The Serre-Swan theorem for normed modules

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Abstract

The aim of this note is to analyse the structure of the L^0-normed L^0-modules over a metric measure space. These are a tool that has been introduced by N. Gigli to develop a differential calculus on spaces verifying the Riemannian Curvature Dimension condition. More precisely, we discuss under which conditions an L^0-normed L^0-module can be viewed as the space of sections of a suitable measurable Banach bundle and in which sense such correspondence can be actually made into an equivalence of categories.

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Introduction

The last few years have seen an increasing interest in the study of the class of metric measure spaces that satisfy the so-called $\mathsf{CD}(K,\infty)$ condition [17, 24, 25], which provides a synthetic notion of having Ricci curvature bounded from below by some constant $K \in \mathbb{R}$. A reinforcement of such condition, which rules out the Finsler geometries, has been introduced in [1, 7], where the definition of $\mathsf{RCD}(K,\infty)$ space appeared. We refer to the surveys [28, 29] for an overview of this topic and a detailed list of references.

An important contribution to the vast literature devoted to this subject is given by the paper [6], in which the author proposed a notion of differential structure, precisely tailored for the family of RCD spaces. More in detail, it is possible to develop a first-order calculus on any abstract metric measure space, while a second-order one can be built only in presence of a lower Ricci curvature bound. An object that plays a fundamental role in such construction is the concept of L^0-normed L^0-module, which we are now going to describe.

Let $(\mathrm{X},\mathsf{d},\mathfrak{m})$ be any given metric measure space. We denote by $L^0(\mathfrak{m})$ the space of all real-valued Borel functions on X (up to $\mathfrak{m}$-a.e. equality). Then an $L^0(\mathfrak{m})$-normed $L^0(\mathfrak{m})$-module is an algebraic module $\mathscr{M}$ over the commutative ring $L^0(\mathfrak{m})$, endowed with a pointwise norm operator $|\cdot| : \mathscr{M} \to L^0(\mathfrak{m})$ that is compatible with the module structure, in a suitable sense (cf. Definition 1.11 below). This terminology has been introduced by Gigli in [6] and further investigated in [8]; it represents a variant of the concept of L^∞-module, due to Weaver [30,31], who was in turn inspired by the works of Sauvageot [21, 22]. An analogue of the L^0-normed L^0-modules is given by the 'randomly normed spaces', for whose discussion we refer to [14].

A key example of L^0-normed L^0-module, which actually served as a motivation for the introduction of this kind of structure, is the space $L^0(TM)$ of measurable vector fields on a given Riemannian manifold M. An important observation is that the module $L^0(TM)$ consists of the (measurable) sections of a suitable vector bundle over M, more specifically of the tangent bundle TM. With this remark in mind, a natural question arises:

Is it possible to view any 'locally finitely-generated' L^0-normed
L^0-module as the space of sections of a suitable vector bundle?

The purpose of the present paper is to address such problem. First of all, we propose a notion of measurable Banach bundle over $(\mathrm{X},\mathsf{d},\mathfrak{m})$ having finite-dimensional fibers; cf. Definition 2.1. It turns out that the set of measurable sections of any such bundle inherits a natural structure of $L^0(\mathfrak{m})$-normed $L^0(\mathfrak{m})$-module that is proper, meaning that it is 'locally finitely-generated' in the sense of Definition 1.9. Then our main result (i.e. Theorem 3.4) can be informally stated in the following way:

The category of measurable Banach bundles over $(\mathrm{X},\mathsf{d},\mathfrak{m})$ is
equivalent to that of proper $L^0(\mathfrak{m})$-normed $L^0(\mathfrak{m})$-modules.

It is worth to study the class of proper modules since they actually occur in many interesting situations – for instance, the tangent module $L^0(T\mathrm{X})$ associated to an $\mathsf{RCD}(K,N)$ space, described in the last paragraph of this introduction, is always proper. We shall refer to our result as the 'Serre-Swan theorem for normed modules', the reason being that its statement is fully analogous to that of a classical theorem, whose two original formulations are due to Serre [23] and Swan [26]. Among the several versions of this theorem one can find in the literature, the one that is more similar in spirit to ours is that for smooth Riemannian manifolds [20]. Such result correlates the smooth vector bundles over a connected Riemannian manifold M with the finitely-generated projective modules over the ring $C^\infty(M)$. In this regard, some further details will be provided in Appendix A, where we will also make a comparison with the present paper.

We conclude this introduction by mentioning some special instances of our result that already appeared in previous works. A structural characterisation of the Hilbert modules, which are L^0-normed L^0-modules whose pointwise norm satisfies a pointwise parallelogram identity, can be found in [6, Theorem 1.4.11]. A refinement of such result for a specific module over finite-dimensional RCD spaces, which we now briefly describe, has been proved in [10].

When working with the differential structure of general metric measure spaces, an essential role is played by the tangent module $L^0(T\mathrm{X})$, whose construction is based upon the well-established theory of the Sobolev spaces $W^{1,2}(\mathrm{X},\mathsf{d},\mathfrak{m})$; we refer to [6, Definitions 2.2.1/2.3.1] for its axiomatisation (actually, the object considered therein is the $L^2(\mathfrak{m})$-normed $L^\infty(\mathfrak{m})$-module $L^2(T\mathrm{X})$, whose relation with the module $L^0(T\mathrm{X})$ will be depicted in Appendix B). Nevertheless, the tangent module $L^0(T\mathrm{X})$ might give no geometric information about the underlying space X (e.g., if there are no non-constant absolutely continuous curves in X, then the module $L^0(T\mathrm{X})$ is trivial). This is not the case when we additionally assume a lower bound on the Ricci curvature: Mondino and Naber showed in [19] that the rescalings around $\mathfrak{m}$-a.e. point x of a finite-dimensional RCD space X converge in the pointed-measured-Gromov-Hausdorff topology to the Euclidean space of dimension $k(x) \leq N$. By 'glueing together' these Euclidean fibers, one obtains the Gromov-Hausdorff tangent bundle $T_{\mathrm{GH}}\mathrm{X}$ (as done in [10, Section 4]), which has – a priori – nothing to do with the purely analytical tangent module $L^0(T\mathrm{X})$. However, by relying upon some rectifiability properties of the RCD spaces [5, 9, 16, 19], it is possible to prove (cf. [10, Theorem 5.1]) that $L^0(T\mathrm{X})$ is isometrically isomorphic to the space of measurable sections of $T_{\mathrm{GH}}\mathrm{X}$.

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1 The language of normed modules

1.1 L^0-modules

Let $\mathbb{X} = (\mathrm{X}, \mathsf{d}, \mathfrak{m})$ be a *metric measure space*, which for our purposes means that

$$\begin{aligned} &(\mathrm{X}, \mathsf{d}) \quad \text{is a complete and separable metric space,} \\ &\mathfrak{m} \neq 0 \quad \text{is a non-negative Radon measure on X.} \end{aligned} \tag{1.1}$$

The space $\mathbb{X}$ will remain fixed for the whole paper.

We denote by $L^0(\mathfrak{m})$ the space of (equivalence classes up to $\mathfrak{m}$-a.e. equality of) Borel functions from X to $\mathbb{R}$. Such space is a topological vector space when endowed with the complete and separable distance $\mathsf{d}_{L^0(\mathfrak{m})}$, defined by

$$\mathsf{d}_{L^0(\mathfrak{m})}(f, g) := \inf_{\delta>0} \Big[\delta + \mathfrak{m}\big(\{|f-g| > \delta\}\big)\Big] \quad \text{for every } f, g \in L^0(\mathfrak{m}), \tag{1.2}$$

which metrizes the convergence in measure; see [3] for the details. From an algebraic point of view, $L^0(\mathfrak{m})$ has a natural structure of commutative topological ring, whose multiplicative identity is given by (the equivalence class of) the function identically equal to 1.

Given any Borel set $A \subseteq \mathrm{X}$, it holds that (χ_A), i.e. the ideal of $L^0(\mathfrak{m})$ generated by χ_A, can be naturally identified with $L^0(\mathfrak{m}_{|A})$ and that the quotient ring $L^0(\mathfrak{m})/(\chi_A)$ is isomorphic to $(\chi_{\mathrm{X}\setminus A})$. Moreover, if $(A_i)_{i=1}^n$ is a family of pairwise disjoint Borel subsets of X, then

$$(\chi_A) \cong (\chi_{A_1}) \oplus \ldots \oplus (\chi_{A_n}), \qquad \text{where we set } A := A_1 \cup \ldots \cup A_n. \tag{1.3}$$

In particular, it holds that $(\chi_A) \oplus (\chi_{\mathrm{X}\setminus A}) \cong L^0(\mathfrak{m})$ for every $A \subseteq \mathrm{X}$ Borel.

We now recall some general terminology about modules over commutative rings: given any module M over a commutative ring R, given a set $S \subseteq M$ and denoted by $\Pi : \bigoplus_{v\in S} R \to M$ the map $(r_v)_{v\in S} \mapsto \sum_{v\in S} r_v \cdot v$, we say that

- S *generates* M provided the map Π is surjective,
- S is *independent* provided the map Π is injective,
- S is a *basis* of M provided the map Π is bijective.

An R-module M is *finitely-generated* provided it is generated by a finite set $\{v_1, \ldots, v_n\} \subseteq M$. Moreover, any R-module M is said to be *free* provided it admits a basis.

We also recall that M is *projective as R-module* if it satisfies the following property: given two modules N, P over R, a surjective module homomorphism $f : N \to P$ and a module homomorphism $g : M \to P$, there exists a module homomorphism $h : M \to N$ such that

$$\begin{tikzcd} & M \arrow[dl, "h"'] \arrow[d, "g"] \\ N \arrow[r, "f"'] & P \end{tikzcd} \tag{1.4}$$

is a commutative diagram. It turns out that M is projective as R-module if and only if there exists a module Q over R such that $M \oplus Q$ is a free R-module; cf. [15, Theorem 3.4].

Hereafter, we shall focus our attention on modules M over the commutative ring $L^0(\mathfrak{m})$. We start by fixing some notation: given any Borel subset A of X, let us define

$$M_{|A} := \chi_A \cdot M = \big\{\chi_A \cdot v \,\big|\, v \in M\big\}. \tag{1.5}$$

It turns out that $M_{|A}$ can be viewed as a module over the ring $L^0(\mathfrak{m}_{|A}) \sim (\chi_A)$.

Definition 1.1 *Let M be an $L^0(\mathfrak{m})$-module. Let $A \subseteq \mathrm{X}$ be a Borel set such that $\mathfrak{m}(A) > 0$. Then some elements $v_1, \ldots, v_n \in M$ form a* local basis *on A if $\chi_A \cdot v_1, \ldots, \chi_A \cdot v_n \in M_{|A}$ is a basis for the $L^0(\mathfrak{m}_{|A})$-module $M_{|A}$. In this case, we say that M has* dimension n *on A. Moreover, we say that M has dimension 0 on A provided $M_{|A} = \{0\}$.*

Notice that the previous notion of dimension is well-posed, because any two bases of $M_{|A}$ must have the same cardinality, as a consequence of the fact that $L^0(\mathfrak{m}_{|A})$ is a non-trivial commutative ring; see for instance [4, Theorem 2.6].

Remark 1.2 Any $L^0(\mathfrak{m})$-module M inherits a natural structure of $\mathbb{R}$-linear space, as granted by the fact that the field $\mathbb{R}$ is (isomorphic to) a subring of $L^0(\mathfrak{m})$. ■

Definition 1.3 (Dimensional decomposition) *Let M be an $L^0(\mathfrak{m})$-module. Then a Borel partition $(E_n)_{n\in\mathbb{N}\cup\{\infty\}}$ of X is said to be a* dimensional decomposition *of M provided*

i) *M has dimension n on E_n for every $n \in \mathbb{N}$ with $\mathfrak{m}(E_n) > 0$,*

ii) *M does not admit any finite basis on any Borel set $A \subseteq E_\infty$ with $\mathfrak{m}(A) > 0$.*

The dimensional decomposition, whenever it exists, is unique up to $\mathfrak{m}$-a.e. equality: i.e. given any other sequence $(F_n)_{n\in\mathbb{N}\cup\{\infty\}}$ satisfying the same properties it holds that $\mathfrak{m}(E_n \Delta F_n) = 0$ for every $n \in \mathbb{N} \cup \{\infty\}$.

