The Calibration Method for Free-Discontinuity Problems on Small Domains

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Introduction

Many variational problems arising in several branches of applied analysis (as image processing, fracture mechanics, theory of nematic liquid crystals) lead to consider minimum problems for functionals which couple a volume and a surface integral, depending on a closed set K and a function u smooth outside K. Following a terminology by E. De Giorgi, variational problems of this kind are called *free-discontinuity problems*, and, in the weak formulation proposed by E. De Giorgi and L. Ambrosio in [13], they appear as minimum problems for functionals of the form

$$F(u) = \int_{\Omega} f(x, u, \nabla u)\, dx + \int_{S_u} \psi(x, u^-, u^+, \nu_u)\, d\mathcal{H}^{n-1}, \tag{1}$$

where Ω is a bounded open subset of $\mathbb{R}^n$, and the unknown function u belongs to the space $SBV(\Omega;\mathbb{R}^N)$ of special functions of bounded variation in Ω with values in $\mathbb{R}^N$. We recall that ∇u denotes the approximate gradient of u, S_u is the set of essential discontinuity points of u, ν_u is the approximate unit normal vector to S_u, and u^-, u^+ the approximate limits of u on the two sides of S_u (for a precise definition see Chapter 1); finally, $\mathcal{H}^{n-1}$ denotes the $(n-1)$-dimensional Hausdorff measure.

A typical example is provided by the so-called Mumford-Shah functional, introduced in [31] in the context of image segmentation, which can be written as

$$MS_{\alpha,\beta}(u) := \int_{\Omega} |\nabla u|^2 dx + \alpha \mathcal{H}^{n-1}(S_u) + \beta \int_{\Omega} |u - g|^2\, dx, \tag{2}$$

where g is a function in $L^\infty(\Omega;\mathbb{R}^N)$, and $\alpha > 0$ and $\beta \geq 0$ are constants.

One of the main features of functionals of the form (1) is that they are in general not convex; therefore, all the equilibrium conditions which can be obtained by infinitesimal variations are necessary for minimality, but in general not sufficient.

G. Alberti, G. Bouchitté, and G. Dal Maso have proposed in [2] a sufficient condition for minimality, which is based on the *calibration method* and applies for functionals of the general form (1) defined on scalar maps.

In this thesis we apply this minimality criterion to identify a wide class of nontrivial minimizers for the homogeneous version of the Mumford-Shah functional (defined on scalar maps)

$$MS(u) := \int_{\Omega} |\nabla u|^2 dx + \mathcal{H}^{n-1}(S_u), \tag{3}$$

which occurs in the theory of inner regularity for minimizers of $MS_{\alpha,\beta}$ and is obtained by taking $\alpha = 1$ and dropping the lower order term in (2). In the last part we develop the theory of calibrations for more general functionals with free discontinuities on vector-valued maps and we describe several applications of this result.

All the applications and the examples shown throughout the thesis share the same purpose: we consider a candidate u satisfying the equilibrium conditions for a functional of the form (1) and we prove by calibration that u is a minimizer of F in a sufficiently small domain; in other words, we show that the equilibrium conditions are also sufficient to guarantee the minimality on small domains, as in many classical problems of the Calculus of Variations.

Before giving the details of the results, let us describe the basic idea behind the calibration method focusing our attention on *Dirichlet minimizers* of (1), that is minimizers with prescribed boundary values. Given a candidate u, if we are able to construct a functional G which is invariant on the class of functions having the same boundary values as u, and satisfies

$$G(u) = F(u), \qquad \text{and} \qquad G(v) \leq F(v) \quad \text{for every admissible } v, \tag{4}$$

then u is a Dirichlet minimizer of F. Indeed, if such a functional exists, for every v with the same boundary values as u we have that

$$F(u) = G(u) = G(v) \leq F(v).$$

In [2] the role of G is carried out by the flux of a suitable divergence-free vectorfield $\varphi : \Omega \times \mathbb{R} \to \mathbb{R}^n \times \mathbb{R}$ through the complete graph Γ_v of v, which is defined as the boundary of the subgraph of v (the set of all points $(x,z) \in \Omega \times \mathbb{R}$ such that $z \leq v(x)$), oriented by the inner normal ν_{Γ_v}. Since φ is divergence-free, from the divergence theorem the flux turns out to be invariant with respect to the boundary values, while suitable further conditions on φ guarantee (4). Consider for instance the case of the homogeneous Mumford-Shah functional, for simplicity in two dimensions, and denote the variables in Ω by (x,y) and the "vertical" variable in $\mathbb{R}$ by z. Then it is enough to require that $\varphi = (\varphi^{xy}, \varphi^z)$ is a bounded regular vectorfield satisfying the following assumptions:

(a1) $\varphi^z(x,y,z) \geq \frac{1}{4}|\varphi^{xy}(x,y,z)|^2$ for $\mathcal{L}^2$-a.e. $(x,y) \in \Omega$ and every $z \in \mathbb{R}$;

(a2) $\varphi^{xy}(x,y,u(x,y)) = 2\nabla u(x,y)$ and $\varphi^z(x,y,u(x,y)) = |\nabla u(x,y)|^2$ for $\mathcal{L}^2$-a.e. $(x,y) \in \Omega$;

(b1) $\left| \int_{t_1}^{t_2} \varphi^{xy}(x,y,z)\, dz \right| \leq 1$ for $\mathcal{H}^1$-a.e. $(x,y) \in \Omega$ and every $t_1 < t_2$ in $\mathbb{R}$;

(b2) $\displaystyle\int_{u^-(x,y)}^{u^+(x,y)} \varphi^{xy}(x,y,z)\, dz = \nu_u(x,y)$ for $\mathcal{H}^1$-a.e. $(x,y) \in S_u$.

Indeed, the flux of φ through Γ_v can be expressed as

$$\int_\Omega [\langle \varphi^{xy}(x,y,v), \nabla v \rangle - \varphi^z(x,y,v)]\, dx\, dy + \int_{S_v} \int_{v^-}^{v^+} \langle \varphi^{xy}(x,y,z), \nu_v \rangle\, dz\, d\mathcal{H}^1, \tag{5}$$

where v, ∇v, $v^\pm$, and ν_v are computed at (x,y); since condition (a1) implies that

$$\langle \varphi^{xy}(x,y,v), \nabla v \rangle - \varphi^z(x,y,v) \leq |\nabla v|^2 \qquad \text{for } \mathcal{L}^2\text{-a.e. } (x,y) \in \Omega, \tag{6}$$

while condition (b1) implies

$$\int_{v^-}^{v^+} \langle \varphi^{xy}(x,y,z), \nu_v \rangle\, dz \leq 1 \qquad \text{for } \mathcal{H}^1\text{-a.e. } (x,y) \in S_v, \tag{7}$$

by (5) we have that the inequality in (4) is satisfied for every admissible v. Moreover, conditions (a2) and (b2) guarantee that the equality holds true in (6) and (7), respectively, so that the equality in (4) is fulfilled for the candidate u. We will say that φ is a *calibration* for u with respect to the functional MS if φ is a vectorfield satisfying conditions (a1), (a2), (b1), (b2), and (c1) φ is divergence-free on $\Omega\times\mathbb{R}$.

Summarizing, if there exists a calibration φ for u with respect to MS, then u is a Dirichlet minimizer of MS.

The first applications of this minimality criterion are contained in [2], where the authors provide several examples of nontrivial minimizers for the Mumford-Shah functional with short and easy proofs. The simple expression of the calibrations in all these examples is related to the fact that they concern only minimizers having either a gradient vanishing almost everywhere or an empty discontinuity set. In the first part of this thesis we deal with candidates having a more complicated structure, that is presenting both a non vanishing gradient and a nonempty discontinuity set.

We recall from [6] and [31] that a Dirichlet minimizer u for MS in $\Omega\subset\mathbb{R}^2$ must satisfy the following equilibrium conditions (which can be globally called the Euler-Lagrange conditions for (3)):

(i) u is harmonic on $\Omega\setminus S_u$;

(ii) the normal derivative of u vanishes on both sides of S_u, where S_u is a regular curve;

(iii) the curvature of S_u (where defined) is equal to the difference of the squares of the tangential derivatives of u on both sides of S_u;

(iv) if S_u is locally the union of finitely many regular arcs, then S_u can present only two kinds of singularities: either a regular arc ending at some point, the so-called “crack-tip”, or three regular arcs meeting with equal angles of $2\pi/3$, the so-called “triple junction”.

In Chapters 2 and 3 we construct calibrations for solutions of the Euler equations with a regular discontinuity set, while in Chapter 4 we consider the case of a triple junction. All our results are in two dimensions. The minimality of the crack-tip has been recently proved by different methods in [7], while the problem of finding a calibration for it is still open.

We point out that we do not know of any general method to find calibrations, but each time, according to the geometry of the discontinuity set of the candidate, we have to perform a different construction. In spite of the lack of a general formula, all our constructions present a rather similar structure.

First of all, in terms of calibrations the presence of both a non vanishing gradient and a nonempty discontinuity set corresponds to a conflict between conditions (a2) and (b2), since (a2) and the Neumann conditions (ii) imply that φ^{xy} is tangential to S_u at the points $(x,y,u^{\pm}(x))$ for $(x,y)\in S_u$, while (b2) requires that its average between $u^-(x,y)$ and $u^+(x,y)$ is normal to S_u for $(x,y)\in S_u$. It is therefore convenient to construct the calibration φ by pieces in order to act differently on the regions around the (usual) graph of u, where φ will be somehow determined by condition (a2), and an “intermediate” region, which will give the main contribution to the integral in (b2). More precisely, we decompose the cylinder $\Omega\times\mathbb{R}$ in a finite union of Lipschitz open sets A_i and define φ in such a way that it agrees on A_i with a suitable divergence-free vectorfield φ_i; in order to satisfy condition (c1) we have clearly to require that the vectorfields φ_i satisfy a compatibility condition along the boundary of the sets A_i.

In a neighbourhood of the graph of u we have to construct a divergence-free vectorfield satisfying (a2) and such that for every $(x,y)\in S_u$ there holds

$$\begin{aligned}&\langle\varphi^{xy}(x,y,z),\nu_u(x,y)\rangle>0 \quad \text{for } u^-<z<u^-+\varepsilon \text{ and for } u^+-\varepsilon<z<u^+,\\ &\langle\varphi^{xy}(x,y,z),\nu_u(x,y)\rangle<0 \quad \text{for } u^+<z<u^++\varepsilon \text{ and for } u^--\varepsilon<z<u^-\end{aligned} \tag{8}$$

for a suitable $\varepsilon>0$. These properties are crucial in order to obtain (b1) and (b2) simultaneously.

The aim of the definition of φ in the remaining region is to make (b2) exactly satisfied, that is to annihilate the tangential contribution and to correct the normal one due to the presence of the field around the graph. Of course, φ has to be carefully chosen in order to preserve conditions (a1) and (b1).

The scheme of our proofs is the following: we define a vectorfield φ depending on some parameters and satisfying conditions (a1), (a2), (b1), and (c1); then we choose the parameters in such a way to fulfil also condition (b2). The Euler conditions are involved in the proof in a rather technical way: in general they concern the definition of φ around the graph, which can be therefore regarded as the crucial point of the construction.

The first examples of calibrations for discontinuous functions which are not locally constant, are presented in Chapter 2. We prove that if u is a function satisfying all Euler conditions for the homogeneous Mumford-Shah functional and whose discontinuity set is a straight line segment connecting two points of $\partial\Omega$, then every point (x_0, y_0) in S_u has an open neighbourhood U such that u is a Dirichlet minimizer of (3) in U, provided the tangential derivatives $\partial_\tau u$ and $\partial^2_{\tau\tau} u$ do not vanish at (x_0, y_0).

In Theorem 2.1 we study the special case

$$u(x,y) := \begin{cases} x & \text{if } y > 0, \\ -x & \text{if } y < 0, \end{cases}$$

which, even if very simple, involves most of the main difficulties. The main idea of the proof is in the definition of φ near the graph of u: in order to verify (a2) and to introduce a normal component satisfying (8) we take as φ^{xy} a suitable "rotation" of the vector $2\nabla u$; in other words, we apply to the vectors $\pm 2e_1$ a suitable orthogonal matrix R depending on x, y, z and satisfying $R(x, y, \pm x) = I$, and we set

$$\varphi(x,y,z) = (\pm 2R(x,y,z)e_1, 1).$$

This construction is then adapted in Theorem 2.4 to the case of a general function u satisfying the Euler conditions and having a rectilinear discontinuity set. Near the graph of u we simply take

$$\varphi(x,y,z) = (2R(u,v,z)\nabla u, |\nabla u|^2),$$

where v is the harmonic conjugate of u, while outside a neighbourhood of the graph we are forced to introduce some additional parameters. We will see that it is actually convenient to perform a change of variables through the mapping $(x,y) \mapsto (u(x,y), v(x,y))$, which is conformal near (x_0, y_0), since we are assuming $\partial_\tau u(x_0, y_0) \neq 0$. The additional assumption $\partial^2_{\tau\tau} u(x_0, y_0) \neq 0$ is instead related to the choice of the field in the region far from the graph and to the proof of (b1): indeed, it guarantees that the parameters appearing in the definition of φ can be chosen in such a way that the function

$$I(x,y,t_1,t_2) := \left| \int_{t_1}^{t_2} \varphi^{xy}(x,y,z)\,dz \right|$$

has a strict maximum at the points $(x, y, u^-(x,y), u^+(x,y))$ with (x,y) ranging in S_u.

These first examples are widely generalized in Chapter 3, where we consider candidates u whose discontinuity set can be any analytic curve and we prove the Dirichlet minimality in a uniform neighbourhood of S_u, without additional technical assumptions. More precisely, in Theorem 3.2 we show that, if u is a function satisfying all Euler conditions for the Mumford-Shah functional and S_u is an analytic curve connecting two points of $\partial\Omega$, then there exists an open neighbourhood U of $S_u \cap \overline{\Omega}$ such that u is a Dirichlet minimizer in U of (3).

We note that the analyticity assumption for S_u does not seem too restrictive, since it has been proved that the regular part of the discontinuity set of a minimizer is at least of class C^∞ and it is a conjecture that it is in fact analytic (see Chapter 1).

The original idea of the new construction essentially relies on the following remark: we can define divergence-free vectorfields on an open set $A \subset \Omega \times \mathbb{R}$ starting from a fibration of A by graphs of harmonic functions. Indeed, if $\{u_t\}_{t \in \mathbb{R}}$ is a family of harmonic functions whose graphs are pairwise disjoint and cover A, then the vectorfield

$$\varphi(x,y,z) = (2\nabla u_t(x,y), |\nabla u_t(x,y)|^2) \tag{9}$$

with $t = t(x, y, z)$ satisfying $z = u_t(x, y)$, turns out to be divergence-free on A; moreover, it automatically fulfils conditions (a1) and (a2).

We use this technique to construct the calibration around the graph of u: we take as $\{u_t\}$ the family $\{u + tv\}$, where v is a suitable harmonic function, and according to formula (9) we define

$$\varphi(x, y, z) = (2\nabla u + 2\tfrac{z-u}{v}\nabla v, |\nabla u + \tfrac{z-u}{v}\nabla v|^2);$$

the function v is chosen in such a way that ∇v is normal to S_u and (8) is verified.

This method of construction reminds of the classical method of Weierstrass fields, where the proof of the minimality of a candidate u is obtained by the construction of a slope field starting from a family of solutions of the Euler equation, whose graphs foliate a neighbourhood of the graph of u.

In Chapter 3 we deal also with a different notion of minimality: in Theorem 3.2 we compare u with perturbations which can be very large, but concentrated in a fixed small domain; we wonder if a minimality property is preserved also on a large domain, when we admit as competitors only perturbations of u with L^∞-norm very small outside a small neighbourhood of S_u.

According to this idea, we will say that a function u is a *Dirichlet graph-minimizer* of the Mumford-Shah functional if there exists a neighbourhood A of the complete graph of u such that $MS(u) \le MS(v)$ for all $v \in SBV(\Omega)$ having the same trace on $\partial\Omega$ as u and whose complete graph is contained in A.

As proved in [2, Example 4.10], any harmonic function $u : \Omega \to \mathbb{R}$ is a Dirichlet graph-minimizer of MS, whatever Ω is. If we consider instead a solution u of the Euler equations presenting some discontinuities, what we discover is that the Dirichlet graph-minimality of u may fail when Ω is too large, even in the case of a rectilinear discontinuity set, as the counterexample at the beginning of Section 3.2 shows. Therefore, to achieve this minimality property we have to add some restrictions on the domain Ω. To this aim we introduce a suitable quantity which seems useful to describe the correct interaction between S_u and Ω. Given an open set U (with Lipschitz boundary) and a portion Γ of ∂U (with nonempty relative interior in ∂U), we define

$$K(\Gamma, U) := \inf\left\{ \int_U |\nabla v(x,y)|^2 dx\, dy : \ v \in H^1(U),\ \int_\Gamma v^2 d\mathcal{H}^1 = 1, \text{ and } v = 0 \text{ on } \partial U \setminus \Gamma \right\}.$$

As shown by the notation, $K(\Gamma, U)$ is a quantity depending only on Γ and U, which describes a kind of "capacity" of the prescribed portion of the boundary with respect to the whole open set. Note that if $U_1 \subset U_2$, and $\Gamma_1 \subset \Gamma_2$, then $K(\Gamma_1, U_1) \ge K(\Gamma_2, U_2)$, which suggests that if $K(\Gamma, U)$ is very large, then U is thin in some sense. The qualitative properties of $K(\Gamma, U)$ are studied in the final part of Section 3.2.

Theorem 3.5, which is the main result of Section 3.2, gives a sufficient condition for the Dirichlet graph-minimality in terms of $K(S_u, \Omega)$ and of the geometrical properties of S_u. More precisely, we assume that Γ is a given analytic curve such that $\Gamma \cap \overline{\Omega}$ connects two points of $\partial\Omega$, and $\Omega \setminus \Gamma$ has two connected components Ω_1, Ω_2 with Lipschitz boundary. We prove that there exists a positive constant $c(\Gamma)$ (depending only on the length and on the curvature of Γ) such that, if u is a function satisfying all Euler conditions in Ω, whose discontinuity set coincides with $\Gamma \cap \Omega$ and such that

$$\min_{i=1,2} K(\Gamma \cap \Omega, \Omega_i) > c(\Gamma) \left(\|\partial_\tau u^-\|^2_{C^1(\Gamma\cap\Omega)} + \|\partial_\tau u^+\|^2_{C^1(\Gamma\cap\Omega)} \right), \qquad (10)$$

then u is a Dirichlet graph-minimizer of MS.

We remark that condition (10) imposes a restriction on the size of Ω depending on the behaviour of u along S_u: if u has large or very oscillating tangential derivatives, we have to take Ω quite small to guarantee that (10) is satisfied. In the special case of a locally constant function u, condition (10) is always fulfilled whatever the domain is; so u is always a Dirichlet graph-minimizer whatever Ω is, in agreement with a result proved in [2].

The proof of Theorem 3.5 is based again on the calibration method. Indeed, to prove the graph-minimality of a candidate u it is enough to show that there exist a suitable neighbourhood A of the complete graph of u, and a bounded vectorfield φ on A satisfying conditions (a1), (a2), (b1), (b2), and (c1) (where now (x,y,z), (x,y,t_i) range in A). Condition (10) guarantees that we can extend to a neighbourhood of Γ_u a slightly modified version of the calibration of Theorem 3.2.

In Chapter 4 we study the minimality of solutions u of the Euler equations whose discontinuity set is given by three line segments meeting at the origin with equal angles; in other words, S_u is a rectilinear triple junction, generating a partition of Ω in three sectors of angle $2\pi/3$, that we call A_0, A_1, A_2. In Theorem 4.1 we prove by calibration that, setting $u_i := u|_{A_i}$ and assuming $u_i \in C^2(\overline{A_i})$, there exists a neighbourhood U of the origin such that u is a Dirichlet minimizer of MS in U. This result generalizes Example 4 in [1] where the function u was piecewise constant.

The proof is quite long and technical, and is split in several steps. The symmetry due to the presence of $2\pi/3$-angles is exploited in the whole construction of the calibration. First of all, since the function u_i has to be harmonic in A_i with null normal derivative at ∂A_i, applying Schwarz reflection principle we obtain that u_i can be harmonically extended to a neighbourhood of the origin, cut by a half-line; moreover, from the Euler condition (iii) it follows that the extension of u_i coincides, up to the sign and to additive constants, with u_j on A_j for every $j \neq i$. Using this remark it is easy to see that each u_i must be either symmetric or antisymmetric with respect to the bisecting line of A_i.

In Sections 4.1 – 4.4 we define φ in the symmetric case and we prove that it is a calibration; in Section 4.5 we adapt the construction to the antisymmetric case.

The crucial point of both constructions is, as usual, the definition of the field near the graph of u, where we apply again the "fibration" technique. Indeed, we fibrate a neighbourhood of the graph of each u_i by a family of harmonic functions of the form $u_i + tv_i$. Unlike the construction of φ in the proof of Theorem 3.2 where we choose ∇v orthogonal to S_u, in this case it is convenient to take as v_i a linear function whose gradient is parallel to the bisecting line of A_i. Thanks to the symmetry, this choice ensures that the tangential contributions to the integral in (b2), given by the regions near u^- and u^+, are always of opposite signs and annihilate each other.

The assumption of C^2-regularity for u_i does not seem too restrictive: indeed, by the regularity results for elliptic problems in non-smooth domains (see [22]), it follows that u_i belongs at least to $C^1(\overline{A_i})$, since u_i solves the Laplace equation with Neumann boundary conditions on a sector of angle $2\pi/3$. Moreover, since u_i is either symmetric or antisymmetric with respect to the bisecting line of A_i, one can see u_i as a solution of the Laplace equation on a $\pi/3$-sector with Neumann boundary conditions or respectively mixed boundary conditions. By the regularity results in [22], it turns out that in the first case u_i belongs to $C^2(\overline{A_i})$, while in the second one u_i can be written (in polar coordinates centred at 0) as $u_i(r,\theta) = \tilde{u}_i(r,\theta) + cr^{3/2}\cos\frac{3}{2}\theta$, with $\tilde{u}_i \in C^2(\overline{A_i})$ and $c \in \mathbb{R}$. So, only the function $r^{3/2}\cos\frac{3}{2}\theta$ is not recovered by our theorem: if we were able to construct a calibration also for this function, then we would recover all possible cases.

Finally we remark that the case where S_u is given by three regular curves (not necessarily rectilinear) meeting at a point with $2\pi/3$-angles, is at the moment an open problem and it does not seem to be achievable with a plain arrangement of the calibration used for the rectilinear case, essentially because of the lack of symmetry.

The last part of the thesis corresponds to Chapter 5 where we generalize the calibration method to functionals of the form (1) defined on vector-valued maps. The basic principle is the same we have explained at the beginning: in order to prove the minimality of a function u, we want to construct a functional G satisfying conditions (4) and invariant on the class of the admissible competitors for u. When u is a vector-valued function, it is convenient to consider a different kind of invariant functional: the calibration is no longer a vectorfield, but a pair of functions $(\mathcal{S}, \mathcal{S}_0)$, where $\mathcal{S} : \Omega \times \mathbb{R}^N \to \mathbb{R}^n$ is suitably regular, while $\mathcal{S}_0$ belongs to $L^1(\Omega)$; the comparison functional for F is given by

$$G(v) := -\int_{\partial\Omega} \langle \mathcal{S}(x,v), \nu_{\partial\Omega} \rangle \, d\mathcal{H}^{n-1} + \int_\Omega \mathcal{S}_0(x)\, dx, \tag{11}$$

where $\nu_{\partial\Omega}$ is the inner unit normal to $\partial\Omega$. It is clear that the functional (11) is constant on the functions having the same values at $\partial\Omega$. Moreover, by the divergence theorem we can rewrite (11) as

$$\int_\Omega d\mu_v + \int_\Omega \mathcal{S}_0(x)\, dx,$$

where μ_v is the divergence (in the sense of distributions) of the composite function $\mathcal{S}(\cdot, v(\cdot))$. A generalized version of the chain rule in BV (which is proved in Lemma 5.2) implies that

$$\mu_v = ([\mathrm{div}_x \mathcal{S}](x,v) + \langle (D_z\mathcal{S}(x,v))^\tau, \nabla v\rangle)\, \mathcal{L}^n + \langle \mathcal{S}(x,v^+) - \mathcal{S}(x,v^-), \nu_v\rangle\, \mathcal{H}^{n-1}\lfloor S_v,$$

where $[\mathrm{div}_x\mathcal{S}]$ denotes the divergence of $\mathcal{S}$ with respect to the variable $x \in \Omega$, and $(D_z\mathcal{S})^\tau$ the transpose of the Jacobian matrix of $\mathcal{S}$ with respect to the variable $z \in \mathbb{R}^N$. Therefore the functional (11) turns out to be equal to

$$\int_\Omega ([\mathrm{div}_x\mathcal{S}](x,v) + \langle (D_z\mathcal{S}(x,v))^\tau, \nabla v\rangle + \mathcal{S}_0(x))\, dx + \int_{S_v} \langle \mathcal{S}(x,v^+) - \mathcal{S}(x,v^-), \nu_v\rangle\, d\mathcal{H}^{n-1}. \qquad (12)$$

By comparing this expression with the functional (1), we find pointwise conditions on $\mathcal{S}_0$, $\mathcal{S}$, and the derivatives of $\mathcal{S}$, which guarantee (4), and then the Dirichlet minimality of a given u. For instance, in the case of the Mumford-Shah functional (3) defined on vector-valued maps, it is enough to require the following conditions:

(a1) $[\mathrm{div}_x\mathcal{S}](x,z) + \mathcal{S}_0(x) \le -\frac{1}{4}|D_z\mathcal{S}(x,z)|^2$ for $\mathcal{L}^n$-a.e. $x \in \Omega$, and for every $z \in \mathbb{R}^N$;

(a2) $[\mathrm{div}_x\mathcal{S}](x,u) + \mathcal{S}_0(x) = -|\nabla u(x)|^2$ and $(D_z\mathcal{S}(x,u))^\tau = 2\nabla u(x)$ for $\mathcal{L}^n$-a.e. $x \in \Omega$;

(b1) $|\mathcal{S}(x,z_1) - \mathcal{S}(x,z_2)| \le 1$ for $\mathcal{H}^{n-1}$-a.e. $x \in \Omega$ and for every $z_1, z_2 \in \mathbb{R}^N$;

(b2) $\mathcal{S}(x,u^+) - \mathcal{S}(x,u^-) = \nu_u$ for $\mathcal{H}^{n-1}$-a.e. $x \in S_u$.

For a precise statement in the case of a general functional of the form (1) we refer to Lemma 5.4 and Lemma 5.5 in Section 5.1.

The connection between the conditions above in the case $N = 1$ and those ones of the scalar formulation by Alberti, Bouchitté, Dal Maso, is studied in Remark 5.8. Here we only observe that, while in the scalar formulation we need condition (c1) to ensure that the comparison functional is invariant with respect to the boundary values, in this new framework this is guaranteed just by the expression of the functional (11); so, there is no condition corresponding to (c1). In fact, in the case $N = 1$, given a calibration $(\mathcal{S}, \mathcal{S}_0)$, the vectorfield $\varphi = (\varphi^x, \varphi^z) : \Omega\times\mathbb{R} \to \mathbb{R}^n\times\mathbb{R}$ defined as

$$\varphi^x(x,z) := \partial_z\mathcal{S}(x,z), \qquad \varphi^z(x,z) := -[\mathrm{div}_x\mathcal{S}](x,z) - \mathcal{S}_0(x)$$

is a calibration in the sense of Alberti, Bouchitté, Dal Maso. Indeed, φ turns out to be divergence-free, and the remaining conditions of the scalar formulation follow from conditions (a1), (a2), (b1), and (b2) stated above. Conversely, given any divergence-free vectorfield $\varphi = (\varphi^x, \varphi^z)$, we can always write φ^x as the derivative with respect to $z \in \mathbb{R}$ of a suitable function $\mathcal{S} : \Omega\times\mathbb{R} \to \mathbb{R}^n$, and using the relation $\partial_z\varphi^z = -\mathrm{div}_x\varphi^x$ (which follows from (c1)), we can deduce that there exists a function $\mathcal{S}_0$ of the variable x such that $\varphi^z(x,z) = -[\mathrm{div}_x\mathcal{S}](x,z) - \mathcal{S}_0(x)$. If we rewrite now conditions (a1), (a2), (b1), and (b2) of the scalar formulation by using these expressions of φ^x and φ^z, we obtain that the pair $(\mathcal{S}, \mathcal{S}_0)$ is a calibration.

The formulation in terms of $(\mathcal{S}, \mathcal{S}_0)$ is related to classical field theory for multiple integrals of the form

$$F_0(u) = \int_\Omega f(x,u,\nabla u)\, dx.$$

In this context a sufficient condition for the minimality of a candidate $u \in C^1(\Omega;\mathbb{R}^N)$ is obtained by comparing F_0 with the integral of a null-lagrangian of divergence type, which is constructed starting from a suitably defined slope field $\mathcal{P}$, called *Weyl field*, and a function $\mathcal{S} \in C^2(\Omega\times\mathbb{R}^N;\mathbb{R}^n)$, the *eikonal map* associated with $\mathcal{P}$ (cf., e.g., [18]). In Section 5.2 we prove that, under suitable assumptions on f and ψ, whenever a Weyl field exists for a function $u \in C^1(\Omega;\mathbb{R}^N)$ (so that u is a Dirichlet minimizer for F_0), then there exists a calibration for u with respect to the functional F (which is given by the eikonal map $\mathcal{S}$ and by $\mathcal{S}_0 \equiv 0$), so u is also a Dirichlet minimizer for F among SBV functions.

Some examples and applications are presented in Section 5.3. In Examples 5.14, 5.16, 5.17, and 5.18 we deal with minimizers of the Mumford-Shah functional, and we generalize some results proved in [2] for the scalar case. A purely vectorial example is given by Example 5.15, where we study the minimality of continuous solutions of the Euler equations for a functional arising in fracture mechanics, which can be defined only on maps from $\Omega \subset \mathbb{R}^n$ into $\mathbb{R}^n$.

Finally, we point out that, as mentioned in [2], one could try to generalize the calibration theory from the scalar case to the vectorial one by replacing divergence-free vectorfields by closed n-forms on $\Omega\times\mathbb{R}^N$, acting on the graphs of the functions v, viewed as suitably defined surfaces in $\Omega\times\mathbb{R}^N$. This could lead to the idea that our choice of writing the calibration in terms of the pair $(\mathcal{S},\mathcal{S}_0)$ is somehow restrictive when $N > 1$. This is not the case at all, since the existence of a calibration expressed via differential forms implies the existence of a calibration expressed in terms of a pair $(\mathcal{S},\mathcal{S}_0)$, as shown in Section 5.4.

The results of Chapter 2 are obtained in collaboration with Gianni Dal Maso and Massimiliano Morini, and are published in [11], while the results of Chapter 3 are achieved in collaboration with Massimiliano Morini and published in [27]. The content of Chapter 4 will appear in [25], while the content of Chapter 5 corresponds to the paper [26].

Chapter 1

Preliminary results

In this chapter we collect some preliminary results which will be useful in the sequel. In Section 1.1 we recall some basic results from the theory of functions with bounded variation. In Sections 1.2 and 1.3 we deal with necessary and sufficient conditions for the minimality of the homogeneous Mumford-Shah functional on scalar maps: in Section 1.2 we write the Euler-Lagrange equations, while in Section 1.3 we present the theory of calibrations.

Let us fix some notation. Given $x, y \in \mathbb{R}^n$, we denote their scalar product by $\langle x, y \rangle$, and the euclidean norm of x by $|x|$. We set $\mathbb{S}^{n-1} := \{x \in \mathbb{R}^n : |x| = 1\}$. Given a set $B \subset \mathbb{R}^n$, we denote the Lebesgue measure of B by $\mathcal{L}^n(B)$ and the $(n-1)$-dimensional Hausdorff measure of B by $\mathcal{H}^{n-1}(B)$. If $a, b \in \mathbb{R}$, the maximum and the minimum of $\{a, b\}$ are denoted by $a \vee b$ and $a \wedge b$, respectively.

1.1 Functions of bounded variation

Let Ω be a bounded open subset of $\mathbb{R}^n$, let $u \in L^1_{loc}(\Omega; \mathbb{R}^N)$, and let $x_0 \in \Omega$. We say that u has an approximate limit at $x_0 \in \Omega$ if there exists $z \in \mathbb{R}^N$ such that

$$\lim_{r \to 0^+} \frac{1}{\mathcal{L}^n(B_r(x_0))} \int_{B_r(x_0)} |u(x) - z| \, dx = 0, \tag{1.1}$$

where $B_r(x_0)$ is the ball of radius r centred at x_0. The set S_u of points where this property does not hold is called the *approximate discontinuity set of* u. For any $x_0 \in \Omega \setminus S_u$ the vector z (which is uniquely determined by (1.1)) is called the *approximate limit* of u at x_0 and denoted by $\tilde{u}(x_0)$.

We say that a function $u : \Omega \to \mathbb{R}^N$ has *bounded variation in* Ω, and we write $u \in BV(\Omega; \mathbb{R}^N)$, if u belongs to $L^1(\Omega; \mathbb{R}^N)$ and its distributional derivative Du is a finite Radon $\mathbb{R}^{nN}$-valued measure in Ω. If Ω has Lipschitz boundary, we can speak about the *trace* of u on $\partial\Omega$, which belongs to $L^1(\partial\Omega, \mathcal{H}^{n-1})$ and will be still denoted by u.

If $u \in BV(\Omega; \mathbb{R}^N)$, then S_u is countably $(\mathcal{H}^{n-1}, n-1)$-rectifiable, that is, it can be covered, up to an $\mathcal{H}^{n-1}$-negligible set, by countably many C^1-hypersurfaces. Moreover, for $\mathcal{H}^{n-1}$-a.e. $x_0 \in S_u$ there exists a triplet $(u^+(x_0), u^-(x_0), \nu_u(x_0)) \in \mathbb{R}^N \times \mathbb{R}^N \times \mathbb{S}^{n-1}$ such that $u^+(x_0) \neq u^-(x_0)$, $\nu_u(x_0)$ is normal to S_u in an approximate sense, and

$$\lim_{r \to 0^+} \frac{1}{\mathcal{L}^n(B_r^{\pm}(x_0))} \int_{B_r^{\pm}(x_0)} |u(x) - u^{\pm}(x_0)| \, dx = 0, \tag{1.2}$$

where $B_r^{\pm}(x_0)$ is the intersection of $B_r(x_0)$ with the half-plane $\{x \in \mathbb{R}^n : \pm\langle x - x_0, \nu_u(x_0)\rangle \geq 0\}$. The triplet $(u^+(x_0), u^-(x_0), \nu_u(x_0))$ is uniquely determined up to a permutation of $(u^+(x_0), u^-(x_0))$ and a change of sign of $\nu_u(x_0)$. Condition (1.2) says that $\nu_u(x_0)$ points from the side of S_u corresponding to $u^-(x_0)$ to the side corresponding to $u^+(x_0)$.

For every $u \in BV(\Omega;\mathbb{R}^N)$, by applying the Radon-Nicodým theorem we can decompose the measure Du as $D^a u + D^s u$, where $D^a u$ is the absolutely continuous part with respect to the Lebesgue measure $\mathcal{L}^n$ and $D^s u$ is the singular part. The density of $D^a u$ with respect to $\mathcal{L}^n$ is denoted by ∇u and agrees with the *approximate gradient of* u. The measure $D^s u$ can be in turn written as $D^j u + D^c u$, where $D^j u$ is the restriction of $D^s u$ to S_u and is called the *jump part*, while $D^c u$ is the restriction to $\Omega \setminus S_u$ and is called the *Cantor part*. The density of $D^j u$ with respect to the measure $\mathcal{H}^{n-1}\lfloor S_u$ is given by the tensor product $(u^+ - u^-) \otimes \nu_u$. We also call the sum $D^a u + D^c u$ the *diffuse part* of the derivative of u and denote it by $\tilde{D}u$.

We say that a function $u : \Omega \to \mathbb{R}^N$ is a *special function of bounded variation*, and we write $u \in SBV(\Omega;\mathbb{R}^N)$, if $u \in BV(\Omega;\mathbb{R}^N)$ and $D^c u = 0$.

Finally, for every $u \in BV(\Omega;\mathbb{R}^N)$ we define as *graph of* u the set

$$\operatorname{graph} u := \{(x, \tilde{u}(x)) : \; x \in \Omega \setminus S_u\}.$$

In the scalar case $N = 1$, for every $u \in BV(\Omega)$ we call 1_u the characteristic function of the *subgraph of* u in $\Omega \times \mathbb{R}$, namely the function defined by $1_u(x,z) := 1$ for $z \le u(x)$ and $1_u(x,z) = 0$ for $z > u(x)$. We define as *complete graph of* u (and we denote it by Γ_u) the measure theoretic boundary of the subgraph of u, that is the singular set of 1_u. We note that, assuming u and S_u sufficiently regular, the complete graph Γ_u consists of the union of the graph of u and of all segments joining $(x, u^-(x))$ and $(x, u^+(x))$ with x ranging in S_u.

For more details we refer to the book [6] by L. Ambrosio, N. Fusco, and D. Pallara, where a self-contained presentation of BV and SBV spaces can be found.

1.2 The Euler-Lagrange equations for the Mumford-Shah functional

Let Ω denote a bounded open subset of $\mathbb{R}^2$ with Lipschitz boundary, and let us consider the homogeneous Mumford-Shah functional

$$MS(u) = \int_\Omega |\nabla u|^2 dx + \mathcal{H}^1(S_u) \tag{1.3}$$

for $u \in SBV(\Omega)$.

In the sequel we will refer to the following definition of minimizers.

Definition 1.1 *An* absolute minimizer of (1.3) in Ω *is a function* $u \in SBV(\Omega)$ *such that*

$$\int_\Omega |\nabla u|^2 dx + \mathcal{H}^1(S_u) \le \int_\Omega |\nabla v|^2 dx + \mathcal{H}^1(S_v) \tag{1.4}$$

for every $v \in SBV(\Omega)$, *while a* Dirichlet minimizer in Ω *is a function* $u \in SBV(\Omega)$ *such that (1.4) is satisfied for every* $v \in SBV(\Omega)$ *with the same trace on* $\partial\Omega$ *as* u.

Let us focus our attention on necessary optimality conditions near a regular portion of $\overline{S}_u$. Let u be a Dirichlet minimizer of MS and let $U \subset \Omega$ be an open set such that $\overline{S}_u \cap U$ is a graph, that is

$$\overline{S}_u \cap U = \{(t, \psi(t)) : \; t \in D\}$$

for some open set $D \subset \mathbb{R}$ and $\psi : D \to \mathbb{R}$. Set $U^+ := \{(t,s) \in U : s > \psi(t)\}$ and $U^- := \{(t,s) \in U : s < \psi(t)\}$. Let $\varphi \in C^1(\overline{U})$ be a function vanishing in a neighbourhood of $\partial U^+ \setminus \overline{S}_u$; by comparing u with the function $v := u + \varepsilon\varphi$, from the minimality of u we obtain that

$$\int_{U^+} \langle \nabla u, \nabla \varphi \rangle \, dx = 0.$$

This means that u is a weak solution of the following problem:

$$\begin{cases} \Delta u = 0 & \text{in } U^+, \\ \partial_\nu u = 0 & \text{on } \partial U^+ \cap \overline{S}_u. \end{cases} \tag{1.5}$$

A similar problem is solved by u in U^-.

The Euler equation (1.5) has been obtained by considering only variations of u and keeping S_u fixed. By considering also variations of S_u we expect to derive a transmission condition for u along S_u, which takes into account the interaction between the bulk and the surface part of the functional. Assume that u belongs to $W^{2,2}(U^+ \cup U^-)$ and suppose that $\overline{S}_u \cap U$ is the graph of a C^2-function (that is, ψ is of class C^2). Then it can be proved that

$$-\mathrm{div}\left(\frac{\nabla\psi}{\sqrt{1+|\nabla\psi|^2}}\right) = |(\nabla u)^+|^2 - |(\nabla u)^-|^2 \qquad \text{on } S_u \cap U, \tag{1.6}$$

where the left-hand side is the curvature of S_u, while at the right-hand side $(\nabla u)^\pm$ denote the traces of ∇u on $\overline{S}_u \cap U$ from $U^\pm$, respectively.

We note that, if ψ is known to be only of class $C^{1,\gamma}$, equation (1.6) actually still holds in a weak sense. Then using (1.6) it is possible to prove that, as soon as we know that $\overline{S}_u \cap U$ is of class $C^{1,\gamma}$, then $\overline{S}_u \cap U$ turns out to be in fact of class C^∞.

The following conjecture is still an open problem.

Conjecture (De Giorgi). *If u is a Dirichlet minimizer of MS, then S_u is analytic near its regular points.*

We conclude this section by some remarks on the regularity of the discontinuity set of a minimizer, which represents a very challenging mathematical problem. In [31] D. Mumford and J. Shah conjectured that, if u is a Dirichlet minimizer of MS, then S_u is locally the union of finitely many $C^{1,1}$ embedded arcs; moreover, they showed that, if the conjecture is true, then only two kinds of singularity can occur inside Ω: either a line terminates at some point, the so-called "crack-tip", or three lines meet forming equal angles of $2\pi/3$, the so-called "triple junction".

In [6, Theorem 8.1] the following regularity result is proved.

Theorem 1.2 *If $u \in SBV(\Omega)$ is a minimizer of MS, there exists an $\mathcal{H}^1$-negligible set $\Sigma \subset \overline{S}_u \cap \Omega$ relatively closed in Ω such that $\Omega \cap \overline{S}_u \setminus \Sigma$ is a curve of class $C^{1,1}$.*

This result is still far from Mumford-Shah conjecture, since we are only able to say that Σ is $\mathcal{H}^1$-negligible, and not that it has locally finite $\mathcal{H}^0$ measure.

1.3 The calibration method for the Mumford-Shah functional

In this section we present the calibration method for the homogeneous Mumford-Shah functional in two dimensions and we briefly recall how this criterion can be adapted to a general functional with free discontinuities defined on scalar maps.

We first introduce a more general notion of minimality which will be useful in the sequel. Let Ω be a fixed bounded open subset of $\mathbb{R}^2$ with Lipschitz boundary, and $\nu_{\partial\Omega}$ its inner unit normal. Let A denote an open subset of $\Omega\times\mathbb{R}$ with Lipschitz boundary, whose closure can be written as

$$\overline{A} = \{(x,y,z) \in \overline{\Omega}\times\mathbb{R} : \ \tau_1(x,y) \le z \le \tau_2(x,y)\},$$

where the two functions $\tau_1, \tau_2 : \overline{\Omega} \to [-\infty, +\infty]$ satisfy $\tau_1 < \tau_2$.

Definition 1.3 *We say that a function $u \in SBV(\Omega)$ is an* absolute $\overline{A}$-minimizer of MS *if the complete graph of u is contained in $\overline{A}$ and $MS(u) \le MS(v)$ for every $v \in SBV(\Omega)$ such that $\Gamma_v \subset \overline{A}$, while u is a* Dirichlet $\overline{A}$-minimizer *if we add the requirement that the competing functions v have the same trace on $\partial\Omega$ as u.*

For every vectorfield $\varphi : \overline{A} \to \mathbb{R}^2\times\mathbb{R}$ we define the maps $\varphi^{xy} : \overline{A} \to \mathbb{R}^2$ and $\varphi^z : \overline{A} \to \mathbb{R}$ by

$$\varphi(x,y,z) = (\varphi^{xy}(x,y,z), \varphi^z(x,y,z)).$$

We shall consider the collection $\mathcal{F}$ of all piecewise C^1-vectorfields $\varphi : \overline{A} \to \mathbb{R}^2\times\mathbb{R}$ with the following property: there exist a finite family $(A_i)_{i\in I}$ of pairwise disjoint open subsets of A with Lipschitz boundary whose closures cover $\overline{A}$, and a family $(\varphi_i)_{i\in I}$ of vectorfields in $C^1(\overline{A}_i; \mathbb{R}^2\times\mathbb{R})$ such that φ agrees at any point with one of the φ_i.

Let $u \in SBV(\Omega)$ be such that $\Gamma_u \subset \overline{A}$. A *calibration* for u on $\overline{A}$ (with respect to the functional MS) is a bounded vectorfield $\varphi \in \mathcal{F}$ satisfying the following properties:

(a1) $\varphi^z(x,y,z) \ge \frac{1}{4}|\varphi^{xy}(x,y,z)|^2$ for $\mathcal{L}^2$-a.e. $x \in \Omega$ and every $z \in [\tau_1, \tau_2]$;

(a2) $\varphi^{xy}(x,y,u(x,y)) = 2\nabla u(x,y)$ and $\varphi^z(x,y,u(x,y)) = |\nabla u(x,y)|^2$ for $\mathcal{L}^2$-a.e. $x \in \Omega$;

(b1) $\displaystyle\left|\int_{t_1}^{t_2} \varphi^{xy}(x,y,z)\,dz\right| \le 1$ for $\mathcal{H}^1$-a.e. $(x,y) \in \Omega$, and every t_1, t_2 in $[\tau_1, \tau_2]$;

(b2) $\displaystyle\int_{u^-(x,y)}^{u^+(x,y)} \varphi^{xy}(x,y,z)\,dz = \nu_u(x,y)$ for $\mathcal{H}^1$-a.e. $(x,y) \in S_u$;

(c1) φ is divergence-free in the sense of distributions in $\overline{A}$.

If also the following condition is satisfied:

(c2) $\langle \varphi^{xy}, \nu_{\partial\Omega}\rangle = 0$ $\mathcal{H}^2$-a.e. on $\partial A \cap (\partial\Omega\times\mathbb{R})$,

then φ is called an *absolute calibration* for u on $\overline{A}$.

We note that, in order to prove condition (c1), it is enough to show that $\mathrm{div}\varphi_i = 0$ in A_i for every $i \in I$, and the following transmission condition is satisfied:

$$\langle \varphi_i, \nu_{\partial A_i}\rangle = \langle \varphi_j, \nu_{\partial A_j}\rangle \qquad \mathcal{H}^2\text{-a.e. on } \partial A_i \cap \partial A_j,$$

where $\nu_{\partial A_i}$ and $\nu_{\partial A_j}$ denote the unit normal vector to ∂A_i and ∂A_j, respectively.

We can now state the fundamental theorem of the calibration method, which is proved in [1] and [2].

Theorem 1.4 *Let $u \in SBV(\Omega)$ be such that $\Gamma_u \subset \overline{A}$. If there exists a calibration for u on $\overline{A}$ (with respect to MS), then u is a Dirichlet $\overline{A}$-minimizer of the homogeneous Mumford-Shah functional. If there exists an absolute calibration for u on $\overline{A}$, then u is an absolute $\overline{A}$-minimizer.*

The following lemma, which allows to construct divergence-free vectorfields starting from families of harmonic functions, will be useful in the construction of the calibrations of Chapters 3 and 4.

Lemma 1.5 *Let U be an open subset of $\mathbb{R}^2$ and I, J be two real intervals. Let $u : U\times J \to I$ be a function of class C^1 such that*

- *$u(\cdot,\cdot;s)$ is harmonic for every $s \in J$;*
- *there exists a C^1-function $t : U\times I \to J$ such that $u(x,y;t(x,y;z)) = z$.*

Then, if we define in $U\times I$ the vectorfield

$$\varphi(x,y,z) := (2\nabla u(x,y;t(x,y;z)), |\nabla u(x,y;t(x,y;z))|^2),$$

where $\nabla u(x,y;t(x,y;z))$ denotes the gradient of u with respect to the variables (x,y) computed at the point $(x,y;t(x,y;z))$, φ is divergence-free in $U\times I$.

Proof. – Let us compute the divergence of φ:

$$\begin{aligned}\mathrm{div}\varphi(x,y,z) = 2\Delta u(x,y;t(x,y;z)) &+ 2\langle\partial_s\nabla u(x,y;t(x,y;z)),\nabla t(x,y;z)\rangle \\ &+ 2\partial_z t(x,y;z)\,\langle\nabla u(x,y;t(x,y;z)),\partial_s\nabla u(x,y;t(x,y;z))\rangle, \qquad (1.7)\end{aligned}$$

where $\Delta u(x,y;t(x,y;z))$ denotes the Laplacian of u with respect to (x,y) computed at $(x,y;t(x,y;z))$, and $\nabla t(x,y;z)$ denotes the gradient of t with respect to (x,y). By differentiating the identity verified by the function t first with respect to z and with respect to (x,y), we derive that

$$\partial_s u(x,y;t(x,y;z))\,\partial_z t(x,y;z) = 1, \qquad \nabla u(x,y;t(x,y;z)) + \partial_s u(x,y;t(x,y;z))\,\nabla t(x,y;z) = 0.$$

Using these identities and substituting in (1.7), we finally obtain

$$\mathrm{div}\varphi(x,y,z) = 2\Delta u(x,y;t(x,y;z)) = 0,$$

since by assumption u is harmonic with respect to (x,y). □

Let us consider now a general functional of the form

$$F(u) := \int_\Omega f(x,u,\nabla u)\,dx + \int_{S_u}\psi(x,u^-,u^+,\nu_u)\,\mathcal{H}^{n-1},$$

where Ω is a bounded open subset of $\mathbb{R}^n$ with Lipschitz boundary, the unknown u belongs to $SBV(\Omega)$, and f, ψ are Borel functions.

Let f^* and $\partial_\xi^- f$ denote the convex conjugate and the subdifferential of f with respect to the last variable. We recall that the subdifferential of a function $g : \mathbb{R}^n \to [0,+\infty]$ at the point $\xi \in \mathbb{R}^n$ is defined as the set of vectors $\eta \in \mathbb{R}^n$ such that $g(\xi) + \langle\eta,\zeta-\xi\rangle \le g(\zeta)$ for every $\zeta \in \mathbb{R}^n$.

As before, let A be an open subset of $\Omega\times\mathbb{R}$ with Lipschitz boundary whose closure can be written as

$$\overline{A} = \{(x,z) \in \overline{\Omega}\times\mathbb{R} : \tau_1(x) \le z \le \tau_2(x)\},$$

where $\tau_1,\tau_2 : \overline{\Omega} \to [-\infty,+\infty]$ satisfy $\tau_1 < \tau_2$.

The regularity assumptions on φ can be weakened by requiring that φ is *approximately regular*, i.e. it is bounded and for every Lipschitz hypersurface M in $\mathbb{R}^{n+1}$ there holds

$$\operatorname*{ap\,lim}_{(x,z)\to(x_0,z_0)} \langle\varphi(x,z),\nu_M(x_0,z_0)\rangle = \langle\varphi(x_0,z_0),\nu_M(x_0,z_0)\rangle \quad \text{for } \mathcal{H}^n\text{-a.e. } (x,z) \in M\cap\overline{A},$$

where $\nu_M(x_0,y_0)$ is the unit normal to M at (x_0,y_0). It is easy to see that, if $\varphi \in \mathcal{F}$, then φ is approximately regular.

Let $u \in SBV(\Omega)$ be such that $\Gamma_u \subset \overline{A}$. A *calibration* for u on $\overline{A}$ with respect to the functional F is an approximately regular vectorfield $\varphi = (\varphi^x,\varphi^z) : \overline{A} \to \mathbb{R}^n\times\mathbb{R}$ satisfying the following conditions:

(a1) $\varphi^z(x,z) \geq f^*(x,z,\varphi^x(x,z))$ for $\mathcal{L}^n$-a.e. $x \in \Omega$ and every $z \in [\tau_1,\tau_2]$;

(a2) $\varphi^x(x,u(x)) \in \partial_\xi^- f(x,u(x),\nabla u(x))$ and $\varphi^z(x,u(x)) = f^*(x,u(x),\varphi^x(x,u(x)))$ for $\mathcal{L}^n$-a.e. $x \in \Omega$;

(b1) $\displaystyle\int_{t_1}^{t_2} \langle \varphi^x(x,z),\nu\rangle\,dz \leq \psi(x,t_1,t_2,\nu)$ for $\mathcal{H}^{n-1}$-a.e. $x \in \Omega$, every $\nu \in \mathbb{S}^{n-1}$, and every $t_1 < t_2$ in $[\tau_1,\tau_2]$;

(b2) $\displaystyle\int_{u^-(x)}^{u^+(x)} \langle \varphi^x(x,z),\nu_u(x)\rangle\,dz = \psi(x,u^-(x),u^+(x),\nu_u(x))$ for $\mathcal{H}^{n-1}$-a.e. $x \in S_u$;

(c1) φ is divergence-free in the sense of distributions in $\overline{A}$.

If also the following condition is satisfied:

(c2) $\langle \varphi^x, \nu_{\partial\Omega}\rangle = 0$ $\mathcal{H}^n$-a.e. on $\partial A \cap (\partial\Omega \times \mathbb{R})$,

then φ is called an *absolute calibration.*

The following theorem is proved in [2].

