EXISTENCE OF SOLUTIONS TO A PHASE–FIELD MODEL OF DYNAMIC FRACTURE WITH A CRACK–DEPENDENT DISSIPATION

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Abstract. We propose a phase–field model of dynamic crack propagation based on the Ambrosio–Tortorelli approximation, which takes in account dissipative effects due to the speed of the crack tips. In particular, adapting the time discretization scheme contained in [4, 14], we show the existence of a dynamic crack evolution satisfying an energy dissipation balance, according to Griffith's criterion.

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1. Introduction

In this paper we present a phase–field model of dynamic brittle fracture based on a suitable adaptation of Griffith's dynamic criterion [15], and different from the one proposed in [4, 13, 14]. Following these papers, we rely on the Ambrosio–Tortorelli functional [2], which provides a good approximation of the corresponding stationary problem.

In the quasi–static setting, namely when the data vary slowly compared to the elastic wave speed of the material, Griffith's criterion [11] states that in the crack growth there is an exact balance between the decrease in stored elastic energy and the energy used to increase the crack. This is turned into a precise definition for a sharp interface model in [9]; in the antiplane case, for a given time–dependent Dirichlet condition $t \mapsto w(t)$, the following energy functional is considered

$$\frac{1}{2}\int_{\Omega\setminus K}|\nabla u|^2\,dx + \mathcal{H}^{d-1}(K), \quad K \text{ compact},\ u \in H^1(\Omega\setminus K),\ u = w(t) \text{ on } \partial\Omega, \tag{1.1}$$

where Ω is an open subset of $\mathbb{R}^d$ and $\mathcal{H}^{d-1}$ is the $d-1$–dimensional Hausdorff measure. The first term in (1.1) is the stored elastic energy, while the second one represents the energy used to produce a crack. In this setting, a quasi–static evolution is a time–dependent pair $t \mapsto (u(t), K(t))$ satisfying the unilateral minimality condition

$$\frac{1}{2}\int_{\Omega\setminus K(t)}|\nabla u(t)|^2\,dx + \mathcal{H}^{d-1}(K(t)) \le \frac{1}{2}\int_{\Omega\setminus K^*}|\nabla u^*|^2\,dx + \mathcal{H}^{d-1}(K^*) \tag{1.2}$$

for every compact set $K^* \supseteq K(t)$ and for every function $u^* \in H^1(\Omega\setminus K^*)$, $u^* = w(t)$ on $\partial\Omega$, complemented by an energy balance which includes the work done by the boundary data.

Searching a minimizer for this functional can be quite hard (for a detailed analysis of (1.1) we refer to [3] and the reference therein). For this reason in [2] the authors introduce a regularized version of (1.1): the set K is replaced by a function $v \in [0,1]$ which takes a value near 0 in a small neighborhood of K, and a value near 1 far from it. More precisely for every $\varepsilon > 0$ they consider

$$\mathcal{E}_\varepsilon(u,v) := \frac{1}{2}\int_\Omega((v^+)^2+\eta_\varepsilon)|\nabla u|^2\,dx, \quad \mathcal{H}_\varepsilon(v) := \frac{1}{4\varepsilon}\int_\Omega|1-v|^2\,dx + \varepsilon\int_\Omega|\nabla v|^2\,dx$$

with $0 < \eta_\varepsilon \ll \varepsilon$, where v^+ denotes the positive part of v, i.e., $v^+ := \max\{v, 0\}$. A minimum point $(u_\varepsilon, v_\varepsilon)$ of $\mathcal{E}_\varepsilon + \mathcal{H}_\varepsilon$ provides a good approximation of a minimizer (u, K) of (1.1), in the sense that u_ε is close to u, v_ε is close to 0 near K, and $\mathcal{E}_\varepsilon(u_\varepsilon, v_\varepsilon) + \mathcal{H}_\varepsilon(u_\varepsilon, v_\varepsilon)$ approximates the energy (1.1). The minimum problem (1.2) is replaced by

$$\mathcal{E}_\varepsilon(u_\varepsilon(t), v_\varepsilon(t)) + \mathcal{H}_\varepsilon(u_\varepsilon(t), v_\varepsilon(t)) \le \mathcal{E}_\varepsilon(u^*, v^*) + \mathcal{H}_\varepsilon(u^*, v^*) \quad \text{for every } u^*, v^* \in H^1(\Omega),\ v^* \le v_\varepsilon(t), \tag{1.3}$$

