

Asymptotic limit of linear parabolic equations with spatio-temporal degenerated potentials

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Abstract

In this paper, we observe how the heat equation in a non-cylindrical domain can arise as the asymptotic limit of a parabolic problem in a cylindrical domain, by adding a potential that vanishes outside the limit domain. This can be seen as a parabolic version of a previous work by the first and last authors, concerning the stationary case [3]. We provide a strong convergence result for the solution by use of energetic methods and Γ -convergence technics. Then, we establish an exponential decay estimate coming from an adaptation of an argument by B. Simon.

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1 Introduction

For $\Omega \subset \mathbb{R}^N$ open and $T > 0$, we define the cylinder $Q_T = \Omega \times (0, T)$. Let $\lambda > 0$ be a positive real parameter. For $f_\lambda \in L^2(Q_T)$, $g_\lambda \in H_0^1(\Omega)$ and $a : Q_T \rightarrow \mathbb{R}^+$ a bounded measurable function, we consider the solution u_λ of the parabolic problem

$$(P_\lambda) \begin{cases} \partial_t u - \Delta u + \lambda a(x, t)u = f_\lambda & \text{in } Q_T, \\ u = 0 & \text{on } \partial\Omega \times (0, T), \\ u(x, 0) = g_\lambda(x) & \text{in } \Omega. \end{cases}$$

Since (P_λ) is a classical parabolic problem, existence and regularity of solutions follow from a standard theory well developed in the literature (see Section 3). In particular, under our assumptions, $u \in L^2(0, T; H_0^1(\Omega))$ is continuous in time (thus the initial condition $u(x, 0) = g_\lambda(x)$ is well defined in $L^2(\Omega)$) and the equation is satisfied in a weak sense (see Section 3 for an exact formulation).

In this paper we are interested in the limit of u_λ when $\lambda \rightarrow +\infty$. In particular, we assume spatial and temporal degeneracies for the potential a , which means that

$$O_a := \text{Int}(\{(x, t) \in Q_T : a(x, t) = 0\}) \neq \emptyset. \quad (1.1)$$

We also assume that ∂O_a has zero Lebesgue measure.

In order to describe the results of this paper, let us start with elementary observations. Assume that, when λ goes to $+\infty$, f_λ converges to f and g_λ converges to g , for instance in L^2 . Assume also that u_λ converges weakly in $L^2(Q_T)$ to some $u \in L^2(Q_T)$.

Under those assumptions it is not very difficult to get the following a priori bound using the equation in (P_λ) (see Lemma 3)

$$\lambda \int_{Q_T} a u_\lambda^2 dx dt \leq C. \quad (1.2)$$

This shows that u_λ converges strongly to 0 in any set of the form $\{a(x, t) > \varepsilon\}$, for any $\varepsilon > 0$.

Then, multiplying the equation in (P_λ) by any $\varphi \in C_0^\infty(O_a)$ we get, after some integration by parts (in this paper we shall denote ∇ for ∇_x , i.e. the gradient in space),

$$\int_{Q_T} u_\lambda \partial_t \varphi - \int_{Q_T} u_\lambda \Delta \varphi = \int_{Q_T} f_\lambda \varphi.$$

Passing to the limit, we obtain that $\partial_t u - \Delta u = f$ in $\mathcal{D}'(O_a)$. Under some suitable extra assumptions on the potential a , we will actually be able to prove that the limit u satisfies the following more precise problem:

$$(P_\infty) \begin{cases} u \in L^2(0, T; H_0^1(\Omega)), & u' \in L^2(0, T; L^2(\Omega)) \\ u = 0 \text{ a.e. in } Q_T \setminus O_a \\ \int_{Q_T} (u'v + \nabla u \nabla v) = \int_{Q_T} f_\lambda v, \\ \text{for all } v \in L^2(0, T; H_0^1(\Omega)) \text{ s.t. } v = 0 \text{ a.e. in } Q_T \setminus O_a \\ u(x, 0) = g(x) \quad \text{in } \Omega. \end{cases}$$

Problem (P_∞) , which arises here naturally as the limit problem associated with the family of problems (P_λ) , is a non standard heat equation since O_a may, in general, not be cylindrical. This type of heat equation in a noncylindrical domain appears in many applications, and different approaches have been developed recently to solve problems related to (P_∞) (see for e.g. [6, 8, 9, 10, 14, 18] and the references therein). As a byproduct of our work, we have obtained an existence and uniqueness result for the problem (P_∞) (see Corollary 1).

In this paper, we study in more detail the convergence of u_λ , when λ goes to infinity. Our first result gives a sufficient condition on the potential a , for which the convergence of u_λ to u is stronger than a weak L^2 convergence. Indeed, assuming a monotonicity condition on the potential a , and using purely energetic and variational methods, we obtain that the convergence holds strongly in $L^2(0, T; H^1(\Omega))$; see Section 5. Our approach can be seen as the continuation of a previous work [3], where the stationary problem has been studied using the theory of Γ -convergence.

Here is our first main result.

Theorem 1. *For all $\lambda > 0$, let u_λ be the solution of (P_λ) with $f_\lambda \in L^2(\Omega \times (0, T))$ and $g_\lambda \in H_0^1(\Omega)$. Assume that $a : \Omega \times [0, T] \rightarrow \mathbb{R}^+$ is a Lipschitz function which satisfies*

$$\partial_t a(x, t) \leq 0 \text{ a.e. in } Q_T. \quad (1.3)$$

Assume also that the initial condition g_λ satisfies

$$\sup_{\lambda > 0} \left(\lambda \int_{\Omega} a(x, 0) g_\lambda(x)^2 dx \right) < +\infty,$$

converges weakly to g in $L^2(\Omega)$, and that f_λ converges weakly to f in $L^2(Q_T)$.

Then u_λ converges strongly to u in $L^2(0, T; H^1(\Omega))$, where u is the unique solution of (P_∞) .

Remark 1. *In particular, condition (1.3) implies that the family of sets $\Omega_a(t) \subset \Omega$, defined for $t > 0$ by $\Omega_a(t) := \{x \in \Omega, (x, t) \in O_a\}$, is increasing in time for the inclusion. In that case, by a slight abuse of terminology, we will often write simply that O_a is increasing in time (for the inclusion).*

Our second result is a quantitative convergence of u_λ to 0, outside O_a (in other words, away from the vanishing region), with very general assumptions on a (only continuous and $O_a \neq \emptyset$), but in the special case when $f_\lambda = 0$ in $Q_T \setminus O_a$. This is obtained using an adaptation of an argument due to Simon [19], and proves that u_λ decays exponentially fast to 0 in the region $Q_T \setminus O_a$. Compared to the standard bound (1.2), this result expresses that u_λ goes to 0 much faster than one could expect. We also take the opportunity of this paper to write a similar estimate for the stationary problem (see Lemma 5 in Section 6).

Theorem 2. *For all $\lambda > 0$, let u_λ be the solution of (P_λ) with $f_\lambda \in L^2(Q_T)$ and $g_\lambda \in H_0^1(\overline{O_a} \cap \{t = 0\})$. Assume that $f_\lambda = 0$ in $Q_T \setminus O_a$. Let $a : \overline{\Omega} \times [0, T] \rightarrow \mathbb{R}^+$*

be a continuous function for which O_a is non empty. For every $\varepsilon > 0$, define $A_\varepsilon := \{(x, t); \text{dist}((x, t), O_a) > \varepsilon\}$. Then, for every $\varepsilon > 0$, there exists a constant $C > 0$ such that

$$\sup_{\lambda > 0} \left(\lambda e^{c_\varepsilon \sqrt{\lambda}} \int_{A_\varepsilon} u_\lambda^2 dx \right) \leq C,$$

where $c_\varepsilon := \varepsilon \min_{(x,t) \in A_{\varepsilon/2}} a(x, t)$.

