Congested traffic dynamics and very degenerate elliptic equations under supercritical Sobolev regularity

Albert Clop, Raffaella Giova, Farhad Hatami, Antonia Passarelli di Napoli

March 22, 2018

Abstract

We study the congested transport dynamics arising from a non-autonomous traffic optimization problem. In this setting, we prove one can find an optimal traffic strategy with support on the trajectories of a DiPerna-Lions flow. The proof follows the scheme introduced by Brasco, Carlier and Santambrogio in the autonomous setting, applied to the case of supercritical Sobolev dependence in the spatial variable. This requires both Lipschitz and weighted Sobolev apriori bounds for the minimizers of a class of integral functionals whose ellipticity bounds are satisfied only away from a ball of the gradient variable.

1 Introduction

Wardrop equilibriums are flow configurations on a network that satisfy a natural mass preservation property, as well as the condition that every actually used path between two points has the same cost. This concept was introduced in [**?**] in finite-dimensional networks, and is well known among geographical economists. While trying to extend the notion of Wardrop equilibrium to the continuous setting, Carlier, Jiménez and Santambrogio proposed in [**?**] a continuous model for congested traffic equilibrium. Since then, the interest in the topic has grown, and connections have been found by researchers in partial differential equations, calculus of variations and traffic engineering.

In the present paper, we recover an explicit way to construct these equilibriums. This construction was first described by Brasco, Carlier and Santambrogio [**?**] in an autonomous setting, as a natural continuation of the model presented in [**?**]. In fact, in that model the traffic cost depends only on the traffic intensity. However, it may happen that the traffic intensity does not determine the traffic cost. For instance, two different locations with the same traffic intensity may have very different road conditions. This partially explains the need for non-autonomous models for congested traffic. Already in [**?**], the authors look at two possible formulations of the traffic problem, as well as a precise proof of their equivalence and dual formulation. Here we study such non autonomous counterparts from the regularity point of view, thus allowing for non-constant same-traffic cost, for which we want to find a reasonably regular solution.

2010 *Mathematics Subject Classification.* 35D10; 35J70; 49K10

Key words and phrases. Optimal transport with congestion, Congested traffic dynamics, Wardrop equilibrium, degenerate PDEs, regularity

Towards finding such traffic optimals, we need to focus on two issues. The first of this issues concerns the regularity of minimizers of certain highly degenerate integral functionals. More precisely, we focus on optimals for minimizing problems of the form

$$\inf_u \int \mathcal{F}(x, Du)\, dx + f(x)\, u\, dx$$

where $f \in L^s_{loc}(\Omega)$ for some $s > n$, and $\mathcal{F}$ is a somewhat special Carathéodory function. In the gradient variable, $\mathcal{F}$ is radial, has growth $p \geq 2$, and is highly degenerate: ellipticity bounds are available only away from a ball on the gradient variable ($\mathcal{F}$ can even vanish on that ball). In the spatial variable, $\mathcal{F}$ enjoys Sobolev dependence $W^{1,s}_{loc}$. Minimizers are shown to be locally Lipschitz, that is, $Du \in L^\infty_{loc}$ (see Theorem **??**). Moreover, in Theorem **??** we show that for smooth boundaries the local Lipschitz estimates for minimizers are indeed global. The supercritical Sobolev smoothnes of $\mathcal{F}$ in the spatial variable is also used to show that minimizers admit distributional derivatives up to second order, at least away from the degeneracy set. This is stated in terms of a weighted apriori bound, which guarantees that

$$(|Du| - 1)_+\, |D^2 u| \in L^2_{loc}. \tag{1.1}$$

{apriori2n

where $(\cdot)_+$ denotes positive part. The weight $(|Du| - 1)_+$ is the distance between Du and the degeneracy set $\{|Du| \leq 1\}$. See Theorem **??** for the precise statement. Theorems **??** and **??** extend, respectively, previous contributions to the L^∞ bounds (for instance [**?**, Theorem 3.1] for the scalar case, and [**?**] in the vectorial setting) and the Sobolev bounds (for instance extending [**?**, Theorem 4.1]), in which Lipschitz regularity in the space variable was assumed for the energy. See also [**?**] for the critical Sobolev regularity in the spatial variable.

The regularity results described above have their own interest. For instance, these results extend the well known theory for the scalar p-laplace equation,

$$-\operatorname{div} |\nabla u|^{p-2}\, \nabla u = f$$

to the much more degenerate equation

$$-\operatorname{div}\left((|\nabla u| - 1)_+^{p-1} \frac{\nabla u}{|\nabla u|} \right) = f.$$

As a matter of fact, the p-laplace equation degenerates only on the singleton $\{|\nabla u| = 0\}$, while in the second case, as it usually happens in traffic congestion problems, the degeneracy set $\{|\nabla u| \leq 1\}$ is much bigger. These highly degenerate equations have already been object of study by many authors, see for instance [**?**, **?**, **?**, **?**, **?**].

The second issue we pay attention to is the application of the above regularity results to the analysis of traffic models. In this direction, we show that the construction of Wardrop equilibriums from [**?**] can be extended to non-autonomous, Sobolev-dependent counterparts with growth $1 < q \leq 3/2$. As in the autonomous case, one first reduces the congested traffic problem to a regularity question on very degenerate elliptic equations. Such reduction is possible because of the non-autonomous duality result [**?**, Theorem 3.1]. After this reduction, as in [**?**], the construction of equilibriums involves the implementation of the so-called Dacorogna-Moser scheme, for which, in turn, DiPerna-Lions theory of flows for weakly differentiable vector fields needs to be used. This is, in fact, the reason why the above mentioned apriori estimates are needed. Following [**?**], one then obtain a DiPerna-Lions flow supporting an explicit traffic configuration that minimizes the cost. Moreover, this configuration is indeed a Wardrop equilibrium, whose support consists of the set of rectifiable trajectories of a reasonably nice velocity field in the sense of DiPerna and Lions. See Theorem **??** in Susbsection **??** for the precise statement of our result. The reason for restricting to traffic problems with growth $1 < q \leq 3/2$ is that our Sobolev bounds (**??**) only allow to use Dacorogna-Moser scheme within this range. Further research is due to fill the missing range $3/2 < q \leq 2$ at the traffic side.

Especially for applications, it is desirable to understand what is the situation when the x dependence is not continuous. We know, thought, that if $s = n = 2$ then $\nabla u \in L^\infty$ may fail even in the linear uniformly elliptic setting [**?**, **?**], while in the case $n > 2$ some regularity results under a Sobolev assumption on the coefficients can be found in [**?**, **?**, **?**, **?**]. Thus, other arguments will be needed. Also, non-autonomous extensions to the more recent orthotropic formulation of the traffic problem (see for instance [**?**, **?**, **?**]) will be of interest.

The paper is structured as follows. In Section **??** we recall the fundamentals of congested traffic theory, and prove our main results in this direction (we are using here the global estimates of Section **??**). In Sections **??** and **??** we prove, respectively, the interior Lipschitz estimates and the interior second order Sobolev estimates for minimizers of certain very degenerate integral functionals. In Section **??** we show that the results in Sections **??** and **??** admit a global version, up to the boundary, under enough boundary regularity. This is needed in the proof of the results at Section **??**.

We will follow the usual convention and denote by c or C a general constant that may vary on different occasions, even within the same line of estimates. Relevant dependencies on parameters and special constants will be suitably emphasized using parentheses or subscripts. All the norms we use on will be the standard Euclidean ones and denoted by $|\cdot|$ in all cases. In particular, for matrices ξ, $\eta \in \mathbb{R}^{n\times m}$ we write $\langle \xi, \eta \rangle := \mathrm{trace}(\xi^T \eta)$ for the usual inner product of ξ and η, and $|\xi| := \langle \xi, \xi \rangle^{\frac{1}{2}}$ for the corresponding euclidean norm. By $B_r(x)$ we will denote the ball in $\mathbb{R}^n$ centered at x of radius r. The integral mean of a function u over a ball $B_r(x)$ will be denoted by $u_{x,r}$, that is

$$u_{x,r} := \frac{1}{|B_r(x)|} \int_{B_r(x)} u(y)\, dy,$$

where $|B_r(x)|$ is the Lebesgue measure of the ball in $\mathbb{R}^n$. If no confusion may arise, we shall omit the dependence on the center.

2 Congested traffic minimization problems

{straffic}

In this section we summarize the models and results in [**?**, **?**] for the reader's convenience.

2.1 The scalar problem

We are given two probability measures μ_0, μ_1 on a bounded domain $\Omega \subset \mathbb{R}^n$, representing both sources and destinations of travellers on a urban network. A *traffic strategy* is a measure Q on the metric space $C([0,1]; \overline{\Omega})$ supported on the subset of rectifiable curves $W^{1,\infty}([0,1]; \overline{\Omega})$. To any traffic strategy Q, one associates the corresponding *traffic intensity* $i = i_Q$, which is a scalar measure defined as follows,

$$\langle i_Q, \varphi \rangle = \int_{C([0,1];\overline{\Omega})} \left(\int_0^1 \varphi(\alpha(t)) \, |\alpha'(t)| \, dt \right) dQ(\alpha),$$

for each $\varphi \in C(\overline{\Omega}; \mathbb{R})$. This measure summarizes all the traffic flow through the region given by the testing function φ.

The total traffic cost of a urban configuration is an integral functional of the traffic intensity. This means that one is given a function

$$H : \overline{\Omega} \times [0, \infty) \to [0, \infty)$$
$$(x, i) \to H(x, i)$$

such that:

$(H0)$ $H(x, 0) = 0$.

$(H1)$ For every fixed $x \in \overline{\Omega}$, $i \mapsto H(x, i)$ is non-decreasing and convex.

$(H2)$ If $(x, i) \in \overline{\Omega} \times [0, \infty)$ then

$$a\, i^q \leqslant H(x, i) \leqslant b\, (i^q + 1) \tag{2.1}$$ {growth}

$(H3)$ If $(x, i) \in \overline{\Omega} \times [0, \infty)$ and $g(x, i) = \partial_i H(x, i)$ then

$$0 \leqslant g(x, i) \leqslant c(i^{q-1} + 1). \tag{2.2}$$ {derivative

Then, an *equilibrium* is any traffic strategy Q that solves the following variational problem,

$$\inf_{Q \in \mathcal{Q}^q(\mu_0, \mu_1)} \int_\Omega H(x, i_Q(x))\, dx. \tag{2.3}$$ {eq72}

Above, $\mathcal{Q}^q(\mu_0, \mu_1)$ denotes the set of all possible measures Q such that $i_Q \in L^q(\Omega)$ and

$$(\pi_0)_\# Q = \mu_0,$$
$$(\pi_1)_\# Q = \mu_1,$$

where $\pi_t(\alpha) = \alpha(t)$ denotes the time evaluation of a path $\alpha \in C([0, 1]; \overline{\Omega})$ at time $t \in [0, 1]$. As usually in transport theory, $f_\# \mu$ denotes the image measure of μ through the map f.

Theorem 1. *The minimization problem* (**??**) *admits a solution if* $\mathcal{Q}^q(\mu_0, \mu_1) \neq \emptyset$.

The above result was proven in [**?**, Theorem 2.10] when $n = 2$ and $1 < q < \infty$, in the autonomous setting, that is, with the extra assumption that $H(x, i) = H(i)$ (see [**?**, Theorem 2] for a non-autonomous counterpart). The proof comes together with a precise description of the relation between the minimizers and Wardrop's equilibrium, see Subsection **??** for details. Moreover, if $H(x, \cdot)$ is strictly convex and Q_1, Q_2 are equilibriums one can see that $i_{Q_1} = i_{Q_2}$, althoug it is false in general that $Q_1 = Q_2$, see [**?**]. Concerning the non-emptiness of $\mathcal{Q}^q(\mu_0, \mu_1)$, we refer to [**?**, Theorem 4.4] as well as to the following result (see also [**?**, Remarks 2.5 and 2.6]).

{Qnonempty

Theorem 2. *If* $\mu_0, \mu_1 \in L^q(\Omega)$, *then* $\mathcal{Q}^q(\mu_0, \mu_1)$ *is non-empty.*

The above Theorem is a consequence of works by De Pascale, Evans and Pratelli. Roughly speaking, one constructs for the given datum μ_0 and μ_1 the trivial traffic strategy

$$Q = \gamma \otimes \delta_{[x,y]}$$

where $d\gamma(x,y)$ is an optimal transport plan between μ_0 and μ_1 for the Monge problem, and $\delta_{[x,y]}$ is the Dirac delta on the line segment $[x,y]$, understood as a measure on $C([0,1];\Omega)$. For this particular Q, one easily sees that i_Q is precisely the transport density associated to γ (see [?] for the definition of transport density). But De Pascale, Evans and Pratelli proved that the transport density inherits the Lebesgue integrability of μ_0, μ_1, and in particular it belongs to L^q (see [?] for $1 < q < 2$, and [?] for $q \geq 2$), as claimed.

It is because of Theorem **??** that, from now on, we will assume that μ_0, μ_1 are absolutely continuous with respect to the Lebesgue measure. Abusing of notation, we will also denote by μ_0, μ_1 their Radon-Nikodym derivatives, so that $d\mu_j(x) = \mu_j(x)\, dx$.

2.2 The vector problem

There is though a different way to prove existence of minimizers at (**??**), with the advantage of being more explicit. The basic idea (see [?] for details) is to compare (**??**) with the following variational problem,

$$\inf_{\sigma \in \Sigma^q(\mu_0,\mu_1)} \int_\Omega \mathcal{H}(x, \sigma(x))\, dx \tag{2.4}$$ {eq73}

where $\mathcal{H}(x,\xi) = H(x,|\xi|)$ for each $\xi \in \mathbb{R}^n$, and H satisfies $(H0)$, $(H1)$, $(H2)$ and $(H3)$. Also, $\Sigma^q(\mu_0,\mu_1)$ denotes the set of all $L^q(\Omega)$ weak solutions of the following Neumann problem

$$\begin{cases} \operatorname{div} \sigma = \mu_0 - \mu_1 & \Omega, \\ \sigma \cdot \vec{n} = 0 & \partial\Omega. \end{cases} \tag{2.5}$$ {beckmann}

In this context, the following result is fundamental.

{filippoeq

Theorem 3. *If $\mu_0, \mu_1 \in L^q$, and $\frac{1}{\mu_0}, \frac{1}{\mu_1} \in L^\infty$, then the infimums at* (**??**) *and* (**??**) *are equal.*

It was proven in [?, Theorem 3.2] in the autonomous setting (see also [?, Section 5] for a non-autonomous counterpart). One of the implications is easy. Indeed, to every $Q \in \mathcal{Q}^p(\mu_0,\mu_1)$ the vector measure σ_Q defined by

$$\langle \sigma_Q, F\rangle = \int_{C([0,1];\overline{\Omega})} \left(\int_0^1 \langle F(\alpha(t)), \alpha'(t)\rangle\, dt \right) dQ(\alpha), \qquad \forall F \in C(\overline{\Omega}, \mathbb{R}^n), \tag{2.6}$$ {eq75}

does the job. The converse direction, much more delicate, requires the superposition principle (see Theorem **??** below).

2.3 Wardrop equilibriums

{wardropeq

As mentioned in [?] (see also [?]), the congestion effects in a given traffic strategy $Q \in \mathcal{Q}^q(\mu_0,\mu_q)$ are captured by the traffic intensity i_Q through the function $g(x,i) = \partial_i H(x,i)$. Namely, one thinks the function $g(x, i_Q(x))$ as the density of a Riemannian metric

$$d_Q(x,y) = \inf_{\alpha \in C^{x,y}} \int_\alpha g(z, i_Q(z))\, |dz| = \inf_{\alpha \in C^{x,y}} \int_0^1 g(\alpha(t), i_Q(\alpha(t)))\, |\alpha'(t)|\, dt$$

where $C^{x,y}$ denotes the subset of $C([0,1];\overline{\Omega})$ of rectifiable curves such that $\alpha(0) = x$ and $\alpha(1) = y$. In this context, a curve $\alpha_0 \in C^{x,y}$ is called a Q-geodesic if

$$d_Q(x,y) = \int_{\alpha_0} g(z, i_Q(z))\,|dz| \tag{2.7}$$ {geod}

With this terminology, a *Wardrop equilibrium* is a traffic strategy $Q \in \mathcal{Q}(\mu_0, \mu_1)$ that satisfies the following two properties:

(a) Q is supported on Q-geodesics, that is,

$$Q\left(\left\{\alpha \in C([0,1];\overline{\Omega}) : d_Q(\alpha(0), \alpha(1)) = \int_\alpha g(z, i_Q(z))\,|dz|\right\}\right) = 1$$

(b) The transport plan $\gamma_Q = (\pi_0, \pi_1)_\# Q$ solves the following Monge problem

$$\inf_{\gamma \in \Pi(\mu_0,\mu_1)} \int_{\Omega\times\Omega} d_Q(x,y)\,d\gamma(x,y)$$

The following result was given in [**?**, Theorem 4.2] in the autonomous case (see [**?**, Theorem 2] for the non-autonomous counterpart).

{wardropth}

Theorem 4. *A traffic strategy $Q \in \mathcal{Q}(\mu_0, \mu_1)$ is a minimizer of* (**??**) *if and only if Q is a Wardrop equilibrium.*

This result was proven in [**?**] in the autonomous case. A sketch for a non-autonomous counterpart was given in [**?**]. In boths cases, the proof is quite involved, as regularity issues need to be handled in detail. Indeed, already at the very beginning, the definition d_Q may have problems, since $i_Q \in L^q$ only implies $g(x, i_Q(x)) \in L^{\frac{q}{q-1}}$ and this may not be enough to define line integrals.

2.4 Duality and Regularity Theory

The transition from congested traffic problems to the classical regularity theory for elliptic PDE relies on the following duality result. It was proven by Brasco and Petrache in [**?**, Theorem 3.1], but we remind it here for the reader's convenience. By $W^{1,r}_*(\Omega)$ we denote the class of functions $\varphi \in W^{1,r}(\Omega)$ such that $\int_\Omega \varphi = 0$.

