_{k,z}(y_{d-k},...,y_1)=1_{|\sum_{i=1}^{d-k}(y_\tau)_{\eta_i^\kappa(y_\tau)}-\sum_{i=1}^{d-k}y_i|\leq 4\tau}d\nu^1_{k,z}(y_{d-k},...,y_1),\end{aligned}$$

for any $z\in\bigotimes_{i=d-k+1}^{d}\tilde{H}_{\eta_i^\kappa(y_\tau)}$. Hence,

$$\begin{aligned}&\limsup_{\lambda\to\infty}A_{12}(\lambda,\varepsilon)\\&\leq\limsup_{\lambda\to\infty}\lambda\mathcal{L}^d\left(\left\{x\in B_R:C(n,\varepsilon,\tau,\zeta)\rho_0^{\frac{1}{4}}\sum_{k=0}^{d-2}\sum_{y_\tau\in S_\tau}\mathbf{I}^{d-k}_{\frac{3}{4}}(\nu^1_{k,x^{2,d-k}_{\eta_i^\kappa(y_\tau)}},\bigotimes_{i=1}^{d-k}\tilde{H}_{\eta_i^\kappa(y_\tau)})(x^{1,d-k}_{\eta_i^\kappa(y_\tau)})>\lambda\right\}\right)\\&+\limsup_{\lambda\to\infty}\lambda\mathcal{L}^d\left(\left\{x\in B_R:C(n,\varepsilon,\zeta)\sum_{k=0}^{d-2}\sum_{y_\tau\in S_\tau}\mathbf{1}_{B_{4\tau}(y_\tau)}(x)\mathbf{M}^{d-k}(\nu^2_{k,x^{2,d-k}_{\eta_i^\kappa(y_\tau)}},\bigotimes_{i=1}^{d-k}\tilde{H}_{\eta_i^\kappa(y_\tau)})(x^{1,d-k}_{\eta_i^\kappa(y_\tau)})>\lambda\right\}\right).\end{aligned}\tag{3.31}$$

We easily derive from the boundedness of $\mathbf{I}^{d-k}_{\frac{3}{4}}(.,X)$ from $\mathcal{M}_b(X)$ to $L^{\frac{d-k}{d-k-\frac{3}{4}},\infty}(X)$ with $X=\bigotimes_{i=1}^{d-k}\tilde{H}_{\eta_i^\kappa(y_\tau)}$ that the first term in the right hand-side of (3.31) equals zero. Thanks to (2.6), we get that the second term in the right hand-side of (3.31) is bounded by

$$\limsup_{\lambda\to\infty}\sum_{k=0}^{d-2}\sum_{y_\tau\in S_\tau}\lambda\mathcal{L}^d\left(\left\{x\in B_{8\tau}(y_\tau):C(n,\varepsilon,\zeta)\mathbf{M}^{d-k}(\nu^2_{k,x^{2,d-k}_{\eta_i^\kappa(y_\tau)}},\bigotimes_{i=1}^{d-k}\tilde{H}_{\eta_i^\kappa(y_\tau)})(x^{1,d-k}_{\eta_i^\kappa(y_\tau)})>\lambda\right\}\right)$$
$$\le C(n,\varepsilon,\zeta)\sum_{k=0}^{d-2}\sum_{y_\tau\in S_\tau}\int_{\tilde{H}_{\eta_d^\kappa(y_\tau)}}\cdots\int_{\tilde{H}_{\eta_1^\kappa(y_\tau)}}\mathbf{1}_{|\sum_{i=d-k+1}^d((y_\tau)_{\eta_i^\kappa(y_\tau)}-x_i)|\le 8\tau}$$
$$\times\, d\nu^{2,s}_{k,\sum_{i=d-k+1}^d x_i}(x_1,...,x_{d-k})d\mathcal{H}^1(x_{d-k+1})...d\mathcal{H}^1(x_d)$$
$$\le C(n,\varepsilon,\zeta)\sum_{k=0}^{d-2}\sum_{y_\tau\in S_\tau}\int_{H_{\eta^\kappa_{d-k}(y_\tau)}}\int_{\tilde{H}_{\eta^\kappa_{d-k}(y_\tau)}}\mathbf{1}_{|y_\tau-(z_1+z_2)|\le 16\tau}d|D^s f_{z_2}^{\eta^\kappa_{d-k}(y_\tau)}|(z_1)d\mathcal{H}^{d-1}(z_2).$$

where $\nu^{2,s}_{k,\sum_{i=d-k+1}^d x_i}(x_1,...,x_{d-k})$ is the singular part of $\nu^2_{k,\sum_{i=d-k+1}^d x_i}(x_1,...,x_{d-k})$.
Thanks to (2.1) in Proposition 3 and definition of η, one has

$$\limsup_{\lambda\to\infty}A_{12}(\lambda,\varepsilon)\le C(n,\varepsilon,\zeta)\sum_{k=0}^{d-2}\sum_{y_\tau\in S_\tau}\int_{\mathbb{R}^d}\mathbf{1}_{B_{20\tau}(y_\tau)}(x)|\langle\eta^\kappa_{d-k}(y_\tau),\eta(x)\rangle|d|\mu|^s(x).$$

Because of $\langle\eta^\kappa_{d-k}(y_\tau),\eta^\kappa(y_\tau)\rangle=0$ for any $k=0,1,..,d-2$, so

$$\begin{aligned}|\langle\eta^\kappa_{d-k}(y_\tau),\eta(x)\rangle|&\le|\langle\eta^\kappa_{d-k}(y_\tau),\eta(x)-\eta^\kappa(x)\rangle|+\langle\eta^\kappa_{d-k}(y_\tau),\eta^\kappa(x)-\eta^\kappa(y_\tau)\rangle|\\&\le|(\eta-\eta^\kappa)(x)|+||\nabla\eta^\kappa||_{L^\infty(\mathbb{R}^d)}|x-y_\tau|,\end{aligned}$$

which implies that

$$\begin{aligned}\limsup_{\lambda\to\infty}A_{12}(\lambda,\varepsilon)&\le C(n,\varepsilon,\zeta)\sum_{y_\tau\in S_\tau}\int_{\mathbb{R}^d}\mathbf{1}_{B_{20\tau}(y_\tau)}(x)\left[|(\eta-\eta^\kappa)(x)|+||\nabla\eta^\kappa||_{L^\infty(\mathbb{R}^d)}\tau\right]d|\mu^s|(x)\\&\overset{(3.13)}{\le}C(n,\varepsilon,\zeta)||(\eta-\eta^\kappa)|\mu^s|||_{\mathcal{M}(\mathbb{R}^d)}+C(n,\varepsilon,\zeta,\kappa)\tau|\mu|^s(\mathbb{R}^d).\end{aligned}$$

Therefore, we get (3.30).
Step 8: Estimate $A_9(\lambda,\varepsilon)$ and finish the proof.
Hence, we derive from (3.25) and (3.24), (3.28), (3.29), (3.30) that

$$\limsup_{\lambda\to\infty}A_9(\lambda,\varepsilon)\le C|\log(\varepsilon)||\mu|^s(\mathbb{R}^d)+C(n,\varepsilon,\zeta)|||\eta-\eta^\kappa||\mu|^s||_{\mathcal{M}(\mathbb{R}^d)}+C(n,\varepsilon,\kappa,\zeta)\tau|\mu|^s(\mathbb{R}^d).$$

Combining this with (3.20) yields

$$\begin{aligned}&\limsup_{\lambda\to\infty}\lambda\mathcal{L}^d\left(\left\{\mathbf{T}^{1,n}_\varepsilon(\mu)>\lambda\right\}\cap B_R\right)\le C|\log(\varepsilon)||\mu|^s(\mathbb{R}^d)\\&\quad+C(n,\varepsilon,\zeta)|||\eta-\eta^\kappa||\mu|^s||_{\mathcal{M}(\mathbb{R}^d)}+C(n,\varepsilon,\kappa,\zeta)\tau|\mu|^s(\mathbb{R}^d)+C(n,\varepsilon)\zeta^\alpha|\mu|^s(\mathbb{R}^d).\end{aligned}$$

At this point, sending $\tau\to0$, then $\kappa\to0$ and $\zeta\to0$, we obtain (3.12). The proof is complete. □

Lemma 4. *Let $\mathbf{K}^{e,\rho}_{\varepsilon,n}$ be in* (3.21)*. Then, for any $e\in S^{d-1}$ there holds*

$$\sum_{j=-\infty}^{\infty}(2^j\rho)^d\int_{S^{d-1}}\sup_{r\in[2^j\rho,2^{j+1}\rho]}|\mathbf{K}^{e,\rho}_{\varepsilon,n}(r\theta)|d\mathcal{H}^{d-1}(\theta)\le C|\log(\varepsilon)|. \tag{3.32}$$

Proof. **1. Case**: $j \geq 1$. For any $r \in [2^j\rho, 2^{j+1}\rho], \theta \in S^{d-1}$, we can estimate

$$
\begin{aligned}
|\mathbf{K}^{e,\rho}_{\varepsilon,n}(r\theta)| &= |\frac{\varepsilon^{-d+1}}{\rho^{\alpha}} \int_{\mathbb{R}^d} [\mathbf{K}_n(y) - \mathbf{K}_n(r\theta)]\, \varphi_{\zeta\rho}(r\theta - y) \frac{\langle \phi^{e,\varepsilon}((r\theta - y)/\rho), \eta^{\kappa}_{y_\tau}\rangle}{|r\theta - y|^{d-\alpha}} dy| \\
&\leq C\frac{\varepsilon^{-d+1}}{\rho^{\alpha}} \int_{\mathbb{R}^d} |\mathbf{K}_n(r\theta - y) - \mathbf{K}_n(r\theta)|\, \frac{\mathbf{1}_{|y|\leq\rho}\mathbf{1}_{|\frac{y}{|y|}-e|\leq\varepsilon}}{|y|^{d-\alpha}} dy.
\end{aligned}
$$

By (2.16), one has for $|y| < r/2$, $|\mathbf{K}_n(r\theta - y) - \mathbf{K}_n(r\theta)| \leq C\frac{|\Omega_n(\theta)||y|}{r^{d+1}} + \frac{C}{r^d}|\Omega_n(r\theta - y) - \Omega_n(r\theta)|$. So,

$$
\begin{aligned}
|\mathbf{K}^{e,\rho}_{\varepsilon,n}(r\theta)| &\leq C\frac{\varepsilon^{-d+1}|\Omega_n(\theta)|}{\rho^{\alpha} r^{d+1}} \int_{\mathbb{R}^d} \frac{\mathbf{1}_{|y|\leq\rho}\mathbf{1}_{|\frac{y}{|y|}-e|\leq\varepsilon}}{|y|^{d-\alpha-1}} dy \\
&\quad + C\frac{\varepsilon^{-d+1}}{\rho^{\alpha} r^{d}} \int_{\mathbb{R}^d} |\Omega_n(r\theta - y) - \Omega_n(r\theta)| \frac{\mathbf{1}_{|y|\leq\rho}\mathbf{1}_{|\frac{y}{|y|}-e|\leq\varepsilon}}{|y|^{d-\alpha}} dy \\
&\leq C\frac{|\Omega_n(\theta)|\rho}{r^{d+1}} + C\frac{\varepsilon^{-d+1}}{\rho^{\alpha} r^{d-\alpha}} \int_{\mathbb{R}^d} |\Omega_n(\theta - y) - \Omega_n(\theta)| \frac{\mathbf{1}_{|y|\leq 2^{-j}}\mathbf{1}_{|\frac{y}{|y|}-e|\leq\varepsilon}}{|y|^{d-\alpha}} dy.
\end{aligned}
$$

Thus,

$$
\begin{aligned}
&(2^j\rho)^d \int_{S^{d-1}} \sup_{r\in[2^j\rho, 2^{j+1}\rho]} |\mathbf{K}^{e,\rho}_{\varepsilon,n}(r\theta)| d\mathcal{H}^{d-1}(\theta) \\
&\leq C2^{-j}||\Omega_n||_{L^1(S^{d-1})} + C\varepsilon^{-d+1}2^{j\alpha} \int_{\mathbb{R}^d} \sup_{|h|\leq 1/2} |h|^{-\alpha_0/2} ||\Omega_n(.-h) - \Omega_n(.)||_{L^1(S^{d-1})} \frac{\mathbf{1}_{|y|\leq 2^{-j}}\mathbf{1}_{|\frac{y}{|y|}-e|\leq\varepsilon}}{|y|^{d-\alpha-\alpha_0/2}} dy \\
&\overset{(3.2),(2.19)}{\leq} C2^{-j} + C2^{-j\alpha_0/2} \leq C2^{-j\alpha_0/2}.
\end{aligned}
$$

which implies

$$
\sum_{j=1}^{\infty} (2^j\rho)^d \int_{S^{d-1}} \sup_{r\in[2^j\rho, 2^{j+1}\rho]} |\mathbf{K}^{e,\rho}_{\varepsilon,n}(r\theta)| d\mathcal{H}^{d-1}(\theta) \leq C. \tag{3.33}
$$

2. Case: $j \leq 0$. We prove that

$$
(2^j\rho)^d \int_{S^{d-1}} \sup_{r\in[2^j\rho, 2^{j+1}\rho]} |\mathbf{K}^{e,\rho}_{\varepsilon,n}(r\theta)| d\mathcal{H}^{d-1}(\theta) \leq C|\log(\varepsilon)| 2^{j\frac{1}{2}\min\{\alpha,1\}}. \tag{3.34}
$$

Let ψ be a smooth function in $\mathbb{R}^d$ such that $\psi(x) = 1$ if $|x| \leq 1$ and $\psi(x) = 0$ if $|x| > 2$. Since

$$
\sum_{i=-\infty}^{\infty} \left(\psi(2^{-i}\rho^{-1}y) - \psi(2^{-i+1}\rho^{-1}y)\right) = 1 \quad \forall \quad y \in \mathbb{R}^d, \tag{3.35}
$$

so for any $r \in (2^j\rho, 2^{j+1}\rho], \theta \in S^{d-1}$,

$$
\begin{aligned}
|\mathbf{K}^{e,\rho}_{\varepsilon,n}(r\theta)| &\leq \sum_{i=-\infty}^{\infty} |\mathbf{K}_{i,n}(r\theta)| + |\mathbf{c}(\varepsilon,\kappa,\tau,\zeta)\varphi_\rho(r\theta)\mathbf{K}_n(r\theta)| \\
&\leq \sum_{i=-\infty}^{1} |\mathbf{K}_{i,n}(r\theta)| + C1_{j=1}(2^j\rho)^{-d}|\Omega_n(\theta)|,
\end{aligned}
$$

where

$$\mathbf{K}_{i,n}(r\theta) = \frac{\varepsilon^{-d+1}}{\rho^\alpha}\int_{\mathbb{R}^d} \mathbf{K}_n(r\theta - y)\varphi_{\zeta\rho}(y)\frac{\langle \phi^{e,\varepsilon}(y/\rho), \eta^\kappa_{y_\tau}\rangle}{|y|^{d-\alpha}}\left(\psi(2^{-i}\rho^{-1}y) - \psi(2^{-i+1}\rho^{-1}y)\right)dy.$$

We have

$$\left(\sum_{i=-\infty}^{j-3} + \sum_{i=j+3}^{1}\right)|\mathbf{K}_{i,n}(r\theta)| \leq C\left(\sum_{i=-\infty}^{j-3} + \sum_{i=j+3}^{1}\right)\frac{\varepsilon^{-d+1}}{\rho^\alpha}\int_{\mathbb{R}^d}\frac{|\Omega_n(r\theta - y)|}{|r\theta - y|^d}\frac{\mathbf{1}_{|\frac{y}{|y|}-e|\leq\varepsilon}}{|y|^{d-\alpha}}\mathbf{1}_{2^{i-1}\rho<|y|<2^{i+1}\rho}dy$$

$$= C\left(\sum_{i=-\infty}^{j-3} + \sum_{i=j+3}^{1}\right)\frac{\varepsilon^{-d+1}}{\rho^\alpha r^{d-\alpha}}\int_{\mathbb{R}^d}\frac{|\Omega_n(r\theta - y)|}{|\theta - y|^d}\frac{\mathbf{1}_{|\frac{y}{|y|}-e|\leq\varepsilon}}{|y|^{d-\alpha}}\mathbf{1}_{2^{i-1}\rho r^{-1}<|y|<2^{i+1}\rho r^{-1}}dy.$$