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where the inequality $v^* \leq v_\varepsilon(t)$ reflects the inclusion $K^* \supseteq K(t)$; we refer to [10] for the convergence of the quasi–static problems. In particular for a quasi–static evolution $t \mapsto (u_\varepsilon(t), v_\varepsilon(t))$ the following properties hold

(Q_1) $u_\varepsilon(t)$ solves $\operatorname{div}[((v_\varepsilon^+(t))^2 + \eta_\varepsilon)\nabla u_\varepsilon(t)] = 0$ in Ω with suitable boundary conditions,
(Q_2) $v_\varepsilon(t)$ minimizes $v^* \mapsto \mathcal{E}_\varepsilon(u_\varepsilon(t), v^*) + \mathcal{H}_\varepsilon(v^*)$ subject to $v^* \leq v_\varepsilon(t)$,
(Q_3) $(u_\varepsilon(t), v_\varepsilon(t))$ satisfies an energy balance.

In the dynamic case the first condition (Q_1) is replaced by the wave equation, while in the energy balance we also need to take in account the kinetic energy term. Developing these principles, in [4] the authors propose the following phase–field model for dynamic crack propagation

(D_1) $u_\varepsilon(t)$ solves $\ddot{u}_\varepsilon(t) - \operatorname{div}[((v_\varepsilon^+(t))^2 + \eta_\varepsilon)\nabla u_\varepsilon(t)] = 0$ in $(0,T) \times \Omega$ with suitable boundary conditions,
(D_2) $v_\varepsilon(t)$ minimizes $v^* \mapsto \mathcal{E}_\varepsilon(u_\varepsilon(t), v^*) + \mathcal{H}_\varepsilon(v^*)$ subject to $v^* \leq v_\varepsilon(t)$ for every $t \in [0,T]$,
(D_3) $(u_\varepsilon(t), v_\varepsilon(t))$ satisfies a dynamic energy balance.

A solution of (D_1)–(D_3) is approximated by a time discretization scheme: to pass from the previous time to the next one, one solves a wave equation for u keeping v fixed, and then a minimum problem for v keeping u fixed. This method is used [14] to prove the existence of a pair (u, v) satisfying (D_1)–(D_3) in the more general linear elastic case, that is when $|\nabla u|^2$ is replaced by $\mathbb{A}eu \cdot eu$, where $\mathbb{A}$ is the elastic tensor and $eu := \frac{1}{2}(\nabla u + \nabla u^T)$ is the symmetrized gradient. For technical reasons a viscoelastic dissipation term is added to (D_1), which means they consider

$$\ddot{u}_\varepsilon(t) - \operatorname{div}[((v_\varepsilon^+(t))^2 + \eta_\varepsilon)\mathbb{A}(eu_\varepsilon(t) + e\dot{u}_\varepsilon(t))] = 0 \quad \text{in } (0,T) \times \Omega.$$

The disadvantage of this term appears when we consider the behaviour of the solution as $\varepsilon \to 0$, a problem which is out of the scope of this paper. If we were able to prove the convergence of the solutions, as $\varepsilon \to 0$, toward the evolution of a dynamic sharp interface, then the energy dissipation balance for the damped wave equation in cracked domains [6] would imply that the limit crack does not depend on time.