{petrache}

Lemma 5. *Let $T \in W^{-1,q}(\Omega)$, $1 < q < \infty$, and $p = \frac{q}{q-1}$. Let $\mathcal{H} : \Omega \times \mathbb{R}^n \to [0,\infty)$ be a Carathéodory function, such that*

$$a\,|\xi|^q \le \mathcal{H}(x,\xi) \le b(|\xi|^q + 1), \qquad (x,\xi) \in \Omega \times \mathbb{R}^n.$$

Assume also that $\xi \mapsto \mathcal{H}(x,\xi)$ is convex, for each fixed x. Then

$$\inf_{\sigma \in \Sigma^q(T)} \int_\Omega \mathcal{H}(x, \sigma(x))\,dx = \sup_{\varphi \in W^{1,p}_*(\Omega)} \left(\langle T, \varphi\rangle - \int_\Omega \mathcal{H}^*(x, \nabla\varphi(x))\,dx\right) \tag{2.8}$$ {dualeq}

where $\mathcal{H}^*(x,\cdot)$ is the Legendre transform of $\mathcal{H}(x,\cdot)$. Moreover, one has

$$\sigma(x) \in \partial\mathcal{H}^*(x, \nabla\varphi(x)) \qquad \text{for a.e.} x \in \Omega,$$

where $\partial\mathcal{H}^*(x,\xi)$ denotes the subdifferential $\partial\mathcal{H}^*(x,\xi) = \{z \in \mathbb{R}^n; \mathcal{H}^*(x,\xi) + \mathcal{H}(x,z) = \xi \cdot z\}$.

For the traffic applications of the present paper, one is left to see if the distribution $T = \mu_0 - \mu_1$ belongs to $W^{-1,q}$. In this direction, if both μ_0 and μ_1 are L^q functions, one can choose for any $\varphi \in W^{1,q'}_*$ an appropriate constant φ_Ω so that the Poincaré inequality applies, and one gets

$$\begin{aligned} \left| \int \varphi \, d(\mu_0 - \mu_1) \right| &= \left| \int (\varphi - \varphi_\Omega) \, d(\mu_0 - \mu_1) \right| \\ &\leq \|\varphi - \varphi_\Omega\|_{L^{q'}} \|\mu_0 - \mu_1\|_{L^q} \\ &\leq C(\Omega, q) \|D\varphi\|_{L^{q'}} \|\mu_0 - \mu_1\|_{L^q}, \end{aligned}$$

and therefore $T \in W^{-1,q}$. Another interesting example is that of the point masses, that is, $\mu_j = \delta_{x_j}$ for $j = 0, 1$. In this case, $\mu_0 - \mu_1 \in W^{-1,q}$ if and only if $1 \leq q < \frac{n}{n-1}$, see [**?**, Example 2.4].

As explained in [**?**] (see also [**?**, Theorem 2.1]), when $\xi \mapsto \mathcal{H}(x, \xi)$ is strictly convex the Legendre transform $\xi \mapsto \mathcal{H}^*(x, \xi)$ becomes C^1-smooth, and therefore the subdifferential $\partial \mathcal{H}^*(x, \xi)$ reduces to a singleton

$$\partial \mathcal{H}^*(x, \xi) = \{\nabla_\xi \mathcal{H}^*(x, \xi)\},$$

whence the optimals σ and φ must necessarily be related by

$$\sigma(x) = \nabla_\xi \mathcal{H}^*(x, \nabla \varphi(x)).$$

In particular, in that strictly convex setting the optimizer $\sigma \in \Sigma^p(T)$ is unique. Moreover, φ can be found by solving the following Neumann problem

$$\begin{cases} -\operatorname{div} \nabla_\xi \mathcal{H}^*(x, \nabla \varphi) = T & \Omega \\ \nabla_\xi \mathcal{H}^*(x, \nabla \varphi) \cdot \vec{n} = 0 & \partial\Omega \end{cases}$$

The following examples explain a bit better the situation:

- If $H(x, i) = \frac{i^q}{q}$, then $\mathcal{H}^*(x, \xi) = \frac{|\xi|^p}{p}$ with $\frac{1}{p} + \frac{1}{q} = 1$. In particular, $\mathcal{H}^*$ is strictly convex, and $\nabla \mathcal{H}^*(x, \xi) = |\xi|^{p-2}\xi$ degenerates at just one point. The relation between optimals $\sigma = |\varphi|^{p-2}\varphi$ also says that φ is a solution of an inhomogeneous Neumann problem for the p-Laplace operator,

$$\begin{cases} \Delta_p \varphi = \mu_0 - \mu_1 \\ \nabla \varphi \cdot \vec{n} = 0 \end{cases}$$

  On the other hand, since $g(x, 0) = \partial_i H(x, 0) = 0$, according to (**??**) and Theorem **??**, this example gives zero cost to zero traffic. This is not completely realistic, as a positive cost must be expected even in absence of traffic.

- If $H(x, i) = \frac{i^q}{q} + \lambda i$ one gets $g(x, 0) = \partial_i H(x, 0) = \lambda$, and so for $\lambda > 0$ this model is more realistic. At the same time, though, $\mathcal{H}^*(x, \xi) = \frac{(|\xi| - \lambda)_+^p}{p}$ is not strictly convex, and therefore

$$\nabla \mathcal{H}^*(x, \xi) = (|\xi| - \lambda)^{p-1} \frac{\xi}{|\xi|}$$

  degenerates not only when $|\xi| = 0$ but on the whole ball $\{|\xi| \leq \lambda\}$ of the gradient variable. Still, $\mathcal{H}^*$ is C^1 in the second variable, and so one gets a Neumann boundary value problem for the following Euler-Lagrange equation

$$-\operatorname{div} \left((|\nabla \varphi| - \lambda)^{p-1} \frac{\nabla \varphi}{|\nabla \varphi|} \right) = \mu_0 - \mu_1$$

  although it signifficantly differs from the p-Laplace equation.

2.5 DiPerna-Lions Theory

{dipernali

Given a vector field $b : [0,1] \times \mathbb{R}^n \to \mathbb{R}^n$, and a measure $\mu_0 \in \mathcal{M}(\mathbb{R}^n)$, a measure valued map

$$\mu : [0,1] \to \mathcal{M}(\mathbb{R}^n)$$
$$t \mapsto \mu_t$$

is said to be a *weak solution* of the Cauchy problem for the continuity equation

$$\begin{cases} \frac{d}{dt}\, u + \operatorname{div}(b\, u) = 0 \\ u(0,\cdot) = \mu_0 \end{cases} \tag{2.9}$$ {conteq}

if for any testing function $\psi \in C_c^\infty([0,1) \times \mathbb{R}^n; \mathbb{R})$ one has that

$$\int \psi(0,x)\, d\mu_0(x) + \iint \frac{\partial \psi}{\partial t}\, d\mu_t\, dt + \iint b \cdot \nabla \psi\, d\mu_t\, dt = 0$$

Equivalently, for each $\varphi \in C_b^\infty(\mathbb{R}^n)$ the real-variable function $t \mapsto \int \varphi\, d\mu_t$ is absolutely continuous with derivative

$$\frac{d}{dt} \int \varphi\, d\mu_t = \int b \cdot \nabla \varphi\, d\mu_t \qquad \text{for a.e. } t \in [0,1].$$

Under enough smoothness, the problem (**??**) has as its unique solution $u(t,\cdot)$ the image measure of the initial datum μ_0 through the flow $X(t,\cdot)$ at time t,

$$u(t,\cdot) = X(t,\cdot)_{\#}(\mu_0).$$

Above, $X(t,\cdot)$ is obtained by solving the following ODE

$$\begin{cases} \dot{X}(t,x) = b(t, X(t,x)), \\ X(0,x) = x. \end{cases} \tag{2.10}$$ {trajODE}

It is well known, though, that such flow X need not be well defined in general. In their celebrated paper, DiPerna and Lions [**?**] developed a systematic way to understand transport and continuity equations for non-smooth vector fields b. One of the main points in their theory was to establish a relation between the notion of well-defined flow and the solvability of initial value problems for scalar conservation laws and the notion of renormalized solution. As in [**?**], we recall that when $\mu_t \in L^1_{loc}$ then μ_t is said to be a renormalized solution of (**??**) if, for every $\beta \in C^1(\mathbb{R})$, the equation

$$\frac{d}{dt}\beta(\mu_t) + b \cdot \nabla(\beta(\mu_t)) + \operatorname{div}(b)\, \mu_t\, \beta'(\mu_t) = 0$$

is satisfied in $(0,1) \times \Omega$ in the sense of distributions. Clearly, renormalized solutions are weak solutions (simply take $\beta(u) = u$) while the converse is false in general. Since the work of DiPerna and Lions, it is known that if b is a Sobolev vector field then weak solutions to the continuity equation are renormalized. The following result summarizes the main ideas of DiPerna - Lions theory.

{Dipernali

Theorem 6. *Let $b \in L^1([0,1]; W^{1,1}_{loc}(\Omega))$ be such that $\operatorname{div}(b) \in L^1([0,1]; L^\infty(\Omega))$, and $b \cdot \vec{n} = 0$ on $\partial\Omega$ in the weak sense. Then there is a unique continuous map*

$$X : [0,1]^2 \to L^1(\Omega; \mathbb{R}^n)$$
$$(t,s) \mapsto X(t,s,\cdot)$$

that leaves $\overline{\Omega}$ invariant, and such that the following holds:

(a) $X(t,s,\cdot)_{\#}dx$ *is absolutely continuous with respect to the Lebesgue measure* dx*, and*

$$e^{-|A(t)-A(s)|} \leq \frac{d}{dx}(X(t,s,\cdot)_{\#}dx) \leq e^{|A(t)-A(s)|}$$

where $A(t) = \int_0^t \|\operatorname{div}(b)(s,\cdot)\|_{L^\infty(\Omega)}\, ds$*.*

(b) If $0 \leq r < s < t \leq 1$ *then* $X(t,r,x) = X(t,s,X(s,r,x))$ *for almost every* $x \in \Omega$*.*

(c) For almost every $x \in \Omega$*,* $X(t,s,\cdot)$ *is an absolutely continuous solution of* (**??**)*, that is,*

$$X(t,s,x) = x + \int_s^t b(r, X(r,s,x))\, dr.$$

(d) If $\mu_0 \in L^p(\Omega)$ *and* $s \in [0,1)$*, then* $u(t,\cdot) = X(t,s,\cdot)_{\#}(\mu_0)$ *is the unique renormalized solution* $u \in C^0([s,1]; L^p(\Omega))$ *of the Cauchy problem* (**??**) *with initial condition* $u(s,\cdot) = \mu_0$*.*

2.6 Superposition principle

{superposit

We say that μ_t is a superposition solution of (**??**) if there exists a measure Q on the metric space $C([0,1];\mathbb{R}^n)$, supported on the subset of absolutely continuous trajectories of b, that is,

$$\int_{C([0,1];\mathbb{R}^n)} \left| \alpha(t) - \alpha(0) - \int_0^t b(s,\alpha(s))\, ds \right| dQ(\alpha) = 0 \qquad \text{for every } t \in [0,1],$$

and such that $\mu_t = (\pi_t)_{\#}Q$ for each $t \in [0,1]$, that is,

$$\int \varphi(x)\, d\mu_t(x) = \int_{C([0,1];\mathbb{R}^n)} \varphi(\alpha(t))\, dQ(\alpha)$$

for each $t \in [0,1]$ and all $\varphi \in C_c(\mathbb{R}^n)$. It can be seen that superposition solutions to (**??**) are indeed weak solutions. Superposition solutions are nicer in the sense that they can be represented as time evaluations of a more intrinsic measure Q on the set of rectifiable trajectories of b. In particular, this obviously allows for vector fields for which uniqueness of solutions in (**??**) may not be true. The following result was proven in [**?**, Theorem 2].

{sprinc}

Theorem 7. *If* $u(t,\cdot) = \mu_t$ *is a non-negative measure valued weak solution of* (**??**)*, and*

$$\int_0^1 \int_{\mathbb{R}^n} \frac{|b(t,x)|}{1+|x|}\, d\mu_t(x)\, dt < \infty \tag{2.11}$$

{globalint

then it is also a superposition solution.

As a corollary, the following result holds.

{uniqsuper

Corollary 8. *If* (**??**) *holds, then for each Borel set* $A \subset \mathbb{R}^n$ *the following statements are equivalent:*

(a) Solutions to (**??**) *are unique for every* $x \in A$*.*

(b) Non-negative measure valued weak solutions to (**??**) *are unique whenever* μ_0 *is supported on* A*.*

As a consequence, positive measure-valued weak solutions to the continuity equation (**??**) are superposition solutions only assuming (**??**). Let us mention that everything above can be reformulated on bounded domains $\Omega \subset \mathbb{R}^n$, by adding boundary conditions on b of Neumann type.

2.7 Traffic applications of the Main Theorem

{trafficapp

One of the fundamental results of [?] consists of proving that at least one of the optimal strategies Q_0 in **??** is supported on the flow of a reasonably nice vector field. In the present paper, we want extend this result from its original autonomous formulation to certain non-autonomous situations. To this end, some regularity needs to be assumed in the spatial x variable. We have in mind the following model,

$$H(x,i) = a(x)\,\frac{i^q}{q} + b\,i \tag{2.12}$$ {Hmodel}

where in principle one can have $1 < q < \infty$, $b > 0$, and $a > 0$ is a measurable function, with $a, \frac{1}{a} \in L^\infty(\Omega)$, and more importantly $a \in W^{1,s}(\Omega)$ for some $s > n$. Of course, other models may fit into the discussion. Here Ω is a domain in $\mathbb{R}^n$ with nice boundary. This is our main result.

{mainthmapp

Theorem 9. *Let $1 < q \le 3/2$. If $x \mapsto H(x,i)$ is as in (**??**), and μ_0, μ_1 are probability densities such that $\frac{1}{\mu_0}, \frac{1}{\mu_1} \in W^{1,\infty}$, then (**??**) admits an equilibrium strategy $Q \in \mathcal{Q}^q(\mu_0,\mu_1)$ of the form*

$$Q = X_{\#}(\mu_0)$$

for some DiPerna - Lions flow map X.

The proof of this theorem repeats the structure of [?]. This means that optimal traffic strategies will be constructed by means of the classical Dacorogna-Moser scheme. Such structures will be proven to have support on the trajectories of a DiPerna-Lions velocity field. This obviously will require the uses of DiPerna-Lions theory (see Subsection **??** for the definition of DiPerna - Lions flow), for which L^∞ and Sobolev estimates are essential, and here is where the results from Sections **??**, **??** and **??** will enter the game.

Let us start by observing that if H is as in (**??**) then

$$\mathcal{H}^*(x,\xi) = \frac{a(x)}{p}\left(\frac{|\xi|-b}{a(x)}\right)_+^p$$

so that

$$D_\xi \mathcal{H}^*(x,\xi) = \left(\frac{|\xi|-b}{a(x)}\right)_+^{p-1}\frac{\xi}{|\xi|}$$
$$D_{\xi\xi}\mathcal{H}^*(x,\xi) = \frac{p-1}{a(x)}\left(\frac{|\xi|-b}{a(x)}\right)_+^{p-2}\frac{\xi\otimes\xi}{|\xi|^2} + \left(\frac{|\xi|-b}{a(x)}\right)_+^{p-1}\left(\frac{1}{|\xi|}\,\mathbf{Id} - \frac{\xi\otimes\xi}{|\xi|^3}\right)$$

This immediately gives the following bounds

$$\begin{aligned}\langle D_{\xi\xi}\mathcal{H}^*(x,\xi)\lambda,\lambda\rangle &\ge c\,(|\xi|-b)_+^{p-1}\,|\xi|^{-1}\,|\lambda|^2\\ |D_{\xi\xi}\mathcal{H}^*(x,\xi)| &\le C\,(|\xi|-b)_+^{p-2}\\ |D_{x\xi}\mathcal{H}^*(x,\xi)| &\le k(x)\,(|\xi|-b)_+^{p-1}\end{aligned} \tag{2.13}$$ {improvedbo

for all $\xi \in \mathbb{R}^n$, $x \in \Omega$ and $\lambda \in \mathbb{R}^n$, and some function $k \in L^s(\Omega)$. In particular, for H as in (**??**), bounds (**??**) guarantee that $\mathcal{F} = \mathcal{H}^*$ satisfies the assumptions **(F0)**–**(F4)** from Section **??**.

Towards the proof of Theorem **??**, we start with the following regularity result. It is a direct consequence of Theorem **??** and its global version from Theorem **??**.

{sigmabdd}

Proposition 10. *Let $\Omega \subset \mathbb{R}^n$ be a bounded domain, with C^3-smooth boundary. Suppose also that $\mu_0, \mu_1 \in L^s$ for some $s > n$. If H is as in (**??**), $q \le 2$, and σ is a minimizer in (**??**), then $\sigma \in L^\infty$.*

Proof. Since H is strictly convex in the second variable, the remarks after Lemma **??** apply. So if σ is a minimizer then

$$\sigma = \nabla_\xi \mathcal{H}^*(x, \nabla u) \tag{2.14}$$ {relations

where u is necessarily a minimizer of

$$\inf_{u \in W^{1,p}(\Omega;\mathbb{R})} \int \mathcal{H}^*(x, \nabla u) + (\mu_0 - \mu_1)\, u\, dx.$$

Since $\mathcal{F} = \mathcal{H}^*$ satisfies conditions **(F0)**–**(F4)** of Section **??**, we are legitimate to use Theorem **??**. As a consequence, and since $\mu_0, \mu_1 \in L^s$ for some $s > n$, we deduce that $u \in W^{1,\infty}_{loc}$. The global boundedness of ∇u is a consequence of the boundary reflection result, see Theorem **??**. So the claim follows. □

It is worth mentioning that the essential condition on H here is that $\mathcal{F} = \mathcal{H}^*$ satisfies **(F0)-(F4)**. Now, we continue with the Sobolev estimates for σ.

{sigmasob}

Proposition 11. *Let $\Omega \subset \mathbb{R}^n$ be a bounded domain, with C^3-smooth boundary. Suppose also that $\mu_0, \mu_1 \in L^s$ for some $s > n$. If σ is a minimizer in (**??**), H is as in (**??**), and $q \leq 3/2$, then $\sigma \in W^{1,1}_{loc}(\Omega)$.*

Proof. We start as in the proof of Proposition **??**. So we have

$$\sigma = \nabla_\xi \mathcal{H}^*(x, \nabla u) \tag{2.15}$$ {relations

where u is necessarily a minimizer of

$$\inf_{u \in W^{1,p}(\Omega;\mathbb{R})} \int \mathcal{H}^*(x, \nabla u) + (\mu_0 - \mu_1)\, u\, dx.$$

Now, we have for $\mathcal{F}(x, \xi) = \mathcal{H}^*(x, \xi)$ the bounds in (**??**). In particular, this means that

$$\begin{aligned} |D\sigma| &\leq |D_{x\xi}\mathcal{H}^*(x, Du)| + |D_{\xi\xi}\mathcal{H}^*(x, Du)\, D^2 u| \\ &\leq k(x)\, (|Du| - b)_+^{p-1} + C\, (|Du| - b)_+^{p-2}\, |D^2 u| \end{aligned}$$

Using now Theorem **??** together with the fact that $k \in L^s$, we see that the first term on the right hand side above belongs to L^s, and so in particular it is locally integrable. Concerning the second, we use Theorem **??** to deduce that if $1 < q \leq \frac{3}{2}$ then $p \geq 3$ and so

$$\int (|Du| - b)_+^{p-2}\, |D^2 u| \leq \left(\int (|Du| - b)_+^2\, |D^2 u|^2 \right)^{\frac{1}{2}} \left(\int (|Du| - b)_+^{2(p-3)} \right)^{\frac{1}{2}}$$

which is easily seen to be finite as a consequence of Theorems **??** and **??**. The claim follows. □

We can now prove Theorem **??**.