The convergence of weak solutions of (P_λ) was already observed in [11] as a starting point for a more detailed analysis about the associated semigroup. This was then used in [11] to analyse the asymptotic behaviour of a non linear periodic-parabolic problem of logistic type, where the equation is the following, also considered before in [14],

$$\partial_t u - \Delta u = \mu u - a(x, t)u^p, \quad (1.4)$$

used in some models of population dynamics. A possible link between our Problem (P_λ) and the non-linear equation (1.4) is coming from the fact that asymptotic limit of the principal eigenvalue for the linear parabolic operator $\partial_t - \Delta + \lambda a(x, t)$ plays a role in the dynamical behaviour of non-linear logistic equation (cf. [5, 11, 14]). We thus believe that the results and technics developed in the present paper could be possibly used in the study of more general equations such as (1.4).

Furthermore, another possible application of our results could be for numerical purposes. Indeed, for the ones who would be interested by computing a numerical solution of the non-cylindrical limiting problem (P_∞) , one could use the cylindrical problem (P_λ) for a large λ , much easier to compute via standard methods. The strong convergence stated in Theorem 1 together with the exponential rate of convergence stated in Theorem 2 give some good estimates about the difference between those two different solutions.

2 The stationary problem

This section concerns only the stationary problem. In particular, throughout the section, all functions u, a, f , etc., will be functions of $x \in \Omega$ (and independent of t).

We assume $\Omega \subset \mathbb{R}^N$ to be an open and bounded domain and $a : \bar{\Omega} \rightarrow \mathbb{R}^+$ be a measurable and bounded non-negative function. We suppose that

$$K_a := \{x \in \bar{\Omega}; a(x) = 0\} \subset \Omega \text{ is a closed set in } \mathbb{R}^N. \quad (2.1)$$

Moreover, we assume that

$$\Omega_a := \text{Int}(K_a) \neq \emptyset. \quad (2.2)$$

Under hypothesis (2.1) we know that

$$H_0^1(K_a) := H^1(\mathbb{R}^N) \cap \{u = 0 \text{ q.e. in } \mathbb{R}^N \setminus K_a\} = H^1(\mathbb{R}^N) \cap \{u = 0 \text{ a.e. in } \mathbb{R}^N \setminus K_a\},$$

and hypothesis (2.2) implies that

$$H_0^1(K_a) \neq \{0\}.$$

Notice that we are working with a functional space of the form $H_0^1(A)$, where A is a closed subspace of \mathbb{R}^N . Therefore, we do not claim that $H_0^1(A) = H_0^1(\text{Int}(A))$, which is true only under more regularity assumptions on the set A .

Furthermore, we define the functionals E_λ and E on $L^2(\Omega)$ as follows.

$$E_\lambda(u) = \begin{cases} \frac{1}{2} \int_\Omega |\nabla u|^2 + \lambda a u^2 \, dx & \text{if } u \in H_0^1(\Omega) \\ +\infty & \text{otherwise.} \end{cases} \quad (2.3)$$

$$E(u) = \begin{cases} \frac{1}{2} \int_\Omega |\nabla u|^2 \, dx & \text{if } u \in H_0^1(K_a) \\ +\infty & \text{otherwise.} \end{cases}$$

The following result was already stated and used in [3]. For the sake of completeness, we reproduce the proof here and refer the reader to [3] for the connection of this result with Γ -convergence and several examples.

Proposition 1. *Let $f_\lambda \in L^2(\Omega)$ be a family of functions indexed by some real parameter $\lambda > 0$ and uniformly bounded in $L^2(\Omega)$. Moreover, assume that f_λ converges to a function $f \in L^2(\Omega)$ in the weak topology of $L^2(\Omega)$, when λ tends to $+\infty$. Then the unique solution of the problem*

$$(P_\lambda^s) \begin{cases} -\Delta u + \lambda a u = f_\lambda \\ u \in H_0^1(\Omega), \end{cases}$$

converges strongly in $H^1(\Omega)$, when $\lambda \rightarrow +\infty$, to the unique solution of the problem

$$(P_\infty^s) \begin{cases} -\Delta u = f \\ u \in H_0^1(K_a). \end{cases}$$

Proof. This is a standard consequence of the Γ -convergence of energies E_λ , which relies on the fact that u_λ is the unique minimizer in $H_0^1(\Omega)$ for

$$v \mapsto E_\lambda(v) - 2 \int_\Omega f_\lambda v,$$

whereas u is the unique minimizer in $H_0^1(K_a)$ for

$$v \mapsto E(v) - 2 \int_\Omega f v.$$

Let us write the full details of the proof. For any $\lambda > 0$, let u_λ be the solution of (P_λ^s) . We first prove that $\{u_\lambda\}_{\lambda>0}$ is compact in $L^2(\Omega)$. This comes from the energy equality

$$\int_\Omega (|\nabla u_\lambda|^2 + \lambda a u_\lambda^2) \, dx = \int_\Omega f_\lambda u_\lambda \, dx,$$

which implies

$$\int_\Omega |\nabla u_\lambda|^2 \leq \|f_\lambda\|_{L^2(\Omega)} \|u_\lambda\|_{L^2(\Omega)} \leq C \|u_\lambda\|_{L^2(\Omega)} \quad \text{with } C \text{ a positive constant.}$$

Thanks to Poincaré's inequality we also have that

$$\|u_\lambda\|_{L^2(\Omega)}^2 \leq C(\Omega) \int_{\Omega} |\nabla u_\lambda|^2 dx,$$

which finally proves that u_λ is uniformly bounded in $H_0^1(\Omega)$.

Now let w be any point in the L^2 -adherence of the family $\{u_\lambda\}_{\lambda>0}$. In other words, there exists a subsequence, still denoted by u_λ , converging strongly in L^2 to w . Since u_λ is bounded in $H^1(\Omega)$, we can assume, up to extracting a further subsequence, that u_λ converges weakly in $H^1(\Omega)$ to a function that must necessarily be w . Now let u be the solution of the limit problem (P_∞^s) . Since u_λ is a minimizer of

$$u \mapsto E_\lambda(u) - 2 \int_{\Omega} f_\lambda u dx, \quad (2.4)$$

and since $au = 0$ we have

$$E_\lambda(u_\lambda) - 2 \int_{\Omega} f_\lambda u_\lambda dx \leq E_\lambda(u) - 2 \int_{\Omega} f_\lambda u dx = E(u) - 2 \int_{\Omega} f_\lambda u dx.$$

Hence, passing to the limsup in the previous inequality, it follows that

$$\begin{aligned} E(w) - 2 \int_{\Omega} fw dx &\leq \liminf_{\lambda} \left(E_\lambda(u_\lambda) - 2 \int_{\Omega} f_\lambda u_\lambda dx \right) \\ &\leq \limsup_{\lambda} \left(E_\lambda(u_\lambda) - 2 \int_{\Omega} f_\lambda u_\lambda dx \right) \\ &\leq E(u) - 2 \int_{\Omega} fu dx, \end{aligned} \quad (2.5)$$

which shows that w is a minimizer, and thus $w = u$. By uniqueness of the adherence point, we infer that the whole sequence u_λ converges strongly in L^2 to u (and weakly in H^1).

It remains to prove the strong convergence in H^1 . To do so, it is enough to prove

$$\|\nabla u_\lambda\|_{L^2(\Omega)} \rightarrow \|\nabla u\|_{L^2(\Omega)}.$$

Due to the weak convergence in $H^1(\Omega)$ (up to subsequences) we already have

$$\|\nabla u\|_{L^2(\Omega)} \leq \liminf_{\lambda} \|\nabla u_\lambda\|_{L^2(\Omega)},$$

and going back to (2.5) we get the reverse inequality, with a limsup.