Theorem 1.4 *Let M be a finitely-generated $L^0(\mathfrak{m})$-module. Then M admits a dimensional decomposition $E_0, \ldots, E_n$.*

The previous result is taken from [13, Theorem 1.1]. As a consequence, we have that:

Proposition 1.5 *Let M be a finitely-generated $L^0(\mathfrak{m})$-module. Then M is projective as a module over $L^0(\mathfrak{m})$.*

Proof. We know from Theorem 1.4 that M admits a dimensional decomposition $E_0, \ldots, E_n$. For every $k = 1, \ldots, n$, let us choose a local basis $v_1^k, \ldots, v_k^k \in M_{|E_k}$ for $M_{|E_k}$. Define M' as the $L^0(\mathfrak{m})$-module given by

$$M' := \bigoplus_{k=1}^n \underbrace{(\chi_{E_k}) \oplus \ldots \oplus (\chi_{E_k})}_{k \text{ times}}.$$

Then we denote by $\Phi : M' \to M$ the following map:

$$\Phi\big((f_i^k)_{1\leq i\leq k\leq n}\big) := \sum_{k=1}^{n}\sum_{i=1}^{k} f_i^k \cdot v_i^k \quad \text{for every } (f_i^k)_{1\leq i\leq k\leq n} = (f_1^1, f_1^2, f_2^2, \ldots, f_1^n, \ldots, f_n^n) \in M'.$$

Hence Φ is a module isomorphism, so that it suffices to prove that M' is projective. Call

$$Q := \bigoplus_{k=1}^{n} \underbrace{(\chi_{\mathrm{X}\setminus E_k}) \oplus \ldots \oplus (\chi_{\mathrm{X}\setminus E_k})}_{k \text{ times}}.$$

It follows from (1.3) that $M' \oplus Q \cong \bigoplus_{k=1}^{n} L^0(\mathfrak{m})^k$, which is a free $L^0(\mathfrak{m})$-module. Therefore one has that M' (and accordingly also M) is projective, as required. □

We would also like to build a dimensional decomposition on $L^0(\mathfrak{m})$-modules that are not necessarily finitely-generated. To do so, we need to endow such modules with some additional topological structures. For this reason, we introduce the following definitions:

Definition 1.6 (Locality/glueing) *Let M be an $L^0(\mathfrak{m})$-module. Then we say that*

i) *M has the* locality property *if for any $v \in M$ and any sequence $(A_n)_{n\in\mathbb{N}}$ of Borel subsets of X it holds that*

$$\chi_{A_n} \cdot v = 0 \quad \textit{for every } n \in \mathbb{N} \quad \Longrightarrow \quad \chi_{\bigcup_{n\in\mathbb{N}} A_n} \cdot v = 0. \tag{1.6}$$

ii) *M has the* glueing property *if for any $(v_n)_{n\in\mathbb{N}} \subseteq M$ and any sequence $(A_n)_{n\in\mathbb{N}}$ of pairwise disjoint Borel subsets of X there exists an element $v \in M$ such that*

$$\chi_{A_n} \cdot v = \chi_{A_n} \cdot v_n \quad \textit{for every } n \in \mathbb{N}. \tag{1.7}$$

As one might expect, neither locality nor glueing are in general granted on (algebraic) modules over the ring $L^0(\mathfrak{m})$. Counterexamples can be easily built by suitably adapting the arguments contained in Example 1.2.4 and Example 1.2.5, respectively, of [6].

Remark 1.7 It directly follows from Theorem 1.4 that any finitely-generated $L^0(\mathfrak{m})$-module has both the locality property and the glueing property. ■

Theorem 1.8 *Let M be an $L^0(\mathfrak{m})$-module with the locality property and the glueing property. Then M admits a dimensional decomposition $(E_n)_{n\in\mathbb{N}\cup\{\infty\}}$.*

Proof. Let $n \in \mathbb{N}$ be fixed. We define the family $\mathcal{F}_n$ as

$$\mathcal{F}_n := \big\{ A \subseteq \mathrm{X} \text{ Borel} \bigm| M \text{ has dimension } n \text{ on } A \big\}.$$

Then we denote by E_n the $\mathfrak{m}$-essential union of the elements of $\mathcal{F}_n$. Since $\mathcal{F}_n$ is closed under taking subsets, we can write E_n as $\bigcup_{i\in\mathbb{N}} A_i$ for some sequence $(A_i)_{i\in\mathbb{N}} \subseteq \mathcal{F}_n$ of pairwise disjoint sets. For any $i \in \mathbb{N}$, choose a local basis $v_1^i, \ldots, v_n^i \in M$ for $M_{|_{A_i}}$ on A_i. Hence take those elements $v_1, \ldots, v_n \in M_{|_{E_n}}$ satisfying $\chi_{A_i} \cdot v_j = \chi_{A_i} \cdot v_j^i$ for all $i \in \mathbb{N}$ and $j = 1, \ldots, n$. We now show that $v_1, \ldots, v_n$ is a local basis for M on E_n:

- Let $v \in M_{|E_n}$ be arbitrary. For every $i \in \mathbb{N}$, there exist functions $f_1^i, \ldots, f_n^i \in L^0(\mathfrak{m}_{|A_i})$ such that $\chi_{A_i} \cdot v = \sum_{j=1}^n f_j^i \cdot v_j^i$. Call $f_j := \sum_{i\in\mathbb{N}} f_j^i \in L^0(\mathfrak{m}_{|E_n})$ for any $j = 1, \ldots, n$. Since $\chi_{A_i} \cdot \big(\sum_{j=1}^n f_j \cdot v_j\big) = \sum_{j=1}^n f_j^i \cdot v_j^i = \chi_{A_i} \cdot v$ is satisfied for every $i \in \mathbb{N}$, it holds that $v = \sum_{j=1}^n f_j \cdot v_j$ by locality property, proving that $v_1, \ldots, v_n$ generate M on E_n.

- Suppose that $\sum_{j=1}^n f_j \cdot v_j = 0$ for some $f_1, \ldots, f_n \in L^0(\mathfrak{m}_{|E_n})$. In particular, we have that $\sum_{j=1}^n (\chi_{A_i} f_j) \cdot v_j^i = \chi_{A_i} \cdot \big(\sum_{j=1}^n f_j \cdot v_j\big) = 0$ for all $i \in \mathbb{N}$, whence $\chi_{A_i} f_j = 0$ holds for any $i \in \mathbb{N}$ and $j = 1, \ldots, n$. This grants that $f_1, \ldots, f_n = 0$, so that $v_1, \ldots, v_n$ are independent on E_n.

Therefore M has dimension n on E_n. Now let us define $E_\infty := \mathrm{X} \setminus \bigcup_{n\in\mathbb{N}} E_n$. It only remains to show item ii). We argue by contradiction: suppose that M is finitely-generated on some Borel set $A \subseteq E_\infty$ of positive $\mathfrak{m}$-measure. Let $v_1, \ldots, v_n$ be generators of $M_{|A}$. Since $A \cap E_n = \emptyset$, the elements $v_1, \ldots, v_n$ cannot form a basis for $M_{|A}$, then there exist $i \in \{1, \ldots, n\}$ and $A_1 \subseteq A$ Borel with $\mathfrak{m}(A_1) > 0$ such that $\chi_{A_1} \cdot v_i$ can be written as an $L^0(\mathfrak{m}_{|A_1})$-linear combination of the $(\chi_{A_1} \cdot v_j)$'s with $j \neq i$. Given that $A_1 \cap E_{n-1} = \emptyset$, we have that $\{v_j : j \neq i\}$ cannot be a basis for $M_{|A_1}$, and so on. By repeating the same argument finitely many times, we finally obtain a Borel set $A_n \subseteq A_1$ that is not $\mathfrak{m}$-negligible and that satisfies $M_{|A_n} = \{0\}$. This cannot hold because $A \cap E_0 = \emptyset$, thus leading to a contradiction. This proves ii). $\square$

In the sequel, we shall mainly focus on the following class of $L^0(\mathfrak{m})$-modules, which strictly contains all the finitely-generated $L^0(\mathfrak{m})$-modules.

Definition 1.9 (Proper modules) *Let M be an $L^0(\mathfrak{m})$-module having the locality property and the glueing property. Denote by $(E_n)_{n\in\mathbb{N}\cup\{\infty\}}$ its dimensional decomposition. Then M is said to be* proper *provided $\mathfrak{m}(E_\infty) = 0$.*

The following result shows that in the category of the $L^0(\mathfrak{m})$-modules having both the locality and the glueing property, any proper $L^0(\mathfrak{m})$-module M is projective. Nevertheless, any such module M needs not be projective as $L^0(\mathfrak{m})$-module.

Proposition 1.10 *Let M be a proper $L^0(\mathfrak{m})$-module. Consider any two $L^0(\mathfrak{m})$-modules N, P with the locality and the glueing property, a surjective module homomorphism $f : N \to P$ and a module homomorphism $g : M \to P$. Then there exists a module homomorphism $h : M \to N$ such that $f \circ h = g$.*

Proof. Fix a dimensional decomposition $(E_n)_{n\in\mathbb{N}}$ of M. Given any $n \in \mathbb{N} \setminus \{0\}$, choose a local basis $v_1^n, \ldots, v_n^n \in M_{|E_n}$ for M on E_n. Since the map f is surjective, for any $n \geq i \geq 1$ there exists $w_i^n \in N$ such that $f(w_i^n) = g(v_i^n)$. Now let $v \in M$ be fixed. Then there is a unique family of functions $(\lambda_i^n)_{n\geq i\geq 1} \subseteq L^0(\mathfrak{m})$ such that each λ_i^n is concentrated on E_n and

$$\chi_{E_n} \cdot v = \sum_{i=1}^n \lambda_i^n \cdot v_i^n \quad \text{for every } n \in \mathbb{N} \setminus \{0\}. \tag{1.8}$$

Since the module N has the glueing property, we know that there exists an element $h(v) \in N$ such that $\chi_{E_n} \cdot h(v) = \sum_{i=1}^n \lambda_i^n \cdot w_i^n$ for every $n \geq 1$. Such element $h(v)$ is also uniquely determined because N has the locality property, therefore we defined a map $h : M \to N$. It follows from its very construction that h is a homomorphism of $L^0(\mathfrak{m})$-modules, whence it only remains to prove that $f \circ h = g$. To this aim, take any $v \in M$ and choose those $(\lambda_i^n)_{n \geq i \geq 1}$ that satisfy (1.8). Then for all $n \geq 1$ we have that

$$\begin{aligned}\chi_{E_n} \cdot (f \circ h)(v) &= f\big(\chi_{E_n} \cdot h(v)\big) = f\left(\sum_{i=1}^n \lambda_i^n \cdot w_i^n\right) = \sum_{i=1}^n \lambda_i^n \cdot f(w_i^n) = \sum_{i=1}^n \lambda_i^n \cdot g(v_i^n) \\ &= \chi_{E_n} \cdot g(v),\end{aligned}$$

whence accordingly $(f \circ h)(v) = g(v)$ by the locality property of P, as required. □

1.2 L^0-normed L^0-modules

The notion of $L^0(\mathfrak{m})$-normed $L^0(\mathfrak{m})$-module has been introduced in [6] and then further investigated in [8], with the aim of building a differential structure over $\mathbb{X}$. We briefly recall here its definition, taken from [8, Definition 1.6]:

Definition 1.11 (L^0-normed L^0-module) *A topological $L^0(\mathfrak{m})$-module $(\mathscr{M}, +, \cdot, \tau)$ is said to be* $L^0(\mathfrak{m})$-normed *provided it is endowed with an operator $|\cdot| : \mathscr{M} \to L^0(\mathfrak{m})$, called* pointwise norm*, which satisfies the following properties:*

i) *One has $|v| \geq 0$ and $|f \cdot v| = |f||v|$ in the $\mathfrak{m}$-a.e. sense for every $v \in \mathscr{M}$ and $f \in L^0(\mathfrak{m})$.*

ii) *Given a Borel probability measure $\mathfrak{m}'$ on X with $\mathfrak{m} \ll \mathfrak{m}' \ll \mathfrak{m}$, we have that*

$$\mathsf{d}_{\mathscr{M}}(v, w) := \int |v - w| \wedge 1 \, \mathrm{d}\mathfrak{m}' \quad \text{for every } v, w \in \mathscr{M} \tag{1.9}$$

is a complete distance on $\mathscr{M}$ that induces the topology τ.

The particular choice of the measure $\mathfrak{m}'$ could change the distance $\mathsf{d}_{\mathscr{M}}$, but does not affect neither the completeness of $\mathsf{d}_{\mathscr{M}}$ nor its induced topology τ.

Remark 1.12 It follows from the definition of $L^0(\mathfrak{m})$-normed $L^0(\mathfrak{m})$-module that the multiplication by L^0-functions $\cdot : L^0(\mathfrak{m}) \times \mathscr{M} \to \mathscr{M}$ and the pointwise norm $|\cdot| : \mathscr{M} \to L^0(\mathfrak{m})$ are continuous operators. ■

Fix any two $L^0(\mathfrak{m})$-normed $L^0(\mathfrak{m})$-modules $\mathscr{M}$ and $\mathscr{N}$. A *module morphism* $\Phi : \mathscr{M} \to \mathscr{N}$ is any $L^0(\mathfrak{m})$-linear operator such that $\big|\Phi(v)\big| \leq |v|$ holds $\mathfrak{m}$-a.e. for every $v \in \mathscr{M}$. The family of all module morphisms from $\mathscr{M}$ to $\mathscr{N}$ will be denoted by $\mathrm{Mor}(\mathscr{M}, \mathscr{N})$.

Lemma 1.13 *Let $\mathscr{M}$ be an $L^0(\mathfrak{m})$-normed $L^0(\mathfrak{m})$-module. Then $\mathscr{M}$ has both the locality property and the glueing property.*

Proof. Locality. Consider any $v \in \mathscr{M}$ and any sequence $(A_n)_n$ of Borel subsets of X such that $\chi_{A_n} \cdot v = 0$ for every $n \in \mathbb{N}$. This means that $\chi_{A_n}|v| = |\chi_{A_n} \cdot v| = 0$ holds $\mathfrak{m}$-a.e. for any $n \in \mathbb{N}$. Let us call $A := \bigcup_n A_n$. Therefore $|\chi_A \cdot v| = \chi_A \, |v| \leq \sum_n \chi_{A_n}|v| = 0$ is satisfied

$\mathfrak{m}$-a.e. in X, showing that $\chi_A \cdot v = 0$. This grants that $\mathscr{M}$ has the locality property.
GLUEING. Let $(v_n)_n \subseteq \mathscr{M}$ and let $(A_n)_n$ be a sequence of pairwise disjoint Borel sets in X. Let us define $w_n := \sum_{k=0}^{n} \chi_{A_k} \cdot v_k$ for every $n \in \mathbb{N}$. Given that $\sum_{k=0}^{\infty} \mathfrak{m}'(A_k) \leq 1$, we know that $\sum_{k \geq n} \mathfrak{m}'(A_k) \to 0$ as $n \to \infty$, whence accordingly

$$\varlimsup_{n,m\to\infty} \mathsf{d}_{\mathscr{M}}(w_n, w_m) \overset{(1.9)}{=} \varlimsup_{n,m\to\infty} \sum_{k=n\wedge m+1}^{n\vee m} \int_{A_k} |v_k| \wedge 1 \, \mathrm{d}\mathfrak{m}' \leq \varlimsup_{n,m\to\infty} \sum_{k>n\wedge m} \mathfrak{m}'(A_k) = 0,$$

which ensures that the sequence $(w_n)_n$ is $\mathsf{d}_{\mathscr{M}}$-Cauchy. Call $v \in \mathscr{M}$ its limit. Given any $k \in \mathbb{N}$, it holds that $\chi_{A_k} \cdot w_n = \chi_{A_k} \cdot v_k$ for all $n \geq k$, whence by passing to the limit as $n \to \infty$ we get that $\chi_{A_k} \cdot v = \chi_{A_k} \cdot v_k$. This shows that $\mathscr{M}$ satisfies the glueing property. □

Remark 1.14 Any proper $L^0(\mathfrak{m})$-normed $L^0(\mathfrak{m})$-module $\mathscr{M}$ is separable.