Theorem 1.6 *Let $u \in SBV(\Omega)$ be such that $\Gamma_u \subset \overline{A}$. If there exists a calibration for u on $\overline{A}$ with respect to F, then u is a Dirichlet $\overline{A}$-minimizer of F, that is $F(u) \leq F(v)$ for every $v \in SBV(\Omega)$ with the same trace on $\partial\Omega$ as u and such that $\Gamma_v \subset \overline{A}$. If there exists an absolute calibration for u on $\overline{A}$ with respect to F, then u is an absolute $\overline{A}$-minimizer of F, that is $F(u) \leq F(v)$ for every $v \in SBV(\Omega)$ such that $\Gamma_v \subset \overline{A}$.*

Chapter 2

Calibrations for minimizers with a rectilinear discontinuity set

In this chapter we show the first examples of calibrations for discontinuous functions, which are not locally constant. In particular, we consider solutions w of the Euler-Lagrange equations for the homogeneous Mumford-Shah functional

$$MS(w) = \int_\Omega |\nabla w(x,y)|^2 dx\, dy + \mathcal{H}^1(S_w), \tag{2.1}$$

and we assume that the discontinuity set S_w is a straight line segment connecting two boundary points of the domain. We prove that, under the additional assumptions that the tangential derivatives $\partial_\tau w$ and $\partial^2_{\tau\tau} w$ of w do not vanish on both sides of S_w, the Euler conditions are also sufficient for the Dirichlet minimality in small domains.

Let Ω be a circle in $\mathbb{R}^2$ with centre on the x-axis, and set

$$\Omega_0 := \{(x,y) \in \Omega : y \neq 0\}, \qquad S := \{(x,y) \in \Omega : y = 0\}.$$

If $w \in C^1(\Omega_0)$ with $\int_{\Omega_0} |\nabla w|^2 dx\, dy < +\infty$, then it is easy to see that w satisfies the Euler conditions for the Mumford-Shah functional (see Section 1.2) if and only if w has one of the following forms:

$$w(x,y) = \begin{cases} u(x,y) & \text{if } y > 0, \\ -u(x,y) + c_1 & \text{if } y < 0, \end{cases} \tag{2.2}$$

or

$$w(x,y) = \begin{cases} u(x,y) + c_2 & \text{if } y > 0, \\ u(x,y) & \text{if } y < 0, \end{cases} \tag{2.3}$$

where $u \in C^1(\Omega)$ is harmonic with normal derivative vanishing on S and c_1, c_2 are real constants. For our purposes, it is enough to consider the case $c_1 = 0$ in (2.2) and $c_2 = 1$ in (2.3).

In both cases we will construct an explicit calibration for w in the cylinder $U \times \mathbb{R}$, where U is a suitable neighbourhood of (x_0, y_0). Since this construction is elementary when $(x_0, y_0) \notin S_w$ (see [2]), we consider only the case $(x_0, y_0) \in S_w$.

In Section 2.1 we consider the special case of the function

$$w(x,y) := \begin{cases} x & \text{if } y > 0, \\ -x & \text{if } y < 0, \end{cases} \tag{2.4}$$

and give in full details the expression of the calibration for w (see Theorem 2.1); then in Theorem 2.3 we adapt the same construction to the function

$$w(x,y) := \begin{cases} x+1 & \text{if } y > 0, \\ x & \text{if } y < 0. \end{cases} \tag{2.5}$$

In Section 2.2 we consider the general cases (2.2) and (2.3): the former case (2.2) is studied in Theorem 2.4 by a suitable change of variables and by adding two new parameters to the construction used in Theorem 2.1; the minor changes for (2.2) are considered in Theorem 2.5.

2.1 A model case

In this section we deal with the minimality of the functions (2.4) and (2.5). The aim of the study of these simpler cases (but we will see that they involve the main difficulties) is to clarify the ideas of the general construction.

Theorem 2.1 *Let $w : \mathbb{R}^2 \to \mathbb{R}$ be the function defined by*

$$w(x,y) := \begin{cases} x & \textit{if } y > 0, \\ -x & \textit{if } y < 0. \end{cases}$$

Then every point $(x_0, y_0) \neq (0,0)$ has an open neighbourhood U such that w is a Dirichlet minimizer in U of the Mumford-Shah functional (2.1).

PROOF. – The result follows from Example 4.10 of [2] if $y_0 \neq 0$. We consider now the case $y_0 = 0$, assuming for simplicity that $x_0 > 0$. We will construct a local calibration of w near $(x_0, 0)$. Let us fix $\varepsilon > 0$ such that

$$0 < \varepsilon < \frac{x_0}{10}, \qquad 0 < \varepsilon < \frac{1}{32}. \tag{2.6}$$

For $0 < \delta < \varepsilon$ we consider the open rectangle

$$U := \{(x,y) \in \mathbb{R}^2 : |x - x_0| < \varepsilon, \, |y| < \delta\}$$

and the following subsets of $U \times \mathbb{R}$ (see Fig. 2.1):

$$\begin{aligned}
A_1 &:= \{(x,y,z) \in U \times \mathbb{R} : x - \alpha(y) < z < x + \alpha(y)\}, \\
A_2 &:= \{(x,y,z) \in U \times \mathbb{R} : b + \kappa(\lambda)\, y < z < b + \kappa(\lambda)\, y + h\}, \\
A_3 &:= \{(x,y,z) \in U \times \mathbb{R} : -h < z < h\}, \\
A_4 &:= \{(x,y,z) \in U \times \mathbb{R} : -b + \kappa(\lambda)\, y - h < z < -b + \kappa(\lambda)\, y\}, \\
A_5 &:= \{(x,y,z) \in U \times \mathbb{R} : -x - \alpha(-y) < z < -x + \alpha(-y)\},
\end{aligned}$$

where

$$\alpha(y) := \sqrt{4\varepsilon^2 - (\varepsilon - y)^2},$$

$$h := \frac{x_0 - 3\varepsilon}{4}, \qquad \kappa(\lambda) := \frac{\lambda}{4} - \frac{1}{\lambda}, \qquad b := 2h + \kappa(\lambda)\, \delta, \qquad \lambda := \frac{1 - 4\varepsilon}{2h}.$$

We will assume that

$$\delta < \frac{x_0 - 3\varepsilon}{8\, |\kappa(\lambda)|}, \tag{2.7}$$

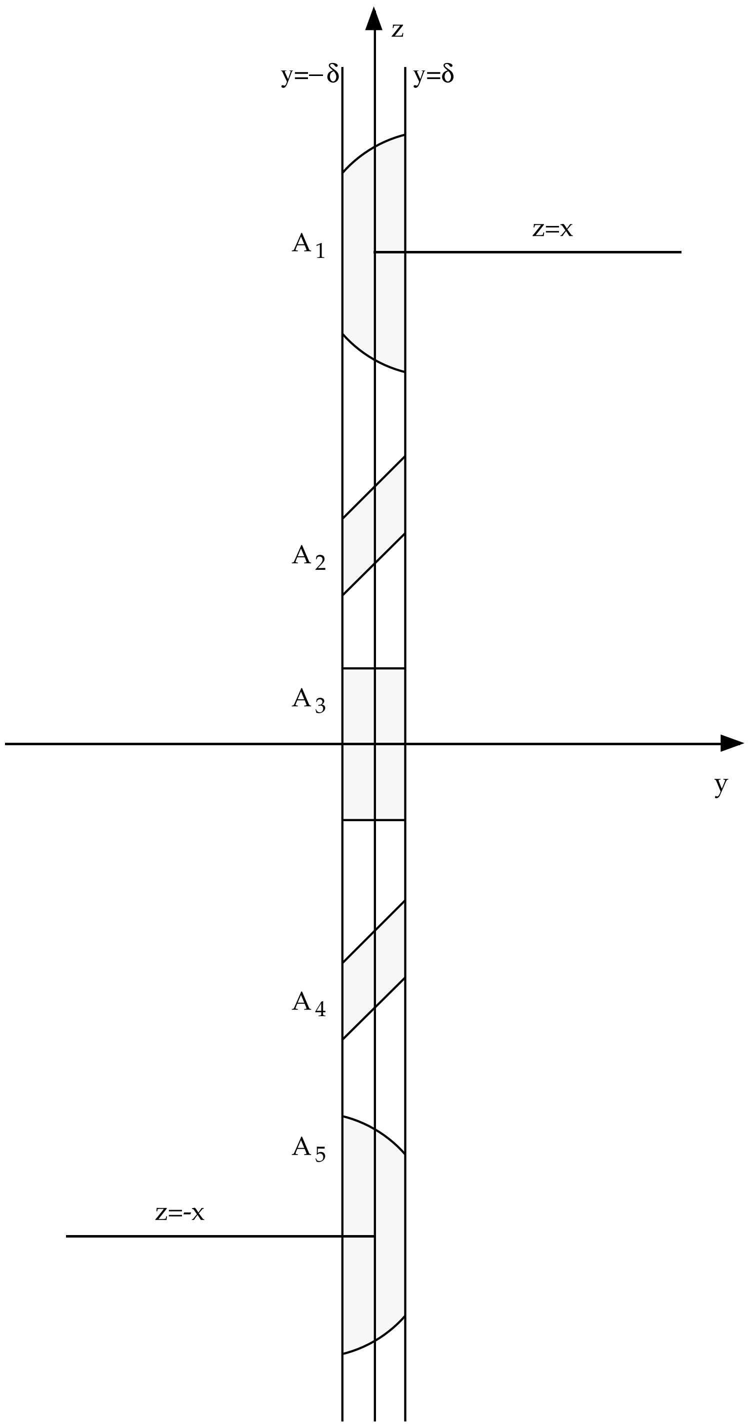

Figure 2.1: Section of the sets $A_1, \ldots, A_5$ at $x = \text{constant}$.

so that the sets $A_1,\dots,A_5$ are pairwise disjoint.

For every $(x,y,z)\in U\times\mathbb{R}$, let us define the vector $\varphi(x,y,z)=(\varphi^x,\varphi^y,\varphi^z)(x,y,z)\in\mathbb{R}^3$ as follows:

$$\begin{cases} \left(\dfrac{2(\varepsilon-y)}{\sqrt{(\varepsilon-y)^2+(z-x)^2}},\dfrac{-2(z-x)}{\sqrt{(\varepsilon-y)^2+(z-x)^2}},1\right) & \text{if } (x,y,z)\in A_1, \\[2ex] \left(0,\lambda,\dfrac{\lambda^2}{4}\right) & \text{if } (x,y,z)\in A_2, \\[2ex] (f(y),0,1) & \text{if } (x,y,z)\in A_3, \\[2ex] \left(0,\lambda,\dfrac{\lambda^2}{4}\right) & \text{if } (x,y,z)\in A_4, \\[2ex] \left(\dfrac{-2(\varepsilon+y)}{\sqrt{(\varepsilon+y)^2+(z+x)^2}},\dfrac{2(z+x)}{\sqrt{(\varepsilon+y)^2+(z+x)^2}},1\right) & \text{if } (x,y,z)\in A_5, \\[2ex] (0,0,1) & \text{otherwise,} \end{cases}$$

where

$$f(y):=-\frac{1}{h}\left(\int_0^{\alpha(y)}\frac{\varepsilon-y}{\sqrt{t^2+(\varepsilon-y)^2}}\,dt-\int_0^{\alpha(-y)}\frac{\varepsilon+y}{\sqrt{t^2+(\varepsilon+y)^2}}\,dt\right).$$

Note that $A_1\cup A_5$ is an open neighbourhood of $\operatorname{graph} w\cap(U\times\mathbb{R})$. The purpose of the definition of φ in A_1 and A_5 (see Fig. 2.2) is to provide a divergence-free vectorfield satisfying condition (a2) of Section 1.3 and such that

$$\begin{aligned} \varphi^y(x,0,z)>0 &\qquad \text{for } |z|<x, \\ \varphi^y(x,0,z)<0 &\qquad \text{for } |z|>x. \end{aligned}$$

These properties are crucial in order to obtain (b1) and (b2) simultaneously.

The role of A_2 and A_4 is to give the main contribution to the integral in (b2). To explain this fact, suppose, for a moment, that $\varepsilon=0$; in this case we would have $A_1=A_5=\emptyset$ and

$$\int_{-x}^{x}\varphi^y(x,0,z)\,dz=1,$$

so that the y-component of equality (b2) would be satisfied.

The purpose of the definition of φ in A_3 is to correct the x-component of φ, in order to obtain (b1).

We shall prove that, for a suitable choice of δ, the vectorfield φ is a calibration for w in the rectangle U.

Inequality (a1) is clearly satisfied in all regions: the only nontrivial case is A_3, where using (2.6) we have

$$|f(y)|\le\frac{4\,(\alpha(y)+\alpha(-y))}{x_0-3\varepsilon}\le\frac{8\sqrt{3}\varepsilon}{x_0-3\varepsilon}<2.$$

On the graph of w we have

$$\varphi(x,y,w(x,y))=\begin{cases}(2,0,1) & \text{if } y>0,\\ (-2,0,1) & \text{if } y<0,\end{cases}$$

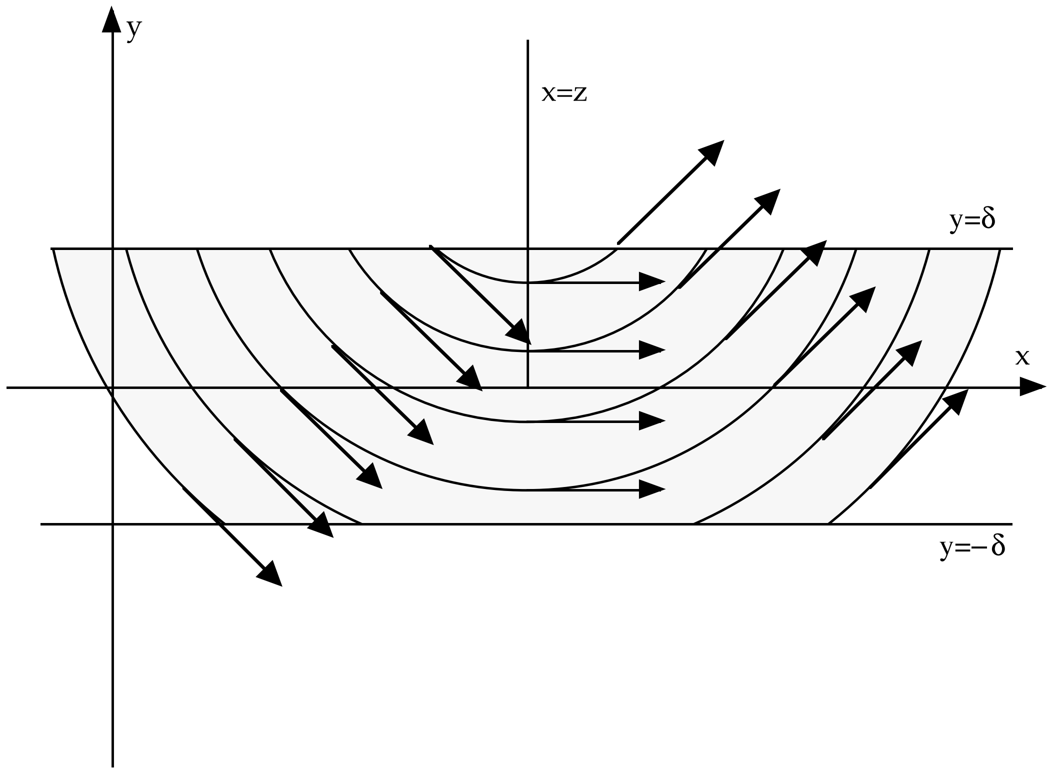

Figure 2.2: Section of the set A_1 at $z = \text{constant}$.

so condition (a2) is satisfied.

Note that for a given $z \in \mathbb{R}$ we have

$$\partial_x \varphi^x(x,y,z) + \partial_y \varphi^y(x,y,z) = 0 \tag{2.8}$$

for every (x,y) such that $(x,y,z) \in A_1 \cup A_5$. This implies φ is divergence-free in $A_1 \cup A_5$. Moreover $\text{div}\varphi = 0$ in the other sets A_i, and the normal component of φ is continuous across ∂A_i: the choice of $\kappa(\lambda)$ ensures that this property holds for ∂A_2 and ∂A_4 (see Fig. 2.3). Therefore φ is divergence-free in the sense of distributions in $U \times \mathbb{R}$.

We now compute

$$\int_{-x}^{x} \varphi^y(x,y,z)\, dz.$$

Let us fix y with $|y| < \delta$. Since $\varphi^y(x,y,z)$ depends on $z - x$, we have

$$\int_{x-\alpha(y)}^{x} \varphi^y(x,y,z)\, dz = \int_{x}^{x+\alpha(y)} \varphi^y(\xi,y,x)\, d\xi. \tag{2.9}$$

Using (2.8) and applying the divergence theorem to the curvilinear triangle

$$T = \{(\xi,\eta) \in \mathbb{R}^2 \,:\, \xi > x,\ \eta < y,\ (\varepsilon - \eta)^2 + (x - \xi)^2 < 4\varepsilon^2\}$$

(see Fig. 2.4), we obtain

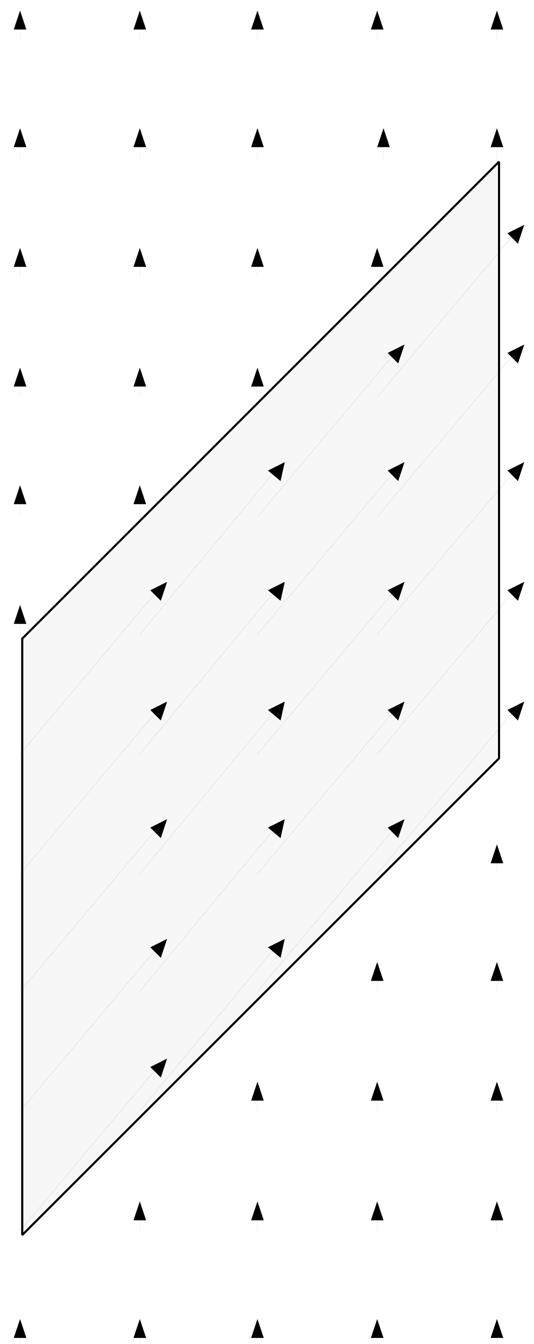

Figure 2.3: Section of the set A_2 at $x =$ constant.

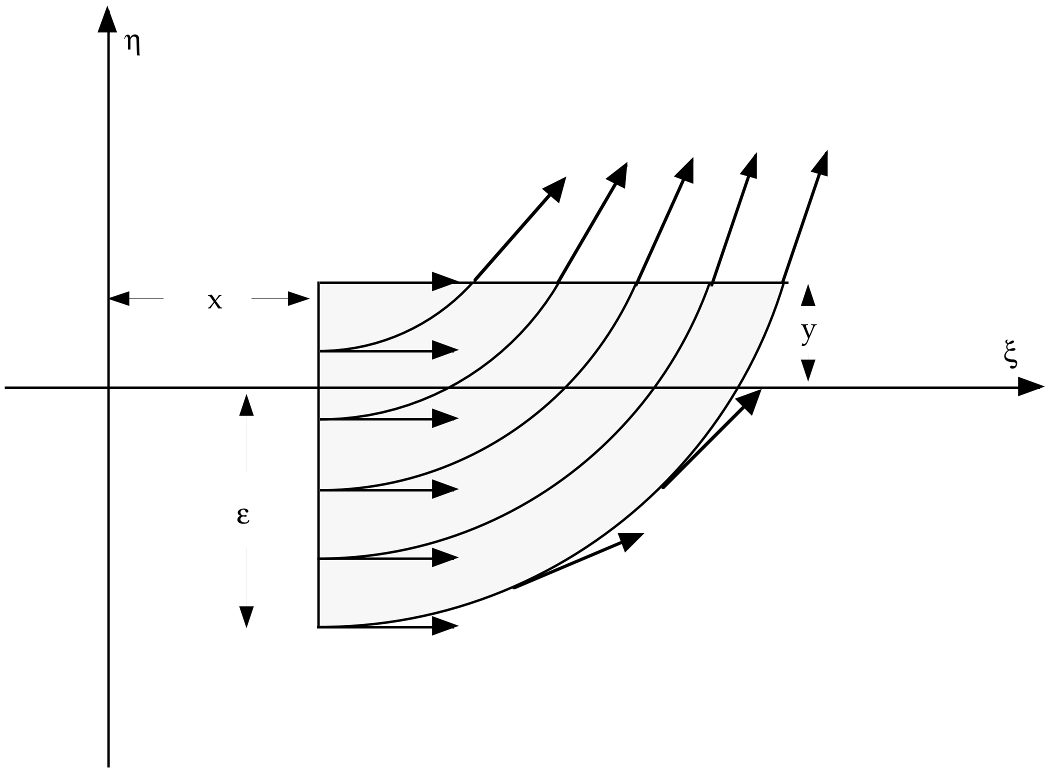

Figure 2.4: The curvilinear triangle T.

$$\int_{x}^{x+\alpha(y)} \varphi^{y}(\xi, y, x)\, d\xi = \int_{-\varepsilon}^{y} \varphi^{x}(x, \eta, x)\, d\eta = 2(y+\varepsilon). \tag{2.10}$$

From (2.9) and (2.10), we get

$$\int_{x-\alpha(y)}^{x} \varphi^{y}(x, y, z)\, dz = 2(y+\varepsilon). \tag{2.11}$$

Similarly we can prove that

$$\int_{-x}^{-x+\alpha(-y)} \varphi^{y}(x, y, z)\, dz = 2(-y+\varepsilon). \tag{2.12}$$

Using the definition of φ in A_2, A_3, A_4, we obtain

$$\int_{-x}^{x} \varphi^{y}(x, y, z)\, dz = 1. \tag{2.13}$$

On the other hand, by the definition of f, we have immediately that

$$\int_{-x}^{x} \varphi^{x}(x, y, z)\, dz = 0. \tag{2.14}$$

From these equalities it follows in particular that condition (b2) is satisfied on the jump set $S_w \cap U = \{(x,y) \in U : y = 0\}$.

Let us begin now the proof of (b1). Let us fix $(x,y) \in U$. For every $t_1 < t_2$ we set

$$I(t_1,t_2) := \int_{t_1}^{t_2} (\varphi^x, \varphi^y)(x,y,z)\, dz.$$

It is enough to consider the case $-x - \alpha(-y) \le t_1 \le t_2 \le x - \alpha(y)$. We can write

$$\begin{aligned}
I(t_1,t_2) &= I(t_1,-x) + I(-x,x) + I(x,t_2),\\
I(t_1,-x) &= I(t_1 \wedge (-x+\alpha(-y)), -x) + I(t_1 \vee (-x+\alpha(-y)), -x+\alpha(-y)),\\
I(x,t_2) &= I(x, t_2 \vee (x-\alpha(y))) + I(x-\alpha(y), t_2 \wedge (x-\alpha(y))).
\end{aligned}$$

Therefore

$$\begin{aligned}
I(t_1,t_2) = I(-x,x) &+ I(t_1 \wedge (-x+\alpha(-y)), -x) + I(x, t_2 \vee (x-\alpha(y)))\\
&+ I(t_1 \vee (-x+\alpha(-y)), t_2 \wedge (x-\alpha(y))) - I(-x+\alpha(-y), x-\alpha(y)). \qquad (2.15)
\end{aligned}$$

Let B be the ball of radius 4ε centred at $(0,-4\varepsilon)$. We want to prove that

$$I(x,t) \in \overline{B} \qquad (2.16)$$

for every t with $x - \alpha(y) \le t \le x + \alpha(y)$. Let us denote the components of $I(x,t)$ by a^x and a^y. Arguing as in the proof of (2.11), we get the identity

$$a^y = 2(\varepsilon - y) - 2\sqrt{(t-x)^2 + (\varepsilon - y)^2} \le 0.$$

As $|\varphi^x| \le 2$, we have also

$$(a^x)^2 \le 4(t-x)^2 = (2(\varepsilon - y) - a^y)^2 - 4(\varepsilon - y)^2.$$

From these estimates it follows that

$$(a^x)^2 + (a^y + 4\varepsilon)^2 \le 16\varepsilon^2,$$

which proves (2.16). In the same way we can prove that

$$I(t,-x) \in \overline{B} \qquad (2.17)$$

for every t with $-x - \alpha(-y) \le t \le -x + \alpha(-y)$.

If $f(y) \ge 0$, we define

$$C := ([0, 2hf(y)] \times [0, \tfrac{1}{2} - 2\varepsilon]) \cup (\{2hf(y)\} \times [0, 1-4\varepsilon]);$$

if $f(y) \le 0$, we simply replace $[0, 2hf(y)]$ by $[2hf(y), 0]$. ¿From the definition of φ in A_2, A_3, A_4, it follows that

$$I(-x+\alpha(-y), x-\alpha(y)) = (2hf(y), 1-4\varepsilon) \qquad (2.18)$$

and

$$I(s_1,s_2) \in C \qquad (2.19)$$

for $-x+\alpha(-y) \le s_1 \le s_2 \le x-\alpha(y)$. Let $D := C - (2hf(y), 1-4\varepsilon)$, i.e.,

$$D = ([-2hf(y),0]\times[-1+4\varepsilon, -\tfrac{1}{2}+2\varepsilon]) \cup (\{0\}\times[-1+4\varepsilon,0]),$$

for $f(y) \ge 0$; the interval $[-2hf(y),0]$ is replaced by $[0,-2hf(y)]$ when $f(y) \le 0$. ¿From (2.15), (2.13), (2.14), (2.16), (2.17), (2.18) and (2.19) we obtain

$$I(t_1,t_2) \in (0,1) + 2\overline{B} + D. \tag{2.20}$$

As $f(0)=0$, we can choose δ so that (2.7) is satisfied and

$$|2hf(y)| = \frac{x_0 - 3\varepsilon}{2}|f(y)| \le \varepsilon \tag{2.21}$$

for $|y| < \delta$. It is then easy to see that, by (2.6), the set $(0,1)+2\overline{B}+D$ is contained in the unit ball centred at $(0,0)$. So that (2.20) implies (b1). □

Remark 2.2 The assumption $(x_0,y_0) \neq (0,0)$ in Theorem 2.1 cannot be dropped. Indeed, there is no neighbourhood U of $(0,0)$ such that w is a Dirichlet minimizer of the Mumford-Shah functional in U.

To see this fact, let ψ be a function defined on the square $Q = (-1,1)\times(-1,1)$ satisfying the boundary condition $\psi = w$ on ∂Q and such that $S_\psi = ((-1,-1/2)\cup(1/2,1))\times\{0\}$. For every ε, let ψ_ε be the function defined on $Q_\varepsilon := \varepsilon Q$ by $\psi_\varepsilon(x,y) := \varepsilon\psi(x/\varepsilon, y/\varepsilon)$. Note that ψ_ε satisfies the boundary condition $\psi_\varepsilon = w$ on ∂Q_ε. Let us compute the Mumford-Shah functional for ψ_ε on Q_ε:

$$\int_{Q_\varepsilon} |\nabla\psi_\varepsilon|^2 dx\,dy + \mathcal{H}^1(S_{\psi_\varepsilon}) = \varepsilon^2 \int_Q |\nabla\psi|^2 dx\,dy + \varepsilon.$$

Since

$$\int_{Q_\varepsilon} |\nabla w|^2 dx\,dy + \mathcal{H}^1(S_w) = 4\varepsilon^2 + 2\varepsilon,$$

we have

$$\int_{Q_\varepsilon} |\nabla\psi_\varepsilon|^2 dx\,dy + \mathcal{H}^1(S_{\psi_\varepsilon}) < \int_{Q_\varepsilon} |\nabla w|^2 dx\,dy + \mathcal{H}^1(S_w)$$

for ε sufficiently small. □

The construction shown in the proof of Theorem 2.1 can be easily adapted to define a calibration for the function w in (2.5).

Theorem 2.3 *Let $w : \mathbb{R}^2 \to \mathbb{R}$ be the function defined by*

$$w(x,y) := \begin{cases} x+1 & \text{if } y > 0, \\ x & \text{if } y < 0. \end{cases}$$

Then every point $(x_0,y_0) \in \mathbb{R}^2$ has an open neighbourhood U such that w is a Dirichlet minimizer in U of the Mumford-Shah functional (2.1).

Proof. – The result follows by Example 4.10 of [2] if $y_0 \neq 0$. We consider now the case $y_0 = 0$; we will construct a local calibration of w near $(x_0,0)$, using the same technique as in Theorem 2.1. We give only the new definitions of the sets $A_1, \dots, A_5$ and of the function φ, and leave to the reader the verification of the fact that this function is a calibration for suitable values of the involved parameters.

Let us fix $\varepsilon > 0$ such that

$$0 < \varepsilon < \frac{1}{24}, \qquad 0 < \varepsilon < \frac{1}{32}. \tag{2.22}$$

For $0 < \delta < \varepsilon$ we consider the open rectangle

$$U := \{(x,y) \in \mathbb{R}^2 : |x - x_0| < \varepsilon, \, |y| < \delta\}$$

and the following subsets of $U \times \mathbb{R}$

$$\begin{aligned}
A_1 &:= \{(x,y,z) \in U \times \mathbb{R} : x + 1 - \alpha(y) < z < x + 1 + \alpha(y)\},\\
A_2 &:= \{(x,y,z) \in U \times \mathbb{R} : b + \kappa(\lambda)\, y + 3h < z < b + \kappa(\lambda)\, y + 4h\},\\
A_3 &:= \{(x,y,z) \in U \times \mathbb{R} : x_0 + 3\varepsilon + 2h < z < x_0 + 3\varepsilon + 3h\},\\
A_4 &:= \{(x,y,z) \in U \times \mathbb{R} : b + \kappa(\lambda)\, y < z < b + \kappa(\lambda)\, y + h\},\\
A_5 &:= \{(x,y,z) \in U \times \mathbb{R} : x - \alpha(-y) < z < x + \alpha(-y)\},
\end{aligned}$$

where

$$\alpha(y) := \sqrt{4\varepsilon^2 - (\varepsilon - y)^2},$$

$$h := \frac{1 - 6\varepsilon}{5}, \qquad \kappa(\lambda) := \frac{\lambda}{4} - \frac{1}{\lambda}, \qquad b := x_0 + 3\varepsilon + \kappa(\lambda)\,\delta, \qquad \lambda := \frac{1 - 4\varepsilon}{2h}.$$

We will assume that

$$\delta < \frac{1 - 6\varepsilon}{10|\kappa(\lambda)|}, \tag{2.23}$$

so that the sets $A_1, \ldots, A_5$ are pairwise disjoint.

For every $(x,y,z) \in U \times \mathbb{R}$, let us define the vector $\varphi(x,y,z) \in \mathbb{R}^3$ as follows:

$$\begin{cases}
\left(\dfrac{2(\varepsilon - y)}{\sqrt{(\varepsilon - y)^2 + (z - x - 1)^2}}, \dfrac{-2(z - x - 1)}{\sqrt{(\varepsilon - y)^2 + (z - x - 1)^2}}, 1 \right) & \text{if } (x,y,z) \in A_1,\\[2ex]
\left(0, \lambda, \dfrac{\lambda^2}{4}\right) & \text{if } (x,y,z) \in A_2,\\[2ex]
(f(y), 0, 1) & \text{if } (x,y,z) \in A_3,\\[2ex]
\left(0, \lambda, \dfrac{\lambda^2}{4}\right) & \text{if } (x,y,z) \in A_4,\\[2ex]
\left(\dfrac{2(\varepsilon + y)}{\sqrt{(\varepsilon + y)^2 + (z - x)^2}}, \dfrac{2(z - x)}{\sqrt{(\varepsilon + y)^2 + (z - x)^2}}, 1 \right) & \text{if } (x,y,z) \in A_5,\\[2ex]
(0,0,1) & \text{otherwise,}
\end{cases}$$

where

$$f(y) := -\frac{2}{h} \left(\int_0^{\alpha(y)} \frac{\varepsilon - y}{\sqrt{t^2 + (\varepsilon - y)^2}}\, dt + \int_0^{\alpha(-y)} \frac{\varepsilon + y}{\sqrt{t^2 + (\varepsilon + y)^2}}\, dt \right)$$

for every $|y| < \delta$. □

2.2 The general case

In this section we denote by Ω a ball in $\mathbb{R}^2$ centred at $(0,0)$ and we consider as u in (2.2) and in (2.3) a generic harmonic function with normal derivative vanishing on S. We add the technical assumption that the first and second order tangential derivatives of u are not zero on S.

Theorem 2.4 *Let $u:\Omega\to\mathbb{R}$ be a harmonic function such that $\partial_y u(x,0)=0$ for $(x,0)\in\Omega$, and let $w:\Omega\to\mathbb{R}$ be the function defined by*

$$w(x,y):=\begin{cases} u(x,y) & \text{for } y>0,\\ -u(x,y) & \text{for } y<0.\end{cases}$$

Assume that $u_0:=u(0,0)\neq 0$, $\partial_x u(0,0)\neq 0$, and $\partial^2_{xx}u(0,0)\neq 0$. Then there exists an open neighbourhood U of $(0,0)$ such that w is a Dirichlet minimizer in U of the Mumford-Shah functional (2.1).

PROOF. – We may assume $u(0,0)>0$ and $\partial_x u(0,0)>0$. We shall give the proof only for $\partial^2_{xx}u(0,0)>0$, and we shall explain at the end the modification needed for $\partial^2_{xx}u(0,0)<0$. Let $v:\Omega\to\mathbb{R}$ be the harmonic conjugate of u that vanishes on $y=0$, i.e., the function satisfying $\partial_x v(x,y)=-\partial_y u(x,y)$, $\partial_y v(x,y)=\partial_x u(x,y)$, and $v(x,0)=0$.

Consider a small neighbourhood U of $(0,0)$ such that the map $\Phi(x,y):=(u(x,y),v(x,y))$ is invertible on U and $\partial_x u>0$ on U. We call Ψ the inverse function $(u,v)\mapsto(\xi(u,v),\eta(u,v))$, which is defined in the neighbourhood $V:=\Phi(U)$ of $(u_0,0)$. Note that, if U is small enough, then $\eta(u,v)=0$ if and only if $v=0$. Moreover,

$$D\Psi=\begin{pmatrix}\partial_u\xi & \partial_v\xi\\ \partial_u\eta & \partial_v\eta\end{pmatrix}=\frac{1}{|\nabla u|^2}\begin{pmatrix}\partial_x u & \partial_x v\\ \partial_y u & \partial_y v\end{pmatrix}, \tag{2.24}$$

where, in the last formula, all functions are computed at $(x,y)=\Psi(u,v)$, and so $\partial_u\xi=\partial_v\eta$, $\partial_v\xi=-\partial_u\eta$ and $\partial_u\eta(u,0)=0$, $\partial_v\eta(u,0)>0$. In particular, ξ and η are harmonic, and

$$\partial^2_{uu}\eta(u,0)=0,\qquad \partial^2_{vv}\eta(u,0)=0. \tag{2.25}$$

On U we will use the coordinate system (u,v) given by Φ. By (2.24) the canonical basis of the tangent space to U at a point (x,y) is given by

$$\tau_u=\frac{\nabla u}{|\nabla u|^2},\qquad \tau_v=\frac{\nabla v}{|\nabla v|^2}. \tag{2.26}$$

For every $(u,v)\in V$, let $G(u,v)$ be the matrix associated with the first fundamental form of U in the coordinate system (u,v), and let $g(u,v)$ be its determinant. By (2.24) and (2.26),

$$g=((\partial_u\eta)^2+(\partial_v\eta)^2)^2=\frac{1}{|\nabla u(\Psi)|^4}. \tag{2.27}$$

We set $\gamma(u,v):=\sqrt[4]{g(u,v)}$.

The calibration $\varphi(x,y,z)$ on $U\times\mathbb{R}$ will be written as

$$\varphi(x,y,z)=\frac{1}{\gamma^2(u(x,y),v(x,y))}\phi(u(x,y),v(x,y),z). \tag{2.28}$$

We will adopt the following representation for $\phi:V\times\mathbb{R}\to\mathbb{R}^3$:

$$\phi(u,v,z)=\phi^u(u,v,z)\tau_u+\phi^v(u,v,z)\tau_v+\phi^z(u,v,z)e_z, \tag{2.29}$$

where e_z is the third vector of the canonical basis of $\mathbb{R}^3$, and τ_u, τ_v are computed at the point $\Psi(u,v)$. We now reformulate the conditions of Section 1.3 in this new coordinate system. It is known from differential geometry (see, e.g., [9, Proposition 3.5]) that, if $X = X^u\tau_u + X^v\tau_v$ is a vectorfield on U, then the divergence of X is given by

$$\mathrm{div}X = \frac{1}{\gamma^2}(\partial_u(\gamma^2 X^u) + \partial_v(\gamma^2 X^v)). \tag{2.30}$$

Using (2.26), (2.27), (2.28), (2.29), and (2.30) it turns out that φ is a calibration if the following conditions are satisfied:

(a1) $(\phi^u(u,v,z))^2 + (\phi^v(u,v,z))^2 \leq 4\phi^z(u,v,z)$ for every $(u,v,z) \in V\times\mathbb{R}$;

(a2) $\phi^u(u,v,\pm u) = \pm 2$, $\phi^v(u,v,\pm u) = 0$, and $\phi^z(u,v,\pm u) = 1$ for every $(u,v) \in V$;

(b1) $\displaystyle\left(\int_s^t \phi^u(u,v,z)\,dz\right)^2 + \left(\int_s^t \phi^v(u,v,z)\,dz\right)^2 \leq \gamma^2(u,v)$ for every $(u,v) \in V$, $s,t \in \mathbb{R}$;

(b2) $\displaystyle\int_{-u}^{u} \phi^u(u,0,z)\,dz = 0$ and $\displaystyle\int_{-u}^{u} \phi^v(u,0,z)\,dz = \gamma(u,0)$ for every $(u,0) \in V$;

(c1) $\partial_u\phi^u + \partial_v\phi^v + \partial_z\phi^z = 0$ for every $(u,v,z) \in V\times\mathbb{R}$.

Given suitable parameters $\varepsilon > 0$, $h > 0$, $\lambda > 0$, that will be chosen later, and assuming

$$V = \{(u,v) : |u - u_0| < \delta, |v| < \delta\}, \tag{2.31}$$

with $\delta < \varepsilon$, we consider the following subsets of $V\times\mathbb{R}$

$$\begin{aligned}
A_1 &:= \{(u,v,z) \in V\times\mathbb{R} : u - \alpha(v) < z < u + \alpha(v)\}, \\
A_2 &:= \{(u,v,z) \in V\times\mathbb{R} : 3h + \beta(u,v) < z < 3h + \beta(u,v) + 1/\lambda\}, \\
A_3 &:= \{(u,v,z) \in V\times\mathbb{R} : -h < z < h\}, \\
A_4 &:= \{(u,v,z) \in V\times\mathbb{R} : -3h + \beta(u,v) - 1/\lambda < z < -3h + \beta(u,v)\}, \\
A_5 &:= \{(u,v,z) \in V\times\mathbb{R} : -u - \alpha(-v) < z < -u + \alpha(-v)\},
\end{aligned}$$

where

$$\alpha(v) := \sqrt{4\varepsilon^2 - (\varepsilon - v)^2},$$

and β is a suitable smooth function satisfying $\beta(u,0) = 0$, which will be defined later. It is easy to see that, if ε and h are sufficiently small, while λ is sufficiently large, then the sets $A_1, \dots, A_5$ are pairwise disjoint, provided δ is small enough. Moreover, since $\gamma(u,0) = \partial_v\eta(u,0) > 0$, by continuity we may assume that

$$\gamma(u,v) > 128\varepsilon \qquad \text{and} \qquad \partial_v\eta(u,v) > 8\varepsilon \tag{2.32}$$

for every $(u,v) \in V$.

For $(u,v)\in V$ and $z\in\mathbb{R}$ the vector $\phi(u,v,z)$ introduced in (2.28) is defined as follows:

$$\begin{cases} \dfrac{2(\varepsilon-v)}{\sqrt{(\varepsilon-v)^2+(z-u)^2}}\tau_u-\dfrac{2(z-u)}{\sqrt{(\varepsilon-v)^2+(z-u)^2}}\tau_v+e_z & \text{in } A_1,\\ -\lambda\sigma(u,v)\dfrac{v}{\sqrt{(u-a)^2+v^2}}\tau_u+\lambda\sigma(u,v)\dfrac{u-a}{\sqrt{(u-a)^2+v^2}}\tau_v+\mu e_z & \text{in } A_2,\\ f(v)\tau_u+e_z & \text{in } A_3,\\ -\lambda\sigma(u,v)\dfrac{v}{\sqrt{(u-a)^2+v^2}}\tau_u+\lambda\sigma(u,v)\dfrac{u-a}{\sqrt{(u-a)^2+v^2}}\tau_v+\mu e_z & \text{in } A_4,\\ -\dfrac{2(\varepsilon+v)}{\sqrt{(\varepsilon+v)^2+(z+u)^2}}\tau_u+\dfrac{2(z+u)}{\sqrt{(\varepsilon+v)^2+(z+u)^2}}\tau_v+e_z & \text{in } A_5,\\ e_z & \text{otherwise,} \end{cases}$$

where

$$a<u_0-11\delta,\qquad \mu>0 \tag{2.33}$$

$$f(v):=-\frac{1}{h}\left(\int_0^{\alpha(v)}\frac{(\varepsilon-v)}{\sqrt{t^2+(\varepsilon-v)^2}}\,dt-\int_0^{\alpha(-v)}\frac{(\varepsilon+v)}{\sqrt{t^2+(\varepsilon+v)^2}}\,dt\right),$$

$$\sigma(u,v):=\frac{1}{2}\gamma(a+\sqrt{(u-a)^2+v^2},0)-2\varepsilon. \tag{2.34}$$

We choose β as the solution of the Cauchy problem

$$\begin{cases} \lambda\sigma(u,v)(-v\,\partial_u\beta+(u-a)\partial_v\beta)=(\mu-1)\sqrt{(u-a)^2+v^2},\\ \beta(u,0)=0. \end{cases} \tag{2.35}$$

Since the line $v=0$ is not characteristic for the equation near $(u_0,0)$, there exists a unique solution $\beta\in C^\infty(V)$, provided V is small enough.

In the coordinate system (u,v) the definition of the field ϕ in A_1, A_3, and A_5 is the same as the definition of φ in the proof of Theorem 2.1. The crucial difference is in the definition on the sets A_2 and A_4, where now we are forced to introduce two new parameters a and μ. Note that the definition given in Theorem 2.1 can be regarded as the limiting case as a tends to $+\infty$.

In order to satisfy condition (a1), it is enough to take the parameter μ such that

$$\frac{\lambda^2}{4}\sigma^2(u,v)\le\mu$$

for every $(u,v)\in V$, and require that

$$|f(v)|\le 2. \tag{2.36}$$

Since

$$|f(v)| \leq \frac{\alpha(v)+\alpha(-v)}{h} \leq \frac{4\,\varepsilon}{h}, \tag{2.37}$$

inequality (2.36) is true if we impose

$$2\,\varepsilon \leq h.$$

Looking at the definition of ϕ on A_1 and A_5, one can check that condition (a2) is satisfied.

By direct computations it is easy to see that ϕ satisfies condition (c1) on A_1 and A_5. Similarly, the vectorfield

$$\left(-\frac{v}{\sqrt{(u-a)^2+v^2}}, \frac{u-a}{\sqrt{(u-a)^2+v^2}}\right)$$

is divergence-free; since $(u-a)^2+v^2$ is constant along the integral curves of this field, by construction the same property holds for σ, so that ϕ satisfies condition (c1) in A_2 and A_4.

In A_3, condition (c1) is trivially satisfied.

Note that the normal component of ϕ is continuous across each ∂A_i: for the region A_3 this continuity is guaranteed by our choice of β. This implies that (c1) is satisfied in the sense of distributions on $V \times \mathbb{R}$.

Arguing as in the proof of (2.11), (2.12), (2.14) in Theorem 2.1, we find that for every $(u,v) \in V$

$$\int_{-u}^{-u+\alpha(-v)} \phi^u(u,v,z)\,dz + \int_{-h}^{h} \phi^u(u,v,z)\,dz + \int_{u-\alpha(v)}^{u} \phi^u(u,v,z)\,dz = 0,$$

$$\int_{-u}^{-u+\alpha(-v)} \phi^v(u,v,z)\,dz + \int_{-h}^{h} \phi^v(u,v,z)\,dz + \int_{u-\alpha(v)}^{u} \phi^v(u,v,z)\,dz = 4\varepsilon.$$

Now, it is easy to see that

$$\int_{-u}^{u} \phi^u(u,v,z)\,dz = -2\sigma(u,v)\frac{v}{\sqrt{(u-a)^2+v^2}}, \tag{2.38}$$

$$\int_{-u}^{u} \phi^v(u,v,z)\,dz = 4\varepsilon + 2\sigma(u,v)\frac{u-a}{\sqrt{(u-a)^2+v^2}}; \tag{2.39}$$

since for $v=0$ we have

$$\sigma(u,0) = \frac{1}{2}\gamma(u,0) - 2\varepsilon,$$

condition (b2) is satisfied.

By continuity, if δ is small enough, we have

$$\int_{-u}^{u} \phi^v(u,v,z)\,dz > \frac{7}{8}\gamma(u,v) \tag{2.40}$$

for every $(u,v) \in V$.

From now on, we regard the pair (ϕ^u, ϕ^v) as a vector in $\mathbb{R}^2$. To prove condition (b1) we set

$$I_{\varepsilon,a}(u,v,s,t) := \int_s^t (\phi^u, \phi^v)(u,v,z)\,dz$$

for every $(u,v) \in V$, and for every $s,t \in \mathbb{R}$. We want to compare the behaviour of the functions $|I_{\varepsilon,a}|^2$ and γ^2; to this aim, we define the function

$$d_{\varepsilon,a}(u,v,s,t) := |I_{\varepsilon,a}(u,v,s,t)|^2 - \gamma^2(u,v).$$

We have already shown (condition (b2)) that

$$d_{\varepsilon,a}(u,0,-u,u)=0. \tag{2.41}$$

We start by proving that, if V is sufficiently small, condition (b1) holds for every $(u,v)\in V$, for s close to $-u$ and t close to u. Using the definition of $\phi(u,v,z)$ on A_1 and A_5, one can compute explicitly $d_{\varepsilon,a}(u,v,s,t)$ for $|s+u|\le\alpha(-v)$ and for $|t-u|\le\alpha(v)$. By direct computations one obtains

$$\nabla_{v,s,t}\, d_{\varepsilon,a}(u,0,-u,u)=0 \tag{2.42}$$

for $(u,0)\in V$.

We now want to compute the Hessian matrix $\nabla^2_{v,s,t}\, d_{\varepsilon,a}$ at the point $(u_0,0,-u_0,u_0)$. By (2.34) and (2.27), after some easy computations, we get

$$\partial^2_{vv}\sigma(u,0)=\frac{1}{2(u-a)}\partial_u\gamma(u,0)=\frac{1}{2(u-a)}\partial^2_{uv}\eta(u,0).$$

Using this equality and the explicit expression of $d_{\varepsilon,a}$ near $(u_0,0,-u_0,u_0)$, we obtain

$$\partial^2_{vv}d_{\varepsilon,a}(u_0,0,-u_0,u_0)=-\frac{8\varepsilon}{(u_0-a)^2}(\partial_v\eta(u_0,0)-4\varepsilon)+\frac{2}{u_0-a}\partial_v\eta(u_0,0)\,\partial^2_{uv}\eta(u_0,0)-\partial^2_{vv}(\gamma^2)(u_0,0).$$

Since η and γ do not depend on a and ε, for every ε satisfying (2.32) we can find a so close to u_0 that

$$\partial^2_{vv}d_{\varepsilon,a}(u_0,0,-u_0,u_0)<0. \tag{2.43}$$

Moreover, we easily obtain that

$$\begin{gathered}\partial^2_{tt}d_{\varepsilon,a}(u_0,0,-u_0,u_0)=\partial^2_{ss}d_{\varepsilon,a}(u_0,0,-u_0,u_0)=8-\frac{4}{\varepsilon}\partial_v\eta(u_0,0),\\ \partial^2_{vt}d_{\varepsilon,a}(u_0,0,-u_0,u_0)=\partial^2_{vs}d_{\varepsilon,a}(u_0,0,-u_0,u_0)=-\frac{4}{u_0-a}(\partial_v\eta(u_0,0)-4\varepsilon),\\ \partial^2_{st}d_{\varepsilon,a}(u_0,0,-u_0,u_0)=8.\end{gathered}$$

By the above expressions, it follows that

$$\det\begin{pmatrix}\partial^2_{vv}d_{\varepsilon,a} & \partial^2_{vt}d_{\varepsilon,a}\\ \partial^2_{vt}d_{\varepsilon,a} & \partial^2_{tt}d_{\varepsilon,a}\end{pmatrix}(u_0,0,-u_0,u_0)=\frac{16}{(u_0-a)^2}\partial_v\eta(u_0,0)(\partial_v\eta(u_0,0)-4\varepsilon)+\frac{c_1(\varepsilon)}{u_0-a}+c_2(\varepsilon),$$

where $c_1(\varepsilon)$, $c_2(\varepsilon)$ are two constants depending only on ε. Then, if ε satisfies (2.32), a can be chosen so close to u_0 that

$$\det\begin{pmatrix}\partial^2_{vv}d_{\varepsilon,a} & \partial^2_{vt}d_{\varepsilon,a}\\ \partial^2_{vt}d_{\varepsilon,a} & \partial^2_{tt}d_{\varepsilon,a}\end{pmatrix}(u_0,0,-u_0,u_0)>0. \tag{2.44}$$

At last, the determinant of the Hessian matrix of $d_{\varepsilon,a}$ at $(u_0,0,-u_0,u_0)$ is given by

$$\det\nabla^2_{v,s,t}\, d_{\varepsilon,a}(u_0,0,-u_0,u_0)=\frac{32}{\varepsilon^2(u_0-a)}(\partial_v\eta(u_0,0))^2\partial^2_{uv}\eta(u_0,0)(\partial_v\eta(u_0,0)-4\varepsilon)+c_3(\varepsilon),$$

where $c_3(\varepsilon)$ is a constant depending only on ε. Since, by (2.24),

$$\partial^2_{uv}\eta(u_0,0)=-\frac{\partial^2_{xx}u(0,0)}{(\partial_x u(0,0))^3},$$

given ε satisfying (2.32), we can choose a so close to u_0 that

$$\det \nabla^2_{v,s,t}\, d_{\varepsilon,a}(u_0, 0, -u_0, u_0) < 0. \tag{2.45}$$

By (2.43), (2.44), and (2.45), we can conclude that, by a suitable choice of the parameters, the Hessian matrix of $d_{\varepsilon,a}$ (with respect to v, s, t) at $(u_0, 0, -u_0, u_0)$ is negative definite. This fact, with (2.41) and (2.42), allows us to state the existence of a constant $\tau > 0$ such that

$$d_{\varepsilon,a}(u, v, s, t) < 0 \tag{2.46}$$

for $|s + u_0| < \tau$, $|t - u_0| < \tau$, $(u, v) \in V$, $v \neq 0$, provided V is sufficiently small. So, condition (b1) is satisfied for $|s + u_0| < \tau$ and $|t - u_0| < \tau$. We can assume $\delta < \tau < \alpha(v)$ for every $(u, v) \in V$.

From now on, since at this point the parameters ε, a have been fixed, we simply write I instead of $I_{\varepsilon,a}$. We now study the more general case $|s + u| < \alpha(-v)$ and $|t - u| < \alpha(v)$.