To avert this problem, we propose a different model which avoids viscoelastic terms on the displacement, and instead takes into account dissipative effects due to the speed of the crack tips. More precisely, given a forcing term $t \mapsto f(t)$, a phase–field dynamic evolution is a pair $t \mapsto (u_\varepsilon(t), v_\varepsilon(t))$ such that

(D_1') $u_\varepsilon(t)$ solves $\ddot{u}_\varepsilon(t) - \operatorname{div}[((v_\varepsilon^+(t))^2 + \eta_\varepsilon)\mathbb{A}eu_\varepsilon(t)] = f(t)$ in $(0,T) \times \Omega$ with suitable boundary conditions,
(D_2') $v_\varepsilon(t)$ minimizes $v^* \mapsto \mathcal{E}_\varepsilon(u_\varepsilon(t), v^*) + \mathcal{H}_\varepsilon(v^*) + \partial\mathcal{G}(\dot{v}_\varepsilon(t))[v^*]$ subject to $v^* \leq v_\varepsilon(t)$ for a.e. $t \in (0,T)$,
(D_3') $(u_\varepsilon(t), v_\varepsilon(t))$ satisfies a dynamic energy dissipation balance,

where the functional $\mathcal{G}$ is defined as

$$\mathcal{G}(\sigma) := \frac{1}{2}\sum_{i=0}^{k} c_i \int_\Omega \sum_{|\alpha|=i} |\partial^\alpha \sigma|^2 \, dx, \quad \sigma \in H^k(\Omega), \quad k \in \mathbb{N}. \tag{1.4}$$

In the case of zero boundary data, the energy dissipation balance reads as

$$\mathcal{F}_\varepsilon(u_\varepsilon(t), \dot{u}_\varepsilon(t), v_\varepsilon(t)) + 2\int_0^t \mathcal{G}(\dot{v}_\varepsilon(s)) \, ds = \mathcal{F}_\varepsilon(u_\varepsilon(0), \dot{u}_\varepsilon(0), v_\varepsilon(0)) + \int_0^t \int_\Omega f(s) \cdot \dot{u}_\varepsilon(s) \, dx \, ds, \tag{1.5}$$

where $\mathcal{F}_\varepsilon$ denotes the total energy, which is given by

$$\mathcal{F}_\varepsilon(v, w, v) := \mathcal{E}_\varepsilon(u, v) + \mathcal{H}_\varepsilon(v) + \frac{1}{2}\int_\Omega |w|^2 \, dx.$$

Notice that, to get the balance (1.5), we also need to take into account the energy dissipated due to the presence of term $\mathcal{G}$.

Adapting the discretization procedure of [4, 14] we show the existence of a dynamic evolution $t \mapsto (u_\varepsilon(t), v_\varepsilon(t))$ which satisfies (D_1')–(D_3') for $k > d/2$, where k is the greatest order of derivatives that appear in $\mathcal{G}$, and d is the dimension of the space. This term $\mathcal{G}$ that appears in (D_2') and (D_3') guarantees that the phase–field function v is more regular in time, more precisely that $\dot{v} \in L^2((0,T); H^k(\Omega))$. When the viscoelastic dissipation is neglected, this regularity is crucial to obtain the energy estimate (3.18), which is one of the key tools in the proof of our existence result. As explained in Remark 2.1, this term is the counterpart, in the phase–field model, of a dissipation depending on the crack speed in the sharp interface model (see, e.g., [17]).

The paper is organized as follows: in Section 2 we fix the notation and list the main assumptions of our model. In Lemma 2.4 we show that the energy dissipation balance (D_3') is equivalent to the stability condition (2.17), and in Theorem 2.6 we state our main existence result. Section 3 is devoted to the study of the time discretization scheme. We construct an approximation in time of the evolution by solving, with an alternating scheme, the wave equation (D_1') and the unilateral minimality condition (D_2'). Next, we show that the discrete evolution satisfies the estimates (3.18), which allow us to pass to the limit as the time step tends to zero. In particular, without requiring any assumption on the greatest order of derivatives appearing in (1.4), we show the existence of a dynamic evolution $t \mapsto (u(t), v(t))$ which satisfies (D_1') and (D_2') at least in a weaker sense, and only an energy dissipation inequality. Finally, in Section 4 we prove that for $k > d/2$ our evolution is more regular in time, and it satisfies the energy identity (1.5).