Thus, we obtain that for any $r \in (2^j\rho, 2^{j+1}\rho]$,

$$\left(\sum_{i=-\infty}^{j-3} + \sum_{i=j+3}^{\infty}\right)|\mathbf{K}_{i,n}(r\theta)| \leq C\sum_{i=-\infty}^{j-3}\frac{2^{(i-j)d}}{\rho^d 2^{j(d-\alpha)}}G_{2^{i-j+1}}(\theta) + C\sum_{i=j+3}^{1}\frac{1}{\rho^d 2^{i(d-\alpha)}}G_{2^{i-j+1}}(\theta)$$

where $G_\vartheta(\theta) = \frac{\varepsilon^{-d+1}}{\vartheta^d}\int_{\mathbb{R}^d}|\Omega_n(\theta - y)|\mathbf{1}_{|\frac{y}{|y|}-e|\leq\varepsilon}\mathbf{1}_{\vartheta/8<|y|<\vartheta}dy$. We need to check that $\vartheta \geq 16$, $\int_{S^{d-1}} G_\vartheta(\theta) \leq C\vartheta^{d-1}$ and if $\vartheta \leq 1/2$, $\int_{S^{d-1}} G_\vartheta(\theta) \leq C$. In fact, if $\vartheta \leq 1/2$, since $\Omega_n(\theta) = \Omega_n(\varsigma\theta)$ for any $\varsigma > 0, \theta \in S^{d-1}$, thus

$$\int_{S^{d-1}}|\mathbf{G}_\vartheta(\theta)| = \frac{\varepsilon^{-d+1}}{\vartheta^d}\int_{\mathbb{R}^d}\frac{5}{2}\int_{4/5<|x|<6/5}|\Omega_n(x - |x|y)|dx\mathbf{1}_{|\frac{y}{|y|}-e|\leq\varepsilon}\mathbf{1}_{\vartheta/8<|y|<\vartheta}dy$$
$$\leq C||\Omega_n||_{L^1(B_2(0))}\frac{\varepsilon^{-d+1}}{\vartheta^d}\int_{\mathbb{R}^d}\mathbf{1}_{|\frac{y}{|y|}-e|\leq\varepsilon}\mathbf{1}_{\vartheta/16<|y|<2\vartheta}dy \overset{(2.19)}{\leq} C.$$

and if $\vartheta \geq 16$, then

$$\begin{aligned}\int_{S^{d-1}}|\mathbf{G}_\vartheta(\theta)|d\mathcal{H}^{d-1}(\theta) &= \frac{\varepsilon^{-d+1}}{\vartheta^d}\int_{\mathbb{R}^d}\frac{5}{2}\int_{4/5<|x|<6/5}|\Omega_n(x - |x|y)|dx\mathbf{1}_{|\frac{y}{|y|}-e|\leq\varepsilon}\mathbf{1}_{\vartheta/16<|y|<2\vartheta}dy\\
&\leq C\frac{\varepsilon^{-d+1}}{\vartheta^d}\int_{\mathbb{R}^d}\int_{4/5<|x|<6/5}|\Omega_n(x - y)|dx\mathbf{1}_{|\frac{y}{|y|}-e|\leq\varepsilon}\mathbf{1}_{\vartheta/16<|y|<2\vartheta}dy\\
&\leq C\frac{\varepsilon^{-d+1}}{\vartheta^d}\int_{\mathbb{R}^d}\int_{|y|-2<|x|<2+|y|}|\Omega_n(x)|dx\mathbf{1}_{|\frac{y}{|y|}-e|\leq\varepsilon}\mathbf{1}_{\vartheta/16<|y|<2\vartheta}dydh\\
&\leq C\frac{\varepsilon^{-d+1}}{\vartheta^d}\int_{\mathbb{R}^d}|y|^{d-1}||\Omega_n||_{L^1(S^{d-1})}\mathbf{1}_{|\frac{y}{|y|}-e|\leq\varepsilon}\mathbf{1}_{\vartheta/16<|y|<2\vartheta}dydh\\
&\leq C\vartheta^{d-1}.\end{aligned}$$

Therefore,

$$\begin{aligned}&\left(\sum_{i=-\infty}^{j-3} + \sum_{i=j+3}^{1}\right)(2^j\rho)^d\int_{S^{d-1}}\sup_{r'\in[2^j\rho,2^{j+1}\rho]}|\mathbf{K}_{i,n}(r'\theta)|d\mathcal{H}^{d-1}(\theta)\\
&\quad\leq C\sum_{i=-\infty}^{j-3}\frac{2^{jd}2^{(i-j)d}}{2^{j(d-\alpha)}} + C\sum_{i=j+3}^{1}\frac{2^{jd}}{2^{i(d-\alpha)}}2^{(i-j)(d-1)}\\
&\quad\leq C\left(2^{j\alpha}\mathbf{1}_{\alpha<1} + 2^j(|j|+1)\mathbf{1}_{\alpha=1} + 2^j\mathbf{1}_{\alpha>1}\right) \leq C2^{j\frac{1}{2}\min\{\alpha,1\}}.\end{aligned} \tag{3.36}$$

We now estimate $\mathbf{K}_{i,n}(r\theta)$ for $i = j-2, ..., j+2$ and $r \in (2^j\rho, 2^{j+1}\rho]$. We can write

$$\mathbf{K}_{i,n}(r\theta) = \sum_{l=-4}^{\infty} \mathbf{K}_{i,n,l}(r\theta) \quad i = j-2, ..., j+2 \leq 2$$

where

$$\mathbf{K}_{i,n,l}(r\theta) = \frac{\varepsilon^{-d+1}}{\rho^{\alpha}} \int_{\mathbb{R}^d} \mathbf{1}_{2^{i-l-1}\rho < |r\theta - y| \leq 2^{i-l}\rho} \mathbf{K}_n(r\theta - y) \varphi_{\zeta\rho}(y) \frac{\langle \phi^{e,\varepsilon}(y/\rho), \eta^{\kappa}_{y_\tau} \rangle}{|y|^{d-\alpha}}$$
$$\times \left(\psi(2^{-i}\rho^{-1}y) - \psi(2^{-i+1}\rho^{-1}y)\right) dy.$$

First we will show that

$$\sum_{i=j-2}^{j+2} (2^j\rho)^d \int_{S^{d-1}} \sup_{r \in [2^j\rho, 2^{j+1}\rho]} |\mathbf{K}_{i,n,l}(r\theta)| d\mathcal{H}^{d-1}(\theta) \leq C 2^{j\alpha} \ \ \forall \ \ l \geq -4. \tag{3.37}$$

In fact, one has

$$(2^j\rho)^d \int_{S^{d-1}} \sup_{r \in [2^j\rho, 2^{j+1}\rho]} |\mathbf{K}_{i,n,l}(r\theta)| d\mathcal{H}^{d-1}(\theta) \leq C (2^j\rho)^d \frac{\varepsilon^{-d+1}}{\rho^{\alpha}} \frac{1}{(2^{i-l}\rho)^d (2^i\rho)^{d-\alpha}}$$
$$\times \int_{S^{d-1}} \sup_{r \in [2^j\rho, 2^{j+1}\rho]} \int_{\mathbb{R}^d} |\Omega_n(r\theta - y)| \mathbf{1}_{|\frac{y}{|y|} - e| \leq \varepsilon} \mathbf{1}_{|r\theta - y| \sim 2^{i-l}\rho} \mathbf{1}_{|y| \sim 2^i\rho} dy d\mathcal{H}^{d-1}(\theta).$$

We change variable to get that

$$(2^j\rho)^d \int_{S^{d-1}} \sup_{r \in [2^j\rho, 2^{j+1}\rho]} |\mathbf{K}_{i,n,l}(r\theta)| d\mathcal{H}^{d-1}(\theta) \leq C (2^j\rho)^d \frac{\varepsilon^{-d+1}}{\rho^{\alpha}} \frac{1}{(2^{i-l}\rho)^d (2^i\rho)^{d-\alpha}}$$
$$\times \int_{S^{d-1}} \sup_{r \in [2^j\rho, 2^{j+1}\rho]} \int_{\mathbb{R}^d} |\Omega_n(\theta - y)| \mathbf{1}_{|\frac{y}{|y|} - e| \leq \varepsilon} \mathbf{1}_{|r\theta - ry| \sim 2^{i-l}\rho} \mathbf{1}_{r|y| \sim 2^i\rho} r^d dy d\mathcal{H}^{d-1}(\theta)$$
$$\leq C \frac{\varepsilon^{-d+1} 2^{j(d+\alpha)}}{2^{(i-l)d}} \int_{S^{d-1}} \int_{\mathbb{R}^d} |\Omega_n(\theta - y)| \mathbf{1}_{|\frac{y}{|y|} - e| \leq \varepsilon} \mathbf{1}_{|\theta - y| \sim 2^{-l}} \mathbf{1}_{||y| - 1| \lesssim 2^{-l}} dy d\mathcal{H}^{d-1}(\theta).$$

It is easy to see that

$$\int_{S^{d-1}} \int_{\mathbb{R}^d} |\Omega_n(\theta - y)| \mathbf{1}_{|\frac{y}{|y|} - e| \leq \varepsilon} \mathbf{1}_{|\theta - y| \sim 2^{-l}} \mathbf{1}_{||y| - 1| \lesssim 2^{-l}} dy d\mathcal{H}^{d-1}(\theta)$$
$$\leq C \inf_{|\vartheta - 1| \lesssim 2^{-l-m}} \int_{S^{d-1}} \int_{\mathbb{R}^d} |\Omega_n(\vartheta\theta - y)| \mathbf{1}_{|\frac{y}{|y|} - e| \leq \varepsilon} \mathbf{1}_{|\vartheta\theta - y| \sim 2^{-l}} \mathbf{1}_{||y| - 1| \lesssim 2^{-l}} dy d\mathcal{H}^{d-1}(\theta)$$

for any $m \geq m(d)$, thus

$$(2^j\rho)^d \int_{S^{d-1}} \sup_{r \in [2^j\rho, 2^{j+1}\rho]} |\mathbf{K}_{i,n,l}(r\theta)| d\mathcal{H}^{d-1}(\theta)$$
$$\leq C \frac{\varepsilon^{-d+1} 2^{j(d+\alpha)}}{2^{(i-l)d}} 2^l \int_{||h| - 1| \lesssim 2^{-l-m(d)}} \int_{\mathbb{R}^d} |\Omega_n(h - y)| \mathbf{1}_{|\frac{y}{|y|} - e| \leq \varepsilon} \mathbf{1}_{|h - y| \sim 2^{-l}} \mathbf{1}_{||y| - 1| \lesssim 2^{-l}} dy dh$$
$$\leq C \frac{\varepsilon^{-d+1} 2^{j(d+\alpha)}}{2^{(i-l)d}} 2^l \int_{\mathbb{R}^d} \left[\int_{\mathbb{R}^d} |\Omega_n(h - y)| \mathbf{1}_{|h - y| \sim 2^{-l}} dh \right] \mathbf{1}_{|\frac{y}{|y|} - e| \leq \varepsilon} \mathbf{1}_{||y| - 1| \lesssim 2^{-l}} dy$$
$$= C \frac{\varepsilon^{-d+1} 2^{j(d+\alpha)}}{2^{(i-l)d}} 2^l ||\Omega_n||_{L^1(S^{d-1})} 2^{-ld} \int_{\mathbb{R}^d} \mathbf{1}_{|\frac{y}{|y|} - e| \leq \varepsilon} \mathbf{1}_{||y| - 1| \lesssim 2^{-l}} dy$$
$$\leq C 2^{j\alpha},$$

since $\int_{\mathbb{R}^d} \mathbf{1}_{|\frac{y}{|y|}-e|\leq\varepsilon}\mathbf{1}_{||y|-1|\lesssim 2^{-l}}dy \leq C\varepsilon^{d-1}2^{-l}$. Thus, this implies (3.37).
Next, condition 3.3 implies $|\int_{\mathbb{R}^d} \mathbf{1}_{r_1<|r\theta-y|\leq r_2}\mathbf{K}_n(r\theta-y)dy| \leq 2c_2$ for any $r_1 < r_2$. Thus, for $l_0 > 100$,

$$\sum_{l=l_0}^{\infty} |\mathbf{K}_{i,n,l}(r\theta)| \leq \sum_{l=l_0}^{\infty} \frac{\varepsilon^{-d+1}}{\rho^{\alpha}} \int_{\mathbb{R}^d} \mathbf{1}_{2^{i-l-1}\rho<|r\theta-y|\leq 2^{i-l}\rho}|\mathbf{K}_n(r\theta-y)||\Theta(y)-\Theta(r\theta)|dy + C\frac{\varepsilon^{-d+1}}{\rho^{\alpha}}|\Theta(r\theta)|,$$

where $\Theta(y) = \varphi_{\zeta\rho}(y)\frac{\langle\phi^{e,\varepsilon}(y/\rho),\eta^{\kappa}_{y_\tau}\rangle}{|y|^{d-\alpha}}\left(\psi(2^{-i}\rho^{-1}y)-\psi(2^{-i+1}\rho^{-1}y)\right)$. Since $|\varphi_{\zeta\rho}(y)| \leq C\mathbf{1}_{|y|>\zeta\rho}$, $|\nabla\varphi_{\zeta\rho}(y)| \leq \frac{C\mathbf{1}_{\zeta\rho<|y|\leq 2\zeta\rho}}{|y|}$, so we easily see that $|\Theta(r\theta)| \leq \frac{C1_{|\theta-e|\leq\varepsilon}}{(2^i\rho)^{d-\alpha}}$, and $|\Theta(y)-\Theta(r\theta)| \leq \frac{C|r\theta-y|}{\varepsilon}\frac{1}{(2^i\rho)^{d-\alpha+1}}$ for any $l > 100$, $2^{i-l-1}\rho < |r\theta-y| \leq 2^{i-l}\rho$ and $r \in [2^j\rho, 2^{j+1}\rho]$, Thus,

$$\begin{aligned}
&\sum_{l=l_0}^{\infty}\sum_{i=j-2}^{j+2}(2^j\rho)^d\int_{S^{d-1}}\sup_{r\in[2^j\rho,2^{j+1}\rho]}|\mathbf{K}_{i,n,l}(r\theta)|d\mathcal{H}^{d-1}(\theta)\\
&\leq C\sum_{l=l_0}^{\infty}\sum_{i=j-2}^{j+2}(2^j\rho)^d\int_{S^{d-1}}\sup_{r\in[2^j\rho,2^{j+1}\rho]}\frac{\varepsilon^{-d+1}}{\rho^{\alpha}}\int_{2^{i-l-1}\rho<|r\theta-y|\leq 2^{i-l}\rho}\frac{|\mathbf{K}_n(r\theta-y)||r\theta-y|}{\varepsilon(2^i\rho)^{d-\alpha+1}}dy\mathcal{H}^{d-1}(\theta)\\
&\quad+\sum_{i=j-2}^{j+2}C(2^j\rho)^d\int_{S^{d-1}}\frac{\varepsilon^{-d+1}}{\rho^{\alpha}}\frac{\mathbf{1}_{|\theta-e|\leq\varepsilon}}{(2^i\rho)^{d-\alpha}}d\mathcal{H}^{d-1}(\theta)\\
&\leq C\sum_{l=l_0}^{\infty}\sum_{i=j-2}^{j+2}(2^j\rho)^d\frac{\varepsilon^{-d+1}}{\rho^{\alpha}}\frac{2^{i-l}\rho}{\varepsilon(2^i\rho)^{d-\alpha+1}}+C2^{j\alpha}\\
&\leq C2^{j\alpha}\varepsilon^{-d}2^{-l_0}+C2^{j\alpha}.
\end{aligned} \tag{3.38}$$

Therefore, it follows from (3.37) and (3.38) that

$$\sum_{i=j-2}^{j+2}(2^j\rho)^d\int_{S^{d-1}}\sup_{r\in[2^j\rho,2^{j+1}\rho]}|\mathbf{K}_{i,n}(r\theta)|d\mathcal{H}^{d-1}(\theta) \leq C2^{j\alpha}(1+l_0+\varepsilon^{-d}2^{-l_0}).$$

