LOCAL UNIQUENESS OF m-BUBBLING SEQUENCES FOR THE GEL'FAND EQUATION.

DANIELE BARTOLUCCI$^{(\dagger)}$, ALEKS JEVNIKAR, YOUNGAE LEE$^{(\ddagger)}$, AND WEN YANG$^{(\sharp)}$

ABSTRACT. We consider the Gel'fand problem,

$$\begin{cases} \Delta w_\varepsilon + \varepsilon^2 h e^{w_\varepsilon} = 0 & \text{in} \quad \Omega, \\ \\ w_\varepsilon = 0 & \text{on} \quad \partial\Omega, \end{cases}$$

where h is a nonnegative function in $\Omega \subset \mathbb{R}^2$. Under suitable assumptions on h and Ω, we prove the local uniqueness of $m-$bubbling solutions for any $\varepsilon > 0$ small enough.

Keywords: Gel'fand equation, local uniqueness, blow up solutions.

1. **Introduction**

We are concerned with the Gel'fand problem,

$$\begin{cases} -\Delta w_n = \varepsilon_n^2 h e^{w_n} & \text{in} \quad \Omega, \\ \\ w_n = 0 & \text{on} \quad \partial\Omega, \end{cases} \tag{1.1}$$

where $\Omega \subset \mathbb{R}^2$ is a smooth and bounded domain, $\lim\limits_{n\to+\infty} \varepsilon_n \to 0$, and

$$h(x) = \hat{h}(x)\exp(-4\pi \sum_{i=1}^{\ell} \alpha_i G(x, p_i)) \geq 0, \;\; \hat{h} > 0, \;\; \hat{h} \in C^\infty(\overline{\Omega}). \tag{1.2}$$

Here p_i's are distinct points in Ω, $\alpha_i > -1$ for any $i = 1, \cdots, \ell$ and $G(x, p)$ is the Green function satisfying,

$$-\Delta G(x, p) = \delta_p \quad \text{in} \quad \Omega, \quad G(x, p) = 0 \quad \text{on} \quad \partial\Omega.$$

The equation in (1.1) and its mean field type analogue (1.3) below, have a long history in pure and applied mathematics, ranging from conformal geometry, thermal ignition models, kinetic and mean field models in statistical mechanics, chemiotaxis dynamics and gauge field theories, see for example [8, 10, 20, 31, 32, 33] and references therein. Therefore a huge work has been done to understand these

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equations. The literature is so large that it is impossible to provide a complete list and we just refer to to [3, 4, 5, 9, 12, 13, 14, 15, 16, 21, 22, 23, 25, 26, 28, 30] and the references quoted therein. Among many other things, an interesting property of these problems is the lack of compactness [9] which in turn causes non uniqueness of solutions for $\varepsilon > 0$. As a consequence the global bifurcation diagram is in general very rich and complicated [14], [26] and solutions may be degenerate. Here we say that a solution of (1.1) is degenerate if the corresponding linearized problem (see (1.11) below) admits a non trivial solution. This is why it is important to understand where we can recover uniqueness and non degeneracy in the bifurcation diagram.

In this paper, we consider a sequence of bubbling solutions w_n of (1.1) with $n \to +\infty$.

Definition 1.1. *A sequence of solutions w_n of* (1.1)*, is said to be an $m-$bubbling (or blow up) sequence if*

$$\varepsilon_n^2 h e^{w_n} \rightharpoonup 8\pi \sum_{j=1}^{m} \delta_{q_j} \quad \text{as} \quad n \to +\infty,$$

weakly in the sense of measures in Ω, where $\{q_1, \cdots, q_m\} \subset \Omega$ are m distinct points satisfying $\{q_1, \cdots, q_m\} \cap \{p_1, \cdots, p_\ell\} = \emptyset$. The points q_j are said to be the blow up points and $\{q_1, \cdots, q_m\}$ the blow up set.

Remark 1.1. *It is well known* [22, 23, 26]*, that if w_n of* (1.1) *is an $m-$bubbling sequence, then $\varepsilon_n^2 \int_\Omega h e^{w_n} \to 8m\pi$ as $n \to +\infty$.*

By letting $\lambda_n := \varepsilon_n^2 \int_\Omega h e^{w_n}$, we see that (1.1) takes the form of the well known mean field equation [10],

$$\begin{cases} -\Delta u_n = \lambda_n \dfrac{h e^{u_n}}{\int_\Omega h e^{u_n}} & \text{in} \quad \Omega, \\ \\ u_n = 0 & \text{on} \quad \partial\Omega. \end{cases} \tag{1.3}$$

Let us shortly discuss some results about (1.3). Let

$$R(x,y) = G(x,y) + \frac{1}{2\pi} \log |x-y|,$$

be the regular part of the Green function $G(x,y)$. For fixed

$$\mathbf{q} = (q_1, \cdots, q_m) \in \Omega \times \cdots \times \Omega,$$

we set,

$$G_j^*(x) = 8\pi R(x, q_j) + 8\pi \sum_{l \neq j} G(x, q_l), \tag{1.4}$$

$$l(\mathbf{q}) = \sum_{j=1}^{m} \left(\Delta \log h(q_j) \right) h(q_j) e^{G_j^*(q_j)}, \tag{1.5}$$

and

$$f_{\mathbf{q},j}(x) = 8\pi \left[R(x,q_j) - R(q_j,q_j) + \sum_{l \neq j} (G(x,q_l) - G(q_j,q_l)) \right] + \log \frac{h(x)}{h(q_j)}. \tag{1.6}$$

We will denote by $B_r(q)$ the ball of radius r centred at $q \in \Omega$. For the case $m \geq 2$ we fix a constant $r_0 \in (0, \frac{1}{2})$ and a family of open sets Ω_j satisfying $\Omega_l \cap \Omega_j = \emptyset$ if $l \neq j$, $\bigcup_{j=1}^m \overline{\Omega}_j = \overline{\Omega}$, $B_{2r_0}(q_j) \subset \Omega_j$, $j = 1, \cdots, m$. Then, let us define,

$$D(\mathbf{q}) = \lim_{r \to 0} \sum_{j=1}^{m} h(q_j) e^{G_j^*(q_j)} \left(\int_{\Omega_j \setminus B_{r_j}(q_j)} e^{\Phi_j(x,\mathbf{q})} \mathrm{d}x - \frac{\pi}{r_j^2} \right), \tag{1.7}$$

where $\Omega_1 = \Omega$ if $m = 1$, $r_j = r\sqrt{8h(q_j)e^{G_j^*}(q_j)}$ and

$$\Phi_j(x, \mathbf{q}) = 8\pi \sum_{l=1}^{m} G(x, q_l) - G_j^*(q_j) + \log h(x) - \log h(q_j). \tag{1.8}$$

To describe the location of the blow up points, we introduce the following function. For $(x_1, \cdots, x_m) \in \Omega \times \cdots \times \Omega$ we define,

$$f_m(x_1, x_2, \cdots, x_m) = \sum_{j=1}^{m} \Big[\log(h(x_j)) + 4\pi R(x_j, x_j) \Big] + 4\pi \sum_{l \neq j} G(x_l, x_j), \tag{1.9}$$

and let $D^2_\Omega f_m$ be its Hessian matrix on Ω. The function f_m is also known in literature as the $m-$vortex Hamiltonian [27]. It is also well known [13, 26] that the blow up points vector $\mathbf{q} = (q_1, \cdots, q_m)$ is a critical point of f_m. In this paper, we deal with critical points of the function f_m whose Hessian matrix $D^2_\Omega f_m$ is non-degenerate.

In view of Remark 1.1, we see that, if u_n is an m-bubbling sequence of (1.3), then $\lambda_n \to 8m\pi$. We say that local uniqueness of m-bubbling sequences of (1.3) holds if the following is true:
for a fixed critical point $\mathbf{q} = (q_1, \cdots, q_m)$ of f_m, if there exists two m-bubbling sequences $u_n^{(1)}, u_n^{(2)}$ of (1.3) with the same λ_n and whose blow up set is $\{q_1, \cdots, q_m\}$, then, if λ_n is close enough to $8\pi m$, it holds $u_n^{(1)} \equiv u_n^{(2)}$.
Actually, Lin and Yan in [24] have initiated the study of the local uniqueness of m-bubbling sequences for the Chern-Simons-Higgs equation. Inspired by that approach, in [6, 7] the authors of this work proved local uniqueness and non-degeneracy of m-bubbling sequences of (1.3). More precisely, we have the following:

Theorem 1A ([6, 7])**.** *Let $\mathbf{q} = (q_1, \cdots, q_m)$ be a critical point of f_m such that $q_j \in \Omega \setminus \{p_1, \cdots, p_\ell\}$, $j = 1, \cdots, m$ and $det(D^2_\Omega f_m(\mathbf{q})) \neq 0$. Assume that either $\ell(\mathbf{q}) \neq 0$ or $D(\mathbf{q}) \neq 0$. Then the following facts hold true:*
(i) Local uniqueness with respect to λ_n: Let $u_n^{(1)}$ and $u_n^{(2)}$ be two sequence of solutions of (1.3) *with fixed λ_n. If λ_n is sufficiently close to $8m\pi$, then $u_n^{(1)} \equiv u_n^{(2)}$.*
(ii) Non-degeneracy with respect to λ_n: If λ_n is sufficiently close to $8m\pi$, then the linearized problem for (1.3),

$$\begin{cases} \Delta\phi_n + \lambda_n \frac{he^{u_n}}{\int_\Omega he^{u_n} dy} \left(\phi_n - \frac{\int_\Omega he^{u_n}\phi_n dy}{\int_\Omega he^{u_n} dy} \right) = 0 & \text{in} \quad \Omega, \\ \phi_n = 0 & \text{on} \quad \partial\Omega, \end{cases} \tag{1.10}$$

admits only the trivial solution $\phi_n \equiv 0$.