Indeed, call $(E_n)_{n\in\mathbb{N}}$ the dimensional decomposition of $\mathscr{M}$. For any $n \in \mathbb{N}$, choose a local basis $v_1^n, \ldots, v_n^n \in \mathscr{M}|_{E_n}$ for $\mathscr{M}|_{E_n}$. Fix a countable dense subset D of $L^0(\mathfrak{m})$. Then the set

$$\left\{ \sum_{n=1}^{\infty} \sum_{i=1}^{n} f_i^n \cdot v_i^n \;\middle|\; (f_i^n)_{1\leq i\leq n} \subseteq D \right\} \subseteq \mathscr{M},$$

which is countable by construction, is dense in $\mathscr{M}$ by Remark 1.12. ■

Definition 1.15 (The category of proper L^0-normed L^0-modules) *The category having the proper $L^0(\mathfrak{m})$-normed $L^0(\mathfrak{m})$-modules as objects and the module morphisms as arrows is denoted by* $\mathbf{NMod}_{\mathrm{pr}}(\mathbb{X})$.

A classical reference for the language of categories we shall make us of is given by [18].

2 The language of measurable Banach bundles

2.1 Measurable Banach bundles

The aim of this section is to propose a notion of measurable Banach bundle, or briefly MBB, over the given metric measure space $\mathbb{X} = (\mathrm{X}, \mathsf{d}, \mathfrak{m})$. An alternative definition of MBB, which does not perfectly fit into our framework, can be found in [10].

Definition 2.1 (MBB) *We define a* measurable Banach bundle *over the space $\mathbb{X}$ as any quadruplet $\overline{\mathbb{T}} = (T, \underline{E}, \pi, \mathsf{n})$, where*

i) *$\underline{E} = (E_n)_{n\in\mathbb{N}}$ is a Borel partition of* X,

ii) *the set $T := \bigsqcup_{n\in\mathbb{N}} E_n \times \mathbb{R}^n$ is called* total space *and is always implicitly endowed with the σ-algebra $\bigcap_{n\in\mathbb{N}} (\iota_n)_* \mathscr{B}(E_n \times \mathbb{R}^n)$, where $\iota_n : E_n \times \mathbb{R}^n \hookrightarrow T$ denotes the inclusion map for every $n \in \mathbb{N}$,*

iii) *the map sending any element* $(x, v) \in T$ *to its base point* $x \in \mathrm{X}$ *is denoted by* $\pi : T \to \mathrm{X}$ *and is called* projection map*,*

iv) $\mathsf{n} : T \to [0, +\infty)$ *is a measurable function with the property that for any* $n \in \mathbb{N}$ *it holds that* $\mathsf{n}(x, \cdot)$ *is a norm on* $\mathbb{R}^n$ *for* $\mathfrak{m}$*-a.e. point* $x \in E_n$*.*

Given $n \in \mathbb{N}$ and $x \in E_n$, we say that $(\overline{\mathbb{T}})_x := \pi^{-1}\{x\} = \{x\} \times \mathbb{R}^n$ is the *fiber* of $\overline{\mathbb{T}}$ over x. We will often implicitly identify the fiber $(\overline{\mathbb{T}})_x$ with the vector space $\mathbb{R}^n$ itself.

Remark 2.2 It is immediate to check that a subset S of the total space T of an MBB $\overline{\mathbb{T}}$ is measurable if and only if $S \cap (E_n \times \mathbb{R}^n)$ is a Borel subset of $E_n \times \mathbb{R}^n$ for any $n \in \mathbb{N}$. ■

We now describe which are the (pre-)morphisms between any two given MBB's.

Definition 2.3 (MBB pre-morphisms) *Let* $\overline{\mathbb{T}}_1 = (T_1, \underline{E}^1, \pi_1, \mathsf{n}_1)$*,* $\overline{\mathbb{T}}_2 = (T_2, \underline{E}^2, \pi_2, \mathsf{n}_2)$ *be MBB's over* $\mathbb{X}$*. Then a measurable map* $\overline{\varphi} : T_1 \to T_2$ *is said to be an* MBB pre-morphism *from* $\overline{\mathbb{T}}_1$ *to* $\overline{\mathbb{T}}_2$ *provided the diagram*

$$\begin{tikzcd} T_1 \arrow[r, "\overline{\varphi}"] \arrow[dr, "\pi_1"'] & T_2 \arrow[d, "\pi_2"] \\ & \mathrm{X} \end{tikzcd} \tag{2.1}$$

is commutative and for $\mathfrak{m}$*-a.e.* $x \in \mathrm{X}$ *it holds that* $\overline{\varphi}|_{(\overline{\mathbb{T}}_1)_x} : \big((\overline{\mathbb{T}}_1)_x, \mathsf{n}_1(x, \cdot)\big) \to \big((\overline{\mathbb{T}}_2)_x, \mathsf{n}_2(x, \cdot)\big)$ *is a linear* 1*-Lipschitz map.*

We declare two MBB pre-morphisms $\overline{\varphi}, \overline{\varphi}' : T_1 \to T_2$ to be equivalent, briefly $\overline{\varphi} \sim \overline{\varphi}'$, if

$$\overline{\varphi}|_{(\overline{\mathbb{T}}_1)_x} = \overline{\varphi}'|_{(\overline{\mathbb{T}}_1)_x} \quad \text{holds for } \mathfrak{m}\text{-a.e. } x \in \mathrm{X}. \tag{2.2}$$

We are now finally in a position to define the category of measurable Banach bundles over $\mathbb{X}$.

Definition 2.4 (The category of MBB's) *The collection of measurable Banach bundles over* $\mathbb{X}$ *and of equivalence classes of MBB pre-morphisms form a category, which we shall denote by* $\mathbf{MBB}(\mathbb{X})$*.*

2.2 The section functor

Once a notion of measurable Banach bundle is given, it is natural to consider its 'measurable sections', namely those maps which assign (in a measurable way) to almost every point of the underlying metric measure space an element of the fiber over such point.

It will turn out that the space $\Gamma(\mathbb{T})$ of all measurable sections of a measurable Banach bundle $\mathbb{T}$ is a proper L^0-normed L^0-module. The correspondence $\mathbb{T} \mapsto \Gamma(\mathbb{T})$ can be made into a functor, called 'section functor', from the category of measurable Banach bundles to the category of proper L^0-normed L^0-modules.

Definition 2.5 (Sections of an MBB) *Let $\overline{\mathbb{T}} = (T, \underline{E}, \pi, \mathsf{n})$ be (a representative of) an MBB over $\mathbb{X}$. Then we call* (measurable) section *of $\overline{\mathbb{T}}$ any measurable right inverse of the projection π, i.e. any measurable map $\overline{s} : \mathrm{X} \to T$ such that $\pi \circ \overline{s} = \mathrm{id}_{\mathrm{X}}$.*

Two given sections $\overline{s}_1, \overline{s}_2 : \mathrm{X} \to T$ are equivalent provided $\overline{s}_1(x) = \overline{s}_2(x)$ for $\mathfrak{m}$-a.e. $x \in \mathrm{X}$. The space of all equivalence classes of sections of $\overline{\mathbb{T}}$ will be denoted by $\Gamma(\overline{\mathbb{T}})$. We add some structure to the set $\Gamma(\overline{\mathbb{T}})$, in order to get an $L^0(\mathfrak{m})$-normed $L^0(\mathfrak{m})$-module:

i) Vector space. Let $s_1, s_2 \in \Gamma(\overline{\mathbb{T}})$ and $\lambda \in \mathbb{R}$. Pick a representative $\overline{s}_i : \mathrm{X} \to T$ of s_i for each $i = 1, 2$. Then we can pointwise define the sections $\overline{s}_1 + \overline{s}_2$ and $\lambda\,\overline{s}_1$ of $\overline{\mathbb{T}}$ as

$$\left.\begin{aligned} (\overline{s}_1 + \overline{s}_2)(x) &:= \overline{s}_1(x) + \overline{s}_2(x) \\ (\lambda\,\overline{s}_1)(x) &:= \lambda\,\overline{s}_1(x) \end{aligned}\right. \qquad \text{for every } x \in \mathrm{X}. \tag{2.3}$$

Therefore we define $s_1 + s_2 \in \Gamma(\overline{\mathbb{T}})$ and $\lambda\, s_1 \in \Gamma(\overline{\mathbb{T}})$ as the equivalence classes of $\overline{s}_1 + \overline{s}_2$ and $\lambda\,\overline{s}_1$, respectively. It can be readily seen that these operations are well-defined and give to $\Gamma(\overline{\mathbb{T}})$ a vector space structure.

ii) Multiplication by L^0-functions. Fix $s \in \Gamma(\overline{\mathbb{T}})$ and $f \in L^0(\mathfrak{m})$. Choose a representative $\overline{s} : \mathrm{X} \to T$ of s and a Borel version $\overline{f} : \mathrm{X} \to \mathbb{R}$ of f. Then the map $\overline{f} \cdot \overline{s} : \mathrm{X} \to T$, which is given by

$$(\overline{f} \cdot \overline{s})(x) := \overline{f}(x)\,\overline{s}(x) \in (\overline{\mathbb{T}})_x \qquad \text{for every } x \in \mathrm{X}, \tag{2.4}$$

is a section of $\overline{\mathbb{T}}$. Hence we define $f \cdot s \in \Gamma(\overline{\mathbb{T}})$ as the equivalence class of $\overline{f} \cdot \overline{s}$. This yields a well-posed bilinear operator $\cdot : L^0(\mathfrak{m}) \times \Gamma(\overline{\mathbb{T}}) \to \Gamma(\overline{\mathbb{T}})$.

iii) Pointwise norm. Consider a section $s \in \Gamma(\overline{\mathbb{T}})$. Pick a representative $\overline{s} : \mathrm{X} \to T$ of s. Define the Borel function $|\overline{s}| : \mathrm{X} \to [0, +\infty)$ as

$$|\overline{s}|(x) := \mathsf{n}\big(\overline{s}(x)\big) \qquad \text{for every } x \in \mathrm{X}. \tag{2.5}$$

Then we denote by $|s| \in L^0(\mathfrak{m})$ the equivalence class of the function $|\overline{s}|$. This gives us a well-defined operator $|\cdot| : \Gamma(\overline{\mathbb{T}}) \to L^0(\mathfrak{m})$.

iv) Topology on $\Gamma(\overline{\mathbb{T}})$. Pick a Borel probability measure $\mathfrak{m}'$ on X such that $\mathfrak{m} \ll \mathfrak{m}' \ll \mathfrak{m}$. Then we define the distance $\mathsf{d}_{\Gamma(\overline{\mathbb{T}})}$ on $\Gamma(\overline{\mathbb{T}})$ as follows:

$$\mathsf{d}_{\Gamma(\overline{\mathbb{T}})}(s_1, s_2) := \int |s_1 - s_2| \wedge 1 \,\mathrm{d}\mathfrak{m}' \qquad \text{for every } s_1, s_2 \in \Gamma(\overline{\mathbb{T}}). \tag{2.6}$$

We denote by τ the topology induced by $\mathsf{d}_{\Gamma(\overline{\mathbb{T}})}$.

It turns out that $\Gamma(\overline{\mathbb{T}})$ is an $L^0(\mathfrak{m})$-normed $L^0(\mathfrak{m})$-module. Furthermore, given any measurable Banach bundle $\mathbb{T}$ over the space $\mathbb{X}$, we define

$$\Gamma(\mathbb{T}) := \Gamma(\overline{\mathbb{T}}) \qquad \text{for one (thus any) representative } \overline{\mathbb{T}} \text{ of } \mathbb{T}. \tag{2.7}$$

Well-posedness of such definition is granted by the fact that $\Gamma(\overline{\mathbb{T}}_1)$ and $\Gamma(\overline{\mathbb{T}}_2)$ are isomorphic as $L^0(\mathfrak{m})$-normed $L^0(\mathfrak{m})$-modules whenever $\overline{\mathbb{T}}_1$ and $\overline{\mathbb{T}}_2$ are equivalent bundles.

Remark 2.6 (Constant sections) In the forthcoming discussion, a key role will be played by those sections of $\mathbb{T}$ that are obtained in this way: for any $n \in \mathbb{N}$ and any vector $v \in \mathbb{R}^n$, we consider the section $\boldsymbol{v} \in \Gamma(\mathbb{T})$ that is identically equal to v on E_n and null elsewhere.