At this point we take $2^{l_0} \sim \varepsilon^{-d}$ and obtain that

$$\sum_{i=j-2}^{j+2}(2^j\rho)^d\int_{S^{d-1}}\sup_{r\in[2^j\rho,2^{j+1}\rho]}|\mathbf{K}_{i,n}(r\theta)|d\mathcal{H}^{d-1}(\theta) \leq C2^{j\alpha}|\log(\varepsilon)|.$$

From this and (3.36) we get (3.34). Then, (3.32) follows from (3.33) and (3.34). The proof is complete. □

4 Regular Lagrangian flows and quantitative estimates with BV vector fields

We first recall some definitions and properties of Regular Lagrangian flows introduced in [18]. Given a vector field $\mathbf{B}(t,x):(0,T)\times\mathbb{R}^d\to\mathbb{R}^d$, we assume the following growth condition:
(R1) The vector field $\mathbf{B}(t,x)$ can be decomposed as

$$\frac{\mathbf{B}(t,x)}{1+|x|} = \tilde{B}_1(t,x)+\tilde{B}_2(t,x)$$

with $\tilde{B}_1 \in L^1((0,T);L^1(\mathbb{R}^d))$ and $\tilde{B}_2 \in L^1((0,T);L^\infty(\mathbb{R}^d))$.
We denot by L^0_{loc} the space of measurable functions endowed with local convergence in measure, and $\mathcal{B}(E_1;E_2)$ the space of bounded functions between the sets E_1 and E_2, $\log L_{loc}(\mathbb{R}^d)$ the space of measurable functions $u:\mathbb{R}^d \to \mathbb{R}$ such that $\int_{B_r} \log(1+|u(x)|)dx$ is finite for any $r>0$.
The following is definition of Regular Lagrangian flow:

Definition 1. *If $\mathbf{B}$ is a vector field satisfying **(R1)**, then for fixed $t_0 \in [0,T)$, a map*

$$X \in C([t_0,T];L^0_{_{loc}}(\mathbb{R}^d)) \cap \mathcal{B}([t_0,T];\log L_{_{loc}}(\mathbb{R}^d))$$

is a regular Lagrangian flow in the renormalized sense relative to $\mathbf{B}$ starting at t_0 if we have the following:
i) *The equation $\partial_t(h(X(t,x))) = (\nabla h)(X(t,x))\mathbf{B}(t,X(t,x))$ holds in $\mathcal{D}'((t_0,T)\times\mathbb{R}^d)$, for every function $h \in C^1(\mathbb{R}^d,\mathbb{R})$ that satisfies $|h(z)| \leq C(1+\log(1+|z|))$ and $|\nabla h(z)| \leq \frac{C}{1+|z|}$ for all $z \in \mathbb{R}^d$,*
ii) *$X(t_0,x) = x$ for $\mathcal{L}^d-a.e\ x \in \mathbb{R}^d$,*
iii) *There exists a constant $L>0$ such that $X(t,.)_{\#}\mathcal{L}^d \leq L\mathcal{L}^d$ for any $t \in [t_0,T]$ i.e $\int_{\mathbb{R}^d} \varphi(X(t,x))dx \leq L\int_{\mathbb{R}^d}\varphi(x)dx$ for all measurable $\varphi:\mathbb{R}^d \to [0,\infty)$. The constant L in **iii)** will be called the compressibility constant of X.*

We define the sub-level of the flow as

$$G_R = \left\{ x \in \mathbb{R}^d : |X(t,x)| \leq R \ \text{ for almost all } \quad t \in [t_0,T] \right\}.$$

The following Lemma gives a basic estimate for the decay of the super-levels of a regular Lagrangian flow. This Lemma was proven in [18].

Lemma 5. *Let $\mathbf{B}$ be a vector field satisfying **(R1)** and let X be a regular Lagrangian flow relative to $\mathbf{B}$ starting at time t_0, with compressibility constant L. Then for all $r,\lambda>0$ we have $\mathcal{L}^d(B_r\backslash G_R) \leq g(r,R)$ where the function g depends only on L, $||\tilde{B}_1||_{L^1((0,T);L^1(\mathbb{R}^d))}$ and $||\tilde{B}_2||_{L^1((0,T);L^\infty(\mathbb{R}^d))}$ and satisfies $g(r,R)\downarrow 0$ for r fixed and $R\uparrow\infty$.*

The following is our main theorem.

Theorem 2. *Let $\mathbf{B} \in L^1([0,T];L^1_{_{loc}}(\mathbb{R}^d,\mathbb{R}^d))$ and $R>1$. Assume that*

$$\mathbf{B}^i = \sum_{j=1}^m \mathbf{K}^i_j \star b_j \quad \textit{in } B_{2R}, \ \textit{with } b_j \in L^1([0,T],BV(\mathbb{R}^d)), \tag{4.1}$$

where $(\mathbf{K}^i_j)_{i,j}$ are singular kernels in $\mathbb{R}^d$ satisfying conditions of singular kernel $\mathbf{K}$ in Theorem 1 with constants $c_1,c_2>0$. Let $t_0 \in [0,T)$, $\mathbf{B}_1,\mathbf{B}_2 \in L^1([0,T];L^1_{_{loc}}(\mathbb{R}^d,\mathbb{R}^d))$ and let X_1,X_2 be regular Lagrangian flows starting at time t_0 associated to $\mathbf{B}_1,\mathbf{B}_2$ resp. with compression constants $L_1,L_2 \leq L_0$ for some $L_0>0$. Assume that $||(\mathbf{B}_1,\mathbf{B}_2)||_{L^1([0,T]\times B_R)} \leq c_R$. Then, if $\mathrm{div}(\mathbf{B}) \in L^1((0,T),\mathcal{M}_b(B_{2R}))$ *and* $(\mathrm{div}(\mathbf{B}))^+ \in L^1((0,T),L^1(B_{2R}))$, *for any $\kappa \in (0,1), r>1$ there exists $\delta_0 = \delta_0(d,T,r,R,c_R,c_1,c_2,L_0,b,\kappa) \in (0,1/100)$ such that*

$$\sup_{t_1\in[t_0,T]} \mathcal{L}^d\left(\left\{x \in B_r : |X_{1t_1}(x) - X_{2t_1}(x)| > \delta^{1/2}\right\}\right) \leq \mathcal{L}^d\left(B_r\backslash G_{1,R}\right) + \mathcal{L}^d\left(B_r\backslash G_{2,R}\right)$$
$$+ \frac{C(d)L_0}{\delta}||\left(\mathbf{B}_1-\mathbf{B},\mathbf{B}_2-\mathbf{B}\right)||_{L^1([0,T]\times B_R)} + \kappa \quad \textit{for any} \quad \delta \in (0,\delta_0). \tag{4.2}$$

where $G_{i,R} = \left\{x \in \mathbb{R}^N : |X_i(s,x)| \leq R \quad \text{for almost all } s \in [t_0,T]\right\}$ for i=1,2.

We derive from Theorem 2 and Lemma 5 that

Corollary 1. *Let* $\mathbf{B} \in L^1([0,T]; L^1_{loc}(\mathbb{R}^d, \mathbb{R}^d))$. *Assume that for any* $R > 0$, *there exist singular kernels* $(\mathbf{K}^i_j)_{i,j}$ $(i = 1,..,d, j = 1,...,m(R))$ *in* $\mathbb{R}^d$ *satisfying conditions of singular kernel* $\mathbf{K}$ *in Theorem 1 with constants* $c_{1R}, c_{2R} > 0$*; and* $b_{jR} \in L^1([0,T], BV(\mathbb{R}^d))$ *such that*

$$\mathbf{B}^i = \sum_{j=1}^{m} \mathbf{K}^i_{jR} \star b_{jR} \quad \text{in } B_{2R}. \tag{4.3}$$

Let $t_0 \in [0,T)$, $\mathbf{B}_1, \mathbf{B}_2 \in L^1([0,T]; L^1_{loc}(\mathbb{R}^d, \mathbb{R}^d))$ *and let* X_1, X_2 *be regular Lagrangian flows starting at time* t_0 *associated to* $\mathbf{B}_1, \mathbf{B}_2$ *resp. with compression constants* $L_1, L_2 \le L_0$ *for some* $L_0 > 0$. *Assume that* $\mathbf{B}_1, \mathbf{B}_2$ *satisfy* $(\mathbf{R}_1)$ *i.e* $\frac{\mathbf{B}_l(t,x)}{|x|+1} = \tilde{B}_{1l}(t,x) + \tilde{B}_{2l}(t,x)$ $l = 1,2$ *with*

$$\sum_{l=1,2} ||\tilde{B}_{1l}||_{L^1((0,T);L^1(\mathbb{R}^d))} + ||\tilde{B}_{2l}||_{L^1((0,T);L^\infty(\mathbb{R}^d))} \le C_0.$$

Then, if $\mathrm{div}(\mathbf{B}) \in L^1((0,T), \mathcal{M}_{loc}(\mathbb{R}^d))$ *and* $(\mathrm{div}(\mathbf{B}))^+ \in L^1((0,T), L^1_{loc}(\mathbb{R}^d))$, *for any* $\kappa \in (0,1), r > 1$ *there exists* $R_0 = R_0(d,T,r,C_0,L_0,\kappa) > 1$, $\delta_0 = \delta_0(d,T,r,C_0,c_{1R_0},c_{2R_0},L_0,b_{R_0},\kappa) \in (0,1/100)$ *such that*

$$\sup_{t_1 \in [t_0,T]} \mathcal{L}^d\left(\left\{x \in B_r : |X_{1t_1}(x) - X_{2t_1}(x)| > \delta^{1/2}\right\}\right) \le \frac{C(d)L_0}{\delta} ||\left(\mathbf{B}_1 - \mathbf{B}, \mathbf{B}_2 - \mathbf{B}\right)||_{L^1([0,T]\times B_{R_0})} + \kappa. \tag{4.4}$$

for any $\delta \in (0,\delta_0)$.

Proof of Theorem 2. Without loss generality, we assume $t_0 = 0$.
Step 1: By proposition 2, there exist unit vectors $\xi_t(x) \in \mathbb{R}^m, \eta_t(x) \in \mathbb{R}^d$ such that $D^s b_t(x) = \xi_t(x) \otimes \eta_t(x)|D^s b_t|(x)$ i.e $D^s_{x_j} b_{tk}(x) = \xi_{tk}(x)\eta_{tj}(x)|D^s b_t|(x)$ for any $k = 1,...,m, j = 1,...,d$.
Let $\eta^\varepsilon_t \in C^\infty((0,T)\times\mathbb{R}^d,\mathbb{R}^d), \xi^\varepsilon_t \in C^\infty((0,T)\times\mathbb{R}^d,\mathbb{R}^m)$ be such that $|\eta^\varepsilon_t| = |\xi^\varepsilon_t| = 1$ and

$$\lim_{\varepsilon\to 0}\int_0^T\int_{\mathbb{R}^d} |\eta_t - \eta^\varepsilon_t| d|D^s b_t| dt + \lim_{\varepsilon\to 0}\int_0^T\int_{\mathbb{R}^d} |\xi_t - \xi^\varepsilon_t| d|D^s b_t| dt = 0.$$

For $\delta \in (0, \frac{1}{100}), 1 < \gamma < |\log(\delta)|, \varepsilon > 0$, and $t \in [0,T]$, let us define the quantity

$$\Phi^{\gamma,\varepsilon}_\delta(t) = \frac{1}{2}\int_D \log\left(1 + \frac{|X_{1t}(x) - X_{2t}(x)|^2 + \gamma\langle\eta^\varepsilon_t(X_{1t}(x)), X_{1t}(x) - X_{2t}(x)|\rangle^2}{\delta^2}\right) dx. \tag{4.5}$$

where $D = B_r \cap G_{1,R} \cap G_{2,R}$. Since $\partial_t X_{jt} = \mathbf{B}_{jt}(X_{jt})$ $j = 1,2$, one has for any $t_1 \in [0,T]$

$$\begin{aligned}
\sup_{t_1\in[0,T]} \Phi^{\gamma,\varepsilon}_\delta(t_1) &= \sup_{t_1\in[0,T]} \int_0^{t_1} \frac{d\Phi^{\gamma,\varepsilon}_\delta(t)}{dt} dt \\
&\le \sup_{t_1\in[0,T]} \int_0^{t_1}\int_D \frac{\langle X_{1t} - X_{2t}, \mathbf{B}_{1t}(X_{1t}) - \mathbf{B}_{2t}(X_{2t})\rangle}{\delta^2 + |X_{1t} - X_{2t}|^2 + \gamma\langle\eta^\varepsilon_t(X_{1t}), X_{1t} - X_{2t}\rangle^2} dxdt \\
&+ \sup_{t_1\in[0,T]} \int_0^{t_1}\int_D \frac{\gamma\langle\eta^\varepsilon(X_{1t}), X_{1t} - X_{2t}\rangle\langle\eta^\varepsilon_t(X_{1t}), \mathbf{B}_{1t}(X_{1t}) - \mathbf{B}_{2t}(X_{2t})\rangle}{\delta^2 + |X_{1t} - X_{2t}|^2 + \gamma\langle\eta^\varepsilon_t(X_{1t}), X_{1t} - X_{2t}\rangle^2} dxdt \\
&+ \sup_{t_1\in[0,T]} \int_0^{t_1}\int_D \frac{\gamma\langle\eta^\varepsilon(X_{1t}), X_{1t} - X_{2t}\rangle\langle(\nabla\eta^\varepsilon_t)(X_{1t})\mathbf{B}_{1t}(X_{1t}), X_{1t} - X_{2t}\rangle}{\delta^2 + |X_{1t} - X_{2t}|^2 + \gamma\langle\eta^\varepsilon_t(X_{1t}), X_{1t} - X_{2t}\rangle^2} dxdt \\
&+ \sup_{t_1\in[0,T]} \int_0^{t_1}\int_D \frac{\gamma\langle\eta^\varepsilon(X_{1t}), X_{1t} - X_{2t}\rangle\langle(\partial_t\eta^\varepsilon_t)(X_{1t}), X_{1t} - X_{2t}\rangle}{\delta^2 + |X_{1t} - X_{2t}|^2 + \gamma\langle\eta^\varepsilon_t(X_{1t}), X_{1t} - X_{2t}\rangle^2} dxdt \\
&= I_1(\delta,\varepsilon,\gamma) + I_2(\delta,\varepsilon,\gamma) + I_3(\delta,\varepsilon,\gamma) + I_4(\delta,\varepsilon,\gamma).
\end{aligned} \tag{4.6}$$

By $||\eta^\varepsilon||_{L^\infty(\mathbb{R}^d)} \leq 1$, and changing variable along the flows with $(X_{jt})_\#\mathcal{L}^d \leq L_0\mathcal{L}^d$ for all $t \in [0,T]$ and $j = 1, 2$, we get

$$\begin{aligned} I_1(\delta,\varepsilon,\gamma) + I_2(\delta,\varepsilon,\gamma) \leq & \frac{CL_0\gamma^{1/2}}{\delta}||\left(\mathbf{B}_1 - \mathbf{B}, \mathbf{B}_2 - \mathbf{B}\right)||_{L^1([0,T]\times B_R)} \\ & + I_5(\delta,\varepsilon,\gamma) + I_6(\delta,\varepsilon,\gamma), \end{aligned} \tag{4.7}$$

and

$$|I_3(\delta,\varepsilon,\gamma)| \leq CL_0\gamma^{1/2}||\nabla\eta^\varepsilon||_{L^\infty}||\mathbf{B}_1||_{L^1([0,T]\times B_R)}, \quad |I_4(\delta,\varepsilon,\gamma)| \leq C\gamma^{1/2}r^dT||\partial_t\eta^\varepsilon||_{L^\infty} \tag{4.8}$$

where

$$I_5(\delta,\varepsilon,\gamma) = \sup_{t_1\in[0,T]}\int_0^{t_1}\int_D \frac{\langle X_{1t} - X_{2t}, \mathbf{B}_t(X_{1t}) - \mathbf{B}_t(X_{2t})\rangle}{\delta^2 + |X_{1t} - X_{2t}|^2 + \gamma\langle\eta_t^\varepsilon(X_{1t}), X_{1t} - X_{2t}\rangle^2}dxdt;$$

$$I_6(\delta,\varepsilon,\gamma) = \sup_{t_1\in[0,T]}\int_0^{t_1}\int_D \frac{\gamma\langle\eta_t^\varepsilon(X_{1t}), X_{1t} - X_{2t}\rangle\langle\eta_t^\varepsilon(X_{1t}), \mathbf{B}_t(X_{1t}) - \mathbf{B}_t(X_{2t})\rangle}{\delta^2 + |X_{1t} - X_{2t}|^2 + \gamma\langle\eta_t^\varepsilon(X_{1t}), X_{1t} - X_{2t}\rangle^2}dxdt.$$