The proof of convergence of the whole sequence follows by uniqueness of the adherent point in $H^1(\Omega)$. \square

Remark 2. Notice that when u is a solution of (P_∞^s) , then $-\Delta u = f$ only in $\text{Int}(K_a)$ and $-\Delta u = 0$ in K_a^c . However, in general $-\Delta u$ has a singular part on ∂K_a . Typically, if K_a is for instance a set of finite perimeter, then in the distributional sense in Ω ,

$$-\Delta u = f \mathbf{1}_{K_a} + \frac{\partial u}{\partial \nu} \mathcal{H}^{N-1}|_{\partial K_a},$$

where ν is the outer normal on ∂K_a and \mathcal{H}^{N-1} is the $N-1$ dimensional Hausdorff measure.

As a consequence of Proposition 1, we easily obtain the following result.

Proposition 2. *Assume that f_λ converges weakly to a function f in $L^2(\Omega)$. For any $\lambda > 0$, let u_λ be the solution of Problem (P_λ^s) . Then, when $\lambda \rightarrow \infty$,*

$$\lambda \int_{\Omega} a u_\lambda^2 dx \rightarrow 0, \quad (2.6)$$

$$\lambda a u_\lambda \rightarrow f \mathbf{1}_{\Omega \setminus K_a} + (\Delta u)|_{\partial K_a} \text{ in } \mathcal{D}'(\Omega), \quad (2.7)$$

where u is the solution of (P_∞^s) . Moreover, the convergence in (2.7) holds in the weak-* topology of H^{-1} .

Proof. Due to Proposition 1 we know that u_λ converges strongly in $H^1(\Omega)$ to u , the solution of Problem (P_∞^s) . In particular, from the fact that

$$\int_{\Omega} |\nabla u_\lambda|^2 dx \rightarrow \int_{\Omega} |\nabla u|^2 dx = \int_{\Omega} u f dx,$$

and

$$\int_{\Omega} u_\lambda f_\lambda dx \rightarrow \int_{\Omega} u f dx,$$

passing to the limit in the following energy equality

$$\int_{\Omega} |\nabla u_\lambda|^2 dx + \lambda \int_{\Omega} a u_\lambda^2 dx = \int_{\Omega} u_\lambda f_\lambda dx, \quad (2.8)$$

we obtain (2.6). Next, let us now prove (2.7). Thus, since u_λ is a solution of (P_λ^s) then, for every test function $\psi \in C_c^\infty(\Omega)$, after integrating by parts in Ω we arrive at

$$\int_{\Omega} u_\lambda (-\Delta \psi) dx + \lambda \int_{\Omega} a u_\lambda \psi dx = \int_{\Omega} f_\lambda \psi dx.$$

Passing to the limit we obtain that $\lambda a u_\lambda \rightarrow f + \Delta u$ in $\mathcal{D}'(\Omega)$. Now returning to (2.8), we can write, for every ψ satisfying $\|\psi\|_{H^1(\Omega)} \leq 1$,

$$\left| \lambda \int_{\Omega} a u_\lambda \psi dx \right| \leq \|f_\lambda\|_2 + \|\nabla u_\lambda\|_2 \leq C.$$

Taking the supremum in ψ we get

$$\|\lambda a u_\lambda\|_{H^{-1}} \leq C.$$

Therefore, $\lambda a u_\lambda$ is weakly-* sequentially compact in H^{-1} and we obtain the convergence by uniqueness of the limit in the distributional sense. \square

3 Existence and regularity of solutions for (P_λ)

In order to define properly a solution for (P_λ) , we first recall the definition of the spaces $L^p(0, T; X)$, with X a Banach space, which consist of all (strongly) measurable functions (see [13, Appendix E.5]) $u : [0, T] \rightarrow X$ such that

$$\|u\|_{L^p(0, T; X)} = \left(\int_0^T \|u(t)\|_X^p dt \right)^{1/p} < +\infty,$$

for $1 \leq p < +\infty$, and

$$\|u\|_{L^\infty(0, T; X)} = \operatorname{ess\,sup}_{t \in (0, T)} \|u(t)\|_X < +\infty.$$

For simplicity we will sometimes use the following notation for $p = 2$ and $X = L^2(\Omega)$:

$$\|\cdot\|_2 \equiv \|\cdot\|_{L^2(0, T; L^2(\Omega))}.$$

We will also use the notation $u(x, t) = u(t)(x)$ for $(x, t) \in \Omega \times (0, T)$.

Next, we will denote by u' the derivative of u in the t variable, intended in the following weak sense: we say that $u' = v$ if

$$\int_0^T \varphi'(t)u(t)dt = - \int_0^T \varphi(t)v(t)dt$$

for all scalar test functions $\varphi \in C_0^\infty(0, T)$. The space $H^1(0, T; X)$ consists of all functions $u \in L^2(0, T; X)$ such that $u' \in L^2(0, T; X)$.

We will often use the following remark.

Remark 3. *By [13, Theorem 3 page 303], if $u \in L^2(0, T; H_0^1(\Omega))$ and $u' \in L^2(0, T; H^{-1}(\Omega))$, then $u \in C([0, T], L^2(\Omega))$ (after possibly being redefined on a set of measure zero). Moreover the mapping $t \mapsto \|u(t)\|_{L^2(\Omega)}^2$ is absolutely continuous and $\frac{d}{dt}\|u(t)\|_{L^2(\Omega)}^2 = 2\langle u'(t), u(t) \rangle_{L^2(\Omega)}$.*

In this section we collect some useful information about the solution u_λ coming from the classical theory of parabolic problems that can be directly found in the literature.

Firstly, the existence and uniqueness of a weak solution u for the problem (P_λ) follows from the standard Galerkin method see [13, Theorems 3 and 4, Section 7.1]. According to this theory, a weak solution means that:

$$(P_\lambda) \begin{cases} u \in L^2(0, T; H_0^1(\Omega)), & u' \in L^2(0, T; H^{-1}(\Omega)) \\ \int_{Q_T} (u'v + \nabla u \cdot \nabla v + \lambda u v) = \int_{Q_T} f_\lambda v \\ \text{for all } v \in L^2(0, T; H_0^1(\Omega)), \\ u(x, 0) = g_\lambda(x). \end{cases}$$

Remember that by Remark 3 above, such weak solution u is continuous in time so that the initial condition makes sense. In the rest of the paper, (P_λ) will always refer to the above precise formulation of the problem that was first stated in the Introduction.

Next, thanks to [13, Theorem 5, Section 7.1], by considering λu as a right hand side term (in $L^2(\Omega \times (0, T))$), we have the following.

Lemma 1. Let $\lambda > 0$, $g_\lambda \in H_0^1(\Omega)$, $f_\lambda \in L^2(0, T; L^2(\Omega))$, and let u_λ be the weak solution to (P_λ) . Then,

$$u_\lambda \in L^2(0, T; H^2(\Omega)) \cap L^\infty(0, T; H_0^1(\Omega)), \quad u'_\lambda \in L^2(0, T; L^2(\Omega)),$$

and u_λ satisfies the following estimate:

$$\begin{aligned} \sup_{0 \leq t \leq T} \|u_\lambda(t)\|_{H_0^1(\Omega)} + \|u_\lambda\|_{L^2(0, T; H^2(\Omega))} + \|u'_\lambda\|_2 \\ \leq C \left(\lambda \|au_\lambda\|_2 + \|f_\lambda\|_2 + \|g_\lambda\|_{H_0^1(\Omega)} \right), \end{aligned} \quad (3.1)$$

where the constant C depends only on Ω and T .

Remark 4. Notice that the bound (3.1) is not very useful when $\lambda \rightarrow +\infty$ since what we usually control is $\sqrt{\lambda}\|au\|_2$ (shown below in Lemma 3) but not $\lambda\|au\|_2$ thus the right hand side blows-up a priori.

Finally, we recall that assuming more regularity on g_λ , f_λ and a yields more regularity on u_λ (but we shall not use it in this paper).