We note that for a fixed constant λ_n in Theorem 1A, the corresponding

$$(\varepsilon_n^{(i)})^2 = \frac{\lambda_n}{\int_\Omega h e^{u_n^{(i)}}}$$

might be different for $i = 1,2$. Interestingly enough, it turns out that if $\varepsilon_n^2 = \frac{\lambda_n}{\int_\Omega h e^{u_n}}$ is regarded as the main fixed parameter instead of λ_n, then the quantities $D(\mathbf{q})$ and $l(\mathbf{q})$ are no longer important as they happen to be if λ_n is held fixed. This difference is caused by the difference in the linearized operators. Indeed, for a fixed λ_n, the corresponding linearized problem is (1.10), but for a fixed ε_n, we have the following linearized problem:

$$\begin{cases} \Delta\phi_n + \varepsilon_n^2 h e^{w_n}\phi_n = 0 & \text{in} \quad \Omega, \\ \\ \phi_n = 0 & \text{on} \quad \partial\Omega. \end{cases} \tag{1.11}$$

The non-degeneracy of the latter problem with respect to $\varepsilon_n > 0$ is already well known [17, 18, 28]:

Theorem 1B ([17, 18, 28]). *Let w_n be an $m-$bubbling sequence of* (1.1) *whose blow up set is $\{q_1, \cdots, q_m\} \subset \Omega \setminus \{p_1, \cdots, p_\ell\}$, where $\mathbf{q} = (q_1, \cdots, q_m)$ is a critical point of f_m such that $det(D^2_\Omega f_m(\mathbf{q})) \neq 0$. If $\varepsilon_n > 0$ is sufficiently small, then the linearized problem* (1.11) *admits only the trivial solution $\phi_n \equiv 0$.*

On the other hand, the local uniqueness of m-bubbling sequences for (1.1) has remained a long standing open problem. Indeed, compared to the non-degeneracy argument, one has to face a truly new difficulty when comparing different bubbling sequences for (1.1), see the discussion later on. The aim of the present paper is to overcome this difficulty and solve the local uniqueness problem. More precisely, motivated by Theorem 1B and [24, 6], it is natural to ask whether or not the local uniqueness of m-bubbling sequences for (1.1) with respect to $\varepsilon_n > 0$ holds as well even if we do not assume that either $\ell(\mathbf{q}) \neq 0$ or $D(\mathbf{q}) \neq 0$. Indeed, it turns out that, whenever there exist two solutions $w^{(1)}_{\varepsilon_n}$ and $w^{(2)}_{\varepsilon_n}$ of (1.1) sharing the same value of $\varepsilon_n > 0$, and the same blow up set $\{q_1, \cdots, q_m\} \subset \Omega \setminus \{p_1, \cdots, p_\ell\}$, then we only need the non-degenerate assumption of the Hessian matrix $D^2_\Omega f_m$ at $\{q_1, \cdots, q_m\}$ to prove that $w^{(1)}_{\varepsilon_n} = w^{(2)}_{\varepsilon_n}$ for any ε_n small enough.

Theorem 1.1. *Let $w^{(i)}_{\varepsilon_n}$, $i = 1,2$ be two $m-$bubbling sequences of* (1.1), *with the same $\varepsilon_n > 0$ and the same blow up set $\{q_1, \cdots, q_m\} \cap \{p_1, \cdots, p_\ell\} = \emptyset$, where $\mathbf{q}$ is a critical point of f_m and assume that $D^2_\Omega f_m$ is non-degenerate. If $\varepsilon_n > 0$ is sufficiently small, then $w^{(1)}_{\varepsilon_n} = w^{(2)}_{\varepsilon_n}$.*

Compared to Theorem 1A, we note that in Theorem 1.1, for $\varepsilon_n > 0$ fixed, then the corresponding values of $\lambda_n^{(i)} = \varepsilon_n^2 \int_\Omega h e^{w_n^{(i)}} dx$ may be different for $i = 1,2$.

To prove Theorem 1.1 we will analyse the asymptotic behavior of the normalized difference,

$$\zeta_n = \frac{w^{(1)}_{\varepsilon_n} - w^{(2)}_{\varepsilon_n}}{\|w^{(1)}_{\varepsilon_n} - w^{(2)}_{\varepsilon_n}\|_{L^\infty(\Omega)}}.$$

By using the refined estimates in [13], we will see that near each blow up point q_j, and after a suitable scaling, then ξ_n converges to an entire solution of the linearized problem associated to the Liouville equation:

$$\Delta v + e^v = 0 \quad \text{in } \mathbb{R}^2, \quad \int_{\mathbb{R}^2} e^v < +\infty. \tag{1.12}$$

The solutions of (1.12) are classified and take the following form, see [12],

$$v(z) = v_{\mu,a}(z) = \log \frac{8e^\mu}{(1 + e^\mu |z + a|^2)^2}, \quad \mu \in \mathbb{R}, \quad a = (a_1, a_2) \in \mathbb{R}^2. \tag{1.13}$$

The linearized operator L relative to $v_{0,0}$ is determined by,

$$L\phi := \Delta\phi + \frac{8}{(1 + |z|^2)^2}\phi \quad \text{in} \quad \mathbb{R}^2. \tag{1.14}$$

It is well known that the kernel of L has real dimension 3 with eigenfunctions Y_0, Y_1, Y_2, where,

$$Y_0(z) = \frac{1 - |z|^2}{1 + |z|^2} = \frac{\partial v_{\mu,a}}{\partial \mu}\Big|_{(\mu,a)=(0,0)},$$
$$Y_1(z) = \frac{z_1}{1 + |z|^2} = -\frac{1}{4}\frac{\partial v_{\mu,a}}{\partial a_1}\Big|_{(\mu,a)=(0,0)},$$
$$Y_2(z) = \frac{z_2}{1 + |z|^2} = -\frac{1}{4}\frac{\partial v_{\mu,a}}{\partial a_2}\Big|_{(\mu,a)=(0,0)}.$$

After a suitable scaling, we shall see that the projections of ξ_n on Y_0, Y_1, Y_2 completely describe the behavior of ξ_n near the blow up point. In addition we can show that the projections of ξ_n on Y_0 associated to each blow up point coincides and that ξ_n will converge to the corresponding (uniquely defined) Fourier coefficient far away from blow up points.
By using a suitably defined Pohozaev identity and in view of the non-degeneracy of $D^2_\Omega f_m$, we will show that the projections on the translation kernels (Y_1, Y_2) associate to each blow up point is zero. On the other side, for the projection on Y_0, we notice that after normalization and away from the blow up points, the limit of the function ξ_n tends to some harmonic function in Ω with Dirichlet boundary condition. As a consequence, we will see that ξ_n converges to 0 outside the blow up area, which proves that the projection on Y_0 is 0. Therefore all those projections vanish which yield the desired result. We point out that this is different from the previous works [24] and [6, 7], where one is bound to use the assumptions about $l(\mathbf{q})$ and $D(\mathbf{q})$ to show that the projection along Y_0 vanishes.

This paper is organized as follows. In section 2, we review some known sharp estimates for blow up solutions of (1.3). In section 3 we prove Theorem 1.1 and leave some technical results in the Appendix.

2. Preliminaries

In this section we shall list some of the results which will be used below. By setting,

$$\lambda_n = \varepsilon_n^2 \int_\Omega h e^{w_n}. \tag{2.1}$$

we see that $u_n := w_n$ is an $m-$bubbling sequence of

$$\begin{cases} \Delta u_n + \lambda_n \frac{he^{u_n}}{\int_\Omega he^{u_n}} = 0 & \text{in} \quad \Omega, \\ \\ u_n = 0 & \text{on} \quad \partial\Omega, \end{cases} \tag{2.2}$$

which blows up at $q_j \notin \{p_1, \cdots, p_\ell\}$, $j = 1, \cdots, m$ and in particular $\lambda_n \to 8\pi m$, see Remark 1.1. Next let us state various well known and sharp estimates for these blow up solutions of (2.2). Let

$$\tilde{u}_n = u_n - \log\left(\int_\Omega he^{u_n}\right),$$

then it is easy to see that,

$$\Delta\tilde{u}_n + \lambda_n h(x)e^{\tilde{u}_n(x)} = 0 \quad \text{in} \quad \Omega, \quad \text{and} \quad \int_\Omega he^{\tilde{u}_n} = 1. \tag{2.3}$$

We denote by,

$$\mu_n = \max_\Omega \tilde{u}_n, \quad \mu_{n,j} = \max_{B_{r_0}(q_j)} \tilde{u}_n = \tilde{u}_n(x_{n,j}) \quad \text{for } j = 1, \cdots, m. \tag{2.4}$$

To give a good description of the bubbling solution around a blow up point, we define,

$$U_{n,j}(x) = \log \frac{e^{\mu_{n,j}}}{\left(1 + \frac{\lambda_n h(x_{n,j})}{8} e^{\mu_{n,j}} |x - x_{n,j,*}|^2\right)^2}, \quad x \in \mathbb{R}^2, \tag{2.5}$$

where the point $x_{n,j,*}$ is chosen to satisfy,

$$\nabla U_{n,j}(x_{n,j}) = \nabla \log h(x_{n,j}),$$

and it is not difficult to check that

$$|x_{n,j} - x_{n,j,*}| = O(e^{-\mu_{n,j}}). \tag{2.6}$$

In $B_{r_0}(x_{n,j})$, we use the following term $\eta_{n,j}$ to capture the difference between the genuine solution $\tilde{u}_n$ and the approximate bubble $U_{n,j}$:

$$\eta_{n,j}(x) = \tilde{u}_n - U_{n,j}(x) - \left(G_j^*(x) - G_j^*(x_{n,j})\right), \quad x \in B_{r_0}(x_{n,j}). \tag{2.7}$$