On the other hand,

$$\begin{aligned} \sup_{t_1\in[0,T]}\Phi_\delta^{\gamma,\varepsilon}(t_1) \geq & \frac{1}{2}|\log(\delta)|\sup_{t_1\in[0,T]}\mathcal{L}^d\left(\left\{x \in D : |X_{1t_1}(x) - X_{2t_1}(x)| > \delta^{1/2}\right\}\right) \\ \geq & \frac{1}{2}|\log(\delta)|\sup_{t_1\in[0,T]}\mathcal{L}^d\left(\left\{x \in B_r : |X_{1t_1}(x) - X_{2t_1}(x)| > \delta^{1/2}\right\}\right) \\ & - \frac{1}{2}|\log(\delta)|\left(\mathcal{L}^d\left(B_r\backslash G_{1,R}\right) + \mathcal{L}^d\left(B_r\backslash G_{2,R}\right)\right). \end{aligned} \tag{4.9}$$

It follows from (4.6),(4.7),(4.8) and (4.9) and $\gamma < |\log(\delta)|$ that for any $t_1 \in [0,T]$

$$\begin{aligned} & \sup_{t_1\in[0,T]}\mathcal{L}^d\left(\left\{x \in D : |X_{1t_1}(x) - X_{2t_1}(x)| > \delta^{1/2}\right\}\right) \leq \mathcal{L}^d\left(B_r\backslash G_{1,R}\right) + \mathcal{L}^d\left(B_r\backslash G_{2,R}\right) \\ & + \frac{C(\varepsilon,\gamma,r,T)}{|\log(\delta)|}\left(L_0||B_1||_{L^1([0,T]\times B_R)} + 1\right) + \frac{CL_0}{\delta}||\left(\mathbf{B}_1 - \mathbf{B}, \mathbf{B}_2 - \mathbf{B}\right)||_{L^1([0,T]\times B_R)} \\ & + \frac{2I_5(\delta,\varepsilon,\gamma)}{|\log(\delta)|} + \frac{2I_6(\delta,\varepsilon,\gamma)}{|\log(\delta)|}. \end{aligned} \tag{4.10}$$

Step 2: We prove that for any $\varepsilon_1 \in (0, 1/100)$,

$$\begin{aligned} & \limsup_{\delta\to0}\frac{I_5(\delta,\varepsilon,\gamma)}{|\log(\delta)|} \leq C(\varepsilon_1)\int_0^T\int_{\mathbb{R}^d}|\eta_t - \eta_t^\varepsilon|d|D^sb_t|dt \\ & + C(L_0)\varepsilon_1|\log(\varepsilon_1)|\int_0^T\int_{\mathbb{R}^d}d|D^sb_t|dt + C(L_0,\varepsilon_1)\gamma^{-1/2}\int_0^T\int_{\mathbb{R}^d}d|D^sb_t|dt. \end{aligned} \tag{4.11}$$

Indeed, thanks to (4.29) in Lemma 7 below with $x_1 = X_{1t}, x_2 = X_{2t} \in B_R$ and changing variable along the flows with $(X_{lt})_{\#}\mathcal{L}^d \le L_0\mathcal{L}^d$ for all $t \in [0,T]$ and $l = 1,2$, we find that

$$\begin{aligned}
\limsup_{\delta\to 0}\frac{I_5(\delta,\varepsilon,\gamma)}{|\log(\delta)|} &\le \limsup_{\delta\to 0}\frac{2L_0}{|\log(\delta)|}\sum_{i,j}\int_0^T\int_{B_R}\frac{\mathbf{P}_1(Db)}{\delta}\wedge \mathbf{T}^1_{\varepsilon_1,i,j}(D^a b_j)dxdt \\
&+ \limsup_{\delta\to 0}\frac{2L_0}{|\log(\delta)|}\sum_{i,j}\int_0^T\int_{B_R}\frac{\mathbf{P}_1(Db)}{\delta}\wedge \mathbf{T}^1_{\varepsilon_1,i,j}(\omega^{\varepsilon}_{tij})dxdt \\
&+ \limsup_{\delta\to 0}\frac{2L_0}{|\log(\delta)|}\int_0^T\int_{B_R}\mathbf{P}_1(Db)dxdt \\
&+ \limsup_{\delta\to 0}\frac{2L_0\varepsilon_1}{|\log(\delta)|}\sum_{i,j}\int_0^T\int_{B_R}\frac{\mathbf{P}_1(Db)}{\delta}\wedge \mathbf{T}^2_{\varepsilon_1,i,j}(Db_{tj})dxdt \\
&+ \limsup_{\delta\to 0}\frac{2L_0\gamma^{-1/2}}{|\log(\delta)|}\sum_{i,j}\int_0^T\int_{B_R}\frac{\mathbf{P}_1(Db)}{\delta}\wedge \mathbf{T}^1_{\varepsilon_1,i,j}(\xi_{tj}|D^s b_{tj}|)dxdt \\
&= (1) + (2) + (3) + (4) + (5), \qquad (4.12)
\end{aligned}$$

where $\sum_{i,j} := \sum_{i=1}^d\sum_{j=1}^m$, $\omega^{\varepsilon}_{tij} := (\eta_t - \eta_t^{\varepsilon})\xi_{tj}|D^s b_{tj}|$ and $\mathbf{T}^1_{\varepsilon_1,i,j}, \mathbf{T}^2_{\varepsilon_1,i,j}$ are defined in Lemma 7 and $\mathbf{P}_1(Db) \in L^1((0,T), L^{q_0}_{loc}(\mathbb{R}^d))$ for some $q_0 > 1$.

Clearly, (3) = 0. We can apply (2.31) in Proposition 7 (and Remark 10) to $\mathbf{T}^1_{\varepsilon_1,i,j}$ and $f = \mathbf{P}_1(Db)$ to get that (1) = 0,

$$(2) \le C(\varepsilon_1)\int_0^T\int_{\mathbb{R}^d}|\eta_t - \eta_t^{\varepsilon}|d|D^s b_t|dt; \quad (5) \le C(\varepsilon_1)\gamma^{-1/2}\int_0^T\int_{\mathbb{R}^d}d|D^s b_t|dt.$$

On the other hand, it is clear to see that $\mathbf{K}^i_j$ and $\Theta_2^{\varepsilon_1,e}$ satisfy Theorem 1. So, we can apply (3.9) in Theorem 1 to $\mathbf{T}^2_{\varepsilon_1,i,j}, f = \mathbf{P}_1(Db)$, (with $\alpha = 1, \varepsilon = \varepsilon_1$) and obtain that

$$(4) \le C\varepsilon_1|\log(\varepsilon_1)|\int_0^T\int_{\mathbb{R}^d}d|D^s b_t|dt.$$

Plugging above estimates into (4.12) gives (4.11).
Step 3: We prove that for any $\varepsilon_2 \in (0, 1/100)$

$$\limsup_{\delta\to 0}\frac{I_6(\delta,\varepsilon,\gamma)}{|\log(\delta)|} \le C(\varepsilon_2)\gamma^{1/2}\int_0^T\int_{\mathbb{R}^d}|\eta_t - \eta_t^{\varepsilon}|d|D^s b_t|dt + C\gamma^{1/2}\varepsilon_2|\log(\varepsilon_2)|\int_0^T\int_{\mathbb{R}^d}d|D^s b_t|dt. \qquad (4.13)$$

Indeed, thanks to (4.30) in Lemma 7 below with $x_1 = X_{1t}, x_2 = X_{2t} \in B_R, \varepsilon_1 = \varepsilon_2$ and changing variable along the flows with $(X_{lt})_\#\mathcal{L}^d \leq L_0\mathcal{L}^d$ for all $t \in [0,T]$ and $l = 1,2$, we find that

$$\begin{aligned}\limsup_{\delta\to 0}\frac{I_6(\delta,\varepsilon,\gamma)}{|\log(\delta)|} &\leq \limsup_{\delta\to 0}\frac{4\gamma^{1/2}L_0}{|\log(\delta)|}\sum_{i,j}\int_0^T\int_{B_R}\frac{\mathbf{P}_2(Db)}{\delta}\wedge \mathbf{T}^1_{\varepsilon_2,i,j}(D^ab_j)dxdt\\ &+\limsup_{\delta\to 0}\frac{4\gamma^{1/2}L_0}{|\log(\delta)|}\sum_{i,j}\int_0^T\int_{B_R}\frac{\mathbf{P}_2(Db)}{\delta}\wedge \mathbf{T}^1_{\varepsilon_2,i,j}(\omega^\varepsilon_{tij})dxdt\\ &+\limsup_{\delta\to 0}\frac{4L_0\gamma^{1/2}}{|\log(\delta)|}\sum_{i,j}\int_0^T\int_{B_R}\mathbf{P}_2(Db)dxdt\\ &+\limsup_{\delta\to 0}\frac{2L_0\gamma^{1/2}\varepsilon_2}{|\log(\delta)|}\sum_{i,j}\int_0^T\int_{B_R}\frac{\mathbf{P}_2(Db)}{\delta}\wedge \mathbf{T}^2_{\varepsilon_2,i,j}(Db_j)dxdt\\ &+\limsup_{\delta\to 0}\frac{C(\varepsilon_2,\gamma)}{|\log(\delta)|}\int_0^T\int_{B_R}\frac{\mathbf{I}_1(\mathbf{1}_{B_{4\lambda}}(\mathrm{div}^a(B_t))^+)}{\delta}\wedge\mathbf{M}(\mathbf{1}_{B_{4\lambda}}(\mathrm{div}^a(B_t))^+)dxdt\\ &=(6)+(7)+(8)+(9)+(10),\end{aligned}$$

where $\omega^\varepsilon_{tij} := (\eta_t-\eta^\varepsilon_t)\xi_{tj}|D^sb_{tj}|$ and $\mathbf{P}_2(Db)\in L^1((0,T),L^{q_0}_{loc}(\mathbb{R}^d))$ for some $q_0>1$. Similarly, we also obtain that $(6)+(8)=0$ and

$$(7)\leq C(\varepsilon_2)\gamma^{1/2}\int_0^T\int_{\overline{B}_\lambda}|\eta_t-\eta^\varepsilon_t|d|D^sb_t|dt;\quad (9)\leq C\gamma^{1/2}\varepsilon_2|\log(\varepsilon_2)|\int_0^T\int_{\overline{B}_\lambda}d|D^sb_t|dt.$$

Moreover, by 2.10 in Lemma (1), one has $(10)=0$. Thus, we get (4.13). Therefore, we derive from (4.10) and (4.11), (4.13) that

$$\begin{aligned}&\sup_{t_1\in[0,T]}\mathcal{L}^d\left(\left\{x\in D:|X_{1t_1}(x)-X_{2t_1}(x)|>\delta^{1/2}\right\}\right)\leq\mathcal{L}^d\left(B_r\backslash G_{1,R}\right)+\mathcal{L}^d\left(B_r\backslash G_{2,R}\right)\\ &+\frac{C(d)L_0}{\delta}||\left(\mathbf{B}_1-\mathbf{B},\mathbf{B}_2-\mathbf{B}\right)||_{L^1([0,T]\times B_R)}+A(\delta)\end{aligned}\tag{4.14}$$

and

$$\begin{aligned}\limsup_{\delta\to 0}A(\delta)&\leq 2\limsup_{\delta\to 0}\frac{I_5(\delta,\varepsilon,\gamma)}{|\log(\delta)|}+2\limsup_{\delta\to 0}\frac{I_6(\delta,\varepsilon,\gamma)}{|\log(\delta)|}\\ &\leq C(\varepsilon_1)\int_0^T\int_{\mathbb{R}^d}|\eta_t-\eta^\varepsilon_t|d|D^sb_t|dt+C\varepsilon_1|\log(\varepsilon_1)|\int_0^T\int_{\mathbb{R}^d}d|D^sb_t|dt\\ &+C(\varepsilon_1)\gamma^{-1/2}\int_0^T\int_{\mathbb{R}^d}d|D^sb_t|dt+C(\varepsilon_2)\gamma^{1/2}\int_0^T\int_{\mathbb{R}^d}|\eta_t-\eta^\varepsilon_t|d|D^sb_t|dt\\ &\qquad+C\gamma^{1/2}\varepsilon_2|\log(\varepsilon_2)|\int_0^T\int_{\mathbb{R}^d}d|D^sb_t|dt.\end{aligned}\tag{4.15}$$

In the right hand side of (4.15), we let $\varepsilon\to 0$, then $\varepsilon_2\to 0$, $\gamma\to\infty$ and $\varepsilon_1\to 0$ to get that $\limsup_{\delta\to 0}A(\delta)\leq 0$. Combining this and (4.14) yields (4.2). The proof is complete. □

Let $\Theta^{\varepsilon,e}_1,\Theta^{\varepsilon,e}_2$ be in Lemma 4. Given $\varepsilon_1\in(0,1/100)$, we have the following identities:

Lemma 6. *For any $i=1,...,d$, $x_1\neq x_2\in B_R(0)$ and $\varepsilon_1\in(0,1/100)$ we have*

$$\mathbf{B}^i_t(x_1)-\mathbf{B}^i_t(x_2)=rA^{regular}_{i1}+rA^{appro}_{i1}+rA^{diff\text{-}1}_{i1}+rA^{diff\text{-}2}_{i1}+r\varepsilon_1A_{i2}+r(\mathbf{e}_1.\eta^\varepsilon_t(x_1))A^{singular}_{i1}\tag{4.16}$$

and [1]

$$\langle \eta^{\varepsilon}_t(x_1), A_1^{singular}\rangle = E^{regular} + E^{appro} + E^{diff\text{-}1} + E^{diff\text{-}2} + +\tilde{\Theta}^{\varepsilon_1,\mathbf{e}_1}_{1,r} \star [\mathrm{div}(\mathbf{B}_t)]\,(x_1) + \tilde{\Theta}^{\varepsilon_1,\mathbf{e}_2}_{1,r} \star [\mathrm{div}(\mathbf{B}_t)]\,(x_2), \tag{4.17}$$

where $\mathbf{e}_1 = -\mathbf{e}_2 = \frac{x_1-x_2}{|x_1-x_2|}$ *and* $r = |x_1 - x_2|$,

$$\Theta^{\varepsilon_1,e}_{l,r}(.) \;=\; \Theta^{\varepsilon_1,e}_{l}(\frac{\cdot}{r}), \qquad \tilde{\Theta}^{\varepsilon_1,e}_{l,r}(.) \;=\; \frac{1}{r}\frac{\varepsilon_1^{-d+1}}{|.|^{d-1}}\Theta^{\varepsilon_1,e}_{l,r}(.), \;\; l \;=\; 1,2; \tag{4.18}$$

$$A_{i1}^{regular} \;:=\; \sum_{j=1}^{m}\Big[\mathbf{K}^i_j \star \tilde{\Theta}^{\varepsilon_1,\mathbf{e}_1}_{1,r} \star (\mathbf{e}_1.D^a b_j)\Big]\,(x_1) \;-\; \Big[\mathbf{K}^i_j \star \tilde{\Theta}^{\varepsilon_1,\mathbf{e}_2}_{1,r} \star (\mathbf{e}_2.D^a b_j)\Big]\,(x_2); \tag{4.19}$$