Let us set

$$m_1(u, v) := \max\{|I(u, v, s, t)| : |s + u| \leq \alpha(-v),\ |t - u| \leq \alpha(v),\ |t - u_0| \geq \tau\}.$$

By the definition of $A_1, \ldots, A_5$, for $\rho = \alpha(\delta) + \delta$ we have $(\phi^u, \phi^v) = 0$ on $(V \times [u_0 - \rho, u_0 + \rho]) \setminus A_1$ and $(V \times [-u_0 - \rho, -u_0 + \rho]) \setminus A_5$. This implies that

$$m_1(u, v) = \max\{|I(u, v, s, t)| : |s + u_0| \leq \rho,\ \tau \leq |t - u_0| \leq \rho\}$$

for $(u, v) \in V$. The function m_1, as supremum of a family of continuous functions, is lower semicontinuous. Moreover, m_1 is also upper semicontinuous; indeed, suppose, by contradiction, that there exist two sequences (u_n), (v_n) converging respectively to u, v, such that $(m_1(u_n, v_n))$ converges to a limit $l > m_1(u, v)$; then, there exist (s_n), (t_n) such that

$$|s_n + u_n| \leq \alpha(-v_n), \qquad |t_n - u_n| \leq \alpha(v_n), \qquad |t_n - u_0| \geq \tau, \tag{2.47}$$

and $m_1(u_n, v_n) = |I(u_n, v_n, s_n, t_n)|$. Up to subsequences, we can assume that (s_n), (t_n) converge respectively to s, t such that, by (2.47),

$$|s + u| \leq \alpha(-v), \qquad |t - u| \leq \alpha(v), \qquad |t - u_0| \geq \tau;$$

hence, we have that

$$m_1(u, v) \geq |I(u, v, s, t)| = \lim_{n\to\infty} |I(u_n, v_n, s_n, t_n)| = l,$$

which is impossible since $l > m_1(u, v)$. Therefore, m_1 is continuous.

Let B be the open ball of radius 4ε centred at $(0, -4\varepsilon)$. Arguing as in (2.16), we can prove that

$$I(u, v, u, t) \in B \tag{2.48}$$

whenever $0 < |t - u| \leq \alpha(v)$. In the same way we can prove that

$$I(u, v, s, -u) \in B \tag{2.49}$$

for $0 < |s + u| \leq \alpha(-v)$. We can write

$$I(u, v, s, t) = I(u, v, s, -u) + I(u, v, -u, u) + I(u, v, u, t). \tag{2.50}$$

So, for $|s + u| \leq \alpha(-v)$, $|t - u| \leq \alpha(v)$, and $|t - u_0| \geq \tau$, by (2.49), (2.38), (2.39), and (2.48), we obtain that

$$I(u, 0, s, t) \in (0, \gamma(u, 0)) + B + \overline{B},$$

hence, by (2.32), $I(u,0,s,t)$ belongs to the open ball of radius $\gamma(u,0)$ centred at $(0,0)$, and so, $m_1(u,0) < \gamma(u,0)$. By continuity, if V is small enough,

$$m_1(u,v) < \gamma(u,v) \tag{2.51}$$

for every $(u,v) \in V$.

Analogously, we define

$$m_2(u,v) := \max\{|I(u,v,s,t)| : |s+u| \le \alpha(-v),\ |s+u_0| \ge \tau,\ |t-u| \le \alpha(v),\}.$$

Arguing as in the case of m_1, we can prove that, if V is small enough,

$$m_2(u,v) < \gamma(u,v) \tag{2.52}$$

for every $(u,v) \in V$.

By (2.51), (2.52), and (2.46), we can conclude that $I(u,v,s,t)$ belongs to the ball centred at $(0,0)$ with radius $\gamma(u,v)$, for $|s+u| \le \alpha(-v)$ and $|t-u| \le \alpha(v)$. More precisely, let $E(u,v)$ be the intersection of this ball with the upper half plane bounded by the horizontal straight line passing through the point $(0, \frac{3}{4}\gamma(u,v))$: by (2.50), (2.40), (2.48), (2.49), and (2.32), we deduce that

$$I(u,v,s,t) \in E(u,v) \tag{2.53}$$

for $|s+u| \le \alpha(-v)$ and $|t-u| \le \alpha(v)$.

We can now conclude the proof of (b1). It is enough to consider the case $-u-\alpha(-v) \le s \le t \le u+\alpha(v)$. We can write

$$\begin{aligned} I(u,v,s,t) &= I(u,v,s\wedge(-u+\alpha(-v)), t\vee(u-\alpha(v))) \\ &\quad + I(u,v,s\vee(-u+\alpha(-v)), t\wedge(u-\alpha(v))) - I(u,v,-u+\alpha(-v),u-\alpha(v)). \end{aligned} \tag{2.54}$$

By (2.53), it follows that

$$I(u,v,s\wedge(-u+\alpha(-v)), t\vee(u-\alpha(v))) \in E(u,v). \tag{2.55}$$

Let $C_1(u,v)$ be the parallelogram having three consecutive vertices at the points

$$(2hf(v),0), \qquad (0,0), \qquad \sigma(u,v)\frac{(-v,u-a)}{\sqrt{(u-a)^2+v^2}},$$

let $C_2(u,v)$ be the segment with endpoints

$$(2hf(v),0), \qquad (2hf(v),0) + 2\sigma(u,v)\frac{(-v,u-a)}{\sqrt{(u-a)^2+v^2}},$$

and let $C(u,v) := C_1(u,v) \cup C_2(u,v)$.

From the definition of φ in A_2, A_3, A_4, it follows that

$$I(u,v,-u+\alpha(-v),u-\alpha(v)) = (2hf(v),0) + 2\sigma(u,v)\frac{(-v,u-a)}{\sqrt{(u-a)^2+v^2}} \tag{2.56}$$

and

$$I(u,v,s_1,s_2) \in C(u,v) \tag{2.57}$$

for $-u+\alpha(-v) \le s_1 \le s_2 \le u-\alpha(v)$. Let

$$D(u,v) := C(u,v) - (2hf(v),0) - 2\sigma(u,v)\frac{(-v,u-a)}{\sqrt{(u-a)^2+v^2}}.$$

From (2.54), (2.55), (2.56), and (2.57) we obtain

$$I(u,v,s,t) \in E(u,v) + D(u,v). \tag{2.58}$$

As $|v| < \delta < 10\delta < u-a$ by (2.33), the angle that the segment $C_2(u,v)$ forms with the vertical is less than $\arctan(1/10)$. Moreover, we may assume that the lenght $2\sigma(u,v)$ of the segment $C_2(u,v)$ is less than $\gamma(u,v)$; indeed, this is true for $v=0$ and, by continuity, it remains true if δ is small enough. By (2.32) and (2.37), we have also that $|2hf(v)| \le \gamma(u,v)/16$. Using these properties and simple geometric considerations, it is possible to prove that $E(u,v)+D(u,v)$ is contained in the ball with centre $(0,0)$ and radius $\gamma(u,v)$. This concludes the proof of (b1).

If $\partial^2_{xx}u(0,0) < 0$, it is enough to change the definition of ϕ in the sets A_2 and A_4, as follows:

$$\lambda\sigma(u,v)\frac{v}{\sqrt{(a-u)^2+v^2}}\tau_u + \lambda\sigma(u,v)\frac{a-u}{\sqrt{(a-u)^2+v^2}}\tau_v + \mu e_z,$$

where $a > u_0 + 11\delta$ and

$$\sigma(u,v) := \frac{1}{2}\gamma(a - \sqrt{(a-u)^2+v^2}, 0) - 2\varepsilon.$$

□

Theorem 2.5 *Let $u:\Omega\to\mathbb{R}$ be a harmonic function such that $\partial_y u(x,0)=0$ for $(x,0)\in\Omega$, and let $w:\Omega\to\mathbb{R}$ be the function defined by*

$$w(x,y) := \begin{cases} u(x,y)+1 & \text{for } y>0,\\ u(x,y) & \text{for } y<0.\end{cases}$$

Assume that $\partial_x u(0,0) \ne 0$ and $\partial^2_{xx}u(0,0) \ne 0$. Then there exists an open neighbourhood U of $(0,0)$ such that w is a Dirichlet minimizer in U of the Mumford-Shah functional (2.1).

Proof. – We will write the calibration φ as in (2.28) and we will adopt the representation (2.29) for ϕ. We will use the same technique as in Theorem 2.4. We give only the new definitions of the sets $A_1,\dots,A_5$ and of the function ϕ when $\partial_x u(0,0) > 0$ and $\partial^2_{xx}u(0,0) > 0$, and leave to the reader the verification of the fact that this function is a calibration for suitable values of the involved parameters. The case $\partial^2_{xx}u(0,0) < 0$ can be treated by the changes introduced at the end of Theorem 2.4.

Let $u_0 := u(0,0)$. Given $\varepsilon > 0$, $h > 0$, $\lambda > 0$, and assuming

$$V := \{(u,v) : |u-u_0| < \delta, |v| < \delta\},$$

we consider the following subsets of $V\times\mathbb{R}$

$$\begin{aligned}
A_1 &:= \{(u,v,z) \in V\times\mathbb{R} : u+1-\alpha(v) < z < u+1+\alpha(v)\},\\
A_2 &:= \{(u,v,z) \in V\times\mathbb{R} : 5h+\beta(u,v) < z < 5h+\beta(u,v)+1/\lambda\},\\
A_3 &:= \{(u,v,z) \in V\times\mathbb{R} : 2h < z < 4h\},\\
A_4 &:= \{(u,v,z) \in V\times\mathbb{R} : h+\beta(u,v) < z < h+\beta(u,v)+1/\lambda\},\\
A_5 &:= \{(u,v,z) \in V\times\mathbb{R} : u-\alpha(-v) < z < u+\alpha(-v)\},
\end{aligned}$$

where

$$\alpha(v) := \sqrt{4\varepsilon^2 - (\varepsilon - v)^2},$$

and β is a suitable smooth function satisfying $\beta(u,0) = 0$, which will be defined later. For $(u,v) \in V$ and $z \in \mathbb{R}$ the vector $\phi(u,v,z)$ is defined as follows:

$$\begin{cases} \dfrac{2(\varepsilon - v)}{\sqrt{(\varepsilon-v)^2 + (z-u-1)^2}}\tau_u - \dfrac{2(z-u-1)}{\sqrt{(\varepsilon-v)^2+(z-u-1)^2}}\tau_v + e_z & \text{in } A_1, \\ -\lambda\sigma(u,v)\dfrac{v}{\sqrt{(u-a)^2+v^2}}\tau_u + \lambda\sigma(u,v)\dfrac{u-a}{\sqrt{(u-a)^2+v^2}}\tau_v + \mu e_z & \text{in } A_2, \\ f(v)\tau_u + e_z & \text{in } A_3, \\ -\lambda\sigma(u,v)\dfrac{v}{\sqrt{(u-a)^2+v^2}}\tau_u + \lambda\sigma(u,v)\dfrac{u-a}{\sqrt{(u-a)^2+v^2}}\tau_v + \mu e_z & \text{in } A_4, \\ \dfrac{2(\varepsilon+v)}{\sqrt{(\varepsilon+v)^2+(z-u)^2}}\tau_u + \dfrac{2(z-u)}{\sqrt{(\varepsilon+v)^2+(z-u)^2}}\tau_v + e_z & \text{in } A_5, \\ e_z & \text{otherwise,} \end{cases}$$

where $a < u_0 - 11\delta$, $\mu > 0$,

$$f(v) := -\frac{1}{h}\left(\int_0^{\alpha(v)} \frac{(\varepsilon - v)}{\sqrt{t^2 + (\varepsilon-v)^2}}\,dt + \int_0^{\alpha(-v)} \frac{(\varepsilon+v)}{\sqrt{t^2+(\varepsilon+v)^2}}\,dt\right),$$

$$\sigma(u,v) := \frac{1}{2}\gamma(a + \sqrt{(u-a)^2+v^2}, 0) - 2\varepsilon,$$

and β is the solution of the Cauchy problem (2.35). $\square$

Chapter 3

Calibrations for minimizers with a regular discontinuity set

In this chapter we consider solutions u of the Euler-Lagrange equations for the homogeneous Mumford-Shah functional (2.1) whose discontinuity set is an analytic curve connecting two boundary points.

Section 3.1 is devoted to the proof of the Dirichlet minimality of u in a uniform small neighbourhood of S_u. In Section 3.2 we deal with a different notion of minimality: instead of comparing u with perturbations which can be very large, but concentrated in a fixed small domain, as in Section 3.1, we consider as competitors perturbations of u with L^∞-norm very small outside a small neighbourhood of S_u, but support possibly coinciding with $\overline{\Omega}$. According to this idea we give the following definition.

Definition 3.1 *We say that* u *is a* Dirichlet graph-minimizer *of the Mumford-Shah functional (2.1) in* Ω *if there exists an open neighbourhood* A *of the complete graph* Γ_u *of* u *such that* u *is a Dirichlet* $\overline{A}$*-minimizer of (2.1).*

In Theorem 3.5 we give a sufficient condition for the graph-minimality in terms of the geometrical properties of S_u (namely, the length and the curvature) and of a sort of capacity of S_u with respect to the domain Ω, which is defined in (3.58) and whose qualitative properties are studied in Subsection 3.2.1. We present also a counterexample when the condition of Theorem 3.5 is violated.

In the sequel the following notation and remarks will be useful. Given any subset U of $\mathbb{R}^2$ and $\delta > 0$, we denote by U_δ the δ-neighbourhood of U, defined by

$$U_\delta := \{(x,y) \in \mathbb{R}^2 : \exists (x_0,y_0) \in U \text{ such that } |(x-x_0, y-y_0)| < \delta\}.$$

Let Γ be a smooth curve in Ω. Fix an orientation of Γ and call ν the corresponding normal vectorfield to Γ. If $\xi \mapsto (x(\xi), y(\xi))$ is a parameterization of Γ by the arc-length, then the (signed) curvature is given by

$$\operatorname{curv}\Gamma(\xi) = -\langle (\ddot{x}(\xi), \ddot{y}(\xi)), \nu(\xi)\rangle; \tag{3.1}$$

since the two vectors in (3.1) are parallel, it follows that

$$[\operatorname{curv}\Gamma(\xi)]^2 = (\ddot{x}(\xi))^2 + (\ddot{y}(\xi))^2. \tag{3.2}$$

We will denote the length of Γ by $l(\Gamma)$, and the L^∞-norm of the function $(\operatorname{curv}\Gamma)$ by $k(\Gamma)$.

3.1 The Dirichlet minimality

In this section we prove that, if we assume that S_u is an analytic curve, then the Euler-Lagrange equations guarantee the Dirichlet minimality of u in small domains. This result generalizes Theorems 2.4 and 2.5 of the previous chapter in several directions: the discontinuity set S_u does not need any more to be rectilinear, there are no additional assumptions on the tangential derivatives of u along S_u, and the Dirichlet minimality of u is proved in a uniform neighbourhood of $S_u \cap \overline{\Omega}$.

Let us give and prove the precise statement of the result.

Theorem 3.2 *Let Ω_0 be a connected open subset of $\mathbb{R}^2$ and Γ be a simple analytic curve in Ω_0 connecting two points of the boundary. Let u be a function in $H^1(\Omega_0 \setminus \Gamma)$ with $S_u = \Gamma$, with different traces at every point of Γ, and satisfying the Euler conditions in Ω_0, that is,*

i) u is harmonic in $\Omega_0 \setminus \Gamma$;

ii) $\partial_\nu u = 0$ on Γ;

iii) $|\nabla u^+|^2 - |\nabla u^-|^2 = \operatorname{curv}\Gamma$ at every point of Γ,

where $\nabla u^\pm$ denote the traces of ∇u on Γ. Finally, let Ω be an open set with Lipschitz boundary, compactly contained in Ω_0, such that $\Omega \cap \Gamma \neq \emptyset$. Then there exists an open neighbourhood U of $\Gamma \cap \overline{\Omega}$ contained in Ω_0 such that u is a Dirichlet minimizer in U of the Mumford-Shah functional (2.1).

PROOF. – In the sequel, the intersection $\Gamma \cap \overline{\Omega}$ will be still denoted by Γ. Let

$$\Gamma : \begin{cases} x = x(s) \\ y = y(s) \end{cases}$$

be a parameterization by the arc-length, where s varies in $[0, l(\Gamma)]$; we choose as orientation the normal vectorfield $\nu(s) = (-\dot{y}(s), \dot{x}(s))$.

By Cauchy-Kowalevski theorem (see [24]) there exist an open neighbourhood U of Γ contained in Ω_0 and a harmonic function ξ defined on U such that

$$\xi(\Gamma(s)) = s \qquad \text{and} \qquad \partial_\nu \xi(\Gamma(s)) = 0.$$

We can suppose U simply connected. Let $\eta : U \to \mathbb{R}^2$ be the harmonic conjugate of ξ that vanishes on Γ, i.e., the function satisfying $\partial_x \eta(x,y) = -\partial_y \xi(x,y)$, $\partial_y \eta(x,y) = \partial_x \xi(x,y)$, and $\eta(\Gamma(s)) = 0$.

Taking U smaller if needed, we can suppose that the map $\Phi(x,y) := (\xi(x,y), \eta(x,y))$ is invertible on U. We call Ψ the inverse function $(\xi, \eta) \mapsto (\tilde{x}(\xi,\eta), \tilde{y}(\xi,\eta))$, which is defined in the open set $V := \Phi(U)$. Note that, if U is small enough, then $(\tilde{x}(\xi,\eta), \tilde{y}(\xi,\eta))$ belongs to Γ if and only if $\eta = 0$. Moreover,

$$D\Psi = \begin{pmatrix} \partial_\xi \tilde{x} & \partial_\eta \tilde{x} \\ \partial_\xi \tilde{y} & \partial_\eta \tilde{y} \end{pmatrix} = \frac{1}{|\nabla \xi|^2} \begin{pmatrix} \partial_x \xi & \partial_x \eta \\ \partial_y \xi & \partial_y \eta \end{pmatrix}, \tag{3.3}$$

where, in the last formula, all functions are computed at $(x,y) = \Psi(\xi,\eta)$, and so

$$\partial_\xi \tilde{x} = \partial_\eta \tilde{y} \qquad \text{and} \qquad \partial_\eta \tilde{x} = -\partial_\xi \tilde{y}. \tag{3.4}$$

In particular, $\tilde{x}$ and $\tilde{y}$ are harmonic.

On U we will use the coordinate system (ξ, η) given by Φ. By (3.3) the canonical basis of the tangent space to U at a point (x,y) is given by

$$\tau_\xi = \frac{\nabla \xi}{|\nabla \xi|^2}, \qquad \tau_\eta = \frac{\nabla \eta}{|\nabla \eta|^2}. \tag{3.5}$$

For every $(\xi,\eta)\in V$, let $G(\xi,\eta)$ be the matrix associated with the first fundamental form of U in the coordinate system (ξ,η), and let $g(\xi,\eta)$ be its determinant. By (3.3) and (3.5),

$$g=((\partial_\xi\tilde{x})^2+(\partial_\xi\tilde{y})^2)^2=\frac{1}{|\nabla\xi(\Psi)|^4}. \tag{3.6}$$

We set $\gamma(\xi,\eta)=\sqrt[4]{g(\xi,\eta)}$.

From now on we will assume that V is symmetric with respect to $\{(\xi,\eta)\in\Phi(U):\eta=0\}$.

Note that we can write the function u in this new coordinate system as

$$u(\xi,\eta)=\begin{cases}u_1(\xi,\eta) & \text{if } (\xi,\eta)\in V,\ \eta<0,\\ u_2(\xi,\eta) & \text{if } (\xi,\eta)\in V,\ \eta>0,\end{cases}$$

where we can suppose that u_1 and u_2 are defined in V (indeed, u_1 is a priori defined only on the set $\{(\xi,\eta)\in V:\eta<0\}$, but it can be extended to V by reflection; an analogous argument applies to u_2), $0<u_1(\xi,0)<u_2(\xi,0)$ for every $(\xi,0)\in V$, and

i) $\partial^2_{\xi\xi}u_i(\xi,\eta)+\partial^2_{\eta\eta}u_i(\xi,\eta)=0$ for $i=1,2$;

ii) $\partial_\eta u_1(\xi,0)=\partial_\eta u_2(\xi,0)=0$;

iii) $(\partial_\xi u_2(\xi,0))^2-(\partial_\xi u_1(\xi,0))^2=\operatorname{curv}\Gamma(\xi)$.

The calibration $\varphi(x,y,z)$ on $U\times\mathbb{R}$ will be written as

$$\varphi(x,y,z)=\frac{1}{\gamma^2(\xi(x,y),\eta(x,y))}\phi(\xi(x,y),\eta(x,y),z), \tag{3.7}$$

where $\phi:V\times\mathbb{R}\to\mathbb{R}^3$ can be represented by

$$\phi(\xi,\eta,z)=\phi^\xi(\xi,\eta,z)\tau_\xi+\phi^\eta(\xi,\eta,z)\tau_\eta+\phi^z(\xi,\eta,z)e_z, \tag{3.8}$$

where e_z is the third vector of the canonical basis of $\mathbb{R}^3$, and τ_ξ, τ_η are computed at the point $\Psi(\xi,\eta)$. We now reformulate the conditions of Section 1.3 in this new coordinate system. It is known from differential geometry (see, e.g., [9, Proposition 3.5]) that, if $X=X^\xi\tau_\xi+X^\eta\tau_\eta$ is a vectorfield on U, then the divergence of X is given by

$$\operatorname{div}X=\frac{1}{\gamma^2}(\partial_\xi(\gamma^2X^\xi)+\partial_\eta(\gamma^2X^\eta)). \tag{3.9}$$

Using (3.5), (3.6), (3.7), (3.8), and (3.9) it turns out that φ is a calibration if the following conditions are satisfied:

(a1) $(\phi^\xi(\xi,\eta,z))^2+(\phi^\eta(\xi,\eta,z))^2\le 4\phi^z(\xi,\eta,z)$ for every $(\xi,\eta,z)\in V\times\mathbb{R}$;

(a2) $\phi^\xi(\xi,\eta,u(\xi,\eta))=2\partial_\xi u(\xi,\eta)$, $\phi^\eta(\xi,\eta,u(\xi,\eta))=2\partial_\eta u(\xi,\eta)$, and $\phi^z(\xi,\eta,u(\xi,\eta))=(\partial_\xi u(\xi,\eta))^2+(\partial_\eta u(\xi,\eta))^2$ for every $(\xi,\eta)\in V$;

(b1) $\left(\int_s^t\phi^\xi(\xi,\eta,z)\,dz\right)^2+\left(\int_s^t\phi^\eta(\xi,\eta,z)\,dz\right)^2\le\gamma^2(\xi,\eta)$ for every $(\xi,\eta)\in V$, $s,t\in\mathbb{R}$;

(b2) $\int_{u_1}^{u_2}\phi^\xi(\xi,0,z)\,dz=0$ and $\int_{u_1}^{u_2}\phi^\eta(\xi,0,z)\,dz=\gamma(\xi,0)=1$ for every $(\xi,0)\in V$;

(c1) $\partial_\xi\phi^\xi+\partial_\eta\phi^\eta+\partial_z\phi^z=0$ for every $(\xi,\eta,z)\in V\times\mathbb{R}$.

Given suitable parameters $\varepsilon > 0$ and $\lambda > 0$, that will be chosen later, we consider the following subsets of $V\times\mathbb{R}$:

$$\begin{aligned}
A_1 &:= \{(\xi,\eta,z) \in V\times\mathbb{R} :\, z < u_1(\xi,\eta) - \varepsilon\},\\
A_2 &:= \{(\xi,\eta,z) \in V\times\mathbb{R} :\, u_1(\xi,\eta) - \varepsilon < z < u_1(\xi,\eta) + \varepsilon\},\\
A_3 &:= \{(\xi,\eta,z) \in V\times\mathbb{R} :\, u_1(\xi,\eta) + \varepsilon < z < \beta_1(\xi,\eta)\},\\
A_4 &:= \{(\xi,\eta,z) \in V\times\mathbb{R} :\, \beta_1(\xi,\eta) < z < \beta_2(\xi,\eta) + 1/\lambda\},\\
A_5 &:= \{(\xi,\eta,z) \in V\times\mathbb{R} :\, \beta_2(\xi,\eta) + 1/\lambda < z < u_2(\xi,\eta) - \varepsilon\},\\
A_6 &:= \{(\xi,\eta,z) \in V\times\mathbb{R} :\, u_2(\xi,\eta) - \varepsilon < z < u_2(\xi,\eta) + \varepsilon\},\\
A_7 &:= \{(\xi,\eta,z) \in V\times\mathbb{R} :\, z > u_2(\xi,\eta) + \varepsilon\},
\end{aligned}$$

where β_1 and β_2 are suitable smooth function such that $u_1(\xi,0) < \beta_1(\xi,0) = \beta_2(\xi,0) < u_2(\xi,0)$, which will be defined later. Since we suppose $u_2 > 0$ on V, if ε is small enough, while λ is sufficiently large, then the sets $A_1,\dots,A_7$ are nonempty and disjoint, provided V is sufficiently small.

The vector $\phi(\xi,\eta,z)$ introduced in (3.7) will be written as

$$\phi(\xi,\eta,z) = (\phi^{\xi\eta}(\xi,\eta,z), \phi^z(\xi,\eta,z)),$$

where $\phi^{\xi\eta}$ is the two-dimensional vector given by the pair (ϕ^ξ,ϕ^η). For $(\xi,\eta) \in V$ and $z \in \mathbb{R}$ we define $\phi(\xi,\eta,z)$ as follows:

$$\begin{cases}
(0,\omega_1(\xi,\eta)) & \text{in } A_1 \cup A_3,\\[2ex]
\left(2\nabla u_1 + 2\frac{z-u_1}{v_1}\nabla v_1, \left|\nabla u_1 + \frac{z-u_1}{v_1}\nabla v_1\right|^2\right) & \text{in } A_2,\\[2ex]
(\lambda\sigma(\xi,\eta)\nabla w, \mu) & \text{in } A_4,\\[2ex]
(0,\omega_2(\xi,\eta)) & \text{in } A_5 \cup A_7,\\[2ex]
\left(2\nabla u_2 + 2\frac{z-u_2}{v_2}\nabla v_2, \left|\nabla u_2 + \frac{z-u_2}{v_2}\nabla v_2\right|^2\right) & \text{in } A_6,
\end{cases}$$

where ∇ denotes the gradient with respect to the variables (ξ,η), the functions v_i are defined by

$$v_1(\xi,\eta) := \varepsilon + M\eta, \quad v_2(\xi,\eta) := \varepsilon - M\eta,$$

and M and μ are positive parameters which will be fixed later, while

$$\omega_i(\xi,\eta) := \frac{\varepsilon^2 M^2}{v_i^2(\xi,\eta)} - (\partial_\xi u_i(\xi,\eta))^2 - (\partial_\eta u_i(\xi,\eta))^2 \tag{3.10}$$

for $i = 1,2$, and for every $(\xi,\eta) \in V$. We choose w as the solution of the Cauchy problem

$$\begin{cases}
\Delta w = 0,\\
w(\xi,0) = -\dfrac{2\varepsilon}{1-2\varepsilon M}\displaystyle\int_0^\xi n(s)(\partial_\xi u_1(s,0) + \partial_\xi u_2(s,0))\,ds,\\
\partial_\eta w(\xi,0) = n(\xi),
\end{cases} \tag{3.11}$$

where n is a positive analytic function that will be chosen later in a suitable way (if V is sufficiently small, w is defined in V). To define σ, we need some further explanations: we call $p(\xi,\eta)$ the solution of the problem

$$\begin{cases} \partial_\eta p(\xi,\eta) = \dfrac{\partial_\xi w}{\partial_\eta w}(p(\xi,\eta),\eta), \\ p(\xi,0) = \xi, \end{cases} \tag{3.12}$$

which is defined in V, provided V is small enough. By applying the Implicit Function theorem, it is easy to see that there exists a function q defined in V (take V smaller, if needed) such that

$$p(q(\xi,\eta),\eta) = \xi. \tag{3.13}$$

At last, we define

$$\sigma(\xi,\eta) := \frac{1}{n(q(\xi,\eta))}(1-2\varepsilon M).$$

We choose β_i, for $i=1,2$, as the solution of the Cauchy problem

$$\begin{cases} \lambda\sigma(\xi,\eta)\partial_\xi w(\xi,\eta)\partial_\xi \beta_i(\xi,\eta) + \lambda\sigma(\xi,\eta)\partial_\eta w(\xi,\eta)\partial_\eta \beta_i(\xi,\eta) - \mu = -\omega_i(\xi,\eta), \\ \beta_i(\xi,0) = \frac{1}{2}(u_1(\xi,0)+u_2(\xi,0)). \end{cases} \tag{3.14}$$

Since the line $\eta=0$ is not characteristic, there exists a unique solution $\beta_i \in C^\infty(V)$, provided V is small enough.

The purpose of the definition of ϕ in A_2 and A_6 is to provide a divergence-free vectorfield satisfying condition (a2) and such that

$$\begin{aligned} &\phi^\eta(\xi,0,z) \geq 0 \quad \text{for } u_1 < z < u_2, \\ &\phi^\eta(\xi,0,z) \leq 0 \quad \text{for } z < u_1 \text{ and } z > u_2. \end{aligned}$$

These properties are crucial in order to obtain (b1) and (b2) simultaneously.

The role of A_4 is to give the main contribution to the integral in (b2). The idea of the construction is to start from the gradient field of a harmonic function w whose normal derivative is positive on the line $\eta=0$, while the tangential derivative is chosen in order to annihilate the ξ-component of ϕ, as required in (b2). Then, we multiply the field by a function σ which is defined first on $\eta=0$ in order to make (b2) true, and then in a neighbourhood of $\eta=0$ by assuming σ constant along the integral curves of the gradient field, so that $\sigma\nabla w$ remains divergence-free.

The other sets A_i are simply regions of transition, where the field is taken purely vertical.

Since

$$\omega_i(\xi,0) = M^2 - (\partial_\xi u_i(\xi,0))^2,$$

condition (a1) is satisfied in $A_1 \cup A_3$ and in $A_5 \cup A_7$ if we require that

$$M > \sup\{|\partial_\xi u_i(\xi,0)| : (\xi,0)\in V,\ i=1,2\},$$

provided V is small enough.

Arguing in a similar way, if we impose that

$$\mu > \sup\left\{\frac{\lambda^2}{4}(1-2\varepsilon M)^2\left(1+\frac{4\varepsilon^2}{(1-2\varepsilon M)^2}(\partial_\xi u_1(\xi,0)+\partial_\xi u_2(\xi,0))^2\right) : (\xi,0)\in V\right\},$$

condition (a1) holds in A_4, provided V is sufficiently small.

In the other cases (a1) is trivial.

Looking at the definition of ϕ on A_2 and A_6, one can check that condition (a2) is satisfied.

Let us prove condition (c1). By Lemma 1.5 it follows that ϕ is divergence-free in $A_2 \cup A_6$, noting that it is constructed starting from the family of harmonic functions $u_i(\xi,\eta)+tv_i(\xi,\eta)$.

In A_4 condition (c1) is true since, as remarked above, ϕ is the product of ∇w with the function σ which is constant along the integral curves of ∇w by construction.

In the other sets condition (c1) is trivially satisfied.

Note that the normal component of ϕ is continuous across each ∂A_i: for the regions A_2, A_6, and for A_4, this continuity is guaranteed by our choice of ω_i and β_i, respectively. This implies that (c1) is satisfied in the sense of distributions on $V\times\mathbb{R}$.

By direct computations we find that

$$\int_{u_1}^{u_2} \phi^\xi \, dz = 2\varepsilon\partial_\xi u_1 + 2\varepsilon\partial_\xi u_2 + \lambda\left(\beta_2-\beta_1+\frac{1}{\lambda}\right)\sigma\partial_\xi w, \tag{3.15}$$

$$\int_{u_1}^{u_2} \phi^\eta \, dz = 2\varepsilon\partial_\eta u_1 + 2\varepsilon\partial_\eta u_2 + \frac{M\varepsilon^2}{\varepsilon+M\eta} + \frac{M\varepsilon^2}{\varepsilon-M\eta} + \lambda\left(\beta_2-\beta_1+\frac{1}{\lambda}\right)\sigma\partial_\eta w, \tag{3.16}$$

for every $(\xi,\eta)\in V$.

By using (3.11) and the definition of σ, we obtain

$$\int_{u_1(\xi,0)}^{u_2(\xi,0)} \phi^\xi(\xi,0,z)\, dz = 0 \tag{3.17}$$

and

$$\int_{u_1(\xi,0)}^{u_2(\xi,0)} \phi^\eta(\xi,0,z)\, dz = 1, \tag{3.18}$$

so condition (b2) is satisfied.

The proof of condition (b1) will be split in two steps: we first prove that condition (b1) holds if s and t respectively belong to a suitable neighbourhood of $u_1(\xi,\eta)$ and $u_2(\xi,\eta)$, whose width is uniform with respect to (ξ,η) in V; then, by a quite simple continuity argument we show that condition (b1) is true if s or t is not too close to $u_1(\xi,\eta)$ or $u_2(\xi,\eta)$ respectively.

For $(\xi,\eta)\in V$ and $s,t\in\mathbb{R}$, we set

$$I(\xi,\eta,s,t) := \int_s^t \phi^{\xi\eta}(\xi,\eta,z)\, dz$$

and we denote its components by I^ξ and I^η.

Step 1.– For a suitable choice of ε and of the function n (see (3.11)) there exists $\delta>0$ such that condition (b1) holds for $|s-u_1(\xi,\eta)|<\delta$, $|t-u_2(\xi,\eta)|<\delta$, and $(\xi,\eta)\in V$, provided V is small enough.

To estimate the vector whose components are given by (3.15) and (3.16), we use suitable polar coordinates. If V is small enough, for every $(\xi,\eta)\in V$ there exist $\rho_{\varepsilon,n}(\xi,\eta)>0$ and $-\pi/2<\theta_{\varepsilon,n}(\xi,\eta)<\pi/2$ such that

$$I^\xi(\xi,\eta,u_1(\xi,\eta),u_2(\xi,\eta)) = \rho_{\varepsilon,n}(\xi,\eta)\sin\theta_{\varepsilon,n}(\xi,\eta), \tag{3.19}$$

$$I^\eta(\xi,\eta,u_1(\xi,\eta),u_2(\xi,\eta)) = \rho_{\varepsilon,n}(\xi,\eta)\cos\theta_{\varepsilon,n}(\xi,\eta). \tag{3.20}$$

In the notation above we have made explicit the dependence on the parameter ε and on the function n which appears in the definition of w (see (3.11)).

In order to prove condition (b1), we want to compare the behaviour of the functions $\rho_{\varepsilon,n}$ and γ for $|\eta|$ small. We have already proved that $\rho_{\varepsilon,n}(\xi,0)=\gamma(\xi,0)=1$; we start computing the first derivative of γ and of $\rho_{\varepsilon,n}$ with respect to the variable η.

CLAIM 1.– There holds that $\partial_\eta(|\nabla_{x,y}\xi(\Psi)|^2)(\xi,0)=-2\,\mathrm{curv}\,\Gamma(\xi)$.

PROOF OF THE CLAIM. By (3.6) we obtain

$$|\nabla_{x,y}\xi(\Psi)|^2=\frac{1}{(\partial_\xi\tilde{x})^2+(\partial_\xi\tilde{y})^2},$$

hence

$$\partial_\eta(|\nabla_{x,y}\xi(\Psi)|^2)=-[(\partial_\xi\tilde{x})^2+(\partial_\xi\tilde{y})^2]^{-2}(2\partial_\xi\tilde{x}\,\partial^2_{\xi\eta}\tilde{x}+2\partial_\xi\tilde{y}\,\partial^2_{\xi\eta}\tilde{y}). \tag{3.21}$$

Using the fact that $(\partial_\xi\tilde{x})^2+(\partial_\xi\tilde{y})^2$ is equal to 1 at $(\xi,0)$, and the equalities in (3.4), we finally get

$$\partial_\eta(|\nabla_{x,y}\xi(\Psi)|^2)(\xi,0)=-2(-\partial_\xi\tilde{x}\,\partial^2_{\xi\xi}\tilde{y}+\partial_\xi\tilde{y}\,\partial^2_{\xi\xi}\tilde{x})=-2\,\mathrm{curv}\,\Gamma(\xi),$$

where the last equality follows from (3.1): therefore the claim is proved.

Since $\gamma=(|\nabla_{x,y}\xi(\Psi)|^2)^{-\frac{1}{2}}$, one has that $\partial_\eta\gamma=-\frac{1}{2}(|\nabla_{x,y}\xi(\Psi)|^2)^{-\frac{3}{2}}\partial_\eta(|\nabla_{x,y}\xi(\Psi)|^2)$; using the previous claim we can conclude that

$$\partial_\eta\gamma(\xi,0)=-\frac{1}{2}\partial_\eta(|\nabla_{x,y}\xi(\Psi)|^2)(\xi,0)=\mathrm{curv}\,\Gamma(\xi).$$

Using the equality

$$\rho^2_{\varepsilon,n}(\xi,\eta)=\left[I^\xi(\xi,\eta,u_1(\xi,\eta),u_2(\xi,\eta))\right]^2+\left[I^\eta(\xi,\eta,u_1(\xi,\eta),u_2(\xi,\eta))\right]^2,$$

we obtain

$$\partial_\eta\rho_{\varepsilon,n}=\frac{1}{\rho_{\varepsilon,n}}\partial_\eta\left(I^\xi(\xi,\eta,u_1,u_2)\right)I^\xi(\xi,\eta,u_1,u_2)+\frac{1}{\rho_{\varepsilon,n}}\partial_\eta\left(I^\eta(\xi,\eta,u_1,u_2)\right)I^\eta(\xi,\eta,u_1,u_2).$$

By (3.17) it follows that the first addend in the expression above is equal to zero at $(\xi,0)$, while by (3.18) it turns out that $I^\eta(\xi,0,u_1,u_2)=\rho_{\varepsilon,n}(\xi,0)=1$; therefore,

$$\partial_\eta\rho_{\varepsilon,n}(\xi,0)=\partial_\eta\left(I^\eta(\xi,0,u_1,u_2)\right). \tag{3.22}$$

By (3.16) it follows that

$$\begin{aligned}\partial_\eta\left(I^\eta(\xi,\eta,u_1,u_2)\right)=2\varepsilon\partial^2_{\eta\eta}u_1+2\varepsilon\partial^2_{\eta\eta}u_2-\frac{\varepsilon^2}{(\varepsilon+M\eta)^2}M^2+\frac{\varepsilon^2}{(\varepsilon-M\eta)^2}M^2\\+\lambda(\partial_\eta\beta_2-\partial_\eta\beta_1)\sigma\partial_\eta w+\lambda(\beta_2-\beta_1+1/\lambda)\partial_\eta(\sigma\partial_\eta w).\end{aligned} \tag{3.23}$$

From (3.14) and the Euler condition iii), we have that

$$\begin{aligned}\lambda(\partial_\eta\beta_2(\xi,0)-\partial_\eta\beta_1(\xi,0))\sigma(\xi,0)\partial_\eta w(\xi,0)&=-\omega_2(\xi,0)+\omega_1(\xi,0)\\&=(\partial_\xi u_2(\xi,0))^2-(\partial_\xi u_1(\xi,0))^2=\mathrm{curv}\,\Gamma(\xi),\end{aligned} \tag{3.24}$$

while

$$\partial_\eta(\sigma\partial_\eta w)(\xi,0)=-\partial_\xi(\sigma\partial_\xi w)(\xi,0)=\partial_\xi(2\varepsilon\partial_\xi u_1(\xi,0)+2\varepsilon\partial_\xi u_2)(\xi,0),$$

where we have used the fact that $\sigma\nabla w$ is divergence-free and the definition of σ and w. Putting this last fact together with (3.23), (3.24), and the harmonicity of u_i, we finally get

$$\partial_\eta \rho_{\varepsilon,n}(\xi,0) = \operatorname{curv}\Gamma(\xi) = \partial_\eta \gamma(\xi,0). \tag{3.25}$$

CLAIM 2.– There holds that $\partial^2_{\eta\eta}(|\nabla_{x,y}\xi(\Psi)|^2)(\xi,0) = 4\,[\operatorname{curv}\Gamma(\xi)]^2$.

PROOF OF THE CLAIM. By differentiating with respect to η the expression in (3.21) and by (3.4), we obtain

$$\begin{aligned}\partial^2_{\eta\eta}(|\nabla_{x,y}\xi(\Psi)|^2) &= -2[(\partial_\xi\tilde{x})^2 + (\partial_\xi\tilde{y})^2]^{-2}[(\partial^2_{\xi\eta}\tilde{x})^2 + \partial_\xi\tilde{x}\,\partial^3_{\xi\eta\eta}\tilde{x} + (\partial^2_{\xi\eta}\tilde{y})^2 + \partial_\xi\tilde{y}\,\partial^3_{\xi\eta\eta}\tilde{y}]\\ &\quad +8[(\partial_\xi\tilde{x})^2 + (\partial_\xi\tilde{y})^2]^{-3}(\partial_\xi\tilde{x}\,\partial^2_{\xi\eta}\tilde{x} + \partial_\xi\tilde{y}\,\partial^2_{\xi\eta}\tilde{y})^2\\ &= -2[(\partial_\xi\tilde{x})^2 + (\partial_\xi\tilde{y})^2]^{-2}[(\partial^2_{\xi\xi}\tilde{y})^2 + (\partial^2_{\xi\xi}\tilde{x})^2 - \partial_\xi\tilde{x}\,\partial^3_{\xi\xi\xi}\tilde{x} - \partial_\xi\tilde{y}\,\partial^3_{\xi\xi\xi}\tilde{y}]\\ &\quad +8[(\partial_\xi\tilde{x})^2 + (\partial_\xi\tilde{y})^2]^{-3}(-\partial_\xi\tilde{x}\,\partial^2_{\xi\xi}\tilde{y} + \partial_\xi\tilde{y}\,\partial^2_{\xi\xi}\tilde{x})^2.\end{aligned}$$

Note that

$$-\partial_\xi\tilde{x}\,\partial^3_{\xi\xi\xi}\tilde{x} - \partial_\xi\tilde{y}\,\partial^3_{\xi\xi\xi}\tilde{y} = (\partial^2_{\xi\xi}\tilde{y})^2 + (\partial^2_{\xi\xi}\tilde{x})^2 - \frac{1}{2}\partial^2_{\xi\xi}((\partial_\xi\tilde{x})^2 + (\partial_\xi\tilde{y})^2).$$

Using (3.1), (3.2), and the fact that $(\partial_\xi\tilde{x})^2 + (\partial_\xi\tilde{y})^2$ is equal to 1 at $(\xi,0)$, we obtain the claim.

By using Claims 1 and 2, we can conclude that

$$\begin{aligned}\partial^2_{\eta\eta}\gamma(\xi,0) &= \left[\frac{3}{4}(|\nabla_{x,y}\xi(\Psi)|^2)^{-\frac{5}{2}}[\partial_\eta(|\nabla_{x,y}\xi(\Psi)|^2)]^2 - \frac{1}{2}(|\nabla_{x,y}\xi(\Psi)|^2)^{-\frac{3}{2}}\partial^2_{\eta\eta}(|\nabla_{x,y}\xi(\Psi)|^2)\right]\Bigg|_{(\xi,0)}\\ &= [\operatorname{curv}\Gamma(\xi)]^2.\end{aligned} \tag{3.26}$$

The second derivative of $\rho_{\varepsilon,n}$ with respect to η is given by

$$\begin{aligned}\partial^2_{\eta\eta}\,\rho_{\varepsilon,n} = \frac{1}{\rho_{\varepsilon,n}}\Big\{&\left[\partial_\eta\left(I^\xi(\xi,\eta,u_1,u_2)\right)\right]^2 + \partial^2_{\eta\eta}\left(I^\xi(\xi,\eta,u_1,u_2)\right)I^\xi(\xi,\eta,u_1,u_2)\\ &+ \left[\partial_\eta\left(I^\eta(\xi,\eta,u_1,u_2)\right)\right]^2 + \partial^2_{\eta\eta}\left(I^\eta(\xi,\eta,u_1,u_2)\right)I^\eta(\xi,\eta,u_1,u_2)\Big\} - \frac{1}{\rho_{\varepsilon,n}}[\partial_\eta(\rho_{\varepsilon,n})]^2.\end{aligned}$$

By the equalities (3.17), (3.18), and (3.22), the expression above computed at $(\xi,0)$ reduces to

$$\partial^2_{\eta\eta}\,\rho_{\varepsilon,n}(\xi,0) = \left[\partial_\eta\left(I^\xi(\xi,\eta,u_1,u_2)\right)\big|_{(\xi,0)}\right]^2 + \partial^2_{\eta\eta}\left(I^\eta(\xi,\eta,u_1,u_2)\right)\big|_{(\xi,0)}\,. \tag{3.27}$$

By differentiating (3.15) and (3.23) with respect to η, we obtain that

$$\partial_\eta\left(I^\xi(\xi,\eta,u_1,u_2)\right)(\xi,0) = [\lambda(\partial_\eta\beta_2 - \partial_\eta\beta_1)\sigma\partial_\xi w + \partial_\eta\sigma\,\partial_\xi w + \sigma\partial^2_{\xi\eta}w]|_{(\xi,0)}, \tag{3.28}$$

and

$$\begin{aligned}\partial^2_{\eta\eta}\left(I^\eta(\xi,\eta,u_1,u_2)\right)(\xi,0) &= \frac{4}{\varepsilon}M^3 + \lambda[\partial^2_{\eta\eta}\beta_2(\xi,0) - \partial^2_{\eta\eta}\beta_1(\xi,0)]\sigma(\xi,0)\partial_\eta w(\xi,0)\\ &\quad + 2\lambda[\partial_\eta\beta_2(\xi,0) - \partial_\eta\beta_1(\xi,0)]\partial_\eta(\sigma\partial_\eta w)(\xi,0) + \partial^2_{\eta\eta}\sigma(\xi,0)\partial_\eta w(\xi,0)\\ &\quad + 2\partial_\eta\sigma(\xi,0)\partial^2_{\eta\eta}w(\xi,0) + \sigma(\xi,0)\partial^3_{\eta\eta\eta}w(\xi,0),\end{aligned} \tag{3.29}$$

while, by using the equation (3.14),

$$\begin{aligned}
[\lambda(\partial^2_{\eta\eta}\beta_2 - \partial^2_{\eta\eta}\beta_1)\sigma\partial_\eta w]|_{(\xi,0)} &= [\partial_\eta\omega_1 - \partial_\eta\omega_2 - \lambda\partial_\eta(\partial_\xi\beta_2 - \partial_\xi\beta_1)\sigma\partial_\xi w - \lambda\partial_\eta(\sigma\partial_\eta w)(\partial_\eta\beta_2 - \partial_\eta\beta_1)]|_{(\xi,0)} \\
&= [-\frac{4}{\varepsilon}M^3 - \lambda\partial_\xi(\partial_\eta\beta_2 - \partial_\eta\beta_1)\sigma\partial_\xi w + \lambda\partial_\xi(\sigma\partial_\xi w)(\partial_\eta\beta_2 - \partial_\eta\beta_1)]|_{(\xi,0)}.
\end{aligned}$$

Since by (3.24) and by the definition of σ we have that

$$\lambda[\partial_\eta\beta_2(\xi,0) - \partial_\eta\beta_1(\xi,0)] = \frac{\operatorname{curv}\Gamma(\xi)}{1-2\varepsilon M},$$

and moreover,

$$\sigma(\xi,0)\partial_\xi w(\xi,0) = -2\varepsilon(\partial_\xi u_1(\xi,0) + \partial_\xi u_2(\xi,0)),$$

we obtain that

$$\begin{aligned}
&[\lambda(\partial^2_{\eta\eta}\beta_2 - \partial^2_{\eta\eta}\beta_1)\sigma\partial_\eta w + 2\lambda(\partial_\eta\beta_2 - \partial_\eta\beta_1)\partial_\eta(\sigma\partial_\eta w)]|_{(\xi,0)} = \\
&\qquad = -\frac{4}{\varepsilon}M^3 + \frac{2\varepsilon}{1-2\varepsilon M}\partial_\xi((\partial_\xi u_1 - \partial_\xi u_2)\operatorname{curv}\Gamma)(\xi,0).
\end{aligned}$$

By using the definition of σ, we can write

$$\begin{aligned}
\partial_\eta\sigma &= -(1-2\varepsilon M)\frac{n'(\xi)}{n^2(\xi)}\partial_\eta q, \\
\partial^2_{\eta\eta}\sigma &= -(1-2\varepsilon M)\left[-2\frac{(n'(\xi))^2}{n^3(\xi)}(\partial_\eta q)^2 + \frac{n''(\xi)}{n^2(\xi)}(\partial_\eta q)^2 + \frac{n'(\xi)}{n^2(\xi)}\partial^2_{\eta\eta}q\right].
\end{aligned}$$

In order to compute the derivatives of q, we differentiate the equality (3.13) with respect to η:

$$\begin{aligned}
\partial_\eta q(\xi,0) &= -\partial_\eta p(\xi,0) = \frac{2\varepsilon}{1-2\varepsilon M}(\partial_\xi u_1(\xi,0) + \partial_\xi u_2(\xi,0)), \\
\partial^2_{\eta\eta}q(\xi,0) &= -2\partial^2_{\xi\eta}p(\xi,0)\partial_\eta q(\xi,0) - \partial^2_{\eta\eta}p(\xi,0) = \left[-\frac{(\partial_\xi w)^2}{(\partial_\eta w)^3}\partial^2_{\xi\eta}w - \frac{1}{\partial_\eta w}\partial^2_{\xi\eta}w\right](\xi,0).
\end{aligned}$$

By the definition of w, we obtain

$$\partial^2_{\eta\eta}q(\xi,0) = -\frac{n'(\xi)}{n(\xi)} - \frac{n'(\xi)}{n(\xi)}\frac{4\varepsilon^2}{(1-2\varepsilon M)^2}(\partial_\xi u_1(\xi,0) + \partial_\xi u_2(\xi,0))^2.$$

Finally, we have

$$\begin{aligned}
\partial^2_{\eta\eta}w(\xi,0) &= -\partial^2_{\xi\xi}w(\xi,0) = \frac{2\varepsilon}{1-2\varepsilon M}[n'(\partial_\xi u_1 + \partial_\xi u_2) + n(\partial^2_{\xi\xi}u_1 + \partial^2_{\xi\xi}u_2)]|_{(\xi,0)}, \\
\partial^3_{\eta\eta\eta}w(\xi,0) &= -\partial^2_{\xi\xi}\partial_\eta w(\xi,0) = -n''(\xi).
\end{aligned}$$

By substituting all information above in (3.28) and in (3.29), and by using (3.27), we finally obtain that

$$\begin{aligned}
\partial^2_{\eta\eta}\rho_{\varepsilon,n}(\xi,0) &= -a_\varepsilon(\xi)\frac{n''(\xi)}{n(\xi)} + h_\varepsilon\left(\xi, \frac{n'(\xi)}{n(\xi)}\right) \\
&= -a_\varepsilon(\xi)\left(\frac{n'(\xi)}{n(\xi)}\right)' + h_\varepsilon\left(\xi, \frac{n'(\xi)}{n(\xi)}\right) - a_\varepsilon(\xi)\left(\frac{n'(\xi)}{n(\xi)}\right)^2, \qquad (3.30)
\end{aligned}$$

where

$$\begin{aligned} a_\varepsilon(\xi) &\to 1 && \text{uniformly in } [0,l(\Gamma)], \\ h_\varepsilon(\xi,\tau) &\to 2\tau^2 && \text{uniformly on the compact sets of } [0,l(\Gamma)]\times\mathbb{R}, \end{aligned} \tag{3.31}$$

as $\varepsilon \to 0$.

Claim 3.– There exists $\overline{\varepsilon} > 0$ such that for every $\varepsilon \in (0,\overline{\varepsilon})$, we can find an analytic function $n : [0,l(\Gamma)] \to (0,+\infty)$ satisfying

$$\partial^2_{\eta\eta}(\rho_{\varepsilon,n} - \gamma)(\xi,0) = -\frac{\pi^2}{16\,l^2(\Gamma)} \qquad \text{and} \qquad \left|\frac{n'(\xi)}{n(\xi)}\right| \le N \quad \forall \xi \in [0,l(\Gamma)], \tag{3.32}$$

where $N := 1 + \max\left\{\dfrac{\pi}{4\,l(\Gamma)}, k(\Gamma)\right\}$ and $k(\Gamma) = \|\operatorname{curv}\Gamma\|_\infty$.