Far away from the blow up points, i.e., $x \in \overline{\Omega} \setminus \bigcup_{j=1}^m B_{r_0}(q_j)$, $\tilde{u}_n$ is well approximated by a sum of Green's function,

$$\phi_n(x) = \tilde{u}_n(x) - \sum_{j=1}^m \rho_{n,j} G(x, x_{n,j}) - \tilde{u}_{n,0}, \tag{2.8}$$

where here and in the rest of this article, we set

$$\tilde{u}_{n,0} = \tilde{u}_n|_{\partial\Omega}, \tag{2.9}$$

and the local mass associated to each blow up point q_j, $1 \le j \le m$ is defined by

$$\rho_{n,j} = \lambda_n \int_{B_{r_0}(q_j)} he^{\tilde{u}_n} \mathrm{d}y. \tag{2.10}$$

The following refined estimates derived by Chen and Lin [13] concerning $\eta_{n,j}$, $1 \le j \le n$ and ϕ_n, play a crucial role in our argument.

Theorem 2A ([13]). *Let u_n be an m-bubbling sequence of* (2.2) *which blows up at the points $q_j \notin \{p_1, \cdots, p_\ell\}$, $j = 1, \cdots, m$. Then we have:*

(a) $\eta_{n,j} = O(\mu_{n,j}^2 e^{-\mu_{n,j}})$ *on* $B_{r_0}(x_{n,j})$,

(b) $\phi_n = o(e^{-\frac{\mu_{n,j}}{2}})$ *in* $C^1(\overline{\Omega} \setminus \bigcup_{j=1}^m B_r(q_j))$.

The interaction between bubbles relative to different blow up points, the difference between each local mass and 8π and the difference between the parameter λ_n and $8\pi m$ play an essential role in the understanding of the blow up behavior. We present these estimates as follows:

Theorem 2B. *Suppose that the assumptions in Theorem 2A hold, then for any $j = 1, \cdots, m$ we have,*

(i) $e^{\mu_{n,j}} h^2(x_{n,j}) e^{G_j^*(x_{n,j})} = e^{\mu_{n,1}} h^2(x_{n,1}) e^{G_1^*(x_{n,1})}(1 + O(e^{-\frac{\mu_{n,1}}{2}}))$,

(ii) $\mu_{n,j} + \tilde{u}_{n,0} + 2\log\left(\frac{\lambda_n h(x_{n,j})}{8}\right) + G_j^*(x_{n,j}) = -\frac{2}{\lambda_n h(x_{n,j})}(\Delta \log h(x_{n,j}))(\mu_{n,j})^2 e^{-\mu_{n,j}}$
$+O(\mu_{n,j} e^{-\mu_{n,j}})$,

(iii) $\nabla\left(\log h(x) + G_j^*(x_{n,j})\right)\big|_{x=x_{n,j}} = O(\mu_{n,j} e^{-\mu_{n,j}})$, $\quad |x_{n,j} - q_j| = O(\mu_{n,j} e^{-\mu_{n,j}})$,

(iv) $|\mu_n - \mu_{n,j}| \le c$ *for* $j = 1, \cdots, m$, $|\tilde{u}_n(x) + \mu_n| \le c_{r_0}$ *for* $x \in \overline{\Omega} \setminus \bigcup_{j=1}^m B_{r_0}(q_j)$,

(v) $\rho_{n,j} - 8\pi = O(\mu_{n,j} e^{-\mu_{n,j}})$,

(vi) *for a fixed small constant* $r > 0$,

$$\begin{aligned}\lambda_n - 8\pi m =& \frac{2l(\mathbf{q})e^{-\mu_{n,1}}}{mh^2(q_1)e^{G_j^*(q_1)}}\left(\mu_{n,1} + \log(\lambda_n h^2(q_1)e^{G_1^*(q_1)}r^2) - 2\right)\\ &+ \frac{8e^{-\mu_{n,1}}}{\pi m h^2(q_1)e^{G_1^*(q_1)}}\left(D(\mathbf{q}) + O(r^\sigma)\right)\\ &+ O(\mu_{n,1}^2 e^{-\frac{3}{2}\mu_{n,1}}) + O(e^{-(1+\frac{\sigma}{2})\mu_{n,1}}),\end{aligned} \tag{2.11}$$

where $\sigma > 0$ is a positive number such that $\hat{h} \in C^{2,\sigma}(\overline{\Omega})$.

Remark 2.1. *The conclusions (i)-(v) and the estimate up to the order $\mu_n e^{-\mu_n}$ on $\lambda_n - 8\pi m$ have been found in* [13]*. More recently* [6] *the authors of this paper derived the higher order estimate in (vi). We remark that, although we will use (vi), we will not make use of the fact that either $l(\mathbf{q}) \neq 0$ or $D(\mathbf{q}) \neq 0$, see* (4.9) *below.*

3. Proof of Theorem 1.1.

To prove Theorem 1.1 we argue by contradiction and assume that (1.1) has two different solutions $w_n^{(1)}$ and $w_n^{(2)}$ with the same ε_n, which blows up at $q_j \notin \{p_1, \cdots, p_\ell\}$, $j = 1, \cdots, m$. We set

$$\lambda_n^{(i)} = \varepsilon_n^2 \int_\Omega h e^{w_n^{(i)}}, \quad i = 1, 2, \tag{3.1}$$

then $u_n^{(i)} := w_n^{(i)}$, $i = 1, 2$ are the $m-$bubbling sequence of

$$\begin{cases} \Delta u_n^{(i)} + \lambda_n^{(i)} \frac{he^{u_n^{(i)}}}{\int_\Omega he^{u_n^{(i)}}} = 0 & \text{in} \quad \Omega, \\ \\ u_n^{(i)} = 0 & \text{on} \quad \partial\Omega, \end{cases} \tag{3.2}$$

respectively, which blow up at $q_j \notin \{p_1, \cdots, p_\ell\}$, $j = 1, \cdots, m$. By Remark 1.1, we have $\lambda_n^{(i)} \to 8\pi m$. We shall use $x_{n,j}^{(i)}, \mu_n^{(i)}, \mu_{n,j}^{(i)}, \tilde{u}_n^{(i)}, U_{n,j}^{(i)}, \eta_{n,j}^{(i)}, x_{n,j,*}^{(i)}, \phi_n^{(i)}, \rho_{n,j}^{(i)}$ to denote $x_{n,j}, \mu_n, \mu_{n,j}, \tilde{u}_n, U_{n,j}, \eta_{n,j}, x_{n,j,*}, \phi_n, \rho_{n,j}$, as defined in section 2, corresponding to $u_n^{(i)}$, $i = 1, 2$ respectively. In order to show that $u_n^{(1)} = u_n^{(2)}$, it is enough to show that,

$$\tilde{u}_n^{(1)} + \log \lambda_n^{(1)} = \tilde{u}_n^{(2)} + \log \lambda_n^{(2)}. \tag{3.3}$$

Indeed, it is not difficult to see from (3.1) that,

$$\tilde{u}_n^{(i)} + \log \lambda_n^{(i)} = u_n^{(i)} + 2 \log \varepsilon, \quad i = 1, 2. \tag{3.4}$$

Therefore, whenever we prove (3.3), then $u_n^{(1)} = u_n^{(2)}$ immediately follows from (3.4). It is useful to set,

$$\hat{u}_n^{(i)} = \tilde{u}_n^{(i)} + \log \lambda_n^{(i)}, \quad \hat{\mu}_{n,j}^{(i)} = \mu_{n,j}^{(i)} + \log \lambda_n^{(i)}, \ j = 1, \cdots, m, \tag{3.5}$$

so that in particular we find,

$$\hat{u}_n^{(i)} = u_n^{(i)} + 2 \log \varepsilon, \quad i = 1, 2. \tag{3.6}$$

Next, let us study the term $\hat{u}_n^{(1)} - \hat{u}_n^{(2)}$. First of all we obtain an estimate about $\|\hat{u}_n^{(1)} - \hat{u}_n^{(2)}\|_{L^\infty(\Omega)}$.

Lemma 3.1. *(i)* $|\hat{\mu}_{n,j}^{(1)} - \hat{\mu}_{n,j}^{(2)}| = O\big(\sum_{i=1}^2 \hat{\mu}_{n,j}^{(i)} e^{-\hat{\mu}_{n,j}^{(i)}}\big)$ *for all* $1 \leq j \leq m$.

(ii) $\|\hat{u}_n^{(1)} - \hat{u}_n^{(2)}\|_{L^\infty(\Omega)} = O\big(\sum_{i=1}^2 \hat{\mu}_{n,1}^{(i)} e^{-\frac{\hat{\mu}_{n,1}^{(i)}}{2}}\big)$.