Proof of Theorem **??**. Although the proof of existence of optimal traffic strategies Q is standard (see [**?**, Theorem 2] or [**?**, Theorem 2.10] for the precise proofs), we write it here for the reader's convenience. Let us take $\sigma \in \Sigma^q(\mu_0, \mu_1)$ to be the unique optimizer at (**??**), and call

$$\hat{\sigma}(t, x) = \frac{\sigma(x)}{(1 - t)\, \mu_0(x) + t\, \mu_1(x)}.$$

Then, $u(t, \cdot) = (1 - t)\, \mu_0 + t\, \mu_1$ is a weak solution of the Cauchy problem

$$\begin{cases} \partial_t u + \operatorname{div}(\hat{\sigma}\, u) = 0, \\ u(0, \cdot) = \mu_0. \end{cases} \tag{2.16}$$ {ce}

Indeed, since $\sigma \in \Sigma^q(\mu_0, \mu_1)$, we know that

$$\int \sigma(x) \cdot \nabla\psi(x)\, dx = \int \psi(x)\, d(\mu_1 - \mu_0)(x) \tag{2.17}$$ {solvingneu

for any $\psi = \psi(x) \in C_c^\infty(\mathbb{R}^n)$. Thus, using that $\frac{1}{\mu_j} \in L^\infty$, equation (**??**) is equivalent to

$$\int \hat{\sigma}(t,x) \cdot \nabla\psi(x)\, u(t,x)\, dx = \int \psi(x)\, d(\mu_1 - \mu_0)(x).$$

We now take any $\varphi = \varphi(t)$ of class $C_c^\infty([0,\infty))$. Then, we multiply both sides in (**??**) by φ and integrate in time on $[0,\infty)$, and obtain

$$\iint \hat{\sigma} \cdot \nabla(\varphi\psi)\, u(t,x)\, dx\, dt = \iint \varphi\psi\, d(\mu_1 - \mu_0)(x)\, dt \tag{2.18}$$ {solving2}

Now, we use Fubini's theorem, and integrate by parts in time on the right hand side, and get that

$$\begin{aligned}
\iint \varphi(t)\psi(x)\, d(\mu_1 - \mu_0)(x)\, dt &= \int \psi(x) \left(\int \varphi(t)\, (\mu_1 - \mu_0)(x)\, dt \right) dx \\
&= \int \psi(x) \left(\int \varphi(t)\, \frac{d}{dt}(u(t,x))\, dt \right) dx \\
&= \int \psi(x) \left(-\varphi(0)\, \mu_0(x) - \int \frac{d\varphi(t)}{dt}\, u(t,x)\, dt \right) dx \\
&= -\int \psi(x)\varphi(0)\, \mu_0(x)\, dx - \iint \psi(x) \frac{d\varphi(t)}{dt}\, u(t,x)\, dt\, dx.
\end{aligned}$$

We end up getting from (**??**) that

$$\iint \hat{\sigma} \cdot \nabla(\varphi\psi)\, u(t,x)\, dx\, dt + \int \psi\varphi(0)\, \mu_0\, dx + \iint \psi \frac{d\varphi}{dt}\, u(t,\cdot)\, dt\, dx = 0,$$

and therefore $u(t,\cdot)$ is a weak solution of (**??**), as claimed. Now, using that $|\Omega| < \infty$, we see that

$$\int_0^1 \int_\Omega |\widehat{\sigma}(t,x)|\, u(t,x)\, dx\, dt = \int_0^1 \int_\Omega |\sigma(x)|\, dx\, dt \le \|\sigma\|_{L^q(\Omega)}\, |\Omega|^{1-\frac{1}{q}} < +\infty.$$

Thus we are legitimate to use the Superposition Principle (i.e. Theorem **??**) and deduce that $u(t,\cdot)$ is indeed a superposition solution. In particular, there exists a probability measure $Q \in \mathcal{P}(C([0,1];\overline{\Omega}))$ with the property that $u(t,\cdot) = (\pi_t)_\# Q$, that is

$$\int_{\overline{\Omega}} \varphi(x)\, u(t,x)\, dx = \int_{C([0,1];\overline{\Omega})} \varphi(\alpha(t))\, dQ(\alpha), \quad \forall \varphi \in C(\overline{\Omega}).$$

It turns out that $Q \in \mathcal{Q}^q(\mu_0, \mu_1)$. Indeed, by construction $u(t,\cdot) = (\pi_t)_\# Q$ at times $t = 0$ and $t = 1$, and so Q fits with the given data μ_0 and μ_1. On the other hand, concerning its traffic intensity i_Q we have

$$\begin{aligned}
\langle i_Q, \varphi \rangle &= \int_{C([0,1];\Omega)} \int_0^1 \varphi(\alpha(t)) |\alpha'(t)|\, dt\, dQ(\alpha) \\
&= \int_0^1 \int_{C([0,1];\Omega)} \varphi(\alpha(t)) |\hat{\sigma}(t,\alpha(t))|\, dQ(\alpha)\, dt \\
&= \int_0^1 \int_\Omega \varphi(x) |\hat{\sigma}(t,x)|\, u(t,x)\, dx\, dt \\
&= \int_0^1 \int_\Omega \varphi(x) |\sigma(x)|\, dx\, dt = \int_\Omega \varphi(x) |\sigma(x)|\, dx,
\end{aligned}$$

thus i_Q is absolutely continuous with density $|\sigma| \in L^q$. Therefore $Q \in \mathcal{Q}^q(\mu_0, \mu_1)$ as claimed. Now it just remains to note that

$$\int H(x, i_Q(x))\, dx = \int H(x, |\sigma(x)|)\, dx = \int \mathcal{H}(x, \sigma(x))\, dx,$$

thus from the optimality of σ and Theorem **??** we get that Q is optimal.

In order to prove that $Q = X_{\#}(\mu_0)$ for some DiPerna-Lions flow X, we need to see that $\hat{\sigma}$ admits such a flow. But this is a consequence of Theorem **??**. Indeed,

- $\hat{\sigma} \in L^\infty(\Omega)$, because $\sigma \in L^\infty$ (by Proposition **??**) and $\frac{1}{\mu_j} \in L^\infty$ (by assumption).
- $\hat{\sigma} \in W^{1,1}(\Omega)$, because $\sigma \in W^{1,1}$ (by Proposition **??**), $\sigma \in L^\infty$ (by Proposition **??**) and $\frac{1}{\mu_j} \in L^\infty \cap W^{1,1}$ (by assumption).
- $\operatorname{div} \hat{\sigma} \in L^\infty$, because by assumption $D(\mu_0 - \mu_1) \in L^\infty$.

By Theorem **??**, $\hat{\sigma}$ induces a well-defined DiPerna-Lions flow X, and $u(t, \cdot) = X(t, \cdot)_{\#}(\mu_0)$ must necessarily be a weak solution of (**??**). However, the fact that $\hat{\sigma} \in L^1([0,1]; L^1(\Omega))$ allows us to use the Superposition Principle (Theorem **??**), and hence all non-negative measure valued weak solutions to (**??**) are superposition solutions. Moreover, Corollary **??** also applies to (**??**), and so there is only one superposition solution. As a consequence,

$$(1-t)\,\mu_0 + t\,\mu_1 = (X_t)_{\#}(\mu_0)$$

for each $t \in [0,1]$. As a consequence, one gets that $(\pi_t)_{\#}Q = (X_t)_{\#}(\mu_0)$ for each time $t \in [0,1]$. Thus $Q = X_{\#}(\mu_0)$, and the theorem is proved. □

3 The L^∞ bounds for Du

{slipschitz

In this section, we are given a bounded domain $\Omega \subset \mathbb{R}^n$, a real number $p \geq 2$, and an integer $N \geq 1$. We look at local minimizers $u : \Omega \subset \mathbb{R}^n \to \mathbb{R}^N$ of inhomogeneous functionals

$$\mathbb{F}(u, \Omega) = \int_\Omega \Big(\mathcal{F}(x, Du) + f(x) \cdot u(x)\Big)\, dx. \tag{3.1}$$

{modenergy

Here, $\mathcal{F}$ is a Carathéodory function with growth p, which is assumed to be *only* asimptotically convex with respect to the gradient variable, and $f : \Omega \subset \mathbb{R}^n \to \mathbb{R}^N$ is given. In this setting, and for $f \equiv 0$, Fonseca, Fusco and Marcellini proved in [**?**] that local minimizers are Lipschitz continuous if

$$|D_{x,\xi}\mathcal{F}(x,\xi)| \leq k(x)\,(1 + |\xi|)^{p-1} \qquad (x,\xi) \in \Omega \times \mathbb{R}^{n\times N}$$

for some function $k \in L^\infty$. More recently, still for $f \equiv 0$, in [**?**] this result was extended to $k \in L^s_{\text{loc}}(\Omega)$ for some $s > n$. Local boundedness for the gradient in the inhomogeneous case was proven in [**?**, Theorem 5.2] for $f \in C^\alpha$ and $\alpha > 0$, and in [**?**, Theorem 2.1] for $f \in L^s$, $s > n$. In both cases, though, the extra assumption that $\mathcal{F}(x,\xi) = \mathcal{F}(\xi)$ was needed.

To be more precise, we have a Carathéodory function

$$\mathcal{F} : \Omega \times \mathbb{R}^{n\times N} \to \mathbb{R}$$
$$(x,\xi) \to \mathcal{F}(x,\xi)$$

This means that, for every fixed $\xi \in \mathbb{R}^{n\times N}$, $x \mapsto \mathcal{F}(x,\xi)$ is measurable, and also that there is a null set $N \subset \Omega$ so that $\xi \mapsto \mathcal{F}(x,\xi)$ is continuous and convex in $\mathbb{R}^{n\times N}$, and C^2 on $\mathbb{R}^{n\times N} \setminus B_{\bar{R}}(0)$. It will be assumed to satisfy the following properties:

(F0) There exist positive constants ℓ, L such that for a.e. $x \in \Omega$ and all $\xi \in \mathbb{R}^{n\times N}$

$$\ell(|\xi|^p - 1) \leq \mathcal{F}(x,\xi) \leq L(|\xi|^p + 1).$$

(F1) There is $F : \overline{\Omega} \times [R, \infty) \to \mathbb{R}$ such that

$$\mathcal{F}(x,\xi) = F(x, |\xi|)$$

for a.e. $x \in \Omega$ and for all $\xi \in \mathbb{R}^{n\times N} \setminus B_R(0)$.

(F2) There exists $\nu > 0$ such that if $\lambda \in \mathbb{R}^{n\times N}$ then

$$\langle D_{\xi\xi}\mathcal{F}(x,\xi)\lambda, \lambda\rangle \geqslant \nu|\xi|^{p-2}|\lambda|^2,$$

for a.e. $x \in \Omega$ and all $\xi \in \mathbb{R}^{nN} \setminus B_R(0)$.

(F3) There exists $L_1 > 0$, such that

$$|D_{\xi\xi}\mathcal{F}(x,\xi)| \leqslant L_1|\xi|^{p-2}$$

for a.e. $x \in \Omega$ and all $\xi \in \mathbb{R}^{nN} \setminus B_R(0)$.

(F4) There exists $s > n$ and a non-negative function $k \in L^s_{\mathrm{loc}}(\Omega)$ such that

$$|D_{x\xi}\mathcal{F}(x,\xi)| \leqslant k(x)|\xi|^{p-1},$$

for a.e. $x \in \Omega$ and for all $\xi \in \mathbb{R}^{nN} \setminus B_R(0)$.

We can now state the main result of this section. {eq121}

Theorem 12. *Let $u \in W^{1,p}_{\mathrm{loc}}(\Omega)$ be a local minimizer of the functional $\mathbb{F}(u,\Omega)$ in (**??**), and assume that the energy density $\mathcal{F}(x,\xi)$ satisfies assumptions* **(F0)-(F4)***. If $f \in L^s_{\mathrm{loc}}(\Omega)$, where s is the exponent appearing in* **(F4)***, then there exists a constant $c = c(n,p,s,\nu,L_1,\rho,R)$, such that*

$$\sup_{B_\rho}|Du| \leqslant c(1 + ||k||_{L^s(B_r)} + ||f||_{L^s(B_r)})^\tau \int_{B_r}(1+|Du|^p)\,dx,$$

for all balls $B_\rho \subset B_r \Subset \Omega$, and an exponent $\tau = \tau(n,s) > 0$.

As usual, the proof of this Theorem is split into two parts: an apriori L^∞ estimate, and an approximation that we will establish in the following two subsections.

3.1 The apriori L^∞ estimate

We introduce some auxiliary notation. For each $\gamma \geq 0$, we will denote

$$\Phi(t) = \Phi_\gamma(t) = \frac{t^2}{(1+t)^2}(1+t)^\gamma. \quad (3.2) \quad \{phigamma\}$$

For such Φ, one easily sees that

$$t\Phi'(t) \leq 2(1+\gamma)\Phi(t). \quad (3.3) \quad \{eq48\}$$

We also introduce the following notation for the positive part of $|Du| - 1$,

$$P = (|Du| - 1)_+$$

so that

$$DP = \chi_{\{|Du|>1\}} \cdot \frac{Du}{|Du|} \cdot D^2u \quad (3.4) \quad \{equality\}$$

The following lemma is an important application in the so called hole-filling method. Its proof can be found for example in [**?**, Lemma 6.1] .

{iter}

Lemma 13. *Let $h:[r,R_0]\to\mathbb{R}$ be a nonnegative bounded function and $0<\vartheta<1$, $A,B\geq 0$ and $\beta>0$. Assume that*

$$h(s)\leq \vartheta h(t)+\frac{A}{(t-s)^{\beta}}+B,$$

for all $r\leq s<t\leq R_0$. Then

$$h(r)\leq \frac{cA}{(R_0-r)^{\beta}}+cB,$$

where $c=c(\vartheta,\beta)>0$.

In this subsection, we prove an apriori estimate that will be used later in the approximation step (see Subsection **??**) for proving Theorem **??**. The precise statement is the following one. Recall that $\mathbb{F}$ is the one defined in (**??**).

{t:Apriori

Theorem 14. *Assume* **(F0)**–**(F4)** *hold, and let $f\in L^s_{\mathrm{loc}}(\Omega)$, where s is the exponent appearing in assumption* **(F4)**. *Fix a ball $B_r(x_0)\Subset\Omega$ and two functions $u,\bar{u}\in W^{1,p}(B_r(x_0);\mathbb{R}^N)$, and define*

$$\widetilde{\mathbb{F}}(v;B_r(x_0)):=\mathbb{F}(v;B_r(x_0))+\int_{B_r(x_0)}\arctan(|v-\bar{u}(x)|^2)\,dx.$$

Let $v\in u+W^{1,p}_0(B_r(x_0);\mathbb{R}^N)$ be a minimizer of $\widetilde{\mathbb{F}}$, satisfying

$$v\in W^{2,2}_{\mathrm{loc}}(B_r(x_0);\mathbb{R}^N)\cap W^{1,\infty}_{\mathrm{loc}}(B_r(x_0);\mathbb{R}^N)\quad\text{ and }\quad |Dv|^{p-2}|D^2v|^2\in L^1_{\mathrm{loc}}(B_r(x_0)).$$

Then, for every $B_{\bar{r}}(\bar{x})\Subset B_r(x_0)$, every $0<\rho<r'\leq\bar{r}$

$$\sup_{B_\rho}|Du|\leqslant C(1+||k||_{L^s(B_{r'})}+||f||_{L^s(B_{r'})})^{\tau}\left[\int_{B_{r'}(x_0)}(1+|Du|)^p\mathrm{d}x\right]^{\frac{1}{p}},$$

for some constant $C=C(n,N,p,s,L_1,\nu,\rho,r')$. Moreover, one has

$$\int_{B_\rho(x_0)}\frac{(|Du|-1)_+^2}{(1+(|Du|-1)_+)^2}\,|Du|^{p-2}|D^2u|^2\,dx\leq C(1+||k||_{L^s(B_{r'})}+||f||_{L^s(B_{r'})})^{\tau}\int_{B_{r'}(x_0)}(1+|Du|^p))\,dx, \tag{3.5}$$ {apriori3}

for some $C=C(n,N,p,s,L,L_1,\nu,\rho,r')$, and an exponent $\tau=\tau(n,s)>0$

In the above result, we are assuming without loss of generality that $\bar{R}=1$ in **(F0)–(F4)**. For the proof of the above result, the integral $\int_{B_r(x_0)}\arctan(|v-\bar{u}|^2)\,dx$ is a perturbation of $\mathbb{F}(v;B_r(x_0))$ that provides no difficulties. Indeed, denoting $g(x,v):=\arctan(|v-\bar{u}(x)|^2)$, we have that g and its derivatives g_{v^α}, $\alpha=1,\dots,m$, are bounded. Thus, for the sake of clarity, we prefer to drop this perturbation term, and to state, and prove an a priori estimate for local minimizers of $\mathbb{F}(\cdot;\Omega)$ only, see Theorem **??** below.