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2. Notation and preliminary results

The space of $m \times d$ matrices with real entries is denoted by $\mathbb{R}^{m\times d}$; in case $m = d$ we denote by $\mathbb{R}^{d\times d}_{sym}$ the subspace of symmetric matrices. The space of linear and continuous operators from $\mathbb{R}^{d\times d}_{sym}$ to itself is denoted by $\mathscr{L}(\mathbb{R}^{d\times d}_{sym}; \mathbb{R}^{d\times d}_{sym})$; the image of $\xi \in \mathbb{R}^{d\times d}_{sym}$ under $\mathbb{A} \in \mathscr{L}(\mathbb{R}^{d\times d}_{sym}; \mathbb{R}^{d\times d}_{sym})$ is denoted by $\mathbb{A}\xi \in \mathbb{R}^{d\times d}_{sym}$.

Let $\Omega \subset \mathbb{R}^d$ be a bounded open set with Lipschitz boundary, let $\partial_{D_1}\Omega$, $\partial_{D_2}\Omega$ be two (possibly empty) Borel subsets of $\partial\Omega$, and let $\partial_{N_1}\Omega$, $\partial_{N_2}\Omega$ be their complements. We denote by $\mathcal{H}^{d-1}$ the $(d-1)$–dimensional Hausdorff measure, and by ν the outer unit normal vector to $\partial\Omega$, which is defined $\mathcal{H}^{d-1}$–a.e. on the boundary.

The partial derivatives with respect to the variable x_i are denoted by ∂_i. Given a function $u\colon \mathbb{R}^d \to \mathbb{R}^m$, we denote its Jacobian matrix by ∇u, whose components are $(\nabla u)_{ij} := \partial_j u_i$. For every multiindex $\alpha = (\alpha_1, \dots, \alpha_d) \in \mathbb{N}^d$, we use $\partial^\alpha u$ to denote the α–derivative of u, which is defined by $\partial^\alpha u := \partial_1^{\alpha_1} \cdots \partial_d^{\alpha_d} u$, with the usual convention $\partial_i^0 u := u$. Given a tensor field $A\colon \mathbb{R}^d \to \mathbb{R}^{m\times m}$, by $\operatorname{div} A$ we mean its divergence with respect to lines, namely $(\operatorname{div} A)_i := \sum_j \partial_j A_{ij}$.

We adopt standard notation for Lebesgue and Sobolev spaces; the boundary values of a Sobolev function are always intended in the sense of traces. To simplify the notation, we shall use $\|\cdot\|_p$ and $\|\cdot\|_{H^k}$ to denote the norms in $L^p(\Omega; \mathbb{R}^m)$ and $H^k(\Omega; \mathbb{R}^m)$, respectively, and $\langle \cdot\,, \cdot \rangle_2$ to denote the scalar product in $L^2(\Omega; \mathbb{R}^m)$.

Given an interval $I \subset \mathbb{R}$ and a Banach space X, the spaces $C^k(I; X)$, $L^p(I; X)$, $W^{k,p}(I; X)$, and $H^k(I; X)$ are defined as usual. Given $u \in L^p(I; X)$ we use $\dot u \in \mathcal{D}'(I; X)$ to denote its distributional time derivative. When dealing with an element $u \in H^1(I; X)$ we always assume u to be the *continuous* representative of its class. Therefore it makes sense to consider the pointwise value $u(t)$ for every $t \in [0, T]$. To simplify the notation, we shall write $\|\cdot\|_{L^p(X)}$ and $\|\cdot\|_{H^k(X)}$ to denote the norms in $L^p(I; X)$ and $H^k(I; X)$, respectively, omitting the domain Ω when X is a Lebesgue or a Sobolev space on Ω.