Lemma 2. Assume that

$$a' \text{ and } f'_\lambda \in L^2(0, T; L^2(\Omega)) \text{ and } g_\lambda \in H^2(\Omega).$$

Let u_λ be the weak solution of problem (P_λ) . Then

$$u_\lambda \in L^\infty(0, T; H^2(\Omega)), \quad u'_\lambda \in L^\infty(0, T; L^2(\Omega)) \cap L^2(0, T; H_0^1(\Omega)), \quad u''_\lambda \in L^2(0, T; H^{-1}(\Omega)),$$

and the quantity

$$\sup_{0 \leq t \leq T} (\|u_\lambda(t)\|_{H^2(\Omega)} + \|u'_\lambda(t)\|_{L^2(\Omega)}) + \|u'_\lambda\|_{L^2(0, T; H_0^1(\Omega))} + \|u''_\lambda\|_{L^2(0, T; H^{-1}(\Omega))}$$

is bounded by

$$C (\|f_\lambda\|_{H^1(0, T; L^2(\Omega))} + \|\lambda au_\lambda\|_{H^1(0, T; L^2(\Omega))} + \|g_\lambda\|_{H^2(\Omega)}),$$

where the constant C depends only on Ω and T .

Remark 5. Notice again that this bound is not very useful when $\lambda \rightarrow +\infty$. However, if λ is fixed, we observe at least that u''_λ is in $L^2(0, T; H^{-1}(\Omega))$.

4 Uniqueness of solution for (P_∞)

In this section we focus on the following problem that will arise as the limit of (P_λ) . Our notion of solution for the problem $\partial_t u - \Delta u = f$ in O_a will precisely be the following :

$$(P_\infty) \begin{cases} u \in L^2(0, T; H_0^1(\Omega)), & u' \in L^2(0, T; L^2(\Omega)) \\ u = 0 \text{ a.e. in } Q_T \setminus O_a \\ \int_{Q_T} (u'v + \nabla u \cdot \nabla v) = \int_{Q_T} f v, \\ \quad \text{for all } v \in L^2(0, T; H_0^1(\Omega)) \text{ s.t. } v = 0 \text{ a.e. in } Q_T \setminus O_a \\ u(x, 0) = g(x) \quad \text{in } \Omega. \end{cases}$$

As a byproduct of Section 5 we will prove the existence of a solution for the problem (P_∞) , as a limit of solutions for (P_λ) . In this section, we prove the uniqueness which follows from a simple energy bound. Notice that a solution u to (P_∞) is continuous in time (see Remark 3) thus the initial condition $u(x, 0) = g(x)$ makes sense in $L^2(\Omega)$.

Proposition 3. *Any solution u of (P_∞) satisfies the following energy bound*

$$\frac{1}{4} \sup_{t \in (0, T)} \|u(t)\|_{L^2(\Omega)}^2 + \|\nabla u\|_2^2 \leq \frac{1}{2} \|g\|_{L^2(\Omega)}^2 + T \|f\|_2. \quad (4.1)$$

Consequently, there exists at most one solution to problem (P_∞) .

Proof. Let u be a solution to (P_∞) , and $s \in (0, T)$. Choosing $v = u \mathbf{1}_{(0, s)}$ (where $\mathbf{1}_{(0, s)}$ is the characteristic function of $(0, s)$) in the weak formulation of the problem, we deduce that

$$\int_0^s \int_\Omega u' u \, dx dt + \int_0^s \int_\Omega |\nabla u|^2 \, dx dt = \int_0^s \int_\Omega f u \, dx dt. \quad (4.2)$$

Now applying Remark 3 and using the fact that $u \in L^2(0, T; H_0^1(\Omega))$ and $u' \in L^2(0, T; L^2(\Omega))$ we obtain that $t \mapsto \|u(t)\|_2^2$ is absolutely continuous, and for a.e. t , there holds

$$\frac{d}{dt} \|u(t)\|_{L^2(\Omega)}^2 = 2 \langle u'(t), u(t) \rangle_{L^2(\Omega)}.$$

Returning to (4.2) we get

$$\frac{1}{2} \|u(s)\|_{L^2(\Omega)}^2 - \frac{1}{2} \|u(0)\|_{L^2(\Omega)}^2 + \int_0^s \int_\Omega |\nabla u|^2 \, dx dt = \int_0^s \int_\Omega f u \, dx dt. \quad (4.3)$$

By Young's inequality we have

$$\begin{aligned} \left| \int_0^s \int_\Omega f u \, dx dt \right| &\leq \frac{\alpha}{2} \|f\|_{L^2(\Omega \times (0, s))}^2 + \frac{1}{2\alpha} \|u\|_{L^2(\Omega \times (0, s))}^2 \\ &\leq \frac{\alpha}{2} \|f\|_{L^2(\Omega \times (0, s))}^2 + \frac{T}{2\alpha} \sup_{t \in (0, T)} \|u(t)\|_{L^2(\Omega)}^2. \end{aligned} \quad (4.4)$$

Setting $\alpha = 2T$, estimating (5.2) by (4.4) and passing to the supremum in $s \in (0, T)$ finally gives

$$\frac{1}{4} \sup_{t \in (0, T)} \|u(t)\|_{L^2(\Omega)}^2 + \|\nabla u\|_2^2 \leq \frac{1}{2} \|g\|_{L^2(\Omega)}^2 + T \|f\|_2^2,$$

as desired.

Now assume that u_1 and u_2 are two solutions of (P_∞) , and set $w := u_1 - u_2$. Then w is a solution of (P_∞) with $f = 0$ and $g = 0$. Therefore, applying (4.1) to w automatically gives $w = 0$, which proves the uniqueness of the solution of (P_∞) . \square

5 Convergence of u_λ

We now analyse the convergence of u_λ , which will follow from energy bounds for u_λ and u'_λ . As already mentioned before, the standard energy bound for the solutions of (P_λ) that is stated in Lemma 1, blows up a priori when λ goes to $+\infty$. Our goal in the sequel is to get better estimates, uniform in λ . The price to pay is the condition $\partial_t a \leq 0$ which implies that O_a is nondecreasing in time (for the set inclusion).

5.1 First energy bound

Lemma 3. *Assume that $g_\lambda \in H_0^1(\Omega)$ and $f_\lambda \in L^2(0, T; L^2(\Omega))$, and let u_λ be the weak solution of problem (P_λ) . Then,*

$$\frac{1}{4} \sup_{t \in (0, T)} \|u_\lambda(t)\|_{L^2(\Omega)}^2 + \|\nabla u_\lambda\|_2^2 + \lambda \int_0^T \int_\Omega a u_\lambda^2 dx dt \leq \|g_\lambda\|_{L^2(\Omega)}^2 + T \|f_\lambda\|_2^2. \quad (5.1)$$

Proof. Let u_λ be a solution of (P_λ) and $s \in (0, T)$. Testing with $v = u \mathbf{1}_{[0, s]}$ in the weak formulation of (P_λ) , we obtain

$$\begin{aligned} \frac{1}{2} \|u_\lambda(s)\|_{L^2(\Omega)}^2 - \frac{1}{2} \|u_\lambda(0)\|_{L^2(\Omega)}^2 + \int_0^s \int_\Omega |\nabla u_\lambda|^2 dx dt + \lambda \int_0^s \int_\Omega a u_\lambda^2 dx dt \\ = \int_0^s \int_\Omega f_\lambda u_\lambda dx dt. \end{aligned} \quad (5.2)$$

Arguing as in the proof of Proposition 3, we obtain for any $\alpha > 0$,

$$\begin{aligned} \frac{1}{2} \|u_\lambda(s)\|_{L^2(\Omega)}^2 + \|\nabla u\|_2^2 + \lambda \int_0^s \int_\Omega a u_\lambda^2 dx dt &\leq \|g_\lambda\|_{L^2(\Omega)}^2 + |\langle f_\lambda, u_\lambda \rangle_{L^2(\Omega \times (0, s))}| \\ &\leq \|g_\lambda\|_{L^2(\Omega)}^2 + \frac{\alpha}{2} \|f_\lambda\|_2^2 + \frac{1}{2\alpha} \|u_\lambda\|_{L^2(\Omega \times (0, s))}^2, \\ &\leq \|g_\lambda\|_{L^2(\Omega)}^2 + \frac{\alpha}{2} \|f_\lambda\|_2^2 + \sup_{t \in (0, T)} \frac{T}{2\alpha} \|u_\lambda(t)\|_{L^2(\Omega)}^2. \end{aligned}$$

Inequality (5.1) follows by taking the supremum in $s \in (0, T)$ and fixing $\alpha := 2T$. \square

5.2 Second energy bound

We now derive a uniform bound on $\|u'_\lambda\|_2$. To this end, we will assume a time-monotonicity condition on a .