{Apriori}

Theorem 15. *Let $\mathcal{F}(x,\xi)$ satisfy conditions* **(F0)**–**(F4)**, *and let $f\in L^s_{\mathrm{loc}}(\Omega)$, where s is the exponent appearing in assumption* **(F4)**. *Assume that $u\in W^{2,2}_{\mathrm{loc}}(\Omega,\mathbb{R}^N)\cap W^{1,\infty}_{\mathrm{loc}}(\Omega,\mathbb{R}^N)$ is a local minimizer of the functional $\mathbb{F}(\cdot;\Omega)$, and that $|Du|^{p-2}|D^2u|^2\in L^1_{\mathrm{loc}}(\Omega)$. Then the estimate*

$$\sup_{B_\rho}|Du|\leqslant C(1+\|k\|_{L^s(B_R)}+\|f\|_{L^s(B_R)})^{\tau}\left[\int_{B_R}(1+|Du|)^p\mathrm{d}x\right]^{\frac{1}{p}}, \tag{3.6}$$ {apriorisu

holds for every concentric balls $B_\rho\subset B_R\Subset\Omega$. Moreover, the following Caccioppoli inequality holds,

$$\int_{B_\rho}\frac{(|Du|-1)_+^2}{(1+(|Du|-1)_+)^2}\,|Du|^{p-2}|D^2u|^2\,dx\leq C(1+\|k\|_{L^s(B_{2R})}+\|f\|_{L^s(B_{2R})})^{\tau}\int_{B_{2R}}(1+|Du|^p)\;dx, \tag{3.7}$$ {apriori3b

for some $C=C(n,N,p,L,L_1,\nu,s,\rho,R)$, an exponent $\tau=\tau(n,s)>0$ and for every concentric balls $B_\rho\subset B_R\subset B_{2R}\Subset\Omega$.

Proof. We will prove the theorem in 3 steps.

Step 1. The first step is to prove that if $\gamma \geqslant 0$ and if $\eta \in C_0^\infty(\Omega)$ is a non-negative cut off function, then one has

$$\begin{aligned} \int_\Omega \eta^2 \Phi(P)\,|Du|^{p-2}|D^2u|^2\,dx \le C(\gamma+1)^2 \int_\Omega \eta^2 k^2 |Du|^{\gamma+p}\,dx \\ + C\int_\Omega |D\eta|^2|Du|^{\gamma+p}\,dx + C(\gamma+1)^2\int_\Omega \eta^2|f|^2|Du|^\gamma\,dx, \end{aligned} \tag{3.8}$$ {eq70}

where Φ is the function defined at (**??**) and $P = (|Du|-1)_+$, with $C = C(n, N, \nu, p, L_1)$. Since u is a local minimizer of $\mathbb{F}(\cdot;\Omega)$, it satisfies the following integral identity

$$\int_\Omega \langle D_\xi \mathcal{F}(x, Du), D\psi\rangle = \int_\Omega f\cdot\psi \qquad \forall \psi \in C_0^\infty(\Omega, \mathbb{R}^N).$$

By our assumption on u and a standard approximation argument, we can choose

$$\psi \equiv \sum_s D_{x_s}\Big(\eta^2\cdot\Phi(P)\cdot D_{x_s}u\Big),$$

where $\eta \in C_0^\infty(\Omega)$. Such a choice, together with an integration by parts in the left hand side of previous identity, yields

$$-\sum_s \int_\Omega \big\langle D_{x_s\xi}\mathcal{F}(x, Du) + D_{\xi\xi}\mathcal{F}(x, Du)\cdot D_{x_s}Du, D\big(\eta^2\cdot\Phi(P)\cdot D_{x_s}u\big)\big\rangle = \int_\Omega f\cdot\psi \tag{3.9}$$ {eq71}

We now use the product rule to calculate the derivatives of $\eta^2\cdot\Phi(P)\cdot D_{x_s}u$. This converts (**??**) into

$$I_1 + I_2 + I_3 + I_4 + I_5 + I_6 + I_7 + I_8 + I_9 = 0,$$

where

$$\begin{aligned} I_1 &= 2\sum_s \int_\Omega \langle D_{\xi\xi}\mathcal{F}(x, Du)\cdot D_{x_s}Du, D\eta\cdot D_{x_s}u\rangle\,\eta\,\Phi(P)\,dx,\\ I_2 &= \sum_s \int_\Omega \langle D_{\xi\xi}\mathcal{F}(x, Du)\cdot D_{x_s}Du, D_{x_s}Du\rangle\,\eta^2\,\Phi(P)\,dx,\\ I_3 &= \sum_s \int_\Omega \langle D_{\xi\xi}\mathcal{F}(x, Du)\cdot D_{x_s}Du, \Phi'(P)\,DP\cdot D_{x_s}u\rangle\,\eta^2\,dx,\\ I_4 &= 2\sum_s \int_\Omega \langle D_{x_s\xi}\mathcal{F}(x, Du), D\eta\cdot D_{x_s}u\rangle\,\eta\,\Phi(P)\,dx,\\ I_5 &= \sum_s \int_\Omega \langle D_{x_s\xi}\mathcal{F}(x, Du), D_{x_s}Du\rangle\,\eta^2\,\Phi(P)\,dx,\\ I_6 &= \sum_s \int_\Omega \langle D_{x_s\xi}\mathcal{F}(x, Du), \Phi'(P)\,DP\cdot D_{x_s}u\rangle\,\eta^2\,dx,\\ I_7 &= 2\sum_s \int_\Omega f\,\eta\,\Phi(P)\,D_{x_s}\eta\cdot D_{x_s}u\,dx\\ I_8 &= \sum_s \int_\Omega f\,\eta^2\,\Phi(P)\,D_{x_sx_s}u\,dx\\ I_9 &= \sum_s \int_\Omega f\,\eta^2\,\Phi'(P)\,D_{x_s}u\cdot D_{x_s}P\,dx \end{aligned}$$

We will estimate each term separately. It is worth pointing out that the integrals I_i, with $i = 1, \dots, 6$ will be estimate with arguments similar to those in [**?**] ,[**?**]. We will report here for the sake of completeness.

For the estimate of I_1 we use assumption **(F3)** and Young's inequality as follows

$$\begin{aligned}
|I_1| \leqslant & 2L_1 \sum_s \int_\Omega \eta\, |D\eta|\, |Du|^{p-2}\, |D_{x_s} Du|\; |D_{x_s} u|\, \Phi(P)\; dx \\
\leqslant & \varepsilon \int_\Omega \eta^2 |Du|^{p-2} |D^2 u|^2 \Phi(P)\, dx + C_\varepsilon(L_1) \int_\Omega |D\eta|^2 |Du|^p \Phi(P)\, dx.
\end{aligned} \tag{3.10}$$ {eq46.1}

In order to estimate I_4,we use assumption **(F4)** and Young's inequality as follows,

$$\begin{aligned}
|I_4| & \leqslant \int_\Omega \eta\, |D\eta|\, k\, |Du|^p\, \Phi(P)\, dx \\
& \leqslant C \int_\Omega \eta^2\, k^2\, |Du|^p\, \Phi(P)\, dx + C \int_\Omega |D\eta|^2\, |Du|^p\, \Phi(P)\, dx.
\end{aligned} \tag{3.11}$$ {eq47.3}

We estimate I_5, using **(F4)** and Young's inequality again. Indeed

$$\begin{aligned}
|I_5| & \leqslant \int_\Omega \eta^2 k\, |Du|^{p-1}\, |D^2 u|\, \Phi(P)\, dx \\
& \leqslant \varepsilon \int_\Omega \eta^2\, |Du|^{p-2}\, |D^2 u|^2\, \Phi(P)\; dx + C_\varepsilon \int_\Omega \eta^2\, k^2\, |Du|^p\, \Phi(P)\, dx.
\end{aligned} \tag{3.12}$$ {eq47.4}

For the estimate of I_6, again by virtue of assumption **(F4)**, we have

$$\begin{aligned}
|I_6| & \leqslant \sum_s \int_\Omega \eta^2\, k\, |Du|^{p-1}\, |D_{x_s} u|\, \Phi'(P)\, |DP|\, dx \\
& \leqslant C \int_\Omega \eta^2\, k\, |Du|^p\, \Phi'(P)\, |DP|\, dx \\
& \leqslant C \int_\Omega \eta^2\, k\, |Du|^p\, \Phi'(P) |D^2 u| \\
& = C \int_{\{|Du| \geq 2\}} \eta^2\, k\, |Du|^p\, \Phi'(P) |D^2 u| + C \int_{\{1<|Du|<2\}} \eta^2\, k\, |Du|^p\, \Phi'(P) |D^2 u|,
\end{aligned} \tag{3.13}$$ {eq47.1}

where we used the equality in (**??**). Noting that

$$|Du| = (|Du|-1)_+ + 1 \leq 2(|Du|-1)_+ \quad \text{on the set } \{|Du| \geq 2\} \tag{3.14}$$ {est2}

and, recalling (**??**), we can estimate the first integral in the right hand side of previous inequality as follows

$$\begin{aligned}
& C \int_{\{|Du| \geq 2\}} \eta^2\, k\, |Du|^p\, \Phi'(P) |D^2 u| \leq 2C \int_{\{|Du| \geq 2\}} \eta^2\, k\, |Du|^{p-1}\, P\Phi'(P) |D^2 u| \\
& \leq 4C(1+\gamma) \int_{\{|Du| \geq 2\}} \eta^2\, k\, |Du|^{p-1}\, \Phi(P) |D^2 u| \\
& \leq \varepsilon \int_{\{|Du| \geq 2\}} \eta^2\, |Du|^{p-2}\, \Phi(P) |D^2 u|^2 + C_\varepsilon (1+\gamma)^2 \int_{\{|Du| \geq 2\}} \eta^2\, k^2\, |Du|^p\, \Phi(P).
\end{aligned} \tag{3.15}$$ {I6b}

After setting $C_\gamma = 2(1+\gamma) > 0$, we multiply and divide the last integrand in (**??**) by $\left(\frac{\delta+P}{C_\gamma}\right)^{1/2}$ with $0 < \delta < 1$, and use Young's inequality, thus obtaining

$$\begin{aligned}
& C \int_{\{1<|Du|<2\}} \eta^2\, \Phi'(P) \left\{ \frac{\delta+P}{C_\gamma}\, |Du|^{p-2}\, |D^2 u|^2 \right\}^{\frac{1}{2}} \times \left\{ \frac{C_\gamma}{\delta+P}\, k^2\, |Du|^{p+2} \right\}^{\frac{1}{2}} dx \\
\leqslant & \varepsilon \int_{\{1<|Du|<2\}} \eta^2\, \Phi'(P)\, \frac{\delta+P}{C_\gamma}\, |Du|^{p-2}\, |D^2 u|^2\, dx \\
& + C_\varepsilon \int_{\{1<|Du|<2\}} \eta^2\, k^2\, \Phi'(P)\, \frac{C_\gamma}{\delta+P}\, |Du|^{p+2}\, dx
\end{aligned}$$

$$
\begin{aligned}
&\le \frac{\varepsilon}{C_\gamma} \int_{\{1<|Du|<2\}} \eta^2\, \Phi'(P)\, P\, |Du|^{p-2}\, |D^2u|^2\, dx \\
&\quad + \frac{\varepsilon\delta}{C_\gamma} \int_{\{1<|Du|<2\}} \eta^2\, \Phi'(P)\, |Du|^{p-2}\, |D^2u|^2\, dx \\
&\quad + C_\varepsilon \int_{\{1<|Du|<2\}} \eta^2\, k^2\, \Phi'(P)\, \frac{C_\gamma}{\delta+P}\, |Du|^{p+2}\, dx \\
&\leqslant \varepsilon \int_{\{1<|Du|<2\}} \eta^2\, \Phi(P)\, |Du|^{p-2}\, |D^2u|^2\, dx \\
&\quad + \varepsilon\delta\tilde{C}_\gamma \int_{\{1<|Du|<2\}} \eta^2\, |D^2u|^2\, |Du|^{p-2}\, dx \\
&\quad + C_\gamma^2 C_\varepsilon \int_{\{1<|Du|<2\}} \eta^2\, k^2\, |Du|^{\gamma+p}\, dx
\end{aligned}
\tag{3.16} \qquad \texttt{\{eq52\}}
$$

where, in the last line we used (**??**), that $\Phi'(P) \le c_\gamma$ as long as $0 < P < 1$ on the set $\{1 \leqslant |Du| \leqslant 2\}$ and the fact that, since $\dfrac{P}{\delta+P} \leqslant 1$, we have

$$
\begin{aligned}
(\delta+P)^{-1}\Phi'(P) &= \Phi'(P) \cdot \frac{P}{\delta+P} \cdot P^{-1} \\
&\leqslant C_\gamma \Phi(P) \cdot P^{-2} = C_\gamma\, (1+P)^{\gamma-2} \\
&= C_\gamma |Du|^{\gamma-2} \qquad\qquad \text{in the set} \quad \{1 < |Du| < 2\}
\end{aligned}
\tag{3.17} \qquad \texttt{\{eq88\}}
$$

Plugging (**??**) and (**??**) into (**??**), we get

$$
\begin{aligned}
|I_6| \leqslant{} & 2\varepsilon \int_\Omega \eta^2 \Phi(P) |Du|^{p-2} |D^2u|^2\, dx + \varepsilon\delta\tilde{C}_\gamma \int_\Omega \eta^2\, |D^2u|^2\, |Du|^{p-2}\, dx \\
& + C_\gamma^2 C_\varepsilon \int_{\{|Du|>1\}} \eta^2\, k^2\, |Du|^{\gamma+p}\, dx.
\end{aligned}
$$

By virtue of the assumption $|D^2u|^2\, |Du|^{p-2} \in L^1_{\mathrm{loc}}(\Omega)$, we can let $\delta \to 0$ in previous estimate thus getting

$$
|I_6| \le 2\varepsilon \int_\Omega \eta^2 \Phi(P) |Du|^{p-2} |D^2u|^2\, dx + C_\varepsilon (1+\gamma)^2 \int_{\{|Du|>1\}} \eta^2\, k^2\, |Du|^{\gamma+p}\, dx, \tag{3.18} \qquad \texttt{\{52.1\}}
$$

where we used that $C_\gamma \sim (\gamma+1)$. For I_7, using Young's inequality, we get

$$
\begin{aligned}
|I_7| &\le 2 \int_\Omega \eta |D\eta| |f| |Du| \Phi(P) = 2 \int_{\{|Du|>1\}} \eta |D\eta| |f| |Du|^{\gamma-1} (|Du|-1)^2 \\
&\le C \int_{\{|Du|>1\}} \eta^2 |f|^2 (|Du|-1)^2 |Du|^{\gamma-p} + C \int_{\{|Du|>1\}} |D\eta|^2 (|Du|-1)^2 |Du|^{\gamma+p-2} \\
&= C \int_{\{|Du|>1\}} \eta^2 |f|^2 \Phi(P) |Du|^{2-p} + C \int_{\{|Du|>1\}} |D\eta|^2 \Phi(P) |Du|^p \\
&\le C \int_\Omega \eta^2 |f|^2 \Phi(P) + C \int_\Omega |D\eta|^2 \Phi(P) |Du|^p\, ,
\end{aligned}
\tag{3.19} \qquad \texttt{\{eq85.2\}}
$$

where we used that $p \ge 2$ and that the set of integration is $\{|Du| > 1\}$. Concerning I_8 and I_9, by (**??**) and Young's inequality, we have

$$
\begin{aligned}
|I_8| + |I_9| \le{} & \int_\Omega \eta^2 |f| |D^2u| \Phi(P) + \int_\Omega \eta^2 |f| |Du| |D^2u| \Phi'(P) \\
={} & \int_{\{|Du|>1\}} \eta^2 |f| |D^2u| \Phi(P) + \int_{\{|Du|\ge 2\}} \eta^2 |f| |Du| |D^2u| \Phi'(P) + \int_{\{1<|Du|<2\}} \eta^2 |f| |Du| |D^2u| \Phi'(P)
\end{aligned}
$$

$$\leq \int_{\{|Du|>1\}} \eta^2|f||D^2u|\Phi(P) + 2\int_{\{|Du|\geq 2\}} \eta^2|f||D^2u|P\Phi'(P) + \int_{\{1<|Du|<2\}} \eta^2|f||Du||D^2u|\Phi'(P)$$

$$\leq C_\gamma \int_{\{|Du|>1\}} \eta^2|f||D^2u|\Phi(P) + \int_{\{1<|Du|<2\}} \eta^2|f||Du||D^2u|\Phi'(P). \tag{3.20}$$ {eq8520}

The first integral in the right hand side of (**??**) can be estimated by Young's inequality as follows

$$\begin{aligned} & C_\gamma \int_{\{|Du|>1\}} \eta^2\,|f|\,|D^2u|\,\Phi(P) \\ & \leq \varepsilon \int_{\{|Du|>1\}} \eta^2\,|D^2u|^2\,\Phi(P) + C_\varepsilon \cdot C_\gamma^2 \int_{\{|Du|>1\}} \eta^2|f|^2\,\Phi(P) \\ & \leq \varepsilon \int_\Omega \eta^2|Du|^{p-2}|D^2u|^2\Phi(P) + C_\varepsilon \cdot C_\gamma^2 \int_\Omega \eta^2|f|^2\Phi(P), \end{aligned} \tag{3.21}$$ {eq85bis}

where we used that $|Du|^{p-2} > 1$ on the set $|Du| > 1$ since $p > 2$ and as before $C_\gamma \sim (\gamma+1)$. To estimate the second integral in (**??**) we argue as we did for I_6 multiplying and dividing for $C_\gamma^{\frac{1}{2}}(\delta+P)^{\frac{1}{2}}$ with $0 < \delta < 1$ and we use Young's inequality, thus getting

$$\begin{aligned} & \int_{\{1<|Du|<2\}} \eta^2|f||Du||D^2u|\Phi'(P) = \int_{\{1<|Du|<2\}} \eta^2\Phi'(P) \left\{ \frac{\delta+P}{C_\gamma}|D^2u|^2 \right\}^{\frac{1}{2}} \left\{ |f|^2|Du|^2 \frac{C_\gamma}{\delta+P} \right\}^{\frac{1}{2}} \\ & \leq \frac{\varepsilon}{C_\gamma} \int_{\{1<|Du|<2\}} \eta^2(\delta+P)|D^2u|^2\Phi'(P) + C_\varepsilon \cdot C_\gamma \int_{\{1<|Du|\leq 2\}} \eta^2 \frac{|Du|^2|f|^2}{(\delta+P)}\Phi'(P) \\ & \leq \frac{\varepsilon}{C_\gamma} \int_{\{1<|Du|\leq 2\}} \eta^2\,|D^2u|^2 P\Phi'(P) + \frac{\epsilon}{C_\gamma}\delta \int_{\Omega\cap\{1<|Du|\leq 2\}} \eta^2|D^2u|^2\Phi'(P) \\ & \quad + C_\varepsilon \cdot C_\gamma \int_{\{1<|Du|\leq 2\}} \eta^2 \frac{|Du|^2|f|^2}{(\delta+P)}\Phi'(P) \\ & \leq \varepsilon \int_{\{1<|Du|<2\}} \eta^2|D^2u|^2\Phi(P) + \delta\varepsilon\tilde{C}_\gamma \int_{\{1<|Du|\leq 2\}} \eta^2|D^2u|^2|Du|^{p-2} \\ & \quad + C_\varepsilon\, C_\gamma^2 \int_{\{1<|Du|\leq 2\}} \eta^2|Du|^\gamma|f|^2 \\ & \leq \varepsilon \int_{\{1<|Du|<2\}} \eta^2|D^2u|^2|Du|^{p-2}\Phi(P) + \delta\varepsilon\tilde{C}_\gamma \int_{\{1<|Du|\leq 2\}} \eta^2|D^2u|^2|Du|^{p-2} \\ & \quad + C_\varepsilon C_\gamma^2 \int_{\{1<|Du|\leq 2\}} \eta^2|Du|^\gamma|f|^2, \end{aligned} \tag{3.22}$$ {eq85ter}

where we used that $\delta + P \leq 2P$ in the set $\{|Du| > 2\}$, (**??**) and that $\Phi'(t) \leq c_\gamma$ as long as $P \leq 1$ in the set $\{1 < |Du| \leq 2\}$ and (**??**) and $|Du|^{p-2} > 1$ in the set $\{|Du| > 1\}$. Inserting (**??**) and (**??**) in (**??**) and letting $\delta \to 0$, we get

$$|I_8| + |I_9| \leq 2\varepsilon \int_\Omega \eta^2|Du|^{p-2}|D^2u|^2\Phi(P) + C_\varepsilon\, C_\gamma^2 \int_{\{|Du|>1\}} \eta^2|Du|^\gamma|f|^2. \tag{3.23}$$ {eq852}