Let $\mathbb{A}\colon \Omega \to \mathscr{L}(\mathbb{R}^{d\times d}_{sym}; \mathbb{R}^{d\times d}_{sym})$ be a tensor field satisfying the following natural assumptions in linear elasticity:

$$\mathbb{A} \in L^\infty(\Omega; \mathscr{L}(\mathbb{R}^{d\times d}_{sym}; \mathbb{R}^{d\times d}_{sym})), \tag{2.1}$$

$$\mathbb{A}(x)\xi_1 \cdot \xi_2 = \xi_1 \cdot \mathbb{A}(x)\xi_2 \quad \text{for a.e. } x \in \Omega \text{ and for every } \xi_1, \xi_2 \in \mathbb{R}^{d\times d}_{sym}, \tag{2.2}$$

$$\mathbb{A}(x)\xi \cdot \xi \geq c_{\mathbb{A}} |\xi|^2 \quad \text{for a.e. } x \in \Omega \text{ and for every } \xi \in \mathbb{R}^{d\times d}_{sym}, \tag{2.3}$$

for a constant $c_{\mathbb{A}} > 0$. Thanks to the Second Korn Inequality (see, e.g., [16]) there exists a constant $K > 0$, depending on Ω, such that

$$\|u\|_{H^1} \leq K(\|u\|_2 + \|eu\|_2) \quad \text{for every } u \in H^1(\Omega; \mathbb{R}^d), \tag{2.4}$$

where $eu := \frac{1}{2}(\nabla u + \nabla u^T)$ is the symmetrized gradient of u, and ∇u^T is the transpose matrix of ∇u. By combining the Korn Inequality (2.4) with the assumption (2.3), we obtain that $\mathbb{A}$ satisfies the following ellipticity condition of integral type:

$$\langle \mathbb{A}eu, eu \rangle_2 \geq c_0 \|\nabla u\|_2^2 - c_1 \|u\|_2^2 \quad \text{for every } u \in H^1(\Omega; \mathbb{R}^d), \tag{2.5}$$

for two positive constants c_0 and c_1.

We introduce the spaces

$$H^1_{D_1}(\Omega;\mathbb{R}^d):=\{u\in H^1(\Omega;\mathbb{R}^d):u=0\text{ on }\partial_{D_1}\Omega\}\quad\text{and}\quad H^1_{D_2}(\Omega):=\{v\in H^1(\Omega):v=0\text{ on }\partial_{D_2}\Omega\},$$

and we denote by $H^{-1}_{D_1}(\Omega;\mathbb{R}^d)$ the dual of $H^1_{D_1}(\Omega;\mathbb{R}^d)$. To simplify the notation, we use $\|\cdot\|_*$ to denote the norm of $H^{-1}_{D_1}(\Omega;\mathbb{R}^d)$, and $\langle\cdot,\cdot\rangle_*$ to denote the duality pair between $H^1_{D_1}(\Omega;\mathbb{R}^d)$ and $H^{-1}_{D_1}(\Omega;\mathbb{R}^d)$. The transpose of the natural injection of $H^1_{D_1}(\Omega;\mathbb{R}^d)\hookrightarrow L^2(\Omega;\mathbb{R}^d)$ induces the injection of $L^2(\Omega;\mathbb{R}^d)$ into $H^{-1}_{D_1}(\Omega;\mathbb{R}^d)$, which is defined by $\langle f,\psi\rangle_*:=\langle f,\psi\rangle_2$ for every $f\in L^2(\Omega;\mathbb{R}^d)$ and $\psi\in H^1_{D_1}(\Omega;\mathbb{R}^d)$.

