

# ANISOTROPIC TUBULAR NEIGHBORHOODS IN EUCLIDEAN SPACES

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ABSTRACT. Let  $E \subset \mathbb{R}^N$  be a compact set and  $C \subset \mathbb{R}^N$  be a convex body with  $0 \in \text{int } C$ . We prove that the topological boundary of the anisotropic enlargement  $E + rC$  is contained in a finite union of Lipschitz surfaces and we investigate the regularity of the volume function  $V_E(r) := |E + rC|$  proving that up to a countable set  $V_E$  is of class  $C^1$ .

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## 1. INTRODUCTION

The study of the tubular neighborhood  $E_r := \{x \in \mathbb{R}^N : \text{dist}(x, E) \leq r\}$  of a convex set  $E$  in  $\mathbb{R}^N$  plays a crucial role in convex geometry. Of course, is not without interest to investigate the tubular neighborhood also for non convex sets, and it turns out that the boundary of  $E_r$  becomes more regular than the boundary of  $E$ , which could be very irregular: more precisely, in 1985 Fu [10] proves that  $\partial E_r$  is a Lipschitz manifold whenever  $E$  is compact in  $\mathbb{R}^N$  and  $r > r_0$  for some  $r_0 > 0$ . The approach of Fu is essentially based on the fact that the sublevels of regular values of a proper and semiconcave function are sets of positive reach: this argument can be applied since the distance function is semiconcave far from  $E$ . The semiconcavity of the distance is strongly related with the smoothness of the ball in  $\mathbb{R}^N$ : notice indeed that  $E_r$  can also be written as  $E_r = E + rB$ , where  $B$  is the unit closed ball centered in the origin. In this paper first of all we investigate the extension of such results to the anisotropic case, that is in the case  $E_r = E + rC$  where  $C$  is a prescribed convex body, i.e. a compact convex set in  $\mathbb{R}^N$  with  $0 \in \text{int } C$ . In this case the appropriate anisotropic distance to  $E$ , which we denote by  $d_E$ , could not be semiconcave outside  $E$ , since we are not assuming any kind of regularity of the boundary of  $C$ , unless locally Lipschitz coming from convexity: notice that we are really interested in enlarging  $E$  with a convex body  $C$ , since this case recovers also the crystalline anisotropy where  $C$  is convex but not necessarily strictly convex nor smooth. We will prove (see Thm. 3.1) that for any  $r > 0$  the boundary of  $E_r$  is contained in a finite union of Lipschitz surfaces when  $E$  is bounded and  $C$  is Lipschitz with  $0 \in \text{int } C$ . Of course since  $C$  is not sufficiently smooth we cannot use the Fu's approach, but the key idea of our proof is very easy: we first prove that enlarging  $C$  by a very small set, like  $\varepsilon K$  with  $\varepsilon > 0$  small and  $K \subset B$ , we still obtain a Lipschitz domain, and then we use the same idea of Rataj

and Winter [14] covering  $\partial E_r$  by a finite union of sets with small diameter. The rectifiability of  $\partial E_r$  is an independent interesting result, but actually we need to prove the regularity of  $\partial E_r$  in order to study the regularity of the volume function  $V_E(r) := |E_r|$  (see [18, 19] for the isotropic case). We prove (see Thm. 5.3) that the set of non-differentiability of  $V_E$  is at most countable, that  $V_E$  is of class  $C^1$  whenever it is differentiable and we find an explicit formula for  $V_E'$ . Our result is a generalization of the isotropic case [12] and our proof is partially based on the so called *anisotropic outer Minkowski content* (see [6] and [13] for details). We also need to base our argument on the existence of a so called *Cahn-Hoffmann vector field* for  $C$  with divergence measure bounded from above (see Prop. 4.1), and this is, in our opinion, an interesting result independent on the rest, since we are not assuming the strict convexity of  $C$ , so that such an existence result holds true also in the crystalline case.

## 2. NOTATION AND PRELIMINARIES

**2.1. Notation.** For any  $A$  subset of  $\mathbb{R}^N$  we will denote by  $|A|$  the Lebesgue measure of  $A$  while  $\mathcal{H}^k(A)$  stands for the  $k$ -dimensional Hausdorff measure of  $A$ , where  $k \in \{0, \dots, N\}$ ; of course  $\mathcal{H}^N$  is the Lebesgue measure. For any  $x \in \mathbb{R}^N$  the euclidean norm of  $x$  will be denoted by  $|x|$  while  $x \cdot y$  stands for the euclidean scalar product in  $\mathbb{R}^N$  between  $x$  and  $y$ . For any  $r > 0$  and  $x \in \mathbb{R}^N$  the ball centered in  $x$  with radius  $r$  will be denoted by  $B_r(x)$ ; we let  $B_r := B_r(0)$  and  $S^{N-1} := \partial B_1$ . We finally denote by  $\omega_k$  the volume of the  $k$ -dimensional unit ball in  $\mathbb{R}^k$ .

**2.2. Convex analysis.** Here we recall some basic notions of convex analysis; for all details we refer to [15]. In this paragraph  $C$  will be a *convex body in  $\mathbb{R}^N$* , that is a compact convex subset of  $\mathbb{R}^N$  with  $0 \in \text{int } C$ . We denote by  $h_C: \mathbb{R}^N \rightarrow \mathbb{R}$  the *support function of  $C$* , that is  $h_C(v) := \max_{x \in C} x \cdot v$ . We will use also the *polar of  $h_C$* , denoted by  $h_C^\circ$  and defined by  $h_C^\circ(v) := \max_{h_C(x) \leq 1} x \cdot v$  for each  $v \in \mathbb{R}^N$ ; it turns out that both  $h_C$  and  $h_C^\circ$  are convex and positively 1-homogeneous. We will need also to consider convex sets for which the support function and its polar are more regular. Let  $C$  be of class  $C^2$ . We say that  $C$  is *elliptic* if the curvature of  $\partial C$  is bounded from below by some positive constant. It turns out that if  $C$  is  $C^2$  and elliptic then both  $h_C$  and  $h_C^\circ$  are in  $C^2(\mathbb{R}^N \setminus \{0\})$ .

A very useful notion related with convexity is given by semiconcavity. Let  $A$  be a subset of  $\mathbb{R}^N$  and let  $f: A \rightarrow \mathbb{R}$ . We say that  $f$  is *concave* if the inequality

$$\lambda f(x) + (1 - \lambda)f(y) \leq f(\lambda x + (1 - \lambda)y)$$

holds true whenever  $x, y \in A$ ,  $\lambda \in [0, 1]$  and  $\{\lambda x + (1 - \lambda)y : \lambda \in [0, 1]\} \subset A$ . A function  $f \in C^0(A)$  is said to be *semiconcave* if there exists  $\alpha > 0$  such that for any  $x, y \in A$  and for any  $\lambda \in [0, 1]$  with  $\{\lambda x + (1 - \lambda)y : \lambda \in [0, 1]\} \subset A$  it holds

$$\lambda f(x) + (1 - \lambda)f(y) \leq f(\lambda x + (1 - \lambda)y) + \frac{\alpha}{2}\lambda(1 - \lambda)|x - y|^2. \quad (2.1)$$

Notice that if  $f$  is semiconcave and smooth enough, for instance of class  $C^2$ , then  $D^2f \leq \alpha I$ , where  $I$  is the identity matrix and the inequality holds in the sense of matrices. A useful class of semiconcave functions can be constructed; we have the following well known proposition, see for instance [8].