We remind that

$$I_2 + I_3 = -I_1 - I_4 - I_5 - I_6 - I_7 - I_8 - I_9. \tag{3.24}$$ {eq81}

We now elaborate on the precise form of $D_{\xi\xi}\mathcal{F}(x,\xi)$ to estimate I_3. To do this, we abuse of notation and for every scalar t we denote $F'(x,t) = \partial_t F(x,t)$ and $F''(x,t) = \partial_{tt}F(x,t)$. By **(F1)**, for every $\xi \in \mathbb{R}^{n\times N} \setminus \{0\}$ one has

$$D_{\xi\xi}\mathcal{F}(x,\xi) = \left(\frac{F''(x,|\xi|)}{|\xi|^2} - \frac{F'(x,|\xi|)}{|\xi|^3} \right) \xi \otimes \xi + \frac{F'(x,|\xi|)}{|\xi|}\,\mathbf{Id}_{\mathbb{R}^{n\times N}}$$

Componentwise,

$$D_{\xi_j^\beta,\xi_i^\alpha}\mathcal{F}(x,\xi) = D_{\xi_j^\beta}\left(F'(x,|\xi|)\,\frac{\xi_i^\alpha}{|\xi|}\right)$$
$$= \left(\frac{F''(x,|\xi|)}{|\xi|^2} - \frac{F'(x,|\xi|)}{|\xi|^3}\right)\xi_i^\alpha \xi_j^\beta + \frac{F'(x,|\xi|)}{|\xi|}\delta_{\xi_i^\alpha \xi_j^\beta}$$

Recalling the equality in (**??**), it is well known that for a.e. $x \in \{|Du| \geq 1\}$, we have

$$\begin{aligned}\sum_s \langle D_{\xi\xi}\mathcal{F}(x,Du)\cdot D_{x_s}Du, DP\cdot D_{x_s}u\rangle &= \sum_{s,i,j,\alpha,\beta} D_{\xi_j^\beta,\xi_i^\alpha}\mathcal{F}(x,Du)u^\alpha_{x_s}u^\beta_{x_s x_j}P_{x_i} \\ = \left(\frac{F''(x,|Du|)}{|Du|} - \frac{F'(x,|Du|)}{|Du|^2}\right)&\sum_\alpha\left(\sum_i u^\alpha_{x_i}(|Du|)_{x_i}\right)^2 + F'(x,|Du|)|D(|Du|)|^2.\end{aligned} \tag{3.25}$$

Thus,

$$\begin{aligned}I_3 &= \int_\Omega \eta^2\Phi'(P)\frac{F''(x,|Du|)}{|Du|}\sum_\alpha\left(\sum_i u^\alpha_{x_i}(|Du|)_{x_i}\right)^2 dx \\ &+ \int_\Omega \eta^2\Phi'(P)\,F'(x,|Du|)\left(|D(|Du|)|^2 - \frac{\sum_\alpha\left(\sum_i u^\alpha_{x_i}(|Du|)_{x_i}\right)^2}{|Du|^2}\right)dx.\end{aligned}$$

Now, if we use the Cauchy-Schwartz inequality, we have

$$\sum_\alpha\left(\sum_i u^\alpha_{x_i}(|Du|)_{x_i}\right)^2 \leq |Du|^2|D(|Du|)|^2.$$

Since

$$\Phi'(t) = (1+t)^{\gamma-3}t(\gamma t+2) \tag{3.26}$$ {Phi'}

is nonnegative for every $t \geq 0$ and, by **(F2)**, $F'(x,|Du|) \geq 0$, then we conclude that

$$I_3 \geq \int_\Omega \eta^2\Phi'(P)\,\frac{F''(x,|Du|)}{|Du|}\sum_\alpha\left(\sum_i u^\alpha_{x_i}(|Du|)_{x_i}\right)^2 dx \geq 0. \tag{3.27}$$ {I3}

Therefore, using that $I_3 \geqslant 0$ together with (**??**) we have

$$I_2 \leqslant |I_1| + |I_4| + |I_5| + |I_6| + |I_7| + |I_8| + |I_9|. \tag{3.28}$$ {eq80}

On the other hand, the ellipticity assumption **(F2)** gives that

$$I_2 \geqslant \nu\int_\Omega \eta^2\cdot\Phi(P)\cdot|Du|^{p-2}\cdot|D^2u|^2\,dx. \tag{3.29}$$ {eq82}

Inserting estimates (**??**), (**??**), (**??**), (**??**), (**??**), (**??**), (**??**) into (**??**), we obtain

$$\begin{aligned}&\nu\int_\Omega \eta^2\cdot\Phi(P)\,|Du|^{p-2}\,|D^2u|^2\,dx \\ &\leq 6\,\varepsilon\int_\Omega \eta^2\Phi(P)\,|Du|^{p-2}\,|D^2u|^2\,dx + C_\varepsilon(L_1)\int_\Omega |D\eta|^2|\,|Du|^{p+\gamma}\,dx \\ &+ C_\varepsilon\int_\Omega \eta^2k^2\,|Du|^{p+\gamma}\,dx + C_\epsilon(\gamma+1)^2\int_\Omega \eta^2k^2\,|Du|^{p+\gamma}\,dx \\ &+ C_\varepsilon(\gamma+1)^2\int_\Omega \eta^2|f|^2|Du|^\gamma\,dx,\end{aligned} \tag{3.30}$$ {eq92}

where we used that $\Phi(P) \leq (1+P)^\gamma$. We now choose $\varepsilon = \frac{\nu}{12}$, and reabsorb the first integral in the right hand side by the left hand side. We obtain

$$\begin{aligned}
&\int_\Omega \eta^2 \cdot \Phi(P)\,|Du|^{p-2}\,|D^2u|^2\,dx \\
&\leq C\int_\Omega |D\eta|^2|\,|Du|^{p+\gamma}\,dx + C\int_\Omega \eta^2 k^2\,|Du|^{p+\gamma}\,dx \\
&+ C(\gamma+1)^2\int_\Omega \eta^2 k^2\,|Du|^{p+\gamma}\,dx + C(\gamma+1)^2\int_\Omega \eta^2|f|^2|Du|^\gamma\,dx \\
&\leq C\int_\Omega |D\eta|^2|\,|Du|^{p+\gamma}\,dx + C(\gamma+1)^2\int_\Omega \eta^2 k^2\,|Du|^{p+\gamma}\,dx \\
&+ C(\gamma+1)^2\int_\Omega \eta^2|f|^2|Du|^\gamma\,dx,
\end{aligned}$$

with $C = C(n, N, p, \nu, L_1)$ that is inequality (**??**).

Step 2. Fix a ball $B_R(x_0) \Subset \Omega$ and radii $0 < \rho < r < t < R$. Due to the local nature of our results, without loss of generality we may suppose $R < 1$. Let $\eta \in C_0^\infty(B_t)$ be a cut off function such that $\eta \equiv 1$ on B_r and $|D\eta| \leqslant \dfrac{C}{t-r}$. Inequality (**??**) can be written as follows

$$\int_\Omega \eta^2\Phi(P)\,|Du|^{p-2}|D^2u|^2\,dx \leq J_1 + J_2 + J_3, \tag{3.31}$$ {step2.1}

where we used the notations

$$\begin{aligned}
J_1 &:= C\int |D\eta|^2|Du|^{\gamma+p}\,dx, \\
J_2 &:= C(\gamma+1)^2\int \eta^2 k^2\,|Du|^{\gamma+p}\,dx, \\
J_3 &:= C(\gamma+1)^2\int \eta^2|f|^2|Du|^\gamma\,dx.
\end{aligned}$$

We estimate J_i, $i = 1, 2, 3$ using the assumptions k, $f \in L^s_{\mathrm{loc}}(\Omega)$, Hölder's inequality and the properties of η. Using Hölder's inequality with exponents $\frac{s}{2}$ and $\frac{s}{s-2}$, we get

$$J_1 \leqslant C\,\frac{|B_R|^{2/s}}{(t-r)^2}\left(\int_{B_t} |Du|^{\frac{s(\gamma+p)}{s-2}}\,dx\right)^{\frac{s-2}{s}} \tag{3.32}$$ {eq94}

and

$$J_2 \leqslant C(\gamma+1)^2\left(\int_{B_t} k^s\,dx\right)^{\frac{2}{s}} \cdot \left(\int_{B_t} |Du|^{\frac{s(\gamma+p)}{s-2}}\,dx\right)^{\frac{s-2}{s}}, \tag{3.33}$$ {eq94.1}

and

$$J_3 \leqslant (\gamma+1)^2\left(\int_{B_t} |f|^s\,dx\right)^{\frac{2}{s}} \cdot \left(\int_{B_t} |Du|^{\frac{s(\gamma+p)}{s-2}}\,dx\right)^{\frac{s-2}{s}}. \tag{3.34}$$ {eq96}

Let us now define the constant E_R as follows

$$E_R^2 := \|k\|^2_{L^s(B_R)} + \|f\|^2_{L^s(B_R)}\,. \tag{3.35}$$ {eq113}

Inserting (**??**), (**??**) and (**??**) into (**??**), and using (**??**), we get

$$\int_\Omega \eta^2\,\Phi(P)\,|Du|^{p-2}|D^2u|^2\,dx \leqslant C(\gamma+1)^2\frac{1+E_R^2}{(t-r)^2}\left(\int_{B_R} |Du|^{\frac{s(\gamma+p)}{s-2}}\,dx\right)^{\frac{s-2}{s}}, \tag{3.36}$$ {eq102a}

Following [?], we now introduce the auxiliary function

$$G(t) = 1 + \int_0^t (1+s)^{\frac{\gamma+p-4}{2}} s\, ds.$$

It is easy to see that

$$\frac{1}{2(\gamma+p)^2}(1+t)^{\frac{\gamma+p}{2}} \le G(t) \le 2(1+t)^{\frac{\gamma+p}{2}}, \qquad G'(t) = t(1+t)^{\frac{\gamma+p-4}{2}}. \tag{3.37}$$ {propG}

Let us denote by

$$2^* = \begin{cases} \frac{2n}{n-2} & \text{if } n>2 \\ \text{any exponent} & \text{if } n=2. \end{cases}$$

Thus, by the Sobolev imbedding Theorem, we get

$$\left(\int_\Omega \Big|\eta\, G(P)\Big|^{2^*} dx\right)^{\frac{2}{2^*}} \le C \int_\Omega \Big|D(\eta\, G(P))\Big|^2 dx \le C\int_\Omega |D\eta|^2 G(P)^2\, dx + C\int_\Omega \eta^2 G'(P)^2 |DP|^2\, dx.$$

Using the properties of $G(t)$ at (**??**) in the previous inequality and recalling that $1+P = |Du|$ on the set $\{|Du|>1\}$, we obtain

$$\begin{aligned} &\frac{1}{(\gamma+p)^4}\left(\int_\Omega \eta^{2^*}|Du|^{\frac{2^*(\gamma+p)}{2}}\, dx\right)^{\frac{2}{2^*}} \\ &\quad\le c\int_\Omega |D\eta|^2|Du|^{\gamma+p}\, dx + c\int_\Omega \eta^2 |Du|^{\gamma+p-4}\, P^2\, |DP|^2\, dx \\ &\quad\le \frac{c}{(t-r)^2}\int_{B_t}|Du|^{\gamma+p}\, dx + c\int_\Omega \eta^2 \Phi(P)\, |Du|^{p-2}|D^2u|^2\, dx + \frac{ct^n}{(t-r)^2}, \end{aligned} \tag{3.38}$$ {apriori3d}

where we also used the properties of the function η. Combining estimates (**??**) and (**??**), using that $\eta \equiv 1$ on B_r, we get

$$\left(\int_{B_r} |Du|^{\frac{2^*(\gamma+p)}{2}}\, dx\right)^{\frac{2}{2^*}} \leqslant C(\gamma+p)^6 \frac{1+E_R^2}{(t-r)^2}\left(\int_{B_t}|Du|^{\frac{s(\gamma+p)}{s-2}}\, dx\right)^{\frac{s-2}{s}} \tag{3.39}$$ {eq100}

Since $s>n$, setting $\theta = \frac{n}{s}$, one has $0<\theta<1$ and moreover

$$\frac{s-2}{s(\gamma+p)} = \frac{1-\theta}{\gamma+p} + \frac{\theta(n-2)}{n(\gamma+p)}$$

and so, the use of the interpolation inequality in the right hand side of (**??**) yields

$$\begin{aligned} &\left(\int_{B_r} |Du|^{\frac{2^*(\gamma+p)}{2}}\, dx\right)^{\frac{2}{2^*}} \\ &\le C(\gamma+p)^6\frac{1+E_R^2}{(t-r)^2}\left(\int_{B_t}|Du|^{\gamma+p}\, dx\right)^{1-\theta}\left(\int_{B_t}|Du|^{\frac{2^*(\gamma+p)}{2}}\, dx\right)^{\frac{2\theta}{2^*}} \end{aligned} \tag{3.40}$$ {eq100b}

We now use Young's inequality with exponents $\frac{1}{\theta}$ and $\frac{1}{1-\theta}$ in the right hand side of (**??**) thus getting

$$\begin{aligned} \left(\int_{B_r} |Du|^{\frac{2^*(\gamma+p)}{2}}\, dx\right)^{\frac{2}{2^*}} &\leqslant \frac{1}{2}\left(\int_{B_r} |Du|^{\frac{2^*(\gamma+p)}{2}}\, dx\right)^{\frac{2}{2^*}} \\ &\qquad + C\left((\gamma+p)^6\frac{1+E_R^2}{(t-r)^2}\right)^{\frac{1}{1-\theta}}\int_{B_R}|Du|^{\gamma+p}\, dx \end{aligned} \tag{3.41}$$ {eq100c}

The iteration Lemma **??** now gives that

$$\left(\int_{B_\rho} |Du|^{\frac{2^*}{2}(\gamma+p)}\, dx\right)^{\frac{2}{2^*}} \leqslant C\left((\gamma+p)^6\frac{1+E_R^2}{(R-\rho)^2}\right)^{\frac{s}{s-n}}\int_{B_R}|Du|^{\gamma+p}\, dx. \tag{3.42}$$ {eq100t}

Step 3. Let us define the decreasing sequence of radii ρ_j, $j \in \mathbb{N}$, by setting

$$\rho_j := \rho + \frac{R-\rho}{2^j},$$

and the increasing sequence of exponents

$$p_j = p\left(\frac{2^*}{2}\right)^j.$$

Since $\gamma \geq 0$ can take any value, estimate (**??**) can be written on every ball B_{ρ_j} as follows

$$\left(\int_{B_{\rho_{j+1}}} |Du|^{p_{j+1}}\,dx\right)^{\frac{1}{p_{j+1}}} \leqslant \left[C\frac{p_j^6(1+E_R)}{(\rho_j-\rho_{j+1})}\right]^{\frac{1}{p_j}\frac{s}{s-n}} \left(\int_{B_{\rho_j}} |Du|^{p_j}\,dx\right)^{\frac{1}{p_j}}. \tag{3.43}$$ {eq102}

Iterating estimate (**??**) we obtain

$$\begin{aligned}\left(\int_{B_{\rho_{j+1}}} |Du|^{p_{j+1}}\,dx\right)^{\frac{1}{p_{j+1}}} &\leqslant \prod_{j=0}^{+\infty}\left[C\frac{p_j^6(1+E_R)}{(\rho_j-\rho_{j+1})}\right]^{\frac{1}{p_j}\frac{s}{s-n}} \left(\int_{B_{\rho_0}} |Du|^{p}\,dx\right)^{\frac{1}{p}}\\ &= \prod_{j=0}^{+\infty}\left[C\frac{2^{j+1}p_j^6\,(1+E_R)}{(R-\rho)}\right]^{\frac{1}{p_j}\frac{s}{s-n}} \left(\int_{B_{\rho_0}} |Du|^{p}\,dx\right)^{\frac{1}{p}},\end{aligned}$$

where we used the definition of ρ_j. It is easy to prove that

$$\prod_{j=0}^{+\infty}\left[C\frac{2^{j+1}p_j^2(1+E_R)}{(R-\rho)}\right]^{\frac{1}{p_j}\frac{s}{s-n}} \leq \widehat{C}\left(\frac{(1+E_R)}{(R-\rho)}\right)^{\tau}$$

for $\tau = \frac{2\,(2^*)^3}{s(2^*-2)^2}$, and a constant $\widehat{C} = \widehat{C}(n, N, p, s, \nu, L, L_1)$. Therefore

$$\left(\int_{B_{\rho_{j+1}}} |Du|^{p_{j+1}}\,dx\right)^{\frac{1}{p_{j+1}}} \leqslant \widehat{C}\frac{(1+E_R)^{\tau}}{(R-\rho)^{\tau}}\left(\int_{B_t} |Du|^{p}\,dx\right)^{\frac{1}{p}},$$

for every $j \in \mathbb{N}$. Now, letting $j \to \infty$, and recalling that $\rho_j \geq \rho$, for every $j \in \mathbb{N}$, we end up with

$$\sup_{B_\rho}|Du| = \lim_{j\to\infty}\left(\int_{B_\rho}|Du|^{p_j}\right)^{\frac{1}{p_j}} \leq \widehat{C}\frac{(1+E_R)^{\tau}}{(R-\rho)^{\tau}}\left(\int_{B_R}|Du|^{p}\right)^{\frac{1}{p}},$$

that gives (**??**).