$$A_{i1}^{appro} := \sum_{j=1}^{m}\Big[\mathbf{K}^i_j \star \tilde{\Theta}^{\varepsilon_1,\mathbf{e}_1}_{1,r} \star ((\mathbf{e}_1.(\eta_t - \eta^{\varepsilon}_t))\xi_{tj}|D^s b_{tj}|)\Big]\,(x_1) + \Big[\mathbf{K}^i_j \star \tilde{\Theta}^{\varepsilon_1,\mathbf{e}_2}_{1,r} \star ((\mathbf{e}_1.(\eta_t - \eta^{\varepsilon}_t))\xi_{tj}|D^s b_{tj}|)\Big]\,(x_2); \tag{4.20}$$

$$A_{i1}^{diff\text{-}1} := \sum_{j=1}^{m}\Big[\mathbf{K}^i_j \star \tilde{\Theta}^{\varepsilon_1,\mathbf{e}_1}_{1,r} \star ((\mathbf{e}_1.(\eta^{\varepsilon}_t - \eta^{\varepsilon}_t(x_1)))\xi_{tj}|D^s b_{tj}|)\Big]\,(x_1) + \Big[\mathbf{K}^i_j \star \tilde{\Theta}^{\varepsilon_1,\mathbf{e}_2}_{1,r} \star ((\mathbf{e}_1.(\eta^{\varepsilon}_t - \eta^{\varepsilon}_t(x_2)))\xi_{tj}|D^s b_{tj}|)\Big]\,(x_2); \tag{4.21}$$

$$A_{i1}^{diff\text{-}2} \;:=\; \sum_{j=1}^{m}(\mathbf{e}_1.(\eta^{\varepsilon}_t(x_2) \;-\; \eta^{\varepsilon}_t(x_1)))\Big[\mathbf{K}^i_j \star \tilde{\Theta}^{\varepsilon_1,\mathbf{e}_2}_{1,r} \star (\xi_{tj}|D^s b_{tj}|)\Big]\,(x_2); \tag{4.22}$$

$$A_{i1}^{singular} := \sum_{j=1}^{m}\Big[\mathbf{K}^i_j \star \tilde{\Theta}^{\varepsilon_1,\mathbf{e}_1}_{1,r} \star (\xi_{tj}|D^s b_{tj}|)\Big]\,(x_1) + \Big[\mathbf{K}^i_j \star \tilde{\Theta}^{\varepsilon_1,\mathbf{e}_2}_{1,r} \star (\xi_{tj}|D^s b_{tj}|)\Big]\,(x_2); \tag{4.23}$$

$$A_{i2} \;:=\; \sum_{j=1}^{m}\Big[\mathbf{K}^i_j \star \tilde{\Theta}^{\varepsilon_1,\mathbf{e}_1}_{2,r} \star Db_j\Big]\,(x_1) \;-\; \Big[\mathbf{K}^i_j \star \tilde{\Theta}^{\varepsilon_1,\mathbf{e}_2}_{2,r} \star Db_j\Big]\,(x_2); \tag{4.24}$$

$$E^{regular} := \sum_{i=1}^{d}\sum_{j=1}^{m} -\Big[\mathbf{K}^i_j \star \tilde{\Theta}^{\varepsilon_1,\mathbf{e}_1}_{1,r} \star \big(D^a_{x_i} b_{tj}\big)\Big]\,(x_1) - \Big[\mathbf{K}^i_j \star \tilde{\Theta}^{\varepsilon_1,\mathbf{e}_2}_{1,r} \star \big(D^a_{x_i} b_{tj}\big)\Big]\,(x_2); \tag{4.25}$$

$$E^{appro} := \sum_{i=1}^{d}\sum_{j=1}^{m}\Big[\mathbf{K}^i_j \star \tilde{\Theta}^{\varepsilon_1,\mathbf{e}_1}_{1,r} \star ((\eta^{\varepsilon}_{ti} - \eta_{ti})\xi_{tj}|D^s b_{tj}|)\Big]\,(x_1) + \Big[\mathbf{K}^i_j \star \tilde{\Theta}^{\varepsilon_1,\mathbf{e}_2}_{1,r} \star ((\eta^{\varepsilon}_{ti} - \eta_{ti})\xi_{tj}|D^s b_{tj}|)\Big]\,(x_2); \tag{4.26}$$

$$E^{diff\text{-}1} := \sum_{i=1}^{d}\sum_{j=1}^{m}\Big[\mathbf{K}^i_j \star \tilde{\Theta}^{\varepsilon_1,\mathbf{e}_1}_{1,r} \star ((\eta^{\varepsilon}_{ti}(x_1) - \eta^{\varepsilon}_{ti})\xi_{tj}|D^s b_{tj}|)\Big]\,(x_1) + \Big[\mathbf{K}^i_j \star \tilde{\Theta}^{\varepsilon_1,\mathbf{e}_2}_{1,r} \star ((\eta^{\varepsilon}_{ti}(x_1) - \eta^{\varepsilon}_{ti})\xi_{tj}|D^s b_{tj}|)\Big]\,(x_2); \tag{4.27}$$

$$E^{diff\text{-}2} \;:=\; \sum_{i=1}^{d}\sum_{j=1}^{m}(\eta^{\varepsilon}_{ti}(x_1) \;-\; \eta^{\varepsilon}_{ti}(x_2))\Big[\mathbf{K}^i_j \star \tilde{\Theta}^{\varepsilon_1,\mathbf{e}_2}_{1,r} \star (\xi_{tj}|D^s b_{tj}|)\Big]\,(x_2). \tag{4.28}$$

[1] Here $A_1^{\mathrm{singular}} = (A_{11}^{\mathrm{singular}}, A_{21}^{\mathrm{singular}}, ..., A_{d1}^{\mathrm{singular}})$

Proof. **Step 1.** By Proposition 4 with $\varepsilon = \varepsilon_1$ we have

$$b_{tj}(x_1 - z) - b_{tj}(x_2 - z) = r\tilde{\Theta}^{\varepsilon_1,\mathbf{e}_1}_{1,r} \star (\mathbf{e}_1.Db_{tj})(x_1 - z) + \varepsilon_1 r\tilde{\Theta}^{\varepsilon_1,\mathbf{e}_1}_{2,r} \star Db_{tj}(x_1 - z)$$
$$- r\tilde{\Theta}^{\varepsilon_1,\mathbf{e}_2}_{1,r} \star (\mathbf{e}_2.Db_{tj})(x_2 - z) - \varepsilon_1 r\tilde{\Theta}^{\varepsilon_1,\mathbf{e}_2}_{2,r} \star Db_{tj}(x_2 - z),$$

for any $z \in \mathbb{R}^d$. So, by (4.1), we get

$$\mathbf{B}^i_t(x_1) - \mathbf{B}^i_t(x_2) = \sum_{j=1}^{m} \left(\mathbf{K}^i_j \star b_{tj}(x_1) - \mathbf{K}^i_j \star b_{tj}(x_2)\right)$$
$$= \sum_{j=1}^{m} r\left[\mathbf{K}^i_j \star \tilde{\Theta}^{\varepsilon_1,\mathbf{e}_1}_{1,r} \star (\mathbf{e}_1.Db_{tj})\right](x_1) + r\varepsilon_1\left[\mathbf{K}^i_j \star \Theta^{\varepsilon_1,\mathbf{e}_1}_{2,r} \star Db_{tj}\right](x_1)$$
$$- r\left[\mathbf{K}^i_j \star \tilde{\Theta}^{\varepsilon_1,\mathbf{e}_2}_{1,r} \star (\mathbf{e}_2.Db_{tj})\right](x_2) - r\varepsilon_1\left[\mathbf{K}^i_j \star \tilde{\Theta}^{\varepsilon_1,\mathbf{e}_2}_{2,r} \star Db_{tj}\right](x_2).$$

Using $Db_{tj} = D^a b_{tj} + \xi_{tj}\eta_t|D^s b_t|$ yields

$$\mathbf{B}^i_t(x_1) - \mathbf{B}^i_t(x_2) = rA^{\text{regular}}_{i1} + r\varepsilon_1 A_{i2} + \sum_{j=1}^{m} r\left[\mathbf{K}^i_j \star \tilde{\Theta}^{\varepsilon_1,\mathbf{e}_1}_{1,r} \star ((\mathbf{e}_1.\eta_t)\xi_{tj}|D^s b_{tj}|)\right](x_1)$$
$$+ r\left[\mathbf{K}^i_j \star \tilde{\Theta}^{\varepsilon_1,\mathbf{e}_2}_{1,r} \star ((\mathbf{e}_1.\eta_t)\xi_{tj}|D^s b_{tj}|)\right](x_2).$$

Since $\mathbf{e}_1.\eta_t = \mathbf{e}_1.(\eta_t - \eta^\varepsilon_t) + \mathbf{e}_1.(\eta^\varepsilon_t - \eta^\varepsilon_t(x_1)) + \mathbf{e}_1.\eta^\varepsilon_t(x_1)$,

$$\mathbf{B}^i_t(x_1) - \mathbf{B}^i_t(x_2) = rA^{\text{regular}}_{i1} + r\varepsilon_1 A_{i2} + rA^{\text{appro}}_{i1} + rA^{\text{diff-1}}_{i1} + rA^{\text{diff-2}}_{i1}$$
$$+ \sum_{j=1}^{m} r\left[\mathbf{K}^i_j \star \tilde{\Theta}^{\varepsilon_1,\mathbf{e}_1}_{1,r} \star ((\mathbf{e}_1.\eta^\varepsilon_t(x_1))\xi_{tj}|D^s b_{tj}|)\right](x_1) + r\left[\mathbf{K}^i_j \star \tilde{\Theta}^{\varepsilon_1,\mathbf{e}_2}_{1,r} \star ((\mathbf{e}_1.\eta^\varepsilon_t(x_1))\xi_{tj}|D^s b_{tj}|)\right](x_2)$$
$$= rA^{\text{regular}}_{i1} + rA^{\text{appro}}_{i1} + rA^{\text{diff-1}}_{i1} + rA^{\text{diff-2}}_{i1} + r\varepsilon_1 A_{i2} + r(\mathbf{e}_1.\eta^\varepsilon_t(x_1))A^{\text{singular}}_{i1},$$

which implies (4.16).

Step 2. We have

$$\langle \eta^\varepsilon_t(x_1), A^{\text{singular}}_1\rangle = \sum_{i=1}^{d}\sum_{j=1}^{m}\left[\mathbf{K}^i_j \star \tilde{\Theta}^{\varepsilon_1,\mathbf{e}_1}_{1,r} \star (\eta^\varepsilon_{ti}(x_1)\xi_{tj}|D^s b_{tj}|)\right](x_1)$$
$$+ \left[\mathbf{K}^i_j \star \tilde{\Theta}^{\varepsilon_1,\mathbf{e}_2}_{1,r} \star (\eta^\varepsilon_{ti}(x_1)\xi_{tj}|D^s b_{tj}|)\right](x_2).$$

Since $\eta^\varepsilon_{ti}(x_1) = (\eta^\varepsilon_{ti}(x_1) - \eta^\varepsilon_t) + (\eta^\varepsilon_{ti} - \eta_{ti}) + \eta_{ti}$, and $\eta_{ti}\xi_{tj}|D^s b_{tj}| = D^s_{x_i} b_{tj} = -D^a_{x_i} b_{tj} + D_{x_i} b_{tj}$, thus

$$\langle \eta^\varepsilon_t(x_1), A^{\text{singular}}_1\rangle = E^{\text{diff-1}} + E^{\text{diff-2}} + E^{\text{appro}} + E^{\text{regular}}$$
$$+ \sum_{i=1}^{d}\sum_{j=1}^{m}\left[\mathbf{K}^i_j \star \tilde{\Theta}^{\varepsilon_1,\mathbf{e}_1}_{1,r} \star D_{x_i} b_{tj}\right](x_1) + \left[\mathbf{K}^i_j \star \tilde{\Theta}^{\varepsilon_1,\mathbf{e}_2}_{1,r} \star D_{x_i} b_{tj}\right](x_2).$$

Using associative and commutativity properties of convolution and $\sum_{i=1}^{d}\sum_{j=1}^{m} \mathbf{K}^i_j \star D_{x_i} b_{tj} = \sum_{i=1}^{d} D_{x_i}\left(\sum_{j=1}^{m} \mathbf{K}^i_j \star b_{tj}\right) = \text{div}(\mathbf{B}_t)$ yields

$$\langle \eta^\varepsilon_t(x_1), A^{\text{singular}}_1\rangle = E^{\text{regular}} + E^{\text{appro}} + E^{\text{diff-1}} + E^{\text{diff-2}}$$
$$+ \tilde{\Theta}^{\varepsilon_1,\mathbf{e}_1}_{1,r} \star \left[\sum_{i=1}^{d}\sum_{j=1}^{m} \mathbf{K}^i_j \star D_{x_i} b_{tj}\right](x_1) + \tilde{\Theta}^{\varepsilon_1,\mathbf{e}_2}_{1,r} \star \left[\sum_{i=1}^{d}\sum_{j=1}^{m} \mathbf{K}^i_j \star D_{x_i} b_{tj}\right](x_2)$$
$$= E^{\text{regular}} + E^{\text{appro}} + E^{\text{diff-1}} + E^{\text{diff-2}} + \tilde{\Theta}^{\varepsilon_1,\mathbf{e}_1}_{1,r} \star [\text{div}(\mathbf{B}_t)](x_1) + \tilde{\Theta}^{\varepsilon_1,\mathbf{e}_2}_{1,r} \star [\text{div}(\mathbf{B}_t)](x_2).$$

This gives (4.17). The proof is complete. □

Lemma 6 implies that

Lemma 7. *We define for* $\varepsilon_1 \in (0, 1/100)$

$$\mathbf{T}^l_{\varepsilon_1,i,j}(\mu_l)(x) = \sup_{\rho\in(0,2R),e\in S^{d-1}} \frac{\varepsilon_1^{-d+1}}{\rho}\left|\left(\frac{1}{|.|^{d-1}}\Theta^{\varepsilon_1,e}_{l,\rho}(.)\right)\star\mathbf{K}\star\mu_l(x)\right| \quad \forall \quad x\in\mathbb{R}^d,$$

with $\mu_2 \in \mathcal{M}_b(\mathbb{R}^d,\mathbb{R}^d), \mu_1 \in \mathcal{M}_b(\mathbb{R}^d)$ *or* $\mu_1 \in \mathcal{M}_b(\mathbb{R}^d,\mathbb{R}^d)$*. There exists* $\mathbf{P}_1(Db)(x,t), \mathbf{P}_2(Db)(x,t) \in L^1((0,T), L^{q_0}_{loc}(\mathbb{R}^d)$ *for some* $q_0 > 1$ *such that* $||\mathbf{P}_1(Db)||_{L^1((0,T),L^{q_0}(B_R(0)))} + ||\mathbf{P}_2(Db)||_{L^1((0,T),L^{q_0}(B_R(0)))} \leq C(R,\varepsilon_1,\varepsilon)||b||_{L^1((0,T),BV(\mathbb{R}^d))}$ *for any* $R>0$ *and for any* $x_1 \neq x_2 \in B_\lambda$*, we have*

$$\begin{aligned} A_1 &:= \frac{|\langle x_1-x_2, \mathbf{B}_t(x_1)-\mathbf{B}_t(x_2)\rangle|}{\delta^2+|x_1-x_2|^2+\gamma\langle\eta^\varepsilon_t(x_1),x_1-x_2\rangle^2} \\ &\leq \sum_{l,i,j}\frac{\mathbf{P}_1(Db)(x_l,t)}{\delta}\wedge\mathbf{T}^1_{\varepsilon_1,i,j}(D^ab_j)(x_l) + \sum_{l,i,j}\frac{\mathbf{P}_1(Db)(x_l,t)}{\delta}\wedge\mathbf{T}^1_{\varepsilon_1,i,j}(\omega^\varepsilon_{tij})(x_l) + \mathbf{P}_1(Db)(x_l,t) \\ &+ \varepsilon_1\sum_{l,i,j}\frac{\mathbf{P}_1(Db)(x_l,t)}{\delta}\wedge\mathbf{T}^2_{\varepsilon_1,i,j}(Db_{tj})(x_l) + \gamma^{-1/2}\sum_{l,i,j}\frac{\mathbf{P}_1(Db)(x_l,t)}{\delta}\wedge\mathbf{T}^1_{\varepsilon_1,i,j}(\xi_{tj}|D^sb_{tj}|)(x_l); \end{aligned} \tag{4.29}$$

$$\begin{aligned} A_2 &:= \frac{\gamma\langle\eta^\varepsilon_t(x_1),x_1-x_2\rangle\langle\eta^\varepsilon_t(x_1),\mathbf{B}_t(x_1)-\mathbf{B}_t(x_2)\rangle}{\delta^2+|x_1-x_2|^2+\gamma\langle\eta^\varepsilon_t(x_1),x_1-x_2\rangle^2} \\ &\leq 2\gamma^{1/2}\sum_{l,i,j}\frac{\mathbf{P}_2(Db)(x_l,t)}{\delta}\wedge\mathbf{T}^1_{\varepsilon_1,i,j}(D^ab_j)(x_l) + 2\gamma^{1/2}\sum_{l,i,j}\frac{\mathbf{P}_2(Db)(x_l,t)}{\delta}\wedge\mathbf{T}^1_{\varepsilon_1,i,j}(\omega^\varepsilon_{tij})(x_l) \\ &\qquad + \gamma^{1/2}\sum_l\mathbf{P}_2(Db)(x_l,t) + \gamma^{1/2}\varepsilon_1\sum_{l,i,j}\frac{\mathbf{P}_2(Db)(x_l,t)}{\delta}\wedge\mathbf{T}^2_{\varepsilon_1,i,j}(Db_{tj})(x_l) \\ &\qquad + C(\varepsilon_1,\gamma)\sum_{l=1}^2\frac{\mathbf{I}_1(\mathbf{1}_{B_{4\lambda}}(\mathrm{div}^a(B_t))^+)(x_l)}{\delta}\wedge\mathbf{M}(\mathbf{1}_{B_{4\lambda}}(\mathrm{div}^a(B_t))^+)(x_l); \end{aligned} \tag{4.30}$$

where $\sum_{l,i,j} := \sum_{l=1}^2\sum_{i=1}^d\sum_{j=1}^m$, $\omega^\varepsilon_{tij} := (\eta_t-\eta^\varepsilon_t)\xi_{tj}|D^sb_{tj}|$.