Proof. (i) In view of Theorem 2B-(ii) we have,

$$\begin{aligned} &\mu_{n,j}^{(i)} + \tilde{u}_{n,0}^{(i)} + 2 \log \frac{\lambda_n^{(i)} h(x_{n,j})}{8} + G_j^*(x_{n,j}) \\ &= -\frac{2}{\lambda_n^{(i)} h(x_{n,j}^{(i)})} (\Delta \log h(x_{n,j}^{(i)}))(\mu_{n,j}^{(i)})^2 e^{-\mu_{n,j}^{(i)}} + O(\mu_{n,j}^{(i)} e^{-\mu_{n,j}^{(i)}}), \quad i = 1, 2, \end{aligned} \tag{3.7}$$

which implies that,

$$\begin{aligned} &\hat{\mu}_{n,j}^{(i)} + \log \frac{\lambda_n^{(i)}}{\int_\Omega he^{u_n^{(i)}}} + 2 \log \frac{h(x_{n,j})}{8} + G_j^*(x_{n,j}) \\ &= -\frac{2}{\lambda_n^{(i)} h(x_{n,j}^{(i)})} (\Delta \log h(x_{n,j}^{(i)}))(\mu_{n,j}^{(i)})^2 e^{-\mu_{n,j}^{(i)}} + O(\mu_{n,j}^{(i)} e^{-\mu_{n,j}^{(i)}}), \quad i = 1, 2, \end{aligned} \tag{3.8}$$

where we used $\tilde{u}_{n,0}^{(i)} = -\log \int_\Omega he^{u_n^{(i)}}$, $i = 1, 2$. By (2.6) and Theorem 2B-(iii), we have

$$|x_{n,j}^{(1)} - x_{n,j}^{(2)}| = O(\sum_{i=1}^{2} \mu_{n,j}^{(i)} e^{-\mu_{n,j}^{(i)}}) \quad \text{and} \quad |x_{n,j,*}^{(1)} - x_{n,j,*}^{(2)}| = O(\sum_{i=1}^{2} \mu_{n,j}^{(i)} e^{-\mu_{n,j}^{(i)}}). \tag{3.9}$$

By (3.8)-(3.9), and the fact that $\varepsilon_n^2 = \frac{\lambda_n^{(i)}}{\int_\Omega he^{u_n^{(i)}}}$, $i = 1, 2$, we get

$$\hat{\mu}_{n,j}^{(1)} - \hat{\mu}_{n,j}^{(2)} = O(\sum_{i=1}^{2} \mu_{n,j}^{(i)} e^{-\mu_{n,j}^{(i)}}), \tag{3.10}$$

which implies the first claim. We remark that the assumptions about $l(\mathbf{q})$ and $D(\mathbf{q})$ are not needed here since $\varepsilon_n^2 = \frac{\lambda_n^{(i)}}{\int_\Omega he^{u_n^{(i)}}}$, $i = 1, 2$.

(ii) Let us fix a small constant $r_0 > 0$. In view of (2.7) and Theorem 2A-(a), we see that for $x \in B_{r_0}(q_j)$, it holds,

$$\begin{aligned} \hat{u}_n^{(1)} - \hat{u}_n^{(2)} &= U_{n,j}^{(1)} - U_{n,j}^{(2)} + G_j^*(x_{n,j}^{(2)}) - G_j^*(x_{n,j}^{(1)}) + \log \frac{\lambda_n^{(1)}}{\lambda_n^{(2)}} + \eta_{n,j}^{(1)} - \eta_{n,j}^{(2)} \\ &= U_{n,j}^{(1)} - U_{n,j}^{(2)} + G_j^*(x_{n,j}^{(2)}) - G_j^*(x_{n,j}^{(1)}) + \log \frac{\lambda_n^{(1)}}{\lambda_n^{(2)}} + O(\sum_{i=1}^{2} (\mu_{n,j}^{(i)})^2 e^{-\mu_{n,j}^{(i)}}). \end{aligned} \tag{3.11}$$

By the definition of $U_{n,j}^{(i)}$, we find that,

$$U_{n,j}^{(1)} - U_{n,j}^{(2)} + \log \frac{\lambda_n^{(1)}}{\lambda_n^{(2)}} = 2 \log \frac{(1 + \frac{h(x_{n,j}^{(2)})}{8} e^{\hat{\mu}_{n,j}^{(2)}} |x - x_{n,j,*}^{(2)}|^2)}{(1 + \frac{h(x_{n,j}^{(1)})}{8} e^{\hat{\mu}_{n,j}^{(1)}} |x - x_{n,j,*}^{(1)}|^2)} + \hat{\mu}_{n,j}^{(1)} - \hat{\mu}_{n,j}^{(2)}. \tag{3.12}$$

In view of (3.9) and (i) above, we also see that,

$$\begin{aligned} & h(x_{n,j}^{(2)}) e^{\hat{\mu}_{n,j}^{(2)}} |x - x_{n,j,*}^{(2)}|^2 - h(x_{n,j}^{(1)}) e^{\hat{\mu}_{n,j}^{(1)}} |x - x_{n,j,*}^{(1)}|^2 \\ &= O(e^{\hat{\mu}_{n,j}^{(1)}}) \left(|x - x_{n,j,*}^{(1)}| |x_{n,j,*}^{(1)} - x_{n,j,*}^{(2)}| + |x_{n,j,*}^{(1)} - x_{n,j,*}^{(2)}|^2 \right) \\ &\quad + O(e^{\hat{\mu}_{n,j}^{(1)}}) \left(|x - x_{n,j,*}^{(1)}|^2 (|\hat{\mu}_{n,j}^{(1)} - \hat{\mu}_{n,j}^{(2)}| + |x_{n,j}^{(1)} - x_{n,j}^{(2)}|) \right), \end{aligned} \tag{3.13}$$

which together with (3.10)-(3.12), implies that,

$$\|\hat{u}_n^{(1)} - \hat{u}_n^{(2)}\|_{L^\infty(B_{r_0}(q_j))} = O(\sum_{i=1}^{2} \hat{\mu}_{n,1}^{(i)} e^{-\frac{\hat{\mu}_{n,1}^{(i)}}{2}}). \tag{3.14}$$

Next, we estimate $\hat{u}_n^{(1)} - \hat{u}_n^{(2)}$ in $\Omega \setminus \bigcup_{j=1}^m B_{r_0}(q_j)$. By (2.8) and Theorem 2A-(a), we have,

$$\hat{u}_n^{(1)} - \hat{u}_n^{(2)} = \sum_{j=1}^{m} \left(\rho_{n,j}^{(1)} G(x, x_{n,j}^{(1)}) - \rho_{n,j}^{(2)} G(x, x_{n,j}^{(2)}) \right) + \phi_n^{(1)} - \phi_n^{(2)} = o(\sum_{i=1}^{2} e^{-\frac{\hat{\mu}_{n,j}^{(i)}}{2}}) \tag{3.15}$$

for $x \in \Omega \setminus \bigcup_{j=1}^{m} B_{r_0}(q_j)$. Clearly (3.14) and (3.15) prove (ii), and so the proof of Lemma 3.1 is completed. □

Let us define,

$$\xi_n(x) = \frac{\hat{u}_n^{(1)}(x) - \hat{u}_n^{(2)}(x)}{\|\hat{u}_n^{(1)} - \hat{u}_n^{(2)}\|_{L^\infty(\Omega)}}, \quad x \in \Omega, \tag{3.16}$$

$$g_n^*(x) = \frac{h(x)}{\|\hat{u}_n^{(1)} - \hat{u}_n^{(2)}\|_{L^\infty(\Omega)}} \left(e^{\hat{u}_n^{(1)}(x)} - e^{\hat{u}_n^{(2)}(x)} \right), \quad x \in \Omega, \tag{3.17}$$

and

$$c_n(x) = \frac{e^{\hat{u}_n^{(1)}} - e^{\hat{u}_n^{(2)}}}{\hat{u}_n^{(1)} - \hat{u}_n^{(2)}} = e^{\hat{u}_n^{(1)}} \left(1 + O(\|\hat{u}_n^{(1)} - \hat{u}_n^{(2)}\|_{L^\infty(\Omega)}) \right), \quad x \in \Omega. \tag{3.18}$$

It is not difficult to see that

$$\begin{cases} \Delta \xi_n + g_n^* = \Delta \xi_n + h c_n \xi_n = 0 & \text{in} \quad \Omega, \\ \\ \xi_n = 0 & \text{on} \quad \partial\Omega. \end{cases} \tag{3.19}$$

Once more, it is worth to point out that $\xi_n = 0$ on $\partial\Omega$ since $\varepsilon_n^2 = \frac{\lambda_n^{(i)}}{\int_\Omega h e^{u_n^{(i)}}}$, $i = 1, 2$.

To study the behavior of ξ_n in $B_\delta(x_{n,j}^{(1)})$, we set

$$\xi_{n,j}(z) = \xi_n(e^{-\frac{\hat{\mu}_{n,j}^{(1)} - \log 8}{2}} z + x_{n,j}^{(1)}), \quad |z| \leq \delta e^{\frac{\hat{\mu}_{n,j}^{(1)} - \log 8}{2}} \quad \text{for} \quad j = 1, \cdots, m. \tag{3.20}$$

Our first estimate is about the limit of $\xi_{n,j}$.