Definition 1 (Assumptions (A)). *We say that Assumptions (A) hold if $a : Q_T \rightarrow \mathbb{R}^+$ is Lipschitz and*

$$\partial_t a(x, t) \leq 0 \text{ for a.e. } (x, t) \in Q_T. \quad (5.3)$$

Lemma 4. *We suppose that Assumptions (A) hold. Then, the solution u_λ of (P_λ) satisfies the estimate:*

$$\begin{aligned} & \int_0^T \int_\Omega (u'_\lambda)^2 dxdt + \sup_{s \in (0, T)} \left(\int_\Omega |\nabla u_\lambda(s)|^2 dx \right) \\ & \leq \int_0^T \int_\Omega f_\lambda^2 dxdt + \int_\Omega |\nabla u_\lambda(0)|^2 dx + \lambda \int_\Omega a(0) g_\lambda^2 dx. \end{aligned} \quad (5.4)$$

Proof. Thanks to Lemma 1, we know that $u'_\lambda \in L^2(0, T; H_0^1(\Omega))$. Consequently, for every $s \in (0, T)$, the function $v := u'_\lambda \mathbf{1}_{(0, s)}$ is an admissible test function in the weak formulation of (P_λ) . Hence, we obtain the identity

$$\int_0^s \int_\Omega (u'_\lambda)^2 dxdt + \int_0^s \int_\Omega \nabla u_\lambda \cdot \nabla u'_\lambda dxdt + \lambda \int_0^s \int_\Omega a u_\lambda u'_\lambda dxdt = \int_0^s \int_\Omega f_\lambda u'_\lambda dxdt,$$

or written differently (applying Remark 3),

$$\begin{aligned} \int_0^s \int_\Omega |u'_\lambda|^2 dxdt + \int_0^s \left(\frac{1}{2} \int_\Omega |\nabla u_\lambda|^2 dx \right)' dt + \lambda \int_0^s \left[\left(\frac{1}{2} \int_\Omega a u_\lambda^2 dx \right)' - \frac{1}{2} \int_\Omega a' u_\lambda^2 dx \right] dt \\ = \int_0^s \int_\Omega f_\lambda u'_\lambda dxdt. \end{aligned}$$

This yields

$$\begin{aligned} \int_0^s \int_\Omega (u'_\lambda)^2 dxdt + \frac{1}{2} \int_\Omega |\nabla u_\lambda(s)|^2 dx + \frac{\lambda}{2} \int_\Omega a(s) u_\lambda(s)^2 dx - \frac{\lambda}{2} \int_0^s \int_\Omega a' u_\lambda^2 dxdt \\ = \int_0^s \int_\Omega f_\lambda u'_\lambda dxdt + \frac{1}{2} \int_\Omega |\nabla u_\lambda(0)|^2 dx + \frac{\lambda}{2} \int_\Omega a(0) u_\lambda(0)^2 dx. \end{aligned}$$

By Young's inequality,

$$\int_0^s \int_\Omega f_\lambda u'_\lambda dxdt \leq \frac{1}{2} \int_0^s \int_\Omega f_\lambda^2 dxdt + \frac{1}{2} \int_0^s \int_\Omega (u'_\lambda)^2 dxdt,$$

so that we obtain

$$\begin{aligned} \int_0^s \int_\Omega (u'_\lambda)^2 dxdt + \int_\Omega |\nabla u_\lambda(s)|^2 dx + \lambda \int_\Omega a(s) u_\lambda(s)^2 dx - \lambda \int_0^s \int_\Omega a' u_\lambda^2 dxdt \\ \leq \int_0^s \int_\Omega |f_\lambda|^2 dxdt + \int_\Omega |\nabla u_\lambda(0)|^2 dx + \lambda \int_\Omega a(0) u_\lambda(0)^2 dx. \end{aligned}$$

Finally, using Assumption (A), the initial condition on $u_\lambda(0)$ and passing to the supremum in s , we conclude that estimate (5.4) holds. \square

5.3 Weak convergence of solutions

Using the previous energy estimates, we first establish the weak convergence of u_λ to the solution u of Problem (P_∞) , under assumption (A), and supposing certain bounds on the right hand side f_λ and on the initial data g_λ .

Proposition 4. *Assume that a satisfies assumption (A). Let (f_λ) be a bounded sequence in $L^2(Q_T)$ and (g_λ) be a bounded sequence in $H_0^1(\Omega)$, satisfying*

$$\sup_\lambda \left(\lambda \int_\Omega a(0)g_\lambda^2 dx \right) < \infty. \quad (5.5)$$

Up to extracting subsequences, we can assume that f_λ converges weakly to a function f in $L^2(Q_T)$, and g_λ converges weakly to a function $g \in H_0^1(\Omega)$.

Let u_λ be the solution of (P_λ) . Then u_λ converges weakly in $L^2(Q_T)$ to the unique solution u of problem (P_∞) .

Proof. We know by Lemma 3 that u_λ is uniformly bounded in $L^2(0, T; H_0^1(\Omega))$, thus converges weakly (up to extracting a subsequence) in $L^2(0, T; H_0^1(\Omega))$ to some function $u \in L^2(0, T; H_0^1(\Omega))$. Under assumption (A), we also know, thanks to Lemma 4, that

$$\|u'_\lambda\|_{L^2(Q_T)} \leq C,$$

so that, u'_λ converges also weakly in $L^2(Q_T)$ (up to extract a further subsequence) to some limit $w \in L^2(Q_T)$, which must be equal to u' by uniqueness of the limit in $\mathcal{D}'(Q_T)$. This shows that $u' \in L^2(Q_T)$.

Next, from (5.1) we know that

$$\sup_\lambda \left(\lambda \int_0^T \int_\Omega a u_\lambda^2 dx dt \right) \leq C,$$

which implies that u must be equal to zero a.e. on any set of the form $\{a > \varepsilon\}$, with $\varepsilon > 0$. By considering the union for $n \in \mathbb{N}^*$ of those sets with $\varepsilon = 1/n$, we obtain that $u = 0$ a.e. on $Q_T \setminus O_a$.

Now let us check that u satisfies the equation in the weak sense. Let v be any test function in $L^2(0, T; H_0^1(\Omega))$ such that $v = 0$ a.e. in $Q_T \setminus O_a$. Then $au_\lambda v = 0$ a.e. in Q_T , and using the fact that u_λ is a solution of (P_λ) , we can write

$$\langle u'_\lambda, v \rangle_{L^2(Q_T)} + \langle \nabla u_\lambda, \nabla v \rangle_{L^2(Q_T)} = \langle f_\lambda, v \rangle_{L^2(Q_T)}.$$

Thus passing to the (weak) limit in u_λ , u'_λ and f_λ we get

$$\langle u', v \rangle_{L^2(Q_T)} + \langle \nabla u, \nabla v \rangle_{L^2(Q_T)} = \langle f, v \rangle_{L^2(Q_T)}.$$

To conclude that u is a solution of (P_∞) it remains to prove that $u(x, 0) = g(x)$ for a.e. $x \in \Omega$. For this purpose, we let $v \in C^1([0, T], H_0^1(\Omega))$ be any function satisfying $v(T) = 0$.