Proof. Set

$$\mathbf{T}^{l,1}_{\varepsilon_1,i,j}(\mu_l)(x) = \sup_{\rho\in(0,2R),e\in S^{d-1}} \varepsilon_1^{-d+1}\left|\left(\frac{1}{|.|^{d-1}}\Theta^{\varepsilon_1,e}_{l,\rho}(.)\right)\star\mathbf{K}\star\mu_l(x)\right| \quad \forall \quad x\in\mathbb{R}^d.$$

1. Thanks to (4.16), we obtain that

$$\begin{aligned} A_1 &= \frac{r\langle\mathbf{e}_1,\mathbf{B}_t(x_1)-\mathbf{B}_t(x_2)\rangle}{\delta^2+r^2+\gamma r^2(\mathbf{e}_1.\eta^\varepsilon_t(x_1))^2} \\ &= \frac{r^2\langle\mathbf{e}_1,A_1^{\text{regular}}\rangle}{\delta^2+r^2+\gamma r^2(\mathbf{e}_1.\eta^\varepsilon_t(x_1))^2} + \frac{r^2\langle\mathbf{e}_1,A_1^{\text{appro}}\rangle}{\delta^2+r^2+\gamma r^2(\mathbf{e}_1.\eta^\varepsilon_t(x_1))^2} + \frac{r^2\langle\mathbf{e}_1,A_1^{\text{diff-1}}\rangle}{\delta^2+r^2+\gamma r^2(\mathbf{e}_1.\eta^\varepsilon_t(x_1))^2} \\ &+ \frac{r^2\langle\mathbf{e}_1,A_1^{\text{diff-2}}\rangle}{\delta^2+r^2+\gamma r^2(\mathbf{e}_1.\eta^\varepsilon_t(x_1))^2} + \frac{r^2\varepsilon_1\langle\mathbf{e}_1,A_2\rangle}{\delta^2+r^2+\gamma r^2(\mathbf{e}_1.\eta^\varepsilon_t(x_1))^2} + \frac{r^2(\mathbf{e}_1.\eta^\varepsilon_t(x_1))\langle\mathbf{e}_1,A_1^{\text{singular}}\rangle}{\delta^2+r^2+\gamma r^2(\mathbf{e}_1.\eta^\varepsilon_t(x_1))^2} \\ &= (1)+(2)+(3)+(4)+(5)+(6), \end{aligned} \tag{4.31}$$

with $r = |x_1-x_2|$. By definition of $\mathbf{T}^l_{\varepsilon_1,i,j}, \mathbf{T}^{l,1}_{\varepsilon_1,i,j}$, we can estimate that

$$
\begin{aligned}
|(1)| &\le \sum_{l,i,j} \frac{\mathbf{T}^{1,1}_{\varepsilon_1,i,j}(D^a b_j)(x_l)}{\delta} \wedge \mathbf{T}^1_{\varepsilon_1,i,j}(D^a b_j)(x_l);\\
|(2)| &\le \sum_{l,i,j} \frac{\mathbf{T}^{1,1}_{\varepsilon_1,i,j}((\eta_t-\eta^\varepsilon_t)\xi_{tj}|D^s b_{tj}|)(x_l)}{\delta} \wedge \mathbf{T}^1_{\varepsilon_1,i,j}((\eta_t - \eta^\varepsilon_t)\xi_{tj}|D^s b_{tj}|)(x_l);\\
|(3)| &\le \sum_{l,i,j} \mathbf{T}^1_{\varepsilon_1,i,j}((\eta^\varepsilon_t - \eta^\varepsilon_t(x_l))\xi_{tj}|D^s b_{tj}|)(x_l)\\
|(4)| &\le ||\nabla \eta^\varepsilon_t||_{L^\infty(\mathbb{R}^d)} \sum_{l,i,j} \mathbf{T}^{1,1}_{\varepsilon_1,i,j}(\xi_{tj}|D^s b_{tj}|)(x_l)\\
|(5)| &\le \varepsilon_1 \sum_{l,i,j} \frac{\mathbf{T}^{2,1}_{\varepsilon_1,i,j}(Db_{tj})(x_l)}{\delta} \wedge \mathbf{T}^2_{\varepsilon_1,i,j}(Db_{tj})(x_l);\\
|(6)| &\le \gamma^{-1/2} \sum_{l,i,j} \frac{\mathbf{T}^{1,1}_{\varepsilon_1,i,j}(\xi_{tj}|D^s b_{tj}|)(x_l)}{\delta} \wedge \mathbf{T}^1_{\varepsilon_1,i,j}(\xi_{tj}|D^s b_{tj}|)(x_l).
\end{aligned}
$$

Set

$$
\begin{aligned}
\mathbf{P}_1(Db)(x,t) = \sum_{i,j} &\mathbf{T}^{1,1}_{\varepsilon_1,i,j}(D^a b_j)(x) + \mathbf{T}^{1,1}_{\varepsilon_1,i,j}((\eta_t-\eta^\varepsilon_t)\xi_{tj}|D^s b_{tj}|)(x) + \mathbf{T}^1_{\varepsilon_1,i,j}((\eta^\varepsilon_t - \eta^\varepsilon_t(x))\xi_{tj}|D^s b_{tj}|)(x)\\
&+ ||\nabla \eta^\varepsilon_t||_{L^\infty(\mathbb{R}^d)} \mathbf{T}^{1,1}_{\varepsilon_1,i,j}(\xi_{tj}|D^s b_{tj}|)(x) + \mathbf{T}^{2,1}_{\varepsilon_1,i,j}(Db_{tj})(x) + \mathbf{T}^{1,1}_{\varepsilon_1,i,j}(\xi_{tj}|D^s b_{tj}|)(x).
\end{aligned}
$$

By Remarks 8 and 9, there exists $q_0 > 1$ such that $||\mathbf{P}_1(Db)||_{L^1((0,T),L^{q_0}(B_R(0)))} \le C(R,\varepsilon_1,\varepsilon)||b||_{L^1((0,T),BV(\mathbb{R}^d))}$ for any $R > 0$. Combining these with (4.31) yields (4.29).
2. Again, thanks to (4.16) we obtain that

$$
\begin{aligned}
A_2 &= \frac{\gamma r(\mathbf{e}_1.\eta^\varepsilon_t(x_1))\langle \eta^\varepsilon_t(x_1), \mathbf{B}_t(x_1) - \mathbf{B}_t(x_2)\rangle}{\delta^2 + r^2 + \gamma r^2(\mathbf{e}_1.\eta^\varepsilon_t(x_1))^2}\\
&= \frac{\gamma r^2(\mathbf{e}_1.\eta^\varepsilon_t(x_1))\langle \eta^\varepsilon_t(x_1), A_1^{\text{regular}}\rangle}{\delta^2 + r^2 + \gamma r^2(\mathbf{e}_1.\eta^\varepsilon_t(x_1))^2} + \frac{\gamma r^2(\mathbf{e}_1.\eta^\varepsilon_t(x_1))\langle \eta^\varepsilon_t(x_1), A_1^{\text{appro}}\rangle}{\delta^2 + r^2 + \gamma r^2(\mathbf{e}_1.\eta^\varepsilon_t(x_1))^2} + \frac{\gamma r^2(\mathbf{e}_1.\eta^\varepsilon_t(x_1))\langle \eta^\varepsilon_t(x_1), A_1^{\text{diff-1}}\rangle}{\delta^2 + r^2 + \gamma r^2(\mathbf{e}_1.\eta^\varepsilon_t(x_1))^2}\\
&+ \frac{\gamma r^2(\mathbf{e}_1.\eta^\varepsilon_t(x_1))\langle \eta^\varepsilon_t(x_1), A_1^{\text{diff-2}}\rangle}{\delta^2 + r^2 + \gamma r^2(\mathbf{e}_1.\eta^\varepsilon_t(x_1))^2} + \frac{\gamma r^2\varepsilon_1(\mathbf{e}_1.\eta^\varepsilon_t(x_1))\langle \eta^\varepsilon_t(x_1), A_2\rangle}{\delta^2 + r^2 + \gamma r^2(\mathbf{e}_1.\eta^\varepsilon_t(x_1))^2} + \frac{\gamma r^2(\mathbf{e}_1.\eta^\varepsilon_t(x_1))^2\langle \eta^\varepsilon_t(x_1), A_1^{\text{singular}}\rangle}{\delta^2 + r^2 + \gamma r^2(\mathbf{e}_1.\eta^\varepsilon_t(x_1))^2}\\
&= (7) + (8) + (9) + (10) + (11) + \frac{\gamma r^2(\mathbf{e}_1.\eta^\varepsilon_t(x_1))^2\langle \eta^\varepsilon_t(x_1), A_1^{\text{singular}}\rangle}{\delta^2 + r^2 + \gamma r^2(\mathbf{e}_1.\eta^\varepsilon_t(x_1))^2}. \qquad (4.32)
\end{aligned}
$$

Plugging (4.17) into (4.32) gives

$$
\begin{aligned}
A_2 &= (7) + (8) + (9) + (10) + (11) + \frac{\gamma r^2(\mathbf{e}_1.\eta^\varepsilon_t(x_1))^2 E^{\text{regular}}}{\delta^2 + r^2 + \gamma r^2(\mathbf{e}_1.\eta^\varepsilon_t(x_1))^2} + \frac{\gamma r^2(\mathbf{e}_1.\eta^\varepsilon_t(x_1))^2 E^{\text{appro}}}{\delta^2 + r^2 + \gamma r^2(\mathbf{e}_1.\eta^\varepsilon_t(x_1))^2}\\
&+ \frac{\gamma r^2(\mathbf{e}_1.\eta^\varepsilon_t(x_1))^2 E^{\text{diff-1}}}{\delta^2 + r^2 + \gamma r^2(\mathbf{e}_1.\eta^\varepsilon_t(x_1))^2} + \frac{\gamma r^2(\mathbf{e}_1.\eta^\varepsilon_t(x_1))^2 E^{\text{diff-2}}}{\delta^2 + r^2 + \gamma r^2(\mathbf{e}_1.\eta^\varepsilon_t(x_1))^2}\\
&+ \frac{\gamma r^2(\mathbf{e}_1.\eta^\varepsilon_t(x_1))^2 \left[\tilde{\Theta}^{\varepsilon_1,\mathbf{e}_1}_{1,r} \star [\text{div}(\mathbf{B}_t)](x_1) + \tilde{\Theta}^{\varepsilon_1,\mathbf{e}_2}_{1,r} \star [\text{div}(\mathbf{B}_t)](x_2)\right]}{\delta^2 + r^2 + \gamma r^2(\mathbf{e}_1.\eta^\varepsilon_t(x_1))^2}\\
&= (7) + (8) + (9) + (10) + (11) + (12) + (13) + (14) + (15) + (16). \qquad (4.33)
\end{aligned}
$$

As above, there exists $\mathbf{P}_2(Db)(x,t) \in L^1((0,T), L^{q_0}_{loc}(\mathbb{R}^d)$ for $q_0 > 1$ such that $||\mathbf{P}_2(Db)||_{L^1((0,T),L^{q_0}(B_R(0)))} \le C(R,\varepsilon_1,\varepsilon)||b||_{L^1((0,T),BV(\mathbb{R}^d))}$ for any $R > 0$ and

$$|(7)| + |(12)| \le 2\gamma^{1/2}\sum_{l,i,j} \frac{\mathbf{P}_2(Db)(x_l,t)}{\delta} \wedge \mathbf{T}^1_{\varepsilon_1,i,j}(D^a b_j)(x_l);$$

$$|(8)| + |(13) \le 2\gamma^{1/2}\sum_{l,i,j} \frac{\mathbf{P}_2(Db)(x_l,t)}{\delta} \wedge \mathbf{T}^1_{\varepsilon_1,i,j}((\eta_t - \eta^\varepsilon_t)\xi_{tj}|D^s b_{tj}|)(x_l);$$

$$|(9)| + |(10)| + |(14)| + |(15)| \le \gamma^{1/2}\sum_{l} \mathbf{P}_2(Db)(x_l,t);$$

$$|(11)| \le \varepsilon_1\gamma^{1/2}\sum_{l,i,j} \frac{\mathbf{P}_2(Db)(x_l,t)}{\delta} \wedge \mathbf{T}^2_{\varepsilon_1,i,j}(Db_{tj})(x_l).$$

For (16), thanks to $\Theta^{\varepsilon,e}_1 \ge 0$ and $\mathrm{div}^s(B_t) \le 0$ we can estimate

$$\begin{aligned}(16) &\le \frac{\gamma r^2(\mathbf{e}_1.\eta^\varepsilon_t(x_1))^2\left[\tilde{\Theta}^{\varepsilon_1,\mathbf{e}_1}_{1,r} \star [(\,\mathrm{div}^a(\mathbf{B}_t))^+]\,(x_1) + \tilde{\Theta}^{\varepsilon_1,\mathbf{e}_2}_{1,r} \star [(\mathrm{div}^a(\mathbf{B}_t))^+]\,(x_2)\right]}{\delta^2 + r^2 + \gamma r^2(\mathbf{e}_1.\eta^\varepsilon_t(x_1))^2} \\ &\le C(\varepsilon_1,\gamma)\sum_{l=1}^{2} \frac{\mathbf{I}_1(\mathbf{1}_{B_{4R}}(\mathrm{div}^a(\mathbf{B}_t))^+)(x_l)}{\delta} \wedge \mathbf{M}(\mathbf{1}_{B_{4R}}(\mathrm{div}^a(\mathbf{B}_t))^+)(x_l).\end{aligned}$$

Combining above inequalities together yields (4.30). The proof is complete. □

5 Well posedness of Regular Lagrangian flows and Transport, Continuity equations

5.1 *Well posedness of Regular Lagrangian flows*: The following results are obtained from Theorem 2, Corollary 1 and Lemma 5 and . Theirs proof are very similar to proofs in Section 6 and 7 in [18].