Lemma 3.2. *There exists $b_{j,0}, b_{j,1}$ and $b_{j,2}$ such that*

$$\xi_{n,j}(z) \to \sum_{i=0}^{3} b_{j,i} \psi_{j,0}(z) \quad \textit{in} \quad C^0_{\mathrm{loc}}(\mathbb{R}^2), \tag{3.21}$$

where

$$\psi_{j,0}(z) = \frac{1 - h(q_j)|z|^2}{1 + h(q_j)|z|^2}, \quad \psi_{j,1}(z) = \frac{\sqrt{h(q_j)} z_1}{1 + h(q_j)|z|^2}, \quad \psi_{j,2}(z) = \frac{\sqrt{h(q_j)} z_2}{1 + h(q_j)|z|^2}. \tag{3.22}$$

Proof. From Lemma 3.1, (2.6) and (a) in Theorem 2A, we have,

$$\Delta \xi_{n,j} + 8h(x_{n,j}^{(1)}) e^{-\hat{\mu}_{n,j}^{(1)}} e^{\hat{u}_n^{(1)}(e^{-\frac{\hat{\mu}_{n,j}^{(1)} - \ln 8}{2}} z + x_{n,j}^{(1)})} \xi_{n,j}(1 + o(1)) = 0,$$

and

$$8h(x_{n,j}^{(1)}) e^{-\hat{\mu}_{n,j}^{(1)}} e^{\hat{u}_n^{(1)}(e^{-\frac{\hat{\mu}_{n,j}^{(1)} - \ln 8}{2}} z + x_{n,j}^{(1)})} \to \frac{8h(q_j)}{(1 + h(q_j)|z|^2)^2} \quad \text{in} \quad C^0_{\mathrm{loc}}(\mathbb{R}^2). \tag{3.23}$$

Therefore, since $|\xi_{n,j}| \leq 1$ and because of the equation (3.19), then we conclude that $\xi_{n,j} \to \xi_j$ in $C^0_{\rm loc}(\mathbb{R}^2)$, where ξ_j satisfies,

$$\Delta \xi_j + \frac{8h(q_j)}{(1+h(q_j)|z|^2)^2}\xi_j = 0 \quad \text{in} \quad \mathbb{R}^2. \tag{3.24}$$

By [1, Proposition 1], there are constants $b_{j,i}$ such that

$$\xi_j = b_{j,0}\psi_{j,0} + b_{j,1}\psi_{j,1} + b_{j,2}\psi_{j,2},$$

which proves the lemma. □

As discussed in the introduction, we shall see in the next lemma that all the Fourier coefficients $b_{j,0}$ take the same value and that $\xi_n(x)$ will converge to such constant (denoted by b_0) far away from the blow up points. Furthermore we will prove that $b_0 = b_{j,1} = b_{j,2} = 0$ for $j = 1, \cdots, m$.

For any $r > 0$, we denote by

$$\Lambda^-_{n,j,r} = re^{-(\hat{\mu}^{(1)}_{n,j} - \log 8)/2}, \text{ and } \Lambda^+_{n,j,r} = re^{(\hat{\mu}^{(1)}_{n,j} - \log 8)/2}, \ j = 1, \cdots, m.$$

Lemma 3.3. *Let ξ_n be defined in (3.16), then we have,*

(*i*) $b_{j,0} = b_0 = 0$ *for* $j = 1, \cdots, m$ *and*

$$\xi_n(x) = o_R(1) + o_n(1) \quad in \quad C^0_{\rm loc}(\overline{\Omega} \setminus \bigcup_{j=1}^m B_{\Lambda^-_{n,j,R}}(x^{(1)}_{n,j}) \ as\ n \to +\infty,$$

where $o_R(1) \to 0$ *as* $R \to +\infty$ *and* $n \to 0$ *as* $n \to +\infty$.

(*ii*) $b_{j,1} = b_{j,2} = 0$ *for* $j = 1, \cdots, m$.

Taking Lemma 3.3 for granted, then we can conclude the proof of Theorem 1.1.

Proof of Theorem 1.1. Let x^*_n be a maximum point of ξ_n, then we have

$$|\xi_n(x^*_n)| = 1. \tag{3.25}$$

In view of Lemma 3.3, we find that $\lim\limits_{n\to+\infty} x^*_n = q_j$ for some j and

$$\lim_{n\to+\infty} e^{\frac{\hat{\mu}^{(1)}_{n,j} - \ln 8}{2}} s_n = +\infty, \quad \text{where } s_n = |x^*_n - x^{(1)}_{n,j}|. \tag{3.26}$$

On the other hand, let

$$\tilde{\xi}_n(x) = \xi_n(s_n x + x^{(1)}_{n,j}).$$

From Theorem 2A and equation (3.19), we can see that $\tilde{\xi}_n$ satisfies,

$$0 = \Delta\tilde{\xi}_n + s_n^2 h c_n \tilde{\xi}_n = \Delta\tilde{\xi}_n + \frac{s_n^2 h(x^{(1)}_{n,j}) e^{\hat{\mu}^{(1)}_{n,j}} \tilde{\xi}_n \left(1 + o(1) + O(s_n|x|)\right)}{(1 + \frac{h(x^{(1)}_{n,j})}{8} e^{\hat{\mu}^{(1)}_{n,j}} |s_n x + x^{(1)}_{n,j} - x^{(1)}_{n,j,*}|^2)^2}.$$

By using (3.25) we find that,

$$\left|\tilde{\xi}_n((x^*_n - x^{(1)}_{n,j})/s_n)\right| = |\xi_n(x^*_n)| = 1. \tag{3.27}$$

Next, in view of (3.26) and the fact that $|\tilde{\tilde{\xi}}_n| \le 1$, we see that $\tilde{\tilde{\xi}}_n \to \tilde{\tilde{\xi}}_0$ on any compact subset of $\mathbb{R}^2 \setminus \{0\}$, where $\tilde{\tilde{\xi}}_n$ satisfies

$$\Delta \tilde{\tilde{\xi}}_n = 0 \quad \text{in} \quad \mathbb{R}^2 \setminus \{0\}. \tag{3.28}$$

Since $|\tilde{\tilde{\xi}}_n| \le 1$, then $x = 0$ is a removable singularity. As a consequence, $\tilde{\tilde{\xi}}_0$ is a bounded harmonic function in $\mathbb{R}^2$ and then from (3.27) we conclude that either $\tilde{\tilde{\xi}}_0 = 1$ or $\tilde{\tilde{\xi}}_0 = -1$. By using $e^{-\frac{\hat{\mu}^{(1)}_{n,j}}{2}} \ll s_n$ and $\lim\limits_{n\to\infty} s_n = 0$, we also find that,

$$|\xi_n(x)| \ge \frac{1}{2} \quad \text{for} \quad s_n \le |x - x^{(1)}_{n,j}| \le 2s_n, \tag{3.29}$$

which is the desired contradiction to Lemma 3.3. This fact concludes the proof of Theorem 1.1. □

Next we prove Lemma 3.3 and divide the discussion into two parts:

(1) We first prove that $b_{j,0} = b_0 = 0$ for $j = 1, \cdots, m$ by Green's identity and O.D.E. type arguments.
(2) Then we use a suitably defined Pohozaev identity to show that $b_{j,1} = b_{j,2} = 0$ for $j = 1, \cdots, m$.

Proof of Lemma 3.3-(i). We recall that ξ_n satisfies,

$$\Delta \xi_n + h e^{\hat{u}^{(1)}_n} \xi_n (1 + o(1)) = 0 \quad \text{in} \quad \Omega. \tag{3.30}$$

It is easy to see that $e^{\hat{u}^{(1)}_n} \to 0$ in $C^0_{\rm loc}(\overline{\Omega} \setminus \{q_1, \cdots, q_m\})$ by (2.8) and Theorem 2B-*(ii)*. Since $\|\xi_n\|_{L^\infty(\Omega)} \le 1$, we see that $\xi_n \to \xi_0$ in $C^0_{\rm loc}(\overline{\Omega} \setminus \{q_1, \cdots, q_m\})$, where the points $\{q_1, \cdots, q_m\}$ are removable singular points of ξ_0 and

$$\Delta \xi_0 = 0 \quad \text{in} \quad \Omega. \tag{3.31}$$

Therefore, in view of the Dirichlet boundary conditions, we find,

$$\xi_0 = 0 \quad \text{in} \quad \Omega, \tag{3.32}$$

which implies that,

$$\xi_n \to b_0 = 0 \quad \text{in} \quad C^0_{\rm loc}(\overline{\Omega} \setminus \{q_1, \cdots, q_m\}). \tag{3.33}$$

Let us fix a small constant $\delta > 0$ and set, $\psi_{n,j}(x) = \frac{1-\frac{h(x_{n,j}^{(1)})}{8}|x-x_{n,j}^{(1)}|^2 e^{\hat{\mu}_{n,j}^{(1)}}}{1+\frac{h(x_{n,j}^{(1)})}{8}|x-x_{n,j}^{(1)}|^2 e^{\hat{\mu}_{n,j}^{(1)}}}$. For $d \in (0,\delta)$ and in view of (2.6), we find that,