For every $\varepsilon>0$ fixed, we define the elastic energy functional $\mathcal{E}\colon H^1(\Omega;\mathbb{R}^d)\times H^1(\Omega)\to[0,+\infty]$ and the phase–field surface energy functional $\mathcal{H}\colon H^1(\Omega)\to[0,+\infty)$ in the following way:

$$\mathcal{E}(u,v):=\frac12\int_\Omega b(v)\mathbb{A}eu\cdot eu\,dx\quad\text{and}\quad\mathcal{H}(v):=\frac{1}{4\varepsilon}\int_\Omega|1-v|^2\,dx+\varepsilon\int_\Omega|\nabla v|^2\,dx,$$

where $b\colon\mathbb{R}\to\mathbb{R}$ is a function satisfying

$$b\in C^1(\mathbb{R})\text{ is convex and non decreasing,}\tag{2.6}$$

$$b(s)\ge\eta>0\text{ for every }s\in\mathbb{R}.\tag{2.7}$$

Notice that these assumptions hold true for $b(s):=(\max\{s,0\})^2+\eta$, $s\in\mathbb{R}$. Furthermore, we define two other functionals $\mathcal{K}\colon L^2(\Omega;\mathbb{R}^d)\to[0,+\infty)$ and $\mathcal{G}\colon H^k(\Omega)\to[0,+\infty)$ as

$$\mathcal{K}(w):=\frac12\int_\Omega|w|^2\,dx\quad\text{and}\quad\mathcal{G}(\sigma):=\frac12\sum_{i=0}^k c_i\int_\Omega\sum_{|\alpha|=i}|\partial^\alpha\sigma|^2\,dx,$$

where $k\in\mathbb{N}$, c_i are positive real numbers, and $|\alpha|:=\alpha_1+\dots+\alpha_d$. In particular $\mathcal{K}(\dot u(t))$ represents the kinetic energy of a function u in $H^1(I;L^2(\Omega;\mathbb{R}^d))$ at time $t\in I$. Finally, we define the total energy $\mathcal{F}\colon H^1(\Omega;\mathbb{R}^d)\times L^2(\Omega;\mathbb{R}^d)\times H^1(\Omega)\to[0,+\infty]$ as

$$\mathcal{F}(u,w,v):=\mathcal{K}(w)+\mathcal{E}(u,v)+\mathcal{H}(v).$$

Throughout the paper we always assume that k is a positive integer, $\mathbb{A}$ and b satisfy (2.1)–(2.3), (2.6), and (2.7), and T, ε, and c_i, $i=0,\dots,k$, are positive real numbers. Given

$$w_1\in H^2((0,T);L^2(\Omega;\mathbb{R}^d))\cap H^1((0,T);H^1(\Omega;\mathbb{R}^d)),\quad w_2\in H^k(\Omega)\text{ with }w_2\le1\text{ on }\partial_{D_2}\Omega,\tag{2.8}$$

$$f\in L^2((0,T);L^2(\Omega;\mathbb{R}^d)),\quad g\in H^1((0,T);H^{-1}_{D_1}(\Omega;\mathbb{R}^d)),\tag{2.9}$$

$$u_0-w_1(0)\in H^1_{D_1}(\Omega;\mathbb{R}^d),\quad u_1\in L^2(\Omega;\mathbb{R}^d),\quad v_0-w_2\in H^1_{D_2}(\Omega)\cap H^k(\Omega)\text{ with }v_0\le1\text{ in }\Omega,\tag{2.10}$$

we search a solution (u,v) of the elastodynamics system

$$\ddot u(t)-\operatorname{div}[b(v(t))\mathbb{A}eu(t)]=f(t)+g(t)\quad\text{in }\Omega\quad\text{for a.e. }t\in(0,T),\tag{2.11}$$

with boundary conditions formally written as

$$u(t)=w_1(t)\quad\text{on }\partial_{D_1}\Omega\quad\text{for a.e. }t\in(0,T),\tag{2.12}$$

$$v(t)=w_2\quad\text{on }\partial_{D_2}\Omega\quad\text{for a.e. }t\in(0,T),\tag{2.13}$$