More precisely, for any $n \in \mathbb{N}$ and any vector $v \in \mathbb{R}^n$, we define $\boldsymbol{v} \in \Gamma(\mathbb{T})$ as the equivalence class of the section $\overline{\boldsymbol{v}} : \mathrm{X} \to T$, given by

$$\overline{\boldsymbol{v}}(x) := \begin{cases} (x, v) & \text{if } x \in E_n, \\ (x, 0) & \text{if } x \in \mathrm{X} \setminus E_n, \end{cases} \tag{2.8}$$

where $\overline{\mathbb{T}} = (T, \underline{E}, \pi, \mathbf{n})$ is any chosen representative of $\mathbb{T}$. ■

Proposition 2.7 *The space* $\Gamma(\mathbb{T})$ *is a proper* $L^0(\mathfrak{m})$*-normed* $L^0(\mathfrak{m})$*-module. More precisely, for any representative* $\overline{\mathbb{T}} = (T, \underline{E}, \pi, \mathbf{n})$ *of the bundle* $\mathbb{T}$ *it holds that* $\underline{E} = (E_n)_{n\in\mathbb{N}}$ *constitutes a dimensional decomposition of* $\Gamma(\mathbb{T})$*.*

Proof. Fix $\overline{\mathbb{T}} = (T, \underline{E}, \pi, \mathbf{n}) \in \mathbb{T}$ and $n \in \mathbb{N}$. Denote by $\mathsf{e}_1, \ldots, \mathsf{e}_n$ the canonical basis of $\mathbb{R}^n$. Then consider the sections $\mathbf{e}_1, \ldots, \mathbf{e}_n \in \Gamma(\mathbb{T})$ defined in Remark 2.6. We claim that

$$\mathbf{e}_1, \ldots, \mathbf{e}_n \quad \text{is a local basis for } \Gamma(\mathbb{T}) \text{ on } E_n. \tag{2.9}$$

Take any $s \in \Gamma(\mathbb{T})$, with representative $\bar{s} : \mathrm{X} \to T$. Since the map $\bar{s}_{|E_n} : E_n \to E_n \times \mathbb{R}^n$ is Borel measurable by Remark 2.2, there exists a Borel function $\bar{c} = (\bar{c}_1, \ldots, \bar{c}_n) : E_n \to \mathbb{R}^n$ such that $\bar{s}(x) = \big(x, \bar{c}(x)\big)$ holds for every $x \in E_n$. Now extend each $\bar{c}_i$ to the whole X by declaring it equal to 0 on the complement of E_n. Hence $\chi_{E_n} \cdot \bar{s} = \sum_{i=1}^n \bar{c}_i \cdot \bar{\mathbf{e}}_i$, where $\bar{\mathbf{e}}_1, \ldots, \bar{\mathbf{e}}_n$ are defined as in (2.8). Calling $c_i \in L^0(\mathfrak{m})$ the equivalence class of $\bar{c}_i$ for every $i = 1, \ldots, n$, we deduce that $\chi_{E_n} \cdot s = \sum_{i=1}^n c_i \cdot \mathbf{e}_i$, which grants that $\mathbf{e}_1, \ldots, \mathbf{e}_n$ generate $\Gamma(\mathbb{T})$ on E_n.

Now suppose that $\sum_{i=1}^n c_i \cdot \mathbf{e}_i = 0$ for some $c_1, \ldots, c_n \in L^0(\mathfrak{m})$. Choose a Borel representative $\bar{c}_i : \mathrm{X} \to \mathbb{R}$ of each c_i, whence $\big(\bar{c}_1(x), \ldots, \bar{c}_n(x)\big) = \big(\sum_{i=1}^n \bar{c}_i \cdot \bar{\mathbf{e}}_i\big)(x) = 0$ holds for $\mathfrak{m}$-a.e. $x \in E_n$, in other words $\chi_{E_n} c_1, \ldots, \chi_{E_n} c_n = 0$. Therefore the sections $\mathbf{e}_1, \ldots, \mathbf{e}_n$ are independent on E_n. This yields (2.9) and accordingly the statement. □

In order to define the functor Γ from $\mathbf{MBB}(\mathbb{X})$ to $\mathbf{NMod}_{\mathrm{pr}}(\mathbb{X})$, it only remains to declare how it behaves on morphisms, namely to associate to any MBB morphism $\varphi \in \mathrm{Mor}(\mathbb{T}_1, \mathbb{T}_2)$ a suitable morphism $\Gamma(\varphi) : \Gamma(\mathbb{T}_1) \to \Gamma(\mathbb{T}_2)$ of proper $L^0(\mathfrak{m})$-normed $L^0(\mathfrak{m})$-modules.

Let $\overline{\mathbb{T}}_i = (T_i, \underline{E}^i, \pi_i, \mathbf{n}_i)$ be representatives of MBB's over $\mathbb{X}$ for $i = 1, 2$. Take a section $\bar{s}$ of $\overline{\mathbb{T}}_1$ and a pre-morphism $\bar{\varphi} : T_1 \to T_2$. Since $\bar{\varphi} \circ \bar{s} : \mathrm{X} \to T_2$ is measurable as composition of measurable maps and $\pi_2 \circ \bar{\varphi} \circ \bar{s} = \pi_1 \circ \bar{s} = \mathrm{id}_{\mathrm{X}}$, we conclude that $\bar{\varphi} \circ \bar{s}$ is a section of $\overline{\mathbb{T}}_2$.

Now let us call $\mathbb{T}_1$, $\mathbb{T}_2$ and φ the equivalence classes of $\overline{\mathbb{T}}_1$, $\overline{\mathbb{T}}_2$ and $\bar{\varphi}$, respectively. Then we define $\Gamma(\varphi) : \Gamma(\mathbb{T}_1) \to \Gamma(\mathbb{T}_2)$ as follows: given any $s \in \Gamma(\mathbb{T}_1)$, we set

$$\Gamma(\varphi)(s) := \text{the equivalence class of } \bar{\varphi} \circ \bar{s}, \qquad \text{where } \bar{s} \text{ is any representative of } s. \tag{2.10}$$

In the next result, we shall prove that $\Gamma(\varphi)$ is actually a module morphism.

Lemma 2.8 *Let $\mathbb{T}_1$, $\mathbb{T}_2$ be two measurable Banach bundles over $\mathbb{X}$ and let $\varphi \in \mathrm{Mor}(\mathbb{T}_1, \mathbb{T}_2)$. Then $\Gamma(\varphi) \in \mathrm{Mor}\big(\Gamma(\mathbb{T}_1), \Gamma(\mathbb{T}_2)\big)$.*

Proof. It suffices to show that for any $s_1, s_2 \in \Gamma(\mathbb{T}_1)$ and $f_1, f_2 \in L^0(\mathfrak{m})$ one has

$$\begin{gathered}\Gamma(\varphi)(f_1 \cdot s_1 + f_2 \cdot s_2) = f_1 \cdot \Gamma(\varphi)(s_1) + f_2 \cdot \Gamma(\varphi)(s_2), \\ \big|\Gamma(\varphi)(s_1)\big| \leq |s_1| \quad \mathfrak{m}\text{-a.e. in } \mathrm{X}.\end{gathered} \tag{2.11}$$

Choose representatives $\overline{\mathbb{T}}_i = (T_i, \underline{E}^i, \pi_i, \mathsf{n}_i)$ of $\mathbb{T}_i$, $\overline{\varphi} : T_1 \to T_2$ of φ and $\overline{s}_i : \mathrm{X} \to T_1$ of s_i for each $i = 1, 2$. Further, choose Borel functions $\overline{f}_1, \overline{f}_2 : \mathrm{X} \to \mathbb{R}$ that are representatives of f_1 and f_2, respectively. Hence for $\mathfrak{m}$-a.e. point $x \in \mathrm{X}$ it holds that

$$\begin{aligned}\big(\overline{\varphi} \circ (\overline{f}_1 \cdot \overline{s}_1 + \overline{f}_2 \cdot \overline{s}_2)\big)(x) &= \overline{\varphi}\big(\overline{f}_1(x)\, \overline{s}_1(x) + \overline{f}_2(x)\, \overline{s}_2(x)\big) \\ &= \overline{f}_1(x)\, (\overline{\varphi} \circ \overline{s}_1)(x) + \overline{f}_2(x)\, (\overline{\varphi} \circ \overline{s}_2)(x),\end{aligned}$$

whence $\Gamma(\varphi)(f_1 \cdot s_1 + f_2 \cdot s_2) = f_1 \cdot \Gamma(\varphi)(s_1) + f_2 \cdot \Gamma(\varphi)(s_2)$, i.e. the first in (2.11).

To prove the second one, observe that for $\mathfrak{m}$-a.e. $x \in \mathrm{X}$ one has that

$$|\overline{\varphi} \circ \overline{s}_1|(x) = \mathsf{n}_2\big((\overline{\varphi} \circ \overline{s}_1)(x)\big) = (\mathsf{n}_2 \circ \overline{\varphi})\big(\overline{s}_1(x)\big) \leq \mathsf{n}_1\big(\overline{s}_1(x)\big) = |\overline{s}_1|(x),$$

so that $\big|\Gamma(\varphi)(s_1)\big| \leq |s_1|$ holds $\mathfrak{m}$-a.e. in X. Therefore the thesis is achieved. □

Definition 2.9 (Section functor) *The covariant functor $\Gamma : \mathbf{MBB}(\mathbb{X}) \to \mathbf{NMod}_{\mathrm{pr}}(\mathbb{X})$, which associates to any object $\mathbb{T}$ of $\mathbf{MBB}(\mathbb{X})$ the object $\Gamma(\mathbb{T})$ of $\mathbf{NMod}_{\mathrm{pr}}(\mathbb{X})$ and to any morphism $\varphi : \mathbb{T}_1 \to \mathbb{T}_2$ the morphism $\Gamma(\varphi) : \Gamma(\mathbb{T}_1) \to \Gamma(\mathbb{T}_2)$, is called* section functor *on $\mathbb{X}$.*

3 Main result: the Serre-Swan theorem

Our main result states that the section functor is actually an equivalence of categories. We shall refer to such result as the *Serre-Swan theorem for normed modules*. First, we prove a technical lemma that provides us with a suitable dense subset of the space of all measurable sections of a measurable Banach bundle. Then such density result (Lemma 3.1) will be needed to show that the section functor is 'essentially surjective' (Proposition 3.2) and fully faithful (Proposition 3.3). Finally, the Serre-Swan theorem (Theorem 3.4) will immediately follow.

Given a measurable Banach bundle $\mathbb{T}$ over $\mathbb{X}$ and any $n \in \mathbb{N}$, we set

$$\mathsf{S}(\mathbb{T}, n) := \left\{ \sum_{i \in \mathbb{N}} \chi_{A_i} \cdot \boldsymbol{q}^i \;\middle|\; (A_i)_{i \in \mathbb{N}} \text{ is a Borel partition of } E_n,\ (q^i)_{i \in \mathbb{N}} \subseteq \mathbb{Q}^n \right\}, \tag{3.1}$$

where the 'constant sections' $\boldsymbol{q}^i \in \Gamma(\mathbb{T})$ are defined as in Remark 2.6. Note that any element of the form $\sum_{i \in \mathbb{N}} \chi_{A_i} \cdot \boldsymbol{q}^i \in \Gamma(\mathbb{T})$ is well-defined since the sets A_i's are pairwise disjoint.

Then we define the family $\mathsf{S}(\mathbb{T}) \subseteq \Gamma(\mathbb{T})$ of 'simple sections' of $\mathbb{T}$ as follows:

$$\mathsf{S}(\mathbb{T}) := \big\{ t \in \Gamma(\mathbb{T}) \;\big|\; \chi_{E_n} \cdot t \in \mathsf{S}(\mathbb{T}, n) \text{ for every } n \in \mathbb{N} \big\}. \tag{3.2}$$

We now show that such class of sections, which is a $\mathbb{Q}$-vector space, is actually dense in $\Gamma(\mathbb{T})$:

Lemma 3.1 *Let $\mathbb{T}$ be a measurable Banach bundle over $\mathbb{X}$. Then $\mathsf{S}(\mathbb{T})$ is dense in $\Gamma(\mathbb{T})$.*

Proof. Let $s \in \Gamma(\mathbb{T})$ and $\varepsilon > 0$ be fixed. Choose any Borel probability measure $\mathfrak{m}'$ on X such that $\mathfrak{m} \ll \mathfrak{m}' \ll \mathfrak{m}$ and define the distance $\mathsf{d}_{\Gamma(\mathbb{T})}$ on $\Gamma(\mathbb{T})$ as in (2.6). We aim to find a simple section $t \in \mathsf{S}(\mathbb{T})$ that satisfies $\mathsf{d}_{\Gamma(\mathbb{T})}(s,t) \leq \varepsilon$. To do so, choose representatives $\overline{\mathbb{T}} = (T, \underline{E}, \pi, \mathbf{n})$ and $\bar{s} : \mathrm{X} \to T$ of $\mathbb{T}$ and s, respectively. We can clearly suppose without loss of generality that $\mathbf{n}(x,\cdot)$ is a norm for every $x \in \mathrm{X}$. Given any $n \in \mathbb{N}$, let us define

$$E_{n,k} := \left\{ x \in E_n \;\middle|\; k-1 < \sup_{q \in \mathbb{Q}^n \setminus \{0\}} \frac{\mathbf{n}(x,q)}{|q|} \leq k \right\} \qquad \text{for every } k \in \mathbb{N}.$$