**Proposition 2.1.** *Let  $A \subset \mathbb{R}^N$  and let  $S \subset \mathbb{R}^M$  be compact. Let  $F \in C^0(S \times A)$ . Then the function  $f: A \rightarrow \mathbb{R}$  defined by  $f(x) := \inf_{s \in S} F(s, x)$  is semiconcave provided  $F(s, \cdot)$  satisfies (2.1) uniformly with respect to  $s$ .*

**2.3. Geometric Measure Theory.** In this paragraph we recall some notions of Geometric Measure Theory we will need; for all details we refer the reader to [2], [9] and [17]. Let  $N \geq 1$  be integer and let  $k \in \mathbb{N}$  with  $k \leq N$ . Let  $S \subset \mathbb{R}^N$ . We say that  $S$  is  $k$ -rectifiable if there exist a bounded set  $B \subset \mathbb{R}^k$  and a Lipschitz function  $f: B \rightarrow \mathbb{R}^N$  such that  $S = f(B)$ ; equivalently, by the Kirszbraun's extension Theorem, we can say that  $S$  is  $k$ -rectifiable if  $S$  is contained in a finite union of Lipschitz surfaces in  $\mathbb{R}^N$ . We say that  $S \subset \mathbb{R}^n$  is *countably  $\mathcal{H}^k$ -rectifiable* if there exist countably many Lipschitz functions  $f_h: \mathbb{R}^k \rightarrow \mathbb{R}^N$  such that

$$\mathcal{H}^k \left( S \setminus \bigcup_{h=0}^{+\infty} f_h(\mathbb{R}^k) \right) = 0.$$

A useful characterization of rectifiability is the Besicovitch-Marstrand-Mattila's Theorem (see, for instance, [2, Thm. 2.63]): a Borel set  $S \subset \mathbb{R}^N$  with  $\mathcal{H}^k(S) < +\infty$  is countably  $\mathcal{H}^k$ -rectifiable if and only if for  $\mathcal{H}^k$ -a.e.  $x \in S$  we have

$$\lim_{\rho \rightarrow 0^+} \frac{\mathcal{H}^k(S \cap B_\rho(x))}{\omega_k \rho^k} = 1. \quad (2.2)$$

It turns out that if  $S$  is countably  $\mathcal{H}^k$ -rectifiable then for  $\mathcal{H}^k$ -almost any point  $x_0 \in S$  it is well defined the *approximate tangent space*  $\text{Tan}^k(S, x_0)$ , that is

$$\lim_{\rho \rightarrow 0^+} \frac{1}{\rho^k} \int_S \phi \left( \frac{x - x_0}{\rho} \right) d\mathcal{H}^k(x) = \int_{\text{Tan}^k(S, x_0)} \phi(y) d\mathcal{H}^k(y), \quad \forall \phi \in C_c^\infty(\mathbb{R}^N).$$

In particular, if  $k = N - 1$  then  $\text{Tan}^{N-1}(S, x_0)^\perp$  is generated by some unit vector denoted by  $\nu_S$ .

Let now  $E \subset \mathbb{R}^N$  be a measurable set and  $\Omega \subset \mathbb{R}^N$  be an open domain; we denote by  $\chi_E$  the characteristic function of  $E$ . We say that  $E$  has *finite perimeter in  $\Omega$*  if  $\chi_E \in BV(\Omega)$ ; the perimeter of  $E$  in  $\Omega$  is defined by  $\mathcal{P}(E; \Omega) := |D\chi_E|(\Omega)$ , where  $|D\chi_E|$  denotes the total variation of  $D\chi_E$ ; we also let  $\mathcal{P}(E) := \mathcal{P}(E; \mathbb{R}^N)$ . For sufficiently smooth boundaries the perimeter coincides with the  $(N - 1)$ -dimensional Hausdorff measure of the topological boundary. The upper and lower  $N$ -dimensional densities of  $E$  at  $x$  are respectively defined by

$$\Theta_N^*(E, x) := \limsup_{\rho \rightarrow 0} \frac{|E \cap B_\rho(x)|}{\omega_N \rho^N}, \quad \Theta_{*N}(E, x) := \liminf_{\rho \rightarrow 0} \frac{|E \cap B_\rho(x)|}{\omega_N \rho^N}.$$

If  $\Theta_N^*(E, x) = \Theta_{*N}(E, x)$  their common value is denoted by  $\Theta_N(E, x)$ . For every  $t \in [0, 1]$  we define  $E^t := \{x \in \mathbb{R}^N : \Theta_N(E, x) = t\}$ . The *essential boundary* of  $E$  is defined as  $\partial^*E := \mathbb{R}^N \setminus (E^0 \cup E^1)$ . It turns out that if  $E$  has finite perimeter in  $\Omega$ , then  $\mathcal{H}^{N-1}(\partial^*E \setminus E^{1/2}) = 0$ , and  $\mathcal{P}(E; \Omega) = \mathcal{H}^{N-1}(\partial^*E \cap \Omega)$ . Moreover, one can define a subset of  $E^{1/2}$  as the set of points  $x$  where there exists a unit vector  $\nu_E(x)$  such that

$$\frac{E - x}{\rho} \rightarrow \{y \in \mathbb{R}^N : y \cdot \nu_E(x) \leq 0\}, \quad \text{in } L_{\text{loc}}^1(\mathbb{R}^N) \text{ as } \rho \rightarrow 0^+,$$

and which is referred to as the outer normal to  $E$  at  $x$ . The set where  $\nu_E(x)$  exists is called the *reduced boundary* and is denoted by  $\mathcal{F}E$ . One can show that  $\mathcal{H}^{N-1}(\partial^*E \setminus \mathcal{F}E) = 0$ , moreover, one has the decomposition  $D\chi_E = (-\nu_E)\mathcal{H}^{N-1} \llcorner \mathcal{F}E$ . Let us collect some elementary properties of sets with countably  $\mathcal{H}^{N-1}$ -rectifiable boundary and with finite perimeter in  $\Omega$ ; for any  $E \subseteq \mathbb{R}^N$  we let  $E^c := \mathbb{R}^N \setminus E$ . Assume that  $E$  has finite perimeter in  $\Omega$  and  $\partial E$  is countably  $\mathcal{H}^{N-1}$ -rectifiable. Then the following relations hold true:  $\mathcal{H}^{N-1}(\mathcal{F}E) = \mathcal{H}^{N-1}(\mathcal{F}E^c)$  and  $\nu_{E^c}(x) = -\nu_E(x)$  for any  $x \in \mathcal{F}E$ .

We finally recall an anisotropic version of the coarea formula which we will need; for details see [11, Thm. 3]. Let  $u \in BV(\Omega)$  and let  $\alpha: \mathbb{R}^N \rightarrow (0, +\infty)$  be a convex and positively one-homogeneous function with  $c^{-1}|v| \leq \alpha(v) \leq c|v|$  for any  $v \in \mathbb{R}^N$  and for some constant  $c > 0$ . Then the following formula holds true:

$$\int \alpha(Du) = \int_{-\infty}^{+\infty} \int_{\Omega \cap \mathcal{F}\{u \leq t\}} \alpha(\nu_{\{u \leq t\}}) d\mathcal{H}^{N-1} dt. \quad (2.3)$$