Testing the equation with this v , using that $u_\lambda(0) = g_\lambda$ and integrating by parts with respect to t we obtain

$$-\langle g_\lambda, v(0) \rangle_{L^2(\Omega)} - \int_0^T \langle u_\lambda, v' \rangle_{L^2(\Omega)} + \int_0^T \langle \nabla u_\lambda, \nabla v \rangle_{L^2(\Omega)} = \int_0^T \langle f_\lambda, v \rangle.$$

Passing to the limit in λ and using the weak convergence of g_λ to g , we get

$$-\langle g, v(0) \rangle_{L^2(\Omega)} - \int_0^T \langle u, v' \rangle_{L^2(\Omega)} + \int_0^T \langle \nabla u, \nabla v \rangle_{L^2(\Omega)} = \int_0^T \langle f, v \rangle_{L^2(\Omega)}.$$

Integrating back again by parts on u yields

$$\langle g, v(0) \rangle_{L^2(\Omega)} = \langle u(0), v(0) \rangle_{L^2(\Omega)},$$

and since $v(0)$ is arbitrary, we deduce that $u(0) = g$ in $L^2(\Omega)$.

Finally, the convergence of u_λ to u holds a priori up to a subsequence, but by uniqueness of the solution for the problem (P_∞) (see Proposition 3), the convergence holds for the whole sequence. \square

Corollary 1. *Let $O_a \subset Q_T$ be open and increasing in time (in the sense of Remark 1), and let $f \in L^2(\Omega \times (0, T))$ and $g \in H_0^1(\Omega)$. Then there exists a (unique) solution for (P_∞) .*

Proof. It suffices to apply Theorem 3 with, for instance $a(x, t) := \text{dist}((x, t), \overline{O_a})$, $f_\lambda = f$ and $g_\lambda = g$. \square

Remark 6. (Convergence in $\mathcal{D}'(\Omega \times (0, T))$). *Under assumptions (A), letting u being the weak limit of u_λ in $L^2(Q_T)$, we already know that*

$$u = 0 \quad \text{a.e. in } Q_T \setminus O_a.$$

Then

$$f_\lambda + \Delta u_\lambda - u'_\lambda \longrightarrow f + \Delta u - u' \quad \text{in } \mathcal{D}'(\Omega \times (0, T)),$$

which implies that

$$\lambda u_\lambda \longrightarrow h \quad \text{in } \mathcal{D}'(\Omega \times (0, T)), \tag{5.6}$$

for some distribution $h = f + \Delta u - u' \in \mathcal{D}'(\Omega \times (0, T))$, supported in O_a^c . Actually, since $u = 0$ in O_a^c , we have

$$\Delta u = 0 \quad \text{and} \quad u' = 0 \quad \text{in } \mathcal{D}'(\overline{O_a^c}).$$

This means that

$$h = 0 \quad \text{in } \mathcal{D}'(O_a) \quad \text{and} \quad h = f \quad \text{in } \mathcal{D}'(\overline{O_a^c}).$$

Notice that, a priori, h could have a singular part supported on ∂O_a . We finally deduce that

$$\lambda u_\lambda \xrightarrow{\lambda \rightarrow +\infty} f \mathbf{1}_{O_a^c} + (\Delta u - u')|_{\partial O_a} \quad \text{in } \mathcal{D}'(\Omega \times (0, T)). \tag{5.7}$$

5.4 Strong convergence in $L^2(0, T; H^1(\Omega))$

We now go further using the same argument as for the stationary problem, and prove a stronger convergence which is one of our main results.

Theorem 3. *Under the same hypotheses as in Proposition 4, denote by u_λ the solution of (P_λ) . Then, u_λ converges strongly in $L^2(0, T; H_0^1(\Omega))$ to the solution u of problem (P_∞) .*

Proof. We have the bound

$$\|u_\lambda\|_{L^2(0, T; H^1(\Omega))} \leq C,$$

and we already know (by Proposition 4) that u_λ converges weakly in $L^2(0, T; H^1(\Omega))$ to u , the unique solution of problem (P_∞) .

Moreover, by the lower semicontinuity of the norm with respect to the weak convergence, there holds

$$\|u\|_{L^2(0, T; H^1(\Omega))} \leq \liminf_\lambda \|u_\lambda\|_{L^2(0, T; H^1(\Omega))}.$$

Hence, to prove the strong convergence we only need to prove the reverse inequality, with a limsup. For this purpose we use the fact that $u(t)$ is a competitor for $u_\lambda(t)$ in the minimization problem solved by u_λ at t fixed. Indeed, for a.e. t fixed, u_λ solves

$$-\Delta u_\lambda + \lambda a u_\lambda = f_\lambda - u'_\lambda,$$

thus, u_λ is a minimizer in $H_0^1(\Omega)$ for the energy

$$v \mapsto E_\lambda(v) - 2 \int_\Omega v(f_\lambda - u'_\lambda),$$

where E_λ is defined by (2.3). Furthermore, due to the bound (5.4) obtained in Lemma 4, since f_λ is bounded in $L^2(Q_T)$ and g_λ is bounded in $L^2(\Omega)$ and satisfies (5.5), we know that u'_λ is bounded in $L^2(Q_T)$, and

$$u'_\lambda \rightarrow u' \quad \text{in weak } L^2(Q_T).$$

We also know that, up to a subsequence, $u_\lambda \rightarrow u$ strongly in $L^2(Q_T)$ (because it is bounded in $H^1(Q_T)$).

Now, using that u is a competitor for u_λ (for a.e. t fixed), we obtain

$$\begin{aligned} \int_\Omega |\nabla u_\lambda|^2 dx - 2 \int_\Omega u_\lambda(f_\lambda - u'_\lambda) &\leq E_\lambda(u_\lambda) - 2 \int_\Omega u_\lambda(f_\lambda - u'_\lambda) \leq E_\lambda(u) - 2 \int_\Omega u(f_\lambda - u'_\lambda) \\ &\leq \int_\Omega |\nabla u|^2 dx - 2 \int_\Omega u(f_\lambda - u'_\lambda). \end{aligned}$$

Integrating in $t \in [0, T]$ and passing to the limsup in λ , we get the desired inequality, which achieves the proof. \square

6 Simon's exponential estimate

6.1 The stationary case

Following a similar argument to [19, Theorem 4.1] we ascertain some strong convergence far from the set $\Omega_a := \text{Int}(K_a)$, where K_a is defined by $K_a := \{x \in \bar{\Omega}; a(x) = 0\}$.

Lemma 5. *Let $a : \bar{\Omega} \rightarrow \mathbb{R}^+$ be a continuous non-negative potential and u_λ be the unique weak solution in $H_0^1(\Omega)$ of $-\Delta u_\lambda + \lambda a u_\lambda = f_\lambda$ in Ω . Introduce $K_a := \{x \in \bar{\Omega}; a(x) = 0\}$ and assume that $\Omega_a := \text{Int}\{a(x) = 0\}$ is non empty (hypothesis (2.2)). Let $\varepsilon > 0$ be fixed, and define*

$$\Omega_\varepsilon := \{x \in \Omega; \text{dist}(x, \Omega_a) > \varepsilon\} \quad \text{and} \quad \delta := \min_{x \in \bar{\Omega}_\varepsilon} a(x) > 0.$$

Then, there exists a constant $C > 0$ such that for all $\lambda > 0$ and for all Lipschitz function $\eta : \Omega \rightarrow \mathbb{R}$ that is equal to 1 in $\Omega_{2\varepsilon}$ and to 0 outside Ω_ε , we have

$$\int_{\Omega_\varepsilon} e^{2\sqrt{\lambda}\frac{\delta}{2}} \mathbf{1}_{\Omega_{2\varepsilon}^c} \text{dist}(x, \Omega_{2\varepsilon}^c) \eta^2 u_\lambda(x) \left(\frac{\lambda\delta}{2} u_\lambda - f_\lambda \right) dx \leq C. \quad (6.1)$$

Proof. To lighten the notation, in this proof we will denote by $\langle \cdot, \cdot \rangle$ the scalar product in $L^2(\Omega)$.