Since $E_n \ni x \mapsto \mathbf{n}(x,q)/|q|$ is Borel for every $q \in \mathbb{Q}^n \setminus \{0\}$, we know that each $E_{n,k}$ is Borel. Moreover, the fact that any two norms on $\mathbb{R}^n$ are equivalent grants that the supremum in the definition of $E_{n,k}$ is finite for every $x \in E_n$, whence for all $n \in \mathbb{N}$ we have that $(E_{n,k})_{k\in\mathbb{N}}$ constitutes a Borel partition of E_n. For any $n,k \in \mathbb{N}$, call $\bar{s}_{n,k} : E_{n,k} \to \mathbb{R}^n$ that Borel map for which $\bar{s}(x) = \big(x, \bar{s}_{n,k}(x)\big)$ for every $x \in E_{n,k}$. It is well-known that there exists a Borel map $\bar{t}_{n,k} : E_{n,k} \to \mathbb{R}^n$ whose image is a finite subset of $\mathbb{Q}^n$ and satisfying

$$\int_{E_{n,k}} \big|k\,\bar{s}_{n,k}(x) - k\,\bar{t}_{n,k}(x)\big| \wedge 1 \,\mathrm{d}\mathfrak{m}'(x) \leq \frac{\varepsilon}{2^{n+k}}. \tag{3.3}$$

Given that $\mathbf{n}(x,c) \leq k\,|c|$ holds for every $x \in E_{n,k}$ and $c \in \mathbb{R}^n$, we deduce from (3.3) that

$$\int_{E_{n,k}} \mathbf{n}\big(x, \bar{s}_{n,k}(x) - \bar{t}_{n,k}(x)\big) \wedge 1 \,\mathrm{d}\mathfrak{m}'(x) \leq \frac{\varepsilon}{2^{n+k}}. \tag{3.4}$$

Now let us denote by $\bar{t} : \mathrm{X} \to T$ the measurable map such that $\bar{t}|_{E_{n,k}} = (\mathrm{id}_{E_{n,k}}, \bar{t}_{n,k})$ holds for every $n,k \in \mathbb{N}$, which is meaningful since $(E_{n,k})_{n,k\in\mathbb{N}}$ is a partition of X. Call $t \in \Gamma(\mathbb{T})$ the equivalence class of $\bar{t}$. Notice that $t \in \mathsf{S}(\mathbb{T})$ by construction. Property (3.4) yields

$$\mathsf{d}_{\Gamma(\mathbb{T})}(s,t) = \int |s-t| \wedge 1 \,\mathrm{d}\mathfrak{m}' = \sum_{n,k\in\mathbb{N}} \int_{E_{n,k}} \mathbf{n}\big(x, \bar{s}_{n,k}(x) - \bar{t}_{n,k}(x)\big) \wedge 1 \,\mathrm{d}\mathfrak{m}'(x) \leq \sum_{n,k\in\mathbb{N}} \frac{\varepsilon}{2^{n+k}} = \varepsilon,$$

which gives the thesis. □

Proposition 3.2 *Let $\mathscr{M}$ be a proper $L^0(\mathfrak{m})$-normed $L^0(\mathfrak{m})$-module. Then there exists a measurable Banach bundle $\mathbb{T}$ over $\mathbb{X}$ such that $\Gamma(\mathbb{T})$ is isomorphic to $\mathscr{M}$.*

Proof. Let $(E_n)_{n\in\mathbb{N}}$ be a dimensional decomposition of the module $\mathscr{M}$. Set $\underline{E} := (E_n)_{n\in\mathbb{N}}$ and take T, π as in the definition of MBB. In order to define $\mathbf{n}$, fix a sequence $(v_n)_{n\in\mathbb{N}} \subseteq \mathscr{M}$ such that the elements $v_1, \ldots, v_n$ form a local basis for $\mathscr{M}$ on E_n for each $n \in \mathbb{N}$. Fix $n \in \mathbb{N}$. We define the linear and continuous operator $P_n : \mathbb{R}^n \to \mathscr{M}$ in the following way:

$$P_n(c) := \chi_{E_n} \cdot (c_1\,v_1 + \ldots + c_n\,v_n) \in \mathscr{M} \qquad \text{for every } c = (c_1, \ldots, c_n) \in \mathbb{R}^n.$$

For any $q \in \mathbb{Q}^n$, choose any Borel representative $\overline{|P_n(q)|} : \mathrm{X} \to [0,+\infty)$ of $|P_n(q)| \in L^0(\mathfrak{m})$. Hence there is a Borel set $N_n \subseteq E_n$, with $\mathfrak{m}(N_n) = 0$, such that for every $x \in E_n \setminus N_n$ it holds

$$\begin{aligned}
\overline{|P_n(q^1)+P_n(q^2)|}(x) &\leq \overline{|P_n(q^1)|}(x) + \overline{|P_n(q^2)|}(x) && \text{for every } q^1, q^2 \in \mathbb{Q}^n,\\
\overline{|P_n(\lambda\, q)|}(x) &= |\lambda|\, \overline{|P_n(q)|}(x) && \text{for every } \lambda \in \mathbb{Q} \text{ and } q \in \mathbb{Q}^n,\\
\overline{|P_n(q)|}(x) &> 0 && \text{for every } q \in \mathbb{Q}^n \setminus \{0\}.
\end{aligned} \tag{3.5}$$

Then let us define

$$\mathbf{n}(x,q) := \overline{|P_n(q)|}(x) \quad \text{for every } x \in E_n \setminus N_n \text{ and } q \in \mathbb{Q}^n. \tag{3.6}$$

We deduce from (3.5) that $\mathbf{n}(x,\cdot)$ is a norm on $\mathbb{Q}^n$ for every $x \in E_n \setminus N_n$. In particular it is uniformly continuous, whence it can be uniquely extended to a uniformly continuous map on the whole $\mathbb{R}^n$, still denoted by $\mathbf{n}(x,\cdot)$. By approximation, we see that such extension is actually a norm on $\mathbb{R}^n$. Finally, we set $\mathbf{n}(x,c) := 0$ for every $x \in N_n$ and $c \in \mathbb{R}^n$. We thus built a function $\mathbf{n} : T \to [0,+\infty)$. We claim that

$$\mathbf{n}|_{E_n \times \mathbb{R}^n} \text{ is a Carathéodory function} \quad \text{for every } n \in \mathbb{N}, \tag{3.7}$$

which grants that each $\mathbf{n}|_{E_n \times \mathbb{R}^n}$ is Borel, so accordingly that $\mathbf{n}$ is measurable by Remark 2.2. First of all, fix $n \in \mathbb{N}$ and notice that the function $\mathbf{n}(x,\cdot) : \mathbb{R}^n \to [0,+\infty)$ is continuous for every $x \in E_n$. Moreover, given any $c \in \mathbb{R}^n$ and a sequence $(q^k)_{k\in\mathbb{N}} \subseteq \mathbb{Q}^n$ converging to c, we have that $\mathbf{n}(x,c) = \lim_k \mathbf{n}(x,q^k) = \lim_k \overline{|P_n(q^k)|}(x)$ for every $x \in E_n \setminus N_n$, whence the function $\mathbf{n}(\cdot,c) : E_n \to [0,+\infty)$ is Borel as pointwise limit of a sequence of Borel functions. Therefore the claim (3.7) is proved. We thus deduce that $\overline{\mathbb{T}} := (T, \underline{E}, \pi, \mathbf{n})$ is an MBB over the space $\mathbb{X}$. Then let us denote by $\mathbb{T}$ the equivalence class of $\overline{\mathbb{T}}$.

In order to get the thesis, we want to exhibit a module isomorphism $I : \Gamma(\mathbb{T}) \to \mathscr{M}$, namely an $L^0(\mathfrak{m})$-linear map preserving the pointwise norm. We proceed as follows: given any $s \in \Gamma(\mathbb{T})$, choose a representative $\bar{s} : \mathrm{X} \to T$. For any $n \in \mathbb{N}$, pick $\bar{c}^n : \mathrm{X} \to \mathbb{R}^n$ Borel such that $\bar{s}(x) = \big(x, \bar{c}^n(x)\big)$ for every $x \in E_n$ and call $c_1^n, \ldots, c_n^n \in L^0(\mathfrak{m})$ those elements for which $(c_1^n, \ldots, c_n^n)$ is the equivalence class of $\bar{c}^n$. Now let us define

$$I(s) := \sum_{n\in\mathbb{N}} \chi_{E_n} \cdot (c_1^n \cdot v_1 + \ldots + c_n^n \cdot v_n) \in \mathscr{M}. \tag{3.8}$$

One can easily see that the resulting map $I : \Gamma(\mathbb{T}) \to \mathscr{M}$ is a (well-defined) $L^0(\mathfrak{m})$-linear and continuous operator. We show that it is surjective: fix any $v \in \mathscr{M}$, whence for each $n \in \mathbb{N}$ there exist $c_1^n, \ldots, c_n^n \in L^0(\mathfrak{m})$ such that $\chi_{E_n} \cdot v = \chi_{E_n} \cdot (c_1^n \cdot v_1 + \ldots + c_n^n \cdot v_n)$. Pick any Borel representative $\bar{c}^n : \mathrm{X} \to \mathbb{R}^n$ of $(c_1^n, \ldots, c_n^n)$ and define $\bar{s} : \mathrm{X} \to T$ as $\bar{s}(x) := \big(x, \bar{c}^n(x)\big)$ for every $n \in \mathbb{N}$ and $x \in E_n$. Hence the equivalence class $s \in \Gamma(\mathbb{T})$ of $\bar{s}$ satisfies $I(s) = v$, thus proving that the map I is surjective. It only remains to prove that $|I(s)| = |s|$ holds $\mathfrak{m}$-a.e. in X for every $s \in \Gamma(\mathbb{T})$. First of all, for any $n \in \mathbb{N}$ and $q \in \mathbb{Q}^n$ one has that $I(\boldsymbol{q}) = P_n(q)$, where the definition of $\boldsymbol{q}$ is taken from Remark 2.6. Therefore

$$|I(\boldsymbol{q})| = |P_n(q)| \overset{(3.6)}{=} \mathbf{n} \circ \boldsymbol{q} \overset{(2.5)}{=} |\boldsymbol{q}| \quad \text{holds } \mathfrak{m}\text{-a.e. in X}. \tag{3.9}$$

We then directly deduce from (3.9) and the $L^0(\mathfrak{m})$-linearity of I that the equality $\big|I(t)\big| = |t|$ is verified $\mathfrak{m}$-a.e. for every simple section $t \in \mathsf{S}(\mathbb{T})$. Recall that $\mathsf{S}(\mathbb{T})$ is dense in $\Gamma(\mathbb{T})$, as seen in Lemma 3.1. Since both I and the pointwise norm are continuous operators, we finally conclude that $\big|I(s)\big| = |s|$ holds $\mathfrak{m}$-a.e. for every $s \in \Gamma(\mathbb{T})$. Therefore I preserves the pointwise norm, thus completing the proof. $\square$

Proposition 3.3 *The section functor* $\Gamma : \mathbf{MBB}(\mathbb{X}) \to \mathbf{NMod}_{\mathrm{pr}}(\mathbb{X})$ *is full and faithful.*

Proof. Faithful. Fix two measurable Banach bundles $\mathbb{T}_1$, $\mathbb{T}_2$ and two different bundle morphisms $\varphi, \psi \in \mathrm{Mor}(\mathbb{T}_1, \mathbb{T}_2)$. Choose a representative $\overline{T}_i = (T_i, \underline{E}^i, \pi_i, \mathsf{n}_i)$ of $\mathbb{T}_i$ for $i = 1, 2$, then representatives $\overline{\varphi}, \overline{\psi} : T_1 \to T_2$ of φ and ψ, respectively. Hence there exist $n \in \mathbb{N}$ and a Borel set $E \subseteq E^1_n$, with $\mathfrak{m}(E) > 0$, such that $\overline{\varphi}|_{(\overline{\mathbb{T}}_1)_x} \neq \overline{\psi}|_{(\overline{\mathbb{T}}_1)_x}$ for every $x \in E$. Let us denote by $\mathsf{e}_1, \ldots, \mathsf{e}_n$ the canonical basis of $\mathbb{R}^n$. Therefore there exists $k \in \{1, \ldots, n\}$ such that

$$\mathfrak{m}\big(\big\{x \in E : \overline{\varphi}(x, \mathsf{e}_k) \neq \overline{\psi}(x, \mathsf{e}_k)\big\}\big) > 0.$$

This means that $\overline{\varphi} \circ \bar{\mathbf{e}}_k$ is not $\mathfrak{m}$-a.e. coincident with $\overline{\psi} \circ \bar{\mathbf{e}}_k$, where the section $\bar{\mathbf{e}}_k$ is defined as in Remark 2.6, whence $\Gamma(\varphi)(\mathbf{e}_k) \neq \Gamma(\psi)(\mathbf{e}_k)$. This implies that $\Gamma(\varphi) \neq \Gamma(\psi)$, thus proving that the functor Γ is faithful.