Proof of the claim. Set $\tau := n'/n$; in order to prove the claim, by (3.30) and (3.26) we study the Cauchy problem

$$\begin{cases} -a_\varepsilon(\xi)\tau' + h_\varepsilon(\xi,\tau) - \tau^2 - [\operatorname{curv}\Gamma(\xi)]^2 = -\dfrac{\pi^2}{16\,l^2(\Gamma)}, \\ \tau(0) = 0, \end{cases} \tag{3.33}$$

and we investigate for which values of ε it admits a solution defined in the whole interval $[0,l(\Gamma)]$, with L^∞-norm less than N. As $\varepsilon \to 0$, by (3.31) we obtain the limit problem

$$\begin{cases} -\tau' + \tau^2 - (\operatorname{curv}\Gamma)^2 = -\dfrac{\pi^2}{16\,l^2(\Gamma)}, \\ \tau(0) = 0. \end{cases} \tag{3.34}$$

By comparing with the solutions τ_1 and τ_2 of the Cauchy problems

$$\begin{cases} -\tau_1' + \tau_1^2 = -\dfrac{\pi^2}{16\,l^2(\Gamma)}, \\ \tau_1(0) = 0, \end{cases} \qquad \begin{cases} -\tau_2' + \tau_2^2 - k^2(\Gamma) = -\dfrac{\pi^2}{16\,l^2(\Gamma)}, \\ \tau_2(0) = 0, \end{cases} \tag{3.35}$$

one easily sees that the solution of (3.34) is defined in $[0,l(\Gamma)]$, with L^∞-norm less than the maximum between $\|\tau_1\|_\infty$ and $\|\tau_2\|_\infty$, which is, by explicit computation, less than $\max\{\pi/(4l(\Gamma)), k(\Gamma)\}$. By the theorem of continuous dependence on the coefficients (see [23]), we can find $\overline{\varepsilon}$ such that, for every $\varepsilon \in (0,\overline{\varepsilon})$, the solution of (3.33) is defined in $[0,l(\Gamma)]$ with L^∞-norm less than N.

For every $\varepsilon \in (0,\overline{\varepsilon})$, we set

$$n_\varepsilon(\xi) := \mathrm{e}^{\int_0^\xi \tau_\varepsilon(s)\,ds}, \tag{3.36}$$

where τ_ε is the solution of (3.33).

From now on we will simply write ρ_ε and θ_ε instead of $\rho_{\varepsilon,n_\varepsilon}$ and $\theta_{\varepsilon,n_\varepsilon}$.

We now want to estimate the angle $\theta_\varepsilon(\xi,\eta)$ by a quantity which is independent of ε. Since by (3.15) and (3.16)

$$\tan\theta_\varepsilon = \frac{2\varepsilon\partial_\xi u_1 + 2\varepsilon\partial_\xi u_2 + \lambda\left(\beta_2 - \beta_1 + \frac{1}{\lambda}\right)\sigma\partial_\xi w}{2\varepsilon\partial_\eta u_1 + 2\varepsilon\partial_\eta u_2 + M\varepsilon^2(\varepsilon+M\eta)^{-1} + M\varepsilon^2(\varepsilon-M\eta)^{-1} + \lambda\left(\beta_2 - \beta_1 + \frac{2}{\lambda}\right)\sigma\partial_\eta w},$$

we have

$$\partial_\eta \theta_\varepsilon(\xi,0) = -\frac{2\varepsilon}{1-2\varepsilon M}(\partial_\xi u_1 + \partial_\xi u_2)\left(\operatorname{curv}\Gamma - 2\varepsilon(\partial_\xi u_1 + \partial_\xi u_2)\frac{n'_\varepsilon(\xi)}{n_\varepsilon(\xi)}\right) + (1-2\varepsilon M)\frac{n'_\varepsilon(\xi)}{n_\varepsilon(\xi)},$$

and so, by Claim 3, if ε is sufficiently small,

$$|\partial_\eta \theta_\varepsilon(\xi,0)| < N \qquad \forall \xi \in [0, l(\Gamma)]. \tag{3.37}$$

Let $\tilde{\theta}(\eta)$ be an arbitrary continuous function with

$$\tilde{\theta}(0) = 0 \qquad \text{and} \qquad \tilde{\theta}'(0) = N; \tag{3.38}$$

by (3.37), it follows that

$$|\theta_\varepsilon(\xi,\eta)| < \tilde{\theta}(\eta)\operatorname{sign}\eta \tag{3.39}$$

for every $(\xi,\eta) \in V$, provided V is sufficiently small.

Given $h > 0$, we consider the vectors

$$\begin{aligned} b_1^h(\xi,\eta,s) &:= \left(0, -2(s-u_1(\xi,\eta))\partial_\eta u_1(\xi,\eta) - h(s-u_1(\xi,\eta))^2\right), \\ b_2^h(\xi,\eta,t) &:= \left(0, 2(t-u_2(\xi,\eta))\partial_\eta u_2(\xi,\eta) - h(t-u_2(\xi,\eta))^2\right) \end{aligned}$$

for $(\xi,\eta) \in V$ and $s,t \in \mathbb{R}$. We denote by $B(r)$ the open ball centred at $(0,-r)$ with radius r.

Let us define $r_\varepsilon^h(\xi,\eta,s,t)$ as the maximum radius r such that the set

$$(\rho_\varepsilon(\xi,\eta)\sin\tilde{\theta}(\eta), \rho_\varepsilon(\xi,\eta)\cos\tilde{\theta}(\eta)) + b_1^h(\xi,\eta,s) + b_2^h(\xi,\eta,t) + B(r)$$

is contained in the ball centred at $(0,0)$ with radius $\gamma(\xi,\eta)$.

Claim 4.– If we define

$$d := \frac{1}{1 + 16\, l^2(\Gamma) N^2/\pi^2}, \tag{3.40}$$

where N is the constant introduced in the previous claim, then there exists $h > 0$ such that for every $\varepsilon \in (0,\overline{\varepsilon})$ (see Claim 3), there exists $\delta \in (0,\varepsilon)$ so that, if V is small enough,

$$\inf\left\{2\, r_\varepsilon^h(\xi,\eta,s,t) : (\xi,\eta) \in V,\ |s-u_1(\xi,\eta)| \le \delta,\ |t-u_2(\xi,\eta)| \le \delta\right\} > \frac{d}{2}. \tag{3.41}$$

Proof of the claim. Let $\overline{\rho}_\varepsilon^h(\xi,\eta,s,t) > 0$ and $-\pi/2 < \overline{\theta}_\varepsilon^h(\xi,\eta,s,t) < \pi/2$ be such that

$$\begin{aligned} &\left(\rho_\varepsilon(\xi,\eta)\sin\tilde{\theta}(\eta), \rho_\varepsilon(\xi,\eta)\cos\tilde{\theta}(\eta)\right) + b_1^h(\xi,\eta,s) + b_2^h(\xi,\eta,t) = \\ &\qquad = \left(\overline{\rho}_\varepsilon^h(\xi,\eta,s,t)\sin\overline{\theta}_\varepsilon^h(\xi,\eta,s,t), \overline{\rho}_\varepsilon^h(\xi,\eta,s,t)\cos\overline{\theta}_\varepsilon^h(\xi,\eta,s,t)\right). \end{aligned} \tag{3.42}$$

To prove Claim 4, it is enough to show that, for every $\varepsilon \in (0,\overline{\varepsilon})$, there exists $\delta \in (0,\varepsilon)$ with the property that

$$\left(1 - \frac{d}{2}\cos\overline{\theta}_\varepsilon^h(\xi,\eta,s,t)\right)\overline{\rho}_\varepsilon^h(\xi,\eta,s,t) < \left(1 - \frac{d}{2}\right)\gamma(\xi,\eta) \tag{3.43}$$

for $|s-u_1(\xi,\eta)| \le \delta$, $|t-u_2(\xi,\eta)| \le \delta$, and $(\xi,\eta) \in V$ with $\eta \neq 0$, provided V is sufficiently small. Indeed, if (3.43) holds, it follows in particular that $\overline{\rho}_\varepsilon^h(\xi,\eta,s,t) < \gamma(\xi,\eta)$, and this inequality with some easy geometric computations implies that

$$2\,r_\varepsilon^h(\xi,\eta,s,t) = \frac{\gamma^2(\xi,\eta) - (\overline{\rho}_\varepsilon^h(\xi,\eta,s,t))^2}{\gamma - \overline{\rho}_\varepsilon^h(\xi,\eta,s,t)\cos\overline{\theta}_\varepsilon^h(\xi,\eta,s,t)};$$

at this point, it is easy to see that, if V is small enough, inequality (3.43) implies that $2\,r_\varepsilon^h(\xi,\eta,s,t) > d/2$, that is Claim 4. So let us prove (3.43).

We set

$$f^{d,h}(\xi,\eta,s,t) := \left(1 - \frac{d}{2}\cos\overline{\theta}_\varepsilon^h(\xi,\eta,s,t)\right)\overline{\rho}_\varepsilon^h(\xi,\eta,s,t) - \left(1-\frac{d}{2}\right)\gamma(\xi,\eta)$$

and we note that $f^{d,h}(\xi,0,u_1(\xi,0),u_2(\xi,0)) = 0$. We will show that

1. $\nabla_{\eta,s,t}\, f^{d,h}(\xi,0,u_1(\xi,0),u_2(\xi,0)) = 0$ if $(\xi,0) \in V$,
2. $\nabla^2_{\eta,s,t}\, f^{d,h}(\xi,0,u_1(\xi,0),u_2(\xi,0))$ is negative definite if $(\xi,0) \in V$,

where $\nabla_{\eta,s,t}\, f^{d,h}$ and $\nabla^2_{\eta,s,t}\, f^{d,h}$ denote respectively the gradient and the Hessian matrix of $f^{d,h}$ with respect to the variables (η,s,t). Equality 1 follows by direct computations and by (3.25). Using (3.42), the equality in (3.32), and (3.38), we obtain

$$\partial^2_{\eta\eta} f^{d,h}(\xi,0,u_1(\xi,0),u_2(\xi,0)) = -\frac{\pi^2}{16\,l^2(\Gamma)}\left(1-\frac{d}{2}\right) + \frac{d}{2}N^2;$$

then by the definition of d,

$$\partial^2_{\eta\eta} f^{d,h}(\xi,0,u_1(\xi,0),u_2(\xi,0)) = -\frac{\pi^2}{32\,l^2(\Gamma)} < 0. \tag{3.44}$$

Moreover we easily obtain that

$$\partial^2_{tt} f^{d,h}(\xi,0,u_1(\xi,0),u_2(\xi,0)) = \partial^2_{ss} f^{d,h}(\xi,0,u_1(\xi,0),u_2(\xi,0)) = -2h\left(1-\frac{d}{2}\right),$$

$$\partial^2_{s\eta} f^{d,h}(\xi,0,u_1(\xi,0),u_2(\xi,0)) = -2\left(1-\frac{d}{2}\right)\partial^2_{\eta\eta} u_1(\xi,0),$$

$$\partial^2_{t\eta} f^{d,h}(\xi,0,u_1(\xi,0),u_2(\xi,0)) = 2\left(1-\frac{d}{2}\right)\partial^2_{\eta\eta} u_2(\xi,0),$$

$$\partial^2_{ts} f^{d,h}(\xi,0,u_1(\xi,0),u_2(\xi,0)) = 0.$$

From the expressions it follows that

$$\det\begin{pmatrix} \partial^2_{\eta\eta} f^{d,h} & \partial^2_{s\eta} f^{d,h} \\ \partial^2_{s\eta} f^{d,h} & \partial^2_{ss} f^{d,h} \end{pmatrix}(\xi,0,u_1(\xi,0),u_2(\xi,0)) = h(2-d)\frac{\pi^2}{32\,l^2(\Gamma)} - (2-d)^2[\partial^2_{\eta\eta} u_1(\xi,0)]^2,$$

and that the determinant of the Hessian matrix of $f^{d,h}$ at $(\xi,0,u_1(\xi,0),u_2(\xi,0))$ is given by

$$\det\nabla^2_{\eta,s,t}\, f^{d,h}(\xi,0,u_1(\xi,0),u_2(\xi,0)) = -h^2(2-d)^2\frac{\pi^2}{32\,l^2(\Gamma)} + h(2-d)^3[(\partial^2_{\eta\eta} u_1(\xi,0))^2 + (\partial^2_{\eta\eta} u_2(\xi,0))^2].$$

By the definition of d, if h satisfies

$$h > \frac{32}{\pi^2}(2-d)l^2(\Gamma)\sum_{i=1}^{2}\|\partial^2_{\eta\eta}u_i\|^2_{L^\infty(\Gamma)}, \tag{3.45}$$

then for every $(\xi,0)\in V$ we have

$$\det\begin{pmatrix} \partial^2_{\eta\eta}f^{d,h} & \partial^2_{s\eta}f^{d,h} \\ \partial^2_{s\eta}f^{d,h} & \partial^2_{ss}f^{d,h} \end{pmatrix}(\xi,0,u_1(\xi,0),u_2(\xi,0)) > 0, \tag{3.46}$$

and

$$\det\nabla^2_{\eta,s,t}\, f^{d,h}(\xi,0,u_1(\xi,0),u_2(\xi,0)) < 0. \tag{3.47}$$

By (3.44), (3.46), and (3.47), we can conclude that the Hessian matrix of $f^{d,h}$ at $(\xi,0,u_1(\xi,0),u_2(\xi,0))$ is negative definite: both (3.43) and Claim 4 are proved.

Claim 5.– For every $r>0$ and $h>0$, there exists $\tilde\varepsilon>0$ with the property that, if $\varepsilon\in(0,\tilde\varepsilon)$, one can find $\delta\in(0,\varepsilon)$ so that

$$I(\xi,\eta,u_2(\xi,\eta),t)\in B(r)+b_2^h(\xi,\eta,t),$$
$$I(\xi,\eta,s,u_1(\xi,\eta))\in B(r)+b_1^h(\xi,\eta,s),$$

provided V is small enough, for every $|t-u_2(\xi,\eta)|\le\delta$, $|s-u_1(\xi,\eta)|\le\delta$.

Proof of the claim. By the definition of ϕ in A_6, we obtain that

$$I^\xi(\xi,\eta,u_2(\xi,\eta),t) = 2(t-u_2(\xi,\eta))\partial_\xi u_2(\xi,\eta),$$

$$I^\eta(\xi,\eta,u_2(\xi,\eta),t) = 2(t-u_2(\xi,\eta))\partial_\eta u_2(\xi,\eta) - M(\varepsilon-M\eta)^{-1}(t-u_2(\xi,\eta))^2.$$

To get the claim, we need to prove that

$$(2(t-u_2)\partial_\xi u_2)^2 + \left(-M(\varepsilon-M\eta)^{-1}(t-u_2)^2 + h(t-u_2)^2 + r\right)^2 < r^2,$$

which is equivalent to

$$(2(t-u_2)\partial_\xi u_2)^2 + \left(-M(\varepsilon-M\eta)^{-1}+h\right)^2(t-u_2)^4 + 2r\left(-M(\varepsilon-M\eta)^{-1}+h\right)(t-u_2)^2 < 0.$$

The conclusion follows by remarking that, if V is small enough, the left-handside is less than

$$\left(4(\partial_\xi u_2)^2 + 2hr - \frac{2Mr}{3\varepsilon}\right)\delta^2 + o(\delta^2),$$

which is negative if ε is sufficiently small. The proof for u_1 is completely analogous.

Let us conclude the proof of the step. By Claim 4, we can find $h>0$ such that (3.41) is satisfied for $\varepsilon\in(0,\overline\varepsilon)$. If we choose r such that $2r<d/4$, by Claim 5 there exists $\tilde\varepsilon>0$ such that for every $\varepsilon\in(0,\tilde\varepsilon)$ there is $\delta\in(0,\varepsilon)$ so that

$$I(\xi,\eta,s,u_1(\xi,\eta)) + I(\xi,\eta,u_2(\xi,\eta),t)\in B(2r)+b_1^h(\xi,\eta,s)+b_2^h(\xi,\eta,t) \tag{3.48}$$

for every $|s - u_1(\xi,\eta)| < \delta$, $|t - u_2(\xi,\eta)| < \delta$, and $(\xi,\eta) \in V$. If we take $\varepsilon \le \min\{\tilde\varepsilon, \overline{\varepsilon}\}$, then by Claim 4 we have that the set

$$B(2r) + (\rho_\varepsilon(\xi,\eta)\sin\tilde\theta(\eta), \rho_\varepsilon(\xi,\eta)\cos\tilde\theta(\eta)) + b_1^h(\xi,\eta,s) + b_2^h(\xi,\eta,t)$$

is contained in the ball centred at $(0,0)$ with radius $\gamma(\xi,\eta)$. Some easy geometric considerations show that the relation between θ_ε and $\tilde\theta$ (see (3.39)) implies that also the set

$$B(2r) + (\rho_\varepsilon(\xi,\eta)\sin\theta_\varepsilon(\eta), \rho_\varepsilon(\xi,\eta)\cos\theta_\varepsilon(\eta)) + b_1^h(\xi,\eta,s) + b_2^h(\xi,\eta,t) \tag{3.49}$$

is contained in the ball centred at $(0,0)$ with radius $\gamma(\xi,\eta)$, if the condition

$$|b_1^h(\xi,\eta,s) + b_2^h(\xi,\eta,t)| < 2r$$

holds (to make this true, take δ and V smaller if needed). Since

$$I(\xi,\eta,s,t) = I(\xi,\eta,s,u_1(\xi,\eta)) + I(\xi,\eta,u_1(\xi,\eta),u_2(\xi,\eta)) + I(\xi,\eta,u_2(\xi,\eta),t),$$

by (3.48), (3.19), and (3.20), it follows that $I(\xi,\eta,s,t)$ belongs to the set (3.49), and then to the ball centred at $(0,0)$ with radius $\gamma(\xi,\eta)$ for every $|s-u_1(\xi,\eta)| < \delta$, $|t - u_2(\xi,\eta)| < \delta$, and $(\xi,\eta) \in V$. This concludes the proof of Step 1.

Step 2.– If ε is sufficiently small and $\delta \in (0,\varepsilon)$, condition (b1) holds for $|s - u_1(\xi,\eta)| \ge \delta$ or $|t - u_2(\xi,\eta)| \ge \delta$, and $(\xi,\eta) \in V$, provided V is small enough.

Let us fix $\delta \in (0,\varepsilon)$ and set

$$m_1(\xi,\eta) := \max\{|I(\xi,\eta,s,t)| : \ u_1(\xi,\eta) - \varepsilon \le s \le t \le u_2(\xi,\eta) + \varepsilon, \ |t - u_2(\xi,\eta)| \ge \delta\}.$$

It is easy to see that the function m_1 is continuous. Let us prove that $m_1(\xi,0) < \gamma(\xi,0) = 1$.

Fixed $(\xi,0) \in V$, $u_1(\xi,0) - \varepsilon \le s \le t \le u_2(\xi,0) + \varepsilon$, with $|t - u_2(\xi,0)| \ge \delta$, we can write

$$I(\xi,0,s,t) = I(\xi,0,s,u_1(\xi,0)) + I(\xi,0,u_1(\xi,0),u_2(\xi,0)) + I(\xi,0,u_2(\xi,0),t). \tag{3.50}$$

Claim 6.– For every $r > 0$ there exists $\varepsilon > 0$ such that

$$I(\xi,0,u_2(\xi,0),t) \in B(r), \qquad I(\xi,0,s,u_1(\xi,0)) \in B(r)$$

for $0 < |s - u_1(\xi,0)| \le \varepsilon$, $0 < |t - u_2(\xi,0)| \le \varepsilon$, and $(\xi,0) \in V$.

Proof of the claim. See the similar proof of Claim 5 above.

By (3.50), (3.17), (3.18), and Claim 6, it follows that

$$I(\xi,0,s,t) \in (0,1) + \overline{B(r)} + B(r) = (0,1) + B(2r) \tag{3.51}$$

for $0 < |s - u_1(\xi,0)| \le \varepsilon$, $\delta \le |t - u_2(\xi,0)| \le \varepsilon$. If $r < 1/4$, the set $(0,1) + B(2r)$ is contained in the open ball centred at $(0,0)$ with radius 1.

It remains to study the case $|s - u_1| \ge \varepsilon$ and the case $|t - u_2| \ge \varepsilon$. Let us consider the latter; the former would be completely analogous. We can write

$$\begin{aligned} I(\xi,0,s,u_1(\xi,0)) &= I(\xi,0,s\wedge(u_1(\xi,0)+\varepsilon),u_1(\xi,0)) + I(\xi,0,s\vee(u_1(\xi,0)+\varepsilon),u_1(\xi,0)+\varepsilon), \\ I(\xi,0,u_2(\xi,0),t) &= I(\xi,0,u_2(\xi,0),u_2(\xi,0)-\varepsilon) + I(\xi,0,u_2(\xi,0)-\varepsilon,t). \end{aligned}$$

Therefore, by (3.50)

$$
\begin{aligned}
I(\xi,0,s,t) = I(\xi,0,u_1(\xi,0),u_2(\xi,0)) + I(\xi,0,s\wedge(u_1(\xi,0)+\varepsilon),u_1(\xi,0)) \\
+ I(\xi,0,u_2(\xi,0),u_2(\xi,0)-\varepsilon) + I(\xi,0,s\vee(u_1(\xi,0)+\varepsilon),t) \\
- I(\xi,0,u_1(\xi,0)+\varepsilon,u_2(\xi,0)-\varepsilon).
\end{aligned} \tag{3.52}
$$

If $-2\varepsilon(\partial_\xi u_1(\xi,0)+\partial_\xi u_2(\xi,0)) \geq 0$, we define

$$
C := [0,-2\varepsilon(\partial_\xi u_1(\xi,0)+\partial_\xi u_2(\xi,0))]\times[0,1-2\varepsilon M];
$$

if $-2\varepsilon(\partial_\xi u_1(\xi,0)+\partial_\xi u_2(\xi,0)) < 0$, we simply replace $[0,-2\varepsilon(\partial_\xi u_1(\xi,0)+\partial_\xi u_2(\xi,0))]$ by $[-2\varepsilon(\partial_\xi u_1(\xi,0)+\partial_\xi u_2(\xi,0)),0]$. From the definition of ϕ in $A_3\cup A_4\cup A_5$, it follows that

$$
I(\xi,0,u_1(\xi,0)+\varepsilon,u_2(\xi,0)-\varepsilon) = (-2\varepsilon(\partial_\xi u_1(\xi,0)+\partial_\xi u_2(\xi,0)),1-2\varepsilon M) \tag{3.53}
$$

and

$$
I(\xi,0,s,t)\in C \tag{3.54}
$$

for $u_1(\xi,0)+\varepsilon \leq s \leq t \leq u_2(\xi,0)-\varepsilon$. Let $D := C - (-2\varepsilon(\partial_\xi u_1(\xi,0)+\partial_\xi u_2(\xi,0)),1-2\varepsilon M)$. Since $I^\eta(\xi,0,u_2(\xi,0),u_2(\xi,0)-\varepsilon) = -M\varepsilon$, from (3.52), (3.17), (3.18), Claim 6, (3.53), and (3.54), we obtain

$$
\begin{aligned}
I(\xi,0,s,t) \in [(0,1)+\overline{B(r)}+B(r)]\cap\{(x,y)\in\mathbb{R}^2 : y<1-\varepsilon M\} + D \\
= [(0,1)+B(2r)]\cap\{(x,y)\in\mathbb{R}^2 : y<1-\varepsilon M\} + D.
\end{aligned}
$$

If $r<1/4$ and if ε is sufficiently small, the set $[(0,1)+B(2r)]\cap\{(x,y)\in\mathbb{R}^2 : y<1-\varepsilon M\}+D$ is contained in the open ball centred at $(0,0)$ with radius 1 and this means that $m_1(\xi,0)<\gamma(\xi,0)$.

Analogously we define

$$
m_2(\xi,\eta) := \max\{|I(\xi,\eta,s,t)| : \ u_1(\xi,\eta)-\varepsilon \leq s \leq t \leq u_2(\xi,\eta)+\varepsilon,\ |s-u_1(\xi,\eta)|\geq\delta\}.
$$

Arguing as in the case of m_1, we can prove that m_2 is continuous and $m_2(\xi,0)<\gamma(\xi,0)$. By continuity, if V is small enough, $m_1(\xi,\eta)<\gamma(\xi,\eta)$ and $m_2(\xi,\eta)<\gamma(\xi,\eta)$, for every $(\xi,\eta)\in V$. This concludes the proof of Step 2.

By Step 1 and Step 2 we deduce that, choosing ε sufficiently small and $n=n_\varepsilon$ (see (3.36)), condition (b1) is true for $u_1(\xi,\eta)-\varepsilon \leq s,t \leq u_2(\xi,\eta)+\varepsilon$ and in fact for every $s,t\in\mathbb{R}$, from the definition of ϕ in A_1 and A_7. □

3.2 The graph-minimality

We start this section with a negative result: if the domain Ω is too large, the Euler conditions do not guarantee the graph-minimality introduced in Definition 3.1, as the following counterexample (proposed by Gianni Dal Maso) shows.

Proposition 3.3 *Let R be the rectangle $(1,1+4l)\times(-l,l)$ and let*

$$
u(x,y) := \begin{cases} x & \text{if } y\geq 0, \\ -x & \text{if } y<0. \end{cases}
$$

Then u satisfies the Euler conditions for the Mumford-Shah functional in R, but it is not a Dirichlet graph-minimizer in R for l large enough.

Proof. – The Euler conditions are obviously satisfied by u in R.

Let R_0 be the rectangle $(0,4)\times(-1,0)$ and let w be any function in $H^1(R_0)$ such that $w(x,0)=x$ for $x\in(0,2)$, and $w(x,y)=0$ for $(x,y)\in\partial R_0\setminus((0,4)\times\{0\})$.

The idea is to perturb u by the rescaled function $v(x,y):=lw(\frac{x-1}{l},\frac{y}{l})$. We define the perturbed function

$$\tilde u(x,y):=\begin{cases} x & \text{on } R_1\setminus T_\varepsilon,\\ -x+\eta\,(x-1) & \text{on } T_\varepsilon,\\ -x+\eta\,v(x,y) & \text{on } R_2,\end{cases}$$

where η is a positive parameter and the rectangles R_1, R_2, and the triangle T_ε are indicated in Fig. 3.1. We want to show that, if we set $c:=\int_{R_0}|\nabla w(x,y)|^2dx\,dy$, for every $l>c$ and for every ε_0, $\eta_0>0$ there

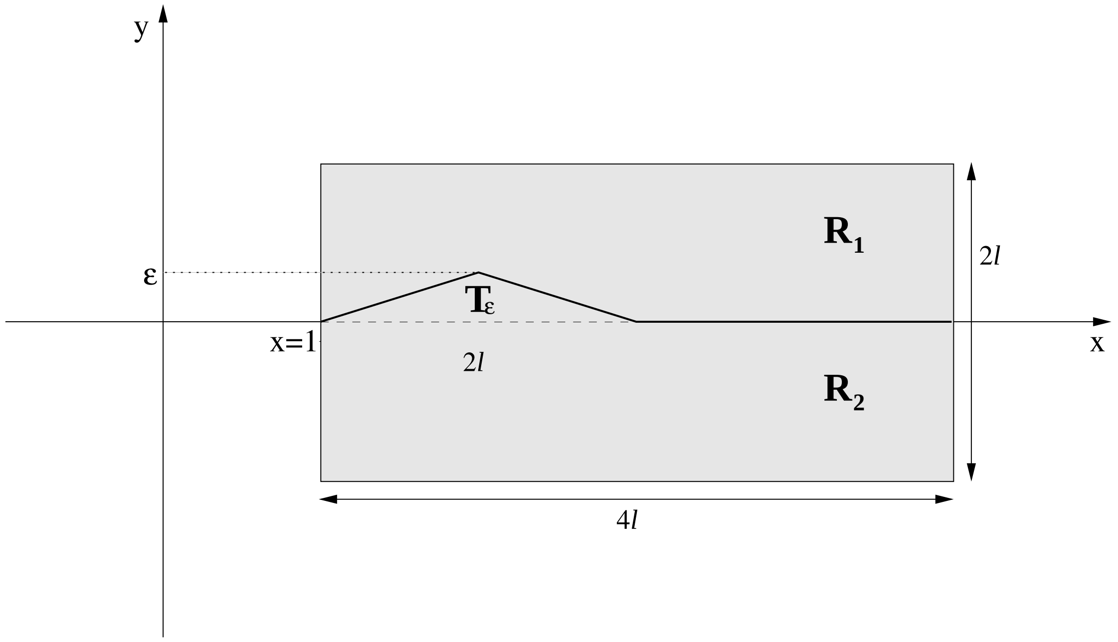

Figure 3.1: the regions R_1, R_2 and T_ε.

exist $\varepsilon<\varepsilon_0$ and $\eta<\eta_0$ such that

$$\int_R|\nabla u(x,y)|^2dx\,dy+\mathcal{H}^1(S_u)>\int_R|\nabla\tilde u(x,y)|^2dx\,dy+\mathcal{H}^1(S_{\tilde u}).$$

By definition, $\tilde u$ satisfies the boundary conditions. Since by the construction of v the function $\tilde u$ is continuous on the interface between T_ε and R_2, then

$$\mathcal{H}^1(S_u)-\mathcal{H}^1(S_{\tilde u})=2l-2\sqrt{l^2+\varepsilon^2}=-\frac{\varepsilon^2}{l}+o(\varepsilon^2). \tag{3.55}$$

On the triangle T_ε, we obtain

$$\int_{T_\varepsilon}|\nabla u(x,y)|^2dx\,dy-\int_{T_\varepsilon}|\nabla\tilde u(x,y)|^2dx\,dy=2l\varepsilon\eta-l\varepsilon\eta^2. \tag{3.56}$$

Finally, since we have that $|\nabla\tilde{u}|^2 = 1+\eta^2|\nabla v|^2 - 2\eta\,\partial_x v$ in R_2, taking into account the boundary conditions of v, we get

$$\begin{aligned}\int_{R_2}|\nabla u(x,y)|^2dx\,dy - \int_{R_2}|\nabla\tilde{u}(x,y)|^2dx\,dy &= -\eta^2\int_{R_2}|\nabla v(x,y)|^2dx\,dy \\ &= -l^2\eta^2\int_{R_0}|\nabla w(x,y)|^2dx\,dy. \end{aligned} \tag{3.57}$$

In order to conclude, by (3.55), (3.56), and (3.57), we have to show that for l large we can choose ε and η arbitrarily close to 0 such that

$$-\frac{\varepsilon^2}{l} - cl^2\eta^2 + 2l\varepsilon\eta - l\varepsilon\eta^2 + o(\varepsilon^2) > 0.$$

If we choose $\eta = \varepsilon/(cl)$, then the equality above reduces to

$$-\frac{\varepsilon^2}{l} + \frac{\varepsilon^2}{c} + o(\varepsilon^2) > 0,$$

which is true if $l > c$. □

As suggested by Proposition 3.3, to get the graph-minimality we have to add some restrictions on the domain Ω. To this aim we introduce a suitable notion of capacity: given an open set U (with Lipschitz boundary) and a portion Γ of ∂U (with nonempty relative interior in ∂U), we define $K(\Gamma, U)$ by the variational problem

$$K(\Gamma,U) := \inf\left\{\int_U |\nabla v(x,y)|^2dx\,dy:\ v\in H^1(U),\ \int_\Gamma v^2 d\mathcal{H}^1 = 1,\ \text{and } v=0 \text{ on } \partial U\setminus\Gamma\right\}. \tag{3.58}$$

First of all, it is easy to see that in the problem above the infimum is attained. Moreover, if $U_1\subset U_2$, and $\Gamma_1\subset\Gamma_2$, then $K(\Gamma_1,U_1)\ge K(\Gamma_2,U_2)$; this suggests that, when $K(\Gamma,U)$ is very large, U has to be thin in some sense. It is convenient to give the following definition.

Definition 3.4 *Given a simple analytic curve Γ, we say that an open set Ω is* Γ-admissible *if it is bounded, $\Gamma\cap\overline{\Omega}$ connects two points of $\partial\Omega$, and $\Omega\setminus\Gamma$ has two connected components having a Lipschitz boundary.*

The following theorem gives a sufficient condition for the graph-minimality in terms of $K(\Gamma,\Omega)$ and of the geometrical properties of the curve. We recall that $l(\Gamma)$ denotes the length of Γ, $\operatorname{curv}\Gamma$ its curvature, and $k(\Gamma)$ the L^∞-norm of $\operatorname{curv}\Gamma$.

Theorem 3.5 *Let Ω_0, Ω, u, and $\Gamma = S_u$ satisfy the same assumptions as in Theorem 3.2; suppose that Ω is Γ-admissible and denote by Ω_1 and Ω_2 the two connected components of $\Omega\setminus\Gamma$, by u_i the restriction of u to Ω_i, and by $\partial_\tau u_i$ its tangential derivative on Γ. There exists an absolute constant $c>0$ (independent of Ω_0, Ω, Γ, and u) such that if*

$$\frac{\min_{i=1,2} K(\Gamma\cap\Omega,\Omega_i)}{1+l^2(\Gamma\cap\Omega)+l^2(\Gamma\cap\Omega)k^2(\Gamma\cap\Omega)} > c\sum_{i=1}^2 \|\partial_\tau u_i\|^2_{C^1(\Gamma\cap\Omega)}, \tag{3.59}$$

then u is a Dirichlet graph-minimizer on Ω.

Remark that condition (3.59) imposes a restriction on the size of Ω depending on the behaviour of u along S_u: if u has large or very oscillating tangential derivatives, we have to take Ω quite small to guarantee that (3.59) is satisfied. In the special case of a locally constant function u, condition (3.59) is always fulfilled; so u is a Dirichlet graph-minimizer whatever Ω is, in agreement with a result of [2].

Proof. – From the definition of d and N (see (3.40) and Claim 3 in the proof of Theorem 3.2) it follows that there is an absolute constant $\tilde{c} > 0$ (independent of Ω_0, Ω, Γ, and u) such that

$$\tilde{c}\,(1 + l^2(\Gamma)k^2(\Gamma)) > \frac{16}{d}. \tag{3.60}$$

The absolute constant c, which appears in (3.59), is defined by

$$c := \max\left\{\tilde{c}, \frac{64}{\pi^2}\right\}. \tag{3.61}$$

Actually, to avoid problems of boundary regularity, we shall work not exactly in Ω, but in a little bit larger set. Let Ω' be a Γ-admissible set such that $\Omega \subset\subset \Omega' \subset\subset \Omega_0$, and

$$\frac{\min_{i=1,2} K(\Gamma \cap \Omega', \Omega'_i)}{1 + l^2(\Gamma \cap \Omega') + l^2(\Gamma \cap \Omega')k^2(\Gamma \cap \Omega')} > c \sum_{i=1}^{2} \|\partial_\tau u_i\|^2_{C^1(\Gamma\cap\Omega')},$$

where Ω'_i denote the connected components of $\Omega' \setminus \Gamma$. This is possible by (3.59) and by the continuity properties of K.

The idea of the proof is to construct first a calibration φ in a cylinder with base an open neighbourhood of $\Gamma \cap \Omega'$, and then to extend φ in a tubular neighbourhood of graph u.

Construction of the calibration around Γ

We essentially recycle the construction of Theorem 3.2, but we need to slightly modify the definition around the graph of u, in order to exploit condition (3.59) and get the extendibility.

To define the calibration $\varphi(x, y, z)$ we use the same notation and the coordinate system (ξ, η) on U (which is supposed to be an open neighbourhood of $\Gamma \cap \Omega'$) introduced in the proof of Theorem 3.2. The vectorfield will be written as

$$\varphi(x, y, z) = \frac{1}{\gamma^2(\xi(x, y), \eta(x, y))}\phi(\xi(x, y), \eta(x, y), z), \tag{3.62}$$

where ϕ can be represented by

$$\phi(\xi, \eta, z) = \phi^\xi(\xi, \eta, z)\tau_\xi + \phi^\eta(\xi, \eta, z)\tau_\eta + \phi^z(\xi, \eta, z)e_z.$$

Given suitable parameters $\varepsilon > 0$ and $\lambda > 0$, we consider the following subsets of $V \times \mathbb{R}$:

$$\begin{aligned}
A_1 &:= \{(\xi, \eta, z) \in V \times \mathbb{R} : u_1(\xi, \eta) - \varepsilon\, v_1(\xi, \eta) < z < u_1(\xi, \eta) + \varepsilon\, v_1(\xi, \eta)\},\\
A_2 &:= \{(\xi, \eta, z) \in V \times \mathbb{R} : u_1(\xi, \eta) + \varepsilon\, v_1(\xi, \eta) < z < u_1(\xi, \eta) + 2\varepsilon\},\\
A_3 &:= \{(\xi, \eta, z) \in V \times \mathbb{R} : u_1(\xi, \eta) + 2\varepsilon < z < \beta_1(\xi, \eta)\},\\
A_4 &:= \{(\xi, \eta, z) \in V \times \mathbb{R} : \beta_1(\xi, \eta) < z < \beta_2(\xi, \eta) + 1/\lambda\},\\
A_5 &:= \{(\xi, \eta, z) \in V \times \mathbb{R} : \beta_2(\xi, \eta) + 1/\lambda < z < u_2(\xi, \eta) - 2\varepsilon\},\\
A_6 &:= \{(\xi, \eta, z) \in V \times \mathbb{R} : u_2(\xi, \eta) - 2\varepsilon < z < u_2(\xi, \eta) - \varepsilon\, v_2(\xi, \eta)\},\\
A_7 &:= \{(\xi, \eta, z) \in V \times \mathbb{R} : u_2(\xi, \eta) - \varepsilon\, v_2(\xi, \eta) < z < u_2(\xi, \eta) + \varepsilon\, v_2(\xi, \eta)\},
\end{aligned}$$

where the functions v_i are defined as

$$v_1(\xi,\eta) := 1 + M\eta, \quad v_2(\xi,\eta) := 1 - M\eta$$

with M positive parameter such that

$$c\,(1 + l^2(\Gamma\cap\Omega') + l^2(\Gamma\cap\Omega')k^2(\Gamma\cap\Omega'))\sum_{j=1}^{2}\|\partial_\tau u_j\|^2_{C^1(\Gamma\cap\Omega')} < M < \min_{j=1,2} K(\Gamma\cap\Omega',\Omega'_i), \tag{3.63}$$

while β_1 and β_2 are the solutions of the Cauchy problems (3.14). Since we suppose $u_2 > 0$ on V, if ε is small enough, while λ is sufficiently large, then the sets $A_1, \dots, A_7$ are nonempty and disjoint, provided V is sufficiently small.

The vector $\phi(\xi,\eta,z)$ introduced in (3.62) will be written as

$$\phi(\xi,\eta,z) = (\phi^{\xi\eta}(\xi,\eta,z), \phi^z(\xi,\eta,z)),$$

where $\phi^{\xi\eta}$ is the two-dimensional vector given by the pair (ϕ^ξ,ϕ^η). We define $\phi(\xi,\eta,z)$ as follows:

$$\begin{cases}
\left(2\nabla u_1 + 2\frac{z-u_1}{v_1}\nabla v_1,\, \left|\nabla u_1 + \frac{z-u_1}{v_1}\nabla v_1\right|^2\right) & \text{in } A_1,\\[2ex]
\left(2\nabla(u_1+\varepsilon v_1) + 2\frac{z-u_1-\varepsilon v_1}{\tilde v_1}\nabla\tilde v_1,\, \left|\nabla(u_1+\varepsilon v_1) + \frac{z-u_1-\varepsilon v_1}{\tilde v_1}\nabla\tilde v_1\right|^2\right) & \text{in } A_2,\\[2ex]
(0,\,\omega_1(\xi,\eta)) & \text{in } A_3,\\[2ex]
(\lambda\sigma(\xi,\eta)\nabla w,\, \mu) & \text{in } A_4,\\[2ex]
(0,\,\omega_2(\xi,\eta)) & \text{in } A_5,\\[2ex]
\left(2\nabla(u_2-\varepsilon v_2) + 2\frac{z-u_2+\varepsilon v_2}{\tilde v_2}\nabla\tilde v_2,\, \left|\nabla(u_2-\varepsilon v_2) + \frac{z-u_2+\varepsilon v_2}{\tilde v_2}\nabla\tilde v_2\right|^2\right) & \text{in } A_6,\\[2ex]
\left(2\nabla u_2 + 2\frac{z-u_2}{v_2}\nabla v_2,\, \left|\nabla u_2 + \frac{z-u_2}{v_2}\nabla v_2\right|^2\right) & \text{in } A_7,
\end{cases}$$

where ∇ denotes the gradient with respect to the variables (ξ,η), the functions $\tilde v_i$ are defined by

$$\tilde v_1(\xi,\eta) := 2\varepsilon + M'\eta, \quad \tilde v_2(\xi,\eta) := 2\varepsilon - M'\eta$$

while

$$\omega_i(\xi,\eta) := \varepsilon^2\left(M + M'\frac{v_i(\xi,\eta)}{\tilde v_i(\xi,\eta)}\right)^2 - (\partial_\xi u_i(\xi,\eta))^2 - (\partial_\eta u_i(\xi,\eta))^2$$

for $i = 1,2$, and for every $(\xi,\eta)\in V$; we take the constant μ sufficiently large in order to get the required inequality between the horizontal and the vertical components of the field (see condition (a1) of Section 1.3), and M' so large that ω_i is positive in V, provided V is small enough. We define w as the solution of the Cauchy problem

$$\begin{cases}
\Delta w = 0,\\
w(\xi,0) = -\dfrac{4\varepsilon}{1-\varepsilon M' - 6\varepsilon^2 M}\displaystyle\int_0^\xi n(s)(\partial_\xi u_1(s,0) + \partial_\xi u_2(s,0))\,ds,\\
\partial_\eta w(\xi,0) = n(\xi),
\end{cases} \tag{3.64}$$

where n is a positive analytic function that must be chosen in a suitable way. We define

$$\sigma(\xi,\eta) := \frac{1}{n(q(\xi,\eta))}(1-\varepsilon M' - 6\varepsilon^2 M),$$

where the function q is constructed in the same way as in (3.13).

Let us prove that for a suitable choice of the involved parameters the vectorfield is a calibration in a suitable neighbourhood U of $\Gamma \cap \Omega'$, which is equivalent to prove that ϕ satisfies (a1), (a2), (b1), (b2), and (c1) of page 37. The proof of conditions (a1), (a2), (b2), and (c1) is the same as in Theorem 3.2. The proof of (b1) is split again in two steps.

STEP 1.– For a suitable choice of ε and of the function n (see (3.64)) there exists $\delta > 0$ such that condition (b1) holds for $|s-u_1(\xi,\eta)| < \delta$, $|t-u_2(\xi,\eta)| < \delta$, and $(\xi,\eta) \in V$, provided V is small enough.

We essentially repeat the proof given in Theorem 3.2: Claims 1, 2, 3, and 4 are still valid with the same proof (up to the obvious changes due to the different definition of ϕ). Claim 5 must be modified as follows.

CLAIM 5.– For $h = \frac{64}{\pi^2} l^2(\Gamma) \sum_{i=1}^2 \|\partial_\xi u_i\|^2_{C^1(\Gamma\cap\Omega')}$, there exist $r \in (0, d/8)$ and $\tilde\delta > 0$ such that for every $\delta \in (0,\tilde\delta)$

$$I(\xi,\eta,u_2(\xi,\eta),t) \in B(r) + b_2^h(\xi,\eta,t),$$
$$I(\xi,\eta,s,u_1(\xi,\eta)) \in B(r) + b_1^h(\xi,\eta,s),$$

provided V is small enough, for every $|t-u_2(\xi,\eta)| \le \delta$, $|s-u_1(\xi,\eta)| \le \delta$.

PROOF OF THE CLAIM. Using the definition of ϕ in A_7, the claim is equivalent to prove

$$(2(t-u_2)\partial_\xi u_2)^2 + \left(-M(1-M\eta)^{-1}+h\right)^2 (t-u_2)^4 + 2r\left(-M(1-M\eta)^{-1}+h\right)(t-u_2)^2 < 0;$$

note that for $a_1 \in (0,1)$ the left-handside is less than

$$\left(4\sum_{i=1}^2 \|\partial_\xi u_i\|^2_{C^1(\Gamma\cap\Omega')} + 2hr - \frac{2r}{1+a_1}M\right)\delta^2 + o(\delta^2),$$

provided V is small enough. To obtain the claim, it is sufficient to prove that

$$\frac{2}{r}\sum_{i=1}^2 \|\partial_\xi u_i\|^2_{C^1(\Gamma\cap\Omega')} < \frac{1}{1+a_1}M - h. \tag{3.65}$$

Since by (3.63), (3.60), and (3.61) we can write

$$M = \left(\frac{16+a_2}{d} + \frac{64}{\pi^2}l^2(\Gamma\cap\Omega')\right)\sum_{i=1}^2 \|\partial_\xi u_i\|^2_{C^1(\Gamma\cap\Omega')},$$

with $a_2 > 0$, the inequality (3.65) is equivalent to

$$\frac{2}{r} < \left(\frac{1}{1+a_1} - 1\right)\frac{64}{\pi^2}l^2(\Gamma\cap\Omega') + \frac{16+a_2}{d}\frac{1}{1+a_1},$$

which is true if a_1 is sufficiently small and r is sufficiently close to $d/8$. The proof for u_1 is completely analogous.

To conclude the proof of the step, let r and h be as in Claim 5. If we choose $\varepsilon < \overline{\varepsilon}$ and $\delta \le \min\{\tilde{\delta}, \varepsilon\}$, by Claim 5 we have that

$$I(\xi,\eta,s,u_1(\xi,\eta)) + I(\xi,\eta,u_2(\xi,\eta),t) \in B(2r) + b_1^h(\xi,\eta,s) + b_2^h(\xi,\eta,t) \tag{3.66}$$

for every $|s-u_1(\xi,\eta)| < \delta$, $|t-u_2(\xi,\eta)| < \delta$, and $(\xi,\eta)\in V$; since h satisfies (3.45) and $2r < d/4$, we can apply Claim 4 to deduce that the set

$$B(2r) + (\rho_\varepsilon(\xi,\eta)\sin\tilde{\theta}(\eta), \rho_\varepsilon(\xi,\eta)\cos\tilde{\theta}(\eta)) + b_1^h(\xi,\eta,s) + b_2^h(\xi,\eta,t)$$

is contained in the ball centred at $(0,0)$ with radius $\gamma(\xi,\eta)$. Some easy geometric considerations show that the relation between θ_ε and $\tilde{\theta}$ (see (3.39)) implies that also the set

$$B(2r) + (\rho_\varepsilon(\xi,\eta)\sin\theta_\varepsilon(\eta), \rho_\varepsilon(\xi,\eta)\cos\theta_\varepsilon(\eta)) + b_1^h(\xi,\eta,s) + b_2^h(\xi,\eta,t) \tag{3.67}$$

is contained in the ball centred at $(0,0)$ with radius $\gamma(\xi,\eta)$, if the condition

$$|b_1^h(\xi,\eta,s) + b_2^h(\xi,\eta,t)| < 2r$$

holds (to make this true, take δ and V smaller if needed). Since

$$I(\xi,\eta,s,t) = I(\xi,\eta,s,u_1(\xi,\eta)) + I(\xi,\eta,u_1(\xi,\eta),u_2(\xi,\eta)) + I(\xi,\eta,u_2(\xi,\eta),t),$$

by (3.48), it follows that $I(\xi,\eta,s,t)$ belongs to the set (3.67), and then to the ball centred at $(0,0)$ with radius $\gamma(\xi,\eta)$ for every $|s-u_1(\xi,\eta)|<\delta$, $|t-u_2(\xi,\eta)|<\delta$, and $(\xi,\eta)\in V$. This concludes the proof of Step 1.

Step 2.– If ε is sufficiently small and $\delta\in(0,\varepsilon)$, condition (b1) holds for $|s-u_1(\xi,\eta)|\ge\delta$ or $|t-u_2(\xi,\eta)|\ge\delta$, and $(\xi,\eta)\in V$, provided V is small enough.

By using condition (3.63), arguing as in the proof of Claim 5, we can prove the following claim.

Claim 6.– There exist $r<1/4$ and $\varepsilon>0$ such that

$$I(\xi,0,u_2(\xi,0),t)\in B(r), \qquad I(\xi,0,s,u_1(\xi,0))\in B(r)$$

for $0<|s-u_1(\xi,0)|\le\varepsilon$, $0<|t-u_2(\xi,0)|\le\varepsilon$, and $(\xi,0)\in V$.

We can conclude the proof of Step 2 in the same way as in Theorem 3.2, with the minor changes due to the different definition of the field.

By Step 1 and Step 2, we conclude that, choosing ε sufficiently small and n in a suitable way, condition (b1) is true for $u_1(\xi,\eta)-\varepsilon\le s,t\le u_2(\xi,\eta)+\varepsilon$. So, φ is a calibration.

Construction of the calibration around the graph of u

Now the matter is to extend the field in a tubular neighbourhood of the graph of u. From now on, we reintroduce the Cartesian coordinates.

Let Γ_i be the curve $\eta=(-1)^ik$, where $k>0$. If k is sufficiently small, for $i=1,2$ the curve Γ_i connects two points of $\partial\Omega_i'$, divides Ω_i' (and then Ω) in two connected components, and the normal vector ν_i to Γ_i which points towards Γ coincides with $(-1)^{i+1}\nabla\eta/|\nabla\eta|$. Set $U' := U\cap\{(x,y)\in\Omega' : |\eta(x,y)|<k\}$ and $U'' := U'\cap\Omega$. Since $\|\nabla\eta\|=1$ on Γ, by (3.63) we can suppose that

$$\frac{M}{1-Mk}\max_{i=1,2}\|\nabla\eta\|_{L^\infty(\Gamma_i)} < \min_{i=1,2} K(\Gamma_i,\Omega_i'\setminus\overline{U'}). \tag{3.68}$$

Chosen δ so small that $(\operatorname{graph} u)_\delta \cap ((U'' \cap \Omega_1) \times \mathbb{R}) \subset A_1$ and $(\operatorname{graph} u)_\delta \cap ((U'' \cap \Omega_2) \times \mathbb{R}) \subset A_7$, we define the vectorfield

$$\hat{\varphi}(x,y,z) = (\hat{\varphi}^{xy}(x,y,z), \hat{\varphi}^z(x,y,z)) \in \mathbb{R}^2 \times \mathbb{R},$$

as follows:

$$\begin{cases} \varphi(x,y,z) & \text{in } \{(x,y,z) \in U'' \times \mathbb{R} : \, u_1(x,y) - \delta < z < u_2(x,y) + \delta\}, \\ \left(2\nabla u + 2\frac{z-u}{\hat{v}_1}\nabla \hat{v}_1, \left|\nabla u + \frac{z-u}{\hat{v}_1}\nabla \hat{v}_1\right|^2\right) & \text{in } (\operatorname{graph} u)_\delta \cap (\Omega_1 \setminus U'') \times \mathbb{R}, \\ \left(2\nabla u + 2\frac{z-u}{\hat{v}_2}\nabla \hat{v}_2, \left|\nabla u + \frac{z-u}{\hat{v}_2}\nabla \hat{v}_2\right|^2\right) & \text{in } (\operatorname{graph} u)_\delta \cap (\Omega_2 \setminus U'') \times \mathbb{R}. \end{cases}$$

The function $\hat{v}_i$ is the solution of the problem

$$\min\left\{\int_{\Omega_i' \setminus \overline{U'}} |\nabla v|^2 dx\, dy - \frac{M}{1-Mk}\int_{\Gamma_i} |\nabla \eta|\, v^2 d\mathcal{H}^1 \,:\, v \in H^1(\Omega_i' \setminus \overline{U'}),\, v|_{\partial(\Omega_i' \setminus \overline{U'}) \setminus \Gamma_i} = 1\right\}. \tag{3.69}$$

Let us show that the problem (3.69) admits a solution. If $\{v_n\}$ is a minimizing sequence, then

$$\sup_n \left\{\int_{\Omega_i' \setminus \overline{U'}} |\nabla v_n|^2 dx\, dy - \frac{M}{1-Mk}\int_{\Gamma_i} |\nabla \eta|\, v_n^2 \, d\mathcal{H}^1\right\} < +\infty. \tag{3.70}$$

We have only to show that $\{v_n\}$ is bounded in $H^1(\Omega_i' \setminus \overline{U'})$. If we put $\overline{v}_n := v_n - 1$, by (3.58) for every $\tau \in (0,1)$ we have

$$\begin{aligned} \int_{\Omega_i' \setminus \overline{U'}} |\nabla v_n|^2 dx\, dy &= \int_{\Omega_i' \setminus \overline{U'}} |\nabla \overline{v}_n|^2 dx\, dy = \left(\int_{\Gamma_i} \overline{v}_n^2 d\mathcal{H}^1\right) \int_{\Omega_i' \setminus \overline{U'}} \left|\nabla\left(\frac{\overline{v}_n}{(\int_{\Gamma_i} \overline{v}_n^2 d\mathcal{H}^1)^{\frac{1}{2}}}\right)\right|^2 dx\, dy \\ &\geq \left(\int_{\Gamma_i} (v_n - 1)^2 d\mathcal{H}^1\right) K(\Gamma_i, \Omega_i' \setminus \overline{U'}) \\ &\geq (1-\tau) K(\Gamma_i, \Omega_i' \setminus \overline{U'}) \int_{\Gamma_i} v_n^2 d\mathcal{H}^1 + K(\Gamma_i, \Omega_i' \setminus \overline{U'}) \left(1 - \frac{1}{\tau}\right) \mathcal{H}^1(\Gamma_i), \end{aligned} \tag{3.71}$$

where we used Cauchy inequality. By (3.68), we can choose τ so small that

$$(1-\tau) K(\Gamma_i, \Omega_i' \setminus \overline{U'}) > \frac{M}{1-Mk} \|\nabla \eta\|_{L^\infty(\Gamma_i)},$$

and substituting (3.71) in (3.70), we obtain

$$\sup_n \int_{\Gamma_i} v_n^2 \, d\mathcal{H}^1 < +\infty.$$

Using again (3.70) and Poincaré inequality, we conclude that $\{v_n\}$ is actually bounded in $H^1(\Omega_i' \setminus \overline{U'})$.