$$\int_{\partial B_d(x_{n,j}^{(1)})} \left(\psi_{n,j}\frac{\partial \xi_n}{\partial \nu} - \xi_n \frac{\partial \psi_{n,j}}{\partial \nu}\right) \mathrm{d}\sigma = \int_{B_d(x_{n,j}^{(1)})} \left(\psi_{n,j}\Delta\xi_n - \xi_n \Delta\psi_{n,j}\right) \mathrm{d}x$$
$$= \int_{B_d(x_{n,j}^{(1)})} \left\{ - \xi_n \psi_{n,j} h \left(\frac{e^{\hat{u}_n^{(1)}} - e^{\hat{u}_n^{(2)}}}{\hat{u}_n^{(1)} - \hat{u}_n^{(2)}}\right) + \xi_n \psi_{n,j} h(x_{n,j}^{(1)}) e^{U_{n,j}^{(1)}} \Big(\frac{1+\frac{h(x_{n,j}^{(1)})}{8}|x-x_{n,j,*}^{(1)}|^2 e^{\hat{\mu}_{n,j}^{(1)}}}{1+\frac{h(x_{n,j}^{(1)})}{8}|x-x_{n,j}^{(1)}|^2 e^{\hat{\mu}_{n,j}^{(1)}}}\Big)^2 \right\} \mathrm{d}x$$
$$= \int_{B_d(x_{n,j}^{(1)})} \rho_n \xi_n \psi_{n,j} \left\{ - h e^{\tilde{u}_n^{(1)}} (1 + O(|\hat{u}_n^{(1)} - \hat{u}_n^{(2)}|)) + h(x_{n,j}^{(1)}) e^{U_{n,j}^{(1)}} (1 + O(e^{-\frac{\hat{\mu}_{n,j}^{(1)}}{2}})) \right\} \mathrm{d}x$$
$$= \int_{B_d(x_{n,j}^{(1)})} \rho_n \xi_n \psi_{n,j} \Big\{ - h e^{U_{n,j}^{(1)} + \eta_{n,j}^{(1)} + G_j^*(x) - G_j^*(x_{n,j}^{(1)})} (1 + O(|\hat{u}_n^{(1)} - \hat{u}_n^{(2)}|))$$
$$+ h(x_{n,j}^{(1)}) e^{U_{n,j}^{(1)}} (1 + O(e^{-\frac{\hat{\mu}_{n,j}^{(1)}}{2}})) \Big\} \mathrm{d}x.$$

Let $\overline{f}(z) = f(e^{-\frac{\mu_{n,j}^{(1)}}{2}} z + x_{n,j}^{(1)})$. By a suitable scaling and by using Theorem 2A, we see that,

$$\int_{\partial B_d(x_{n,j}^{(1)})} \left(\psi_{n,j}\frac{\partial \xi_n}{\partial \nu} - \xi_n \frac{\partial \psi_{n,j}}{\partial \nu}\right) \mathrm{d}\sigma$$
$$= \int_{B_{\Lambda_{n,j,d}^+}(0)} \rho_n \overline{\xi_n}(z)\, \overline{\psi_{n,j}}(z) \frac{O(1)(e^{-\frac{\hat{\mu}_{n,j}^{(1)}}{2}}|z| + |\overline{\hat{u}_n^{(1)}} - \overline{\hat{u}_n^{(2)}}| + e^{-\frac{\hat{\mu}_{n,j}^{(1)}}{2}})}{(1 + \frac{\rho_n h(x_{n,j}^{(1)})}{8}|z + e^{\frac{\mu_{n,j}^{(1)}}{2}}(x_{n,j}^{(1)} - x_{n,j,*}^{(1)})|^2)^2} \mathrm{d}z.$$

In view of Lemma 3.1-(ii), we obtain

$$\int_{\partial B_d(x_{n,j}^{(1)})} \left(\psi_{n,j}\frac{\partial \xi_n}{\partial \nu} - \xi_n \frac{\partial \psi_{n,j}}{\partial \nu}\right) \mathrm{d}\sigma = o\left(\frac{1}{\hat{\mu}_{n,j}^{(1)}}\right). \tag{3.34}$$

Let $\xi_{n,j}^*(r) = \int_0^{2\pi} \xi_n(r,\theta)\mathrm{d}\theta$, where $r = |x - x_{n,j}^{(1)}|$. Then (3.34) yields,

$$(\xi_{n,j}^*)'(r)\psi_{n,j}(r) - \xi_{n,j}^*(r)\psi_{n,j}'(r) = \frac{o\left(\frac{1}{\hat{\mu}_{n,j}^{(1)}}\right)}{r}, \ \forall r \in (\Lambda_{n,j,R}^-, \delta].$$

For any $R > 0$ large enough and for any $r \in (\Lambda_{n,j,R}^-, \delta)$, we also obtain that,

$$\psi_{n,j}(r) = -1 + O\left(\frac{e^{-\hat{\mu}_{n,j}^{(1)}}}{r^2}\right), \ \ \psi_{n,j}'(r) = O\left(\frac{e^{-\hat{\mu}_{n,j}^{(1)}}}{r^3}\right).$$

and so we conclude that,

$$(\xi^*_{n,j})'(r) = \frac{o\left(\frac{1}{\hat{\mu}^{(1)}_{n,j}}\right)}{r} + O\left(\frac{e^{-\hat{\mu}^{(1)}_{n,j}}}{r^3}\right) \quad \text{for all} \quad r \in (\Lambda^-_{n,j,R}, \delta). \tag{3.35}$$

Integrating (3.35), we obtain,

$$\xi^*_{n,j}(r) = \xi^*_{n,j}(\Lambda^-_{n,j,R}) + o(1) + o\Big(\frac{1}{\hat{\mu}^{(1)}_{n,j}}\Big)R + O(R^{-2}) \quad \text{for all} \quad r \in (\Lambda^-_{n,j,R}, \delta). \tag{3.36}$$

By using Lemma 3.2, we find,

$$\xi^*_{n,j}(\Lambda^-_{n,j,R}) = -2\pi b_{j,0} + o_R(1) + o_n(1),$$

where $\lim\limits_{R\to+\infty} o_R(1) = 0$ and $\lim\limits_{n\to+\infty} o_n(1) = 0$ and then (3.36) shows that,

$$\xi^*_{n,j}(r) = -2\pi b_{j,0} + o_R(1) + o_n(1)(1 + O(R)), \quad \text{for all} \quad r \in (\Lambda^-_{n,j,R}, \delta). \tag{3.37}$$

In view of (3.33), we see that,

$$\xi^*_{n,j} = o_n(1) \;\text{ in }\; C_{\rm loc}(M \setminus \{q_1 \cdots q_m\}),$$

which implies that $b_{j,0} = 0$ for $j = 1, \cdots, m$. This fact, together with (3.33), completes the proof of Lemma 3.3-(i). □

For $j = 1, \cdots, m$, let

$$\phi_{n,j}(y) = \frac{\lambda^{(1)}_n}{m}\left[(R(y, x^{(1)}_{n,j}) - R(x^{(1)}_{n,j}, x^{(1)}_{n,j})) + \sum_{l\neq j}(G(y, x^{(1)}_{n,l}) - G(x^{(1)}_{n,j}, x^{(1)}_{n,l}))\right], \tag{3.38}$$

$$v^{(i)}_{n,j}(y) = \hat{u}^{(i)}_n(y) - \phi_{n,j}(y), \quad i = 1, 2. \tag{3.39}$$

Recall the definition of ξ_n given before (3.17). Our aim is to show that all $b_{j,i} = 0$, see Lemma 3.2. This is done by exploiting the following Pohozaev identity to derive a subtle estimate for ξ_n.

Lemma 3.4 ([24, 6]). *We have for $i = 1, 2$ and fixed small $r \in (0, \delta)$, it holds,*

$$\begin{aligned}
&\int_{\partial B_r(x^{(1)}_{n,j})} \langle \nu, D\xi_n\rangle D_i v^{(1)}_{n,j} + \langle \nu, Dv^{(2)}_{n,j}\rangle D_i\xi_n {\rm d}\sigma \\
&- \frac{1}{2}\int_{\partial B_r(x^{(1)}_{n,j})} \langle D(v^{(1)}_{n,j} + v^{(2)}_{n,j}), D\xi_n\rangle \frac{(x - x^{(1)}_{n,j})_i}{|x - x^{(1)}_{n,j}|}{\rm d}\sigma \\
= &- \int_{\partial B_r(x^{(1)}_{n,j})} h(x)\frac{e^{\hat{u}^{(1)}_n} - e^{\hat{u}^{(2)}_n}}{\|\hat{u}^{(1)}_n - \hat{u}^{(2)}_n\|_{L^\infty(M)}}\frac{(x - x^{(1)}_{n,j})_i}{|x - x^{(1)}_{n,j}|}{\rm d}\sigma \\
&+ \int_{B_r(x^{(1)}_{n,j})} h(x)\frac{e^{\hat{u}^{(1)}_n} - e^{\hat{u}^{(2)}_n}}{\|\hat{u}^{(1)}_n - \hat{u}^{(2)}_n\|_{L^\infty(M)}} D_i(\phi_{n,j} + \log h){\rm d}x.
\end{aligned} \tag{3.40}$$

We will need an estimate about both sides of (3.40). This is the outcome of rather delicate computations whose proof is given in the appendix.

Lemma 3.5. *For $j = 1, \cdots, m$, we define*

$$R^*_{n,j}(x) = \sum_{h=1}^{2} \partial_{y_h} R(y,x)|_{y=x^{(1)}_{n,j}} b_{j,h} \tilde{B}_j, \quad G^*_{n,k}(x) = \sum_{h=1}^{2} \partial_{y_h} G(y,x)|_{y=x^{(1)}_{n,k}} b_{k,h} \tilde{B}_k, \tag{3.41}$$

where

$$\tilde{B}_j = 8\sqrt{\frac{2}{h(x^{(1)}_{n,j})}} \int_{\mathbb{R}^2} \frac{|z|^2}{(1+|z|^2)^3} \mathrm{d}z.$$

Then

$$R.H.S \text{ of } (3.40) = \tilde{B}_j \left(e^{-\frac{\hat{\mu}^{(1)}_{n,j}}{2}} \sum_{h=1}^{2} D^2_{hi}(\phi_{n,j} + \log h)(x^{(1)}_{n,j}) b_{j,h} \right) + o(e^{-\frac{\hat{\mu}^{(1)}_{n,j}}{2}}), \tag{3.42}$$

and

$$L.H.S \text{ of } (3.40) = -8\pi \left[e^{-\frac{\hat{\mu}^{(1)}_{n,j}}{2}} D_i R^*_{n,j}(x^{(1)}_{n,j}) + \sum_{k \neq j} e^{-\frac{\hat{\mu}^{(1)}_{n,k}}{2}} D_i G^*_{n,k}(x^{(1)}_{n,j}) \right] + o(e^{-\frac{\hat{\mu}^{(1)}_{n,j}}{2}}). \tag{3.43}$$

In view of these estimates we are now able to prove Lemma 3.3-(ii).