Step 4. In this Step we are going to establish estimate (**??**). To this aim we write (**??**) for $\gamma = 0$,

$$\begin{aligned}\int_\Omega \eta^2\Phi(P)\,|Du|^{p-2}\,|D^2u|^2\,dx \leq C\int_\Omega \eta^2k^2\,|Du|^p\,dx\\ + C\int_\Omega |D\eta|^2\,|Du|^p\,dx + C\int_\Omega \eta^2|f|^2\,dx,\end{aligned} \tag{3.44}$$ {eq700}

for every $\eta \in C_0^\infty(\Omega)$. Choosing radii $0 < \rho < R$, η a cut off function between B_ρ and B_R and using estimate (**??**), we obtain

$$\begin{aligned}\int_{B_\rho}\Phi(P)\,|Du|^{p-2}|D^2u|^2\,dx &\leq C\sup_{B_R}|Du|^p\int_{B_R}k^2\,dx + \frac{C|B_R|}{(R-\rho)^2}\sup_{B_R}|Du|^p + C\int_{B_R}|f|^2\,dx\\ &\leq C(1+E_{2R})^{\tau'}\int_{B_{2R}}|Du|^p\,dx,\end{aligned} \tag{3.45}$$ {eq701}

for a constant $C = C(n, N, p, L, L_1, \nu)$ and an exponent $\tau' = \tau'(s, p, n)$. The proof is finished. □

3.2 The approximation

{appsubsect

In this section, we use the apriori estimate of Theorem **??** to prove Theorem **??**. In particular, we show that local minimizers $u \in W^{1,p}_{loc}(\Omega;\mathbb{R}^N)$ of the functional in (**??**),

$$\mathbb{F}(u,\Omega) = \int_\Omega \mathcal{F}(x,Du) + f(x)\cdot u\,dx$$

are locally Lipschitz. To this end, we state first the following approximation result, which we take from [**?**, **?**]. It shows that one can find a sequence of uniformly elliptic integrands $\mathcal{F}_m$ that approximate the given $\mathcal{F}$. The approximants can be chosen to be Lipschitz in the x variable, and also to have ellipticity bounds on all the domain of the ξ variable, although these bounds may depend on m (see conditions ($\tilde{\mathbf{F}}_\mathbf{m}\mathbf{2}$)-($\tilde{\mathbf{F}}_\mathbf{m}\mathbf{4}$) below). Furthermore, these ellipticity conditions may be assumed uniform in m away from a ball of the ξ variable (see conditions ($\tilde{\mathbf{F}}\mathbf{0}$)-($\tilde{\mathbf{F}}\mathbf{4}$) below). We recall that, without loss of generality, we assumed that the radius R appearing in the assumptions **(F0)**–**(F4)** is equal to 1.

{apprcupgu

Proposition 16. *Let $\mathcal{F} : \Omega\times\mathbb{R}^{n\times N} \to [0,+\infty)$ be a Carathéodory function, convex and C^2 with respect to the second variable, and satisfying assumptions* **(F0)**–**(F4)**. *Fixed an open set $\Omega' \Subset \Omega$, there exists a sequence $\mathcal{F}_m : \Omega'\times\mathbb{R}^{n\times N} \to [0,+\infty)$ of Carathéodory functions, C^2 and convex in the second variable, such that $\mathcal{F}_m$ converges to $\mathcal{F}$ pointwise a.e. on Ω' and everywhere in $\mathbb{R}^{n\times N}$. Moreover, each $\mathcal{F}_m$ can be chosen so that the following properties are satisfied:*

($\tilde{\mathbf{F}}\mathbf{0}$) *there exist constants $\tilde{L}, \tilde{c}_1, \tilde{c}_2 > 0$ such that for all $(x,\xi)\in\Omega'\times\mathbb{R}^{n\times N}$*

$$\tilde{c}_1|\xi|^p - \tilde{c}_2 \le \mathcal{F}_m(x,\xi) \le \tilde{L}(1+|\xi|)^p,$$

($\tilde{\mathbf{F}}\mathbf{1}$) *for every $x\in\Omega'$ and $\xi\in\mathbb{R}^{nN}\setminus B_2(0)$ one has $\mathcal{F}_m(x,\xi) = F_m(x,|\xi|)$,*

($\tilde{\mathbf{F}}\mathbf{2}$) *there exists $\tilde{\nu} = \tilde{\nu}(\nu,p)$ such that for every $x\in\Omega'$, $\xi\in\mathbb{R}^{n\times N}\setminus B_2(0)$ and $\lambda\in\mathbb{R}^{n\times N}$*

$$\tilde{\nu}(1+|\xi|)^{p-2}|\lambda|^2 \le \langle D_{\xi\xi}\mathcal{F}_m(x,\xi)\lambda,\lambda\rangle,$$

($\tilde{\mathbf{F}}\mathbf{3}$) *there exists $\tilde{L}_1 > 0$ such that for all $(x,\xi)\in\Omega'\times\mathbb{R}^{nN}\setminus B_2(0)$*

$$|D_{\xi\xi}\mathcal{F}_m(x,\xi)| \le \tilde{L}_1(1+|\xi|)^{p-2},$$

($\tilde{\mathbf{F}}\mathbf{4}$) *for every $x\in\Omega'$ and $\xi\in\mathbb{R}^{nN}\setminus B_2(0)$,*

$$|D_{\xi x}\mathcal{F}_m(x,\xi)| \le 2^{p-1}k_m(x)(1+|\xi|)^{p-1},$$

where $k_m\in C^\infty(\Omega')$ is a non-negative function, such that $k_m\to k$ in $L^s(\Omega')$,

Moreover, the above properties can be extended to all $\xi\in\mathbb{R}^{n\times N}$ in the following way:

($\tilde{\mathbf{F}}_\mathbf{m}\mathbf{2}$) *There is $\mu_m > 0$ such that for all $(x,\xi)\in\Omega'\times\mathbb{R}^{n\times N}$ and for all $\lambda\in\mathbb{R}^{n\times N}$*

$$\mu_m(1+|\xi|)^{p-2}|\lambda|^2 \le \langle D_{\xi\xi}\mathcal{F}_m(x,\xi)\lambda,\lambda\rangle.$$

($\tilde{\mathbf{F}}_\mathbf{m}\mathbf{3}$) *there exists $\gamma_m > 0$ such that for all $(x,\xi)\in\Omega'\times\mathbb{R}^{nN}$*

$$|D_{\xi\xi}\mathcal{F}_m(x,\xi)| \le \gamma_m(1+|\xi|)^{p-2}.$$

($\mathbf{\tilde{F}_m4}$) *There is $\Lambda_m > 0$ such that for every $x \in \Omega'$ and $\xi \in \overline{B_2(0)}$*

$$|D_{\xi x}\mathcal{F}_m(x,\xi)| \leq \Lambda_m(1+|\xi|)^{p-1}.$$

We now recall a regularity result for minimizers of functionals of the form

$$\inf_w \int_\Omega (\mathcal{F}(x,Dw) + \arctan(|w-\bar{u}|^2))\,dx$$

where $\mathcal{F}$ has standard growth conditions and smooth dependence on the x-variable, and $\bar{u}$ is fixed. In absence of the perturbation term $\arctan(|w-\bar{u}|^2)$, this regularity result is well known. We refer to [**?**] for the higher differentiability result, and to [**?**, Theorem 1.1] as far as the Lipschitz continuity of the local minimizers is concerned. In presence of the perturbation term, the proofs can be easily adapted because of the boundedness of the function $\arctan(|w-\bar{u}|^2)$ and of its derivative with respect to the variable w. More precisely, we have the following.

{acefus}

Theorem 17. *Let $\mathcal{F}:\Omega\times\mathbb{R}^{nN}\to[0,+\infty)$, $\mathcal{F}\in C^2(\Omega\times\mathbb{R}^{nN})$, and define the functional*

$$\int_\Omega \mathcal{F}(x,Dw) + \arctan(|w-\bar{u}|^2)\,dx \tag{3.46}$$ {funzionale

with $\bar{u}\in C^2(\Omega;\mathbb{R}^N)$. Assume that there exists $p\geq 2$ such that for every $x\in\Omega$ and every $\xi,\lambda\in\mathbb{R}^{nN}$,

$$c_1|\xi|^p - c_2 \leq g(x,\xi) \leq L(1+|\xi|)^p,$$

$$\nu\,(1+|\xi|)^{p-2}|\lambda|^2 \leq \langle D_{\xi\xi}g(x,\xi)\lambda,\lambda\rangle,$$

$$|D_{\xi\xi}g(x,\xi)| \leq L_1\,(1+|\xi|)^{p-2},$$

$$|D_{\xi x}g(x,\xi)| \leq K(1+|\xi|)^{p-1},$$

*with positive constants c_1, c_2, L, L_1, ν, K. Then any local minimizer v of (**??**) is in $W^{2,2}_{loc}(\Omega;\mathbb{R}^N)$ and*

$$(1+|Dv|^2)^{\frac{p-2}{2}}|D^2v|^2 \in L^1_{loc}(\Omega).$$

Moreover, if there exists $F:\Omega\times[0,+\infty)\to[0,+\infty)$ such that $\mathcal{F}(x,\xi)=F(x,|\xi|)$, then one also has $v\in W^{1,\infty}_{loc}(\Omega;\mathbb{R}^N)$.

We are now ready for proving Theorem **??**.

Proof of Theorem **??**. Let $u\in W^{1,p}_{\mathrm{loc}}(\Omega)$ be a local minimizer of the functional $\mathbb{F}(u,\Omega)$, and let $B_r\subset\Omega$ be a fixed ball. We consider the sequence of energy densities $\mathcal{F}_m(x,\xi)$ obtained after applying Proposition **??** to the integrand $\mathcal{F}$. For a standard sequence of mollifiers ρ_ε, we set $u_\varepsilon = u*\rho_\varepsilon$, $f_\varepsilon = f*\rho_\varepsilon$. We define

$$\mathbb{F}_{\varepsilon,m}(w,B_r) := \int_{B_r} \mathcal{F}_m(x,Dw) + f_\varepsilon(x)w + \arctan|w-u_\varepsilon|^2\ dx.$$

The lower semi-continuity and strict convexity of $\mathbb{F}_{\varepsilon,m}$ ensures that the minimization problem

$$\min\left\{\mathbb{F}_{\varepsilon,m}(w;B_r) : w\in u+W^{1,p}_0(B_r,\mathbb{R}^N)\right\} \tag{3.47}$$ {eq104}

has a unique solution $v_{\varepsilon,m} \in u + W_0^{1,p}(B_r, \mathbb{R}^N)$. By the growth of $\mathcal{F}_m$ (Proposition **??** , condition ($\tilde{\mathbf{F}}\mathbf{0}$)) and the minimality of $v_{\varepsilon,m}$, there exists a constant C independent of m and such that

$$\begin{aligned}\int_{B_r} |Dv_{\varepsilon,m}|^p\,dx &\le C\int_{B_r} \big(1+\mathcal{F}_m(x, Dv_{\varepsilon,m})\big)\,dx \\ &\le C\left[\mathbb{F}_{\varepsilon,m}(v_{\varepsilon,m}, B_r) + \int_{B_r} \big(1 - f_\varepsilon(x)\cdot v_{\varepsilon,m}\big)\,dx\right] \\ &\le C\left[\mathbb{F}_{\varepsilon,m}(u, B_r) + \int_{B_r} \Big(1 - f_\varepsilon(x)\cdot v_{\varepsilon,m}\Big)\,dx\right] \\ &= C\int_{B_r} \Big(1+\mathcal{F}_m(x, Du) + \arctan|u-u_\varepsilon|^2\Big)\,dx - C\int_{B_r} f_\varepsilon(x)(v_{\varepsilon,m}-u)\,dx \\ &\le C\int_{B_r} \left(1+|Du|^p + \left(\frac{\pi}{2}\right)^2\right)dx + C\int_{B_r} |f_\varepsilon(x)||v_{\varepsilon,m}-u|\,dx.\end{aligned}$$

We now use Young's inequality, and Poincaré inequality, and the previous estimate yields

$$\begin{aligned}\int_{B_r} |Dv_{\varepsilon,m}|^p\,dx &\le C\int_{B_r} \big(1+|Du|^p\big)\,dx + c_\alpha\int_{B_r} |f_\varepsilon(x)|^{p'}\,dx + \alpha\int_{B_r} |v_{\varepsilon,m}-u|^p\,dx \\ &\le C\int_{B_r} \big(1+|Du|^p\big)\,dx + c_\alpha\int_{B_r} |f_\varepsilon(x)|^{p'}\,dx + \alpha\, c_{n,p,r}\int_{B_r} |Dv_{\varepsilon,m}-Du|^p\,dx, \\ &\le C\int_{B_r} \big(1+|Du|^p\big)\,dx + c_\alpha\int_{B_r} |f_\varepsilon(x)|^{p'}\,dx + \alpha\, c_{n,p,r}\int_{B_r} |Du|^p\,dx + \alpha\, c_{n,p,r}\int_{B_r} |Dv_{\varepsilon,m}|^p\,dx,\end{aligned}$$

Choosing $\alpha = \frac{1}{2c_{n,p,r}}$, we can reabsorb the last integral in the right hand side of previous inequality by the left hand side, getting

$$\int_{B_r} |Dv_{\varepsilon,m}|^p\,dx \le C\int_{B_r} \big(1+|Du|^p\big)\,dx + C\int_{B_r} |f_\varepsilon(x)|^{p'}\,dx$$

for a constant C independent of ϵ and m and so with right hand side independent of m. Therefore, by weak compactness, we deduce that $\{v_{\epsilon,m}\}_m$ weakly converges to a function $v_\epsilon \in u + W_0^{1,p}(B_r; \mathbb{R}^N)$ as $m \to +\infty$, up to a subsequence. Set now

$$\mathbb{F}_\varepsilon(w, B_r) := \int_{B_r} \big(\mathcal{F}(x, Dw) + f_\varepsilon(x)w + \arctan|w-u_\varepsilon|^2\big)\,dx.$$

For every fixed $\epsilon > 0$, one can see that $\mathbb{F}_{\epsilon,m}$ Γ-converges to $\mathbb{F}_\epsilon$ as $m \to \infty$ (see Theorem 5.14 and Corollary 7.20 in [?]). As a consequence, v_ϵ is a minimizer of $\mathbb{F}_\epsilon$. Now, the lower semicontinuity of the L^p norm implies that

$$\int_{B_r} |Dv_\varepsilon|^p\,dx \le \liminf_{m\to\infty}\int_{B_r} |Dv_{\varepsilon,m}|^p\,dx \le C\int_{B_r} \big(1+|Du|^p\big)\,dx + c_{n,p,r}\int_{B_r} |f_\varepsilon(x)|^{p'}\,dx$$

From $p' < 2 < s$ and the strong convergence of f_ε to f in L^s, it turns out that $\int_{B_r} |f_\epsilon(x)|^{p'}\,dx \leqslant C$, with C is independent of ϵ and so

$$\int_{B_r} |Dv_\varepsilon|^p\,dx \leqslant C. \tag{3.48}$$ {bound1}

As before, by compactness there exists $v \in u + W_0^{1,p}(B_r; \mathbb{R}^N)$ such that v_ϵ weakly converges to v in $W_0^{1,p}(B_r, \mathbb{R}^N)$. Also, again by the lower semicontinuity of the norm,

$$\int_{B_r} |Dv|^p\,dx \le \lim_{\varepsilon\to 0}\int_{B_r} |Dv_\varepsilon|^p\,dx \leqslant C. \tag{3.49}$$ {bound2}

Observe that, as $\varepsilon \to 0$, the functionals $\mathbb{F}_\varepsilon$ Γ-converge to

$$\mathbb{F}_0(w, B_r) := \int_{B_r} \mathcal{F}(x, Dw) + f(x)\cdot w + \arctan|w-u|^2\,dx,$$

whence v is a minimizer of $\mathbb{F}_0$, and therefore $\mathbb{F}_0(v, B_r) \leq \mathbb{F}_0(u, B_r)$. This, together with the minimality of u, implies that

$$\mathbb{F}(u, B_r) \leq \mathbb{F}(v, B_r) \leq \mathbb{F}_0(v, B_r) \leq \mathbb{F}_0(u, B_r) = \mathbb{F}(u, B_r).$$

Hence the above inequalities are equalities, and as a consequence

$$\int_{B_r} \arctan |u - v|^2 \, dx = 0, \qquad \Rightarrow \qquad u = v \qquad \text{a.e. in } B_r.$$

The functionals $\mathbb{F}_{\varepsilon,m}$ satisfy the assumptions of Theorem **??** for every ε and m. Therefore their minimizers $v_{\varepsilon,m}$ belong to $W^{2,2}_{\text{loc}} \cap W^{1,\infty}$, and so we are legitimate to use the a priori estimate (**??**) of Theorem **??**, thus getting

$$\sup_{B_\rho} |Dv_{\varepsilon,m}| \leqslant C(1 + ||k_\varepsilon||_{L^s(B_r)} + ||f_\varepsilon||_{L^s(B_r)})^\tau \left(\int_{B_{\rho'}} (1 + |Dv_{\varepsilon,m}|^p) \, dx \right)^{\frac{1}{p}}, \tag{3.50}$$ {conv}

for every $B_\rho \subset B_{\rho'} \Subset B_r$ and for a constant C independent of ε and m. By virtue of (**??**) and (**??**), and since $k_\epsilon \to k$ and $f_\varepsilon \to f$ strongly in L^s, passing to the limit, first as $m \to +\infty$, and then as $\varepsilon \to 0$ in estimate (**??**), we conclude that

$$\sup_{B_\rho} |Du| \leqslant C(1 + \|k\|_{L^s(B_r)} + \|f\|_{L^s(B_r)})^\tau \left(\int_{B_{\rho'}} (1 + |Du|^p + |f|^s) \, dx \right)^\gamma.$$

This finishes the proof. □

4 The regularity of D^2u

{ssobolev}

In this section, we establish the integrability of second order distributional derivatives of the local minimizers of the functional $\mathbb{F}(\cdot, \Omega)$. For this, we will need the following Proposition, that is inspired by a result in [?]. The result we present here goes a bit further, as it states convergence on the set $\{|Df| > 1\}$, and not only on $\{|Df| > t\}$ for each $t > 1$ and this improvement is due to the Lipschitz regularity of the minimizers proven in Theorem **??**. This has also consequences in Theorem **??**. The precise statement is as follows.