The solution of (3.69) satisfies

$$\begin{cases} \Delta \hat{v}_i = 0 & \text{in } \Omega_i' \setminus \overline{U'}, \\ \partial_\nu \hat{v}_i = \dfrac{M}{1-Mk} |\nabla \eta| \hat{v}_i & \text{on } \Gamma_i, \\ \hat{v}_i = 1 & \text{on } \partial(\Omega_i' \setminus \overline{U'}) \setminus \Gamma_i, \end{cases} \tag{3.72}$$

and so, in particular, belongs to $C^\infty(\overline{\Omega_i \setminus U''})$. By a truncation argument, it is easy to see that $\hat{v}_i \geq 1$, so $\hat{\varphi}$ is well defined.

Since $\hat{\varphi}$ is a calibration in the set $\{(x,y,z) \in U'' \times \mathbb{R} : u_1(x,y) - \delta < z < u_2(x,y) + \delta\}$, it remains to prove only that the field is globally divergence-free in the sense of distributions and that conditions (a1), (a2), (b1) are verified in the regions $(\operatorname{graph} u)_\delta \cap (\Omega_i \setminus U'') \times \mathbb{R}$. First of all, note that by Lemma 1.5 the field $\hat{\varphi}$ is divergence-free in the regions $(\operatorname{graph} u)_\delta \cap (\Omega_i \setminus U'') \times \mathbb{R}$, since it is constructed starting from the family of harmonic functions $u(x,y) + t\hat{v}_i(x,y)$. To complete the proof, we need to check that the normal components of the traces of φ and of the extension field are equal on the surface of separation, i.e.,

$$\langle \varphi^{xy}, \nu_i \rangle = \left\langle 2\nabla u + 2\frac{z-u}{\hat{v}_i}\nabla \hat{v}_i, \nu_i \right\rangle \qquad \text{on } \Gamma_i, \tag{3.73}$$

where $\nu_i = (-1)^{i+1}\nabla\eta/|\nabla\eta|$. Using the definition of φ, we obtain that

$$\langle \varphi^{xy}, \nu_i \rangle = \left((-1)^{i+1}\partial_\eta u + \frac{z-u}{1-Mk}M \right) |\nabla\eta|;$$

since $\langle \nabla u, \nu_i \rangle = (-1)^{i+1}\partial_\eta u |\nabla\eta|$, the equality (3.73) is equivalent to

$$\frac{M}{1-Mk}|\nabla\eta| = \frac{1}{\hat{v}_i}\langle \nabla\hat{v}_i, \nu_i \rangle,$$

which is true by (3.72).

Conditions (a1) and (a2) are obviously satisfied, while condition (b1) is true if we take δ satisfying

$$\delta \leq \sup \left\{ \left(4|\nabla u| + 2\frac{|\nabla \hat{v}_i|}{\hat{v}_i} \right)^{-1} : (x,y) \in \Omega_i \setminus U'',\ i = 1,2 \right\}.$$

Therefore, with this choice of δ, the vectorfield $\hat{\varphi}$ is a calibration. □

3.2.1 Some properties of $K(\Gamma, U)$

In this subsection we investigate some qualitative properties of the quantity $K(\Gamma, U)$ and we shall compute it explicitly in a very particular case. Let us start by a very simple result.

Proposition 3.6 *Let Γ be a simple analytic curve and $\tilde{\Gamma}$ be an extension of Γ, whose endpoints do not coincide with the endpoints of Γ. If $\Gamma_\delta^\pm$ are the two connected components of $\Gamma_\delta \setminus \tilde{\Gamma}$ (which are well defined if δ is sufficiently small), then*

$$\lim_{\delta \to 0^+} K(\Gamma, \Gamma_\delta^\pm) = +\infty.$$

Proof. – For convenience we set

$$W^\pm(\delta) := \left\{ v \in H^1(\Gamma_\delta^\pm) : \int_\Gamma v^2 d\mathcal{H}^1 = 1,\ v = 0 \text{ on } \partial(\Gamma_\delta^\pm) \setminus \Gamma \right\}.$$

Suppose by contradiction that there exists a sequence $\{\delta_n\}$ decreasing to 0 such that $\sup_n K(\Gamma, \Gamma_{\delta_n}^+) = c < +\infty$; this implies the existence of a sequence $\{v_n\}$ such that

$$v_n \in W^+(\delta_n) \qquad \text{and} \qquad \int_{\Gamma_{\delta_n}^+} |\nabla v_n(x,y)|^2 dx\, dy \leq c$$

for every integer n. From now on, we regard v_n as a function belonging to $H^1(\Gamma^+_{\delta_1})$ which vanishes on $\Gamma^+_{\delta_1} \setminus \Gamma^+_{\delta_n}$. By Poincaré inequality it follows immediately that $\{v_n\}$ is bounded in $H^1(\Gamma^+_{\delta_1})$, and so it admits a weakly convergent subsequence $\{v_{n_k}\}$. Let us call v the limit of the subsequence; since v_{n_k} vanishes on $\Gamma^+_{\delta_1} \setminus \Gamma^+_{\delta_{n_k}}$ for every k, then v must vanish a.e.; on the other hand, since $\int_\Gamma v^2_{n_k} d\mathcal{H}^1 = 1$, by the compactness of the trace operator, we have that $\int_\Gamma v^2 d\mathcal{H}^1 = 1$, and this is clearly impossible. □

We remark that by Theorem 3.5 and Proposition 3.6, if U_0 is a neighbourhood of Γ and $u \in SBV(U_0)$ satisfies the Euler conditions in U_0 with $S_u = \Gamma$, then there exists a neighbourhood U of Γ contained in U_0 such that u is a Dirichlet graph-minimizer in U. Actually, taking U smaller if needed, by Theorem 3.2 we get also the Dirichlet minimality.

Proposition 3.7 (Characterization of $K(\Gamma,U)$) *Let U be an open set with Lipschitz boundary and Γ be a subset of ∂U with nonempty relative interior in ∂U. The constant $K(\Gamma,U)$ is the first eigenvalue of the problem*

$$\begin{cases} \Delta u = 0 & \text{on } U, \\ \partial_\nu u = \lambda u & \text{on } \Gamma, \\ u = 0 & \text{on } \partial U \setminus \Gamma. \end{cases} \tag{3.74}$$

Moreover, it is the unique eigenvalue with a positive eigenfunction.

Proof. – If u is a solution of (3.58), then it is harmonic and there exists a Lagrange multiplier λ such that

$$\int_U \langle \nabla u, \nabla \varphi \rangle \, dx\, dy = \lambda \int_\Gamma u\varphi \, d\mathcal{H}^1 \qquad \forall \varphi \in C^\infty(U): \ \varphi = 0 \text{ on } \partial U \setminus \Gamma, \tag{3.75}$$

which means, by Green formula, that $\partial_\nu u = \lambda u$ on Γ. Using (3.75), one can easily see that $K(\Gamma,U)$ is in fact the minimal eigenvalue of (3.74) and that it has a positive eigenfunction (indeed, if u is a solution also $|u|$ is). Let u be a positive function belonging to the eigenspace of $K(\Gamma,U)$ and v another positive eigenfunction associated with the eigenvalue μ; by Green formula we have

$$\int_\Gamma v \, \partial_\nu u \, d\mathcal{H}^1 - \int_\Gamma u \, \partial_\nu v \, d\mathcal{H}^1 = 0,$$

therefore

$$(K(\Gamma,U) - \mu) \int_\Gamma uv \, d\mathcal{H}^1 = 0.$$

Since both u and v are positive, from the last equality it follows that $\mu = K(\Gamma,U)$. □

Proposition 3.8 *If $U = (0,a)\times(0,b)$ and $\Gamma = (0,a)\times\{0\}$, then*

$$K(\Gamma,U) = \frac{\pi}{a \tanh\left(\frac{\pi b}{a}\right)}. \tag{3.76}$$

Proof. – The function

$$v(x,y) = \sin\left(\frac{\pi}{a}x\right) \sinh\left(\frac{\pi}{a}(b-y)\right)$$

is positive and satisfies (3.74) with $\lambda = \dfrac{\pi}{a\tanh\left(\frac{\pi b}{a}\right)}$. Then, by Proposition 3.7, this quantity coincides with $K(\Gamma,U)$. □

Proposition 3.9 *Let* $g : [0, a_0] \to [0, +\infty)$ *be a Lipschitz function and denote the graph of* g *by* Γ. *Given* $0 \leq a_1 < a_2 \leq a_0$ *and* $b > 0$, *if we set* $\Gamma(a_1, a_2) := \operatorname{graph} g|_{(a_1,a_2)}$ *and*

$$R(a_1, a_2, b) := \{(x, y) : x \in (a_1, a_2),\ y \in (g(x), g(x) + b)\},$$

then

$$\lim_{|a_2 - a_1| \to 0} K\left(\Gamma(a_1, a_2), R(a_1, a_2, b)\right) = +\infty \qquad \textit{uniformly with respect to } b.$$

Proof. – The idea is to transform the region $R(a_1, a_2, b)$ into the rectangle $(0, a_2 - a_1)\times(0, b)$ by a suitable diffeomorphism in order to use (3.76).

Let $\psi : (0, a_2 - a_1)\times(0, b) \to R(a_1, a_2, b)$ be the map defined by $\psi(x, y) = (x + a_1, y + g(x + a_1))$. Let $v \in H^1(R(a_1, a_2, b))$ be such that $v = 0$ on $\partial R(a_1, a_2, b) \setminus \Gamma(a_1, a_2)$ and

$$\int_{\Gamma(a_1,a_2)} v^2 d\mathcal{H}^1 = \int_0^{a_2-a_1} v^2(\psi(x, 0))\sqrt{1 + (g'(x))^2}\, dx = 1. \tag{3.77}$$

If we call $\tilde{v}(x, y) := v(\psi(x, y))$, then $\tilde{v} \in H^1((0, a_2 - a_1)\times(0, b))$, $\tilde{v} = 0$ on the boundary of the rectangle except $(0, a_2 - a_1)\times\{0\}$, and by (3.77) there exists $\lambda > 0$ such that $\lambda^2 \leq \sqrt{1 + \|g'\|^2_\infty}$ and

$$\lambda^2 \int_0^{a_2-a_1} \tilde{v}^2(x, 0)\, dx = 1.$$

Therefore, since $J\psi \equiv 1$,

$$\begin{aligned}
\int_{R(a_1,a_2,b)} |\nabla v(x, y)|^2 dx\, dy &= \int_{(0,a_2-a_1)\times(0,b)} |\nabla v(\psi(x, y))|^2 dx\, dy \\
&\geq (1 + \|g'\|_\infty + \|g'\|^2_\infty)^{-1} \int_{(0,a_2-a_1)\times(0,b)} |\nabla \tilde{v}(x, y)|^2 dx\, dy \\
&\geq \lambda^{-2}(1 + \|g'\|_\infty + \|g'\|^2_\infty)^{-1} K\Big((0, a_2 - a_1)\times\{0\}, (0, a_2 - a_1)\times(0, b)\Big) \\
&\geq (1 + \|g'\|^2_\infty)^{-3/2} \frac{\pi}{2(a_2 - a_1)\tanh\left(\frac{\pi b}{a_2-a_1}\right)},
\end{aligned}$$

where the last inequality follows by the estimate on λ and by (3.76). Since v is arbitrary, using the fact that $0 < \tanh t \leq 1$ for every $t > 0$, we obtain that

$$K\left(\Gamma(a_1, a_2), R(a_1, a_2, b)\right) \geq (1 + \|g'\|_\infty)^{-3/2} \frac{\pi}{2(a_2 - a_1)};$$

so, the conclusion is clear. □

We have already remarked (see Proposition 3.6) that the graph-minimality is guaranteed in small neighbourhoods of the discontinuity set Γ. As a consequence of Proposition 3.9, we obtain that the graph-minimality holds also in the open sets, which are narrow along the direction parallel to Γ and may be very large along the normal direction. This is made precise by the following corollary.

Corollary 3.10 *Let* g *be a positive function, analytic on* $[0, a_0]$, *that is* g *admits an analytic extension, and denote the graph of* g *by* Γ. *For every* $M > 0$ *there exists* $h = h(M, \Gamma)$ *such that, if* Ω *is* Γ-*admissible (see Definition 3.4) and* $\Omega \subset (a_1, a_1 + h)\times\mathbb{R}$ *with* $a_1 \in [0, a_0 - h]$, *and if* u *is a function in* $SBV(\Omega)$ *with* $S_u = \Gamma\cap\Omega$, *with different traces at every point of* $\Gamma\cap\Omega$, *satisfying the Euler conditions in* Ω, *and* $\sum_{i=1}^2 \|\partial_\tau u_i\|_{C^1(\Gamma\cap\Omega)} \leq M$ *(where* u_i *is as above the restriction of* u *to the connected component* Ω_i *of* $\Omega\setminus\Gamma$*), then* u *is a Dirichlet graph-minimizer in* Ω *(see Fig. 3.2).*

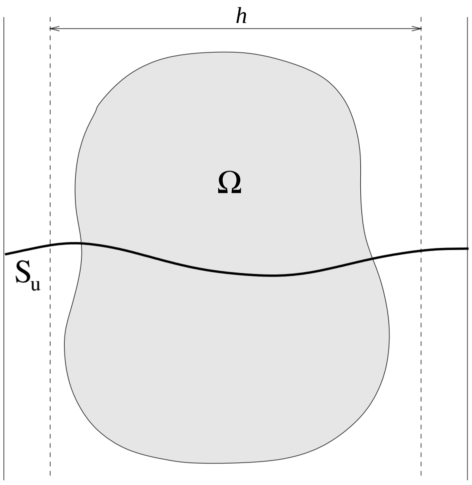

Figure 3.2: if the thickness of Ω is less than h, then u is a Dirichlet graph-minimizer in Ω.

Proof. – By Proposition 3.9 there exists $h > 0$ such that for every $a_1, a_2 \in [0, a_0]$ with $0 < a_2 - a_1 \leq h$ and for every $b > 0$,

$$\frac{K(\Gamma(a_1, a_2), R(a_1, a_2, b))}{1 + l^2(\Gamma) + l^2(\Gamma)k^2(\Gamma)} > c\,M^2.$$

If $\Omega \subset (a_1, a_1 + h) \times \mathbb{R}$, then we can choose $b > 0$ so large that, assuming that Ω_1 is the upper component, $\Omega_1 \subset R(a_1, a_1 + h, b)$. Then by the monotonicity properties of $K(\Gamma, A)$, it follows that

$$\frac{K(\Gamma \cap \Omega, \Omega_1)}{1 + l^2(\Gamma) + l^2(\Gamma)k^2(\Gamma)} > c\,M^2 \geq c \sum_{i=1}^{2} \|\partial_\tau u_i\|^2_{C^1(\Gamma \cap \Omega)}.$$

Applying the same argument to Ω_2, the conclusion follows from Theorem 3.5. □

Chapter 4

Calibrations for minimizers with a triple junction

In this chapter we study the Dirichlet minimality of solutions of the Euler-Lagrange equations for the Mumford-Shah functional (2.1) whose discontinuity set presents a triple junction.

The precise statement of the result is the following.

Theorem 4.1 *Let $\Omega := B(0,1)$ be the open disc in $\mathbb{R}^2$ with radius 1 centred at the origin, and let (A_0, A_1, A_2) be the partition of Ω defined as follows:*

$$A_i := \left\{(r\cos\theta, r\sin\theta) \in \Omega :\ 0 \le r < 1,\ \frac{2}{3}\pi(2-i) < \theta \le \frac{2}{3}\pi(3-i)\right\} \qquad \forall\, i = 0, 1, 2.$$

Let $S_{i,j} := \overline{A_i} \cap \overline{A_j}$ for every $i < j$. Let $u_i \in C^2(\overline{A_i})$ be a harmonic function in A_i, satisfying the Neumann conditions on $\partial A_i \cap \Omega$ and such that $|\nabla u_i| = |\nabla u_j|$ on $S_{i,j}$ for every $i < j$. If u is the function in $SBV(\Omega)$ defined by $u := u_i$ a.e. in each A_i and $u_0(0,0) < u_1(0,0) < u_2(0,0)$, then there exists a neighbourhood U of the origin such that u is a Dirichlet minimizer in U of the Mumford-Shah functional.

The proof is very long and technical and is split in several steps. First of all, the symmetry due to the $2\pi/3$-angles allows to deduce from the other Euler conditions that each u_i must be either symmetric or antisymmetric with respect to the bisecting line of A_i. In Section 4.1 we construct an explicit calibration φ in the case u_i symmetric and we prove that φ satisfies conditions (a1), (a2), (b2), and (c1) (see Section 1.3); in Sections 4.2 and 4.3 we show some estimates, which will be useful in Section 4.4 to prove condition (b1); finally, in Section 4.5 we adapt the calibration to the antisymmetric case.

4.1 Construction of the calibration

Let $\{e^x, e^y\}$ be the canonical basis in $\mathbb{R}^2$ and for $i = 1, 2$ consider the vectors $\tau_i = (-1/2, (-1)^i\sqrt{3}/2)$, $\nu_i = ((-1)^i\sqrt{3}/2, 1/2)$, which are tangent and normal to the set $S_{i-1,i}$ (see Fig. 4.1). As $u_0(0,0) < u_1(0,0) < u_2(0,0)$, there exists an open neighbourhood U of $(0,0)$ such that the function u belongs to $SBV(U)$, the discontinuity set S_u of u on U coincides with $\bigcup_{i<j}(S_{i,j} \cap U)$, and the oriented normal vector ν_u to S_u is given by

$$\nu_u(x,y) = \begin{cases} \nu_1 & \text{for } (x,y) \in S_{0,1}, \\ \nu_2 & \text{for } (x,y) \in S_{1,2}, \\ e^y & \text{for } (x,y) \in S_{0,2}; \end{cases}$$

by the assumptions on u_i, the function u satisfies the Euler conditions for (2.1) in U. We will construct a local calibration $\varphi = (\varphi^{xy}, \varphi^z) : U \times \mathbb{R} \to \mathbb{R}^2 \times \mathbb{R}$ for u.

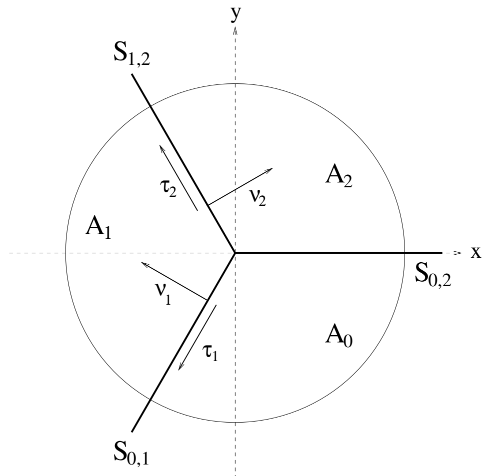

Figure 4.1: the triple junction.

Applying Schwarz reflection principle with respect to $S_{0,1}$ and $S_{0,2}$, the function u_0 can be harmonically extended to $U \setminus S_{1,2}$, and analogously u_1 and u_2 can be extended to $U \setminus S_{0,2}$ and $U \setminus S_{0,1}$, respectively. By the hypothesis on u_i and by Cauchy-Kowalevski theorem (see [24]) the extension of u_0 coincides, up to the sign and to additive constants, with u_1 on A_1 and with u_2 on A_2; analogously, the extension of u_1 coincides, up to the sign and to an additive constant, with u_2 on A_2. Since the composition of the three reflections with respect to $S_{0,1}$, $S_{1,2}$, and $S_{0,2}$ coincides with the reflection with respect to the bisecting line of the sector A_0, by the previous remarks we can deduce that u_0 is either symmetric or antisymmetric with respect to the bisecting line of A_0.

We consider first the case u_0 symmetric (the antisymmetric case will be studied in Section 4.5). Then also u_1, u_2 are symmetric with respect to the bisecting line of A_1, A_2, respectively, and the extensions of u_0, u_1, u_2 by reflection are well defined and harmonic in the whole set U.

In order to define the calibration for u, let $\varepsilon > 0$, $l_i \in (u_{i-1}(0,0), u_i(0,0))$ for $i = 1, 2$, and $\lambda > 0$ be suitable parameters that will be chosen later, and consider the following subsets of $U \times \mathbb{R}$:

$$
\begin{array}{lll}
G_i & := & \{(x,y,z) \in U \times \mathbb{R} : u_i(x,y) - \varepsilon < z < u_i(x,y) + \varepsilon\} \quad \text{for } i = 0,1,2, \\
K_i & := & \{(x,y,z) \in U \times \mathbb{R} : l_i + \alpha_i(x,y) < z < l_i + 2\lambda + \beta_i(x,y)\} \quad \text{for } i = 1,2, \\
H_i & := & \{(x,y,z) \in U \times \mathbb{R} : l_i + \lambda/2 < z < l_i + 3\lambda/2\} \quad \text{for } i = 1,2,
\end{array}
$$

where α_i and β_i are suitable Lipschitz functions such that $\alpha_i(0,0) = \beta_i(0,0) = 0$, which will be defined later. If ε and λ are sufficiently small, then for every i, j the sets G_i, K_j are nonempty and disjoint, while for every i the set H_i is compactly contained in K_i, provided U is small enough (see Fig. 4.2).

The aim of the definition of the calibration φ in G_i is to provide a divergence-free vectorfield satisfying condition (a2) and such that

$$
\begin{array}{ll}
\langle \varphi^{xy}(s\tau_i, z), \nu_i \rangle > 0 & \text{for } u_{i-1} < z < u_{i-1} + \varepsilon \text{ and for } u_i - \varepsilon < z < u_i, \\
\langle \varphi^{xy}(s\tau_i, z), \nu_i \rangle < 0 & \text{for } u_{i-1} - \varepsilon < z < u_{i-1} \text{ and for } u_i < z < u_i + \varepsilon,
\end{array}
$$

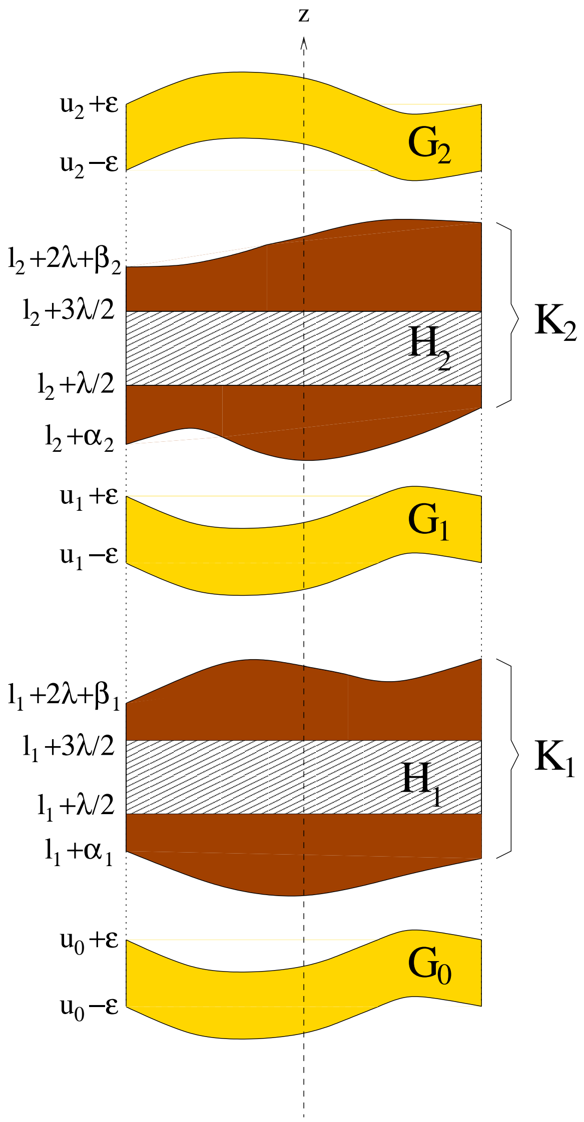

Figure 4.2: section of the sets G_i, K_i, H_i at $x =$ constant.

for $i = 1, 2$ and $s \geq 0$, and analogously

$$\begin{aligned} &\langle \varphi^{xy}(s,0,z), e^y\rangle > 0 \quad \text{for } u_0 < z < u_0+\varepsilon \text{ and for } u_2-\varepsilon < z < u_2, \\ &\langle \varphi^{xy}(s,0,z), e^y\rangle < 0 \quad \text{for } u_0-\varepsilon < z < u_0 \text{ and for } u_2 < z < u_2+\varepsilon; \end{aligned}$$

these properties are crucial in order to obtain (b1) and (b2) simultaneously. Such a field can be obtained by applying the technique shown in Lemma 1.5, starting from the family of harmonic functions $u_i + tv_i$, where we choose as v_i the linear functions defined by

$$v_0(x,y) := \langle \tau_2, (x,y)\rangle + \varepsilon, \quad v_1(x,y) := \langle e^x, (x,y)\rangle + \varepsilon, \quad v_2(x,y) := \langle \tau_1, (x,y)\rangle + \varepsilon.$$

So for every $(x,y,z) \in G_i$, $i=0,1,2$, we define the vector $\varphi(x,y,z)$ as

$$\left(2\nabla u_i + 2\,\frac{z-u_i(x,y)}{v_i(x,y)}\nabla v_i, \left|\nabla u_i + \frac{z-u_i(x,y)}{v_i(x,y)}\nabla v_i\right|^2\right).$$

The role of K_i is to give the exact contribution to the integral in (b2). In order to annihilate the tangential contribution on S_u given by the field in G_i, we insert in K_i the region H_i and for every $(x,y,z) \in H_i$, $i=1,2$, we define $\varphi(x,y,z)$ as

$$\left(-\frac{2\varepsilon}{\lambda}\left(\nabla u_{i-1} + \nabla u_i\right), \mu\right)$$

where μ is a positive constant which will be suitably chosen later. By the harmonicity of u_i this field is divergence-free and, as $\partial_\nu u_i = 0$ on S_u for every i, its horizontal component is purely tangential on S_u. So, it remains to correct only the normal contribution to the integral in (b2) due to the field in G_i. To realize this purpose on the two segments $S_{i-1,i}$, $i=1,2$, we could require that $\alpha_i(s\tau_i) = \beta_i(s\tau_i) = 0$ for every $s \geq 0$ (see the definition of K_i) and define $\varphi(x,y,z)$ for $(x,y,z) \in K_i \setminus \overline{H_i}$ as

$$\left(\frac{1}{\lambda}\, g\Big(\langle \tau_i, (x,y)\rangle\Big)\,\nu_i, \mu\right), \tag{4.1}$$

where g is a function of real variable chosen in such a way that (b2) is satisfied for $(x,y) \in S_{i-1,i}$, i.e.,

$$g(t) := 1 - \sqrt{3}\frac{\varepsilon^2}{v_0(t,0)} \qquad \forall t \in \mathbb{R},$$

as we will see later in (4.19). Note that the two-dimensional field $g\left(\langle \tau_i, (x,y)\rangle\right)\nu_i$ is divergence-free, since it is with respect to the orthonormal basis $\{\tau_i, \nu_i\}$, hence φ is divergence-free in $K_i \setminus \overline{H_i}$; moreover, since $\varphi^z \equiv \mu$ on K_i, the normal component of φ is continuous across the boundary of H_i, so that φ turns out to be divergence-free in the sense of distributions in the whole set K_i. Actually it is crucial to add a component along the direction τ_i to the field in (4.1) in order to make (b1) true, as it will be clear in the proof of Step 2 (see Section 4.3); this component has to be chosen in such a way that it is zero on $S_{i-1,i}$ (so that (b2) remains valid on these segments) and that it depends only on $\langle \nu_i, (x,y)\rangle$ (so that the field remains divergence-free). Therefore we replace in (4.1) the vector $g\left(\langle \tau_i, (x,y)\rangle\right)\nu_i$ by

$$\phi_i(x,y) := (-1)^{i+1} f\Big(\langle \nu_i, (x,y)\rangle\Big)\,\tau_i + g\Big(\langle \tau_i, (x,y)\rangle\Big)\,\nu_i, \tag{4.2}$$

where f is an even smooth function of real variable such that $f(0)=0$ and which will be chosen later in a suitable way (see (4.74)). From this definition it follows that

$$\phi_2^x(x,y) = -\phi_1^x(x,-y), \qquad \phi_2^y(x,y) = \phi_1^y(x,-y), \tag{4.3}$$

so that

$$\phi_1(x,0) + \phi_2(x,0) = 2\phi_1^y(x,0)e^y,$$

i.e., if we assume that $\alpha_i(x,0) = \beta_i(x,0)$ for every $x \geq 0$, the contribution given by the fields (4.2) to the integral in (b2) computed at a point of $S_{0,2}$ is purely normal, as required in (b2), but its modulus is in general different from what we need to obtain exactly the normal vector e^y. In order to correct it, we multiply ϕ_i by a function σ_i which is first defined on $S_{i-1,i} \cup S_{0,2}$ (more precisely, σ_i is taken equal to 1 on $S_{i-1,i}$ and to the correcting factor on $S_{0,2}$); then, we extend it to a neighbourhood of $(0,0)$ by assuming σ_i constant along the integral curves of ϕ_i, so that $\sigma_i\phi_i$ remains divergence-free.

The integral curves of ϕ_i can be represented as the curves $\{(x,y) \in U : y = \psi_i(x,s)\}$, where $\psi_i(x,s)$ is the solution of the problem

$$\begin{cases} \partial_x \psi_i(x,s) \phi_i^x(x, \psi_i(x,s)) - \phi_i^y(x, \psi_i(x,s)) = 0, \\ \psi_i(s,s) = 0, \end{cases} \tag{4.4}$$

which is defined in a sufficiently small neighbourhood of $(0,0)$. By applying the Implicit Function theorem, it is easy to see that if U is small enough, then there exists a unique smooth function h_i defined in U such that

$$h_i(0,0) = 0, \qquad \psi_i(x, h_i(x,y)) = y. \tag{4.5}$$

Note that the curve $\{(x,y) \in U : h_i(x,y) = s\}$ coincides with the integral curve $\{(x,y) \in U : y = \psi_i(x,s)\}$ and that $(h_i(x,y), 0)$ gives the intersection point of the integral curve passing through (x,y) with the x-axis; in other words, the level lines of h_i provide a different representation of the integral curves of ϕ_i in terms of their intersection point with the x-axis.

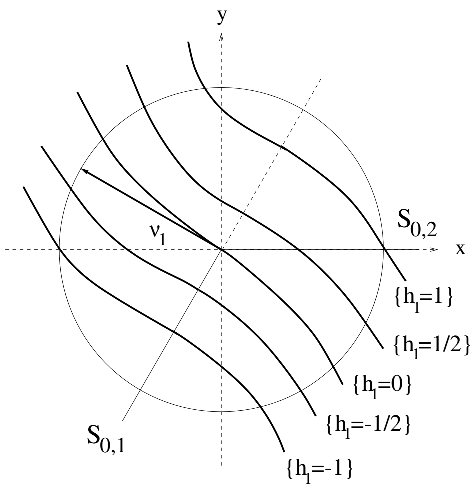

Figure 4.3: integral curves of the field ϕ_1.

We state here some properties of h_i and ψ_i for further references. Since $\psi_i(s,s) = 0$, we have that

$$h_i(s,0) = s \tag{4.6}$$

for every s such that $(s,0) \in U$. By (4.4) and by differentiating the initial condition in (4.4) with respect to s, we obtain

$$\partial_x \psi_i(0,0) = \frac{\phi_i^y(0,0)}{\phi_i^x(0,0)} = \frac{\nu_i^y}{\nu_i^x} = \frac{(-1)^i}{\sqrt{3}}, \qquad \partial_s \psi_i(0,0) = -\partial_x \psi_i(0,0) = \frac{(-1)^{i+1}}{\sqrt{3}}. \tag{4.7}$$

By differentiating the equation in (4.4) with respect to x and to s, and by using (4.2), it is easy to see that

$$\partial^2_{xx} \psi_i(0,0) = \partial^2_{xs} \psi_i(0,0) = 0, \tag{4.8}$$

while by differentiating twice with respect to s the initial condition $\psi_i(s,s)=0$, we obtain that

$$\partial^2_{ss}\psi_i(0,0)=-2\partial^2_{xs}\psi_i(0,0)=0. \tag{4.9}$$

By (4.7) and (4.8), the curve $\{h_i=0\}$ (which coincides with $\{y=\psi_i(x,0)\}$) is tangent to ν_i at 0, which may be an inflection point. Moreover, since $\partial_x\psi_i(0,0)\neq 0$, by continuity the function $\psi_i(\cdot,s)$ is strictly monotone in a small neighbourhood of 0 for s sufficiently small; by this fact and by comparing the values of the function $\psi_i(\cdot,h_i(s\tau_i))$ at the points $h_i(s\tau_i)$ and $s\tau_i^x$, it is easy to see that

$$h_i(s\tau_i)\le 0 \tag{4.10}$$

for every $s\ge 0$ such that $s\tau_i\in U$, provided U is small enough. Remark that by (4.6) and (4.10) it follows that the segment $S_{0,2}$ is all contained in the region $\{h_i\ge 0\}$, while $S_{i-1,i}$ in the region $\{h_i\le 0\}$.

At last, we set

$$\sigma_i(x,y):=\begin{cases}1 & \text{if } h_i(x,y)\le 0,\\ \dfrac{g(h_i(x,y))}{2\phi_i^y(h_i(x,y),0)} & \text{if } h_i(x,y)>0;\end{cases}$$

since by definition $\phi_i^y(0,0)=g(0)\nu_i^y=g(0)/2$, the function σ_i is continuous across the curve $\{h_i=0\}$. Moreover, remark that from (4.3) it follows that $\psi_2(x,s)=-\psi_1(x,s)$, $h_2(x,y)=h_1(x,-y)$, and then

$$\sigma_2(x,y)=\sigma_1(x,-y). \tag{4.11}$$

For every $(x,y,z)\in K_i\setminus\overline{H_i}$, $i=1,2$, we define $\varphi(x,y,z)$ as

$$\left(\frac{1}{\lambda}\sigma_i(x,y)\phi_i(x,y),\mu\right).$$

In the remaining regions of transition it is convenient to take φ purely vertical. In order to make φ divergence-free in the whole set $U\times\mathbb{R}$, we need the normal component of φ to be continuous across the boundary of G_i and H_i. To guarantee this continuity across ∂G_i, we are forced to take as third component of φ the function

$$\omega(x,y,z):=\begin{cases}\dfrac{\varepsilon^2}{v_0^2(x,y)}-|\nabla u_0|^2 & \text{for } z<l_1+\lambda,\\[2ex] \dfrac{\varepsilon^2}{v_1^2(x,y)}-|\nabla u_1|^2 & \text{for } l_1+\lambda\le z<l_2+\lambda,\\[2ex] \dfrac{\varepsilon^2}{v_2^2(x,y)}-|\nabla u_2|^2 & \text{for } z\ge l_2+\lambda.\end{cases} \tag{4.12}$$

Finally, we define the functions α_i,β_i in such a way that the normal component of φ turns out to be continuous also across the boundary of K_i; more precisely, for $i=1,2$ we choose α_i as the solution of the Cauchy problem

$$\begin{cases}\dfrac{1}{\lambda}\sigma_i(x,y)\langle\phi_i(x,y),\nabla\alpha_i(x,y)\rangle-\mu=-\dfrac{\varepsilon^2}{v_{i-1}^2(x,y)}+|\nabla u_{i-1}(x,y)|^2,\\ \alpha_i(s\tau_i)=0,\ \alpha_i(s,0)=0 \quad \text{for } s\ge 0,\end{cases}$$

while β_i as the solution of

$$\begin{cases}\dfrac{1}{\lambda}\sigma_i(x,y)\langle\phi_i(x,y),\nabla\beta_i(x,y)\rangle-\mu=-\dfrac{\varepsilon^2}{v_i^2(x,y)}+|\nabla u_i(x,y)|^2,\\ \beta_i(s\tau_i)=0,\ \beta_i(s,0)=0 \quad \text{for } s\ge 0.\end{cases}$$

Since σ_i is not C^1 near the curve $\{h_i = 0\}$, we cannot expect a C^1-solution. Nevertheless, if U is small enough, then α_i, β_i are Lipschitz functions defined in U, and the possible discontinuity points of $\nabla\alpha_i, \nabla\beta_i$ concentrate only on the curve $\{h_i = 0\}$; indeed, if U is sufficiently small, the Cauchy problems

$$\begin{cases} \dfrac{1}{\lambda}\langle \phi_i(x,y), \nabla\tilde{\alpha}_i(x,y)\rangle - \mu = -\dfrac{\varepsilon^2}{v_{i-1}^2(x,y)} + |\nabla u_{i-1}(x,y)|^2, \\ \tilde{\alpha}_i(s\tau_i) = 0 \quad (s \in \mathbb{R}), \end{cases} \tag{4.13}$$

and

$$\begin{cases} \dfrac{g(h_i(x,y))}{2\lambda\phi_i^y(h_i(x,y),0)}\langle \phi_i(x,y), \nabla\hat{\alpha}_i(x,y)\rangle - \mu = -\dfrac{\varepsilon^2}{v_{i-1}^2(x,y)} + |\nabla u_{i-1}(x,y)|^2, \\ \hat{\alpha}_i(s,0) = 0 \quad (s \in \mathbb{R}), \end{cases} \tag{4.14}$$

admit a unique solution $\tilde{\alpha}_i, \hat{\alpha}_i \in C^\infty(U)$, since the lines $\{s\tau_i : s \in \mathbb{R}\}$ and $\{(s,0) : s \in \mathbb{R}\}$ are not characteristic for these equations. Since the curve $\{h_i = 0\}$, which coincides with the curve $\{y = \psi_i(x,0)\}$, is a characteristic line of both equations (4.13) and (4.14) (use (4.4) and $g(0)/(2\lambda\phi_i^y(0,0)) = 1$), the functions $\tilde{\alpha}_i, \hat{\alpha}_i$ assume the same value on the curve $\{h_i = 0\}$. So, α_i can be regarded as the function defined by

$$\alpha_i(x,y) := \begin{cases} \tilde{\alpha}_i(x,y) & \text{if } h_i(x,y) \le 0, \\ \hat{\alpha}_i(x,y) & \text{if } h_i(x,y) > 0, \end{cases}$$

and therefore α_i is C^∞ in $U \setminus \{h_i = 0\}$, and all derivatives of α_i have finite limits on both sides of $\{h_i = 0\}$. The same argument works for β_i.

The complete definition of the field is therefore the following: for every $(x,y,z) \in U \times \mathbb{R}$, the vector $\varphi(x,y,z) = (\varphi^{xy}, \varphi^z)(x,y,z) \in \mathbb{R}^2 \times \mathbb{R}$ is given by

$$\begin{cases} \left(2\nabla u_i + 2\,\frac{z-u_i(x,y)}{v_i(x,y)}\nabla v_i, \left|\nabla u_i + \frac{z-u_i(x,y)}{v_i(x,y)}\nabla v_i\right|^2\right) & \text{in } G_i \quad (i = 0,1,2), \\ \left(\frac{1}{\lambda}\sigma_i(x,y)\phi_i(x,y), \mu\right) & \text{in } K_i \setminus \overline{H_i} \quad (i = 1,2), \\ \left(-\frac{2\varepsilon}{\lambda}\left(\nabla u_{i-1} + \nabla u_i\right), \mu\right) & \text{in } H_i \quad (i = 1,2), \\ (0, \omega(x,y,z)) & \text{otherwise.} \end{cases}$$

Condition (a1) is trivial in G_i for all i.

Since $\nabla u_i(0,0) = 0$ for all i (this fact easily follows by the assumptions on the regularity of u_i and by the Euler conditions), we have that

$$\frac{\varepsilon^2}{v_i^2(0,0)} - |\nabla u_i(0,0)|^2 = 1 > 0;$$

then, if U is small enough,

$$\frac{\varepsilon^2}{v_i^2(x,y)} - |\nabla u_i(x,y)|^2 > 0$$

for every $(x,y) \in U$ and for every $i = 0,1,2$, and so ω is always positive.

Arguing in a similar way, if we impose that $\mu > 1/(4\lambda^2)$, condition (a1) holds in K_i, provided U is sufficiently small.

By construction conditions (a2) and (c1) are satisfied.

By direct computations we find that for every $(x,y)\in U$

$$\int_{u_{i-1}}^{u_i}\varphi^{xy}\,dz=\frac{\varepsilon^2}{v_{i-1}}\nabla v_{i-1}-\frac{\varepsilon^2}{v_i}\nabla v_i+\frac{1}{\lambda}(\beta_i-\alpha_i+\lambda)\sigma_i\phi_i, \tag{4.15}$$

for $i=1,2$, while

$$\int_{u_0}^{u_2}\varphi^{xy}\,dz=\frac{\varepsilon^2}{v_0}\nabla v_0-\frac{\varepsilon^2}{v_2}\nabla v_2+\frac{1}{\lambda}\sum_{i=1}^{2}(\beta_i-\alpha_i+\lambda)\sigma_i\phi_i. \tag{4.16}$$

Note that for $i=1,2$

$$v_{i-1}(s\tau_i)=v_i(s\tau_i)=v_0(s,0)=-\frac{s}{2}+\varepsilon\quad\forall s\in\mathbb{R}, \tag{4.17}$$

$$\nabla v_{i-1}(x,y)-\nabla v_i(x,y)=\sqrt{3}\nu_i\qquad\forall(x,y)\in U. \tag{4.18}$$

As $h_i(s\tau_i)\le 0$ for every $s\ge 0$ by (4.10), we have that $\sigma_i(s\tau_i)=1$ for every $s\ge 0$, while by definition $\alpha_i(s\tau_i)=\beta_i(s\tau_i)=0$. From these facts, (4.15), (4.17), (4.18), and the definition of ϕ_i, we obtain

$$\int_{u_{i-1}(s\tau_i)}^{u_i(s\tau_i)}\varphi^{xy}(s\tau_i,z)\,dz=\sqrt{3}\frac{\varepsilon^2}{v_0(s,0)}\nu_i+(-1)^{i+1}f(0)\tau_i+g(s)\nu_i=\nu_i, \tag{4.19}$$

where the last equality follows from the definition of g and the fact that $f(0)=0$. Analogously, by the equalities

$$v_0(s,0)=v_2(s,0)\qquad\forall s\in\mathbb{R}, \tag{4.20}$$

$$\nabla v_0(x,y)-\nabla v_2(x,y)=\sqrt{3}e^y\quad\forall(x,y)\in U, \tag{4.21}$$

by the definition of α_i and β_i, and by (4.3), (4.11), (4.16), we have

$$\begin{aligned}\int_{u_0(s,0)}^{u_2(s,0)}\varphi^{xy}(s,0,z)\,dz &= \sqrt{3}\frac{\varepsilon^2}{v_0(s,0)}e^y+2\sigma_1(s,0)\phi_1^y(s,0)e^y\\ &= \sqrt{3}\frac{\varepsilon^2}{v_0(s,0)}e^y+g(s)e^y \;=\; e^y,\end{aligned} \tag{4.22}$$

where the two last equalities follow from (4.6) and from the definition of σ_1 and g. So condition (b2) is satisfied.

The proof of condition (b1) will be split in the next three sections: in Section 4.2 we prove that condition (b1) holds if t_1 and t_2 belong to suitable neighbourhoods of $u_{i-1}(0,0)$ and $u_i(0,0)$, respectively; then, in Section 4.3 we prove condition (b1) for t_1 and t_2 belonging to suitable neighbourhoods of $u_0(0,0)$ and $u_2(0,0)$, respectively; finally, in Section 4.4, by a continuity argument we show that condition (b1) is true in all other cases.

4.2 Estimates for t_1 and t_2 near u_{i-1} and u_i

For $(x,y)\in U$ and $t_1,t_2\in\mathbb{R}$, we set

$$I(x,y,t_1,t_2):=\int_{t_1}^{t_2}\varphi^{xy}(x,y,z)\,dz \tag{4.23}$$

and we denote its absolute value by ρ. In this section, we will show that $\rho(x,y,t_1,t_2) \leq 1$ in a neighbourhood of the point $(0,0,u_{i-1}(0,0),u_i(0,0))$ for $i=1,2$, so that the following step will be proved.

STEP 1.– For a suitable choice of the parameter ε, there exists $\delta > 0$ such that condition (b1) holds for $|t_1 - u_{i-1}(0,0)| < \delta$, $|t_2 - u_i(0,0)| < \delta$ with $i=1,2$, provided U is small enough.

Note that ρ is a continuous function, but its derivatives with respect to x,y may be discontinuous at the points (x,y,t_1,t_2) such that $h_1(x,y)=0$ or $h_2(x,y)=0$; indeed, the curve $\{h_i=0\}$ is the boundary of the different regions of definition of the functions σ_i, α_i, and β_i, whose derivatives may present therefore some discontinuities. Nevertheless, if we set $N_i := \{(x,y) \in U : h_i(x,y) < 0\}$ and $P_i := \{(x,y) \in U : h_i(x,y) > 0\}$, the restrictions of σ_i, α_i, and β_i to the sets N_i and P_i can be extended up to the boundary $\{h_i = 0\}$ as C^∞-functions; so, along the curve $\{h_i=0\}$ the traces of the derivatives of σ_i, α_i, and β_i are defined. Then, also the traces of the derivatives of ρ with respect to x,y are defined at the points (x,y,t_1,t_2) with $h_1(x,y)=0$ or $h_2(x,y)=0$.

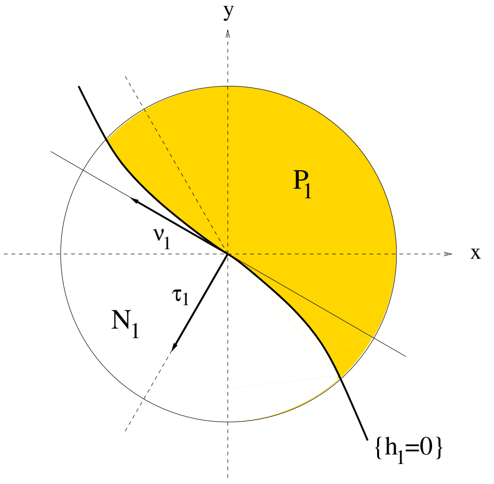

Figure 4.4: the regions P_1 and N_1.

Since we want to study the behaviour of ρ in a neighbourhood of $(0,0,u_{i-1}(0,0),u_i(0,0))$, we can suppose $|t_1 - u_{i-1}(0,0)| \leq \varepsilon$ and $|t_2 - u_i(0,0)| \leq \varepsilon$, so that the possible discontinuities of the derivatives of ρ concentrate only on the curve $\{h_i=0\}$. We study separately the two regions N_i and P_i.

Consider first the case $(x,y) \in \overline{N_i}$, which is the region containing $S_{i-1,i}$. We will study the derivatives of ρ at the points of the form

$$q_i(s) := (s\tau_i, u_{i-1}(s\tau_i), u_i(s\tau_i)), \qquad s \geq 0.$$

We have already shown (condition (b2)) that $\rho(q_i(s)) = 1$ for every $s \geq 0$; we want to prove that

$$\nabla\rho(q_i(s)) = 0 \qquad \forall s \geq 0 \tag{4.24}$$

(where now ∇ denotes the gradient with respect to x,y,t_1,t_2) and that the Hessian matrix of ρ with respect to ν_i, t_1, t_2 is negative definite at $q_i(0)$.

Let I^{τ_i} and I^{ν_i} be the components of the integral in (4.23) along the directions τ_i and ν_i, respectively. Since by definition

$$\rho(x,y,t_1,t_2) = [(I^{\tau_i}(x,y,t_1,t_2))^2 + (I^{\nu_i}(x,y,t_1,t_2))^2]^{1/2},$$

the gradient of ρ is given by

$$\nabla\rho = \frac{1}{\rho}(I^{\tau_i}\nabla I^{\tau_i} + I^{\nu_i}\nabla I^{\nu_i}). \tag{4.25}$$

Note that (4.19) implies that

$$I^{\tau_i}(q_i(s)) = 0 \qquad \text{and} \qquad I^{\nu_i}(q_i(s)) = 1 \qquad \forall s \geq 0, \tag{4.26}$$

hence

$$\nabla\rho(q_i(s)) = \nabla I^{\nu_i}(q_i(s)). \tag{4.27}$$

By the definition of φ in G_i and by (4.15) we can compute explicitly the expression of I^{ν_i} at (x, y, t_1, t_2):

$$\begin{aligned} I^{\nu_i} = {} & -2(t_1 - u_{i-1})\partial_{\nu_i}u_{i-1} + 2(t_2 - u_i)\partial_{\nu_i}u_i + \frac{1}{\lambda}(\beta_i - \alpha_i + \lambda)\sigma_i\phi_i^{\nu_i} \\ & + \frac{\sqrt{3}}{2v_{i-1}}(\varepsilon^2 - (t_1 - u_{i-1})^2) + \frac{\sqrt{3}}{2v_i}(\varepsilon^2 - (t_2 - u_i)^2), \end{aligned} \tag{4.28}$$

where

$$\phi_i^{\tau_i}(x,y) = (-1)^{i+1}f(\langle\nu_i,(x,y)\rangle) \qquad \text{and} \qquad \phi_i^{\nu_i}(x,y) = g(\langle\tau_i,(x,y)\rangle). \tag{4.29}$$

By differentiating (4.28) with respect to the direction ν_i we obtain

$$\begin{aligned} \partial_{\nu_i}I^{\nu_i} = {} & 2(\partial_{\nu_i}u_{i-1})^2 - 2(\partial_{\nu_i}u_i)^2 - 2(t_1 - u_{i-1})\partial^2_{\nu_i\nu_i}u_{i-1} + 2(t_2 - u_i)\partial^2_{\nu_i\nu_i}u_i \\ & + \frac{1}{\lambda}\partial_{\nu_i}(\beta_i - \alpha_i)\sigma_i\phi_i^{\nu_i} + \frac{1}{\lambda}(\beta_i - \alpha_i + \lambda)(\partial_{\nu_i}\sigma_i\phi_i^{\nu_i} + \sigma_i\partial_{\nu_i}\phi_i^{\nu_i}) \\ & - \frac{3}{4v_{i-1}^2}(\varepsilon^2 - (t_1 - u_{i-1})^2) + \frac{3}{4v_i^2}(\varepsilon^2 - (t_2 - u_i)^2) \\ & + \frac{\sqrt{3}}{v_{i-1}}(t_1 - u_{i-1})\partial_{\nu_i}u_{i-1} + \frac{\sqrt{3}}{v_i}(t_2 - u_i)\partial_{\nu_i}u_i. \end{aligned} \tag{4.30}$$

By the Euler conditions, $\partial_{\nu_i}u_{i-1}(s\tau_i) = \partial_{\nu_i}u_i(s\tau_i) = 0$ for every $s \geq 0$. Moreover, since $|\nabla u_{i-1}| = |\nabla u_i|$ on U (see the remark at the beginning of the proof), in the region $\overline{N_i}$ the function $\beta_i - \alpha_i$ coincides with the solution ξ_i of the problem

$$\begin{cases} \dfrac{1}{\lambda}\phi_i^{\tau_i}\partial_{\tau_i}\xi_i + \dfrac{1}{\lambda}\phi_i^{\nu_i}\partial_{\nu_i}\xi_i = \dfrac{\varepsilon^2}{v_{i-1}^2} - \dfrac{\varepsilon^2}{v_i^2}, \\ \xi_i(s\tau_i) = 0 \quad (s \geq 0). \end{cases} \tag{4.31}$$

As $\partial_{\tau_i}\xi_i(s\tau_i) = 0$ and $v_{i-1}(s\tau_i) = v_i(s\tau_i)$ for every $s \geq 0$ (see (4.17)), we have that

$$\partial_{\nu_i}(\beta_i - \alpha_i)(s\tau_i) = \partial_{\nu_i}\xi_i(s\tau_i) = 0. \tag{4.32}$$

By definition $\partial_{\nu_i}\phi_i^{\nu_i} \equiv 0$ and $\sigma_i(x,y) = 1$ for every $(x,y) \in \overline{N_i}$; using these remarks and the first equality in (4.17), we can deduce that

$$\partial_{\nu_i}I^{\nu_i}(q_i(s)) = 0 \tag{4.33}$$

for every $s>0$, and the equality holds also for the trace of $\partial_{\nu_i} I^{\nu_i}$ at $q_i(0)$. Since the derivatives of I^{ν_i} with respect to t_1 and t_2 are given by

$$\partial_{t_1} I^{\nu_i} = -2\partial_{\nu_i} u_{i-1} - \frac{\sqrt{3}}{v_{i-1}}(t_1 - u_{i-1}), \qquad \partial_{t_2} I^{\nu_i} = 2\partial_{\nu_i} u_i - \frac{\sqrt{3}}{v_i}(t_2 - u_1), \tag{4.34}$$

by the Euler conditions it follows that

$$\partial_{t_1} I^{\nu_i}(q_i(s)) = \partial_{t_2} I^{\nu_i}(q_i(s)) = 0. \tag{4.35}$$

As $I^{\nu_i}(q_i(s)) = 1$ for every $s \geq 0$, equalities (4.35) imply that $\partial_{\tau_i} I^{\nu_i}(q_i(s)) = 0$. By this fact, (4.27), (4.33), and (4.35), equality (4.24) is proved.