Full. Fix measurable Banach bundles $\mathbb{T}_1$, $\mathbb{T}_2$ and a module morphism $\Phi : \Gamma(\mathbb{T}_1) \to \Gamma(\mathbb{T}_2)$. We aim to show that there exists a bundle morphism $\varphi \in \mathrm{Mor}(\mathbb{T}_1, \mathbb{T}_2)$ such that $\Phi = \Gamma(\varphi)$. Since the ideas of the proof are similar in spirit to those that have been used for proving Proposition 3.2, we will omit some details. Choose a representative $\overline{\mathbb{T}}_i = (T_i, \underline{E}^i, \pi_i, \mathsf{n}_i)$ of $\mathbb{T}_i$ for $i = 1, 2$. We define the Borel sets $F_{n,m} \subseteq \mathrm{X}$ as

$$F_{n,m} := E^1_n \cap E^2_m \qquad \text{for every } n, m \in \mathbb{N}.$$

For any $n \in \mathbb{N}$ and $q \in \mathbb{Q}^n$, consider the section $\boldsymbol{q} \in \Gamma(\mathbb{T}_1)$ as in Remark 2.6 and choose a representative $\overline{\Phi(\boldsymbol{q})} : \mathrm{X} \to T_2$ of $\Phi(\boldsymbol{q}) \in \Gamma(\mathbb{T}_2)$. Given any $n, m \in \mathbb{N}$, there exists a Borel subset $N_{n,m}$ of $F_{n,m}$, with $\mathfrak{m}(N_{n,m}) = 0$, such that for every $x \in F_{n,m} \setminus N_{n,m}$ it holds

$$\begin{aligned}
\overline{\Phi(\boldsymbol{q}^1 + \boldsymbol{q}^2)}(x) = \overline{\Phi(\boldsymbol{q}^1)}(x) + \overline{\Phi(\boldsymbol{q}^2)}(x) &\qquad \text{for every } q^1, q^2 \in \mathbb{Q}^n,\\
\overline{\Phi(\lambda\, \boldsymbol{q})}(x) = \lambda\, \overline{\Phi(\boldsymbol{q})}(x) &\qquad \text{for every } \lambda \in \mathbb{Q} \text{ and } q \in \mathbb{Q}^n,\\
\mathsf{n}_2\big(\overline{\Phi(\boldsymbol{q})}(x)\big) \leq \mathsf{n}_1\big(\overline{\boldsymbol{q}}(x)\big) &\qquad \text{for every } q \in \mathbb{Q}^n.
\end{aligned} \tag{3.10}$$

Then let us define

$$\overline{\varphi}(x, q) := \begin{cases} \overline{\Phi(\boldsymbol{q})}(x) & \text{for every } x \in F_{n,m} \setminus N_{n,m} \text{ and } q \in \mathbb{Q}^n,\\ 0_{\mathbb{R}^m} & \text{for every } x \in N_{n,m} \text{ and } q \in \mathbb{Q}^n. \end{cases} \tag{3.11}$$

Property (3.10) grants that $\overline{\varphi}(x, \cdot) : \big(\mathbb{Q}^n, \mathsf{n}_1(x, \cdot)\big) \to \big(\mathbb{R}^m, \mathsf{n}_2(x, \cdot)\big)$ is a $\mathbb{Q}$-linear 1-Lipschitz operator for all $x \in F_{n,m}$, whence it can be uniquely extended to an $\mathbb{R}$-linear 1-Lipschitz operator $\overline{\varphi}(x, \cdot) : \big(\mathbb{R}^n, \mathsf{n}_1(x, \cdot)\big) \to \big(\mathbb{R}^m, \mathsf{n}_2(x, \cdot)\big)$. This defines a map $\overline{\varphi} : T_1 \to T_2$. To show that such map is an MBB pre-morphism, it only remains to check its measurability, which amounts to proving that $\overline{\varphi}|_{F_{n,m}\times\mathbb{R}^n} : F_{n,m}\times\mathbb{R}^n \to F_{n,m}\times\mathbb{R}^m$ is Borel for every $n,m\in\mathbb{N}$. We actually show that each $\overline{\varphi}|_{F_{n,m}\times\mathbb{R}^n}$ is a Carathéodory map: for any $x\in F_{n,m}$ we have that $\overline{\varphi}(x,\cdot)$ is continuous by its very construction, while for any vector $c\in\mathbb{R}^n$ we have that the map $F_{n,m}\ni x\mapsto\overline{\varphi}(x,c)\in\mathbb{R}^m$ is Borel as pointwise limit of the Borel maps $\chi_{F_{n,m}}\,\overline{\Phi(\boldsymbol{q}^k)}$, where $(q^k)_{k\in\mathbb{N}}\subseteq\mathbb{Q}^n$ is any sequence converging to c. Hence let us define $\varphi\in\mathrm{Mor}(\mathbb{T}_1,\mathbb{T}_2)$ as the equivalence class of the MBB pre-morphism $\overline{\varphi}$.

We conclude by proving that $\Gamma(\varphi)=\Phi$. For any $n\in\mathbb{N}$ and $q\in\mathbb{Q}^n$, we have that a representative of $\Gamma(\varphi)(\boldsymbol{q})$ is given by the map $\overline{\varphi}\circ\overline{\boldsymbol{q}}$, which $\mathfrak{m}$-a.e. coincides in E^1_n with $\overline{\Phi(\boldsymbol{q})}$, whence $\Gamma(\varphi)(\boldsymbol{q})=\Phi(\boldsymbol{q})$. Since both $\Gamma(\varphi)$ and Φ are $L^0(\mathfrak{m})$-linear, we thus immediately deduce that $\Gamma(\varphi)(t)=\Phi(t)$ for every $t\in\mathsf{S}(\mathbb{T}_1)$. Finally, the density of $\mathsf{S}(\mathbb{T}_1)$ in $\Gamma(\mathbb{T}_1)$, proven in Lemma 3.1, together with the continuity of $\Gamma(\varphi)$ and Φ, grant that $\Gamma(\varphi)=\Phi$, as required. Therefore the section functor Γ is full. □

We now collect the last two results, thus obtaining the main theorem of the paper:

Theorem 3.4 (Serre-Swan) *Let $\mathbb{X}=(\mathrm{X},\mathsf{d},\mathfrak{m})$ be a metric measure space. Then the section functor $\Gamma:\mathbf{MBB}(\mathbb{X})\to\mathbf{NMod}_{\mathrm{pr}}(\mathbb{X})$ on $\mathbb{X}$ is an equivalence of categories.*

Proof. By [2, Proposition 7.25] it suffices to prove that the functor Γ is fully faithful and 'essentially surjective', the latter meaning that for each object $\mathscr{M}$ of $\mathbf{NMod}_{\mathrm{pr}}(\mathbb{X})$ there exists an object $\mathbb{T}$ of $\mathbf{MBB}(\mathbb{X})$ such that $\Gamma(\mathbb{T})$ and $\mathscr{M}$ are isomorphic. Therefore Proposition 3.2 and Proposition 3.3 yield the thesis. □

4 Some further constructions

4.1 Hilbert modules and measurable Hilbert bundles

An important class of L^0-normed L^0-modules is that of *Hilbert modules*, defined as follows:

Definition 4.1 (Hilbert module) *Let $\mathscr{M}$ be an $L^0(\mathfrak{m})$-normed $L^0(\mathfrak{m})$-module. Then we say that $\mathscr{M}$ is a* Hilbert module *provided it satisfies the* pointwise parallelogram rule*, i.e.*

$$|v+w|^2+|v-w|^2=2\,|v|^2+2\,|w|^2\quad\mathfrak{m}\text{-}a.e.\quad\text{for every } v,w\in\mathscr{M}. \tag{4.1}$$

We shall denote by $\mathbf{HNMod}_{\mathrm{pr}}(\mathbb{X})$ the subcategory of $\mathbf{NMod}_{\mathrm{pr}}(\mathbb{X})$ made of those modules that are Hilbert modules. Our goal is to characterise those measurable Banach bundles that correspond to the Hilbert modules via the section functor Γ. As one might expect, such bundles are precisely the ones given by the following definition:

Definition 4.2 (Measurable Hilbert bundle) *Let $\mathbb{T}$ be a measurable Banach bundle over the space $\mathbb{X}$. Then we say that $\mathbb{T}$ is a* measurable Hilbert bundle*, or briefly* MHB*, provided for one (thus any) representative $\overline{\mathbb{T}}=(T,\underline{E},\pi,\mathsf{n})$ of $\mathbb{T}$ it holds that $\mathsf{n}(x,\cdot)$ is a norm induced by a scalar product for $\mathfrak{m}$-a.e. point $x\in\mathrm{X}$.*

Given any such point $x \in \mathrm{X}$, we denote the associated scalar product on $(\overline{\mathbb{T}})_x$ by

$$\big\langle (x,v),(x,w) \big\rangle_x := \frac{\mathbf{n}(x,v+w)^2 - \mathbf{n}(x,v)^2 - \mathbf{n}(x,w)^2}{2} \tag{4.2}$$

for every $(x,v),(x,w) \in (\overline{\mathbb{T}})_x$.

We shall denote by $\mathbf{MHB}(\mathbb{X})$ the subcategory of $\mathbf{MBB}(\mathbb{X})$ made of those bundles that are measurable Hilbert bundles. Therefore we can easily prove that:

Proposition 4.3 *Let $\mathbb{T}$ be a measurable Banach bundle over $\mathbb{X}$. Then $\mathbb{T}$ is a measurable Hilbert bundle if and only if $\Gamma(\mathbb{T})$ is a Hilbert module.*

Proof. Choose any representative $\overline{\mathbb{T}} = (T, \underline{E}, \pi, \mathbf{n})$ of the measurable Banach bundle $\mathbb{T}$.
NECESSITY. Suppose that $\mathbb{T}$ is a measurable Hilbert bundle. This means that $\mathbf{n}(x,\cdot)$ satisfies the parallelogram rule for $\mathfrak{m}$-a.e. $x \in \mathrm{X}$. Now let $s_1, s_2 \in \Gamma(\mathbb{T})$ be fixed and choose some representatives $\overline{s}_1, \overline{s}_2 : \mathrm{X} \to T$. Hence for $\mathfrak{m}$-a.e. $x \in \mathrm{X}$ it holds that

$$\begin{aligned} |\overline{s}_1 + \overline{s}_2|^2(x) + |\overline{s}_1 - \overline{s}_2|^2(x) &= \big(\mathbf{n} \circ (\overline{s}_1 + \overline{s}_2)\big)^2(x) + \big(\mathbf{n} \circ (\overline{s}_1 - \overline{s}_2)\big)^2(x) \\ &= 2\,(\mathbf{n} \circ \overline{s}_1)^2(x) + 2\,(\mathbf{n} \circ \overline{s}_2)^2(x) \\ &= 2\,|\overline{s}_1|^2(x) + 2\,|\overline{s}_2|^2(x), \end{aligned}$$

which grants that $|s_1 + s_2|^2 + |s_1 - s_2|^2 = 2\,|s_1|^2 + 2\,|s_2|^2$ holds $\mathfrak{m}$-a.e. in X. Therefore $\Gamma(\mathbb{T})$ is a Hilbert module by arbitrariness of $s_1, s_2 \in \Gamma(\mathbb{T})$.
SUFFICIENCY. Suppose that $\Gamma(\mathbb{T})$ is Hilbert module. Let $n \in \mathbb{N}$ be fixed. For any $q \in \mathbb{Q}^n$, consider $\overline{\boldsymbol{q}} : \mathrm{X} \to T$ and $\boldsymbol{q} \in \Gamma(\mathbb{T})$ as in Remark 2.6. Then there exists an $\mathfrak{m}$-negligible Borel subset N_n of E_n such that $\mathbf{n}(x,\cdot)$ is a norm and the equality

$$|\overline{\boldsymbol{q}}_1 + \overline{\boldsymbol{q}}_2|^2(x) + |\overline{\boldsymbol{q}}_1 - \overline{\boldsymbol{q}}_2|^2(x) = 2\,|\overline{\boldsymbol{q}}_1|^2(x) + 2\,|\overline{\boldsymbol{q}}_2|^2(x) \quad \text{for every } q_1, q_2 \in \mathbb{Q}^n$$

is satisfied for every point $x \in E_n \setminus N_n$. This implies that

$$\mathbf{n}(x,q_1+q_2)^2 + \mathbf{n}(x,q_1-q_2)^2 = 2\,\mathbf{n}(x,q_1)^2 + 2\,\mathbf{n}(x,q_2)^2 \quad \text{for every } x \in E_n \setminus N_n \text{ and } q_1, q_2 \in \mathbb{Q}^n.$$

Therefore $\mathbf{n}(x,\cdot)$ satisfies the parallelogram rule for every $x \in E_n \setminus N_n$ by continuity, so that accordingly $\mathbb{T}$ is a measurable Hilbert bundle. □

As a consequence of Proposition 4.3, we finally conclude that

Theorem 4.4 (Serre-Swan for Hilbert modules) *The section functor Γ restricts to an equivalence of categories between $\mathbf{MHB}(\mathbb{X})$ and $\mathbf{HNMod}_{\mathrm{pr}}(\mathbb{X})$.*

Remark 4.5 It has been proved in [6, Theorem 1.4.11] that any separable Hilbert module (thus in particular any proper Hilbert module by Remark 1.14) is exactly the space of sections of a suitable measurable Hilbert bundle. Moreover, as pointed out in [6, Remark 1.4.12], this theory of Hilbert modules coincides with that of direct integral of Hilbert spaces (cf. [27]). More precisely, under this identification, a Hilbert module corresponds to a measurable field of Hilbert spaces. ■

Let $\mathscr{H}_1$, $\mathscr{H}_2$ be two given Hilbert modules over $\mathbb{X}$. Then we can consider their tensor product $\mathscr{H}_1 \otimes \mathscr{H}_2$, which is a Hilbert module over $\mathbb{X}$ as well (cf. [6, Section 1.5]).