Proof of Lemma 3.3-(ii). From Lemma 3.4-3.5, we have for $i = 1, 2$

$$\begin{aligned} &\tilde{B}_j \sum_{h=1}^{2} \left(e^{-\frac{\hat{\mu}^{(1)}_{n,j}}{2}} D^2_{hi}(\phi_{n,j} + \log h)(x^{(1)}_{n,j}) b_{j,h} \right) \\ &= -8\pi \sum_{k \neq j} e^{-\frac{\hat{\mu}^{(1)}_{n,k}}{2}} \sum_{h=1}^{2} D_{x_i} \partial_{y_h} G(y,x)|_{(y,x)=(x^{(1)}_{n,j}, x^{(1)}_{n,k})} b_{kh} \tilde{B}_k \\ &\quad - 8\pi e^{-\frac{\hat{\mu}^{(1)}_{n,j}}{2}} \sum_{h=1}^{2} D_{x_i} \partial_{y_h} R(y,x)|_{(y,x)=(x^{(1)}_{n,j}, x^{(1)}_{n,j})} b_{jh} \tilde{B}_j + o(e^{-\frac{\hat{\mu}^{(1)}_{n,j}}{2}}). \end{aligned} \tag{3.44}$$

Set

$$\mathbf{b} = (\hat{b}_{1,1}\tilde{B}_1, \hat{b}_{1,2}\tilde{B}_1, \cdots, \hat{b}_{m,1}\tilde{B}_m, \hat{b}_{m,2}\tilde{B}_m),$$

where

$$\hat{b}_{kh} = \lim_{n \to +\infty} (e^{\frac{\hat{\mu}^{(1)}_{n,j} - \hat{\mu}^{(1)}_{n,k}}{2}}) b_{kh}.$$

Then by Theorem 2B-(iii) and sending n to $+\infty$, we obtain that

$$D^2 f_m(q_1, \cdots, q_m) \cdot \mathbf{b} = 0, \tag{3.45}$$

where f_m is defined in (1.9). By the non-degeneracy assumption $\det(D^2_\Omega f_m(\mathbf{q})) \neq 0$, then we can immediately conclude that $\mathbf{b} = 0$, i.e.,

$$b_{j,1} = b_{j,2} = 0, \quad \forall j = 1, \cdots, m.$$

This fact concludes the proof of Lemma 3.3. □

4. Appendix

This section is devoted to the proof of Lemma 3.5. First of all, we prove an estimate which will be used later on.

Lemma 4.1.

$$\xi_n(x)=\sum_{j=1}^{m}A_{n,j}^{(1)}G(x_{n,j}^{(1)},x)+\sum_{j=1}^{m}\sum_{h=1}^{2}b_{n,j,h}\partial_{y_h}G(y,x)\big|_{y=x_{n,j}^{(1)}}+o(e^{-\frac{\hat{\mu}_{n,1}^{(1)}}{2}}) \tag{4.1}$$

holds in $C^1(\overline{\Omega}\setminus\bigcup_{j=1}^m B_\theta(x_{n,j}^{(1)}))$ *with suitable small constant* θ*, where* $\partial_{y_h}G(y,x)=\frac{\partial G(y,x)}{\partial y_h}$, $y=(y_1,y_2)$,

$$A_{n,j}^{(1)}=\int_{\Omega_j}g_n^*(y)\mathrm{d}y,\quad\text{and}\quad b_{n,j,h}=e^{-\frac{1}{2}\hat{\mu}_{n,j}^{(1)}}\frac{b_{j,h}8\sqrt{2}}{\sqrt{h(q_j)}}\int_{\mathbb{R}^2}\frac{|z|^2}{(1+|z|^2)^3}\mathrm{d}z.$$

Remark 4.1. In fact, we can show that $A_{n,j}^{(1)}=o(e^{-\frac{\hat{\mu}_{n,j}^{(1)}}{2}})$, $j=1,\cdots,m$ by the same argument in [6, Lemma 4.4], [7, Lemma 5.4], [18, (4.5)]. As a consequence, (4.1) can be written as follows:

$$\xi_n(x)=\sum_{j=1}^{m}\sum_{h=1}^{2}b_{n,j,h}\partial_{y_h}G(y,x)\big|_{y=x_{n,j}^{(1)}}+o(e^{-\frac{\hat{\mu}_{n,1}^{(1)}}{2}})\quad\text{in}\quad C^1(\overline{\Omega}\setminus\bigcup_{j=1}^{m}B_\theta(x_{n,j}^{(1)})). \tag{4.2}$$

Proof. By the Green representation formula,

$$\begin{aligned}\xi_n(x)&=\int_\Omega G(y,x)g_n^*(y)\mathrm{d}y\\&=\sum_{j=1}^{m}A_{n,j}^{(1)}G(x_{n,j}^{(1)},x)+\sum_{j=1}^{m}\int_{\Omega_j}(G(y,x)-G(x_{n,j}^{(1)},x))g_n^*(y)\mathrm{d}y.\end{aligned} \tag{4.3}$$

For $x\in\overline{\Omega}\setminus\bigcup_{j=1}^m B_\theta(x_{n,j}^{(1)})$, we see from Theorem 2A-(a) and Theorem 2B-(iv) that,

$$\begin{aligned}&\int_{\Omega_j}(G(y,x)-G(x_{n,j}^{(1)},x))g_n^*(y)\mathrm{d}y\\&=\int_{B_r(x_{n,j}^{(1)})}\langle\partial_yG(y,x)\big|_{y=x_{n,j}^{(1)}},y-x_{n,j}^{(1)}\rangle g_n^*(y)\mathrm{d}y\\&\quad+O(1)\left(\int_{B_r(x_{n,j}^{(1)})}\frac{|y-x_{n,j}^{(1)}|^2e^{\hat{\mu}_{n,j}^{(1)}}}{(1+e^{\hat{\mu}_{n,j}^{(1)}}|y-x_{n,j,*}^{(1)}|^2)^2}\mathrm{d}y\right)+O(e^{-\hat{\mu}_{n,j}^{(1)}})\\&=\int_{B_r(x_{n,j}^{(1)})}\langle\partial_yG(y,x)\big|_{y=x_{n,j}^{(1)}},y-x_{n,j}^{(1)}\rangle g_n^*(y)\mathrm{d}y+O(e^{-\hat{\mu}_{n,j}^{(1)}})\end{aligned} \tag{4.4}$$

for suitable $r>0$. In addition, we have

$$\begin{aligned}&\int_{B_r(x_{n,j}^{(1)})}\langle\partial_y G(y,x)|_{y=x_{n,j}^{(1)}},y-x_{n,j}^{(1)}\rangle g_n^*(y)\mathrm{d}y\\&=16\sqrt{2}e^{-\frac{\hat{\mu}_{n,j}^{(1)}}{2}}\int_{B^+_{\Lambda_{n,j,r}}(0)}\frac{\langle\partial_y G(y,x)|_{y=x_{n,j}^{(1)}},z\rangle h(x_{n,j}^{(1)})\xi_{n,j}(z)}{\left(1+h(x_{n,j}^{(1)})|z+O(e^{-\frac{\hat{\mu}_{n,j}^{(1)}}{2}})|^2\right)^2}\mathrm{d}z+o(e^{-\frac{\hat{\mu}_{n,j}^{(1)}}{2}})\end{aligned}\tag{4.5}$$

By Lemma 3.1, we have for any $x\in\overline{\Omega}\setminus\bigcup_{j=1}^m B_\theta(x_{n,j}^{(1)})$ we get

$$\int_{B_r(x_{n,j}^{(1)})}\langle\partial_y G(y,x)|_{y=x_{n,j}^{(1)}},y-x_{n,j}^{(1)}\rangle g_n^*(y)\mathrm{d}y=\sum_{h=1}^2 b_{n,j,h}\partial_{y_h}G(y,x)|_{y=x_{n,j}^{(1)}}+o(e^{-\frac{\hat{\mu}_{n,j}^{(1)}}{2}}).\tag{4.6}$$

From (4.3) and (4.6), we conclude that (4.1) holds in $C^0(\overline{\Omega}\setminus\bigcup_{j=1}^m B_\theta(x_{n,j}^{(1)}))$. The proof of the fact that (4.1) holds in $C^1(\overline{\Omega}\setminus\bigcup_{j=1}^m B_\theta(x_{n,j}^{(1)}))$ is similar and we omit the proof. □

Now we will provide the proof of Lemma 3.5. We divide it into two parts.