Now we need to compute the trace of the Hessian matrix of ρ with respect to ν_i, t_1, t_2 at the point $q_i(0)$; using (4.26) (4.33), (4.35) and (4.24), the Hessian matrix at $q_i(0)$ reduces to

$$\nabla^2_{\nu_i,t_1,t_2}\rho(q_i(0)) = [\nabla_{\nu_i,t_1,t_2} I^{\tau_i} \otimes \nabla_{\nu_i,t_1,t_2} I^{\tau_i} + \nabla^2_{\nu_i,t_1,t_2} I^{\nu_i}](q_i(0)), \tag{4.36}$$

where ∇_{ν_i,t_1,t_2} denotes the gradient with respect to ν_i, t_1, t_2 and $\otimes$ the tensor product. As before, we know the explicit expression of I^{τ_i}:

$$\begin{aligned} I^{\tau_i} = -2(t_1 - u_{i-1})\partial_{\tau_i} u_{i-1} + 2(t_2 - u_i)\partial_{\tau_i} u_i + \frac{1}{\lambda}(\beta_i - \alpha_i + \lambda)\sigma_i \phi_i^{\tau_i} \\ - \frac{1}{2v_{i-1}}(\varepsilon^2 - (t_1 - u_{i-1})^2) + \frac{1}{2v_i}(\varepsilon^2 - (t_2 - u_i)^2), \end{aligned} \tag{4.37}$$

hence, using the Euler conditions, (4.32), and the fact that $\sigma_i \equiv 1$ in $\overline{N_i}$, it results that

$$\partial_{\nu_i} I^{\tau_i}(q_i(0)) = \frac{1}{2}\partial_{\nu_i} v_{i-1}(0,0) - \frac{1}{2}\partial_{\nu_i} v_i(0,0) + \partial_{\nu_i}\phi_i^{\tau_i}(0,0) = \frac{\sqrt{3}}{2}, \tag{4.38}$$

where the last equality follows by (4.18) and by the equality

$$\partial_{\nu_i}\phi_i^{\tau_i}(0) = (-1)^{i+1} f'(0) = 0. \tag{4.39}$$

By differentiating (4.30) and by using the Euler conditions, (4.32), the constancy of σ_i in $\overline{N_i}$, and the fact that $\partial^2_{\nu_i\nu_i}\phi_i^{\nu_i} \equiv 0$, we have

$$\partial^2_{\nu_i\nu_i} I^{\nu_i}(q_i(0)) = \frac{1}{\lambda}\phi_i^{\nu_i}(0,0)\partial^2_{\nu_i\nu_i}(\beta_i - \alpha_i)(0,0) + \frac{3}{2\varepsilon}\partial_{\nu_i} v_{i-1}(0,0) - \frac{3}{2\varepsilon}\partial_{\nu_i} v_i(0,0) = -\frac{\sqrt{3}}{2\varepsilon}, \tag{4.40}$$

where the last equality follows from

$$\frac{1}{\lambda}\phi_i^{\nu_i}(0,0)\partial^2_{\nu_i\nu_i}(\beta_i - \alpha_i)(0,0) = -\frac{2\sqrt{3}}{\varepsilon}, \tag{4.41}$$

which can be obtained by differentiating (4.31). Using (4.36), (4.38), and (4.40), we obtain that

$$\partial^2_{\nu_i\nu_i}\rho(q_i(0)) = [\partial_{\nu_i} I^{\tau_i}(q_i(0))]^2 + \partial^2_{\nu_i\nu_i} I^{\nu_i}(q_i(0)) = \frac{3}{4} - \frac{\sqrt{3}}{2\varepsilon} < 0, \tag{4.42}$$

provided ε is sufficiently small. Since $\partial_{t_1} I^{\tau_i}(q_i(0)) = 0$ (this can be easily proved using the fact that $\nabla u_{i-1}(0,0) = \nabla u_i(0,0) = 0$), by (4.36) we have that

$$\partial^2_{\nu_i t_1}\rho(q_i(0)) = \partial^2_{\nu_i t_1} I^{\nu_i}(q_i(0)), \qquad \partial^2_{t_1 t_1}\rho(q_i(0)) = \partial^2_{t_1 t_1} I^{\nu_i}(q_i(0)).$$

By differentiating (4.34) and by using the Euler conditions, it turns out that

$$\partial^2_{\nu_i t_1} I^{\nu_i}(q_i(0)) = -2\partial^2_{\nu_i\nu_i} u_{i-1}(0,0), \qquad \partial^2_{t_1 t_1} I^{\nu_i}(q_i(0)) = -\frac{\sqrt{3}}{\varepsilon},$$

so that

$$\det\begin{pmatrix} \partial^2_{\nu_i\nu_i}\rho & \partial^2_{\nu_i t_1}\rho \\ \partial^2_{\nu_i t_1}\rho & \partial^2_{t_1 t_1}\rho \end{pmatrix}(q_i(0)) = \frac{3}{2\varepsilon^2}\left(1-\frac{\sqrt{3}}{2}\varepsilon\right) - 4(\partial^2_{\nu_i\nu_i}u_{i-1}(0,0))^2.$$

Arguing in a similar way, one can find that

$$\partial^2_{\nu_i t_2}\rho(q_i(0)) = 2\partial^2_{\nu_i\nu_i}u_i(0,0), \quad \partial^2_{t_2 t_2}\rho(q_i(0)) = -\frac{\sqrt{3}}{\varepsilon}, \quad \partial^2_{t_1 t_2}\rho(q_i(0)) = 0,$$

so that

$$\det\nabla^2_{\nu_i,t_1,t_2}\rho(q_i(0)) = -\frac{3\sqrt{3}}{2\varepsilon^3}\left(1-\frac{\sqrt{3}}{2}\varepsilon\right) + \frac{4\sqrt{3}}{\varepsilon}[(\partial^2_{\nu_i\nu_i}u_{i-1}(0,0))^2 + (\partial^2_{\nu_i\nu_i}u_i(0,0))^2].$$

Since for ε sufficiently small it results that

$$\det\begin{pmatrix} \partial^2_{\nu_i\nu_i}\rho & \partial^2_{\nu_i t_1}\rho \\ \partial^2_{\nu_i t_1}\rho & \partial^2_{t_1 t_1}\rho \end{pmatrix}(q_i(0)) > 0, \qquad \det\nabla^2_{\nu_i,t_1,t_2}\rho(q_i(0)) < 0, \tag{4.43}$$

then, by (4.42) and (4.43) the Hessian matrix of ρ at $q_i(0)$ is negative definite.

At this point we have all the ingredients we need in order to compare the value of ρ on $S_{i-1,i}$ with its value at a point (x,y,t_1,t_2) for $(x,y)\in\overline{N_i}$ and $|t_1-u_{i-1}(0,0)|\le\varepsilon$, $|t_2-u_i(0,0)|\le\varepsilon$.

Remark that since the curve $\{h_i=0\}$ may have an inflection point at the origin, the set $\overline{N_i}$ might be not convex. If the segment joining (x,y) with its orthogonal projection on $S_{i-1,i}$ (which is a point of the form $s\tau_i$ with $s\ge 0$) is all contained in $\overline{N_i}$, then we can consider the restriction of ρ to the segment joining (x,y,t_1,t_2) with $q_i(s)$ and write its Taylor expansion of second order centred at $q_i(s)$. By (4.24) and the fact that the Hessian matrix of ρ is negative definite at $q_i(0)$ (and then, by continuity in a small neighbourhood), we have that there exist $\delta, C>0$ such that, if U is small enough and $|t_1-u_{i-1}(0,0)|<\delta$, $|t_2-u_i(0,0)|<\delta$, then

$$\rho(x,y,t_1,t_2)\le 1 - C|\langle\nu_i,(x,y)\rangle|^2 - C(t_1-u_{i-1}(s\tau_i))^2 - C(t_2-u_i(s\tau_i))^2 \le 1.$$

In the general case, since the curve $\{y=\psi_i(x,0)\}$ is C^2 with null second derivative at 0, one can find $s>0$, $a\in\mathbb{R}$ such that the segment joining (x,y) with $s\tau_i+a\nu_i$ is all contained in $\overline{N_i}$ and the ratio $|(x,y)-s\tau_i-a\nu_i|/a^2$ is infinitesimal as $a\to 0$. Since $s>0$, the segment joining $s\tau_i+a\nu_i$ with its projection $s\tau_i$ on $S_{i-1,i}$ is all contained in $\overline{N_i}$, so that we can apply to this point the estimate above; if we call L the L^∞-norm of the gradient of ρ, we obtain that

$$\begin{aligned}\rho(x,y,t_1,t_2) &\le \rho(s\tau_i+a\nu_i,t_1,t_2) + L|(x,y)-s\tau_i-a\nu_i| \\ &\le 1-a^2\left(C-L\frac{|(x,y)-s\tau_i-a\nu_i|}{a^2}\right) - C(t_1-u_{i-1}(s\tau_i))^2 - C(t_2-u_i(s\tau_i))^2,\end{aligned}$$

which is less than 1, provided U is small enough. So we have proved that, if ε is sufficiently small, then there exists $\delta>0$ such that

$$\rho(x,y,t_1,t_2)\le 1 \qquad \text{for } (x,y)\in\overline{N_i},\ |t_1-u_{i-1}(0,0)|<\delta,\ |t_2-u_i(0,0)|<\delta, \tag{4.44}$$

provided U is sufficiently small.

Suppose now $(x,y) \in \overline{P_i}$, $|t_1 - u_{i-1}(0,0)| \leq \varepsilon$, $|t_2 - u_i(0,0)| \leq \varepsilon$. In order to show that $\rho \leq 1$ also in this case, we will compute the traces of the gradient and of the Hessian matrix of ρ at the point $q_i(0)$. The main difference with respect to the previous case is that in the region $\overline{P_i}$ the function $\beta_i - \alpha_i$ coincides with the solution η_i of the problem

$$\begin{cases} \dfrac{1}{\lambda}\sigma_i(x,y)\langle\phi_i(x,y), \nabla\eta_i(x,y)\rangle = \dfrac{\varepsilon^2}{v_{i-1}^2(x,y)} - \dfrac{\varepsilon^2}{v_i^2(x,y)}, \\ \eta_i(s,0) = 0 \quad (s \geq 0), \end{cases} \tag{4.45}$$

while the function σ_i is defined as

$$\sigma_i(x,y) = \frac{g(h_i(x,y))}{2\phi_i^y(h_i(x,y),0)} \qquad \forall (x,y) \in \overline{P_i}. \tag{4.46}$$

By (4.26) and (4.25) it follows that

$$\nabla\rho(q_i(0)) = \nabla I^{\nu_i}(q_i(0)). \tag{4.47}$$

By (4.28) we obtain the following expression for the gradient of I^{ν_i} with respect to τ_i, ν_i computed at the point $q_i(0)$:

$$\nabla_{\tau_i,\nu_i} I^{\nu_i}(q_i(0)) = g(0)\nabla\sigma_i(0,0) + \nabla\phi_i^{\nu_i}(0,0) + \frac{\sqrt{3}}{2}\tau_i, \tag{4.48}$$

where we have used the Euler conditions, the fact that $\nabla(\beta_i - \alpha_i)(0,0) = 0$ by (4.45), and that

$$\nabla v_{i-1}(x,y) + \nabla v_i(x,y) = -\tau_i \qquad \forall (x,y) \in U.$$

It follows immediately by (4.29) that

$$\nabla\phi_i^{\nu_i}(x,y) = g'(\langle\tau_i, (x,y)\rangle)\tau_i \tag{4.49}$$

and by the definition of g that

$$g'(t) = \sqrt{3}\varepsilon^2 \frac{\partial_x v_0(t,0)}{v_0^2(t,0)} = -\frac{\sqrt{3}}{2}\varepsilon^2 \frac{1}{v_0^2(t,0)} \tag{4.50}$$

for all $t \in \mathbb{R}$. By differentiating (4.46), we obtain that

$$\nabla\sigma_i(x,y) = \frac{1}{2}p(h_i(x,y))\nabla h_i(x,y), \tag{4.51}$$

where we have set

$$p(t) := \frac{g'(t)}{\phi_i^y(t,0)} - \frac{g(t)}{[\phi_i^y(t,0)]^2}\partial_x\phi_i^y(t,0).$$

To compute the gradient of h_i it is enough to differentiate the second equality in (4.5): this provides

$$\partial_x\psi_i(x,h_i) + \partial_s\psi_i(x,h_i)\partial_x h_i = 0, \qquad \partial_s\psi_i(x,h_i)\partial_y h_i = 1; \tag{4.52}$$

by (4.7) we have that

$$\nabla h_i(0,0) = -2\tau_i. \tag{4.53}$$

Since

$$\partial_x\phi_i^y(x,y) = (-1)^{i+1}\frac{3}{4}f'(\langle\nu_i,(x,y)\rangle) - \frac{1}{4}g'(\langle\tau_i,(x,y)\rangle),$$

we find that $p(0) = 3g'(0)/g(0)$, and substituting in (4.51), we have that

$$\nabla\sigma_i(0,0) = -3\frac{g'(0)}{g(0)}\tau_i. \tag{4.54}$$

Since the partial derivatives of I^{ν_i} with respect to t_1 and t_2 are still given by (4.34), they are equal to 0 at the point $q_i(0)$, as in the previous case. Then, by (4.47), (4.48), (4.49), (4.54), and (4.50), we deduce that

$$\nabla\rho(q_i(0)) = \left(\frac{3\sqrt{3}}{2}\tau_i, 0, 0\right). \tag{4.55}$$

To conclude the study of ρ in this region, we write the Hessian matrix of ρ with respect to ν_i, t_1, t_2, which still satisfies (4.36). Differentiating (4.37) and using the Euler conditions, the fact that $\nabla(\beta_i - \alpha_i)(0,0) = 0$, $\phi_i^{\tau_i}(0,0) = 0$ and (4.39), we obtain that (4.38) still holds. Differentiating (4.30) and computing the result at $q_i(0)$, we have that

$$\partial^2_{\nu_i\nu_i} I^{\nu_i}(q_i(0)) = \frac{1}{\lambda} g(0)\partial^2_{\nu_i\nu_i}(\beta_i - \alpha_i)(0,0) + g(0)\partial^2_{\nu_i\nu_i}\sigma_i(0,0) + \frac{3}{2\varepsilon}(\partial_{\nu_i} v_{i-1}(0,0) - \partial_{\nu_i} v_i(0,0)), \tag{4.56}$$

where we have used in particular that $\partial_{\nu_i}\sigma_i(0,0) = 0$ by (4.54) and that $\partial^2_{\nu_i\nu_i}\phi_i^{\nu_i} \equiv 0$. In order to compute the second derivative of $\beta_i - \alpha_i$ with respect to the direction ν_i, we differentiate (4.45) with respect to x and with respect to y; using the fact that $\partial_x(\beta_i - \alpha_i)(s,0) = 0$ for every $s \geq 0$, we obtain

$$\partial^2_{xx}(\beta_i - \alpha_i)(0,0) = 0, \qquad \partial^2_{xy}(\beta_i - \alpha_i)(0,0) = \frac{6}{\varepsilon}(-1)^{i+1}\frac{\lambda}{g(0)}, \tag{4.57}$$

$$\partial^2_{yy}(\beta_i - \alpha_i)(0,0) = -\frac{2\sqrt{3}}{\varepsilon}\frac{\lambda}{g(0)} + \sqrt{3}(-1)^{i+1}\partial^2_{xy}(\beta_i - \alpha_i)(0,0) = \frac{4\sqrt{3}}{\varepsilon}\frac{\lambda}{g(0)}. \tag{4.58}$$

By the relation $\partial^2_{\nu_i\nu_i} = \frac{3}{4}\partial^2_{xx} + \frac{\sqrt{3}}{2}(-1)^i\partial^2_{xy} + \frac{1}{4}\partial^2_{yy}$, it follows that

$$\partial^2_{\nu_i\nu_i}(\beta_i - \alpha_i)(0,0) = -\frac{2\sqrt{3}}{\varepsilon}\frac{\lambda}{g(0)}.$$

Since $\partial_{\nu_i} h_i(0,0) = 0$ by (4.53), from (4.51) we obtain that

$$\partial^2_{\nu_i\nu_i}\sigma_i(0,0) = \frac{1}{2}\left(\frac{g'(0)}{\phi_i^y(0,0)} - \frac{g(0)}{[\phi_i^y(0,0)]^2}\partial_x\phi_i^y(0,0)\right)\partial^2_{\nu_i\nu_i} h_i(0,0) = \frac{3}{2}\frac{g'(0)}{g(0)}\partial^2_{\nu_i\nu_i} h_i(0). \tag{4.59}$$

By differentiating twice with respect to the direction ν_i the second equality in (4.5), we obtain that

$$(\nu_i^x)^2\partial^2_{xx}\psi_i(x,h_i) + 2\nu_i^x\partial^2_{xs}\psi_i(x,h_i)\partial_{\nu_i} h_i + \partial^2_{ss}\psi_i(x,h_i)(\partial_{\nu_i} h_i)^2 + \partial_s\psi_i(x,h_i)\partial^2_{\nu_i\nu_i} h_i = 0;$$

since $\partial_{\nu_i} h_i(0,0) = 0$ by (4.53) and $\partial^2_{xx}\psi_i(0,0) = 0$ by (4.8), we can conclude that $\partial^2_{\nu_i\nu_i} h_i(0,0) = 0$ and then, by (4.59) also the limit of $\partial^2_{\nu_i\nu_i}\sigma_i$ at $(0,0)$ is equal to 0. Taking (4.17) and (4.56) into account, we can conclude that

$$\partial^2_{\nu_i\nu_i} I^{\nu_i}(q_i(0)) = -\frac{\sqrt{3}}{2\varepsilon},$$

i.e., (4.40) is still satisfied. Since it is easy to see that also the other second derivatives of ρ remain unchanged, we can conclude that the Hessian matrix of ρ with respect to ν_i, t_1, t_2 is negative definite at $q_i(0)$.

If the segment joining (x,y,t_1,t_2) with $q_i(0)$ is all contained in $\overline{P_i}$, then we consider the Taylor expansion of second order centred at $q_i(0)$ of the function ρ restricted to this segment; since the component of (x,y) along τ_i is less or equal than 0, by (4.55) and by the fact that the Hessian matrix of ρ with respect to ν_i, t_1, t_2 is negative definite, we have that there exists $\delta > 0$ such that $\rho(x,y,t_1,t_2) \leq 1$ for $|t_1 - u_{i-1}(0,0)| < \delta$, $|t_2 - u_i(0,0)| < \delta$, provided U is small enough. In the general case, we can find $s \leq 0$, $a \in \mathbb{R}$ such that the segments joining (x,y) with $s\tau_i + a\nu_i$, and $s\tau_i + a\nu_i$ with $(0,0)$ are all contained in $\overline{P_i}$, and $|(x,y) - s\tau_i - a\nu_i|/a^2$ is infinitesimal as $a \to 0$. Arguing as for the region $\overline{N_i}$, this is enough to obtain the same conclusion. So we have proved that, if ε is small enough, there exists $\delta > 0$ such that

$$\rho(x,y,t_1,t_2) \leq 1 \qquad \text{for } (x,y) \in \overline{P_i},\ |t_1 - u_{i-1}(0,0)| < \delta,\ |t_2 - u_i(0,0)| < \delta, \tag{4.60}$$

provided U is sufficiently small.

By (4.44) and (4.60) Step 1 is proved.

4.3 Estimates for t_1 and t_2 near u_0 and u_2

This section is devoted to the proof of the following step.

Step 2.– For a suitable choice of the function f (see (4.2)), there exists $\delta > 0$ such that condition (b1) holds for $|t_1 - u_0(0,0)| < \delta$, $|t_2 - u_2(0,0)| < \delta$, provided U is small enough.

In order to prove the step, we want to show that the function ρ, introduced at the beginning of Section 4.2, is less or equal than 1 in a neighbourhood of the point $(0,0,u_0(0,0),u_2(0,0))$. We can assume that $|t_1 - u_0(0,0)| \leq \varepsilon$, $|t_2 - u_2(0,0)| \leq \varepsilon$. Since now the derivatives of ρ may be discontinuous on the curves $\{h_1 = 0\}$ and $\{h_2 = 0\}$, we have to consider separately four different cases, one for (x,y) belonging to each one of the regions $N_1 \cap N_2$, $N_1 \cap P_2$, $N_2 \cap P_1$, and $P_1 \cap P_2$.

Let I^x and I^y be the components of the integral in (4.23) with respect to e^x and e^y, that are the tangent and the normal direction, respectively, to the third part of the discontinuity set $S_{0,2}$.

Consider first the case $(x,y) \in \overline{P_1} \cap \overline{P_2}$, which is the region containing $S_{0,2}$; as before, we will study the derivatives of ρ at the points of the form

$$q_0(x) := (x, 0, u_0(x,0), u_2(x,0)), \qquad x \geq 0.$$

Condition (4.22) implies that $\rho(q_0(x)) = 1$ for every $x \geq 0$; we want to prove that

$$\nabla\rho(q_0(x)) = 0 \qquad \forall x \geq 0 \tag{4.61}$$

and that the Hessian matrix of ρ with respect to y, t_1, t_2 is negative definite at $q_i(0)$. By the definition of ρ, it follows that

$$\nabla\rho = \frac{1}{\rho}(I^x \nabla I^x + I^y \nabla I^y).$$

Since $I^x(q_0(x)) = 0$ and $I^y(q_0(x)) = 1$ for every $x \geq 0$, we have that

$$\nabla\rho(q_0(x)) = \nabla I^y(q_0(x)).$$

By (4.16) and by the definition of φ in G_i we can write the explicit expression of I^y at (x,y,t_1,t_2):

$$I^y = -2(t_1 - u_0)\partial_y u_0 + 2(t_2 - u_2)\partial_y u_2 + \frac{1}{\lambda}\sum_{i=1}^{2}(\beta_i - \alpha_i + \lambda)\sigma_i \phi_i^y$$
$$+ \frac{\sqrt{3}}{2v_0}(\varepsilon^2 - (t_1 - u_0)^2) + \frac{\sqrt{3}}{2v_2}(\varepsilon^2 - (t_2 - u_2)^2), \tag{4.62}$$

and by differentiating with respect to y, we obtain

$$\partial_y I^y = 2(\partial_y u_0)^2 - 2(\partial_y u_2)^2 - 2(t_1-u_0)\partial^2_{yy}u_0 + 2(t_2-u_2)\partial^2_{yy}u_2 + \frac{1}{\lambda}\sum_{i=1}^{2}[\partial_y(\beta_i-\alpha_i)\sigma_i\phi_i^y + (\beta_i-\alpha_i+\lambda)\partial_y(\sigma_i\phi_i^y)] - \frac{3}{4v_0^2}(\varepsilon^2-(t_1-u_0)^2) + \frac{3}{4v_2^2}(\varepsilon^2-(t_2-u_2)^2) + \frac{\sqrt{3}}{v_0}(t_1-u_0)\partial_y u_0 + \frac{\sqrt{3}}{v_2}(t_2-u_2)\partial_y u_2. \quad (4.63)$$

Since in the region $\overline{P_1}\cap\overline{P_2}$ the functions $\beta_i-\alpha_i$ coincide with the solutions of the problems (4.45), it results that $\partial_y(\beta_i-\alpha_i)(x,0)=0$ for $i=1,2$. Moreover, differentiating (4.11) and the second equality in (4.3) with respect to y, we have that

$$\partial_y\sigma_2(x,y) = -\partial_y\sigma_1(x,-y), \qquad \partial_y\phi_2^y(x,y) = -\partial_y\phi_1^y(x,-y), \quad (4.64)$$

and then, using again (4.3) and (4.11),

$$\phi_1^y(x,0)\partial_y\sigma_1(x,0) = -\phi_2^y(x,0)\partial_y\sigma_2(x,0), \quad \sigma_1(x,0)\partial_y\phi_1^y(x,0) = -\sigma_2(x,0)\partial_y\phi_2^y(x,0).$$

By the Euler conditions, $\partial_y u_0(x,0)=\partial_y u_2(x,0)=0$ for every $x\geq 0$; using all these remarks and (4.20), we deduce that $\partial_y I^y(q_0(x))=0$ for every $x>0$ and the equality holds also for the trace of $\partial_y I^y$ at $q_0(0)$. Since we have that

$$\partial_{t_1}I^y = -2\partial_y u_0 - \frac{\sqrt{3}}{v_0}(t_1-u_0), \qquad \partial_{t_2}I^y = 2\partial_y u_2 - \frac{\sqrt{3}}{v_2}(t_2-u_2), \quad (4.65)$$

by the Euler conditions it follows that $\partial_{t_1}I^y(q_0(x)) = \partial_{t_2}I^y(q_0(x)) = 0$. As $I^y(q_0(x))=1$ for every $x\geq 0$, this implies that $\partial_x I^y(q_0(x))=0$. Thus we have obtained equality (4.61).

By (4.61) and (4.22) the Hessian matrix of ρ computed at $q_0(0)$ reduces to

$$\nabla^2_{y,t_1,t_2}\rho(q_0(0)) = [\nabla_{y,t_1,t_2}I^x\otimes\nabla_{y,t_1,t_2}I^x + \nabla^2_{y,t_1,t_2}I^y](q_0(0)). \quad (4.66)$$

As before, we know that

$$I^x = -2(t_1-u_0)\partial_x u_0 + 2(t_2-u_2)\partial_x u_2 + \frac{1}{\lambda}\sum_{i=1}^{2}(\beta_i-\alpha_i+\lambda)\sigma_i\phi_i^x - \frac{1}{2v_0}(\varepsilon^2-(t_1-u_0)^2) + \frac{1}{2v_2}(\varepsilon^2-(t_2-u_2)^2),$$

hence, by the Euler condition, the fact that $\partial_y(\beta_i-\alpha_i)(0,0)=0$ for $i=1,2$, and (4.21), it results that

$$\partial_y I^x(q_0(0)) = \frac{\sqrt{3}}{2} + \sum_{i=1}^{2}\partial_y(\sigma_i\phi_i^x)(0,0) = \frac{\sqrt{3}}{2} + 2\partial_y\phi_1^x(0,0) + 2\phi_1^x(0,0)\partial_y\sigma_1(0,0),$$

where we have also used the first equalities in (4.3) and in (4.64), and the relation $\partial_y\phi_2^x(x,y) = \partial_y\phi_1^x(x,-y)$. From (4.54) we obtain that

$$\partial_y\sigma_1(0,0) = \frac{3\sqrt{3}}{2}\frac{g'(0)}{g(0)}.$$

Then, using the definition of ϕ_1^x and (4.50), we can conclude that

$$\partial_y I^x(0,0) = \frac{\sqrt{3}}{2} - 3g'(0) = 2\sqrt{3}. \tag{4.67}$$

By differentiating (4.63) with respect to y and by using the Euler condition and the fact that $\partial_y(\beta_i - \alpha_i)(0,0) = 0$ for $i = 1, 2$, we obtain

$$\partial^2_{yy} I^y(q_0(0)) = \frac{1}{\lambda}\sum_{i=1}^{2}[\partial^2_{yy}(\beta_i - \alpha_i)\phi_i^y + \partial^2_{yy}(\sigma_i\phi_i^y)](0,0) + \frac{3\sqrt{3}}{2\varepsilon}.$$

Equality (4.58) implies that

$$\frac{1}{\lambda}\sum_{i=1}^{2}[\partial^2_{yy}(\beta_i - \alpha_i)\sigma_i\phi_i^y](0,0) = \frac{4\sqrt{3}}{\varepsilon}. \tag{4.68}$$

In order to write explicitly $\partial^2_{yy}\sigma_i$ at $(0,0)$, we differentiate the y-component in (4.51) with respect to y and we pass to the limit, taking into account that $\partial_y h_i(0) = (-1)^{i+1}\sqrt{3}$ by (4.53):

$$\partial^2_{yy}\sigma_1(0,0) = \frac{3}{2}p'(0) + \frac{1}{2}p(0)\partial^2_{yy}h_i(0).$$

By differentiating with respect to y the second equality in (4.52), we obtain that

$$\partial^2_{yy}h_1(0,0) = -(\partial_y h_1(0,0))^2\frac{\partial^2_{ss}\psi_1(0,0)}{\partial_s\psi_1(0,0)} = 0,$$

where the last equality follows by (4.9). Since

$$p'(0) = 2\frac{g''(0)}{g(0)} + 3\frac{[g'(0)]^2}{g^2(0)} - 4\frac{\partial^2_{xx}\phi_1^y(0,0)}{g(0)}, \tag{4.69}$$

while

$$\partial^2_{xx}\phi_1^y(0,0) = -\frac{3\sqrt{3}}{8}f''(0) + \frac{1}{8}g''(0), \quad \partial^2_{yy}\phi_1^y(0,0) = -\frac{\sqrt{3}}{8}f''(0) + \frac{3}{8}g''(0), \tag{4.70}$$

and $g''(0) = -\sqrt{3}/(2\varepsilon)$, we can write that

$$\begin{aligned}\frac{1}{\lambda}\sum_{i=1}^{2}(\beta_i - \alpha_i + \lambda)\partial^2_{yy}(\sigma_i\phi_i^y)(0,0) &= (2\phi_1^y\partial^2_{yy}\sigma_1 + 4\partial_y\sigma_1\partial_y\phi_1^y + 2\partial^2_{yy}\phi_1^y)(0,0) \\ &= 2\sqrt{3}f''(0) + 3g''(0) \\ &= 2\sqrt{3}f''(0) - \frac{3\sqrt{3}}{2\varepsilon}. \end{aligned} \tag{4.71}$$

Substituting (4.68) and (4.71) in the expression of $\partial^2_{yy}I^y$, we find that

$$\partial^2_{yy}I^y(q_0(0)) = 2\sqrt{3}f''(0) + \frac{4\sqrt{3}}{\varepsilon}. \tag{4.72}$$

From (4.66), (4.67), and (4.72), we finally obtain that

$$\partial^2_{yy}\rho(q_0(0)) = [\partial_y I^x(q_0(0))]^2 + \partial^2_{yy}I^y(q_0(0)) = 12 + \frac{4\sqrt{3}}{\varepsilon} + 2\sqrt{3}f''(0). \tag{4.73}$$

As in the previous step, we can compute explicitly the other elements of the Hessian matrix of ρ and we find that

$$\det\begin{pmatrix} \partial^2_{yy}\rho & \partial^2_{yt_1}\rho \\ \partial^2_{yt_1}\rho & \partial^2_{t_1t_1}\rho \end{pmatrix}(q_0(0)) = -\frac{6}{\varepsilon}f''(0) - \frac{12\sqrt{3}}{\varepsilon} - \frac{12}{\varepsilon^2} - 4(\partial^2_{yy}u_0(0,0))^2,$$

$$\det\nabla^2_{y,t_1,t_2}\rho(q_0(0)) = \frac{6\sqrt{3}}{\varepsilon^2}f''(0) + \frac{36}{\varepsilon^2} + \frac{12\sqrt{3}}{\varepsilon^3} + \frac{4\sqrt{3}}{\varepsilon}[(\partial^2_{yy}u_0(0,0))^2 + (\partial^2_{yy}u_2(0,0))^2].$$

If we impose the following condition on the second derivative of f at 0:

$$f''(0) < -2\sqrt{3} - \frac{2}{\varepsilon} - \frac{2\varepsilon}{3}[(\partial^2_{yy}u_0(0,0))^2 + (\partial^2_{yy}u_2(0,0))^2], \tag{4.74}$$

then the Hessian matrix of ρ is negative definite at $q_0(0)$.

To conclude, we restrict ρ to the segment joining (x,y,t_1,t_2) with $q_0(x)$ and we write its Taylor expansion of second order centred at $q_0(x)$; using (4.61) and choosing f satisfying (4.74) (so that the Hessian matrix of ρ is negative definite at $q_0(0)$, and then by continuity in a small neighbourhood), we obtain that there exists $\delta > 0$ such that

$$\rho(x,y,t_1,t_2) \le 1 \qquad \text{for } (x,y) \in \overline{P_1} \cap \overline{P_2},\ |t_1 - u_0(0,0)| < \delta,\ |t_2 - u_2(0,0)| < \delta, \tag{4.75}$$

provided U is sufficiently small.

Let us consider the set $\overline{N_1} \cap \overline{N_2}$: in this region $\sigma_1 = \sigma_2 = 1$, while the functions $\beta_i - \alpha_i$ coincide with the solutions of the problems (4.31). By (4.22) the gradient of ρ at the point $q_0(0)$ is given by

$$\nabla\rho(q_0(0)) = \nabla I^y(q_0(0)). \tag{4.76}$$

By (4.62) we derive the explicit expression for the gradient of I^y with respect to x, y; using the Euler condition, the fact that $\nabla(\beta_i - \alpha_i)(0,0) = 0$, the constancy of σ_i and the equality

$$\nabla v_0(x,y) + \nabla v_2(x,y) = -e^x \qquad \forall (x,y) \in U, \tag{4.77}$$

we obtain that

$$\nabla_{x,y}I^y(q_0(0)) = \sum_{i=1}^{2}\nabla\phi_i^y(0,0) + \frac{\sqrt{3}}{2}e^x = -\frac{1}{2}g'(0)e^x + \frac{\sqrt{3}}{2}e^x = \frac{3\sqrt{3}}{4}e^x.$$

Since the partial derivatives of I^y with respect to t_1 and t_2 are still given by (4.65), they are equal to 0 at $q_0(0)$, as in the previous case. Therefore, we have that

$$\nabla\rho(q_0(0)) = \left(\frac{3\sqrt{3}}{4}e^x, 0, 0\right). \tag{4.78}$$

If $(x,y) \neq (0,0)$ belongs to $\overline{N_1} \cap \overline{N_2}$ and the segment joining (x,y) with $(0,0)$ is all contained in $\overline{N_1} \cap \overline{N_2}$, then by the Mean Value theorem, (4.78) and the fact that x is strictly negative, we can conclude that there exists $\delta > 0$ such that

$$\rho(x,y,t_1,t_2) \le 1 \qquad \text{for } |t_1 - u_0(0,0)| < \delta,\ |t_2 - u_2(0,0)| < \delta, \tag{4.79}$$

provided U is sufficiently small. If the segment joining (x,y) with $(0,0)$ is not contained in $\overline{N_1} \cap \overline{N_2}$, then we can find a regular curve connecting (x,y) and $(0,0)$, along which we can repeat the same estimate as above.

At last consider the set $\overline{N_2} \cap \overline{P_1}$, since the case $\overline{N_1} \cap \overline{P_2}$ is completely analogous. In this region, σ_1 is defined by (4.46), while σ_2 is identically equal to 1; the function $\beta_1 - \alpha_1$ coincides with the solution of the problem (4.45) for $i = 1$, while $\beta_2 - \alpha_2$ with the one of (4.31) for $i = 2$. Equality (4.76) still holds, as well as the fact that $\nabla(\beta_i - \alpha_i)(0,0) = (0,0)$ for all i; since $\nabla\sigma_1$ is given by the formula (4.54) and $\nabla\sigma_2 \equiv 0$, by (4.2), (4.50), (4.62), and (4.77) we have that

$$
\begin{aligned}
\nabla_{x,y} I^y(q_0(0)) &= \sum_{i=1}^{2} \nabla\phi_i^y(0,0) + \phi_1^y(0,0)\nabla\sigma_1(0,0) + \frac{\sqrt{3}}{2}e^x \\
&= \frac{3\sqrt{3}}{4}(e^x + \tau_1) \quad = \quad -\frac{3\sqrt{3}}{4}\tau_2,
\end{aligned}
$$

hence

$$
\nabla\rho(q_0(0)) = \left(-\frac{3\sqrt{3}}{4}\tau_2, 0, 0\right).
$$

Since the gradient of ρ vanishes along the direction $(\nu_2, 0, 0)$, we need to compute the Hessian matrix of ρ with respect to ν_2, t_1, t_2 at the point $q_0(0)$; from the equality $\nabla_{\nu_2,t_1,t_2} I^y(q_0(0)) = 0$, we have that

$$
\nabla^2_{\nu_2,t_1,t_2}\rho(q_0(0)) = [\nabla_{\nu_2,t_1,t_2} I^x \otimes \nabla_{\nu_2,t_1,t_2} I^x + \nabla^2_{\nu_2,t_1,t_2} I^y](q_0(0)). \tag{4.80}
$$

Using the fact that $\nabla u_0(0,0) = \nabla u_2(0,0) = 0$ and $\nabla(\beta_i - \alpha_i)(0,0) = 0$, we obtain

$$
\begin{aligned}
\partial_{\nu_2} I^x(q_0(0)) &= \sum_{i=1}^{2} \partial_{\nu_2}\phi_i^x(0,0) + \partial_{\nu_2}\sigma_1(0,0)\phi_1^x(0,0) + \frac{1}{2}\partial_{\nu_2}(v_0 - v_2) \\
&= \partial_y\phi_1^x(0,0) - \frac{9}{4}g'(0) + \frac{\sqrt{3}}{4} = \sqrt{3},
\end{aligned}
$$

where the second equality follows from (4.54) and from the fact that $\partial_{\nu_2}\phi_1^x + \partial_{\nu_2}\phi_2^x = \partial_y\phi_1^x$ at $(0,0)$. If we differentiate (4.62) twice with respect to the direction ν_2 and we compute the result at the point $q_0(0)$, we obtain

$$
\partial^2_{\nu_2\nu_2} I^y(0,0) = \left(\frac{1}{\lambda}\sum_{i=1}^{2}\partial^2_{\nu_2\nu_2}(\beta_i - \alpha_i)\sigma_i\phi_i^y + \sum_{i=1}^{2}\partial^2_{\nu_2\nu_2}\phi_i^y + \partial^2_{\nu_2\nu_2}\sigma_1\phi_1^y + 2\partial_{\nu_2}\sigma_1\partial_{\nu_2}\phi_1^y\right)(0,0) + \frac{3\sqrt{3}}{4\varepsilon}. \tag{4.81}
$$

From (4.57) and (4.58), and from (4.41) it follows respectively that

$$
\partial^2_{\nu_2\nu_2}(\beta_1 - \alpha_1)(0,0) = \frac{4\sqrt{3}}{\varepsilon^3}\frac{\lambda}{g(0)}, \quad \partial^2_{\nu_2\nu_2}(\beta_2 - \alpha_2)(0,0) = -\frac{2\sqrt{3}}{\varepsilon}\frac{\lambda}{g(0)}. \tag{4.82}
$$

Since by (4.51) we have that $\partial_{\nu_2}\sigma_1(x,y) = \frac{1}{2}p(h_1(x,y))\partial_{\nu_2}h_1(x,y)$, then

$$
\partial^2_{\nu_2\nu_2}\sigma_1(0,0) = \frac{1}{2}p'(0)(\partial_{\nu_2}h_1(0,0))^2 + \frac{1}{2}p(0)\partial^2_{\nu_2\nu_2}h_1(0,0).
$$

Some easy computations show that $\partial^2_{\nu_2\nu_2}h_1(0,0) = 0$; using (4.53) it results that

$$
\partial^2_{\nu_2\nu_2}\sigma_1(0,0) = \frac{3}{2}p'(0) = \frac{9}{2}\frac{[g'(0)]^2}{g^2(0)} + \frac{9}{4}\sqrt{3}\frac{f''(0)}{g(0)}, \tag{4.83}
$$

where the last equality follows by (4.69) and by the first equality in (4.70). At last, by using (4.3) and (4.70), we obtain that

$$\sum_{i=1}^{2} \partial^2_{\nu_2\nu_2}\phi^y_i(0,0) = \frac{3}{4}\partial^2_{xx}\phi^y_1(0,0) + \frac{1}{4}\partial^2_{yy}\phi^y_1(0,0) = -\frac{5}{8}\sqrt{3}f''(0) + \frac{3}{8}g''(0), \tag{4.84}$$

and by substituting (4.82), (4.83), and (4.84) in (4.81), we deduce that

$$\partial^2_{\nu_2\nu_2} I^y(q_0(0)) = \frac{\sqrt{3}}{2}f''(0) + \frac{\sqrt{3}}{\varepsilon},$$

hence

$$\partial^2_{\nu_2\nu_2}\rho(q_0(0)) = 3 + \frac{\sqrt{3}}{\varepsilon} + \frac{\sqrt{3}}{2}f''(0).$$

By differentiating (4.65) with respect to ν_2 and by (4.80), we obtain

$$\partial^2_{\nu_2 t_1}\rho(q_0(0)) = -2\partial_{\nu_2}\partial_y u_0(0,0) = -\partial^2_{yy}u_0(0,0), \qquad \partial^2_{\nu_2 t_2}\rho(q_0(0)) = 2\partial_{\nu_2}\partial_y u_2(0,0) = \partial^2_{yy}u_2(0,0).$$

At this point, it is easy to see that, if f satisfies the condition

$$f''(0) < -2\sqrt{3} - \frac{2}{\varepsilon} - \frac{\varepsilon}{6}[(\partial^2_{yy}u_0(0,0))^2 + (\partial^2_{yy}u_2(0,0))^2] \tag{4.85}$$

then the Hessian matrix of ρ with respect to ν_2, t_1, t_2 is negative definite at the point $q_0(0)$. Arguing as for the region $\overline{P_i}$ in the previous section, it can be proved that, if f satisfies (4.85), then there exists $\delta > 0$ such that

$$\rho(x,y,t_1,t_2) \le 1 \qquad \text{for } (x,y) \in \overline{N_2} \cap \overline{P_1},\ |t_1 - u_0(0,0)| < \delta,\ |t_2 - u_2(0,0)| < \delta, \tag{4.86}$$

provided U is sufficiently small.

Since condition (4.74) implies (4.85), if we require that (4.74) holds, then by (4.75), (4.79), and (4.86), we can conclude that Step 2 is true.

4.4 Proof of condition (b1)

In this section we complete the proof of condition (b1). To this aim it is enough to check condition (b1) in the three cases studied in the following step, as it will be clear at the end of the section.

STEP 3.– If ε is sufficiently small, $\delta \in (0,\varepsilon)$, and U is sufficiently small, condition (b1) is true for $t_1 \le t_2$ whenever one of the following three conditions is satisfied:

1) $|t_1 - u_0(0,0)| \ge \delta$ and $|t_1 - u_1(0,0)| \ge \delta$;

2) $|t_2 - u_1(0,0)| \ge \delta$ and $|t_2 - u_2(0,0)| \ge \delta$;

3) $|t_1 - u_0(0,0)| \ge \delta$ and $|t_2 - u_2(0,0)| \ge \delta$.

Let us fix $\delta \in (0,\varepsilon)$ and set

$$M_1(x,y) := \max\{|I(x,y,t_1,t_2)| : \ u_0(x,y) - \varepsilon \le t_1 \le t_2 \le u_2(x,y) + \varepsilon, \\ |t_1 - u_0(0,0)| \ge \delta, \ \ |t_1 - u_1(0,0)| \ge \delta\}.$$

It is easy to see that the function M_1 is continuous. Let us prove that $M_1(0,0) < 1$. For simplicity of notation, from now on we will denote $I(0,0,t_1,t_2)$ simply by $I(t_1,t_2)$ and $u_i(0,0)$ by u_i.

Let t_1, t_2 be such that $u_0 - \varepsilon \le t_1 \le t_2 \le u_2 + \varepsilon$ with $|t_1 - u_0| \ge \delta$ and $|t_1 - u_1| \ge \delta$. Suppose furthermore that $|t_1 - u_1| \le \varepsilon$; then, we can write

$$\begin{aligned} I(t_1,t_2) &= I(t_1,u_1) + I(u_1,u_2) + I(u_2,t_2), \\ I(u_2,t_2) &= I(u_2, t_2 \vee (u_2-\varepsilon)) + I(u_2-\varepsilon, t_2 \wedge (u_2-\varepsilon)). \end{aligned}$$

Therefore, we have

$$I(t_1,t_2) = I(t_1,u_1) + I(u_1,u_2) + I(u_2, t_2 \vee (u_2-\varepsilon)) - I(t_2 \wedge (u_2-\varepsilon), u_2-\varepsilon). \tag{4.87}$$

From the definition of φ in G_1, G_2 it follows that

$$\begin{aligned} I(s_1,u_1) &= -\frac{1}{\varepsilon}(s_1-u_1)^2 e^x \quad \text{for } |s_1-u_1| \le \varepsilon, \\ I(u_2,s_2) &= \frac{1}{\varepsilon}(s_2-u_2)^2 \tau_1 \quad \text{for } |s_2-u_2| \le \varepsilon; \end{aligned} \tag{4.88}$$

using condition (b2), we have that

$$I(t_1,u_1) + I(u_1,u_2) + I(u_2, t_2 \vee (u_2-\varepsilon)) \in \nu_2 - \frac{\delta^2}{\varepsilon} e^x + R_1, \tag{4.89}$$

where R_1 is the parallelogram spanned by the vectors $\varepsilon\tau_1$ and $-\left(\varepsilon - \frac{\delta^2}{\varepsilon}\right) e^x$. Let C be the intersection of the half-plane $\{(x,y) \in \mathbb{R}^2 : \langle \nu_2, (x,y) \rangle \ge 1 - \sqrt{3}\varepsilon\}$ with the open ball centred at 0 with radius 1; some elementary geometric considerations show that

$$\nu_2 - \frac{\delta^2}{\varepsilon} e^x + R_1 \subset C. \tag{4.90}$$

If T_i is the segment joining 0 with $g(0)\nu_i$, then from the definition of φ in K_i, it follows that

$$I(u_{i-1}+\varepsilon, u_i-\varepsilon) = g(0)\nu_i, \tag{4.91}$$

and

$$I(s_1,s_2) \in T_i \tag{4.92}$$

for $u_{i-1} + \varepsilon \le s_1 \le s_2 \le u_i - \varepsilon$, $i = 1,2$. Let $D := -T_2$; from (4.87), (4.89), (4.90), and (4.92), we deduce that

$$I(t_1,t_2) \in C + D;$$

since $g(0) = 1 - \sqrt{3}\varepsilon$, the set $C + D$ is contained in the open ball centred at 0 with radius 1. This concludes the proof when $|t_1 - u_1| \le \varepsilon$.

If $|t_2 - u_1| \le \varepsilon$, we consider the decomposition

$$\begin{aligned} I(t_1,t_2) &= I(t_1,u_0) + I(u_0,u_1) + I(u_1,t_2), \\ I(t_1,u_0) &= I(t_1 \wedge (u_0+\varepsilon), u_0) + I(t_1 \vee (u_0+\varepsilon), u_0+\varepsilon), \end{aligned}$$

and the proof is completely analogous.

When $|t_1 - u_1| > \varepsilon$ and $|t_2 - u_1| > \varepsilon$, we can write

$$\begin{aligned} I(t_1,t_2) &= I(t_1,u_0) + I(u_0,u_2) + I(u_2,t_2), \\ I(t_1,u_0) &= I(t_1 \wedge (u_0+\varepsilon), u_0) + I(t_1 \vee (u_0+\varepsilon), u_0+\varepsilon), \\ I(u_2,t_2) &= I(u_2, t_2 \vee (u_2-\varepsilon)) + I(u_2-\varepsilon, t_2 \wedge (u_2-\varepsilon)); \end{aligned}$$

therefore, we have

$$I(t_1,t_2) = I(t_1 \wedge (u_0+\varepsilon), u_0) + I(u_0,u_2) + I(u_2, t_2 \vee (u_2-\varepsilon)) \\ + I(t_1 \vee (u_0+\varepsilon), t_2 \wedge (u_2-\varepsilon)) - I(u_0+\varepsilon, u_2-\varepsilon). \quad (4.93)$$

Since from the definition of φ in G_0 it follows that

$$I(s_0,u_0) = -\frac{1}{\varepsilon}(s_0-u_0)^2\tau_2 \qquad \text{for } |s_0-u_0| \le \varepsilon, \quad (4.94)$$

using condition (b2) and (4.88), we have that

$$I(t_1 \wedge (u_0+\varepsilon), u_0) + I(u_0,u_2) + I(u_2, t_2 \vee (u_2-\varepsilon)) \in e^y - \frac{\delta^2}{\varepsilon}\tau_2 + R_2, \quad (4.95)$$

where R_2 is the parallelogram spanned by the vectors $\varepsilon\tau_1$ and $-\left(\varepsilon - \frac{\delta^2}{\varepsilon}\right)\tau_2$. Let E be the parallelogram having as consecutive sides T_1 and T_2, and let F be the set $E - g(0)e^y$; as $I(u_1-\varepsilon, u_1+\varepsilon) = 0$, from (4.91) it follows that

$$I(u_0+\varepsilon, u_2-\varepsilon) = g(0)e^y = (1-\sqrt{3}\varepsilon)e^y, \quad (4.96)$$

and from (4.92),

$$I(s_1,s_2) \in E \quad (4.97)$$

for every $u_0+\varepsilon \le s_1 \le s_2 \le u_2-\varepsilon$, with $|s_1-u_1| > \varepsilon$ and $|s_2-u_1| > \varepsilon$. From (4.93), (4.95), (4.96), (4.97), we obtain that

$$I(t_1,t_2) \in e^y - \frac{\delta^2}{\varepsilon}\tau_2 + R_2 + F.$$

The set $e^y - \frac{\delta^2}{\varepsilon}\tau_2 + R_2 + F$ is a polygon, since it is the sum of two polygons, and it is possible to prove that, if $\varepsilon < \sqrt{3}$, its vertices are all contained in the open ball with centre 0 and radius 1. Then, under this condition, the whole set $e^y - \frac{\delta^2}{\varepsilon}\tau_2 + R_2 + F$ is contained in this ball; this concludes the proof of the inequality $M_1(0,0) < 1$.

By continuity, choosing U small enough, we obtain that $M_1(x,y) < 1$ for every $(x,y) \in U$, which proves 1).

To prove 2) and 3), we define analogously

$$M_2(x,y) := \max\{|I(x,y,t_1,t_2)| : \; u_0(x,y)-\varepsilon \le t_1 \le t_2 \le u_2(x,y)+\varepsilon, \\ |t_2 - u_1(0,0)| \ge \delta, \;\; |t_2 - u_2(0,0)| \ge \delta\},$$

$$M_3(x,y) := \max\{|I(x,y,t_1,t_2)| : \; u_0(x,y)-\varepsilon \le t_1 \le t_2 \le u_2(x,y)+\varepsilon, \\ |t_1 - u_0(0,0)| \ge \delta, \;\; |t_2 - u_2(0,0)| \ge \delta\}.$$

It is easy to see that the functions M_2 and M_3 are continuous and, arguing as in the case of M_1, we can prove that $M_2(0,0) < 1$ and $M_3(0,0) < 1$, which yield 2) and 3) by continuity. Step 3 is proved.

Conclusion.– As in Step 3, we simply write u_i instead of $u_i(0,0)$. Let us show that, if f satisfies (4.74), and ε and U are sufficiently small, then condition (b1) is true for $u_0(x,y)-\varepsilon \le t_1 < t_2 \le u_2(x,y)+\varepsilon$ and in fact for every $t_1, t_2 \in \mathbb{R}$, since $\varphi^{xy}(x,y,z) = 0$ for $z \le u_0(x,y)-\varepsilon$ and for $z \ge u_2(x,y)+\varepsilon$.

We start by considering the case $|t_1 - u_0| < \delta$. If $|t_2 - u_1| < \delta$, the conclusion follows from Step 1. If $|t_2 - u_1| \geq \delta$, the result is a consequence of Step 2 when $|t_2 - u_2| < \delta$, and of Step 3.2) in the other case.