{anto}

Proposition 18. *Let $p \geq 2$, $N \geq 1$, and let $f_k, f \in W^{1,p}(\Omega; \mathbb{R}^N)$ be given, and denote $P_k = (|Df_k| - 1)_+$. Assume that:*

(a) $f_k \rightharpoonup f$ in $W^{1,p}(\Omega; \mathbb{R}^N)$,

(b) $P_k \in L^\infty$ and

$$\|P_k\|_{L^\infty(\Omega)} \leq M \tag{4.1}$$ {uniformly}

for some M independent of k.

(c) Assume that

$$\int_\Omega \frac{P_k^p}{(1 + P_k)^2} |DP_k|^2 \, dx \leq N \tag{4.2}$$ {0.1}

for some N independent of k.

Then one has $(|Df| - 1)_+^{\frac{p}{2}+1} \in W^{1,2}(\Omega)$. Moreover, there exists a not relabeled subsequence f_k such that

$$\lim_{k \to \infty} |Df_k| = |Df| \quad \text{strongly in } L^{p+2}(\Omega \cap \{|Df| > 1\}) \ ,$$

and

$$\lim_{k \to \infty} |Df_k| = |Df| \quad \text{a.e. in } \Omega \cap \{|Df| > 1\}.$$

Proof. First, it is immediate to see that

$$\int |D(P_k^{\frac{p+2}{2}})|^2 = c(p)\int P_k^p\,|DP_k|^2$$
$$\leq c(p,M)\int \frac{P_k^p}{(1+P_k)^2}\,|DP_k|^2 \leq c(p,M,N)$$

By compactness, there exist $\varphi \in W^{1,2}$ such that $P_k^{\frac{p+2}{2}} \rightharpoonup \varphi$ in $W^{1,2}$. In particular, the convergence is strong in L^2. As a consequence, $\varphi \geq 0$ almost everywhere. Using also that $r \mapsto r^{\frac{2}{2+p}}$ is $\frac{2}{2+p}$-Hölder continuous on $[0,\infty)$, we can deduce that

$$\int |P_k - \varphi^{\frac{2}{p+2}}|^{p+2} \leq c(p)\int |P_k^{\frac{p+2}{2}} - \varphi|^2$$

and therefore $P_k \to \varphi^{\frac{2}{p+2}}$ strongly in L^{p+2}. From now on, let us denote $P = \varphi^{\frac{2}{p+2}}$, for each $t \geq 0$, consider the set

$$A(t) = \{P > t\}.$$

The integrability of P guarantees that $t \mapsto |A(t)|$ is a right-continuous functions on $[0,\infty)$. Moreover, since $\|P_k - P\|_{p+2} \to 0$ as $k \to \infty$, the convergence also occurs in measure. Therefore, recalling the definition of P_k

$$\lim_{k\to+\infty}\int |P_k - P|^{p+2}$$
$$= \lim_{k\to+\infty}\int_{\{|Df_k|\leq 1\}} |P|^{p+2} + \lim_{k\to+\infty}\int_{\{|Df_k|>1\}} ||Df_k| - 1 - P|^{p+2} = 0$$

that, in particular, implies

$$\lim_{k\to+\infty}\int_{\{|Df_k|>1\}} ||Df_k| - 1 - P|^{p+2} = 0. \tag{4.3}$$ {strong2bis}

Moreover by the convergence in measure of P_k to P, for every $t > 0$, we have

$$0 = \lim_{k\to+\infty} |\{x \in \Omega :\ \big|\big(|Df_k| - 1\big)_+ - P| \geq t\}\big|$$
$$= \lim_{k\to+\infty} |\{x \in \Omega :\ |Df_k| \leq 1 \text{ and } \big|P| \geq t\}\big|$$
$$+ \lim_{k\to+\infty} |\{x \in \Omega :\ |Df_k| > 1 \text{ and } \big||Df_k| - 1 - P| \geq t\}\big|$$

that, in particular, yields

$$0 = \lim_{k\to+\infty} |\{x \in \Omega :\ |Df_k| \leq 1 \text{ and } \big|P| \geq t\}\big| \tag{4.4}$$ {measure1}

For $t > 0$, we write

$$\int_{\{P\geq t\}} \big||Df_k| - 1 - P\big|^{p+2}$$
$$= \int_{\{|Df_k|\leq 1 \text{ and } P\geq t\}} \big||Df_k| - 1 - P\big|^{p+2} + \int_{\{|Df_k|>1 \text{ and } P\geq t\}} \big||Df_k| - 1 - P\big|^{p+2}$$
$$= I_{1,k} + I_{2,k} \tag{4.5}$$ {split}

By (**??**) we have that

$$\lim_{k\to+\infty} I_{2,k} \leq \lim_{k\to+\infty}\int_{\{|Df_k|>1\}} ||Df_k| - 1 - P|^{p+2} = 0 \tag{4.6}$$ {I2}

Since by virtue of the assumption (b), we have $||P||_\infty \leq c(M)$, we get

$$\begin{aligned}
\lim_{k\to+\infty} I_{1,k} &= \lim_{k\to+\infty} \int_{\{|Df_k|\leq 1 \text{ and } P\geq t\}} ||Df_k|-1-P|^{p+2} \\
&\leq \lim_{k\to+\infty} \int_{\{|Df_k|\leq 1 \text{ and } P\geq t\}} (2+|P|)^{p+2} \\
&\leq c(M,p) \lim_{k\to+\infty} |\{|Df_k|\leq 1 \text{ and } P\geq t\}| = 0,
\end{aligned} \tag{4.7}$$
{I1}

by the equality in (**??**). Inserting (**??**) and (**??**) in (**??**), we conclude

$$\lim_{k\to+\infty} \int_{\{P\geq t\}} \big||Df_k|-1-P\big|^{p+2} = 0 \tag{4.8}$$
{strongGG}

for every $t>0$. Note now that

$$B(0) := \{x\in\Omega \,:\, P>0\} = \bigcup_{n\in\mathbb{N}} \left\{x\in\Omega \,:\, P>\frac{1}{n}\right\} =: \bigcup_{n\in\mathbb{N}} B\left(\frac{1}{n}\right)$$

and

$$|\{x\in\Omega \,:\, P>0\}| = \lim_{n\to+\infty} \left|\left\{x\in\Omega \,:\, P>\frac{1}{n}\right\}\right|.$$

Since $B\left(\frac{1}{n}\right) \subset B(0)$, for every $n\in\mathbb{N}$,

$$\|\chi_{B(0)}-\chi_{B(1/n)}\|_{L^1(\Omega)} = \|\chi_{B(0)\setminus B(1/n)}\|_{L^1(\Omega)} = |B(0)-B(1/n)| \to 0.$$

Therefore

$$\begin{aligned}
&\int_{\{P>0\}} \big||Df_k|-1-P\big|^{p+2} \\
&= \int_{\{P>0\}} \big||Df_k|-1-P\big|^{p+2} - \int_{\{P>1/n\}} \big||Df_k|-1-P\big|^{p+2} \\
&+ \int_{\{P>1/n\}} \big||Df_k|-1-P\big|^{p+2} \\
&= \int \big||Df_k|-1-P\big|^{p+2} (\chi_{B(0)}-\chi_{B(1/n)}) \\
&+ \int_{\{P>1/n\}} \big||Df_k|-1-P\big|^{p+2} \\
&\leq C(M)\|\chi_{B(0)}-\chi_{B(1/n)}\|_{L^1(\Omega)} + \int_{\{P>1/n\}} \big||Df_k|-1-P\big|^{p+2}.
\end{aligned} \tag{4.9}$$
{split1}

Passing to the limit first as $k\to\infty$ and using (**??**) with $t=1/n$, and then as $n\to\infty$ in (**??**), we conclude that

$$\lim_{k\to\infty} \int_{\{P>0\}} \big||Df_k|-1-P\big|^{p+2} = 0 \tag{4.10}$$

From previous equality we deduce that

$$|Df_k| \to P+1 \qquad \text{strongly in} \quad L^{p+2}(\{P>0\}) \tag{4.11}$$
{stronG}

and, of course, modulo subsequences, also weakly and almost everywhere. By assumption, (f_k) is weakly convergent in $W^{1,p}(\Omega;\mathbb{R}^N)$ to f, so, by the essential uniqueness of the weak limit we conclude that

$$|Df| = P+1 \qquad\qquad \text{a.e. in } \{P>0\}. \tag{4.12}$$
{strongGGG

Therefore

$$\{P > 0\} = \{x \in \Omega \, : \, |Df| > 1\}.$$

By (**??**) and (**??**) and the equality above, there exists a subsequence (not relabeled) of f_k such that

$$|Df_k| \to_{k\to\infty} |Df| \qquad \text{strongly in } L^{p+2}(\{x \in \Omega \, : \, |Df| > 1\}). \tag{4.13}$$ {strongG4}

The other assertions follows easily. □

We can now proceed with the main result in this section. In what follows we shall use the notation

$$\mathcal{G}(t) := 1 + \int_0^t (1+s)^{\frac{p-4}{2}} s \, ds.$$

{eq134}

Theorem 19. *Let $u \in W^{1,p}(\Omega; \mathbb{R}^{n\times N}$ be a local minimizer of the functional $\mathbb{F}(\cdot, \Omega)$ in (**??**), and so that assumptions* **(F0)-(F4)** *are satisfied with $R = 1$. If $f \in L^s_{\text{loc}}(\Omega)$, then*

$$\mathcal{G}((|Du| - 1)_+) \in W^{1,2}_{\text{loc}}(\Omega) \tag{4.14}$$ {tesi}

and the following Caccioppoli type inequality holds,

$$\int_{B_\rho(x_0)} |D(\mathcal{G}((|Du| - 1)_+))|^2 \le C(1 + \|k\|_{L^s} + \|f\|_{L^s})^\tau \int_{B_r(x_0)} (1 + |Du|^p) \, dx, \tag{4.15}$$ {apriori3}

for every ball $B_r(x_0) \Subset \Omega$, every $0 < \rho < r$, for some $C = C(n, N, p, s, c_1, c_2, L, L_1, \nu, \rho, r)$ and for some $\tau = \tau(s, n) > 0$.

Let us mention that in (**??**) the term on the left hand side is equivalent to

$$\int_{B_\rho(x_0)} |D(\mathcal{G}((|Du| - 1)_+))|^2 = \int_{B_\rho(x_0)} (|Du| - 1)_+^2 \, |Du|^{p-4} \, |D^2u|^2 \, dx$$

so that the above result is, in fact, a weighted bound for D^2u with the weight $(|Du| - 1)_+^2 \, |Du|^{p-4}$.

Proof. Let $v_{\varepsilon,m}$ be the solution of the problem (**??**). The functionals $\mathbb{F}_{\varepsilon,m}$, for every ϵ and m, satisfy the assumptions of Theorem **??**. Therefore their minimizers $v_{\varepsilon,m}$ belong to $W^{2,2}_{\text{loc}} \bigcap W^{1,\infty}$ and are such that $|Dv_{\epsilon,m}|^{p-2}|D^2v_{\varepsilon,m}|^2 \in L^1$. As a consequence, we are legitimate to use the a priori estimate (**??**) of Theorem **??**, thus getting

$$\int_{B_\rho(x_0)} \frac{P_{\varepsilon,m}^2}{(1 + P_{\varepsilon,m})^2} \, |Dv_{\varepsilon,m}|^{p-2}|D^2v_{\varepsilon,m}|^2 \, dx \le C(1 + ||k_\varepsilon||_{L^s} + ||f_\varepsilon||_{L^s})^\tau \int_{B_{2R}(x_0)} (1 + |Dv_{\varepsilon,m}|^p) \, dx,$$

with a constant C independent of m and of ε. As usually, $P_{\varepsilon,m} = (|Dv_{\varepsilon,m}| - 1)_+$. Now using estimate (**??**) and the fact that $||k_\varepsilon||_{L^s} + ||f_\varepsilon||_{L^s} \to ||k||_{L^s} + ||f||_{L^s}$, we obtain

$$\int_{B_\rho(x_0)} |D(\mathcal{G}(P_{\varepsilon,m}))|^2 \, dx = \int_{B_\rho(x_0)} \frac{P_{\varepsilon,m}^2}{(1 + P_{\varepsilon,m})^2} \, |Dv_{\varepsilon,m}|^{p-2}|D^2v_{\varepsilon,m}|^2 \, \mathrm{d}x \le C \tag{4.16}$$ {weak1}

with a constant C independent of m and of ε. Furthermore, we have that

$$||P_{\varepsilon,m}||_{L^\infty(B_\rho)} \le \sup_{B_\rho} |Dv_{\varepsilon,m}| \le C,$$

with a constant C independent of m and ε. By Proposition **??**,

$$|Dv_{\varepsilon,m}| \to |Dv_\varepsilon| \text{ strongly in } \; L^p(B_\rho \cap \{|Dv_\varepsilon| > 1\}), \tag{4.17}$$ {strong1}

and

$$|Dv_{\varepsilon,m}| \to |Dv_\varepsilon| \text{ a.e. in } B_\rho \cap \{|Dv_\varepsilon| > 1\}. \tag{4.18}$$ {strong2}

If we now set $w_{\varepsilon,m} = \mathcal{G}(P_{\varepsilon,m})$, then from (**??**) we deduce that (up to a subsequence) one has $w_{\varepsilon,m} \to w_\varepsilon$ as $m \to \infty$ for some $w_\varepsilon \in W^{1,2}(B_\rho)$, with weak convergence in $W^{1,2}(B_\rho)$, strong convergence in $L^2(B_\rho)$, and a.e. convergence in B_ρ. The latter, together with (**??**), implies that $w_\varepsilon = \mathcal{G}((|Dv_\varepsilon| - 1)_+)$ almost everywhere on $B_\rho \cap \{|Dv_\varepsilon| > 1\}$. But then the lower semicontinuity of the norm implies that

$$\int_{B_\rho(x_0)} |D(\mathcal{G}((|Dv_\varepsilon| - 1)_+))|^2 \leq C, \tag{4.19}$$ {weak4}

with a constant C independent of ε. We now argue for the sequence v_ε as we did for $v_{\varepsilon,m}$, and the theorem follows . □

5 Boundary regularity

{sboundary}

In this section we show that a typical reflection method extends the previous interior estimates up to the boundary. To this end, we will proceed as in [**?**]. We will start with the following auxilliary standard result.

{eq118}

Lemma 20. *Let $B \subset \mathbb{R}^n$ be a ball centered at the origin. Set $B^+ = B \cap \{x_n > 0\}$ and $B^- = B \cap \{x_n < 0\}$, and let $f : B \longrightarrow \mathbb{R}$ be such that $f \in W^{1,p}(B^+)$ and $f \in W^{1,p}(B^-)$. If f is continuous on B, then $f \in W^{1,p}(B)$.*

This result is classical, and we omit its proof. The main result of this section states that Theorem **??** gives global bounds when Ω has nice boundary.

{globalbdd}

Theorem 21. *Suppose Ω is a bounded domain with $\partial\Omega \in C^{3,1}$. Let $\mathcal{F}$ be as in Theorem **??**, and let $u \in W^{1,p}_{\text{loc}}(\Omega)$ be a weak solution of following Neumann boundary problem*

$$\begin{cases} -\operatorname{div}(D_\xi \mathcal{F}(x, \nabla u(x))) = f, & \text{in } \Omega, \\ D_\xi \mathcal{F}(x, \nabla u(x)) \cdot \nu = 0, & \text{on } \partial\Omega, \end{cases} \tag{5.1}$$ {eq122}

for some $f \in L^s(\Omega)$, and $s > n$. Then $Du \in L^\infty(\Omega)$.

Proof. Let $x_0 \in \partial\Omega$, and let V be a neighborhood of x_0 in $\mathbb{R}^n$. We write $x = (x', x_n)$ with $x' \in \mathbb{R}^{n-1}$. We denote $B = \{|x| < 1\}$, $B^+ = \{x \in B; x_n > 0\}$. Since Ω is $C^{3,1}$, there exist a diffeomorphism

$$\psi : B^+ \to V \cap \Omega$$

which is onto, and which extends to ∂B^+ in a C^1-smooth way, with $D\psi(y) \neq 0$ for $y \in \partial B^+$. In particular, we can assume that $\psi(\partial B^+) = \partial\Omega \cap V$. Set $\mathcal{R}$ to be reflection in $\mathbb{R}^n$ with respect to the hyperplane $\{x_n = 0\}$, and $B^- = \mathcal{R}B^+$, $B = B^+ \cup B^-$. We also set $L = \partial B^+ \cap \partial B^-$. Define

$$\Psi(y) =: \begin{cases} \psi(y), & \text{if } y \in B^+, \\ \psi \circ \mathcal{R}(y), & \text{if } y \in B^-, \end{cases}$$

By construction, Ψ is continuous on B, and smooth on $B \setminus L$, with

$$D\Psi(y) =: \begin{cases} D\psi(y), & \text{if } y \in B^+, \\ D\psi(\mathcal{R}(y)) \cdot \mathcal{R}, & \text{if } y \in B^-, \end{cases}$$

In particular, the jacobian $J(y,\Psi) = \det D\Psi(y)$ is well defined on $B \setminus L$. Let us set

$$\widehat{u}(y) = u(\Psi(y)), \qquad \widehat{f}(y) = f(\Psi(y)) \cdot |J(y,\Psi)|. \tag{5.2}$$ {eq123}

Let φ be a smooth testing function with $\mathrm{supp}(\varphi) \subset (\overline{\Omega} \cap \overline{V})$. Without loss of generality, we can assume that φ vanishes on $\partial(V \cap \Omega) \setminus \partial\Omega$. Using that u solves the problem (**??**),

$$\int_\Omega \langle \mathcal{A}(x, \nabla u), \nabla\varphi \rangle \, dx = \int_\Omega f \, \varphi \, dx,$$

where $\mathcal{A}(x,\xi) = D_\xi \mathcal{F}(x,\xi)$. After the change of variables $x = \psi(y)$, this reads

$$\int_{B^+} \langle \mathcal{A}(\psi, \nabla u(\psi)), \nabla\varphi(\psi) \rangle \cdot |J\psi| \, dx = \int_{B^+} f(\psi)\varphi(\psi) \cdot |J\psi| \, dx, \tag{5.3}$$ {eq117}

and similarly, with the change of variable $x = \psi(\mathcal{R}y)$,

$$\int_{B^-} \langle \mathcal{A}(\psi(\mathcal{R}), \nabla u(\psi(\mathcal{R}))), \nabla\varphi(\psi(\mathcal{R})) \rangle \cdot |J\psi(\mathcal{R})| \, dx = \int_{B^-} f(\psi(\mathcal{R}))\varphi(\psi(\mathcal{R})) \cdot |J\psi(\mathcal{R})| \, dx. \tag{5.4}$$ {eq124}

Using the chain rule, (**??**) becomes

$$\int_{B^+} \langle \mathcal{A}\left(\Psi, \nabla(u \circ \Psi) \cdot (D\Psi)^{-1}\right), \nabla(\varphi \circ \Psi) \cdot (D\Psi)^{-1} \rangle \cdot |J\Psi| \, dx = \int_{B^+} f(\Psi)\varphi(\Psi) \cdot |J\Psi| \, dx, \tag{5.5}$$ {eq127}

while for (**??**) we get

$$\int_{B^-} \langle \mathcal{A}\left(\Psi, \nabla(u \circ \Psi) \cdot (D\Psi)^{-1}\right), \nabla(\varphi \circ \Psi) \cdot (D\Psi)^{-1} \rangle \cdot |J\Psi| \, dx = \int_{B^-} f(\Psi)\varphi(\Psi) \cdot |J\Psi| \, dx. \tag{5.6}$$ {eq128}

Putting things together, we get

$$\int_{B^+ \cup B^-} \langle \mathcal{B}(y, \nabla\widehat{u}), \nabla\widehat{\varphi} \rangle \, dx = \int_{B^+ \cup B^-} \widehat{f} \cdot \widehat{\varphi} \, dx, \tag{5.7}$$ {eq116}

where we have defined

$$\mathcal{B}(y,\xi) := \mathcal{A}\Big(\Psi(y), \xi \cdot \mathcal{M}(y)\Big) \cdot \mathcal{M}^t(y) \cdot \mathcal{J}(y), \qquad \forall x \in B, \tag{5.8}$$ {eq130.1}

as well as $\widehat{\varphi(y)} := \varphi(\Psi(y))$, $\mathcal{M}(y) = (D\Psi(y))^{-1}$ and $\mathcal{J}(x) = |J(y,\Psi)|$. It is clear that (**??**) holds not only for functions of the form $\widehat{\varphi} = \varphi(\Psi)$, but also for any function $\widehat{\varphi} \in W_0^{1,p}(B)$.