**2.4. Anisotropic Minkowski content.** We now briefly recall the theory of Minkowski content; for details see [2] (isotropic case), [1] and [19] (outer Minkowski content), [6] and [13] (anisotropic outer Minkowski content). Let  $C \subset \mathbb{R}^N$  be a convex body. Define, for any  $S \subset \mathbb{R}^N$  closed, the *anisotropic Minkowski content* of  $S$  as

$$\mathcal{M}_C(S) := \lim_{\varepsilon \rightarrow 0^+} \frac{|S + \varepsilon C|}{2\varepsilon}$$

whenever such a limit exists. For each closed set  $E \subset \mathbb{R}^N$  we also define the *anisotropic outer Minkowski content* of  $S$  as

$$S\mathcal{M}_C(E) := \lim_{\varepsilon \rightarrow 0^+} \frac{|(E + \varepsilon C) \setminus E|}{\varepsilon}$$

whenever such a limit exists. To be more precise, let us recall what has been proved in [13] (based also on [1] and [6]). First, we have the following existence and characterization result for  $\mathcal{M}_C$ ; let

$$\phi_C(v) := \frac{h_C(v) + h_C(-v)}{2}.$$

**Theorem 2.2.** [13, Thm. 3.7] *Let  $S \subset \mathbb{R}^N$  be compact and  $(N-1)$ -rectifiable. Then  $\mathcal{M}_C(S)$  exists and*

$$\mathcal{M}_C(S) = \int_S \phi_C(\nu_S) d\mathcal{H}^{N-1}.$$

Let now  $\mathcal{O}$  be the class of Borel sets  $E$  of  $\mathbb{R}^N$  such that:

- (a)  $\partial E$  is a countably  $\mathcal{H}^{N-1}$ -rectifiable bounded set;
- (b) there exist  $\gamma > 0$  and a probability measure  $\eta$  in  $\mathbb{R}^N$  absolutely continuous with respect to  $\mathcal{H}^{N-1}$  such that  $\eta(B_r(x)) \geq \gamma r^{N-1}$  for all  $x \in \partial E$  and for all  $r \in (0, 1)$

and let  $\mathcal{O}'_C$  be the class of Borel sets  $E$  of  $\mathbb{R}^N$  such that:

- (a')  $\partial E$  is a countably  $\mathcal{H}^{N-1}$ -rectifiable bounded set and

$$\mathcal{M}_C(\partial E) = \int_{\partial E} \phi_C(\nu_E) d\mathcal{H}^{N-1};$$

- (b') there exist  $\gamma > 0$  and a probability measure  $\eta$  in  $\mathbb{R}^N$  such that  $\eta(B_r(x)) \geq \gamma r^{N-1}$ , for all  $x \in \partial E$  and for all  $r \in (0, 1)$ .

Notice that any  $k$ -rectifiable compact set in  $\mathbb{R}^N$  satisfies properties (b) and (b'): see for instance [1, Rem. 1]. The following existence and characterization result for  $\mathcal{SM}_C$  holds true:

**Theorem 2.3.** [13, Thm. 4.4] *The class  $\mathcal{O}$  and the class  $\mathcal{O}'_C$  are stable under finite unions, and for any  $E \in \mathcal{O}$  (or  $\mathcal{O}'_C$ ) it holds*

$$\mathcal{SM}_C(E) = \int_{\mathcal{F}E} h_C(\nu_E) d\mathcal{H}^{N-1} + 2 \int_{\partial E \cap E^0} \phi_C(\nu_E) d\mathcal{H}^{N-1}. \quad (2.4)$$

**2.5. Anisotropic tubular neighborhoods.** In this paragraph we will introduce all the objects we want to investigate. Let  $N \geq 1$  be integer. Let  $E \subset \mathbb{R}^N$  be compact and  $C \subset \mathbb{R}^N$  be a compact Lipschitz set with  $0 \in \text{int } C$ . For any  $r > 0$  denote  $E_r := E + rC$ . Moreover, let

$$E'_r := \bigcup_{s < r} E_s.$$

It is convenient to introduce the anisotropic distance from  $E$ , that is

$$d_E(x) := \text{ess inf}_{y \in E} h_C^\circ(x - y).$$

Notice that  $E_r = \{d_E \leq r\}$  and  $E'_r = \{d_E < r\}$ . It turns out (for details see [6]) that  $d_E$  is Lipschitz continuous and, if  $C$  is a convex body,

$$h_C(\nabla d_E) = 1 \quad \text{a.e. on } \{d_E > 0\}. \quad (2.5)$$

Finally, let  $V_E: [0, +\infty) \rightarrow \mathbb{R}$  be given by  $V_E(r) := |E_r|$ . Note that for  $C = B_1$ , it is also named *volume function of  $E$*  (see also [18, 19]). It is easy to see that  $V_E$  is continuous.

### 3. REGULARITY OF THE BOUNDARIES

In this section we prove that  $\partial E_r$  and  $\partial E'_r$  are sufficiently smooth, in the sense of geometric measure theory.

**Theorem 3.1.** *For any  $r > 0$  the sets  $\partial E_r, \partial E'_r$  are finite union of Lipschitz surfaces.*

*Proof.* We divide the proof in two steps.

*Step 1:* Let  $K \subset \mathbb{R}^N$  be a bounded set. We claim that for  $\varepsilon$  positive and sufficiently small the set  $C + \varepsilon K$  is a Lipschitz set.

Without loss of generality we can assume  $K \subset B_1$ . For any  $\xi \in \mathbb{R}^N$ ,  $\xi \neq 0$ , we let

$$\xi^\perp := \{x \in \mathbb{R}^N : x \cdot \xi = 0\} \quad \text{and} \quad S_\eta^\xi := \{x \in \mathbb{R}^N : |\pi_\xi(x)| < \eta \text{ and } x \cdot \xi > 0\}$$

where  $\pi_\xi$  denotes the orthogonal projection on  $\xi^\perp$ . Since  $C$  is Lipschitz and compact we can write its boundary locally as a graph of a Lipschitz function in a uniform way: precisely, we can find  $r > 0$  such that  $B_r \subset C$  and such that for any  $z \in \partial C$  there exists a Lipschitz function  $f_z : B_r \cap z^\perp \rightarrow \mathbb{R}$  with

$$\{x + f_z(x)\hat{z} : x \in B_r \cap z^\perp\} = \partial C \cap S_r^z, \quad \hat{z} := z/|z|.$$

Let  $\varepsilon < r/2$  and fix  $x_0 \in \partial(C + \varepsilon K)$ . There exists  $k_0 \in K$  such that  $x_0 \in \partial C + \varepsilon k_0$ , thus  $x_0 = z_0 + \varepsilon k_0$  for some  $z_0 \in \partial C$ . For any  $x \in B_{r/2} \cap z_0^\perp$  and any  $k \in K$  let

$$g(x) := \sup\{f_{z_0}(\pi_{z_0}(x - \varepsilon k)) + \varepsilon k \cdot \hat{z}_0 : k \in K\}.$$

Therefore

$$\{x + g(x)\hat{z}_0 : x \in B_{r/2} \cap z_0^\perp\} = \partial(C + \varepsilon K) \cap S_{r/2}^{z_0}.$$

It is sufficient now to notice that  $g$  is Lipschitz continuous with the same Lipschitz constant  $L$  of  $f_{z_0}$ : indeed for any  $x, y \in B_{r/2} \cap z_0^\perp$  it holds

$$g(x) - g(y) \leq \sup_{k \in K} \{f_{z_0}(\pi_{z_0}(x - \varepsilon k)) - f_{z_0}(\pi_{z_0}(y - \varepsilon k))\} \leq L|x - y|.$$

*Step 2:* Now it is relatively easy to conclude the proof for  $\partial E'_r$ ; the rectifiability of  $\partial E_r$  follows since  $\partial E_r \subseteq \partial E'_r$ . The idea is to use the same argument as in the proof of [14, Prop. 2.3]. If  $r > 0$  by step 1 we can say that for any  $x \in \mathbb{R}^N$  the set  $rC + (B_{r'}(x) \cap E)$  has Lipschitz boundary for  $r' < r$  sufficiently small (apply step 1 to  $rC$  instead of  $C$ ). We cover now  $E$ , which has compact closure, with balls  $B_{r'}(x_1), \dots, B_{r'}(x_d)$  and we let  $E_i := E \cap B_{r'}(x_i)$ . Then

$$\partial E'_r \subseteq \bigcup_{i=1}^d \partial(E_i)_r$$

that is  $\partial E'_r$  is contained in a finite union of Lipschitz surfaces, and this yields the conclusion.  $\square$