We consider now the case $|t_1 - u_0| \geq \delta$. If $|t_1 - u_1| \geq \delta$, the conclusion follows from Step 3.1). If $|t_1 - u_1| < \delta$, the result is a consequence of Step 1 when $|t_2 - u_2| < \delta$, and of Step 3.3) in the other case.

This concludes the proof of condition (b1) and then, of Theorem 4.1 in the case u_0 symmetric. □

4.5 The antisymmetric case

In this section we show how the construction of the calibration for u_i symmetric can be adapted to the antisymmetric case.

If the function u_0 is antisymmetric with respect to the bisecting line of A_0, then the reflection of u_0 with respect to the $S_{0,1}$ and to $S_{0,2}$ provides an extension of u_0, which is harmonic only on $\Omega \setminus S_{1,2}$ and which is multi-valued on $S_{1,2}$, since the traces of the tangential derivatives of u_0 on $S_{1,2}$ have different signs. Since u_1, u_2 coincide, up to the sign and to additive constants, with the reflections of u_0 with respect to $S_{0,1}$ and $S_{0,2}$, respectively, they are antisymmetric with respect to the bisecting line of A_1 and A_2, respectively, and then, their extensions by reflection are harmonic only on $\Omega \setminus S_{0,2}$ and $\Omega \setminus S_{0,1}$, respectively.

The calibration φ can be defined as before, just replacing the sets G_0, G_1, G_2 with

$$\begin{aligned}
\tilde{G}_0 &= \{(x,y,z) \in (U \setminus S_{1,2}) \times \mathbb{R} : u_0(x,y) - \varepsilon < z < u_0(x,y) + \varepsilon\}, \\
\tilde{G}_1 &= \{(x,y,z) \in (U \setminus S_{0,2}) \times \mathbb{R} : u_1(x,y) - \varepsilon < z < u_1(x,y) + \varepsilon\}, \\
\tilde{G}_2 &= \{(x,y,z) \in (U \setminus S_{0,1}) \times \mathbb{R} : u_2(x,y) - \varepsilon < z < u_2(x,y) + \varepsilon\},
\end{aligned}$$

and the sets H_1, H_2 with

$$\begin{aligned}
\tilde{H}_1 &= \{(x,y,z) \in (U \setminus (S_{1,2} \cup S_{0,2})) \times \mathbb{R} : l_1 + \lambda/2 < z < l_1 + 3\lambda/2\}, \\
\tilde{H}_2 &= \{(x,y,z) \in (U \setminus (S_{0,1} \cup S_{0,2})) \times \mathbb{R} : l_2 + \lambda/2 < z < l_2 + 3\lambda/2\}.
\end{aligned}$$

Since u_0 is harmonic in $\Omega \setminus S_{1,2}$, the field φ is divergence-free in $\tilde{G}_0$ by Lemma 1.5. Moreover, the normal component of φ is continuous across the boundary of G_0 since $\partial_{\nu_2} u_0 = \partial_{\nu_2} v_0 = 0$ on $S_{1,2}$. The same argument works for the sets $\tilde{G}_1, \tilde{G}_2$. By the harmonicity of u_0 and u_1, the field is divergence-free in $\tilde{H}_1$ and the normal component of φ is continuous across the boundary of H_1 since $\partial_{\nu_2} u_0 = 0$ on $S_{1,2}$ and $\partial_y u_1 = 0$ on $S_{0,2}$. Therefore, condition (c1) is still satisfied in the sense of distributions on $U \times \mathbb{R}$.

It is easy to see that conditions (a1), (a2), and (b2) are satisfied.

The proof of Step 1, Step 2, and Step 3 can be easily adapted; indeed, even if now the function $|I(x,y,t_1,t_2)|$ may present some discontinuities when $(x,y) \in S_{i,j}$, we can write U as the union of finitely many Lipschitz open subsets U_i such that $|I|$ is $C^2(\overline{U_i} \times \mathbb{R}^2)$ and study the behaviour of $|I|$ separately in each $\overline{U_i}$. So, it results that also condition (b1) is true. □

Chapter 5

The calibration method for functionals on vector-valued maps

The purpose of this chapter is to present and develop a generalization of the calibration method to functionals with free discontinuities defined on vector-valued maps.

In the sequel Ω is a fixed bounded open subset of $\mathbb{R}^n$ with Lipschitz boundary, $\nu_{\partial\Omega}$ is its inner unit normal, while U is a closed subset of $\overline{\Omega}\times\mathbb{R}^N$. The letter x usually denotes the variable in Ω (or $\mathbb{R}^n$), while y or z is the variable in $\mathbb{R}^N$. We will consider functionals of the form

$$F(u) = \int_\Omega f(x,u,\nabla u)\,dx + \int_{S_u} \psi(x,u^-,u^+,\nu_u)\,d\mathcal{H}^{n-1}, \tag{5.1}$$

where $f : \Omega\times\mathbb{R}^N\times\mathbb{R}^{nN} \to [0,+\infty]$, and $\psi : \Omega\times\mathbb{R}^N\times\mathbb{R}^N\times\mathbb{S}^{n-1} \to [0,+\infty]$ are Borel functions, $\mathbb{S}^{n-1} := \{v \in \mathbb{R}^n : |v| = 1\}$, and the unknown function u belongs to the space $SBV(\Omega;\mathbb{R}^N)$ of special functions of bounded variation in Ω. Since the triplet (u^+,u^-,ν_u) is uniquely determined up to a permutation of (u^+,u^-) and a change of sign of ν_u (see Section 1.1), we will assume that ψ satisfies the condition $\psi(x,y,z,\nu) = \psi(x,z,y,-\nu)$.

We start this chapter with the proof of a generalized chain rule in BV, which will be useful in the following. If $u \in BV(\Omega;\mathbb{R}^N)$ and $\mathcal{S}$ is a Lipschitz continuous function from $\mathbb{R}^N$ into $\mathbb{R}^M$, it is known that $\mathcal{S}\circ u$ belongs to $BV(\Omega;\mathbb{R}^M)$. When in addition $\mathcal{S} \in C^1(\mathbb{R}^N;\mathbb{R}^M)$, the following chain rule formula can be written:

$$\begin{aligned} \tilde{D}(\mathcal{S}\circ u) &= D\mathcal{S}(\tilde{u}(x))\tilde{D}u(x) \quad \text{on } \Omega\setminus S_u, \\ D^j(\mathcal{S}\circ u) &= [\mathcal{S}(u^+) - \mathcal{S}(u^-)]\otimes\nu_u\mathcal{H}^{n-1}\lfloor S_u, \end{aligned} \tag{5.2}$$

(see Theorem 3.96 in [6]). Following an idea by [32], we generalize formula (5.2) to the case of a function $\mathcal{S}$, which may depend also on the variable x and is only piecewise C^1 in the sense of the following definition.

Definition 5.1 *We say that a Lipschitz continuous function $\mathcal{S} : U \to \mathbb{R}^M$ is* piecewise C^1 *if $\mathcal{S}$ can be written as*

$$\mathcal{S}(x,y) = \sum_{\alpha\in A} \mathcal{S}^\alpha(x,y)1_{U^\alpha}(x,y), \tag{5.3}$$

where $(U^\alpha)_{\alpha\in A}$ is a finite family of pairwise disjoint Borel sets such that $\cup_{\alpha\in A}U^\alpha = U$, and $(\mathcal{S}^\alpha)_{\alpha\in A}$ is a family of Lipschitz continuous functions belonging to $C^1(\overline{\Omega}\times\mathbb{R}^N;\mathbb{R}^M)$.

Lemma 5.2 *Let $\mathcal{S}: U \to \mathbb{R}^M$ be a Lipschitz continuous function, piecewise C^1 in the sense of Definition 5.1, and satisfying (5.3), and let $u \in BV(\Omega;\mathbb{R}^N)$ be such that* $\operatorname{graph} u \subset U$. *Then, $v := \mathcal{S}(\cdot, u(\cdot))$ belongs to $BV(\Omega;\mathbb{R}^M)$ and*

$$\begin{aligned} \tilde{D}v &= \sum_{\alpha\in A} 1_{U^\alpha}(x,\tilde{u})(D_x\mathcal{S}^\alpha(x,\tilde{u})\mathcal{L}^n + D_y\mathcal{S}^\alpha(x,\tilde{u})\tilde{D}u) \quad \text{on } \Omega\setminus S_u, & (5.4)\\ D^j v &= [\mathcal{S}(x,u^+) - \mathcal{S}(x,u^-)]\otimes \nu_u \mathcal{H}^{n-1}\lfloor S_u. & (5.5)\end{aligned}$$

Proof. – Since the function $\mathcal{S}$ can be extended to a Lipschitz function on the whole $\overline{\Omega}\times\mathbb{R}^N$, by Theorem 3.101 in [6] we have that the function $v = \mathcal{S}(\cdot, u(\cdot))$ belongs to $BV(\Omega;\mathbb{R}^M)$ and formula (5.5) holds true.

Since $\mathcal{S}^\alpha$ is globally Lipschitz and of class C^1 on $\overline{\Omega}\times\mathbb{R}^N$, by Theorem 3.96 in [6] the function $v^\alpha := \mathcal{S}^\alpha(\cdot, u(\cdot))$ belongs to $BV(\Omega;\mathbb{R}^M)$ and the diffuse part of its derivative satisfies the following equality:

$$\tilde{D}v^\alpha = D_x\mathcal{S}^\alpha(x,\tilde{u})\mathcal{L}^n + D_y\mathcal{S}^\alpha(x,\tilde{u})\tilde{D}u. \tag{5.6}$$

Consider now the set

$$E^\alpha := \{x\in\Omega\setminus S_u:\ \tilde{v}(x) = \tilde{v}^\alpha(x)\}.$$

Since v and v^α are both BV functions and their jump sets are both contained in S_u, by the locality property of the derivative of a BV function (see Remark 3.93 in [6]) it follows that $Dv\lfloor E^\alpha = Dv^\alpha\lfloor E^\alpha$. Since $E^\alpha \subset \Omega\setminus S_u$, the previous equality can be rewritten as

$$\tilde{D}v\lfloor E^\alpha = \tilde{D}v^\alpha\lfloor E^\alpha. \tag{5.7}$$

If we define

$$P^\alpha := \{x\in\Omega\setminus S_u:\ (x,\tilde{u}(x))\in U^\alpha\},$$

since $P^\alpha\subset E^\alpha$, by (5.7) and (5.6) we can conclude that

$$\tilde{D}v\lfloor P^\alpha = \tilde{D}v^\alpha\lfloor P^\alpha = D_x\mathcal{S}^\alpha(x,\tilde{u})\mathcal{L}^n\lfloor P^\alpha + D_y\mathcal{S}^\alpha(x,\tilde{u})\tilde{D}u\lfloor P^\alpha,$$

which immediately gives formula (5.4). □

The plan of the chapter is the following: in Section 5.1 we present the calibration method for functionals of the form (5.1) on vector-valued maps; Section 5.2 is devoted to the link between calibration theory and classical field theory; Section 5.3 contains some applications to the Mumford-Shah functional (for vector-valued functions) and to functionals arising in fracture mechanics; finally, in Section 5.4 we reformulate the theory of calibrations in terms of differential forms and show that this formulation does not lead to new results.

5.1 Calibrations for functionals on vector-valued maps

According to Definitions 1.1 and 1.3, we consider the following definition of minimizers of F.

Definition 5.3 *An* absolute minimizer of (5.1) in Ω *is a function $u\in SBV(\Omega;\mathbb{R}^N)$ such that $F(u)\le F(v)$ for all $v\in SBV(\Omega;\mathbb{R}^N)$, while a* Dirichlet minimizer in Ω *is a function $u\in SBV(\Omega;\mathbb{R}^N)$ such that $F(u)\le F(v)$ for all $v\in SBV(\Omega;\mathbb{R}^N)$ with the same trace on $\partial\Omega$ as u. A function u is a* U-minimizer *if the graph of u is contained in U and $F(u)\le F(v)$ for all $v\in SBV(\Omega;\mathbb{R}^N)$ whose graph is contained in U, while u is a* Dirichlet U-minimizer *if we add the requirement that the competing functions v have the same trace on $\partial\Omega$ as u.*

Before proving the key lemma about calibrations, we fix some further notation.

Given two functions $\mathcal{S} : U \to \mathbb{R}^n$, and $u : \Omega \to \mathbb{R}^N$, we will denote the divergence of the composite function $\mathcal{S}(\cdot, u(\cdot))$ by $\mathrm{div}_x[\mathcal{S}(x, u(x))]$, while the divergence of $\mathcal{S}$ with respect to the variable x computed at the point $(x, u(x))$ by $[\mathrm{div}_x \mathcal{S}](x, u(x))$. The Jacobian matrix of $\mathcal{S}$ with respect to y will be denoted by $D_y\mathcal{S}$ and its transpose by $(D_y\mathcal{S})^\tau$. Note that if $\mathcal{S}$ and u are sufficiently regular,

$$\mathrm{div}_x[\mathcal{S}(x, u(x))] = [\mathrm{div}_x \mathcal{S}](x, u) + \langle (D_y\mathcal{S}(x, u))^\tau, \nabla u\rangle.$$

As in Section 1.3, we call f^* and $\partial_\xi^- f$ the convex conjugate and the subdifferential of f with respect to the last variable. It is well known that, if g is any function from $\mathbb{R}^{nN}$ into $[0, +\infty]$, $\langle \xi, \eta\rangle - g^*(\eta) \le g(\xi)$ for every $\xi, \eta \in \mathbb{R}^{nN}$, and the equality holds if and only if $\eta \in \partial_\xi^- g(\xi)$. Moreover, if g is convex and differentiable, then $\partial_\xi^- g(\xi) = \{\partial_\xi g(\xi)\}$. Using these properties, we can prove the following lemma.

Lemma 5.4 *Let F be the functional defined in (5.1). Let $\mathcal{S} \in C^1(\overline{\Omega} \times \mathbb{R}^N; \mathbb{R}^n)$ be Lipschitz continuous and let $\mathcal{S}_0 \in L^1(\Omega)$. Assume that the following conditions are satisfied:*

(a1) $[\mathrm{div}_x \mathcal{S}](x, y) + \mathcal{S}_0(x) \le -f^*(x, y, (D_y\mathcal{S}(x, y))^\tau)$ *for $\mathcal{L}^n$-a.e. $x \in \Omega$ and for every y with $(x, y) \in U$;*

(b1) $\langle \mathcal{S}(x, z) - \mathcal{S}(x, y), \nu\rangle \le \psi(x, y, z, \nu)$ *for $\mathcal{H}^{n-1}$-a.e. $x \in \Omega$, for every $\nu \in \mathbb{S}^{n-1}$, and for every y, z with $(x, y) \in U$, $(x, z) \in U$.*

Then for every $u \in SBV(\Omega; \mathbb{R}^N)$ such that $\mathrm{graph}\, u \subset U$ *we have that* $\mathrm{div}_x[\mathcal{S}(\cdot, u(\cdot))]$ *is a Radon measure on Ω, which will be denoted as μ_u, and*

$$F(u) \ge \int_\Omega d\mu_u + \int_\Omega \mathcal{S}_0(x)\, dx. \tag{5.8}$$

Moreover, equality holds in (5.8) for a given u if and only if

(a2) $[\mathrm{div}_x \mathcal{S}](x, u) + \mathcal{S}_0(x) = -f^*(x, u, (D_y\mathcal{S}(x, u))^\tau)$ *and* $(D_y\mathcal{S}(x, u))^\tau \in \partial_\xi^- f(x, u, \nabla u)$ *for $\mathcal{L}^n$-a.e. $x \in \Omega$;*

(b2) $\langle \mathcal{S}(x, u^+) - \mathcal{S}(x, u^-), \nu_u\rangle = \psi(x, u^-, u^+, \nu_u)$ *for $\mathcal{H}^{n-1}$-a.e. $x \in S_u$,*

where u, $u^\pm$, ∇u, and ν_u are always computed at x.

Proof. – Let $u \in SBV(\Omega; \mathbb{R}^N)$ be such that $\mathrm{graph}\, u \subset U$. By Theorem 3.96 in [6] the function $S(\cdot, u(\cdot))$ belongs to $SBV(\Omega; \mathbb{R}^n)$, and therefore, its divergence is a Radon measure on Ω. Moreover, we have that

$$D_{x_i}[\mathcal{S}_i(x, u)] = \partial_{x_i}\mathcal{S}_i(x, u)\, \mathcal{L}^n + D_y\mathcal{S}_i(x, u)\partial_{x_i} u\, \mathcal{L}^n + [\mathcal{S}_i(x, u^+) - \mathcal{S}_i(x, u^-)](\nu_u)_i\, \mathcal{H}^{n-1}\lfloor S_u,$$

so that the measure μ_u can be written as

$$\begin{aligned}
\mu_u(x) &= \sum_{i=1}^n D_{x_i}[\mathcal{S}_i(x, u(x))] \\
&= [\mathrm{div}_x \mathcal{S}](x, u)\, \mathcal{L}^n + \sum_i D_y\mathcal{S}_i(x, u)\partial_{x_i} u\, \mathcal{L}^n + \sum_i [\mathcal{S}_i(x, u^+) - \mathcal{S}_i(x, u^-)](\nu_u)_i\, \mathcal{H}^{n-1}\lfloor S_u \\
&= [\mathrm{div}_x \mathcal{S}](x, u)\, \mathcal{L}^n + \langle (D_y\mathcal{S}(x, u))^\tau, \nabla u\rangle\, \mathcal{L}^n + \langle \mathcal{S}(x, u^+) - \mathcal{S}(x, u^-), \nu_u\rangle\, \mathcal{H}^{n-1}\lfloor S_u,
\end{aligned}$$

and the functional at the right-hand side of (5.8) has the following expression

$$\int_\Omega d\mu_u + \int_\Omega \mathcal{S}_0(x)\,dx = \int_\Omega ([\mathrm{div}_x\mathcal{S}](x,u) + \langle (D_y\mathcal{S}(x,u))^\tau, \nabla u\rangle + \mathcal{S}_0(x))\,dx + \int_{S_u} \langle \mathcal{S}(x,u^+) - \mathcal{S}(x,u^-), \nu_u\rangle\, d\mathcal{H}^{n-1}. \quad (5.9)$$

Using assumption (a1) we obtain that for $\mathcal{L}^n$-a.e. $x\in\Omega$

$$\begin{aligned}[\mathrm{div}_x\mathcal{S}](x,u) + \langle (D_y\mathcal{S}(x,u))^\tau, \nabla u\rangle + \mathcal{S}_0(x) &\le -f^*(x,u,(D_y\mathcal{S}(x,u))^\tau) + \langle (D_y\mathcal{S}(x,u))^\tau, \nabla u\rangle \\ &\le f(x,u,\nabla u),\end{aligned}$$

and consequently

$$\int_\Omega ([\mathrm{div}_x\mathcal{S}](x,u) + \langle (D_y\mathcal{S}(x,u))^\tau, \nabla u\rangle + \mathcal{S}_0(x))\,dx \le \int_\Omega f(x,u,\nabla u)\,dx. \quad (5.10)$$

Moreover, equality holds in (5.10) if and only if $(D_y\mathcal{S}(x,u))^\tau \in \partial_\xi^- f(x,u,\nabla u)$ and $[\mathrm{div}_x\mathcal{S}](x,u)+\mathcal{S}_0(x) = -f^*(x,u,(D_y\mathcal{S}(x,u))^\tau)$, which is condition (a2).

As for the second integral in (5.9), condition (b1) implies that

$$\int_{S_u} \langle \mathcal{S}(x,u^+) - \mathcal{S}(x,u^-), \nu_u\rangle\, d\mathcal{H}^{n-1} \le \int_{S_u} \psi(x,u^-,u^+,\nu_u)\, d\mathcal{H}^{n-1}. \quad (5.11)$$

Moreover, equality holds in (5.11) if and only if (b2) is satisfied.

The statement follows now from (5.9), (5.10), and (5.11). $\square$

The assumption of C^1-regularity for $\mathcal{S}$ is often too strong for many applications. We prove now a new version of Lemma 5.4 under weaker regularity assumptions for $\mathcal{S}$.

Lemma 5.5 *Let F be the functional defined in (5.1). Let $\mathcal{S} : U \to \mathbb{R}^n$ be a Lipschitz continuous function, piecewise C^1 in the sense of Definition 5.1, and satisfying (5.3). Let $\mathcal{S}_0 \in L^1(\Omega)$. Assume that the following conditions are satisfied:*

- *(a1) $[\mathrm{div}_x\mathcal{S}^\alpha](x,y) + \mathcal{S}_0(x) \le -f^*(x,y,(D_y\mathcal{S}^\alpha(x,y))^\tau)$ for every $\alpha\in A$, for $\mathcal{L}^n$-a.e. $x\in\Omega$, and for every $y\in\mathbb{R}^N$ with $(x,y)\in U^\alpha$;*
- *(b1) $\langle \mathcal{S}(x,z) - \mathcal{S}(x,y), \nu\rangle \le \psi(x,y,z,\nu)$ for $\mathcal{H}^{n-1}$-a.e. $x\in\Omega$, for every $\nu\in\mathbb{S}^{n-1}$, and for every y,z with $(x,y)\in U$, $(x,z)\in U$.*

Then for every $u\in SBV(\Omega;\mathbb{R}^N)$ such that $\mathrm{graph}\, u \subset U$ *we have that $\mathrm{div}_x[\mathcal{S}(\cdot,u(\cdot))]$ is a Radon measure on Ω, which will be denoted as μ_u, and*

$$F(u) \ge \int_\Omega d\mu_u + \int_\Omega \mathcal{S}_0(x)\,dx. \quad (5.12)$$

Moreover, equality holds in (5.8) for a given u if and only if

- *(a2) $[\mathrm{div}_x\mathcal{S}^\alpha](x,u) + \mathcal{S}_0(x) = -f^*(x,u,(D_y\mathcal{S}^\alpha(x,u))^\tau)$ and $(D_y\mathcal{S}^\alpha(x,u))^\tau \in \partial_\xi^- f(x,u,\nabla u)$ for every $\alpha\in A$, for $\mathcal{L}^n$-a.e. $x\in\Omega$ such that $(x,u(x))\in U^\alpha$:*
- *(b2) $\langle \mathcal{S}(x,u^+) - \mathcal{S}(x,u^-), \nu_u\rangle = \psi(x,u^-,u^+,\nu_u)$ for $\mathcal{H}^{n-1}$-a.e. $x\in S_u$,*

where u, $u^\pm$, ∇u, and ν_u are always computed at x.

Proof. – Let $u \in SBV(\Omega;\mathbb{R}^N)$ be such that $\operatorname{graph} u \subset U$. By Lemma 5.2 the function $S(\cdot,u(\cdot))$ belongs to $SBV(\Omega;\mathbb{R}^n)$, and therefore, its divergence is a Radon measure on Ω. By (5.4) and (5.5) we have that the measure μ_u can be written as

$$\mu_u(x) = \sum_{\alpha\in A} 1_{U^\alpha}(x,u)[\operatorname{div}_x\mathcal{S}^\alpha](x,u)\,\mathcal{L}^n + \sum_{\alpha\in A} 1_{U^\alpha}(x,u)\langle (D_y\mathcal{S}^\alpha(x,u))^\tau, \nabla u\rangle\,\mathcal{L}^n \\ + \langle \mathcal{S}(x,u^+) - \mathcal{S}(x,u^-), \nu_u\rangle\,\mathcal{H}^{n-1}\lfloor S_u.$$

The proof of Lemma 5.4 can be now repeated simply replacing $[\operatorname{div}_x\mathcal{S}]$ with $\sum_{\alpha\in A} 1_{U^\alpha}[\operatorname{div}_x\mathcal{S}^\alpha]$, and $D_y\mathcal{S}$ with $\sum_{\alpha\in A} 1_{U^\alpha} D_y\mathcal{S}^\alpha$. □

Definition 5.6 *We say that a pair of functions* $(\mathcal{S},\mathcal{S}_0)$ *is a* calibration for $u \in SBV(\Omega;\mathbb{R}^N)$ on U with respect to the functional (5.1) *if* $\mathcal{S} : U \to \mathbb{R}^n$ *is a Lipschitz continuous function, piecewise* C^1 *in the sense of Definition 5.1,* $\mathcal{S}_0 \in L^1(\Omega)$*, and they satisfy assumptions (a1), (b1), (a2), and (b2) in Lemma 5.5.*

We can now prove the main result of this section.

Theorem 5.7 *Let* u *be a function in* $SBV(\Omega;\mathbb{R}^N)$ *whose graph is contained in* U*. Assume that there exists a calibration* $(\mathcal{S},\mathcal{S}_0)$ *for* u *on* U *with respect to the functional (5.1). Then* u *is a Dirichlet* U*-minimizer of* F*. If, in addition, the normal component of* $\mathcal{S}$ *at* $\partial U \cap (\partial\Omega\times\mathbb{R}^N)$ *does not depend on* y*, namely for* $\mathcal{H}^{n-1}$*-a.e.* $x \in \partial\Omega$ *there exists a constant* $a(x) \in \mathbb{R}$ *such that*

$$\langle \mathcal{S}(x,y), \nu_{\partial\Omega}(x)\rangle = a(x) \qquad \text{for every } y \text{ such that } (x,y) \in U, \tag{5.13}$$

then u *is also an absolute* U*-minimizer of* F*.*

Proof. – Let v be a function in $SBV(\Omega;\mathbb{R}^N)$ such that $v = u$ on $\partial\Omega$ and $\operatorname{graph} v \subset U$. Then the definition of the measure μ_v and the divergence theorem imply that

$$\int_\Omega d\mu_v = -\int_{\partial\Omega} \langle \mathcal{S}(x,v), \nu_{\partial\Omega}\rangle\, d\mathcal{H}^{n-1}.$$

If v has the same trace on $\partial\Omega$ as u, from this identity it follows that

$$\int_\Omega d\mu_v = \int_\Omega d\mu_u, \tag{5.14}$$

and by applying Lemma 5.5 we obtain

$$F(v) \geq \int_\Omega d\mu_v + \int_\Omega \mathcal{S}_0(x)\,dx = \int_\Omega d\mu_u + \int_\Omega \mathcal{S}_0(x)\,dx = F(u).$$

We have thus proved that u is a Dirichlet U-minimizer of F.

If we assume, in addition, that (5.13) holds true, then $\int_\Omega d\mu_v = -\int_\Omega a\,d\mathcal{H}^{n-1}$ for every $v \in SBV(\Omega;\mathbb{R}^N)$ whose graph is contained in U; so, the equality (5.14) is fulfilled even if the traces of u and v on $\partial\Omega$ differ. This proves that u is an absolute U-minimizer of F. □

Remark 5.8 It is natural to wonder what is the link in the case $N = 1$ between our vectorial theory and the calibration method for the scalar case, recalled in Section 1.3, which involves a divergence-free vectorfield φ.

Let $N = 1$. Let us suppose that $(\mathcal{S}, \mathcal{S}_0)$ is a calibration for u and assume furthermore that $\mathcal{S}$ is globally C^1. Take the vectorfield $\varphi = (\varphi^x, \varphi^z) : U \to \mathbb{R}^n \times \mathbb{R}$ defined by $\varphi^x(x,z) := \partial_z \mathcal{S}(x,z)$ and $\varphi^z(x,z) := -[\mathrm{div}_x \mathcal{S}](x,z) - \mathcal{S}_0(x)$. Then φ satisfies all the assumptions of Section 1.3. Indeed, by Remark 2.3 in [2] φ is approximately regular on U. Moreover, conditions (a1) and (a2) on $(\mathcal{S}, \mathcal{S}_0)$ clearly imply that φ satifies (a1) and (a2) of Section 1.3, respectively. By definition of φ we have that

$$\int_{t_1}^{t_2} \varphi^x(x,z)\, dz = \mathcal{S}(x,t_2) - \mathcal{S}(x,t_1),$$

so that conditions (b1) and (b2) on $(\mathcal{S}, \mathcal{S}_0)$ imply conditions (b1) and (b2) of Section 1.3, respectively. If $\mathcal{S}$ is C^2 and $\mathcal{S}_0$ is C^1, then it is trivial that φ is C^1 and $\mathrm{div}\varphi = 0$; in the general case, one can prove that φ is divergence-free by an approximation argument.

Analogously it is easy to see that, if φ is a bounded Lipschitz C^1-vectorfield satisfying the calibration conditions of Section 1.3, then we can construct a calibration $(\mathcal{S}, \mathcal{S}_0)$. Take indeed

$$\mathcal{S}(x,z) := \int_{\tau(x)}^{z} \varphi^x(x,t)\, dt \qquad \text{and} \qquad \mathcal{S}_0(x) := \langle \varphi^x(x,\tau(x)), \nabla\tau(x)\rangle - \varphi^z(x,\tau(x)),$$

where τ is any smooth function satisfying $(x,\tau(x)) \in U$ for every $x \in \overline{\Omega}$.

5.2 An application related to classical field theory

We recall now some classical results from field theory for multiple integrals of the form

$$F_0(u) = \int_\Omega f(x,u,\nabla u)\, dx, \tag{5.15}$$

where $u \in C^1(\overline{\Omega}; \mathbb{R}^N)$ and $f \in C^2(\overline{\Omega} \times \mathbb{R}^N \times \mathbb{R}^{nN})$.

We will call *extremals* of F_0 or f*-extremal* the solutions u of class C^2 of the Euler equations for the integral F_0, i.e.

$$\sum_{i=1}^{n} D_{x_i}[\partial_{\xi_{ij}} f(x,u(x),\nabla u(x))] - \partial_{u_j} f(x,u(x),\nabla u(x)) = 0, \qquad 1 \le j \le N. \tag{5.16}$$

In the classical field theory for multiple integrals several sufficient conditions for the minimality of an f-extremal have been proposed. Among the others, we recall Weyl field theory, which is strictly related to the calibration theory for vector-valued functionals and ensures that a given f-extremal u is in fact a minimizer of F_0 among all functions of class C^1, with the same boundary values as u and whose graph is contained in a suitable neighbourhood of the graph of u. It consists in the construction of a suitable slope field $\mathcal{P}$, called Weyl field, and of a smooth function $\mathcal{S}$, called the eikonal map associated with the field, satisfying the system of equations (5.17) – (5.18). This set of conditions arises from the comparison of F_0 with an invariant functional of divergence type, which is nothing but the functional

$$\int_\Omega \mathrm{div}_x[\mathcal{S}(x,v)]\, dx,$$

where $\mathcal{S}$ is the eikonal map (see, e.g. [18, Chapter 7, Section 4]).

We will show via calibrations that, if a Weyl field exists for an f-extremal u (and then there exists a neighbourhood U of the graph of u such that u minimizes F_0 among C^1-functions with the same boundary values as u and with graph contained in U), then u is also a Dirichlet U-minimizer of the functional (5.1) in the sense of Definition 5.3, provided U is a sufficiently small neighbourhood of the graph of u and the function ψ satisfies the estimate (5.20); moreover, if $\mathcal{S}$ is the eikonal map associated with the Weyl field, then the pair $(\mathcal{S}, \mathcal{S}_0)$ with $\mathcal{S}_0 \equiv 0$ is a calibration for u on U.

Definition 5.9 *Let U be a closed domain in $\overline{\Omega}\times\mathbb{R}^N$. A mapping $p: U \to U\times\mathbb{R}^{nN}$ is called a* slope field *on U if it is of class C^1 and of the form*

$$p(x,y) = (x,y,\mathcal{P}(x,y)) \qquad \text{for every } (x,y)\in U\,;$$

we denote $\mathcal{P}(x,y) = (\mathcal{P}_{ij}(x,y))$ as the slope function *of the field p. We say that* a map $u \in C^1(\overline{\Omega};\mathbb{R}^N)$ fits the slope field p *if* graph $u \subset U$ *and*

$$\partial_{x_i} u_j(x) = \mathcal{P}_{ij}(x,u(x)) \qquad \text{for every } x\in\overline{\Omega}.$$

Finally, a slope field p is said to be a Weyl field *if there is a map $\mathcal{S}\in C^2(U;\mathbb{R}^n)$ such that $\{\mathcal{S},\mathcal{P}\}$ solves the Weyl equations:*

$$[\mathrm{div}_x\mathcal{S}](x,y) = f(x,y,\mathcal{P}(x,y)) - \langle \mathcal{P}(x,y), \partial_\xi f(x,y,\mathcal{P}(x,y))\rangle, \tag{5.17}$$

$$\partial_{y_j}\mathcal{S}_i(x,y) = \partial_{\xi_{ij}} f(x,y,\mathcal{P}(x,y)). \tag{5.18}$$

The function $\mathcal{S}$ is called the eikonal map *associated with p.*

The main results in Weyl field theory can be stated as follows. For a proof we refer to [18].

Theorem 5.10 **(1)** *Assume that the function f satisfies*

$$f(x,y,\xi) - f(x,y,\eta) - \langle \xi-\eta, \partial_\xi f(x,y,\eta)\rangle \geq 0$$

for every $(x,y)\in U$ and $\xi,\eta\in\mathbb{R}^{nN}$, and let $u\in C^2(\overline{\Omega};\mathbb{R}^N)$ fit a Weyl field $p: U\to U\times\mathbb{R}^{nN}$ with the eikonal map $\mathcal{S}: U\to\mathbb{R}^n$. Then u is a minimizer of F_0 among all $v\in C^1(\overline{\Omega};\mathbb{R}^N)$ such that $v|_{\partial\Omega} = u|_{\partial\Omega}$ and graph $v\subset U$*; in particular, u is an f-extremal. Moreover, if there is a constant $\mu>0$ such that*

$$\sum_{i,j,h,k} \partial^2_{\xi_{ij}\xi_{hk}} f(x,y,\xi)\zeta_{ij}\zeta_{hk} \geq \mu|\zeta|^2 \qquad \forall (x,y)\in\overline{\Omega}\times\mathbb{R}^N,\ \xi,\zeta\in\mathbb{R}^{nN}, \tag{5.19}$$

then u is a strict minimizer of F_0 in the same class.

(2) *Vice-versa, if f satisfies the strict convexity condition (5.19), then every f-extremal fits at least locally a Weyl field and is therefore locally minimizing F_0. In other words, for every $x_0\in\Omega$ there exist $\varepsilon>0$ and an open neighbourhood A of x_0 such that u minimizes F_0 among all $v\in C^1(\overline{A};\mathbb{R}^N)$ such that $v|_{\partial A} = u|_{\partial A}$ and* graph $v \subset \{(x,y)\in\overline{A}\times\mathbb{R}^N :\ |y-u(x_0)|\leq\varepsilon\}$.

Let us now state and prove a similar result for free-discontinuity problems.

Theorem 5.11 *Let $f:\overline{\Omega}\times\mathbb{R}^N\times\mathbb{R}^{nN}\to[0,+\infty]$ be a function of class C^2 satisfying (5.19) and let $\psi:\Omega\times\mathbb{R}^N\times\mathbb{R}^N\times\mathbb{S}^{n-1}\to[0,+\infty]$ be a Borel function satisfying*

$$\psi(x,y,z,\nu)\geq c\,\theta(|y-z|), \tag{5.20}$$

where c is a positive constant, while θ is such that $\lim_{t\to0^+}\frac{\theta(t)}{t} = +\infty$. Let u be an f-extremal. Then for every $x_0\in\Omega$ there exist $\varepsilon>0$, an open neighbourhood A (with Lipschitz boundary) of x_0, and a pair $(\mathcal{S},\mathcal{S}_0)$ such that $(\mathcal{S},\mathcal{S}_0)$ is a calibration for u with respect to the functional (5.1) on the set

$$U := \{(x,y)\in\overline{A}\times\mathbb{R}^N :\ |y-u(x_0)|\leq\varepsilon\}; \tag{5.21}$$

therefore u is a Dirichlet U-minimizer of the functional (5.1).

PROOF. – Let u be an f-extremal. By the second part of Theorem 5.10 for every $x_0 \in \Omega$ there exist $\varepsilon > 0$ and an open neighbourhood A (with Lipschitz boundary) of x_0 such that u fits a Weyl fiel in the set (5.21). Denote the Weyl field by $p(x,y) = (x, y, \mathcal{P}(x,y))$ and the eikonal map associated with p by $\mathcal{S}$.

We claim that, if we take $\mathcal{S}_0(x) := 0$ for every $x \in \Omega$, then the pair $(\mathcal{S}, \mathcal{S}_0)$ is a calibration for u on U with respect to the functional F defined in (5.1), provided ε is sufficiently small. Let us prove it. Since f is convex, for every $\eta \in \mathbb{R}^{nN}$ we have that

$$f(x,y,\eta) - \langle \eta, \partial_\xi f(x,y,\eta)\rangle = -f^*(x,y,\partial_\xi f(x,y,\eta));$$

this fact, jointly with (5.17), implies that

$$\begin{aligned} [\mathrm{div}_x \mathcal{S}](x,y) &= -f^*(x,y,\partial_\xi f(x,y,\mathcal{P}(x,y))) \\ &= -f^*(x,y,(D_y\mathcal{S}(x,y))^\tau), \end{aligned} \tag{5.22}$$

where the second equality follows from (5.18). Therefore, condition (a1) is satisfied.

Condition (a2) follows from (5.22) and (5.18), using the fact that u fits the field $\mathcal{P}$, hence $\mathcal{P}(x,u(x)) = \nabla u(x)$ for every $x \in \overline{\Omega}$.

If we call L the L^∞-norm of the Jacobian matrix of $\mathcal{S}$ on U, then we have that

$$\langle \mathcal{S}(x,z) - \mathcal{S}(x,y), \nu\rangle \le L\,|z-y| \tag{5.23}$$

for every $x \in \overline{\Omega}$, $y, z \in \mathbb{R}^N$ such that $(x,y) \in U$, $(x,z) \in U$, and $\nu \in \mathbb{S}^{n-1}$. By the assumption on the function θ there exists $\delta > 0$ such that $\theta(t) \ge Lt/c$ for every $t \in (0,\delta)$; then from (5.20) it follows that

$$\psi(x,y,z,\nu) \ge L|y-z| \qquad \text{for } |y-z| < \delta. \tag{5.24}$$

Taking $\varepsilon < \delta/2$, from (5.23) and (5.24) we have that condition (b1) is satisfied.

Since $S_u = \emptyset$, condition (b2) is trivial.

The conclusion follows now from Theorem 5.7. □

As made precise in the next proposition, when the function f depends only on the variables x, ξ, we are able to prove the minimality of an f-extremal u on the whole domain $\overline{\Omega}$ and to give an estimate of the width ε of the neighbourhood of the graph of u where the minimality holds.

Proposition 5.12 *In addition to the assumptions of Theorem 5.11, suppose that $f = f(x,\xi)$. Let u be an f-extremal. For every $(x,y) \in \overline{\Omega} \times \mathbb{R}^N$ define*

$$\mathcal{S}(x,y) := [\partial_\xi f(x, \nabla u(x))]^\tau (y - u(x)) + \sigma(x), \tag{5.25}$$

where $\sigma : \overline{\Omega} \to \mathbb{R}^n$ is a solution of the equation $\mathrm{div}\sigma = f(x, \nabla u)$. Then the pair $(\mathcal{S}, \mathcal{S}_0)$ with $\mathcal{S}_0 \equiv 0$ is a calibration for u with respect to the functional (5.1) on the set

$$U := \{(x,y) \in \overline{\Omega} \times \mathbb{R}^N : \ |y - u(x)| \le \varepsilon(x)\}, \tag{5.26}$$

where

$$\varepsilon(x) < \frac{1}{2} \inf \left\{ t > 0 : \ c\,\frac{\theta(t)}{t} < |\partial_\xi f(x, \nabla u(x))| \right\}, \tag{5.27}$$

and c, θ are the quantities appearing in (5.20). Therefore u is a Dirichlet U-minimizer of the functional (5.1).

Proof. – Note that by the assumption on θ, the infimum in (5.27) is strictly positive for every $x \in \overline{\Omega}$.

Let us prove that $(\mathcal{S}, \mathcal{S}_0)$ satisfies all the conditions in Lemma 5.4.

By direct computations we have that $D_y\mathcal{S}(x,y) = [\partial_\xi f(x,\nabla u)]^\tau$; using the Euler equations (5.16), the definition of σ, and the convexity of f, we find out that

$$\begin{aligned}[\mathrm{div}_x\mathcal{S}](x,y) &= \sum_{ij} D_{x_i}(\partial_{\xi_{ji}} f(x,\nabla u))(y_j - u_j) - \langle [\partial_\xi f(x,\nabla u)]^\tau, \nabla u\rangle + \mathrm{div}\sigma \\ &= -\langle [\partial_\xi f(x,\nabla u)]^\tau, \nabla u\rangle + f(x,\nabla u) \\ &= -f^*(x, [\partial_\xi f(x,\nabla u)]^\tau).\end{aligned}$$

Conditions (a1) and (a2) are therefore satisfied.

By the definition of $\mathcal{S}$ we obtain

$$|\mathcal{S}(x,z) - \mathcal{S}(x,y)| \leq |\partial_\xi f(x,\nabla u(x))| \cdot |z-y|;$$

since $|z-y| \leq 2\varepsilon(x)$, (5.27) implies that

$$|\partial_\xi f(x,\nabla u(x))| \cdot |z-y| \leq c\,\theta(|z-y|);$$

so condition (b1) follows now from (5.20).

Condition (b2) is trivial since S_u is empty. This concludes the proof.

We notice that the thesis can be proved also in the following way: if we define $\mathcal{P}(x,y) := \nabla u(x)$ for every $(x,y) \in \overline{\Omega}\times\mathbb{R}^N$, it is easy to see that the field $p(x,y) := (x,y,\mathcal{P}(x,y))$ is a Weyl field, $\mathcal{S}$ is the eikonal map associated with p, and u fits p. Then we can follow the proof of Theorem 5.11: the check of condition (a1), (a2), (b2) remains the same, while the estimate on the size of $\varepsilon(x)$ is given by a more careful proof of condition (b1). □

Remark 5.13 When the functional (5.1) satisfies some special further conditions, it is enough to prove the Dirichlet minimality of a given u on a neighbourhood of its graph to conclude that u is in fact a Dirichlet minimizer on the whole cylinder $\overline{\Omega}\times\mathbb{R}$, reducing the domain Ω if needed. For istance, in addition to the assumptions of Proposition 5.12, suppose that the two following conditions are satisfied:

(1) $f(x,\xi) \geq f(x,(I - e_j\otimes e_j)\,\xi)$ for every $x \in \Omega$, $\xi \in \mathbb{R}^{nN}$, $j = 1,\dots,N$, where $\{e_1,\dots,e_N\}$ is the canonical basis of $\mathbb{R}^N$;

(2) $\psi(x,y,z,\nu) \geq \psi(x, T_a^b(y), T_a^b(z), \nu)$ for every $(x,y) \in \Omega\times\mathbb{R}^N$, $\xi \in \mathbb{R}^{nN}$, $\nu \in \mathbb{S}^{n-1}$, $a,b \in \mathbb{R}^N$, where we have set
$$T_a^b : \mathbb{R}^N \to \mathbb{R}^N, \qquad (T_a^b)_j(y) := (y_j \wedge a_j) \vee b_j.$$

If u is an f-extremal, then by Proposition 5.12 we know that u is a Dirichlet U-minimizer of F, where U is the set (5.26). We want to show that for every $x_0 \in \Omega$ there exists an open neighbourhood A (with Lipschitz boundary) of x_0 such that u is a Dirichlet minimizer of F in A.

First of all, we can find an open neighbourhood A (with Lipschitz boundary) of x_0 and two vectors $m, M \in \mathbb{R}^N$ such that $|M - m| < \varepsilon(x)$ for every $x \in \overline{A}$ and

$$m_j \leq u_j(x) \leq M_j \qquad \forall x \in \overline{A},\ 1 \leq j \leq N. \tag{5.28}$$

Let v be a function in $SBV(A;\mathbb{R}^N)$ with the same trace on ∂A as u and define $\hat{v} := T_m^M(v)$, which still belongs to $SBV(A;\mathbb{R}^N)$. Note that $\nabla\hat{v}_j = 1_{\{m_j<v_j<M_j\}}\nabla v_j$ for every j, so that, if we call $J_0(x)$ the set of all indexes j such that $v_j(x) \notin (m_j, M_j)$, the matrix $\nabla\hat{v}(x)$ can be written as

$$\nabla\hat{v}(x) = \nabla v(x) - \sum_{j\in J_0}(e_j\otimes e_j)\,\nabla v(x).$$

By using iteratively condition (1), we obtain that $f(x, \nabla\hat{v}) \le f(x, \nabla v)$, which implies

$$\int_A f(x, \nabla\hat{v})\, dx \le \int_A f(x, \nabla v)\, dx. \tag{5.29}$$

Since $S_{\hat{v}} \subset S_v$, and $\hat{v}^- = T_m^M(v^-)$, $\hat{v}^+ = T_m^M(v^+)$ on $S_{\hat{v}}$, by condition (2) we obtain

$$\int_{S_{\hat{v}} \cap A} \psi(x, \hat{v}^-, \hat{v}^+, \nu_{\hat{v}})\, d\mathcal{H}^{n-1} \le \int_{S_v \cap A} \psi(x, v^-, v^+, \nu_v)\, d\mathcal{H}^{n-1}. \tag{5.30}$$

On the other hand, by (5.28) the function $\hat{v}$ has the same trace on ∂A as u, and its graph is contained in the set

$$\{(x,y) \in \overline{A} \times \mathbb{R}^N : |y - u(x)| \le \varepsilon(x)\}.$$

Since u is a Dirichlet minimizer on this set, we have that

$$\int_A f(x, \nabla u)\, dx \le \int_A f(x, \nabla\hat{v})\, dx + \int_{S_{\hat{v}}} \psi(x, \hat{v}^-, \hat{v}^+, \nu_{\hat{v}})\, d\mathcal{H}^{n-1}. \tag{5.31}$$

Therefore by (5.29), by (5.30), and (5.31), u is a Dirichlet minimizer of F in A.

The same result can be achieved by calibration: indeed, we can extend the function $\mathcal{S}$ in (5.25) to the whole $\overline{\Omega} \times \mathbb{R}^N$ simply by taking $\hat{\mathcal{S}}(x,y) := \mathcal{S}(x, T_m^M(y))$; it is easy to see that assumptions (1) – (2) guarantee that the pair $(\hat{\mathcal{S}}, \mathcal{S}_0)$ provides a calibration for u on $\overline{A} \times \mathbb{R}^N$.

We conclude the remark with some comments on conditions (1) – (2). Condition (1) ensures that the functional decreases when any row of the matrix ∇u is annihilated, which is what occurs when a component of u is truncated. For istance, (1) is fulfilled for $f(\xi) = \sum_{ij} \varphi_{ij}(\xi_{ij})$ where φ_{ij} are convex and positive, and $\varphi_{ij}(0) = 0$. As for condition (2), note that it is satisfied whenever ψ depends on y, z only through the distance $|z - y|$.

5.3 Some further applications

In this section we present some examples and applications. In Examples 5.14, 5.16, 5.17, and 5.18 we deal with minimizers of the Mumford-Shah functional, and we generalize some results proved in [2] for the scalar case. Example 5.15 is a purely vectorial example, since it involves a functional arising in fracture mechanics which can be defined only on maps from $\Omega \subset \mathbb{R}^n$ into $\mathbb{R}^n$.

Example 5.14 Let $u : \Omega \to \mathbb{R}^N$ be a harmonic function. It is well known that u is an extremal of the functional $\int_\Omega |\nabla u|^2$, and a Dirichlet minimizer of it. We can prove via calibrations that u is a Dirichlet minimizer also of the homogeneous Mumford-Shah functional

$$MS(u) = \int_\Omega |\nabla u|^2 dx + \mathcal{H}^{n-1}(S_u), \tag{5.32}$$

if the following condition is satisfied:

$$\operatorname*{osc}_\Omega u \cdot \sup_\Omega |\nabla u| \le 1, \tag{5.33}$$

where $\operatorname{osc} u$ denotes the modulus of the vector in $\mathbb{R}^N$ whose components are the oscillations of the components of u. When (5.33) is not fulfilled, u is still a Dirichlet U-minimizer of the functional MS, where

$$U := \left\{ (x,y) \in \overline{\Omega} \times \mathbb{R}^N : |y - u(x)| \le \frac{1}{4|\nabla u(x)|} \right\}. \tag{5.34}$$

This second result directly follows from Proposition 5.12, where $f(\xi) = |\xi|^2$ and $\psi \equiv 1$. Moreover, a calibration is given by $(\mathcal{S}, \mathcal{S}_0)$ with $\mathcal{S}_0 \equiv 0$ and

$$\mathcal{S}(x,y) = 2[\nabla u(x)]^\tau (y - u(x)) + \sigma(x),$$

where $\sigma : \Omega \to \mathbb{R}^n$ is a solution of the equation $\mathrm{div}\sigma = |\nabla u|^2$. Since u is harmonic in Ω, it is easy to see that we can take $\sigma(x) := [\nabla u(x)]^\tau u(x)$, so that

$$\mathcal{S}(x,y) = 2[\nabla u(x)]^\tau \left(y - \frac{u(x)}{2} \right). \tag{5.35}$$

As for the Dirichlet minimality of u, we can show that, under the assumption (5.33), the calibration $(\mathcal{S}, \mathcal{S}_0)$ can be extended to the whole $\overline{\Omega} \times \mathbb{R}^N$, applying a similar argument to the one used in Remark 5.13.

We recall that, in the case of the functional (5.32), conditions (a1), (a2), (b1), and (b2) in Lemma 5.5 become

(a1) $[\mathrm{div}_x \mathcal{S}^\alpha](x,y) + \mathcal{S}_0(x) \le -\frac{1}{4}|D_y \mathcal{S}^\alpha(x,y)|^2$ for every $\alpha \in A$, for $\mathcal{L}^n$-a.e. $x \in \Omega$, and for every $y \in \mathbb{R}^N$ with $(x,y) \in U^\alpha$;

(a2) $[\mathrm{div}_x \mathcal{S}^\alpha](x,u) + \mathcal{S}_0(x) = -|\nabla u(x)|^2$ and $(D_y \mathcal{S}^\alpha(x,u))^\tau = 2\nabla u(x)$ for every $\alpha \in A$, and for $\mathcal{L}^n$-a.e. $x \in \Omega$ such that $(x, u(x)) \in U^\alpha$;

(b1) $|\mathcal{S}(x,z) - \mathcal{S}(x,y)| \le 1$ for $\mathcal{H}^{n-1}$-a.e. $x \in \Omega$ and for every $y, z \in \mathbb{R}^N$ such that $(x,y) \in U$, $(x,z) \in U$;

(b2) $\mathcal{S}(x,u^+) - \mathcal{S}(x,u^-) = \nu_u$ for $\mathcal{H}^{n-1}$-a.e. $x \in S_u$,

where $\mathcal{S}(x,y) = \sum_{\alpha \in A} \mathcal{S}^\alpha(x,y) 1_{U^\alpha}(x,y)$.

Let m_j and M_j be the infimum and the supremum of u_j in Ω, respectively (then $\mathrm{osc}\, u_j = M_j - m_j$). Let T be the function from $\mathbb{R}^N$ into $\mathbb{R}^N$ defined as $T_j(y) = (y_j \vee m_j/2) \wedge M_j/2$. Define

$$\hat{\mathcal{S}}(x,y) := 2[\nabla u(x)]^\tau \, T\left(y - \frac{u(x)}{2} \right).$$

It is easy to see that $(\hat{\mathcal{S}}, \mathcal{S}_0)$ satisfies conditions (a1) and (a2). Condition (b2) is trivial. Finally, for every $y, z \in \mathbb{R}^N$ we have

$$|\hat{\mathcal{S}}(x,z) - \hat{\mathcal{S}}(x,y)| \le 2|\nabla u(x)| \cdot |T(z - u/2) - T(y - u/2)|. \tag{5.36}$$

Since $T_j(z - u/2)$ and $T_j(y - u/2)$ belong to the interval $[m_j/2, M_j/2]$ for every $1 \le j \le N$, we deduce that $|T(z - u/2) - T(y - u/2)| \le |M - m|/2$; so, condition (b1) follows from (5.36) and (5.33).

These two minimality results generalize those obtained in [1] for scalar harmonic functions. Note that the minimality of u can be proved by applying the scalar argument to each component u_j, but this provides a more restrictive condition on the size of the domains where the minimality holds. Indeed, by the scalar result in [1], since u_j is harmonic for every j, if

$$\operatorname*{osc}_{\Omega} u_j \cdot \sup_{\Omega} |\nabla u_j| \le \frac{1}{N} \qquad 1 \le j \le N, \tag{5.37}$$

then

$$\int_\Omega |\nabla u_j|^2 dx \le \int_\Omega |\nabla v_j|^2 dx + \frac{1}{N} \mathcal{H}^{n-1}(S_{v_j})$$

for every $v_j \in SBV(\Omega)$ with the same boundary values as u_j; summing over j, we obtain the Dirichlet minimality of u in Ω. On the other hand, it is easy to see that condition (5.37) is stronger than (5.33). Analogous remarks hold for the Dirichlet minimality of u in a neighbourhood of its graph.