Proposition 8. *(Uniqueness) Let* $\mathbf{B}$ *be a vector fields as in Corollary 1 satisfying assumption* $(\mathbf{R_1})$*. Assume that* $\mathrm{div}(\mathbf{B}) \in L^1((0,T),\mathcal{M}_{loc}(\mathbb{R}^d))$, $(\mathrm{div}(\mathbf{B}))^+ \in L^1((0,T),L^1_{loc}(\mathbb{R}^d))$*. If there exist the regular Lagrangian flows* X_1, X_2 *associated to* $\mathbf{B}$ *starting at time* t*, then we have* $X_1 \equiv X_2$*.*

Proposition 9. *(Stability) Let* $\mathbf{B}_n$ *be a sequence of vector fields satisfying assumption* $(\mathbf{R_1})$ *converging in* $L^1_{loc}([0,T]\times\mathbb{R}^d)$ *to a vector field* $\mathbf{B}$ *which satisfies as in Proposition 8. Assume that there exist* X_n *and* X *regular Lagrangian flows starting at time* t *associated* $\mathbf{B}_n$ *and* $\mathbf{B}$ *resp. and denote by* L_n *and* L *the compression constants of the flows. Assume that for some decomposition* $\frac{\mathbf{B}_n}{1+|x|} = \tilde{B}_{n,1} + \tilde{B}_{n,2}$ *as in assumption* $(\mathbf{R_1})$*, we have* $L_n + ||\tilde{B}_{n,1}||_{L^1((0,T),L^1(\mathbb{R}^d)} + ||\tilde{B}_{n,2}||_{L^1((0,T),L^\infty(\mathbb{R}^d)} \le C\ \forall\ n \in \mathbb{N}$*, for some constant* $C > 0$*. Then, for any compact set* K*,*

$$\lim_{n\to\infty}\sup_{s\in[t,T]}\int_K |X_n(s,x) - X(s,x)| \wedge 1dx = 0. \tag{5.1}$$

Proposition 10. *(Compactness) Let* $\mathbf{B}_n \in C^1_b([0,T]\times\mathbb{R}^d,\mathbb{R}^d)$ *converge in* $L^1_{loc}([0,T]\times\mathbb{R}^d)$ *to a vector field* $\mathbf{B}$ *which satisfies as in Proposition 8. Let* X_n *be the flow starting at time* t *associated* $\mathbf{B}_n$ *and denote by* L_n *the compression constants of the flow. Assume that for some decomposition* $\frac{\mathbf{B}_n}{1+|x|} = \tilde{B}_{n,1} + \tilde{B}_{n,2}$ *as in assumption* $(\mathbf{R_1})$*, we have* $L_n + ||\tilde{B}_{n,1}||_{L^1((0,T),L^1(\mathbb{R}^d)} +$

$||\tilde{B}_{n,2}||_{L^1((0,T),L^\infty(\mathbb{R}^d)} \leq C \ \forall \ n \in \mathbb{N}$, *for some constant* $C > 0$. *Then, there exists a regular Lagrangian flow* X *starting at time* t *associated to* B *such that for any compact set* K,

$$\lim_{n\to\infty} \sup_{s\in[t,T]} \int_K |X_n(s,x) - X(s,x)| \wedge 1dx = 0. \tag{5.2}$$

Proposition 11. *(Existence) Let* $\mathbf{B}$ *be as in Proposition 8. Assume that* $\operatorname{div}(\mathbf{B}) \geq a(t)$ *in* $(0,T) \times \mathbb{R}^d$ *with* $a \in L^1((0,T))$. *Then, for all* $t \in [0,T)$ *there exists a regular Lagrangian flow* $X := X(.,t,.)$ *associated to* $\mathbf{B}$ *starting at time* t. *Moreover, the flow* X *satisfies* $X \in C(D_T; L^0_{loc}(\mathbb{R}^d)) \cap \mathcal{B}(D_T; \log L_{loc}(\mathbb{R}^d))$ *where* $D_T = \{(s,t) : 0 \leq t \leq s \leq T\}$ *and for every* $0 \leq t \leq \tau \leq s \leq T$, *there holds* $X(s,\tau,X(\tau,t,x)) = X(s,t,x)$ *for* $\mathcal{L}^d$*-a.e* $x \in \mathbb{R}^d$.

In the previous proposition we assume the condition $\operatorname{div}(\mathbf{B}) \geq a(t)$ in order to be sure to have a smooth approximating sequence with equi-bounded compression constants.

Proposition 12. *(Properties of the Jacobian) Let* $\mathbf{B}$ *be as in Proposition* (11), $X : X(.,t,.)$ *the regular Lagrangian flow associated to* $\mathbf{B}$ *starting at time* t. *Assume that* $\operatorname{div}(\mathbf{B}) \in L^1((0,T), L^\infty(\mathbb{R}^d))$. *Then, the function* $JX(s,t,x) = \exp\left(\int_t^s \operatorname{div}(\mathbf{B})(\tau, X(\tau,t,x))d\tau\right)$ *satisfies*

$$\int_{\mathbb{R}^d} \phi(x)dx = \int_{\mathbb{R}^d} \phi(X(s,t,x))JX(s,t,x)dx \quad \forall \quad \phi \in L^1(\mathbb{R}^d) \tag{5.3}$$

and $\partial_s JX(s,t,x) = JX(s,t,x)\operatorname{div}(\mathbf{B})(\tau, X(s,t,x))$ *for all* $s \in (t,T)$. *Moreover,* $\exp(-L) \leq JX(s,t,x) \leq \exp(L)$ *with* $L = ||\operatorname{div}(\mathbf{B})||_{L^1((0,T),L^\infty(\mathbb{R}^d))}$ *and* $JX \in C(D_T; L^\infty(\mathbb{R}^d) - w\star) \cap C(D_T; L^1_{loc}(\mathbb{R}^d))$ *where* $D_T = \{(s,t) : 0 \leq t \leq s \leq T\}$. *Besides, for any* $0 \leq t \leq s \leq T$, $X^{-1}(t,s,.)(x)$ *exists almost everywhere* $x \in \mathbb{R}^d$. *The function* JX *is called the Jacobian of the flow* X.

5.2 *Well posedness of Transport and continuity equations*: Next, we will connect the Regular Lagrangian flows to the transport and continuity equations. We first recall definition of renormalized solution of (1.2), it was first introduced in [30].

Definition 2. *Let* $u_0 : \mathbb{R}^d \to \mathbb{R}$ *be a measurable function, let* $\mathbf{B} \in L^1_{loc}((0,T) \times \mathbb{R}^d; \mathbb{R}^d)$ *be a vector field such that* $\operatorname{div}(\mathbf{B}) \in L^1_{loc}((0,T) \times \mathbb{R}^d)$ *and let* $G, F \in L^1_{loc}((0,T) \times \mathbb{R}^d)$. *A measure function* $u : [0,T] \times \mathbb{R}^d \to \mathbb{R}$ *is a renormalized solution of* (1.2) *if for every function* $\beta : \mathbb{R} \to \mathbb{R}$ *satisfying* $\beta \in C^1_b(\mathbb{R})$ *and* $\beta'(z)z \in L^\infty(\mathbb{R})$, $\beta(0) = 0$ *we have that*

$$\partial_t \beta(u) + \operatorname{div}(\mathbf{B}\beta(u)) + \operatorname{div}(\mathbf{B})\left(u\beta'(u) - \beta(u)\right) = Gu\beta'(u) + F\beta'(u)$$

and $\beta(u)(t=0) = \beta(u_0)$ *in the sense of distributions.*

We have the following proposition:

Proposition 13. *Let* $\mathbf{B}$ *be as in Proposition* (11), X *be the regular Lagrangian flow associated to* $\mathbf{B}$ *starting at time* 0 *in Proposition* (11). *Assume that* $\operatorname{div}(\mathbf{B}) \in L^1((0,T), L^\infty(\mathbb{R}^d))$. *Let* $G, F \in L^1((0,T) \times \mathbb{R}^d)$ *and let* $u_0 : \mathbb{R}^d \to \mathbb{R}$ *be a measurable function. Then, there exists a unique renormalized solution* $u : [0,T] \times \mathbb{R}^d \to \mathbb{R}$ *of* (1.2) *starting from* u_0. *Furthermore, for any* $(t,x) \in [0,T] \times \mathbb{R}^d$ *we have*

$$\begin{aligned} u(t,x) = &\frac{u_0(\overline{x})}{JX(t,\overline{x})} \exp\left(\int_0^t G(s, X(s,\overline{x}))ds\right) \\ &+ \frac{1}{JX(t,\overline{x})} \int_0^t f(\tau, X(\tau,\overline{x})) \exp\left(\int_\tau^t G(s, X(s,\overline{x}))ds\right) JX(\tau,\overline{x})d\tau, \end{aligned} \tag{5.4}$$

with $\overline{x} = X^{-1}(t,.)(x)$, $JX(t,\overline{x}) := JX(t,0,\overline{x})$.

Proof of previous proposition is very similar to [23][Proof of Theorem 2.7]. It is left to the reader.

6 Appendix

First we show the formula (1.3).

Lemma 8. *The function u given in* (1.3) *is a solution to* (1.2).

Proof. Clearly $u(0,x) = u_0(x)$. We can write for any $(t,y) \in (0,T) \times \mathbb{R}^d$

$$V(t) := u(t, X(t,y)) = \left[u_0(y) + \int_0^t F(\tau, X(\tau,y)) \exp\left(\int_0^\tau (\text{div}(\mathbf{B}) - G)(s, X(s,y)) ds \right) d\tau \right]$$
$$\times \exp\left(- \int_0^t (\text{div}(\mathbf{B}) - G)(s, X(s,y)) ds \right).$$

Differentiating in time yields

$$\frac{d}{dt} V(t) = \partial_t u(t, X(t,y)) + \frac{dX(t,y)}{dt}.(\nabla u)(t, X(t,y)) = (\partial_t u + \mathbf{B}.\nabla u)(t,x),$$

and

$$\frac{d}{dt} V(t) = -(\text{div}(\mathbf{B}) - G)(t, X(t,y)) u(t, X(t,y)) + F(t, X(t,y))$$
$$= [-(\text{div}(\mathbf{B}) - G) u + F](t,x),$$

with $x = X(t,y)$. Hence, $(\partial_t u + \mathbf{B}.\nabla u)(t,x) = [-(\text{div}(\mathbf{B}) - G) u + F](t,x)$ which implies that u is a solution to (1.2). The proof is complete. □

Proof of Proposition 1. 1. In [30], Diperna-Lions showed that there exist two different regular Lagrangian flows X_1, X_2 associated to the following vector field

$$\mathbf{B}(x) = \left(\mathbf{B}^1(x), \mathbf{B}^2(x)\right) = \left(-\operatorname{sign}(x_2) \left[\frac{x_1}{|x_2|^2} \mathbf{1}_{|x_1| \le |x_2|} + \mathbf{1}_{|x_1| > |x_2|} \right], - \left[\frac{1}{|x_2|} \mathbf{1}_{|x_1| \le |x_2|} + \mathbf{1}_{|x_1| > |x_2|} \right] \right)$$

(starting at 0) such that for any $x \in \mathbb{R}^2$, $X_1, X_2 \in W^{1,p}(-T,T)$ for any $T < \infty, p < 2$, $X_1, X_2 \in L^\infty_{loc}(\mathbb{R}^2; C(\mathbb{R})) \cap C(\mathbb{R}^2; L^p_{loc}(\mathbb{R}))$ and $X(t,.)_\# \mathcal{L}^d = \mathcal{L}^d$ for any $t \in [0,T]$, $X_j(t+s,.) = X_j(t, X(s,.))$ a.e on $\mathbb{R}^2$, for all $t, s \in \mathbb{R}^d$.
Clearly, $\frac{|\mathbf{B}(x)|}{|x|+1} \in L^1(\mathbb{R}^2) + L^\infty(\mathbb{R}^2)$ and $\text{div}(\mathbf{B}) = 0$.
2. Therefore, it is enough to show that there exist functions $\Omega_1, ..., \Omega_m \in (L^\infty \cap BV)(S^1)$ such that $\Omega_l(\theta) = \Omega_l(t\theta)$ for $\theta \in S^1, t > 0$, $\int_{S^1} \Omega_j = 0$ and for any $R > 1$ we have

$$\partial_l \mathbf{B}^i = \sum_{j=1}^m \left(\frac{\Omega_j^i(.)}{|.|^2} \right) \star \mu_{jR}^l \text{ in } \mathcal{D}'(B_R) \tag{6.1}$$

for some $\mu_{jR}^l \in \mathcal{M}_b(\mathbb{R}^2)$ $l = 1,2$ and $j = 1, ..., m$.
Let $K_1(x_1,x_2) = c(d) \frac{x_2^2 - x_1^2}{(x_1^2 + x_2^2)^2}$, $K_2(x_1,x_2) = c(d) \frac{x_1^2 - x_2^2}{(x_1^2 + x_2^2)^2}$ be kernels of operators $\mathcal{R}_1^2, \mathcal{R}_2^2$, where $\mathcal{R}_1, \mathcal{R}_2$ are the Riesz transforms in $\mathbb{R}^2$. Let $\chi \in C_c^\infty([0,\infty)$ be such that $\chi = 1$ in $[0,2)$ and $\chi = 0$ in $(4,\infty)$ and set $\chi_r(x) = \chi(\frac{|x|}{r})$. Put $b_0(x_1,x_2) = -\mathbf{1}_{|y_1| > |y_2|} \in BV_{loc}(\mathbb{R}^2)$. Fix $R > 100$, we have

$$\partial_1 \mathbf{B}^1(x) = -\frac{\operatorname{sign}(x_2)}{|x_2|} \mathbf{1}_{|x_1| \le |x_2|} + \chi_{8R} \frac{\operatorname{sign}(x_2)|x_1|}{|x_2|^2} d\delta_{|x_1| = |x_2|}(x_1) d\mathcal{L}^1(x_2) + \chi_R \partial_1 b_0(x_1,x_2)$$
$$= \mathbf{K}_1 \star \delta_0(x) + \sum_{j=1,2} \mathcal{R}_j^2(\chi_R \partial_1 b_0)(x) + \sum_{j=1,2} \mathcal{R}_j^2(\chi_{8R}\nu)(x) \text{ in } \mathcal{D}'(B_R). \tag{6.2}$$

where $\mathbf{K}_1(y) = \frac{\Omega_1(y)}{|y|^2}$, $\Omega_1(y_1,y_2) = -\frac{|y|}{y_2}\mathbf{1}_{|y_1|\leq|y_2|} \in (L^\infty \cap BV)(B_2(0)\backslash B_1(0))$ with $\int_{S^1}\Omega_1 = 0$, $\chi_R\partial_1 b_0 \in \mathcal{M}_b(\mathbb{R}^2)$ and $\nu(x_1,x_2) = \frac{\operatorname{sign}(x_2)|x_1|}{|x_2|^2} d\delta_{|x_1|=|x_2|}(x_1) d\mathcal{L}^1(x_2)$ in $\mathcal{D}'(B_{2R})$.
Since $\chi_{8R} = (\chi_{8R} - \chi_{|x|}) + (\chi_{|x|} - \chi_{|x|/8}) + \chi_{|x|/8}$,

$$\sum_{j=1,2}\mathcal{R}_j^2(\chi_{8R}\nu)(x) = \sum_{j=1,2}\chi_R(x)\mathcal{R}_j^2(\chi_{|x|/8}\nu)(x) + \sum_{j=1,2}\chi_R(x)\mathcal{R}_j^2((\chi_{8R}-\chi_{|x|})\nu)(x) + \sum_{j=1,2}\mathcal{R}_j^2((\chi_{|x|}-\chi_{|x|/2})\nu)(x) \text{ in } \mathcal{D}'(B_R), \tag{6.3}$$

and

$$\chi_R(x)\mathcal{R}_j^2(\chi_{|x|/8}\nu)(x), \chi_R(x)\mathcal{R}_j^2((\chi_{8R}-\chi_{|x|})\nu)(x) \in (L^\infty \cap L^1)(\mathbb{R}^2)\ j=1,2. \tag{6.4}$$

We now show that there exists $\tilde{\Omega}_j \in (L^\infty \cap BV)(B_2(0)\backslash B_1(0))$ such that $\tilde{\Omega}_j(\theta) = \tilde{\Omega}_j(r\theta)$ for any $r > 0, \theta \in S^1$, $\int_{S^1}\tilde{\Omega}_j(\theta) = 0$ and

$$\mathcal{R}_j^2((\chi_{|x|}-\chi_{|x|/2})\nu)(x) = \frac{\tilde{\Omega}_j(x)}{|x|^2} = \frac{\tilde{\Omega}_j(.)}{|.|^2} \star \delta_0(x)\ \ \forall\ x \in \mathbb{R}^2 \tag{6.5}$$