#### 4. CONSTRUCTION OF A CAHN-HOFFMANN VECTOR FIELD FOR $C$

First of all we recall that  $q \in L^\infty(\Omega; \mathbb{R}^N)$  is said to be a *divergence measure vector field* if the distribution  $\operatorname{div} n$  is a Radon measure on  $\Omega$ . It turns out that  $\operatorname{div} n$  is absolutely continuous with respect to  $\mathcal{H}^{N-1}$  (see, for instance, [16, Thm. 3.2-b]). Moreover, we recall some basic results of the theory of viscosity solutions; for details we refer to [7]. Let  $\operatorname{Sym}_N(\mathbb{R})$

be the set of all symmetric  $N \times N$  matrices with real entries, let  $\Omega$  be a subset of  $\mathbb{R}^N$  and let  $F: \Omega \times \mathbb{R} \times \mathbb{R}^n \times \text{Sym}_N(\mathbb{R}) \rightarrow \mathbb{R}$  be a continuous function such that the following monotonicity condition holds:

$$F(x, r, p, X) \leq F(x, s, p, Y)$$

whenever  $r \leq s$  and  $Y \leq X$  in the sense of matrices. Let  $u: \Omega \rightarrow \mathbb{R}$  be upper semicontinuous. We say that  $u$  is a *viscosity subsolution* of the equation  $F(x, u, Du, D^2u) = 0$  on  $\Omega$  if for any  $\phi \in C^2(\Omega)$  and for any  $\bar{x} \in \Omega$  local maximum point of  $u - \phi$  it holds

$$F(\bar{x}, u(\bar{x}), D\phi(\bar{x}), D^2\phi(\bar{x})) \leq 0.$$

Let now  $u: \Omega \rightarrow \mathbb{R}$  be lower semicontinuous. We say that  $u$  is a *viscosity supersolution* of the equation  $F(x, u, Du, D^2u) = 0$  on  $\Omega$  if for any  $\phi \in C^2(\Omega)$  and for any  $\bar{x} \in \Omega$  local minimum point of  $u - \phi$  it holds

$$F(\bar{x}, u(\bar{x}), D\phi(\bar{x}), D^2\phi(\bar{x})) \geq 0.$$

If  $u$  is both a viscosity subsolution and supersolution then  $u$  is called *viscosity solution* of  $F(x, u, Du, D^2u) = 0$  on  $\Omega$ .

We are ready to start the construction of a Cahn-Hoffmann vector field for  $C$  in the smooth case.

**Proposition 4.1.** *Assume that  $C$  is a convex body of class  $C^2$  and elliptic. Let  $n := \nabla h_C(\nabla d_E)$ . Then  $n \in L^\infty(\mathbb{R}^N; \mathbb{R}^N)$  is a divergence measure vector field,*

$$\|n\|_\infty \leq \max_{z \in C} |z| \quad (4.1)$$

and

$$\text{div } n \leq \frac{N-1}{r} \quad (4.2)$$

in the distributional sense out of  $E_r$ .

*Proof.* First of all we point out that the assumptions on  $C$  guarantee that both  $h_C$  and  $h_C^\circ$  are in  $C^2(\mathbb{R}^N \setminus \{0\})$ . From the standard fact that  $h_C^2/2$  and  $(h_C^\circ)^2/2$  are Legendre-Fenchel convex conjugates, so that their gradients  $h_C \nabla h_C$  and  $h_C^\circ \nabla h_C^\circ$  are inverse mappings, we deduce that for any  $z \in \mathbb{R}^N \setminus \{0\}$

$$\nabla h_C(\nabla h_C^\circ(z)) = \frac{z}{h_C^\circ(z)}. \quad (4.3)$$

For the sake of simplicity we will denote  $d := d_E$ .

*Step 1.* The proof of (4.1) is easy: indeed, if we fix  $x \in \mathbb{R}^N$  with  $d(x) > 0$  and  $y \in \overline{E^1}$  is such that  $d(x) = h_C^\circ(x - y)$  then formula (4.3) reads as

$$n(x) = \frac{x - y}{h_C^\circ(x - y)}$$

from which we immediately get (4.1) since  $C = \{x \in \mathbb{R}^N : h_C^\circ(x) \leq 1\}$ .

*Step 2.* We prove (4.2). First of all, it turns out that  $d$  is a viscosity supersolution of

$$-\operatorname{div} \nabla h_C(\nabla d) = -\frac{N-1}{r} \quad (4.4)$$

in  $\mathbb{R}^N \setminus E_r$ . This is a variant of a classical result, see [3]. The proof is quite straightforward. Indeed, if  $\phi$  is a smooth function which touches the graph of  $d$  from below at a point  $\bar{x} \notin E_r$  (that is,  $\phi \leq d$ ,  $\phi(\bar{x}) = d(\bar{x})$ ) then by definition of  $d$ ,  $\phi$  also touches the graph of  $x \mapsto h_C^\circ(x - \bar{y})$  from below at  $\bar{x}$ , where  $\bar{y} \in E$  is a point of minimal distance to  $\bar{x}$ . Being both functions smooth at  $\bar{x}$ , it follows that  $\nabla \phi(\bar{x}) = \nabla h_C^\circ(\bar{x} - \bar{y})$  and  $D^2 \phi(\bar{x}) \leq D^2 h_C^\circ(\bar{x} - \bar{y})$ . In particular,

$$\begin{aligned} -\operatorname{div} \nabla h_C(\nabla \phi)(\bar{x}) &= -D^2 h_C(\nabla \phi(\bar{x})) : D^2 \phi(\bar{x}) \\ &\geq -D^2 h_C(\nabla h_C^\circ(\bar{x} - \bar{y})) : D^2 h_C^\circ(\bar{x} - \bar{y}) \\ &= -\operatorname{div} \nabla h_C(\nabla h_C^\circ)(\bar{x} - \bar{y}). \end{aligned}$$

Combining (4.3) with the Euler's identity, for any  $z \in \mathbb{R}^N \setminus \{0\}$  we obtain, also by direct computation,

$$\operatorname{div} \nabla h_C(\nabla h_C^\circ(z)) = \operatorname{div} \frac{z}{h_C^\circ(z)} = \frac{N h_C^\circ(z) - z \cdot \nabla h_C^\circ(z)}{|h_C^\circ(z)|^2} = \frac{N-1}{h_C^\circ(z)}$$

and therefore finally

$$-\operatorname{div} \nabla h_C(\nabla \phi)(\bar{x}) \geq -\frac{N-1}{h_C^\circ(\bar{x} - \bar{y})}.$$

We find that not only  $d$  is a viscosity supersolution of (4.4) out of  $E_r$ , but the more precise inequality

$$-\operatorname{div} \nabla h_C(\nabla \phi)(\bar{x}) \geq -\frac{N-1}{d(\bar{x})}$$

holds. Since  $h_C^\circ \in C^2(\mathbb{R}^N \setminus \{0\})$  by Proposition 2.1 we can say that  $d$  is (locally) semiconcave out of  $E_r$ , and in particular  $D^2 d \leq c$  in both the viscosity and distributional sense. It is not obvious however to deduce from these facts that

$$-\operatorname{div} \nabla h_C(\nabla d) = D^2 h_C(\nabla d) : D^2 d \leq (N-1)/r$$

out of  $E_r$  in the sense of distributions, as the left-hand side is the product of a  $L^\infty$ , yet discontinuous function, and a Radon measure.