Proof of Lemma 3.5-(3.42). By Theorem 2B-(iv), we have

$$\int_{\partial B_r(x_{n,j}^{(1)})}he^{\hat{u}_n}\xi_n(1+o(1))\frac{(x-x_{n,j}^{(1)})_i}{|x-x_{n,j}^{(1)}|}\mathrm{d}\sigma=O(e^{-\hat{\mu}_{n,j}^{(1)}}).\tag{4.7}$$

For the second term on the right hand side of (3.40), it is not difficult to see that,

$$\begin{aligned}D_i(\phi_{n,j}+\log h)&=D_i(\phi_{n,j}+\log h)(x_{n,j}^{(1)})+\sum_{h=1}^2 D^2_{ih}(\phi_{n,j}+\log h)(x_{n,j}^{(1)})(x-x_{n,j}^{(1)})_h\\&\quad+O(|x-x_{n,j}^{(1)}|^2).\end{aligned}\tag{4.8}$$

Since $\mathbf{q}$ is a critical point of f_m, by (2.6) and Theorem 2B-(iii)-(vi), we find that,

$$D_i(\phi_{n,j}+\log h)(x_{n,j}^{(1)})=D_i(G_j^*+\log h)(x_{n,j}^{(1)})+O(\hat{\mu}_{n,j}^{(1)}e^{-\hat{\mu}_{n,j}^{(1)}})=O(\hat{\mu}_{n,j}^{(1)}e^{-\hat{\mu}_{n,j}^{(1)}}).\tag{4.9}$$

From (4.8) and (4.9) we have,

$$\begin{aligned}&\int_{B_r(x_{n,j}^{(1)})}he^{\hat{u}_n}\xi_n(1+o(1))D_i(\phi_{n,j}+\log h)\mathrm{d}x\\&=\int_{B_{\Lambda^+_{n,j,r}}(0)}\frac{16\sqrt{2}h(x_{n,j}^{(1)})\xi_n(x_{n,j}^{(1)}+e^{-\frac{(\hat{\mu}_{n,j}^{(1)}-\log 8)}{2}}z)}{(1+h(x_{n,j}^{(1)})|z|^2)^2}\left(\sum_{h=1}^2 D^2_{hi}(\phi_{n,j}+\log h)(x_{n,j}^{(1)})e^{-\frac{\hat{\mu}_{n,j}^{(1)}}{2}}z_h\right)dz\\&\quad+o(e^{-\frac{\hat{\mu}_{n,j}^{(1)}}{2}}),\end{aligned}\tag{4.10}$$

and then combined with Lemma 3.1, we conclude that

$$\begin{aligned}
&\int_{B_r(x_{n,j}^{(1)})} he^{\hat{u}_n}\xi_n(1+o(1))D_i(\phi_{n,j}+\log h)\mathrm{d}x\\
&=\int_{B_{\Lambda^+_{n,j,r}}(0)} \frac{8\sqrt{2}h(x_{n,j}^{(1)})\sqrt{h(x_{n,j}^{(1)})}|z|^2}{(1+h(x_{n,j}^{(1)})|z|^2)^3}\mathrm{d}z\left(\sum_{h=1}^{2} b_{j,h}D^2_{hi}(\phi_{n,j}+\log h)(x_{n,j}^{(1)})e^{-\frac{\hat{\mu}_{n,j}^{(1)}}{2}}\right)\\
&\quad+o(e^{-\frac{\hat{\mu}_{n,j}^{(1)}}{2}})\\
&=\tilde{B}_j\left(\sum_{h=1}^{2} b_{j,h}D^2_{hi}(\phi_{n,j}+\log h)(x_{n,j}^{(1)})e^{-\frac{\hat{\mu}_{n,j}^{(1)}}{2}}\right)+o(e^{-\frac{\hat{\mu}_{n,j}^{(1)}}{2}}).
\end{aligned} \tag{4.11}$$

Clearly (3.42) follows from (4.7) and (4.11). □

Proof of Lemma 3.5-(3.43). By the definition of $G^*_{n,k}(x)$, we have

$$\Delta G^*_{n,k}(x)=0 \quad \text{in} \quad B_r(x_{n,j}^{(1)})\setminus B_\theta(x_{n,j}^{(1)}), \quad \forall \theta\in(0,r).$$

Then for any $x\in B_r(x_{n,j}^{(1)})\setminus B_\theta(x_{n,j}^{(1)})$, we have

$$\begin{aligned}
0&=\Delta G^*_{n,k}D_i\log\frac{1}{|x-x_{n,j}^{(1)}|}+\Delta\log\frac{1}{|x-x_{n,j}^{(1)}|}D_iG^*_{n,k}\\
&=\mathrm{div}\Bigg(\nabla G^*_{n,k}D_i\log\frac{1}{|x-x_{n,j}^{(1)}|}+\nabla\log\frac{1}{|x-x_{n,j}^{(1)}|}D_iG^*_{n,k}\\
&\quad-\nabla G^*_{n,k}\cdot\nabla\log\frac{1}{|x-x_{n,j}^{(1)}|}e_i\Bigg),
\end{aligned} \tag{4.12}$$

where $e_i=\frac{x_i}{|x|}$, $i=1,2$. It implies that

$$\int_{\partial B_r(x_{n,j}^{(1)})}\frac{\nabla_i G^*_{n,k}}{|x-x_{n,j}^{(1)}|}\mathrm{d}\sigma=\int_{\partial B_\theta(x_{n,j}^{(1)})}\frac{\nabla_i G^*_{n,k}}{|x-x_{n,j}^{(1)}|}\mathrm{d}\sigma. \tag{4.13}$$

In view of (4.2), we have

$$\xi_n(x)=\sum_{k=1}^{m}e^{-\frac{\hat{\mu}_{n,k}^{(1)}}{2}}G^*_{n,k}(x)+o(e^{-\frac{\hat{\mu}_{n,j}^{(1)}}{2}}) \quad \text{in} \quad \mathrm{C}^1(B_r(x_{n,j}^{(1)})\setminus B_\theta(x_{n,j}^{(1)})). \tag{4.14}$$

By using Theorem 2B, we find that,

$$\frac{\lambda_n^{(1)}}{m}=\rho_{n,j}^{(1)}+O(\hat{\mu}_{n,j}^{(1)}e^{-\hat{\mu}_{n,j}^{(1)}})=8\pi+O(\hat{\mu}_{n,j}^{(1)}e^{-\hat{\mu}_{n,j}^{(1)}}),$$

which together with Theorem 2A-(b), implies that for $i = 1, 2$,

$$\begin{aligned}
\nabla v_{n,j}^{(i)}(x) &= \nabla(\tilde{u}_n^{(i)} - \phi_{n,j}) = \nabla \left(\tilde{u}_n^{(i)} - \frac{\lambda_n^{(1)}}{m} R(x, x_{n,j}^{(1)}) - \frac{\lambda_n^{(1)}}{m} \sum_{k \neq j} G(x, x_{n,j}^{(1)}) \right) \\
&= \nabla \left(\tilde{u}_n^{(i)} - \frac{\lambda_n^{(1)}}{m} \sum_{k=1}^{m} G(x, x_{n,k}^{(1)}) - \frac{\lambda_n^{(1)}}{2\pi m} \log |x - x_{n,j}^{(1)}| \right) \\
&= \nabla \phi_n^{(i)} - 4 \frac{(x - x_{n,j}^{(1)})}{|x - x_{n,j}^{(1)}|^2} + o(e^{-\frac{\hat{\mu}_{n,j}^{(1)}}{2}}) \\
&= -4 \frac{(x - x_{n,j}^{(1)})}{|x - x_{n,j}^{(1)}|^2} + o(e^{-\frac{\hat{\mu}_{n,j}^{(1)}}{2}}) \quad \text{in} \quad C^1(B_r(x_{n,j}^{(1)}) \setminus B_\theta(x_{n,j}^{(1)})).
\end{aligned} \tag{4.15}$$

By using $D_i D_h(\log |z|) = \frac{\delta_{ih}|z|^2 - 2z_i z_h}{|z|^4}$, we see that,

$$\int_{\partial B_\theta(x_{n,j}^{(1)})} \frac{\nabla_i G_{n,j}^*}{|x - x_{n,j}^{(1)}|} \mathrm{d}\sigma = 2\pi D_i \sum_{h=1}^{2} \partial_{y_h} R(y, x)|_{x=y=x_{n,j}^{(1)}} b_{j,h} \tilde{B}_j + o_\theta(1). \tag{4.16}$$

From (4.13)-(4.16), we get for any $\theta \in (0, r)$,

$$\begin{aligned}
&\text{L.H.S. of (3.40)} \\
&= -8\pi \left[e^{-\frac{\hat{\mu}_{n,j}^{(1)}}{2}} D_i R_{n,j}^*(x_{n,j}^{(1)}) + \sum_{k \neq j} e^{-\frac{\hat{\mu}_{n,k}^{(1)}}{2}} D_i G_{n,k}^*(x_{n,j}^{(1)}) \right] + o_\theta(1) e^{-\frac{\hat{\mu}_{n,j}^{(1)}}{2}},
\end{aligned} \tag{4.17}$$

where $o_\theta(1) \to 0$ as $\theta \to 0$. This fact concludes the proof of Lemma 3.5. □

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Daniele Bartolucci, Department of Mathematics, University of Rome *"Tor Vergata"*, Via della ricerca scientifica n.1, 00133 Roma, Italy.
E-mail address: bartoluc@mat.uniroma2.it

Aleks Jevnikar, Department of Mathematics, University of Pisa, Largo Bruno Pontecorvo 5, 56127 Pisa, Italy.
E-mail address: aleks.jevnikar@dm.unipi.it

Youngae Lee, Department of Mathematics Education, Teachers College, Kyungpook National University, Daegu, South Korea
E-mail address: youngaelee@knu.ac.kr

Wen Yang, Wuhan Institute of Physics and Mathematics, Chinese Academy of Sciences, P.O. Box 71010, Wuhan 430071, P. R. China
E-mail address: wyang@wipm.ac.cn