Example 5.15 In this example we consider a functional related to Griffith and Barenblatt theories of fracture mechanics of the form

$$H(u) := \mu \int_\Omega |e(u)|^2 dx + \frac{\lambda}{2}\int_\Omega (\mathrm{div} u)^2\, dx + \int_{S_u} \theta(|u^+ - u^-|)\, d\mathcal{H}^{n-1}$$

where u is a function from $\Omega \subset \mathbb{R}^n$ into $\mathbb{R}^n$, $e(u)$ denotes the symmetrized gradient of u, θ is a positive function satisfying $\lim_{t\to 0^+} \theta(t)/t = +\infty$, and μ, λ are real parameters. In the context of fracture mechanics, Ω is a reference configuration of an elastic body, possibly subject to fracture, and u parameterizes its displacement; the bulk term represents the energy relative to the elastic deformation outside the fracture, while the surface integral is the energy needed to produce the crack.

The functional H is clearly of the form (5.1) with $f(\xi) = \mu\,|(\xi^\tau + \xi)/2|^2 + \frac{\lambda}{2}(\mathrm{tr}\,\xi)^2$ and $\psi(y,z) = \theta(|z-y|)$. However, since the bulk term in H involves only the symmetric part of the matrix ∇u, the appropriate setting for the minimum problem for H is not exactly the space $SBV(\Omega;\mathbb{R}^n)$, but the space $SBD(\Omega)$ of special functions with bounded deformation (for a complete overview on the properties of this space see [5]). Even if the calibration method has been developed only for SBV functions, we can actually prove by calibration that, if u is an f-extremal, i.e. $u \in C^1(\overline{\Omega};\mathbb{R}^n) \cap C^2(\Omega;\mathbb{R}^n)$ and u solves the equation

$$\mu\Delta u + (\mu+\lambda)\nabla(\mathrm{div} u) = 0 \qquad \text{on } \Omega, \tag{5.38}$$

then u minimizes H among all functions $v \in SBD(\Omega)$ with the same trace on $\partial\Omega$ as u, and whose graph is contained in the set

$$U := \{(x,y) \in \overline{\Omega}\times\mathbb{R}^n : \ |y - u(x)| \le \varepsilon(x)\},$$

where

$$\varepsilon(x) < \frac{1}{2}\inf\left\{t > 0 : \ \frac{\theta(t)}{t} < |2\mu e(u)(x) + \lambda \mathrm{div} u(x) I|\right\}.$$

Indeed, since $\partial_{\xi_{ij}} f(\xi) = \mu(\xi_{ji}+\xi_{ij}) + \lambda(\mathrm{tr}\,\xi)\delta_{ij}$, Proposition 5.12 implies that u is a Dirichlet U-minimizer of H in the class $SBV(\Omega;\mathbb{R}^n)$ and a calibration is given by $(\mathcal{S},\mathcal{S}_0)$ with $\mathcal{S}_0 \equiv 0$ and

$$\mathcal{S}(x,y) = [2\mu e(u)(x) + \lambda \mathrm{div} u(x) I]\left(y - \frac{u(x)}{2}\right); \tag{5.39}$$

this last fact follows from formula (5.25) where we have taken $\sigma(x) := [\mu e(u)(x) + \frac{\lambda}{2}\mathrm{div} u(x) I]\, u(x)$, which is a solution of $\mathrm{div}\sigma = f(\nabla u)$ thanks to (5.38).

On the other hand, we can show that the pair $(\mathcal{S},\mathcal{S}_0)$ provides a calibration also in the space $SBD(\Omega)$ in the following sense: consider the functional

$$H_1(v) := -\int_{\partial\Omega} \langle \mathcal{S}(x,v), \nu_{\partial\Omega}\rangle\, d\mathcal{H}^{n-1},$$

which is the same used as comparison functional in the proof of Theorem 5.7; then, H_1 is well defined on $SBD(\Omega)$, is invariant on SBD functions having the same trace on $\partial\Omega$, and satisfies the equality $H_1(u) = H(u)$ and the inequality $H_1(v) \le H(v)$ for every $v \in SBD(\Omega)$. This implies that u is a Dirichlet minimizer of the functional H in the class of SBD functions.

Let us prove the properties of H_1 stated above. If we set for simplicity of notation $A(x) := 2\mu e(u)(x) + \lambda \mathrm{div} u(x) I$, by (5.39) the functional H_1 can be rewritten as

$$H_1(v) = -\frac{1}{2}\int_{\partial\Omega} \langle A(2v-u), \nu_{\partial\Omega}\rangle\, d\mathcal{H}^{n-1},$$

whence it is clear that it is well defined on $SBD(\Omega)$ and invariant on the class of functions in $SBD(\Omega)$ having the same trace on $\partial\Omega$. By the generalized Green's formula in $SBD(\Omega)$ we have that

$$\begin{aligned} -\frac{1}{2}\int_{\partial\Omega}\langle A(2v-u),\nu_{\partial\Omega}\rangle\,d\mathcal{H}^{n-1} &= \frac{1}{2}\int_{\Omega}\langle 2v-u,\mathrm{div}A\rangle\,dx+\frac{1}{2}\int_{\Omega}A\,d(2Ev-Eu) \\ &= \frac{1}{2}\int_{\Omega}\langle A,2e(v)-e(u)\rangle\,dx+\int_{S_v}\langle A(v^+-v^-),\nu_v\rangle\,d\mathcal{H}^{n-1}, \end{aligned} \tag{5.40}$$

where the last equality follows by the fact that $\mathrm{div}A=0$, by the decomposition theorem for the measures Ev,Eu and by the remark that $S_u=\emptyset$. Using the definition of the matrix A and (5.40) it is easy to see that

$$H_1(u)=\frac{1}{2}\int_{\Omega}\langle A,e(u)\rangle\,dx=H(u), \tag{5.41}$$

while, using also the elementary inequality $2\langle\xi,\eta\rangle\le|\xi|^2+|\eta|^2$ for every $\xi,\eta\in\mathbb{R}^{n^2}$, we obtain

$$\begin{aligned} \int_{\Omega}\langle A,e(v)\rangle\,dx &= 2\mu\int_{\Omega}\langle e(u),e(v)\rangle\,dx+\lambda\int_{\Omega}\mathrm{div}u\,\mathrm{div}v\,dx \\ &\le \mu\int_{\Omega}|e(v)|^2dx+\frac{\lambda}{2}\int_{\Omega}(\mathrm{div}v)^2dx+H(u). \end{aligned} \tag{5.42}$$

Since the graph of v is contained in U, we have that $\langle A(v^+-v^-),\nu_v\rangle\le\theta(|v^+-v^-|)$ $\mathcal{H}^{n-1}$-a.e. on S_v, so that

$$\int_{S_v}\langle A(v^+-v^-),\nu_v\rangle\,d\mathcal{H}^{n-1}\le\int_{S_v}\theta(|v^+-v^-|)\,d\mathcal{H}^{n-1}. \tag{5.43}$$

By (5.40), (5.41), (5.42), and (5.43), we deduce that $H_1(v)\le H(v)$ for every $v\in SBD(\Omega)$ whose graph is contained in U.

We conclude this example by noticing that the existence of a weak solution in $W^{1,2}(\Omega;\mathbb{R}^n)$ for the Dirichlet boundary value problem associated with the equation (5.38) is guaranteed if $\mu>0$ and $2\mu+3\lambda>0$; moreover, the additional requirements of regularity for u are always satisfied in any open subset $\Omega'\subset\subset\Omega$ (see [10]).

Example 5.16 Let Ω be a product of the form $(0,a)\times V$, where V is a regular domain in $\mathbb{R}^{n-1}$, and let u be the step function defined as $u(x):=0$ for $0<x_1<c$, and $u(x)=h$ for $c<x_1<a$, where $c\in(0,a)$ and $h\in\mathbb{R}^N$, $h\neq 0$. Then, u is a Dirichlet minimizer of the Mumford-Shah functional (5.32) in Ω if $|h|^2\ge a$.

This result generalizes Example 4.12 in [1], where u is a scalar step function.

We prove the statement by calibration. Let $\{e_1,\dots,e_n\}$ be the canonical basis of $\mathbb{R}^n$. A calibration for u is given by the pair $(\mathcal{S},\mathcal{S}_0)$ with $\mathcal{S}_0\equiv 0$ and

$$\mathcal{S}(x,y):=\begin{cases}0 & \text{if } \langle y,\frac{h}{|h|}\rangle\le\frac{\lambda}{2}\langle x,e_1\rangle,\\ 2\lambda\left(\langle y,\frac{h}{|h|}\rangle-\frac{\lambda}{2}\langle x,e_1\rangle\right)e_1 & \text{if } \frac{\lambda}{2}\langle x,e_1\rangle\le\langle y,\frac{h}{|h|}\rangle\le\frac{\lambda}{2}\langle x,e_1\rangle+\frac{\lambda}{2}a,\\ a\lambda^2e_1 & \text{if } \langle y,\frac{h}{|h|}\rangle\ge\frac{\lambda}{2}\langle x,e_1\rangle+\frac{\lambda}{2}a,\end{cases} \tag{5.44}$$

where $\lambda:=1/\sqrt{a}$. Some direct computations show that

$$\begin{aligned} |D_y\mathcal{S}(x,y)|^2 &= \begin{cases}4\lambda^2 & \text{if } \frac{\lambda}{2}\langle x,e_1\rangle\le\langle y,\frac{h}{|h|}\rangle\le\frac{\lambda}{2}\langle x,e_1\rangle+\frac{\lambda}{2}a,\\ 0 & \text{otherwise,}\end{cases} \\ \mathrm{div}\mathcal{S}(x,y) &= \begin{cases}-\lambda^2 & \text{if } \frac{\lambda}{2}\langle x,e_1\rangle\le\langle y,\frac{h}{|h|}\rangle\le\frac{\lambda}{2}\langle x,e_1\rangle+\frac{\lambda}{2}a,\\ 0 & \text{otherwise,}\end{cases} \end{aligned}$$

so that condition (a1) is trivially satisfied, while condition (a2) is true if $|h| \geq \frac{\lambda}{2}x_1 + \frac{\lambda}{2}a$ for every $x_1 \in [c,a)$, which is guaranteed by the assumption $|h|^2 \geq a$.

One easily checks that the vector $\mathcal{S}(x,z) - \mathcal{S}(x,y)$ can always be written as μe_1 with $|\mu| \leq 1$ (μ depending on x,y,z), so that condition (b1) is fulfilled. As for condition (b2), since $|h| \geq \frac{\lambda}{2}(c+a)$ by the assumption $|h|^2 \geq a$, we have that $\mathcal{S}(x,h) - \mathcal{S}(x,0) = a\lambda^2 e_1 - 0 = e_1$ for every $x \in S_u$.

We note that the minimality of u can be proved by applying the scalar result to one component of u. Take, indeed, $j \in \{1,\dots,N\}$ such that $h_j \neq 0$; we know that if $h_j^2 \geq a$, then

$$\mathcal{H}^{n-1}(S_{u_j}) \leq \int_\Omega |\nabla v_j|^2 dx + \mathcal{H}^{n-1}(S_{v_j})$$

for every $v \in SBV(\Omega)$ with the same boundary values as u. Now, the left-hand side coincides with $MS(u)$, while the right-hand side is less than or equal to $MS(v)$, since $S_{v_j} \subset S_v$. So, the Dirichlet minimality of u is shown, but under the stronger condition $h_j^2 \geq a$.

Actually, since the Mumford-Shah functional is invariant by rotation (and then u is a Dirichlet minimizer if and only if $R \circ u$ is a Dirichlet minimizer, where R is any rotation in $\mathbb{R}^N$), the scalar result can be exploited in a more efficient way. Let R be a rotation in $\mathbb{R}^N$ transforming the vector $h/|h|$ in e_1 and let $\hat{u} := R \circ u$. Applying the argument above to the first component of $\hat{u}$, we have that $\hat{u}$ is a Dirichlet minimizer of MS if $|h|^2 \geq a$, which is the same condition we have found via vectorial calibration theory. We also note that the calibration (5.44) can be obtained starting from the vectorfield which calibrates $\hat{u}_1$ simply replacing the one-dimensional vertical variable by the component of the vector y along $h/|h|$ and following the instructions of Remark 5.8.

Example 5.17 Let $\Omega := B(0,r)$ be the open ball in $\mathbb{R}^2$ centred at the origin with radius r, and let (A_1, A_2, A_3) be the partition of Ω defined as follows:

$$A_i := \left\{ x = (\rho\cos\theta, \rho\sin\theta) :\ 0 \leq \rho < r,\ \frac{2}{3}\pi(i-1) \leq \theta < \frac{2}{3}\pi i \right\}.$$

Let $u \in SBV(\Omega;\mathbb{R}^N)$ be the function defined as $u := a_i$ in each A_i, where a_1, a_2, a_3 are three distinct vectors in $\mathbb{R}^N$. In [2, Example 4.14] it is proved that, when $N = 1$, u is a Dirichlet minimizer of the Mumford-Shah functional (5.32) if the values a_i are sufficiently far apart, more precisely if

$$\min\{|a_1 - a_2|, |a_2 - a_3|, |a_3 - a_1|\} \geq \sqrt{2r}. \tag{5.45}$$

This result can be generalized to the vectorial case $N > 1$, where beside condition (5.45) we require that

$$\max\{|a_1 - a_2|, |a_2 - a_3|, |a_3 - a_1|\} \geq \sqrt{(2+\sqrt{3})r}. \tag{5.46}$$

Note that when $N = 1$ condition (5.46) is implied by (5.45): indeed, without loss of generality we can assume that $a_1 \leq a_2 \leq a_3$, so that the maximum in (5.46) is $a_3 - a_1$; then by (5.45) we obtain

$$a_3 - a_1 = (a_3 - a_2) + (a_2 - a_1) \geq 2\sqrt{2r} > \sqrt{(2+\sqrt{3})r}.$$

We prove the statement by calibration. For every i,j we call S_{ij} the interface between A_i and A_j, which is oriented by the normal ν_{ij} pointing from A_i to A_j and we suppose that the maximum in (5.46) is given by $|a_1 - a_2|$. Let $\mathcal{S}_0 \equiv 0$ and

$$\mathcal{S}(x,y) := [\sigma_1(x,y) \vee 0]\,\nu_{31} + [\sigma_2(x,y) \vee 0]\,\nu_{32},$$

where

$$\sigma_1(x,y) := 1 - \frac{|y-a_1|^2}{r - \langle \nu_{31}, x\rangle}, \qquad \sigma_2(x,y) := 1 - \frac{|y-a_2|^2}{r - \langle \nu_{32}, x\rangle}.$$

For any $r' < r$ the function $\mathcal{S}$ is Lipschitz in $\overline{B(0,r')}\times\mathbb{R}^N$. By direct computations we have that

$$|D_y\mathcal{S}(x,y)|^2 = 4\frac{|y-a_1|^2}{(r-\langle\nu_{31},x\rangle)^2}1_{\{\sigma_1>0\}} + 4\frac{|y-a_2|^2}{(r-\langle\nu_{32},x\rangle)^2}1_{\{\sigma_2>0\}} + 4\frac{\langle y-a_1, y-a_2\rangle}{(r-\langle\nu_{31},x\rangle)(r-\langle\nu_{32},x\rangle)}1_{\{\sigma_1>0,\,\sigma_2>0\}}, \tag{5.47}$$

while

$$[\mathrm{div}_x\mathcal{S}](x,y) = -\frac{|y-a_1|^2}{(r-\langle\nu_{31},x\rangle)^2}1_{\{\sigma_1>0\}} - \frac{|y-a_2|^2}{(r-\langle\nu_{32},x\rangle)^2}1_{\{\sigma_2>0\}}. \tag{5.48}$$

Condition (a1) is therefore fulfilled if and only if $\langle y-a_1, y-a_2\rangle \le 0$ for every y such that there exists $x\in B(0,r')$ so that $\sigma_1(x,y)>0$ and $\sigma_2(x,y)>0$. Taking into account the definition of σ_1,σ_2, this is equivalent to require the following: if y belongs to the intersection of the ball centred at a_1 with radius $(r-\langle\nu_{31},x\rangle)$ and the ball centred at a_2 with radius $(r-\langle\nu_{32},x\rangle)$, then the angle spanned by the two vectors $y-a_1$ and $y-a_2$ is greater or equal to $\pi/2$. Some elementary geometric considerations show that this is guaranteed if

$$|a_1-a_2|^2 \ge (2r-\langle\nu_{31},x\rangle-\langle\nu_{32},x\rangle) \qquad \forall x\in B(0,r'),$$

which is implied by condition (5.46).

From (5.45) it follows that $\sigma_2(x,a_1)\le 0$, so that by (5.47) and (5.48) we have $|D_y\mathcal{S}(x,a_1)|^2=0$ and $[\mathrm{div}_x\mathcal{S}](x,a_1)=0$. Since (5.45) implies analogously that $\sigma_1(x,a_2)\le 0$, and $\sigma_1(x,a_3)\le 0$, $\sigma_2(x,a_3)\le 0$, we deduce that condition (a2) is satisfied.

Let $(x,y),(x,z)\in\overline{B(0,r')}\times\mathbb{R}^N$. If neither (x,y) nor (x,z) belongs to $\{\sigma_1>0,\sigma_2>0\}$, then it is easy to check that the vector $\mathcal{S}(x,z)-\mathcal{S}(x,y)$ can be written as a linear combination $\lambda_1\nu_{31}-\lambda_2\nu_{32}$ with either $\lambda_1,\lambda_2\in[0,1]$ or $\lambda_1,\lambda_2\in[-1,0]$ (depending on x,y,z); since ν_{31} and $-\nu_{32}$ span an angle equal to $2\pi/3$, the modulus of $\mathcal{S}(x,z)-\mathcal{S}(x,y)$ is in this case less than or equal to 1. If $(x,y)\in\{\sigma_1>0,\sigma_2>0\}$, only two cases can occur: either $\mathcal{S}(x,z)-\mathcal{S}(x,y)$ is a linear combination of ν_{31} and $-\nu_{32}$ of the same kind as before (so, the same conclusion holds), or $\mathcal{S}(x,z)-\mathcal{S}(x,y)$ can be written as $\mu_1\nu_{31}+\mu_2\nu_{32}$ with $\mu_i\in[0,\sigma_i(x,y)]$ (depending on x,y,z). In this second case, we obtain

$$|\mathcal{S}(x,z)-\mathcal{S}(x,y)|^2 \le \sigma_1^2(x,y)+\sigma_2^2(x,y)+\sigma_1(x,y)\sigma_2(x,y) \le (\sigma_1(x,y)+\sigma_2(x,y))^2.$$

It is easy to see that, under condition (5.46), $\sigma_1(x,y)+\sigma_2(x,y)\le 1$ for every $(x,y)\in\{\sigma_1>0,\sigma_2>0\}$, so that (b1) is always satisfied.

Finally, using (5.45) we have that $\mathcal{S}(x,a_2)-\mathcal{S}(x,a_1)=\nu_{32}-\nu_{31}=\nu_{12}$ for every $x\in S_{12}$, $\mathcal{S}(x,a_3)-\mathcal{S}(x,a_2)=-\nu_{32}=\nu_{23}$ for every $x\in S_{23}$, while $\mathcal{S}(x,a_1)-\mathcal{S}(x,a_3)=\nu_{31}$ for every $x\in S_{31}$; so, we can conclude that (b2) holds true for every $x\in S_u$.

We have thus proved that under conditions (5.45) – (5.46), u is a Dirichlet minimizer of MS in $B(0,r')$ for every $r'<r$. By an approximation argument this implies the Dirichlet minimality of u in the whole $B(0,r)$.

As in the previous example, the minimality of u can be proved by using the scalar result in [2]: indeed, even if S_{u_j} is strictly contained in S_u for every j, one can always find a rotation R in $\mathbb{R}^N$ tranforming the range of u in a set of three vectors which differ each other for the same component and apply the scalar result to this component. This procedure leads to the following condition: u is a Dirichlet minimizer if

$$\max_{v\in\mathbb{R}^N,|v|=1}\ \min\{|\langle a_1-a_2,v\rangle|,|\langle a_2-a_3,v\rangle|,|\langle a_3-a_1,v\rangle|\}\ge\sqrt{2r},$$

which is always more restrictive than (5.45) – (5.46), except when the vectors a_i-a_j are collinear.

Example 5.18 In this example we deal with the complete Mumford-Shah functional

$$MS_{\alpha,\beta}(u) := \int_\Omega |\nabla u|^2 dx + \alpha \mathcal{H}^1(S_u) + \beta \int_\Omega |u-g|^2 \, dx, \tag{5.49}$$

where $\Omega \subset \mathbb{R}^2$, g is a given function in $L^\infty(\Omega;\mathbb{R}^N)$, and α, β are positive parameters.

Let $\{\Gamma_i\}_{i\in I}$ be a finite family of simple and connected curves of class C^2 such that for every i Γ_i is either a closed curve contained in Ω or it orthogonally meets $\partial\Omega$. Suppose also that $\Gamma_i \cap \Gamma_h = \emptyset$ if $i \neq h$. If g is a piecewise constant function, whose discontinuity set coincides with $\cup_{i\in I}\Gamma_i$, then for large values of β the function g itself is an absolute minimizer of (5.49).

We prove the statement by calibration. We recall that conditions (a1), (a2), (b1), and (b2) in Lemma 5.5 read for the functional (5.49) as

(a1) $[\mathrm{div}_x \mathcal{S}^\gamma](x,y) + \mathcal{S}_0(x) \leq -\frac{1}{4}|D_y\mathcal{S}^\gamma(x,y)|^2 + \beta|y-g(x)|^2$ for every $\gamma \in A$, for $\mathcal{L}^2$-a.e. $x \in \Omega$, and for every $y \in \mathbb{R}^N$ with $(x,y) \in U^\gamma$;

(a2) $[\mathrm{div}_x \mathcal{S}^\gamma](x,u) + \mathcal{S}_0(x) = -|\nabla u(x)|^2 + \beta|u-g|^2$ and $(D_y\mathcal{S}^\gamma(x,u))^\tau = 2\nabla u(x)$ for every $\gamma \in A$, and for $\mathcal{L}^2$-a.e. $x \in \Omega$ such that $(x,u(x)) \in U^\gamma$;

(b1) $|\mathcal{S}(x,z) - \mathcal{S}(x,y)| \leq \alpha$ for $\mathcal{H}^1$-a.e. $x \in \Omega$ and for every $y,z \in \mathbb{R}^N$ such that $(x,y) \in U$, $(x,z) \in U$;

(b2) $\mathcal{S}(x,u^+) - \mathcal{S}(x,u^-) = \alpha\nu_u$ for $\mathcal{H}^1$-a.e. $x \in S_u$,

where $\mathcal{S}(x,y) = \sum_{\gamma\in A}\mathcal{S}^\gamma(x,y)1_{U^\gamma}(x,y)$.

Let $\{E_j\}_{j\in J}$ be the partition of Ω generated by the family $\{\Gamma_i\}_{i\in I}$. Then the function g can be written as

$$g(x) = \sum_{j\in J} a_j 1_{E_j}(x),$$

where $a_j \in \mathbb{R}^N$ and $a_j \neq a_k$ if $j \neq k$. For $j < k$ we call S_{jk} the interface between E_j and E_k, oriented by the normal ν_{jk} pointing from E_j to E_k (in other words, S_{jk} is the set of all $x \in S_g$ such that $g^-(x) = a_j$ and $g^+(x) = a_k$). In this way we have simply relabelled the curves Γ_i.

For every $j < k$ we can construct a C^1-vectorfield $\psi_{jk} : \overline{\Omega} \to \mathbb{R}^n$ such that it agrees with ν_{jk} on S_{jk}, is supported on a neighbourhood of S_{jk}, is tangent to the boundary of Ω, and $|\psi_{jk}| \leq 1$ everywhere. Since the curves S_{jk} are disjoint, the functions ψ_{jk} can be constructed in such a way that their supports are still disjoint; moreover, if S_{jk} is closed, we can also assume that the support of ψ_{jk} is relatively compact in Ω. Finally, for every $j < k$ we define the functions $\lambda_{jk} : \mathbb{R}^N \to \mathbb{R}$ as

$$\lambda_{jk}(y) := \sigma\left(\frac{\langle y - a_j, a_k - a_j\rangle}{|a_k - a_j|^2}\right),$$

where $\sigma : \mathbb{R} \to [0,\alpha]$ is a nondecreasing function of class C^2 such that $\sigma(t) := \frac{1}{3}\alpha t^3$ for $t \in [0,1/8]$, $\sigma(t) := \alpha + \frac{1}{3}\alpha(t-1)^3$ for $t \in [7/8,1]$, $\sigma'(t) \in [0,2\alpha]$ for every t, and $|\sigma''(t)| \leq 16\alpha$ for every t.

Now we set

$$\mathcal{S}(x,y) := \sum_{(j,k):j<k} \lambda_{jk}(y)\,\psi_{jk}(x), \qquad \mathcal{S}_0(x) := -\,\alpha \sum_{(j,k):j<k} \mathrm{div}\psi_{jk}(x) 1_{E_k}(x),$$

and we claim that the pair $(\mathcal{S},\mathcal{S}_0)$ is a calibration for g when β is large enough.

First of all, independently of the choice of σ, the function $\mathcal{S}$ has vanishing normal component on $\partial\Omega$ because of the choice of ψ_{jk}, so that condition (5.13) of Theorem 5.7 is satisfied.

Using the fact that the supports of the functions ψ_{jk} are disjoint, and that $|\psi_{jk}| \leq 1$, while λ_{jk} takes values only on $[0,\alpha]$, it is easy to see that condition (b1) is fulfilled.

Since S_g is the union of the disjoint curves $\{S_{jk}\}_{j<k}$, for every $x \in S_g$ there exists one and only one pair (j,k) with $j < k$ such that $x \in S_{jk}$, so that

$$\mathcal{S}(x, g^+(x)) - \mathcal{S}(x, g^-(x)) = (\lambda_{jk}(a_k) - \lambda_{jk}(a_j))\, \psi_{jk}(x) = (\sigma(1) - \sigma(0))\, \nu_{jk}(x) = \alpha\nu_g(x).$$

Therefore, also condition (b2) is satisfied.

By direct computations we obtain that

$$[\mathrm{div}_x\mathcal{S}](x,y) = \sum_{(j,k):j<k} \lambda_{jk}(y)\, \mathrm{div}\psi_{jk}(x),$$

while

$$D_y\mathcal{S}(x,y) = \sum_{(j,k):j<k} \sigma'\left(\frac{\langle y - a_j, a_k - a_j\rangle}{|a_k - a_j|^2}\right) \psi_{jk}(x) \otimes \frac{a_k - a_j}{|a_k - a_j|^2}.$$

If $x \in E_h$ for any $h \in J$, then

$$\begin{aligned} [\mathrm{div}_x\mathcal{S}](x, g(x)) &= [\mathrm{div}_x\mathcal{S}](x, a_h) = \sum_{j<h} \lambda_{jh}(a_h)\, \mathrm{div}\psi_{jh}(x) + \sum_{k>h} \lambda_{hk}(a_h)\, \mathrm{div}\psi_{hk}(x) \\ &= \alpha \sum_{j<h} \mathrm{div}\psi_{jh}(x), \end{aligned}$$

where the last equality follows from the fact that $\lambda_{jh}(a_h) = \sigma(1) = \alpha$, while $\lambda_{hk}(a_h) = \sigma(0) = 0$. Arguing analogously, since $\sigma'(0) = \sigma'(1) = 0$, we have that $D_y\mathcal{S}(x, g(x)) = 0$, so that, taking into account the definition of $\mathcal{S}_0$, condition (a2) is satisfied.

It remains to prove condition (a1). Let $(x,y) \in \overline{\Omega}\times\mathbb{R}^N$. If x does not belong to any of the supports of the functions ψ_{jk}, then $[\mathrm{div}_x\mathcal{S}](x,y) = 0$, $\mathcal{S}_0(x) = 0$, and $D_y\mathcal{S}(x,y) = 0$, so (a1) is trivially satisfied. If x belongs to the support of ψ_{jk} for any $j < k$, then

$$[\mathrm{div}_x\mathcal{S}](x,y) = \lambda_{jk}(y)\, \mathrm{div}\psi_{jk}(x), \qquad \mathcal{S}_0(x) = -\alpha\, \mathrm{div}\psi_{jk}(x) 1_{E_k}(x),$$

$$D_y\mathcal{S}(x,y) = \sigma'\left(\frac{\langle y - a_j, a_k - a_j\rangle}{|a_k - a_j|^2}\right) \psi_{jk}(x) \otimes \frac{a_k - a_j}{|a_k - a_j|^2};$$

if we write the vector $y - a_j$ as the sum $v + t(a_k - a_j)$ where $v \in \mathbb{R}^N$ is orthogonal to $a_k - a_j$, and $t \in \mathbb{R}$, condition (a1) turns to be equivalent to

$$\mathrm{div}\psi_{jk}(x)(\sigma(t) - \alpha 1_{E_k}(x)) \leq -\frac{1}{4}|\psi_{jk}(x)|^2|\sigma'(t)|^2 + \beta|v + t(a_k - a_j) + a_j - g(x)|^2. \tag{5.50}$$

Since we are assuming that x is in the support of ψ_{jk}, x belongs either to E_j or to E_k. When $x \in E_j$, inequality (5.50) reduces to

$$\mathrm{div}\psi_{jk}(x)\, \sigma(t) \leq -\frac{1}{4}|\psi_{jk}(x)|^2|\sigma'(t)|^2 + \beta|v|^2 + \beta|a_k - a_j|^2 t^2,$$

which is implied by

$$\mathrm{div}\psi_{jk}(x)\, \sigma(t) \leq -\frac{1}{4}|\psi_{jk}(x)|^2|\sigma'(t)|^2 + \beta|a_k - a_j|^2 t^2. \tag{5.51}$$

So, let us prove (5.51) for every $t \in \mathbb{R}$ and $x \in E_j$. Since in (5.51) the equality holds for $t = 0$, it is enough to show the following inequality

$$\mathrm{div}\psi_{jk}(x)\, \sigma'(t) < -\frac{1}{4}|\psi_{jk}(x)|^2 2\sigma'(t)\, \sigma''(t) + 2\beta|a_k - a_j|^2 t \qquad \text{for } t > 0, \tag{5.52}$$

and the opposite inequality for $t<0$. Since $\sigma' \equiv 0$ for $t>1$, inequality (5.52) is trivially satisfied for $t>1$. For $0<t\leq 1$, (5.52) follows immediately from

$$-\|\mathrm{div}\psi_{jk}\|_\infty \sigma'(t) > \frac{1}{2}\sigma'(t)|\sigma''(t)| - 2\beta|a_k-a_j|^2 t,$$

which is satisfied (taking into account the structure of the function σ) for

$$\beta|a_k-a_j|^2 > 8\alpha\|\mathrm{div}\psi_{jk}\|_\infty + 64\alpha^2.$$

The same condition implies also the opposite inequality for $t<0$. Moreover, the same argument can be applied in the case $x\in E_k$.

In conclusion, condition (a1) is fulfilled for $\beta>\beta_0$, where β_0 is defined by

$$\beta_0 := \max_{(j,k):j<k} \frac{1}{|a_k-a_j|^2}\left(8\alpha\|\mathrm{div}\psi_{jk}\|_\infty + 64\alpha^2\right). \tag{5.53}$$

We conclude this example by noticing that this result generalizes Example 5.5 in [2], where g is the characteristic function of a regular set. As in the previous examples, the vectorial statement can be proved by applying the scalar result to one suitable component of g, but this leads to a worse estimate on β_0.

5.4 Calibrations in terms of closed differential forms

In this section we develop the theory of calibrations in terms of differential forms. The scalar method presented in Section 1.3 involves a divergence-free vectorfield on $\Omega\times\mathbb{R}$ (and its flux through the complete graph of the maps u), which is now replaced by a closed n-form on $\Omega\times\mathbb{R}^N$, acting on the graphs of the maps u, viewed as suitably defined n-surfaces in $\Omega\times\mathbb{R}^N$.

As we will see, this formulation is indeed not preferable to the one described in Section 5.1, since it leads to the same kind of conditions, requiring a greater technical effort.

For simplicity we restrict our discussion to piecewise smooth functions $u\in SBV(\Omega;\mathbb{R}^N)$ in the sense of the following definition.

Definition 5.19 *We say that a function $u\in SBV(\Omega;\mathbb{R}^N)$ is* piecewise smooth, *and we write $u\in\mathcal{A}(\Omega)$, if the following conditions are satisfied: up to an $\mathcal{H}^{n-1}$-negligible set, S_u is a finite union of pairwise disjoint $(n-1)$-dimensional boundaryless C^1-manifolds of $\mathbb{R}^n$; u is C^1 on $\Omega\setminus S_u$ up to S_u, that is $u\in C^1(\Omega\setminus S_u;\mathbb{R}^N)$ and there exist the limits of u and ∇u on both sides of (the regular part of) S_u.*

For $u\in\mathcal{A}(\Omega)$ we define the n-surfaces

$$\Sigma_u := \{(x,y)\in\Omega\times\mathbb{R}^N : x\in S_u \text{ and } \exists\, t\in[0,1] \text{ such that } y = tu^+(x)+(1-t)u^-(x)\},$$

$$\Gamma_u := \mathrm{graph}\, u\cup\Sigma_u.$$

Using notation from [19], let us consider an n-form

$$\omega : \Omega\times\mathbb{R}^N \to \wedge^n\mathbb{R}^{n+N},$$

$$\omega(x,y) = \sum_{|\alpha|+|\beta|=n} \omega_{\alpha\beta}(x,y)\, dx^\alpha\wedge dy^\beta,$$

whose coefficients $\omega_{\alpha\beta}$ are of class C^1, and for $u\in\mathcal{A}(\Omega)$ the following functional

$$\int_{\Gamma_u}\omega, \tag{5.54}$$

where the orientation of Γ_u will be defined later in a precise way.

If ω is a closed form, then the functional (5.54) is constant on the functions u which take the same value on $\partial\Omega$. Moreover, if F is the functional (5.1), and if

$$\begin{aligned} \int_{\Gamma_v} \omega &\leq F(v) \qquad \text{for every } v \in \mathcal{A}(\Omega), \\ \text{and} \qquad & \\ \int_{\Gamma_u} \omega &= F(u) \qquad \text{for a given } u \in \mathcal{A}(\Omega), \end{aligned} \tag{5.55}$$

then u is a Dirichlet minimizer of F in the class $\mathcal{A}(\Omega)$.

Let us now look for pointwise conditions on the coefficients of the form ω which guarantee (5.55).

By definition we have that

$$\int_{\Gamma_u} \omega = \int_{\text{graph } u} \omega + \int_{\Sigma_u} \omega. \tag{5.56}$$

On the graph of u we consider the natural orientation given by the parameterization $x \in \Omega \setminus S_u \mapsto (x, u(x))$, so that

$$\int_{\text{graph } u} \omega = \sum_{|\alpha|+|\beta|=n} \int_{\Omega} \omega_{\alpha\beta}(x, u(x))\mu_{\alpha\beta}(x)\, dx, \tag{5.57}$$

where

$$\mu_{\alpha\beta}(x) := \epsilon(\alpha) \det\left(\frac{\partial u_\beta}{\partial x_{\hat{\alpha}}}(x)\right).$$

In the previous formula $\hat{\alpha}$ denotes the increasing complement of α in $\{1, \ldots, n\}$, $\epsilon(\alpha)$ is the sign of permutation of $(1, \ldots, n)$ into $(\alpha, \hat{\alpha})$, and $\frac{\partial u_\beta}{\partial x_{\hat{\alpha}}}$ is the $|\beta|\times|\beta|$ matrix $\frac{\partial u_{\beta_i}}{\partial x_{\hat{\alpha}_j}}$.

On Σ_u we consider the orientation given by the following parameterization: since $u \in \mathcal{A}(\Omega)$, without loss of generality, we may assume that S_u is an $(n-1)$-dimensional C^1-manifold of $\mathbb{R}^n$ without boundary and that S_u can be covered by just one parameter patch $\gamma : S \to S_u$, where S is an $(n-1)$-dimensional domain (the general case can be easily obtained by summing over the C^1-pieces). Assume that γ yields ν_u as orientation, that is the vector

$$\eta(\gamma(\sigma)) := \sum_{i=1}^{n} (-1)^{n-i} \det\left(\frac{d\gamma_{\hat{\imath}}}{d\sigma}(\sigma)\right) e_i$$

(where $\{e_1, \ldots, e_n\}$ is the canonical basis of $\mathbb{R}^n$) satisfies

$$\frac{\eta(\gamma(\sigma))}{|\eta(\gamma(\sigma))|} = \nu_u(\gamma(\sigma)) \qquad \forall \sigma \in S.$$

We consider as parameterization of Σ_u the function $\phi = (\phi^x, \phi^y) : S\times[0,1] \to \Omega\times\mathbb{R}^N$ defined as $\phi^x(\sigma, t) := \gamma(\sigma)$, $\phi^y(\sigma, t) := tu^+(\gamma(\sigma)) + (1-t)u^-(\gamma(\sigma))$ for every $(\sigma, t) \in S\times[0,1]$, so that the second integral in (5.56) is given by

$$\int_{\Sigma_u} \omega = \sum_{|\alpha|+|\beta|=n} \int_0^1 \int_S \omega_{\alpha\beta}(\phi(\sigma,t)) \det\left(\frac{\partial\phi_{\alpha\beta}}{\partial(\sigma,t)}(\sigma,t)\right) d\sigma dt, \tag{5.58}$$

where $\phi_{\alpha\beta} = (\phi^x_{\alpha_1}, \ldots, \phi^x_{\alpha_p}, \phi^y_{\beta_1}, \ldots, \phi^y_{\beta_q})$ for $|\alpha| = p$ and $|\beta| = q = n - p$. By direct computations one can find that

$$\det\left(\frac{\partial\phi_{\hat{0}0}}{\partial(\sigma,t)}\right) = 0,$$

while for every $1 \le i \le n$, $1 \le j \le N$

$$\det\left(\frac{\partial \phi_{\hat{\imath}j}}{\partial(\sigma,t)}\right) = (u_j^+ - u_j^-)\det\left(\frac{d\gamma_{\hat{\imath}}}{d\sigma}\right) = (-1)^{n-i}(u_j^+ - u_j^-)(\nu_u)_i|\eta|,$$

where all the functions at the right-hand side are computed at $\gamma(\sigma)$. Finally, by straightforward computations, if we set $a := \hat{\alpha}$, for $|a| = |\beta| = q \ge 2$ it results that

$$\det\left(\frac{\partial \phi_{\alpha\beta}}{\partial(\sigma,t)}\right) = \sum_{m,k=1}^{q} \epsilon(\alpha, a_{\widehat{k}})(-1)^{n-q+m-a_k}(u_{\beta_m}^+ - u_{\beta_m}^-)\det\left(\frac{\partial(tu^+ + (1-t)u^-)_{\beta_{\widehat{m}}}}{\partial x_{a_{\widehat{k}}}}\right)(\nu_u)_{a_k}|\eta|,$$

where $\beta_{\widehat{m}}$, $a_{\widehat{k}}$ are the increasing complement of β_m in $\{\beta_1, \dots, \beta_q\}$ and of a_k in $\{a_1, \dots, a_q\}$, respectively, while $\epsilon(\alpha, a_{\widehat{k}})$ is the sign of permutation of $(\alpha, a_{\widehat{k}})$ in $\widehat{a_k}$; again all the functions at the right-hand side are computed at $\gamma(\sigma)$. Set $w^t := tu^+ + (1-t)u^-$ and substitute all the above expressions in formula (5.58); since $|\eta|\, d\sigma$ is the area element of the manifold S_u parameterized by γ, we obtain

$$\begin{aligned}
\int_{\Sigma_u} \omega \;&=\; \sum_{i,j} \int_0^1 \int_{S_u} (-1)^{n-i}\omega_{\hat{\imath}j}(x, w^t)(u_j^+ - u_j^-)(\nu_u)_i \, d\mathcal{H}^{n-1} dt \\
&+ \sum_{\substack{|a|=|\beta|=q \\ q\ge 2}} \int_0^1 \int_{S_u} \omega_{\hat{a}\beta}(x, w^t) \sum_{m,k=1}^{q} \epsilon(\alpha, a_{\widehat{k}})(-1)^{n-q+m-a_k}(u_{\beta_m}^+ - u_{\beta_m}^-)\det\left(\frac{\partial w^t_{\beta_{\widehat{m}}}}{\partial x_{a_{\widehat{k}}}}\right)(\nu_u)_{a_k}\, d\mathcal{H}^{n-1} dt \\
&=: \int_{S_u} g_\omega(x, u^-, u^+, \nabla u^-, \nabla u^+, \nu_u)\, d\mathcal{H}^{n-1},
\end{aligned} \tag{5.59}$$

where the last equality follows from changing the order of integration and calling g_ω the integrand with respect to $\mathcal{H}^{n-1}$. Now we wonder what kind of conditions on $\omega_{\alpha\beta}$ guarantee that

$$g_\omega(x, u^-, u^+, \nabla u^-, \nabla u^+, \nu_u) \le \psi(x, u^-, u^+, \nu_u) \qquad \text{on } S_u \tag{5.60}$$

for every admissible u. The answer is given by the following proposition.

Proposition 5.20 *Inequality (5.60) holds true for every $u \in \mathcal{A}(\Omega)$ if and only if the following conditions are satisfied:*

(b0') $\omega_{\alpha\beta} \equiv 0$ for every α, β such that $|\beta| \ge 2$, $|\alpha| + |\beta| = n$;

(b1') $\sum_{i,j} \int_0^1 (-1)^{n-i}\omega_{\hat{\imath}j}(x, tz + (1-t)y)(z_j - y_j)\nu_i \, dt \le \psi(x, y, z, \nu)$ for every $x \in \Omega$, for every $y, z \in \mathbb{R}^N$, and for every $\nu \in \mathbb{S}^{n-1}$.

Moreover, the equality holds for a given u if and only if

(b2') $\sum_{i,j} \int_0^1 (-1)^{n-i}\omega_{\hat{\imath}j}(x, tu^+ + (1-t)u^-)(u_j^+ - u_j^-)(\nu_u)_i \, dt = \psi(x, u^-, u^+, \nu_u)$ for every $x \in S_u$.

Proof. – Let $(x, y) \in \Omega \times \mathbb{R}^N$, and let us prove that $\omega_{\alpha\beta}(x, y) = 0$ for $|\hat{\alpha}| = |\beta| = 2$. By renumbering the coordinates of x and y, we may suppose that $\beta = (1, 2)$ and $a = \hat{\alpha} = (1, 2)$. Given $C \in \mathbb{R}$, we can construct $u \in \mathcal{A}(\Omega)$ such that $x \in S_u$, $\nabla u^-(x) = \nabla u^+(x)$ (hence $\nabla w^t(x) = \nabla u^-(x)$ for every $t \in [0,1]$), and $\partial_{x_i} w_j^t(x) = 0$ for every $(i,j) \neq (1,1)$ and $\partial_{x_1} w_1^t(x) = C$. With this choice we have that

$$g_\omega(x, u^-, u^+, \nabla u^-, \nabla u^+, \nu_u) = \sum_{i,j} \int_0^1 (-1)^{n-i} \omega_{\hat{\imath} j}(x, w^t)(u_j^+ - u_j^-)(\nu_u)_i \, dt + C \sum_{i\neq 1, j\neq 1} \int_0^1 (-1)^i \omega_{(\widehat{1,i})(1,j)}(x, w^t)(u_j^+ - u_j^-)(\nu_u)_i \, dt.$$

Since the value of C is arbitrary and independent of $u^-(x), u^+(x), \nu_u(x)$, inequality (5.60) implies that

$$\sum_{i\neq 1, j\neq 1} \int_0^1 (-1)^i \omega_{(\widehat{1,i})(1,j)}(x, w^t)(u_j^+ - u_j^-)(\nu_u)_i \, dt = 0 \tag{5.61}$$

whatever are the values of $u^-(x), u^+(x), \nu_u(x)$. Choosing $\nu_u(x)$ such that $(\nu_u(x))_i = 0$ for every $i \neq 2$, $(\nu_u(x))_2 = 1$, we have that (5.61) is equivalent to

$$\sum_{j\neq 1} \int_0^1 \omega_{(\widehat{1,2})(1,j)}(x, w^t)(u_j^+ - u_j^-) \, dt = 0 \tag{5.62}$$

whatever are the values of $u^-(x), u^+(x)$. Choosing $u^-(x) = y$, while $u_j^+(x) = y_j$ for every $j \neq 2$, $u_2^+(x) = y_2 + c$ with $c \neq 0$, we obtain that (5.62) is equivalent to

$$c \int_0^1 \omega_{(\widehat{1,2})(1,2)}(x, y_1, y_2 + ct, y_3, \ldots, y_N) \, dt = 0 \tag{5.63}$$

for every $c \neq 0$. By a change of variables, (5.63) can be rewritten as

$$\int_{y_2}^{y_2+c} \omega_{(\widehat{1,2})(1,2)}(x, y_1, s, y_3, \ldots, y_N) \, ds = 0. \tag{5.64}$$

Since (5.64) has to be true for every $c \neq 0$, this implies that $\omega_{(\widehat{1,2})(1,2)}(x,y) = 0$.

Using the fact that the coefficients $\omega_{\alpha\beta} \equiv 0$ for every $|\beta| = 2$, we can repeat the same proof to show that $\omega_{\alpha\beta} \equiv 0$ for every $|\hat{\alpha}| = |\beta| = 3$, and so on.

We have thus proved that (5.60) implies condition (b0'). At this point, it is trivial that (5.60) implies also condition (b1'), and that the equality holds in (5.60) for a given u if and only if also (b2') is satisfied. □

Summarizing, if conditions (b0') and (b1') hold true, by Proposition 5.20 inequality (5.60) is satisfied, hence by (5.59) we have that

$$\int_{\Sigma_u} \omega \leq \int_{S_u} \psi(x, u^-, u^+, \nu_u) \, d\mathcal{H}^{n-1} \tag{5.65}$$

for every $u \in \mathcal{A}(\Omega)$, while the equality holds in (5.65) for a given u if and only if also (b2') is verified.

Assuming that ω satisfies condition (b0'), formula (5.57) reduces to

$$\begin{aligned} \int_{\operatorname{graph} u} \omega &= \int_\Omega \left(\omega_{\hat{0}0}(x, u(x)) + \sum_{i,j} (-1)^{n-i} \omega_{\hat{\imath} j}(x, u(x)) \partial_{x_i} u_j(x) \right) dx \\ &= \int_\Omega \left(\omega_{\hat{0}0}(x, u(x)) + \langle A_\omega(x, u(x)), \nabla u(x) \rangle \right) dx, \end{aligned}$$

where in the last equality $(A_\omega(x,y))_{ji} := (-1)^{n-i}\omega_{\hat{\imath}j}(x,y)$. It is easy to see that, if we require the following condition:

(a1') $\omega_{\hat{0}0}(x,y) \leq -f^*(x,y,A_\omega(x,y))$ for $\mathcal{L}^n$-a.e. $x \in \Omega$ and every $y \in \mathbb{R}^N$,

then

$$\int_{\mathrm{graph}\, u} \omega \leq \int_\Omega f(x,u,\nabla u)\, dx$$

for every $u \in \mathcal{A}(\Omega)$; moreover, the equality holds for a given u if and only if

(a2') $(A_\omega)_{ij}(x,u(x)) \in \partial_{\xi_{ij}} f(x,u(x),\nabla u(x))$ and $\omega_{\hat{0}0}(x,u(x)) = -f^*(x,u(x),A_\omega(x,u(x)))$ for $\mathcal{L}^n$-a.e. $x \in \Omega$.

Therefore by (5.56) we can conclude that (5.55) is guaranteed if conditions (a1'), (a2'), (b0'), (b1'), and (b2') are satisfied. In other words, we have proved the following theorem.

Theorem 5.21 *Let u be a function in $\mathcal{A}(\Omega)$. Assume that there exists a closed n-differential form $\omega : \Omega\times\mathbb{R}^N \to \wedge^n\mathbb{R}^{n+N}$ with coefficient of class C^1 and satisfying condition (a1'), (a2'), (b0'), (b1'), and (b2'). Then u is a Dirichlet minimizer of the functional (5.1) in the class $\mathcal{A}(\Omega)$.*

We conclude this section by proving that, if $u \in \mathcal{A}(\Omega)$ and there exists a differential form ω which calibrates u in the sense of Theorem 5.21, then there exists a calibration $(\mathcal{S},\mathcal{S}_0)$ for u in the sense of Definition 5.6.

Proposition 5.22 *Let u be a function in $\mathcal{A}(\Omega)$ and let $\omega : \Omega\times\mathbb{R}^N \to \wedge^n\mathbb{R}^{n+N}$ be a closed n-differential form satisfying all the assumptions of Theorem 5.21. Then there exists a calibration $(\mathcal{S},\mathcal{S}_0)$ for u, with $\mathcal{S} \in C^2(\Omega\times\mathbb{R}^N;\mathbb{R}^n)$ and $\mathcal{S}_0 \in C^1(\Omega)$.*

Proof. – First of all, we notice that from condition (b0') it follows that

$$\omega(x,y) = \omega_{\hat{0}0}(x,y)\, dx + \sum_{i,j} \omega_{\hat{\imath}j}(x,y)\, dx^{\hat{\imath}} \wedge dy^j.$$

Since ω is a closed form, by computing explicitly the exterior derivative of ω, we obtain that the coefficients $\omega_{\hat{0}0}, \omega_{\hat{\imath}j}$ satisfy the two following equations:

$$\sum_{i=1}^n (-1)^{n-i}\frac{\partial\omega_{\hat{\imath}j}}{\partial x_i}(x,y) - \frac{\partial\omega_{\hat{0}0}}{\partial y_j}(x,y) = 0 \qquad 1 \leq j \leq N, \tag{5.66}$$

$$(-1)^{n-i}\frac{\partial\omega_{\hat{\imath}j}}{\partial y_k}(x,y) = (-1)^{n-i}\frac{\partial\omega_{\hat{\imath}k}}{\partial y_j}(x,y) \qquad 1 \leq i \leq n,\ 1 \leq j,k \leq N. \tag{5.67}$$

The last condition is equivalent to require that for every i the vector $((-1)^{n-i}\omega_{\hat{\imath}j}(x,y))_{j=1,\dots,N}$ is the gradient with respect to y of a function of class C^2; more precisely, there exists a function $\mathcal{S} \in C^2(\Omega\times\mathbb{R}^N;\mathbb{R}^n)$ such that

$$\partial_{y_j}\mathcal{S}_i(x,y) = (-1)^{n-i}\omega_{\hat{\imath}j}(x,y) \qquad 1 \leq i \leq n,\ 1 \leq j \leq N. \tag{5.68}$$

Equation (5.66) can be therefore rewritten as

$$0 = \sum_{i=1}^n \frac{\partial^2\mathcal{S}_i}{\partial x_i \partial y_j}(x,y) - \frac{\partial\omega_{\hat{0}0}}{\partial y_j}(x,y) = \partial_{y_j}\left[\sum_{i=1}^n \partial_{x_i}\mathcal{S}_i(x,y) - \omega_{\hat{0}0}(x,y)\right] \qquad 1 \leq j \leq N,$$

and then there exists a function $\mathcal{S}_0 : \Omega \to \mathbb{R}$ of class C^1 such that $\omega_{\hat{0}0}(x,y) = [\mathrm{div}_x \mathcal{S}](x,y) + \mathcal{S}_0(x)$. By substituting this equality and (5.68) in conditions (a1') and (a2'), we directly obtain that the pair $(\mathcal{S}, \mathcal{S}_0)$ satisfies conditions (a1) and (a2) of Lemma 5.4. Since the left-hand side in (b1') can be rewritten as

$$
\begin{aligned}
&\sum_{i,j} \int_0^1 (-1)^{n+i} \omega_{\hat{i}j}(x, tz + (1-t)y)(z_j - y_j)\nu_i \, dt \\
&\quad = \sum_{i,j} \int_0^1 \partial_{y_j} \mathcal{S}_i(x, tu^+ + (1-t)u^-)(u_j^+ - u_j^-)(\nu_u)_i \, dt \\
&\quad = \sum_{i=1}^n \int_0^1 \frac{d}{dt}[\mathcal{S}_i(x, tu^+ + (1-t)u^-)](\nu_u)_i \, dt \\
&\quad = \sum_{i=1}^n [\mathcal{S}_i(x,u^+) - \mathcal{S}_i(x,u^-)](\nu_u)_i \\
&\quad = \langle \mathcal{S}(x,u^+) - \mathcal{S}(x,u^-), \nu_u \rangle,
\end{aligned}
$$

condition (b1') implies that the function $\mathcal{S}$ satisfies condition (b1) of Lemma 5.4, and in the same way (b2') implies (b2). □

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