We pick now  $R > r$ ,  $\lambda > 0$ , and we introduce  $u^\lambda$  a solution of the problem

$$\min \left\{ \int_{E_R \setminus E_r} h_C(Du) + \left( \frac{N-1}{r} + \lambda \right) \int_{E_R \setminus E_r} u(x) dx : \right. \\ \left. u \in BV(E_{2R} \setminus E_{r/2}), u \geq d, u = d \text{ if } d \geq R \text{ or } d \leq r \right\}. \quad (4.5)$$



Notice that we can easily apply on the functional in (4.5) direct method of the Calculus of Variations: we have lower semicontinuity in the strong convergence of  $L^1$  essentially by Reshetnyak's lower semicontinuity and we have strong  $L^1$ -compactness of sequences bounded in energy since  $h_C(v) \geq c|v|$  for some  $c > 0$ . Moreover, observe that by truncation arguments we clearly have  $r \leq u^\lambda \leq R$  in  $E_R \setminus E_r$ . Standard density estimates for the level sets of  $u^\lambda$  show also that  $u^\lambda$  is a.e. equal to a lower and a upper-semicontinuous function. We assume that  $u^\lambda$  is upper-semicontinuous, and is a.e. equal to its lower-semicontinuous envelope. We check then that  $u^\lambda$  is a strict viscosity subsolution of (4.4) in  $\{u^\lambda > d\}$ , in the following sense: if  $\phi \geq u^\lambda$ ,  $\phi$  smooth,  $\phi(\bar{x}) = u^\lambda(\bar{x})$ , then if  $\nabla\phi(\bar{x}) \neq 0$  one has

$$-\operatorname{div} \nabla h_C(\nabla\phi)(\bar{x}) \leq -\frac{N-1}{r} - \lambda.$$

The proof is easy and quite standard. Possibly replacing  $\phi$  with  $\phi + \eta|\cdot - \bar{x}|^2$ ,  $\eta$  small, we may assume that  $\bar{x}$  is the only contact point. Then, one checks that  $\{\phi - \delta < u^\lambda\}$  has nonempty interior and goes to  $\{\bar{x}\}$  in the Hausdorff distance as  $\delta \rightarrow 0$ . We denote  $H_\lambda = (N-1)/r + \lambda$ . For  $\delta > 0$  small we have

$$\begin{aligned} \int_{E_R \setminus E_r} h_C(Du^\lambda) + H_\lambda \int_{E_R \setminus E_r} u^\lambda dx \\ \leq \int_{E_R \setminus E_r} h_C(D(u^\lambda \wedge (\phi - \delta))) + H_\lambda \int_{E_R \setminus E_r} (u^\lambda \wedge (\phi - \delta)) dx. \end{aligned}$$

Moreover, since for any open set  $A$  the functional

$$u \mapsto \int_A h_C(Du)$$

satisfies the generalized coarea formula (2.3) and it is convex, we get submodularity (see [5, Prop. 3.2]), which reads as

$$\int_A h_C(D(u^\lambda \wedge (\phi - \delta))) + \int_A h_C(D(u^\lambda \vee (\phi - \delta))) \leq \int_A h_C(Du^\lambda) + \int_A h_C(\nabla\phi) dx.$$

Therefore, we obtain that (letting  $A$  a small open set containing  $\{\phi - \delta < u^\lambda\}$ , for  $\delta$  small)

$$-H_\lambda \int_{\{\phi - \delta < u^\lambda\}} (\phi - \delta - u^\lambda) dx \leq \int_A h_C(\nabla\phi) dx - \int_A h_C(D(u^\lambda \vee (\phi - \delta))).$$

If  $\nabla\phi(\bar{x}) \neq 0$  then one may assume that  $\nabla\phi \neq 0$  in  $A$ , so that it follows

$$\begin{aligned} H_\lambda \int_{\{\phi - \delta < u^\lambda\}} (u^\lambda - (\phi - \delta)) dx &\leq \int_A \nabla h_C(\nabla\phi) \cdot (D(\phi - \delta - (u^\lambda \vee (\phi - \delta)))) \\ &= \int_A \operatorname{div} \nabla h_C(\nabla\phi) (u^\lambda - (\phi - \delta))^+ dx \\ &= \int_{\{\phi - \delta < u^\lambda\}} \operatorname{div} \nabla h_C(\nabla\phi) (u^\lambda - (\phi - \delta)) dx. \end{aligned}$$

We deduce that  $\operatorname{div} \nabla h_C(\nabla \phi)(\bar{x}) \geq H_\lambda$ , as claimed, otherwise one reaches a contradiction for small  $\delta$ .

Now, we can deduce that  $u^\lambda \leq d$  (so that in particular  $u^\lambda = d$ ), using a standard comparison result for viscosity sub and supersolution (with one possibly discontinuous). We sketch the argument, see [4] and [7] for details. Let  $m := \max\{u^\lambda - d\}$  and assume by contradiction that  $m > 0$ . For  $\delta > 0$  small, we consider

$$m_\delta := \max_{x,y} \left\{ u^\lambda(y) - d(x) - \frac{|x-y|^2}{2\delta} \right\} \geq m$$

which is reached at  $(x_\delta, y_\delta)$ . We have that  $x_\delta$  is a point of maximum of  $u_\delta^\lambda - d$  where

$$u_\delta^\lambda(x) := \max_y \left\{ u^\lambda(y) - \frac{|x-y|^2}{2\delta} \right\} \geq u^\lambda(x) \quad (4.6)$$

is a sup-convolution. In particular, if  $x \in \{u_\delta^\lambda > d + m/2\}$ , a point  $\bar{y}$  which reaches the maximum in (4.6) is such that  $u^\lambda(\bar{y}) > d(\bar{y})$  as soon as  $\delta < m/L^2$  ( $L$  denoting the Lipschitz constant of  $d$ ), and in this case  $u_\delta^\lambda$  is still a strict subsolution of (4.4) in  $\{u_\delta^\lambda > d + m/2\}$ : take  $\phi$  smooth with  $\phi \geq u_\delta^\lambda$  on  $\{u_\delta^\lambda > d + m/2\}$  and with  $\phi(\bar{x}) = u_\delta^\lambda(\bar{x})$  for some  $\bar{x} \in \{u_\delta^\lambda > d + m/2\}$ , and use

$$\psi(y) := \phi(y + \bar{x} - \bar{y}) + \frac{|\bar{x} - \bar{y}|^2}{2\delta}$$

as a test function in the definition of strict subsolution of (4.4) applied to  $u^\lambda$ . Now, since  $u_\delta^\lambda$  is (near  $x_\delta$ ) semiconvex while  $d$  is semiconcave, we can invoke Jensen's Lemma (see [7] for details), and find that there are points  $x_n \rightarrow x_\delta$  which are local maximum points of

$$x \mapsto u_\delta^\lambda(x) - d(x) + p_n \cdot x - \alpha_n \frac{|x - x_\delta|^2}{2},$$

with  $p_n \rightarrow 0$ ,  $\alpha_n \rightarrow 0$ ,  $u_\delta^\lambda(x_n) > d(x_n) + m/2$ ; notice that we have to add the term  $\alpha_n \frac{|x - x_\delta|^2}{2}$  since, in order to apply Jensen's Lemma, we need  $x_\delta$  be a strict local maximum of the function we perturb with the linear term  $p_n \cdot x$ . By Aleksandrov's Theorem (see again [7] for details) we can also assume that  $u_\delta^\lambda$  and  $d$  are both twice differentiable at  $x_n$ . In particular, for  $n$  large

$$\nabla u_\delta^\lambda(x_n) = \nabla d(x_n) - p_n + \alpha_n(x_n - x_\delta) \neq 0$$

and  $D^2 u_\delta^\lambda(x_n) \leq D^2 d(x_n) + \alpha_n I$  so that

$$\begin{aligned} \frac{N-1}{r} + \lambda &\leq D^2 h_C(\nabla u_\delta^\lambda(x_n)) : D^2 u_\delta^\lambda(x_n) \\ &\leq D^2 h_C(\nabla d(x_n) - p_n + \alpha_n(x_n - x_\delta)) : D^2 d(x_n) \\ &\quad + \alpha_n \operatorname{Tr}(D^2 h_C(\nabla d(x_n) - p_n + \alpha_n(x_n - x_\delta))) \leq \frac{N-1}{r} + o(1) \end{aligned}$$

where  $o(1) \rightarrow 0$  as  $n \rightarrow \infty$ . Since  $\lambda > 0$  this yields a contradiction. Hence  $u^\lambda = d$  for any  $\lambda > 0$ , and it follows that  $d$  is the only minimizer of (4.5) for any  $\lambda > 0$ , and in the limit is also a minimizer for  $\lambda = 0$ .