Let $\varepsilon > 0$ be fixed. For any function ψ supported in Ω_ε and for any Lipschitz function $\rho : \Omega \rightarrow \mathbb{R}$ satisfying $|\nabla\rho|^2 \leq \delta/2$, a direct calculation shows the following identities

$$\begin{aligned} \Delta(e^{-\sqrt{\lambda}\rho}) &= -\sqrt{\lambda}e^{-\sqrt{\lambda}\rho}\Delta\rho + \lambda|\nabla\rho|^2e^{-\sqrt{\lambda}\rho}, \\ \Delta(e^{-\sqrt{\lambda}\rho}\psi) &= \psi\Delta(e^{-\sqrt{\lambda}\rho}) + e^{-\sqrt{\lambda}\rho}\Delta\psi + 2\nabla(e^{-\sqrt{\lambda}\rho}) \cdot \nabla\psi, \end{aligned}$$

from which we obtain that

$$\langle e^{\sqrt{\lambda}\rho}\psi, (-\Delta + \lambda a)e^{-\sqrt{\lambda}\rho}\psi \rangle = \langle \psi, (\sqrt{\lambda}\Delta\rho - \lambda|\nabla\rho|^2 + 2\sqrt{\lambda}\nabla\rho \cdot \nabla - \Delta + \lambda a)\psi \rangle.$$

Then, expanding $\Delta(\rho\psi)$ and rearranging terms as follows:

$$\Delta(\rho\psi) = \Delta\rho\psi + 2\nabla\rho\nabla\psi + \Delta\psi\rho = \Delta\rho\psi + \nabla\rho\nabla\psi + \nabla(\rho\nabla\psi),$$

it is not difficult to see that

$$\begin{aligned} \langle \psi, (\Delta\rho + 2\nabla\rho \cdot \nabla)\psi \rangle &= \langle \psi, \Delta(\rho\psi) \rangle - \langle \psi, \rho\Delta\psi \rangle \\ &= \langle \psi, \Delta(\rho\psi) \rangle - \langle \rho\psi, \Delta\psi \rangle \\ &= 0, \end{aligned}$$

since ψ is compactly supported in Ω . Therefore, by positivity of $-\Delta$ and using the assumption $|\nabla\rho|^2 \leq \delta/2$ we get

$$\begin{aligned} \langle e^{\sqrt{\lambda}\rho}\psi, (-\Delta + \lambda a)e^{-\sqrt{\lambda}\rho}\psi \rangle &\geq \langle \psi, [\lambda(a - |\nabla\rho|^2)]\psi \rangle \\ &\geq \frac{\lambda\delta}{2} \langle \psi, \psi \rangle = \frac{\lambda\delta}{2} \|\psi\|_2^2. \end{aligned} \quad (6.2)$$

Next, we apply (6.2) with the choice $\psi = e^{\sqrt{\lambda}\rho}\eta u_\lambda(x)$, where η is a function equal to 1 in $\Omega_{2\varepsilon}$ and 0 outside Ω_ε . Since by construction $\|u_\lambda\|_2 + \|\nabla u_\lambda\|_2 \leq C\|f_\lambda\|_2$, (6.2) implies

$$\begin{aligned}
\frac{\lambda\delta}{2} \int_{\Omega_\varepsilon} e^{2\sqrt{\lambda}\rho}\eta^2 u_\lambda^2 dx &= \frac{\lambda\delta}{2} \|\psi\|_2^2 \leq \int_{\Omega} e^{2\sqrt{\lambda}\rho}\eta u_\lambda (-\Delta + \lambda a)(\eta u_\lambda) dx \\
&\leq \int_{\Omega_\varepsilon \setminus \Omega_{2\varepsilon}} e^{2\sqrt{\lambda}\rho} u_\lambda (-u_\lambda \Delta \eta - 2\nabla \eta \cdot \nabla u_\lambda) dx \\
&\quad + \int_{\Omega} e^{2\sqrt{\lambda}\rho} \eta^2 u_\lambda (-\Delta u_\lambda + \lambda a u_\lambda) dx \\
&\leq \int_{\Omega_\varepsilon \setminus \Omega_{2\varepsilon}} e^{2\sqrt{\lambda}\rho} u_\lambda (-u_\lambda \Delta \eta - 2\nabla \eta \cdot \nabla u_\lambda) dx + \int_{\Omega} e^{2\sqrt{\lambda}\rho} \eta^2 u_\lambda f_\lambda dx \\
&\leq C(\|f_\lambda\|_2) e^{2\sqrt{\lambda}M} + \int_{\Omega} e^{2\sqrt{\lambda}\rho} \eta^2 u_\lambda f_\lambda dx, \tag{6.3}
\end{aligned}$$

where

$$M := \sup_{x \in \Omega \setminus \Omega_{2\varepsilon}} \rho(x),$$

and the constant C in (6.3) depends on the derivatives of η and ε . Now we take the particular choice

$$\rho(x) := \sqrt{\frac{\delta}{2}} \mathbf{1}_{\Omega_{2\varepsilon}} \text{dist}(x, \Omega_{2\varepsilon}^c),$$

which satisfies all our needed assumptions (i.e. ρ is Lipschitz with $|\nabla \rho|^2 \leq \delta/2$ and supported in Ω_ε). In this case $M = 0$ thus (6.3) simply implies

$$\frac{\lambda\delta}{2} \int_{\Omega_\varepsilon} e^{2\sqrt{\lambda}\rho} \eta^2 u_\lambda^2 dx \leq C + \int_{\Omega} e^{2\sqrt{\lambda}\rho} \eta^2 u_\lambda f_\lambda dx,$$

or differently,

$$\int_{\Omega_\varepsilon} e^{2\sqrt{\lambda}\rho} \eta^2 u_\lambda(x) \left(\frac{\lambda\delta}{2} u_\lambda - f_\lambda \right) dx \leq C,$$

which ends the proof. \square

Remark 7. *The previous lemma can be used for instance in the following two particular cases: first in the particular case when $f = 0$ in $\Omega \setminus \Omega_a$. Thus, we get the useful rate of convergence of $u_\lambda \rightarrow 0$ as $\lambda \rightarrow 0$ far from Ω_a :*

$$\int_{\Omega_{2\varepsilon}} \lambda e^{2\sqrt{\lambda}\frac{\delta}{2} \text{dist}(x, \Omega_{2\varepsilon}^c)} u_\lambda^2 dx \leq C.$$

This is much better compared to the usual and simple energy bound:

$$\lambda \int_{\Omega} a u_\lambda^2 \leq C.$$

Another application is when u_λ is an eigenfunction (this is actually the original framework of Simon [19]), i.e. when $f_\lambda = \sigma(\lambda)u_\lambda$ and with $\sigma(\lambda)$ standing for the first eigenvalue associated with u_λ . In this case, since we are assuming that the potential a might vanish in a subdomain, we have that $\sigma(\lambda)$ is bounded (cf. [2] for further details). Consequently, thanks to this bound for λ large enough $\frac{\lambda\delta}{2} - \sigma(\lambda) \geq 1$ which implies

$$\int_{\Omega_{2\varepsilon}} e^{2\sqrt{\lambda\frac{\delta}{2}}\text{dist}(x,\Omega_{2\varepsilon}^c)} u_\lambda^2 dx \leq C.$$

6.2 The parabolic case

We now extend the previous decay estimate to the parabolic problem.