Indeed, we have

$$\begin{aligned}
&\mathcal{R}_j^2((\chi_{|x|}-\chi_{|x|/2})\nu)(x)\\
&= \lim_{\varepsilon\to 0}\int_{\mathbb{R}}\left(K_j^\epsilon(x_1-|y_2|,x_2-y_2) + K_j^\epsilon(x_1+|y_2|,x_2-y_2)\right)(\chi_{|x|}-\chi_{|x|/2})(y_2,y_2)\frac{dy_2}{y_2}\\
&= \frac{1}{|x|^2}\lim_{\varepsilon\to 0}\int_{\mathbb{R}}\left(K_j^\epsilon(\theta_1-|y_2|,\theta_2-y_2) + K_j^\epsilon(\theta_1+|y_2|,\theta_2-y_2)\right)\left[\chi\left(\sqrt{2}|y_2|\right) - \chi\left(8\sqrt{2}|y_2|\right)\right]\frac{dy_2}{y_2}\\
&:= \frac{\tilde{\Omega}_j(\theta)}{|x|^2},
\end{aligned}$$

with $K_j^\epsilon(.) = \mathbf{1}_{|.|>\epsilon}K_j^\epsilon(.), \theta = x/|x|$.
Clearly, $\tilde{\Omega}_j(\theta_1,-\theta_2) = -\tilde{\Omega}_j(\theta_1,\theta_2)$ and $\tilde{\Omega}_j(-\theta_1,\theta_2) = \tilde{\Omega}_j(\theta_1,\theta_2)$, so, $\int_{S^1}\tilde{\Omega}_j = 0$.
To prove $\tilde{\Omega}_j \in (L^\infty \cap BV)(S^1)$, we can assume that $\theta_2, \theta_1 \geq 0$ and $\theta_1 \neq \theta_2$. So, we can write

$$\tilde{\Omega}_j(\theta) = a_j^1(\theta) + a_j^2(\theta)$$

where

$$\begin{aligned}
a_j^1(\theta) &= \lim_{\varepsilon\to 0}\int_{\frac{1}{4\sqrt{2}}}^{2\sqrt{2}} K_j^\epsilon(\theta_1-y_2,\theta_2-y_2)\left[\chi\left(\sqrt{2}y_2\right) - \chi\left(8\sqrt{2}y_2\right)\right]\frac{dy_2}{y_2}\\
a_j^2(\theta) &= \int_{-\infty}^{0} K_j(\theta_1-|y_2|,\theta_2-y_2)\left[\chi\left(\sqrt{2}|y_2|\right) - \chi\left(8\sqrt{2}|y_2|\right)\right]\frac{dy_2}{y_2}\\
&+ \int_{\mathbb{R}} K_j(\theta_1+|y_2|,\theta_2-y_2)\left[\chi\left(\sqrt{2}|y_2|\right) - \chi\left(8\sqrt{2}|y_2|\right)\right]\frac{dy_2}{y_2}.
\end{aligned}$$

Clearly, $a_j^1(\theta) \in C^\infty(S^1 \cap \{\theta_1,\theta_2 \geq 0\}\backslash\{(1/\sqrt{2},1/\sqrt{2})\})$ $a_j^2(\theta) \in C_b^\infty(S^1 \cap \{\theta_1,\theta_2 \geq 0\})$. Using the fact that

$$\left|\int_{\frac{1}{4\sqrt{2}}}^{2\sqrt{2}} K_j^\epsilon(\theta_1-y_2,\theta_2-y_2)dy_2\right| + \left|\int_{\frac{1}{4\sqrt{2}}}^{2\sqrt{2}} K_j^\epsilon(\theta_1-y_2,\theta_2-y_2)(\theta_2-y_2)dy_2\right| \leq C\ \forall\varepsilon > 0.$$

and Taylor's Formula : for any $y_2, \theta_2 \geq 0$,

$$|\tilde{f}(y_2) - \tilde{f}(\theta_2) - \tilde{f}'(\theta_2)(y_2 - \theta_2)| \leq C|y_2 - \theta_2|^2, \tilde{f}(y_2) = y_2^{-1}\left[\chi\left(\sqrt{2}y_2\right) - \chi\left(8\sqrt{2}y_2\right)\right],$$

we find that

$$|a_j^1(\theta)| \leq C + C\int_{\frac{1}{4\sqrt{2}}}^{2\sqrt{2}} |K_j(\theta_1 - y_2, \theta_2 - y_2)|\theta_2 - y_2|^2 dy_2 \leq C.$$

Thus, we obtain that $\tilde{\Omega}_j \in (L^\infty \cap BV)(S^1)$, and in particular, $\tilde{\Omega}_j \in (L^\infty \cap BV)(B_2(0)\backslash B_1(0))$. Therefore, we derive from (6.2),(6.3),(6.4) and (6.5) that there exist functions $\Omega_1, ..., \Omega_m \in (L^\infty \cap BV)(S^1)$ such that $\Omega_j(\theta) = \Omega_j(r\theta)$ for $\theta \in S^1, r > 0$, $\int_{S^1} \Omega_j = 0$ and for any $R > 1$ we have $\partial_1 \mathbf{B}^1 = \sum_{j=1}^m \left(\frac{\Omega_j(.)}{|.|^2}\right) \star \mu_{jR}$ in $\mathcal{D}'(B_R)$ for some $\mu_{1R}, ..., \mu_{mR} \in \mathcal{M}_b(\mathbb{R}^2)$. Similarly, we can do this for $\partial_2 \mathbf{B}_1, \partial_1 \mathbf{B}^2, \partial_2 \mathbf{B}^2$. The proof is complete. □

To prove Lemma 3, we need to have the following result:

Lemma 9. *Let $e \in S^{d-1}$. For any $z_1, z_1' \in \tilde{H}_e, y_2, y_2', z_2, z_2' \in H_e$, $\varepsilon > 0, \rho > 0$, there holds*

$$\begin{aligned} M &= \int_{\tilde{H}_e} |f(y_2' + y_1) - f(y_2' + z_1')| \mathbf{1}_{|(z_1+z_2)-(y_1+y_2)| \leq \varepsilon}\ 1 \wedge \left(\frac{\rho}{|(z_1' + z_2') - (y_1 + y_2)|}\right)^{d+2} d\mathcal{H}^1(y_1) \\ &\leq \frac{C\rho^2}{\varepsilon} \int_{\tilde{H}_e} 1 \wedge \left(\frac{\rho}{|(z_1' + z_2') - (z + y_2)|}\right)^{d-\frac{1}{2}} d|Df^e_{y_2'}|(z) \\ &+ C\rho \int_{\tilde{H}_e} \mathbf{1}_{|(z_1+z_2)-(z+y_2)| \leq 4\varepsilon}\ 1 \wedge \left(\frac{\rho}{|(z_1' + z_2') - (z + y_2)|}\right)^{d+\frac{1}{2}} d|Df^e_{y_2'}|(z). \end{aligned} \tag{6.6}$$

Proof. Since $|f(y_2' + y_1) - f(y_2' + z_1')| \leq \int_{\tilde{H}_e} \mathbf{1}_{|z - z_1'| \leq 2|z_1' - y_1|} d|Df^e_{y_2'}|(z)$, so, $M \leq \int_{\tilde{H}_e} V d|Df^e_{y_2'}|(z)$, where

$$V = \int_{\tilde{H}_e} \mathbf{1}_{|z - z_1'| \leq 2|z_1' - y_1|} 1_{|(z_1+z_2)-(y_1+y_2)| \leq \varepsilon} \min\left\{1, \left(\frac{\rho}{|(z_1' + z_2') - (y_1 + y_2)|}\right)^{d+2}\right\} d\mathcal{H}^1(y_1).$$

Note that if $|z - z_1'| \leq 2|z_1' - y_1|$, then $|(z_1' + z_2') - (z + y_2)| \leq 4|(z_1' + z_2') - (y_1 + y_2)|$ and $|(z_1 + z_2) - (z + y_2)| \leq |(z_1 + z_2) - (y_1 + y_2)| + 3|(z_1' + z_2') - (y_1 + y_2)|$. Thus, we can estimate

$$\begin{aligned} V &= \int_{\tilde{H}_e} \mathbf{1}_{|(z_1'+z_2')-(y_1+y_2)| \leq \varepsilon} + \int_{\tilde{H}_e} \mathbf{1}_{|(z_1'+z_2')-(y_1+y_2)| > \varepsilon} \\ &\leq C\mathbf{1}_{|(z_1+z_2)-(z+y_2)| \leq 4\varepsilon}\ 1 \wedge \left(\frac{\rho}{|(z_1' + z_2') - (z + y_2)|}\right)^{d+\frac{1}{2}} \int_{\tilde{H}_e} 1 \wedge \left(\frac{\rho}{|z_1' - y_1|}\right)^{\frac{3}{2}} d\mathcal{H}^1(y_1) \\ &+ \frac{C\rho}{\varepsilon}\ 1 \wedge \left(\frac{\rho}{|(z_1' + z_2') - (z + y_2)|}\right)^{d-\frac{1}{2}} \int_{\tilde{H}_e} 1 \wedge \left(\frac{\rho}{|z_1' - y_1|}\right)^{\frac{3}{2}} d\mathcal{H}^1(y_1) \\ &\leq C\rho\mathbf{1}_{|(z_1+z_2)-(z+y_2)| \leq 4\varepsilon} 1 \wedge \left(\frac{\rho}{|(z_1' + z_2') - (z + y_2)|}\right)^{d+\frac{1}{2}} + \frac{C\rho^2}{\varepsilon} 1 \wedge \left(\frac{\rho}{|(z_1' + z_2') - (z + y_2)|}\right)^{d-\frac{1}{2}} \end{aligned}$$

which implies (6.6). The proof is complete. □

Proof of Lemma 3. We first observe that

$$M \leq \sum_{k=0}^{d-2} \int_{\tilde{H}_{e_1}} ... \int_{\tilde{H}_{e_d}} 1 \wedge \left(\frac{\rho}{|\sum_{i=1}^{d}(x_i - y_i)|} \right)^{d+2} \mathbf{1}_{|\sum_{i=1}^{d}(y_{0i}-y_i)|\leq\varepsilon}$$
$$\times \left| f(\sum_{i=1}^{d-k-1} y_i + \sum_{i=d-k+1}^{d} x_i + y_{d-k}) - f(\sum_{i=1}^{d-k-1} y_i + \sum_{i=d-k+1}^{d} x_i + x_{d-k}) \right| d\mathcal{H}^1(y_d)...d\mathcal{H}^1(y_1).$$

Applying Lemma 9 to $e = e_{d-k}$, $y_2' = \sum_{i=1}^{d-k-1} y_i + \sum_{i=d-k+1}^{d} x_i$, $y_2 = \sum_{i\neq d-k} y_i$, $z_1 = y_{0,d-k}, z_2 = \sum_{i\neq d-k} y_{0,i}, z_1' = x_{d-k}, z_2' = \sum_{i\neq d-k} x_i$ yields

$$M \leq \sum_{k=0}^{d-2} \frac{C\rho^2}{\varepsilon} \int_{\tilde{H}_{e_1}} ... \int_{\tilde{H}_{e_d}} 1 \wedge \left(\frac{\rho}{|\sum_{i=0}^{d}(x_i - y_i)|} \right)^{d-\frac{1}{2}}$$
$$\times d\mathcal{H}^1(y_d)...d\mathcal{H}^1(y_{d-k+1}) d|Df^{e_{d-k}}_{\sum_{i=1}^{d-k-1} y_i + \sum_{i=d-k+1}^{d} x_i}|(y_{d-k}) d\mathcal{H}^1(y_{d-k-1})...d\mathcal{H}^1(y_1)$$
$$+ \sum_{k=0}^{d-2} C\rho \int_{\tilde{H}_{e_1}} ... \int_{\tilde{H}_{e_d}} 1 \wedge \left(\frac{\rho}{|\sum_{i=0}^{d}(x_i - y_i)|} \right)^{d+\frac{1}{2}} \mathbf{1}_{|\sum_{i=1}^{d}(y_{0i}-y_i)|\leq\varepsilon}$$
$$\times d\mathcal{H}^1(y_d)...d\mathcal{H}^1(y_{d-k+1}) d|Df^{e_{d-k}}_{\sum_{i=1}^{d-k-1} y_i + \sum_{i=d-k+1}^{d} x_i}|(y_{d-k}) d\mathcal{H}^1(y_{d-k-1})...d\mathcal{H}^1(y_1). \quad (6.7)$$

It is clear to see that

$$\int_{\tilde{H}_{e_{d-k+1}}} ... \int_{\tilde{H}_{e_d}} 1 \wedge \left(\frac{\rho}{|\sum_{i=0}^{d}(x_i - y_i)|} \right)^{d-\frac{1}{2}} d\mathcal{H}^1(y_d)...d\mathcal{H}^1(y_{d-k+1}) \leq C\rho^k \, 1 \wedge \left(\frac{\rho}{|\sum_{i=1}^{d-k}(x_i - y_i)|} \right)^{d-k-\frac{3}{4}},$$

and

$$\int_{\tilde{H}_{e_{d-k+1}}} ... \int_{\tilde{H}_{e_d}} 1 \wedge \left(\frac{\rho}{|\sum_{i=0}^{d}(x_i - y_i)|} \right)^{d+\frac{1}{2}} \mathbf{1}_{|\sum_{i=1}^{d}(y_{0i}-y_i)|\leq\varepsilon} d\mathcal{H}^1(y_d)...d\mathcal{H}^1(y_{d-k+1})$$
$$\leq C\rho^k \mathbf{1}_{|\sum_{i=1}^{d}(y_{0i}-x_i)|\leq 2\varepsilon} \mathbf{1}_{|\sum_{i=1}^{d-k}(y_{0i}-y_i)|\leq 2\varepsilon} 1 \wedge \left(\frac{\rho}{|\sum_{i=1}^{d-k}(x_i - y_i)|} \right)^{d-k+\frac{1}{4}}$$
$$+ C\frac{\rho^{k+1}}{\varepsilon} \, 1 \wedge \left(\frac{\rho}{|\sum_{i=0}^{d-k}(x_i - y_i)|} \right)^{d-k-\frac{3}{4}}.$$

Combining these with (6.7) we find that

$$M \leq \sum_{k=0}^{d-2} \frac{C\rho^{k+2}}{\varepsilon} \int_{\tilde{H}_{e_1}} ... \int_{\tilde{H}_{e_{d-k}}} 1 \wedge \left(\frac{\rho}{|\sum_{i=1}^{d-k}(x_i - y_i)|} \right)^{d-k-\frac{3}{4}} d\nu^1_{k, \sum_{i=d-k+1}^{d} x_i}(y_{d-k}, ..., y_1)$$
$$+ \sum_{k=0}^{d-2} C\rho^{k+1} \mathbf{1}_{|\sum_{i=1}^{d}(y_{0i}-x_i)|\leq 2\varepsilon} \int_{\tilde{H}_{e_1}} ... \int_{\tilde{H}_{e_{d-k}}} 1 \wedge \left(\frac{\rho}{|\sum_{i=1}^{d-k}(x_i - y_i)|} \right)^{d-k+\frac{1}{4}} d\nu^2_{k, \sum_{i=d-k+1}^{d} x_i}(y_{d-k}, .., y_1).$$

Hence, using the fact that for any $\omega \in \mathcal{M}^+(\bigotimes_{i=1}^{d-k} \tilde{H}_{e_i})$,

$$\int_{\tilde{H}_{e_1}} ... \int_{\tilde{H}_{e_{d-k}}} 1 \wedge \left(\frac{\rho}{|\sum_{i=1}^{d-k}(x_i - y_i)|} \right)^{d-k+\frac{1}{4}} d\omega(y_{d-k}, ..., y_1) \leq C\rho^{d-k} \mathbf{M}(\omega, \bigotimes_{i=1}^{d-k} \tilde{H}_{e_i})(\sum_{i=1}^{d-k} x_i),$$

one gets the first inequality of Lemma 3. Similarly, we also have second one. The proof is complete. □

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