Finally, we have shown that the functional in (4.5) is minimized by  $d$ , including for  $\lambda = 0$ . But then, the Euler-Lagrange equation for the problem is easily derived: using perturbations  $d + \delta\phi$  with  $\delta > 0$  small,  $\phi$  smooth, nonnegative, with compact support in  $E_R \setminus E_r$ , we readily find

$$\int_{E_R \setminus E_r} \left( \nabla h_C(\nabla d) \cdot \nabla \phi + \frac{N-1}{r} \phi \right) dx \geq 0,$$

that is precisely (4.2) in the distributional sense.  $\square$

We are ready to prove essentially the same result stated in Proposition 4.1 for a general convex body  $C$ .

**Theorem 4.2.** *Let  $C$  be a convex body. There exists a divergence measure vector field  $n \in L^\infty(\mathbb{R}^N; \mathbb{R}^N)$  such that a.e. on  $\mathbb{R}^N$  we have*

$$n \in \partial h_C(\nabla d_E) \tag{4.7}$$

and

$$\operatorname{div} n \leq \frac{N-1}{r} \tag{4.8}$$

in the distributional sense out of  $E_r$ .

*Proof.* We use again the notation  $d = d_E$ . We prove (4.7) and (4.8) approximating  $C$  by smooth, elliptic, uniformly bounded and convex sets  $C^\sigma$ , with  $C^\sigma \supseteq C$ , and using Proposition 4.1. Let  $E_r^\sigma := E + rC^\sigma$  and denote by  $d^\sigma$  the anisotropic distance from  $C^\sigma$ . Then  $n^\sigma := \nabla h_{C^\sigma}(\nabla d^\sigma) \in C^\sigma$  is well defined a.e., and (4.2) reads

$$\operatorname{div} n^\sigma \leq \frac{N-1}{r} \tag{4.9}$$

out of  $E_r^\sigma$ . As  $\sigma \rightarrow 0^+$  we can assume, up to a subsequence, since  $\|n^\sigma\|_\infty$  remains bounded by (4.1), that  $n^\sigma \xrightarrow{*} n$  in  $L^\infty(\mathbb{R}^N; \mathbb{R}^N)$  and we have for any nonnegative  $C^1$  function  $\phi$  with compact support in  $\mathbb{R}^N \setminus E_r$ , for  $\sigma$  small enough (using the Hausdorff convergence of  $E_r^\sigma$  to  $E_r$ ),

$$-\frac{N-1}{r} \int \phi dx \leq \int n^\sigma \cdot \nabla \phi dx \rightarrow \int n \cdot \nabla \phi dx$$

as  $\sigma \rightarrow 0^+$ , showing that in  $\mathbb{R}^N \setminus E_r$ ,  $\operatorname{div} n$  is a measure bounded from above by  $(N-1)/r$ , so that we get (4.8). On the other hand, if  $\eta : \mathbb{R}_+ \rightarrow \mathbb{R}$  is any smooth nonincreasing function with  $\eta(t) = 1$  for  $t \leq r$ ,  $\eta(t) = 0$  for  $t$  large, one has (since  $n^\sigma = \nabla h_{C^\sigma}(\nabla d^\sigma) \in \partial h_{C^\sigma}(-\eta'(d^\sigma)\nabla d^\sigma)$ , using that  $\nabla h_{C^\sigma}$  is zero-homogeneous and always contained in  $\partial h_{C^\sigma}(0)$ ):

$$\int n^\sigma \cdot (-\nabla(\eta \circ d^\sigma)) dx = \int h_{C^\sigma}(-\nabla(\eta \circ d^\sigma)) dx$$

Since  $h_{C^\sigma} \geq h_C$ , we easily see that, from  $\eta \circ d^\sigma \rightarrow \eta \circ d$  in any  $L^p$  and using standard lower semicontinuity results for integral functionals,

$$\int h_C(-\nabla(\eta \circ d)) dx \leq \liminf_{\sigma \rightarrow 0^+} \int h_C(-\nabla(\eta \circ d^\sigma)) dx \leq \liminf_{\sigma \rightarrow 0^+} \int h_{C^\sigma}(-\nabla(\eta \circ d^\sigma)) dx,$$

that is

$$\int h_C(-\nabla(\eta \circ d)) dx \leq \liminf_{\sigma \rightarrow 0^+} \int n^\sigma \cdot (-\nabla(\eta \circ d^\sigma)) dx. \quad (4.10)$$

On the other hand (using (4.9)),

$$\begin{aligned} \int n^\sigma \cdot (-\nabla(\eta \circ d^\sigma)) dx &= \int n^\sigma \cdot (-\nabla(\eta \circ d)) dx - \int n^\sigma \cdot \nabla(\eta \circ d^\sigma - \eta \circ d) dx \\ &\leq \int n^\sigma \cdot (-\nabla(\eta \circ d)) dx + \frac{N-1}{r} \int (\eta \circ d^\sigma - \eta \circ d) dx \end{aligned}$$

since we have assumed that  $C^\sigma \supseteq C$ , so that  $d^\sigma \leq d$  and  $\eta \circ d^\sigma - \eta \circ d \geq 0$ . Since  $d^\sigma \rightarrow d$  uniformly,  $n^\sigma \xrightarrow{*} n$  and  $\nabla(\eta \circ d) \in L^1(\mathbb{R}^N; \mathbb{R}^N)$ , we deduce that

$$\limsup_{\sigma \rightarrow 0^+} \int n^\sigma \cdot (-\nabla(\eta \circ d^\sigma)) dx \leq \int n \cdot (-\nabla(\eta \circ d)) dx$$

which together with (4.10) yields

$$\int h_C(-\nabla(\eta \circ d)) dx \leq \int n \cdot (-\nabla(\eta \circ d)) dx.$$

Since  $n \in C$  a.e. we obtain (4.7) and this ends the proof.  $\square$

## 5. REGULARITY OF THE VOLUME FUNCTION

In this section we investigate the regularity of the volume function  $V_E$ . Our result extends [12, Eq. (2.20)], where an expression for  $V'_E$  has been given whenever  $C$  is strictly convex. In what follows  $n$  is given as in Theorem 4.2.