Remark 4.6 Suppose that $\mathscr{H}_1$ and $\mathscr{H}_2$ are proper, with dimensional decomposition $(E^1_n)_{n\in\mathbb{N}}$ and $(E^2_m)_{m\in\mathbb{N}}$, respectively. Then it can be readily checked that $\mathscr{H}_1 \otimes \mathscr{H}_2$ has dimension equal to nm on $E^1_n \cap E^2_m$ for any $n, m \in \mathbb{N}$. In particular, the dimensional decomposition $(E_k)_{k\in\mathbb{N}}$ of $\mathscr{H}_1 \otimes \mathscr{H}_2$ is given by

$$E_k := \bigcup_{\substack{n,m\in\mathbb{N}:\\ nm=k}} E^1_n \cap E^2_m \qquad \text{for every } k \in \mathbb{N}, \tag{4.3}$$

so that $\mathscr{H}_1 \otimes \mathscr{H}_2$ is a proper module as well. ■

On the other hand, we now define the tensor product of two MHB's in the following way:

Definition 4.7 (Tensor product of MHB's) *Let $\mathbb{T}_1$, $\mathbb{T}_2$ be measurable Hilbert bundles over $\mathbb{X}$. Choose two representatives $\overline{\mathbb{T}}_1$ and $\overline{\mathbb{T}}_2$, say $\overline{\mathbb{T}}_i = (T_i, \underline{E}^i, \pi_i, \mathsf{n}_i)$ for $i = 1, 2$. Let us define $\underline{E} = (E_k)_{k\in\mathbb{N}}$ as in (4.3) and T, π accordingly. Given $n, m \in \mathbb{N}$ and $x \in E^1_n \cap E^2_m$ such that $\mathsf{n}_1(x, \cdot)$, $\mathsf{n}_2(x, \cdot)$ are norms induced by a scalar product, we define*

$$\mathsf{n}(x, c) := \bigg(\sum_{j,j'=1}^{n} \sum_{\ell,\ell'=1}^{m} c_{(j-1)m+\ell}\, c_{(j'-1)m+\ell'} \big\langle (x, \mathsf{e}_j), (x, \mathsf{e}_{j'}) \big\rangle_{1,x} \big\langle (x, \mathsf{f}_\ell), (x, \mathsf{f}_{\ell'}) \big\rangle_{2,x} \bigg)^{1/2} \tag{4.4}$$

for every $c = (c_1, \ldots, c_{nm}) \in \mathbb{R}^{nm}$, where $\mathsf{e}_1, \ldots, \mathsf{e}_n$ and $\mathsf{f}_1, \ldots, \mathsf{f}_m$ denote the canonical bases of $\mathbb{R}^n$ and $\mathbb{R}^m$, respectively, while $\langle \cdot, \cdot \rangle_{i,x}$ stays for the scalar product on $(\overline{\mathbb{T}}_i)_x$ as in (4.2). Then we define the tensor product $\mathbb{T}_1 \otimes \mathbb{T}_2$ *as the equivalence class of $(T, \underline{E}, \pi, \mathsf{n})$, which turns out to be a measurable Hilbert bundle over $\mathbb{X}$.*

Given any real number $\lambda \in \mathbb{R}$, we shall write $\lceil \lambda \rceil \in \mathbb{Z}$ to indicate the smallest integer number that is greater than or equal to λ.

Theorem 4.8 *Let $\mathbb{T}_1$, $\mathbb{T}_2$ be measurable Hilbert bundles over $\mathbb{X}$. Then*

$$\Gamma(\mathbb{T}_1) \otimes \Gamma(\mathbb{T}_2) = \Gamma(\mathbb{T}_1 \otimes \mathbb{T}_2). \tag{4.5}$$

Proof. We build an operator $\iota : \Gamma(\mathbb{T}_1) \otimes \Gamma(\mathbb{T}_2) \to \Gamma(\mathbb{T}_1 \otimes \mathbb{T}_2)$ in the following way: first of all, fix $s^1 \in \Gamma(\mathbb{T}_1)$ and $s^2 \in \Gamma(\mathbb{T}_2)$. Choose representatives $\overline{\mathbb{T}}_i = (T_i, \underline{E}^i, \pi_i, \mathsf{n}_i)$ and $\overline{s}^i : \mathrm{X} \to T_i$ for $i = 1, 2$. Given $n, m \in \mathbb{N}$, $x \in E^1_n \cap E^2_m$ and called $\overline{s}^1(x) = (x, v)$, $\overline{s}^2(x) = (x, w)$, we define

$$\overline{s}(x) := (x, c), \qquad \text{where } c_k := v_{\lceil k/m \rceil}\, w_{k - m\lceil k/m \rceil + m} \text{ for all } k = 1, \ldots, nm.$$

Hence the equivalence class $\iota(s^1 \otimes s^2)$ of $\overline{s}$ is a section of $\mathbb{T}_1 \otimes \mathbb{T}_2$. Simple computations yield

$$\big|\iota(s^1 \otimes s^2)\big| = \sqrt{|s^1|\,|s^2|} = |s^1 \otimes s^2| \quad \mathfrak{m}\text{-a.e. on } \mathrm{X}.$$

Therefore ι can be uniquely extended to the whole $\Gamma(\mathbb{T}_1) \otimes \Gamma(\mathbb{T}_2)$ by linearity and continuity, thus obtaining an $L^0(\mathfrak{m})$-linear operator $\iota : \Gamma(\mathbb{T}_1) \otimes \Gamma(\mathbb{T}_2) \to \Gamma(\mathbb{T}_1 \otimes \mathbb{T}_2)$ that preserves the pointwise norm. In order to conclude, it only remains to check that such ι is surjective. Fix $n, m \in \mathbb{N}$ and call $(\mathsf{e}_i)_{i=1}^n$, $(\mathsf{f}_j)_{j=1}^m$ and $(\mathsf{g}_k)_{k=1}^{nm}$ the canonical bases of $\mathbb{R}^n$, $\mathbb{R}^m$ and $\mathbb{R}^{nm}$, respectively. Denote by $\mathbf{e}_i \in \Gamma(\mathbb{T}_1)$, $\mathbf{f}_j \in \Gamma(\mathbb{T}_2)$ and $\mathbf{g}_k \in \Gamma(\mathbb{T}_1 \otimes \mathbb{T}_2)$ the associated constant sections. It is then easy to realise that

$$\chi_{E_n^1 \cap E_m^2} \cdot \mathbf{g}_k = \iota\big((\chi_{E_n^1 \cap E_m^2} \cdot \mathbf{e}_{\lceil k/m \rceil}) \otimes (\chi_{E_n^1 \cap E_m^2} \cdot \mathbf{f}_{k - m\lceil k/m \rceil + m})\big) \qquad \text{for every } k = 1, \ldots, nm.$$

Hence the set $(\chi_{E_n^1 \cap E_m^2} \cdot \mathbf{g}_k)_{k=1}^{nm}$, which forms a local basis for $\Gamma(\mathbb{T}_1 \otimes \mathbb{T}_2)$ on $E_n^1 \cap E_m^2$, is contained in the range of the map ι. This grants that ι is surjective, as required. $\square$

4.2 Pullbacks and duals

Let $\mathbb{X} = (\mathrm{X}, \mathsf{d}_\mathrm{X}, \mathfrak{m}_\mathrm{X})$ and $\mathbb{Y} = (\mathrm{Y}, \mathsf{d}_\mathrm{Y}, \mathfrak{m}_\mathrm{Y})$ be fixed metric measure spaces.

Let $f : \mathrm{X} \to \mathrm{Y}$ be any map *of bounded compression*, namely a Borel map such that

$$f_* \mathfrak{m}_\mathrm{X} \leq C\, \mathfrak{m}_\mathrm{Y} \qquad \text{for some constant } C > 0. \tag{4.6}$$

Given any $L^0(\mathfrak{m}_\mathrm{Y})$-normed $L^0(\mathfrak{m}_\mathrm{Y})$-module $\mathscr{M}$, there is a natural way to define its *pullback module* $f^*\mathscr{M}$, which is an $L^0(\mathfrak{m}_\mathrm{X})$-normed $L^0(\mathfrak{m}_\mathrm{X})$-module (see [11] or [12] for the details).

Remark 4.9 Since we are dealing with normed modules over L^0, so that there is no integrability requirement on the pointwise norm of the elements of our modules, it is natural to ask whether the hypothesis (4.6) on f can be weakened. Actually, this is the case: one can build the pullback module under the only assumption that $f_* \mathfrak{m}_\mathrm{X} \ll \mathfrak{m}_\mathrm{Y}$. Indeed, choose any Borel representative A of $\big\{ \frac{\mathrm{d} f_* \mathfrak{m}_\mathrm{X}}{\mathrm{d}\mathfrak{m}_\mathrm{Y}} > 0 \big\}$, where $\frac{\mathrm{d} f_* \mathfrak{m}_\mathrm{X}}{\mathrm{d}\mathfrak{m}_\mathrm{Y}}$ denotes the Radon-Nikodým derivative of $f_* \mathfrak{m}_\mathrm{X}$ with respect to $\mathfrak{m}_\mathrm{Y}$. Then define $\mathfrak{m}'_\mathrm{Y} := (f_* \mathfrak{m}_\mathrm{X})|_A + \mathfrak{m}_\mathrm{Y}|_{\mathrm{X} \setminus A}$. Hence $\mathfrak{m}_\mathrm{Y} \ll \mathfrak{m}'_\mathrm{Y} \ll \mathfrak{m}_\mathrm{Y}$, which grants that $L^0(\mathfrak{m}'_\mathrm{Y}) = L^0(\mathfrak{m}_\mathrm{Y})$ and accordingly that $\mathscr{M}$ is an $L^0(\mathfrak{m}'_\mathrm{Y})$-normed $L^0(\mathfrak{m}'_\mathrm{Y})$-module. Moreover, it holds that $f_* \mathfrak{m}_\mathrm{X} \leq \mathfrak{m}'_\mathrm{Y}$, which says that f is a map of bounded compression when the target Y is endowed with the measure $\mathfrak{m}'_\mathrm{Y}$, so that it makes sense to consider $f^*\mathscr{M}$. $\blacksquare$

We now define what is the pullback of a measurable Banach bundle over $\mathbb{Y}$:

Definition 4.10 (Pullback of an MBB) *Let $\mathbb{T}$ be a measurable Banach bundle over $\mathbb{Y}$. Choose a representative $\overline{\mathbb{T}} = (T, \underline{E}, \pi, \mathbf{n})$ of $\mathbb{T}$. Let us set*

$$\begin{aligned} \underline{E}' &:= \big(f^{-1}(E_n)\big)_{n \in \mathbb{N}} && \textit{and } T', \pi' \textit{ accordingly,} \\ \mathbf{n}'(x, v) &:= \mathbf{n}\big(f(x), v\big) && \textit{for every } (x, v) \in T'. \end{aligned} \tag{4.7}$$

Then we define the pullback bundle $f^*\mathbb{T}$ *as the equivalence class of $(T', \underline{E}', \pi', \mathbf{n}')$, which turns out to be a measurable Banach bundle over $\mathbb{X}$.*

Theorem 4.11 *Let $\mathbb{T}$ be a measurable Banach bundle over $\mathbb{Y}$. Then*

$$f^*\Gamma(\mathbb{T}) = \Gamma(f^*\mathbb{T}). \tag{4.8}$$

Proof. We aim to build a linear map $f^* : \Gamma(\mathbb{T}) \to \Gamma(f^*\mathbb{T})$ such that

$$\begin{aligned} |f^*s| = |s| \circ f \quad & \text{for every } s \in \Gamma(\mathbb{T}), \\ \big\{f^*s \,:\, s \in \Gamma(\mathbb{T})\big\} \quad & \text{generates } \Gamma(f^*\mathbb{T}). \end{aligned} \tag{4.9}$$

Pick a representative $\overline{\mathbb{T}} = (T, \underline{E}, \pi, \mathbf{n})$ of $\mathbb{T}$ and define $(T', \underline{E}', \pi', \mathbf{n}')$ as in (4.7). Take $s \in \Gamma(\mathbb{T})$, with representative $\overline{s} : \mathrm{Y} \to T$. Given any $x \in \mathrm{X}$, let us define $\overline{s}'(x) := (x, v)$, where v is the unique vector for which $\overline{s}\big(f(x)\big) = \big(f(x), v\big)$. It clearly holds that $\overline{s}' : \mathrm{X} \to T'$ is a section of the MBB $(T', \underline{E}', \pi', \mathbf{n}')$. Then we define f^*s as the equivalence class of $\overline{s}'$. We thus built a map $f^* : \Gamma(\mathbb{T}) \to \Gamma(f^*\mathbb{T})$, which is linear and satisfies the first in (4.9) by construction.

Now fix $n \in \mathbb{N}$ and $q \in \mathbb{Q}^n$. Denote by $\boldsymbol{q} \in \Gamma(\mathbb{T})$ and $\boldsymbol{q}' \in \Gamma(f^*\mathbb{T})$ the constant sections associated to q. It is then easy to check that $\boldsymbol{q}' = f^*\boldsymbol{q}$. This grants that

$$\mathsf{S}(f^*\mathbb{T}) \subseteq \left\{ \sum_{i \in \mathbb{N}} \chi_{A_i} \cdot f^*s_i \;\middle|\; (A_i)_i \text{ is a Borel partition of X, } (s_i)_i \subseteq \Gamma(\mathbb{T}) \right\}.$$

Since $\mathsf{S}(f^*\mathbb{T})$ is dense in $\Gamma(f^*\mathbb{T})$ by Lemma 3.1, we finally conclude that the second in (4.9) is verified as well. Therefore the statement is achieved. □

We now introduce the notions of dual module and of dual bundle.

Let $\mathscr{M}$ be an $L^0(\mathfrak{m})$-normed $L^0(\mathfrak{m})$-module. We define the *dual module* $\mathscr{M}^*$ as the space of all $L^0(\mathfrak{m})$-linear continuous maps $T : \mathscr{M} \to L^0(\mathfrak{m})$, endowed with the pointwise norm

$$|T|_* := \operatorname*{ess\,sup}_{\substack{v \in \mathscr{M}: \\ |v| \leq 1}} \big|T(v)\big| \in L^0(\mathfrak{m}) \qquad \text{for every } T \in \mathscr{M}^*. \tag{4.10}$$

It can be readily proven that $\mathscr{M}^*$ has a natural structure of $L^0(\mathfrak{m})$-normed $L^0(\mathfrak{m})$-module.