Lemma 6. *Let $a : \overline{Q_T} \rightarrow \mathbb{R}^+$ be a continuous non-negative potential such that O_a is non empty, $f_\lambda \in L^2(Q_T)$, $g_\lambda \in H_0^1(\overline{O_a} \cap \{t=0\})$ and let u_λ be the solution of (P_λ) .*

For every $\varepsilon > 0$, we define

$$A_\varepsilon := \{(x, t) \in \overline{Q_T}; \text{dist}((x, t), O_a) > \varepsilon\} \quad \text{and} \quad \delta := \min_{(x,t) \in A_\varepsilon} a(x, t) > 0. \quad (6.4)$$

Then, for any $\lambda \geq 4$, and for any Lipschitz function $\eta : Q_T \rightarrow \mathbb{R}$ equal to 1 in $A_{2\varepsilon}$ and 0 outside A_ε , there exists a constant $C > 0$ such that

$$\int_{A_\varepsilon} e^{2\sqrt{\lambda\frac{\delta}{2}}\mathbf{1}_{A_{2\varepsilon}}\text{dist}(x,A_{2\varepsilon}^c)} \eta^2 u_\lambda(x) \left(\frac{\lambda\delta}{2} u_\lambda - f_\lambda \right) dx \leq C.$$

Proof. Let $\varepsilon > 0$ be fixed. We consider any function ψ supported in A_ε and any Lipschitz function $\rho : Q_T \rightarrow \mathbb{R}$ satisfying $|\partial_t \rho| + |\nabla \rho| \leq \delta/2$. Then we notice that for any fixed t , the functions $x \mapsto \rho(x, t)$, $x \mapsto \psi(x, t)$ and $x \mapsto a(x, t)$ satisfy all the conditions required to prove the key estimate (6.2) in the stationary case. Thus, for any fixed t , there holds

$$\langle e^{\sqrt{\lambda}\rho}\psi, (-\Delta + \lambda a)e^{-\sqrt{\lambda}\rho}\psi \rangle_{L^2(\Omega)} \geq \frac{\lambda\delta}{2} \|\psi\|_{L^2(\Omega)}^2.$$

Integrating this inequality over $t \in (0, T)$ yields

$$\langle e^{\sqrt{\lambda}\rho}\psi, (-\Delta + \lambda a)e^{-\sqrt{\lambda}\rho}\psi \rangle_{L^2(Q_T)} \geq \frac{\lambda\delta}{2} \|\psi\|_{L^2(Q_T)}^2.$$

Next we also compute

$$\partial_t (e^{-\sqrt{\lambda}\rho}\psi) = -\sqrt{\lambda}e^{-\sqrt{\lambda}\rho}\psi\partial_t\rho + e^{-\sqrt{\lambda}\rho}\partial_t\psi,$$

so that,

$$\begin{aligned} \langle e^{\sqrt{\lambda}\rho}\psi, \partial_t(e^{-\sqrt{\lambda}\rho}\psi) \rangle_{L^2(Q_T)} &= -\sqrt{\lambda}\langle \psi, \psi\partial_t\rho \rangle_{L^2(Q_T)} + \langle \psi, \partial_t\psi \rangle_{L^2(Q_T)} \\ &= -\sqrt{\lambda}\langle \psi, \psi\partial_t\rho \rangle_{L^2(Q_T)} + \frac{1}{2}\|\psi(T)\|_{L^2(\Omega)}^2 - \frac{1}{2}\|\psi(0)\|_{L^2(\Omega)}^2 \\ &\geq -\sqrt{\lambda}\frac{\delta}{2}\|\psi\|_{L^2(Q_T)}^2 - \frac{1}{2}\|\psi(0)\|_{L^2(\Omega)}^2. \end{aligned}$$

Gathering the previous estimates, we obtain

$$\langle e^{\sqrt{\lambda}\rho}\psi, (\partial_t - \Delta + \lambda a)e^{-\sqrt{\lambda}\rho}\psi \rangle_{L^2(Q_T)} \geq \frac{(\lambda - \sqrt{\lambda})\delta}{2} \|\psi\|_2^2 - \frac{1}{2} \|\psi(0)\|_{L^2(\Omega)}^2. \quad (6.5)$$

Next, we apply (6.5) with the choice $\psi = e^{\sqrt{\lambda}\rho}\eta u_\lambda(x)$, where η is a function equal to 1 in $A_{2\varepsilon}$ and 0 outside A_ε . We assume that $\lambda \geq 4$ so that $\lambda - \sqrt{\lambda} \geq \lambda/2$. Due to the assumptions, $g \in H_0^1(\overline{O_a} \cap \{t = 0\})$ and, then, $\|\psi(0)\|_{L^2(\Omega)} = 0$. Thus, (6.5) implies

$$\begin{aligned} \frac{\lambda\delta}{4} \int_{A_\varepsilon} e^{2\sqrt{\lambda}\rho} \eta^2 u_\lambda^2 dx &= \frac{\lambda\delta}{4} \|\psi\|_2^2 \leq \int_{\Omega \times (0, T)} e^{2\sqrt{\lambda}\rho} \eta u_\lambda (\partial_t - \Delta + \lambda a)(\eta u_\lambda) dx \\ &\leq \int_{A_\varepsilon \setminus A_{2\varepsilon}} e^{2\sqrt{\lambda}\rho} u_\lambda (u_\lambda \partial_t \eta - u_\lambda \Delta \eta - 2\nabla \eta \cdot \nabla u_\lambda) dx \\ &\quad + \int_{\Omega \times (0, T)} e^{2\sqrt{\lambda}\rho} \eta^2 u_\lambda (\partial_t u_\lambda - \Delta u_\lambda + \lambda a u_\lambda) dx \\ &= \int_{A_\varepsilon \setminus A_{2\varepsilon}} e^{2\sqrt{\lambda}\rho} u_\lambda (u_\lambda \partial_t \eta - u_\lambda \Delta \eta - 2\nabla \eta \cdot \nabla u_\lambda) dx \\ &\quad + \int_{\Omega \times (0, T)} e^{2\sqrt{\lambda}\rho} \eta^2 u_\lambda f_\lambda dx \\ &\leq C(\|u_\lambda\|_2^2) e^{2\sqrt{\lambda}M} + \int_{A_\varepsilon} e^{2\sqrt{\lambda}\rho} \eta^2 u_\lambda f_\lambda dx, \end{aligned} \quad (6.6)$$

where

$$M := \sup_{x \in Q_T \setminus \Omega_{2\varepsilon}} \rho(x),$$

and the constant C in (6.6) depends on the derivatives of η and on ε . Now we take the particular choice

$$\rho(x, t) := \sqrt{\frac{\delta}{2}} \mathbf{1}_{A_{2\varepsilon}} \text{dist}(x, A_{2\varepsilon}^c),$$

which satisfies all our needed assumptions (i.e. ρ is Lipschitz with $|\partial_t \rho| + |\nabla \rho| \leq \delta/2$ and supported in A_ε). In this case, $M = 0$ so that that (6.6) reduces to

$$\frac{\lambda\delta}{4} \int_{A_\varepsilon} e^{2\sqrt{\lambda}\rho} \eta^2 u_\lambda^2 dx \leq C + \int_{A_\varepsilon} e^{2\sqrt{\lambda}\rho} \eta^2 u_\lambda f_\lambda dx,$$

and hence

$$\int_{A_\varepsilon} e^{2\sqrt{\lambda}\rho} \eta^2 u_\lambda(x) \left(\frac{\lambda\delta}{4} u_\lambda - f_\lambda \right) dx \leq C.$$

□

We end this section by noticing that Theorem 2 follows directly from Lemma 6.

Corollary 2. *In the particular case when $f = 0$ in $Q_T \setminus O_a$ we get the useful rate of convergence of $u_\lambda \rightarrow 0$ as $\lambda \rightarrow 0$ far from O_a :*

$$\lambda e^{4\varepsilon\sqrt{\lambda}\frac{\delta}{2}} \int_{A_{2\varepsilon}} u_\lambda^2 dx \leq C.$$

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