**Proposition 5.1.** *It turns out that for any  $r > 0$*

$$\lim_{s \rightarrow r^+} \int_{\mathcal{F}E_s} h_C(\nu_{E_s}) d\mathcal{H}^{N-1} = \int_{\mathcal{F}E_r} h_C(\nu_{E_r}) d\mathcal{H}^{N-1} \quad (5.1)$$

and

$$\lim_{s \rightarrow r^-} \int_{\mathcal{F}E_s} h_C(\nu_{E_s}) d\mathcal{H}^{N-1} = \int_{\mathcal{F}E_r} h_C(\nu_{E_r}) d\mathcal{H}^{N-1} - \operatorname{div} n(E_r \setminus E'_r). \quad (5.2)$$

*Proof.* Let us prove (5.1). The easy part is the estimate from below: since  $D\chi_{E_s} \xrightarrow{*} D\chi_{E_r}$ , as measures as  $s \rightarrow r^+$ , applying Reshetnyak's lower semicontinuity we have

$$\begin{aligned} &\int_{\mathcal{F}E_r} h_C(\nu_{E_r}) d\mathcal{H}^{N-1} \\ &= \int h_C \left( \frac{dD\chi_{E_r}}{d|D\chi_{E_r}|} \right) d|D\chi_{E_r}| \\ &\leq \liminf_{s \rightarrow r^+} \int h_C \left( \frac{dD\chi_{E_s}}{d|D\chi_{E_s}|} \right) d|D\chi_{E_s}| = \liminf_{s \rightarrow r^+} \int_{\mathcal{F}E_s} h_C(\nu_{E_s}) d\mathcal{H}^{N-1}. \end{aligned}$$

Now we divide the rest of the proof in some steps.

*Step 1.* We claim that for each continuous function  $\phi: [0, 1] \rightarrow \mathbb{R}$  we have

$$\lim_{k \rightarrow +\infty} \int_0^1 \phi(t) \int_{\mathcal{F}E_{r+t/k}} h_C(\nu_{E_{r+t/k}}) d\mathcal{H}^{N-1} dt = \int_{\mathcal{F}E_r} h_C(\nu_{E_r}) d\mathcal{H}^{N-1} \int_0^1 \phi(t) dt. \quad (5.3)$$

For simplicity of notation we let

$$f(t) := \int_{\mathcal{F}E_t} h_C(\nu_{E_t}) d\mathcal{H}^{N-1}.$$

First of all, combining (2.5) with the coarea formula, for any positive integer  $k$  we obtain

$$\int_0^1 f(r+t/k) dt = \int_0^1 \int_{\{r < d_E < r+t/k\}} h_C(\nabla d_E) d\mathcal{H}^{N-1} dt = \frac{|E_r + 1/kC| - |E_r|}{1/k}$$

and therefore using Theorem 3.1 we are able to pass to the limit applying [6, Thm. 3.4] and thus

$$\lim_{k \rightarrow +\infty} \int_0^1 f(r+t/k) dt = f(r).$$

Of course, for any  $c > 0$  we also have, by a simple change of variable,

$$\lim_{k \rightarrow +\infty} \int_0^c f(r+t/k) dt = cf(r)$$

from which, for each bounded open interval  $I$ ,

$$\lim_{k \rightarrow +\infty} \int_I f(r+t/k) dt = |I|f(r). \quad (5.4)$$

Now using (5.4) it is easy to get (5.3) whenever  $\phi \geq 0$ . Indeed, by Fubini's Theorem

$$\begin{aligned} \int_0^1 \phi(t) f(r+t/k) dt &= \int_0^1 dt \int_0^{\phi(t)} ds f(r+t/k) = \int_0^{\max \phi} ds \int_{\{\phi > s\}} dt f(r+t/k) \\ &\rightarrow \int_0^{\max \phi} ds |\{\phi > s\}| f(r) = f(r) \int_0^1 \phi(t) dt. \end{aligned}$$

For a general continuous function  $\phi$  it is sufficient to apply the previous argument to  $\phi^+$  and  $\phi^-$ .

*Step 2:* Consider  $\eta: \mathbb{R} \rightarrow \mathbb{R}$  a smooth nondecreasing function with  $\eta \equiv 1$  on  $\mathbb{R}_-$  and  $\eta(t) = 0$  for  $t \geq 1$ . Then, letting, for  $k \geq 0$ ,  $\phi_k(x) := \eta(k(d_E(x) - r))$  and  $\phi_k^\varepsilon(x) := \eta(k(d_E(x) - r - \varepsilon))$ , one has, using (4.8),

$$\int n \cdot \nabla(\phi_k - \phi_k^\varepsilon) dx \leq \frac{N-1}{r} \int |\phi_k^\varepsilon - \phi_k| dx \rightarrow \frac{N-1}{r} |E_{r+\varepsilon} \setminus E_r| \quad (5.5)$$

as  $k \rightarrow +\infty$ . On the other hand, using the very definition of  $n$  and the coarea formula,

$$\begin{aligned} \int n \cdot \nabla \phi_k dx &= \int k\eta'(k(d_E - r)) n \cdot \nabla d_E dx \\ &= \int k\eta'(k(d_E - r)) h_C(\nabla d_E) dx = \int_r^{r+1/k} k\eta'(k(s-r)) \int_{\mathcal{F}E_s} h_C(\nu_{E_s}) d\mathcal{H}^{N-1} ds \\ &= \int_0^1 \eta'(s) \int_{\mathcal{F}E_{r+t/k}} h_C(\nu_{E_{r+t/k}}) d\mathcal{H}^{N-1} ds \end{aligned}$$

and since (5.3) it follows, by the very definition of  $\eta$ ,

$$\lim_{k \rightarrow +\infty} \int n \cdot \nabla \phi_k dx = - \int_{\mathcal{F}E_r} h_C(\nu_{E_r}) d\mathcal{H}^{N-1}. \quad (5.6)$$

Similarly,

$$\lim_{k \rightarrow +\infty} \int n \cdot \nabla \phi_k^\varepsilon dx = - \int_{\mathcal{F}E_{r+\varepsilon}} h_C(\nu_{E_{r+\varepsilon}}) d\mathcal{H}^{N-1}.$$

Using (5.6) and the very definition of  $\operatorname{div} n$  we easily get

$$\int_{\mathcal{F}E_r} h_C(\nu_{E_r}) d\mathcal{H}^{N-1} - \int_{\mathcal{F}E_s} h_C(\nu_{E_s}) d\mathcal{H}^{N-1} = \operatorname{div} n(E_r \setminus E_s) \quad (5.7)$$

while passing to the limit in (5.5) as  $k \rightarrow +\infty$  we deduce

$$\int_{\mathcal{F}E_{r+\varepsilon}} h_C(\nu_{E_{r+\varepsilon}}) d\mathcal{H}^{N-1} \leq \int_{\mathcal{F}E_r} h_C(\nu_{E_r}) d\mathcal{H}^{N-1} + \frac{N-1}{r} |E_{r+\varepsilon} \setminus E_r|. \quad (5.8)$$

Passing to the limit in (5.8) as  $\varepsilon \rightarrow 0^+$  we get

$$\limsup_{\varepsilon \rightarrow 0^+} \int_{\mathcal{F}E_{r+\varepsilon}} h_C(\nu_{E_{r+\varepsilon}}) d\mathcal{H}^{N-1} \leq \int_{\mathcal{F}E_r} h_C(\nu_{E_r}) d\mathcal{H}^{N-1}$$

so that the proof of (5.1) is complete. Finally, (5.2) follows from (5.7) since  $\operatorname{div} n$  is a measure and  $E_r \setminus E_s \searrow E_r \setminus E'_r$  as  $s \rightarrow r^-$ .  $\square$

**Remark 5.2.** We observe that since  $\operatorname{div} n$  is a Radon measure on  $\mathbb{R}^N$  and since  $(E_r \setminus E'_r) \cap (E_s \setminus E'_s) = \emptyset$  whenever  $r \neq s$ , we can say that

$$J := \{r > 0 : \operatorname{div} n(E_r \setminus E'_r) > 0\}$$

is at most countable. Indeed,

$$\#\{r > 0 : \operatorname{div} n(E_r \setminus E'_r) > 1\} < +\infty, \quad \#\{r > 0 : \operatorname{div} n(E_r \setminus E'_r) \in (2^{-(k+1)}, 2^{-k}]\} < +\infty$$

for any  $k \in \mathbb{N}$ .