Definition 4.12 (Dual bundle) *Let $\mathbb{T}$ be a measurable Banach bundle over $\mathbb{X}$. Choose a representative $\overline{\mathbb{T}} = (T, \underline{E}, \pi, \mathbf{n})$ of $\mathbb{T}$. Let us set*

$$\mathbf{n}^*(x, v) := \begin{cases} \sup_{w \in (\overline{\mathbb{T}})_x \setminus \{0\}} \frac{|v \cdot w|}{\mathbf{n}(x,w)} & \text{if } \mathbf{n}(x, \cdot) \text{ is a norm,} \\ 0 & \text{otherwise.} \end{cases} \tag{4.11}$$

Then we define the dual bundle $\mathbb{T}^*$ *as the equivalence class of $(T, \underline{E}, \pi, \mathbf{n}^*)$, which turns out to be a measurable Banach bundle over $\mathbb{X}$.*

Theorem 4.13 *Let $\mathbb{T}$ be a measurable Banach bundle over $\mathbb{X}$. Then*

$$\Gamma(\mathbb{T})^* = \Gamma(\mathbb{T}^*). \tag{4.12}$$

Proof. Consider the operator $\iota : \Gamma(\mathbb{T}^*) \to \Gamma(\mathbb{T})^*$ defined as follows: given any $s^* \in \Gamma(\mathbb{T}^*)$, we call $\iota(s^*) : \Gamma(\mathbb{T}) \to L^0(\mathfrak{m})$ the map sending (the equivalence class of) any section $\overline{s}$ to the function $\mathrm{X} \ni x \mapsto \overline{s}^*(x) \cdot \overline{s}(x) \in \mathbb{R}$, where $\overline{s}^*$ is any representative of s^*. One can easily deduce from its very construction that ι is a module morphism that preserves the pointwise norm.

To conclude, it only remains to show that the map ι is surjective. Let $T \in \Gamma(\mathbb{T})^*$ be fixed. For any $n \in \mathbb{N}$, denote by $\mathsf{e}_1^n, \ldots, \mathsf{e}_n^n$ the canonical basis of $\mathbb{R}^n$ and by $\mathbf{e}_1^n, \ldots, \mathbf{e}_n^n \in \Gamma(\mathbb{T})$ the associated constant sections. Hence let us define $s^* \in \Gamma(\mathbb{T}^*)$ as

$$s^*(x) := \sum_{n \in \mathbb{N}} \Big(x, \big(T\mathbf{e}_1^n(x), \ldots, T\mathbf{e}_n^n(x) \big) \Big) \quad \text{for } \mathfrak{m}\text{-a.e. } x \in \mathrm{X}.$$

Simple computations show that $\iota(s^*) = T$. Hence ι is surjective, concluding the proof. □

A Comparison with the Serre-Swan theorem for smooth manifolds

We point out the main analogies and differences between our work and the Serre-Swan theorem for smooth manifolds, for whose presentation we refer to [20, Chapter 11].

The result in the smooth case can be informally stated as follows: *the category of smooth vector bundles over a connected manifold M is equivalent to the category of finitely-generated projective $C^\infty(M)$-modules.*

In our non-smooth setting we had to replace 'smooth' with 'measurable', in a sense, and this led to these discrepancies with the case of manifolds:

i) The fibers of a measurable Banach bundle need not have the same dimension (still, they are finite dimensional), while on a connected manifold any smooth vector bundle must have constant dimension by topological reasons.

ii) In the definition of measurable Banach bundle we do not speak about the analogous of the 'trivialising diffeomorphisms', the reason being that one can always patch together countably many measurable maps still obtaining a measurable map. Hence there is no loss of generality in requiring the total space of the bundle to be of the form $\bigsqcup_{n \in \mathbb{N}} E_n \times \mathbb{R}^n$ and its measurable subsets to be those sets whose intersection with each $E_n \times \mathbb{R}^n$ is a Borel set.

iii) Given that we want to correlate the measurable Banach bundles with the $L^0(\mathfrak{m})$-normed $L^0(\mathfrak{m})$-modules, which are naturally equipped with a pointwise norm $|\cdot|$, we also require the existence of a function n that assigns a norm to (almost) every fiber of our bundle. A similar structure is not treated in the smooth case.

iv) The Serre-Swan theorem for smooth manifolds deals with modules that are finitely-generated and projective. In our context, any finitely-generated module is automatically projective, as seen in Proposition 1.5. Moreover, the flexibility of $L^0(\mathfrak{m})$ actually allowed us to extend the result to all proper modules, that are not necessarily 'globally' finitely-generated but only 'locally' finitely-generated, in a sense.

B A variant for L^p-normed L^∞-modules

The original presentation of the concept of L^0-normed L^0-module, which has been proposed in [6], follows a different line of thought with respect to the one presented here. In [6] it is first given the notion of *L^p-normed L^∞-module*, then by suitably completing such objects one obtains the class of L^0-normed L^0-modules. The role of this completion is to 'remove any integrability requirement'. On the other hand, the axiomatisation of L^0-normed L^0-modules that we presented in Subsection 1.2 is taken from [8].

Our choice of using the language of L^0-normed L^0-modules, instead of L^p-normed L^∞-modules, is only a matter of practicity and is not due to any theoretical reason. Indeed, in this appendix we show that all the results we obtained so far can be suitably reformulated for L^p-normed L^∞-modules.

Let $\mathbb{X} = (\mathrm{X}, \mathsf{d}, \mathfrak{m})$ be a given metric measure space. Fix an exponent $p \in [1, \infty]$. In order to keep a distinguished notation, we shall indicate by $\mathscr{M}^p$ the $L^p(\mathfrak{m})$-normed $L^\infty(\mathfrak{m})$-modules, for whose definition and properties we refer to [6] or [8], while the $L^0(\mathfrak{m})$-normed $L^0(\mathfrak{m})$-modules will be denoted by $\mathscr{M}^0$. The category of $L^p(\mathfrak{m})$-normed $L^\infty(\mathfrak{m})$-modules is denoted by $\mathbf{NMod}^p(\mathbb{X})$ and that of $L^0(\mathfrak{m})$-normed $L^0(\mathfrak{m})$-modules by $\mathbf{NMod}^0(\mathbb{X})$. Moreover, the subcategories of $\mathbf{NMod}^p(\mathbb{X})$ and $\mathbf{NMod}^0(\mathbb{X})$ that consist of all proper modules will be called $\mathbf{NMod}^p_{\mathrm{pr}}(\mathbb{X})$ and $\mathbf{NMod}^0_{\mathrm{pr}}(\mathbb{X})$, respectively. Observe that we added the exponent 0 to the notation of Definition 1.15. Similarly, we shall denote by Γ_0 the section functor Γ that has been introduced in Definition 2.9.

The following results look upon the relation that subsists between the class of $L^p(\mathfrak{m})$-normed $L^\infty(\mathfrak{m})$-modules and that of $L^0(\mathfrak{m})$-normed $L^0(\mathfrak{m})$-modules. First of all, it has been proved in [8, Theorem/Definition 1.7] that

Theorem B.1 (L^0-completion) *Let $\mathscr{M}^p$ be an $L^p(\mathfrak{m})$-normed $L^\infty(\mathfrak{m})$-module. Then there exists a unique couple $(\mathscr{M}^0, \iota)$, called* L^0-completion *of $\mathscr{M}^p$, where $\mathscr{M}^0$ is an $L^0(\mathfrak{m})$-normed $L^0(\mathfrak{m})$-module and $\iota : \mathscr{M}^p \to \mathscr{M}^0$ is a linear map with dense image that preserves the pointwise norm. Uniqueness has to be intended up to unique isomorphism.*

It can be easily seen that the local dimension of a module is invariant under taking the L^0-completion, namely for any Borel set $E \subseteq \mathrm{X}$ with $\mathfrak{m}(E) > 0$ and for any $n \in \mathbb{N}$ it holds

$$\mathscr{M}^p \text{ has dimension } n \text{ on } E \quad \Longleftrightarrow \quad \mathscr{M}^0 \text{ has dimension } n \text{ on } E. \tag{B.1}$$

Given two $L^p(\mathfrak{m})$-normed $L^\infty(\mathfrak{m})$-modules $\mathscr{M}^p$, $\mathscr{N}^p$ and a module morphism $\Phi : \mathscr{M}^p \to \mathscr{N}^p$, there exists a unique module morphism $\widetilde{\Phi} : \mathscr{M}^0 \to \mathscr{N}^0$ extending Φ, where $\mathscr{M}^0$ and $\mathscr{N}^0$ denote the L^0-completions of $\mathscr{M}^p$ and $\mathscr{N}^p$, respectively.

Definition B.2 (L^0-completion functor) *We define the* L^0-completion functor *as the functor $\mathsf{C}^p : \mathbf{NMod}^p(\mathbb{X}) \to \mathbf{NMod}^0(\mathbb{X})$ that assigns to any $\mathscr{M}^p$ its L^0-completion $\mathscr{M}^0$ and to any module morphism $\Phi : \mathscr{M}^p \to \mathscr{N}^p$ its unique extension $\widetilde{\Phi} : \mathscr{M}^0 \to \mathscr{N}^0$.*

Conversely, given any $L^0(\mathfrak{m})$-normed $L^0(\mathfrak{m})$-module $\mathscr{M}^0$, one has that

$$\mathscr{M}^p := \big\{ v \in \mathscr{M}^0 \;\big|\; |v| \in L^p(\mathfrak{m}) \big\} \quad \text{has a structure of } L^p(\mathfrak{m})\text{-normed } L^\infty(\mathfrak{m})\text{-module.} \tag{B.2}$$

Moreover, it holds that the L^0-completion of $\mathscr{M}^p$ is the original module $\mathscr{M}^0$.

Definition B.3 (L^p-restriction functor) *The* L^p-restriction functor *is defined as that functor* $\mathsf{R}^p : \mathbf{NMod}^0(\mathbb{X}) \to \mathbf{NMod}^p(\mathbb{X})$ *that assigns to any* $\mathscr{M}^0$ *its 'restriction'* $\mathscr{M}^p$*, as in* (B.2)*, and to any module morphism* $\widetilde{\Phi} : \mathscr{M}^0 \to \mathscr{N}^0$ *its restriction* $\Phi := \widetilde{\Phi}|_{\mathscr{M}^p} : \mathscr{M}^p \to \mathscr{N}^p$*, which turns out to be a morphism of* $L^p(\mathfrak{m})$*-normed* $L^\infty(\mathfrak{m})$*-modules.*

We can finally collect all of the properties described so far in the following statement:

Theorem B.4 ($\mathbf{NMod}^p(\mathbb{X})$ is equivalent to $\mathbf{NMod}^0(\mathbb{X})$) *Both the functors* C^p *and* R^p *are equivalence of categories, one the inverse of the other.*

Property (B.1) ensures that $\mathscr{M}^p$ and $\mathsf{C}^p(\mathscr{M}^p)$ have the same dimensional decomposition, thus in particular the above functors naturally restrict to $\mathsf{C}^p_{\rm pr} : \mathbf{NMod}^p_{\rm pr}(\mathbb{X}) \to \mathbf{NMod}^0_{\rm pr}(\mathbb{X})$ and $\mathsf{R}^p_{\rm pr} : \mathbf{NMod}^0_{\rm pr}(\mathbb{X}) \to \mathbf{NMod}^p_{\rm pr}(\mathbb{X})$. Therefore:

Corollary B.5 ($\mathbf{NMod}^p_{\rm pr}(\mathbb{X})$ is equivalent to $\mathbf{NMod}^0_{\rm pr}(\mathbb{X})$) *The functors* $\mathsf{C}^p_{\rm pr}$ *and* $\mathsf{R}^p_{\rm pr}$ *are equivalence of categories, one the inverse of the other.*

Now fix a measurable Banach bundle $\mathbb{T}$ over $\mathbb{X}$. Then let us define

$$\Gamma_p(\mathbb{T}) := \big\{ s \in \Gamma_0(\mathbb{T}) \;\big|\; |s| \in L^p(\mathfrak{m}) \big\}. \tag{B.3}$$

The space $\Gamma_p(\mathbb{T})$ can be viewed as an $L^p(\mathfrak{m})$-normed $L^\infty(\mathfrak{m})$-module. Moreover, given any two measurable Banach bundles $\mathbb{T}_1$, $\mathbb{T}_2$ over $\mathbb{X}$ and a bundle morphism $\varphi \in \mathrm{Mor}(\mathbb{T}_1, \mathbb{T}_2)$, let us define $\Gamma_p(\varphi) \in \mathrm{Mor}\big(\Gamma_p(\mathbb{T}_1), \Gamma_p(\mathbb{T}_2)\big)$ as

$$\Gamma_p(\varphi) := \Gamma_0(\varphi)|_{\Gamma_p(\mathbb{T}_1)} : \Gamma_p(\mathbb{T}_1) \to \Gamma_p(\mathbb{T}_2). \tag{B.4}$$

Hence such construction induces an L^p-section functor $\Gamma_p : \mathbf{MBB}(\mathbb{X}) \to \mathbf{NMod}^p_{\rm pr}(\mathbb{X})$. Then

$$\begin{tikzcd} \mathbf{MBB}(\mathbb{X}) \arrow[r, "\Gamma_0"] \arrow[dr, "\Gamma_p"'] & \mathbf{NMod}^0_{\rm pr}(\mathbb{X}) \arrow[d, "\mathsf{R}^p_{\rm pr}"] \\ & \mathbf{NMod}^p_{\rm pr}(\mathbb{X}) \end{tikzcd} \tag{B.5}$$

is a commutative diagram. We can finally conclude that

Theorem B.6 (Serre-Swan for L^p-normed L^∞-modules) *It holds that the* L^p*-section functor* $\Gamma_p : \mathbf{MBB}(\mathbb{X}) \to \mathbf{NMod}^p_{\rm pr}(\mathbb{X})$ *on* $\mathbb{X}$ *is an equivalence of categories.*

Proof. It follows from Theorem 3.4, from Corollary B.5 and from the fact that the diagram in (B.5) commutes. □

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