Arguing as in [**?**], we now claim that there exists a matrix function $x \mapsto \mathcal{O}(x)$, such that

$$\mathcal{O} \in C^{1,1}(B), \qquad \mathcal{O}^t \cdot \mathcal{O} = I, \tag{5.9}$$ {eq114}

and moreover

$$[D\psi(x')]^{-1} = \mathcal{R}[D\psi(x')]^{-1}\mathcal{O}(x'), \qquad \forall x' \in \partial B^+ \cap \partial B^-. \tag{5.10}$$ {eq119}

Since $D\psi$ has positive determinant, we can write $D\psi = OU$, with O orthogonal and U symmetric and positive definite. To see this, note that $O = D\psi \, U^{-1}$ and $[D\psi]^t D\psi = U^2$, therefore it suffices to choose $O = D\psi([D\psi]^t D\psi)^{-\frac{1}{2}}$. Such a choice certainly gives an O wich is orthogonal,

$$\left(D\psi([D\psi]^t D\psi)^{-\frac{1}{2}}\right)^t = \left(([D\psi]^t D\psi)^{-\frac{1}{2}}\right)^t (D\psi)^t = \left(([D\psi]^t D\psi)^{-\frac{1}{2}}\right)(D\psi)^t,$$

so that O is indeed orthogonal. Let us then set $\mathcal{O}$ to be the matrix O in the decomposition $D\psi = OU$. Proving that (**??**) holds deserves some more effort, as it depends on the precise choice of the parametrization ψ. Let us choose ψ having the following form,

$$\psi(x', x_n) = (x', g(x')) - x_n\,(\nabla g(x'), -1) \tag{5.11}$$ {eq120}

where $g \in C^{3,1}(\mathbb{R}^{n-1})$ is restricted to L. With this choice, we obtain

$$D\psi(x) = \begin{pmatrix} \mathbf{Id}_{n-1} - x_n \cdot D^2 g(x') & -\nabla g(x')^t \\ \nabla g(x') & 1 \end{pmatrix}$$

Especially, $\det D\psi(x', 0) = 1 - |\nabla g(x')|^2$. As a consequence, we have the equality

$$[D\psi]^t D\psi = ((D\psi)\mathcal{R})^2 \qquad \text{at points } x = (x', 0) \in L. \tag{5.12}$$ {eq120}

Therefore, using our choice for matrix $\mathcal{O}$, together with (**??**), we get

$$\mathcal{R}(D\psi)^{-1}\mathcal{O} = \mathcal{R}(D\psi)^{-1} \cdot D\psi \cdot ((D\psi)^t D\psi)^{-\frac{1}{2}} = \mathcal{R}(((D\psi)\mathcal{R})^2)^{-\frac{1}{2}} = (D\psi)^{-1}$$

as claimed.

We now observe that

$$\begin{aligned} \mathcal{B}(x,\xi) &= D_\xi \mathcal{F}(\Psi(x), \xi \cdot \mathcal{M}(x))\, \mathcal{M}(x)^t\, \mathcal{J}(x) \\ &= \nabla_\xi \left(\mathcal{F}(\Psi(x), \mathcal{M}(x)^t\, \xi)\, \mathcal{J}(x) \right) \\ &= \nabla_\xi \mathcal{G}(x,\xi) \end{aligned}$$

where $\mathcal{G}(x,\xi) = \mathcal{F}(\Psi(x), \mathcal{M}(x)^t\, \xi)\, \mathcal{J}(x) = F(\Psi(x), |\mathcal{M}(x)^t \xi|)\, \mathcal{J}(x)$. We now justify that $\mathcal{G}$ satisfies the assumptions **(F0)**–**(F4)** of Theorem **??**. Condition **(F0)** is clear, since

$$\begin{aligned} \frac{1}{c} \leq &\mathcal{J}(x) \leq c \\ \frac{1}{c}\,|\xi| \leq &|M(x)^t \xi| \leq c\,|\xi| \end{aligned} \tag{5.13}$$ {MJbounds}

where $c = \|M\|_\infty = \max\{\|(D\psi)^{-1}\|_\infty, \|J(\cdot,\psi)\|_\infty\}$. Concerning **(F1)**, even though one cannot say that $\mathcal{G}$ is genuinely radial, we can certainly say that $\mathcal{G}(x,\xi) = G(x, |\mathcal{M}(x)^t \xi|)$. Having in mind that

$$D_\xi(|\mathcal{M}(x)^t \xi|) = \frac{\mathcal{M}(x) \cdot \mathcal{M}(x)^t \xi}{|\mathcal{M}(x)^t \xi|}$$

and the bounds (**??**), one can show that Theorem **??** still holds (with a similar proof) if one replaces **(F1)** by $\mathcal{G}(x,\xi) = G(x, |\mathcal{M}(x)^t \xi|)$. With respect to **(F2)** and **(F3)**, one only needs to recall (**??**). Finally, we prove **(F4)**. First, at points $x \notin L$, we can clearly differentiate $D_\xi \mathcal{G}$ in x and use the chain rule to obtain

$$|D_{x\xi}\mathcal{G}(x,\xi)| \leq k(\Psi(x))\, |\mathcal{M}(x)^t \xi|^{p-1} |\mathcal{M}(x)^t| \mathcal{J}(x) + |D_\xi \mathcal{F}(\Psi(x), \mathcal{M}(x)^t \xi)\, D_x(\mathcal{M}(x)\, \mathcal{J}(x))$$

Thus, arguing separately for $x \in B^+$ and $x \in B^-$, on which Ψ is separately bilipschitz, one sees that

$$|D_{x,\xi}\mathcal{G}(x,\xi)| \leq \tilde{k}(x)|\xi|^{p-1}$$

for a function $\tilde{k} \in L^s$. Concerning the points $x_0 \in L$, we recall that $D\psi$ is continuous in B^+ up to ∂B^+.

Having in mind also that $D_\xi\mathcal{F}(x,\xi\mathcal{O}) = \mathcal{A}(x,\xi)\cdot\mathcal{O}$, we get at points $x_0 = (x_0', 0) \in L$ the following equality

$$\begin{aligned}
\lim_{x\to x_0, x\in B^+} D_\xi\mathcal{G}(x,\xi) &= \lim_{x\to x_0, x\in B^+} D_\xi\mathcal{F}\left(\psi(x), \xi\cdot\mathcal{M}(x)\right)\cdot\mathcal{M}(x)^t\cdot|\mathcal{J}(x)| \\
&= D_\xi\mathcal{F}\left(\psi(x_0), \xi\cdot\mathcal{M}(x_0)\right)\cdot\mathcal{M}(x_0)^t\cdot|\mathcal{J}(x_0)| \\
&= D_\xi\mathcal{F}\left(\psi(x_0), \xi\cdot\mathcal{M}(x_0)\right)\cdot(\mathcal{R}\,\mathcal{M}(x_0)\,\mathcal{O}(x_0))^t\cdot|\mathcal{J}(x_0)| \\
&= D_\xi\mathcal{F}\left(\psi(x_0), \xi\cdot\mathcal{M}(x_0)\right)\cdot\mathcal{O}(x_0)^t\,\mathcal{R}\,\mathcal{M}(x_0))^t\cdot|\mathcal{J}(x_0)| \\
&= D_\xi\mathcal{F}\left(\psi(x_0), \xi\cdot\mathcal{M}(x_0)\cdot\mathcal{O}(x_0)^t\right)\,\mathcal{R}\,\mathcal{M}(x_0))^t\cdot|\mathcal{J}(x_0)| \\
&= D_\xi\mathcal{F}\left(\psi(x_0), \xi\cdot\mathcal{R}\,\mathcal{M}(x_0)\right)\,\mathcal{R}\,\mathcal{M}(x_0))^t\cdot|\mathcal{J}(x_0)| \\
&= D_\xi\mathcal{F}\left(\psi(\mathcal{R}x_0), \xi\cdot\mathcal{R}\,\mathcal{M}(\mathcal{R}x_0)\right)\,\mathcal{R}\,\mathcal{M}(\mathcal{R}x_0))^t\cdot|\mathcal{J}(\mathcal{R}x_0)| \\
&= \lim_{y\to x_0, x\in B^-} D_\xi\mathcal{G}(y,\xi),
\end{aligned}$$

and so $y \mapsto D_\xi\mathcal{G}(y,\xi)$ is continuous on L. It then follows from Lemma **??** that $x \mapsto D_\xi\mathcal{G}(x,\xi)$ is $W^{1,s}$ on B. Thus **(F4)** follows.
We have just shown that $\mathcal{G}$ satisfies the assumptions of Theorem **??**. Now, since $f \in L^s(\Omega)$ implies $\widehat{f} \in L^s$ we can deduce from Theorem **??** that $|D\widehat{u}| \in L^\infty(\frac{1}{2}B)$. But using again the bilipschitz character of ψ we obtain that $|Du| \in L^\infty(\psi(\frac{1}{2}B^+))$. In particular, Du is bounded up to $\partial\Omega$. The claim follows. □

Acknowledgement. R.Giova and A.Passarelli di Napoli have been partially supported by the Gruppo Nazionale per l'Analisi Matematica, la Probabilità e le loro Applicazioni (GNAMPA) of the Istituto Nazionale di Alta Matematica (INdAM).
R. Giova has been partially supported by Università degli Studi di Napoli Parthenope through the project "sostegno alla Ricerca individuale"(2015 - 2017).

References

[1] L. Ambrosio, *Lecture notes on optimal transport problems.* Mathematical aspects of evolving interfaces (Funchal, 2000), 1–52, Lecture Notes in Math., 1812, Springer, Berlin, 2003.

[2] L. Ambrosio, G. Crippa, *Existence uniqueness stability and differentiability properties of the flow associated to weakly differentiable vector fields, in Transport equations and multi-d hyperbolic conservation laws*, in Lecture notes of the Unione Matematica Italiana (2008).

[3] P. Bousquet, L. Brasco, C. Leone, A. Verde *On the Lipschitz character of orthotropic p-harmonic functions* preprint (2017) http://cvgmt.sns.it/paper/3402/

[4] L. Brasco, *Global L^∞ gradient estimates for solutions to a certain degenerate elliptic equation.* Nonlinear Anal. 74 (2011), no. 2, 516–531.

[5] L. Brasco, G. Carlier, *On certain anisotropic elliptic equations arising in congested optimal transport: local gradient bounds*, Adv. Calc. Var., 7 (2014), 379–407.

[6] L. Brasco, G. Carlier, F. Santambrogio, *Congested traffic dynamics, weak flows and very degenerate elliptic equations* . J. Math. Pures Appl. (9) 93 (2010), no. 6, 652–671.

[7] L. Brasco, C. Leone, G. Pisante, A. Verde, *Sobolev and Lipschitz regularity for local minimizers of widely degenerate anisotropic functionals.* Nonlinear Anal. 153 (2017), 169-199

[8] L. Brasco, M. Petrache, *A continuous model of transportation revisited.* Zap. Nauchn. Sem. S.-Peterburg. Otdel. Mat. Inst. Steklov. (POMI) 411 (2013), Teoriya Predstavleni, Dinamicheskie Sistemy, Kombinatornye Metody. XXII, 5–37, 240; translation in J. Math. Sci. (N.Y.) 196 (2013), no. 2, 119–137.

[9] G. Carlier, C. Jimenez, F. Santambrogio, *Optimal transportation with traffic congestion and Wardrop equilibria.* SIAM J. Control Optim. 47 (2008), no. 3, 1330–1350.

[10] G. Carlier, F. Santambrogio, *A continuous theory of traffic congestion and Wardrop equilibria.* Journal of Mathematical Sciences, vol. 181, no.6, march, 2012, 792–804.

[11] P. Celada, G. Cupini, M. Guidorzi, *Existence and regularity of minimizers of nonconvex integrals with p-q growth,* ESAIM Control Optim. Calc. Var. 13 (2007), 343-358.

[12] A. Clop, D. Faraco, J. Mateu, J. Orobitx, X. Zhong, *Beltrami equations with coefficient in the Sobolev space* $W^{1,p}$, Publicacions Matemtiques 53 (1), 197–230.

[13] M. Colombo, A. Figalli, *Regularity results for very degenerate elliptic equations.* J. Math. Pures Appl. (9) 101 (2014), no. 1, 94–117.

[14] G. Cupini, F. Giannetti, R. Giova, A. Passarelli di Napoli, *Higher integrability for minimizers of asymptotically convex integrals with discontinuous coefficients*, Nonlinear Anal. 154 (2017) , pp. 7 - 24

[15] G. Cupini, M. Guidorzi, E. Mascolo, *Regularity of minimizers of vectorial integrals with p-q growth.* Nonlinear Anal. 54 (2003), no. 4, 591–616.

[16] G. Dal Maso,*An introduction to* Γ*-convergence*,Progr. Nonlinear Differential Equations Appl., no. 8, Birkhäuser, Boston, 1993.

[17] L. De Pascale, L.C. Evans, A. Pratelli, *Integral estimates for transport densities.* Bull. London Math. Soc. 36 (2004), no. 3, 383–395.

[18] L. De Pascale, A. Pratelli, *Sharp summability for Monge transport density via interpolation.* ESAIM Control Optim. Calc. Var. 10 (2004), no. 4, 549–552.

[19] R. J. DiPerna, P.L. Lions, *Ordinary differential equations, transport theory and Sobolev spaces.* Invent. Math. 98 (1989), no. 3, 511–547.

[20] M. Eleuteri, P. Marcellini, E. Mascolo, *Lipschitz estimates for systems with ellipticity conditions at infinity,* Ann. Mat. Pura Appl., 195 (2016), 1575–1603.

[21] M. Feldman, R.J. McCann, *Uniqueness and transport density in Monge's mass transportation problem.* Calc. Var. Partial Differential Equations 15 (2002), no. 1, 81–113.

[22] I. Fonseca, Irene, N.Fusco, P. Marcellini, *An existence result for a nonconvex variational problem via regularity.* ESAIM Control Optim. Calc. Var. 7 (2002), 69–95.

[23] M. Giaquinta, G. Modica, *Remarks on the regularity of the minimizers of certain degenerate functionals*, Manuscripta Math. 57 (1986) 55-99.

[24] R. Giova, *Higher differentiability for* n*-harmonic systems with Sobolev coefficients* J. Diff. Equations 259 (2015) 5667-5687.

[25] R. Giova, A. Passarelli di Napoli, *Higher differentiability of a priori bounded minimizers of convex variational integrals with discontinuous coefficients* - Advances in Calculus of Variations (2017) DOI: doi.org/10.1515/acv-2016-0059

[26] E. Giusti. *Direct methods in the calculus of variations.* World Scientific, 2003.

[27] A. Passarelli di Napoli, *Higher differentiability of minimizers of variational integrals with Sobolev coefficients.* Adv. Calc. Var. 7 (2014), no. 1, 59-89.

[28] A. Passarelli di Napoli, *Higher differentiability of solutions of elliptic systems with Sobolev coefficientes: the case* $p = n = 2$. Potential Anal. 41 (2014), no. 3, 715-735

[29] A. Passarelli di Napoli, *Regularity results for non-autonomous variational integrals with discontinuous coefficients* Atti Accad. Naz. Lincei Rend. Lincei Mat. Appl. 26 (2015), no. 4, 475–496

[30] F. Santambrogio, V. Vespri, *Continuity in two dimensions for a very degenerate elliptic equation.* Nonlinear Anal. 73 (2010), no. 12, 3832–3841.

[31] J.G. Wardrop, *Some theoretical aspects of road traffic research*, Proc. Inst. Civ. Eng. 2, no.2, 325–378, 1952.

A. Clop, F. Hatami
Departament de Matemàtiques,
Facultat de Ciències, Campus de la U.A.B.
08193-Bellaterra (CATALONIA)
E-mail address: albertcp@mat.uab.cat, fhatami@mat.uab.cat

R. Giova
Università degli Studi di Napoli "Parthenope"
Palazzo Pacanowsky - Via Generale Parisi, 13
80132 Napoli, Italy
E-mail address: raffaella.giova@uniparthenope.it

A. Passarelli di Napoli
Università degli Studi di Napoli "Federico II"
Dipartimento di Mat. e Appl. "R. Caccioppoli"
Via Cintia 80126 Napoli, Italy
E-mail address: antpassa@unina.it