We are ready to prove the main regularity result.

**Theorem 5.3.** *Let  $C$  be a convex body. Then  $V_E \in C^1((0, +\infty) \setminus J)$  and*

$$V'_E(r) = \int_{\mathcal{F}E_r} h_C(\nu_{E_r}) d\mathcal{H}^{N-1}, \quad \forall r \in (0, +\infty) \setminus J. \quad (5.9)$$

*Proof.* Notice that from the fact that  $\partial C$  is locally Lipschitz and compact we easily deduce that

$$\theta_C := \inf\{\Theta_N^*(C, x) : x \in \partial C\} > 0.$$

As a consequence, we obtain  $\partial E_r \cap E_r^0 = \emptyset$ : indeed, if  $x \in \partial E_r$  then  $x \in y + r\partial C$  for some  $y \in \partial E$ , hence  $\Theta_N^*(E_r, x) \geq \theta_C > 0$ . Now, Theorem 3.1 and Theorem 2.2 imply that  $E_r \in \mathcal{O}'_C$  for all  $r > 0$ . Then, by Theorem 2.3 it follows that for any  $r > 0$

$$\lim_{s \rightarrow 0^+} \frac{V_E(r+s) - V_E(r)}{s} = \mathcal{SM}_C(E_r) = \int_{\mathcal{F}E_r} h_C(\nu_{E_r}) d\mathcal{H}^{N-1}.$$

This proves that the right derivative exists at any  $r > 0$  and that formula (5.9) holds true whenever  $V_E$  is differentiable at  $r$ . Furthermore, the function

$$(0, +\infty) \setminus J \ni r \mapsto \int_{\mathcal{F}E_r} h_C(\nu_{E_r}) d\mathcal{H}^{N-1}$$

is continuous since (5.1) and (5.2), which proves that if formula (5.9) holds true then  $V_E \in C^1((0, +\infty) \setminus J)$ . Thus it remains to compute the left derivative of  $V_E$ . We divide the rest of the proof in some steps.

*Step 1.* We claim that for any  $r \in (0, +\infty) \setminus J$

$$\int_{\mathcal{F}E_r} h_C(\nu_{E_r}) d\mathcal{H}^{N-1} \leq \liminf_{s \rightarrow 0^+} \frac{|E_r \setminus E_{r-s}|}{s}. \quad (5.10)$$

Let  $C^* := -C$ , that is the symmetrical of  $C$  with respect to the origin; notice that  $h_{C^*}^\circ(-v) = h_C^\circ(v)$  for all  $v \in \mathbb{R}^N$ . We also introduce the corresponding anisotropic distance to  $E_r^c$ :

$$d^*(x) := \operatorname{ess\,inf}_{z \in E_r^c} h_{C^*}^\circ(x - z)$$

where we have denoted  $E_r^c := (E_r')^c$ . Let  $s \in (0, r)$ . Notice that  $E_r \setminus E_{r-s} = \{x : r - s < d_E(x) \leq r\}$ . Let  $x \in \mathbb{R}^N$  with  $d^*(x) < s$ . By definition there exist  $\varepsilon > 0$  and  $z_\varepsilon \in E_r^c$  such that  $h_{C^*}^\circ(x - z_\varepsilon) = s - \varepsilon$ . Then, for any  $y \in E$  we obtain, by the subadditivity of  $h_C^\circ$ ,

$$\begin{aligned} h_C^\circ(x - y) &\geq h_C^\circ(z_\varepsilon - y) - h_C^\circ(z_\varepsilon - x) \\ &= h_C^\circ(z_\varepsilon - y) - h_{C^*}^\circ(x - z_\varepsilon) \geq r - s + \varepsilon \end{aligned}$$

that is  $d_E(x) > r - s$ . Thus  $\{d^*(x) < s, d_E(x) \leq r\} \subseteq E_r \setminus E_{r-s}$ . Taking into account Lemma 3.1 we can say that  $|\{d^* = s\}| = 0$  and  $|E_r^c| = |E_r^c|$ , hence

$$|(E_r^c + sC^*) \setminus E_r^c| = |\{d^*(x) < s, d_E(x) \leq r\}| \leq |E_r \setminus E_{r-s}|.$$

Passing to the limit as  $s \rightarrow 0^+$  we deduce that

$$\mathcal{SM}_{C^*}(E_r^c) \leq \liminf_{s \rightarrow 0^+} \frac{|E_r \setminus E_{r-s}|}{s}.$$

Noticing that  $E_r'^c \in \mathcal{O}'_{C^*}$  by (2.4) we get

$$\mathcal{SM}_{C^*}(E_r'^c) = \int_{\mathcal{F}E_r'^c} h_{C^*}(\nu_{E_r'^c}) d\mathcal{H}^{N-1} + 2 \int_{\partial E_r'^c \cap (E_r'^c)^0} \phi_{C^*}(\nu_{E_r'^c}) d\mathcal{H}^{N-1}.$$

From  $\mathcal{F}E_r = \mathcal{F}E_r'^c$ ,  $\nu_{E_r'^c} = -\nu_{E_r}$ ,  $\partial E_r'^c = \partial E_r'$  and  $(E_r'^c)^0 = E_r'^1 = E_r^1$  it follows

$$\mathcal{SM}_{C^*}(E_r'^c) = \int_{\mathcal{F}E_r} h_C(\nu_{E_r}) d\mathcal{H}^{N-1} + 2 \int_{\partial E_r' \cap E_r^1} \phi_C(\nu_{E_r}) d\mathcal{H}^{N-1}.$$

Notice now that if  $r \notin J$  then  $\mathcal{H}^{N-1}(\partial E_r' \cap E_r^1) = 0$  since  $\partial E_r' \cap E_r^1 \subseteq E_r \setminus E_r'$  and  $\text{div } n$  is absolutely continuous with respect to  $\mathcal{H}^{N-1}$ . Therefore, we get (5.10).

*Step 2.* We prove now that for any  $r \in (0, +\infty) \setminus J$

$$\limsup_{s \rightarrow 0^+} \frac{|E_r \setminus E_{r-s}|}{s} \leq \int_{\mathcal{F}E_r} h_C(\nu_{E_r}) d\mathcal{H}^{N-1}. \quad (5.11)$$

For any  $s \in (0, r)$  we have, using coarea formula and (5.8),

$$\begin{aligned} |E_r \setminus E_{r-s}| &= \int_{r-s}^s \int_{\mathcal{F}E_t} h_C(\nu_{E_t}) d\mathcal{H}^{N-1} dt \\ &= \int_0^s \int_{\mathcal{F}E_{r-s+u}} h_C(\nu_{E_{r-s+u}}) d\mathcal{H}^{N-1} du \\ &\leq \int_0^s \int_{\mathcal{F}E_{r-s}} h_C(\nu_{E_{r-s}}) d\mathcal{H}^{N-1} du + \int_0^s \frac{N-1}{r-s} |E_{r-s+u} \setminus E_{r-s}| du \\ &= s \int_{\mathcal{F}E_{r-s}} h_C(\nu_{E_{r-s}}) d\mathcal{H}^{N-1} + o(s). \end{aligned}$$

Therefore, by (5.2) we obtain

$$\limsup_{s \rightarrow 0^+} \frac{|E_r \setminus E_{r-s}|}{s} \leq \lim_{t \rightarrow r^-} \int_{\mathcal{F}E_t} h_C(\nu_{E_t}) d\mathcal{H}^{N-1} = \int_{\mathcal{F}E_r} h_C(\nu_{E_r}) d\mathcal{H}^{N-1} - \text{div } n(E_r \setminus E_r').$$

Formula (5.11) follows since  $\text{div } n(E_r \setminus E_r') = 0$  whenever  $r \notin J$  and this ends the proof.  $\square$

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