$$b(v(t))\mathbb{A}eu(t)\nu=0\quad\text{on }\partial_{N_1}\Omega\quad\text{for a.e. }t\in(0,T),\tag{2.14}$$

and initial conditions

$$u(0)=u_0,\quad\dot u(0)=u_1,\quad v(0)=v_0.\tag{2.15}$$

We also require that such solution satisfies the following crack stability condition

$$v(t)\in\operatorname*{argmin}_{\substack{v^*-w_2\in H^1_{D_2}(\Omega)\cap H^k(\Omega)\\ v^*\le v(t)}}\{\mathcal{E}(u(t),v^*)+\mathcal{H}(v^*)+\partial\mathcal{G}(\dot v(t))[v^*]\}\quad\text{for a.e. }t\in(0,T)\tag{2.16}$$

and the identity

$$\partial_v\mathcal{E}(u(t),v(t))[\dot v(t)]+\partial\mathcal{H}(v(t))[\dot v(t)]+2\mathcal{G}(\dot v(t))=0\quad\text{for a.e. }t\in(0,T),\tag{2.17}$$

where the derivatives $\partial_v\mathcal{E}$, $\partial\mathcal{H}$, and $\partial\mathcal{G}$ take the form

$$\partial_v\mathcal{E}(u,v)[\varphi]=\frac12\int_\Omega b'(v)\varphi\mathbb{A}eu\cdot eu\,dx\quad\text{for every }u\in H^1(\Omega;\mathbb{R}^d)\text{ and }v,\varphi\in H^1(\Omega)\cap L^\infty(\Omega),$$

$$\partial\mathcal{H}(v)[\varphi] = \frac{1}{2\varepsilon}\int_\Omega (v-1)\varphi\,dx + 2\varepsilon\int_\Omega \nabla v\cdot\nabla\varphi\,dx \quad \text{for every } v,\varphi\in H^1(\Omega),$$
$$\partial\mathcal{G}(\sigma)[\rho] = \sum_{i=0}^k c_i\int_\Omega \sum_{|\alpha|=i}\partial^\alpha\sigma\partial^\alpha\rho\,dx \quad \text{for every } \sigma,\rho\in H^k(\Omega).$$

Remark 2.1. We give an idea of the meaning, in the phase–field model, of the term $\mathcal{G}(\dot v)$ by comparing it with a dissipation, in the sharp interface model, which depends on the velocity of the crack tip. We consider just an example in the particular case $d=2$ and $k=0$ of a rectilinear crack moving along the axis x, $K(t) := \{(\sigma,0) : \sigma\le s(t)\}$ with $s\in C^1([0,T])$, $s(0)=0$, and $\dot s(t)\ge 0$ for every $t\in[0,T]$. In view of the analysis done in [2], the sequence $v_\varepsilon(t)$ which best approximate $K(t)$ takes the following form:

$$v_\varepsilon(t,x) := \Psi\left(\frac{\operatorname{dist}(x,K(t))}{\varepsilon}\right), \quad (t,x)\in[0,T]\times\mathbb{R}^2.$$

Here $\Psi\colon\mathbb{R}\to[0,1]$ is a C^1 function such that $\Psi(s)=0$ for $|s|\le\delta$, $0<\delta<1$, and $\Psi(s)=1$ for $|s|\ge 1$. Notice that the function v_ε, which belongs to $C^1([0,T]\times\mathbb{R}^2)$, is constantly 0 in a $\varepsilon\delta$–neighborhood of $K(t)$, and takes the value 1 outside a ε–neighborhood of $K(t)$. Moreover, its time derivative has the following form

$$\dot v_\varepsilon(t,x) = -\frac{\dot s(t)}{\varepsilon}\partial_1\Phi\left(\frac{x-(s(t),0)}{\varepsilon}\right) \quad \text{for every } (t,x)\in[0,T]\times\mathbb{R}^2,$$

where $\Phi(y) := \Psi(\operatorname{dist}(y,K(0)))$. In particular for every $t\in[0,T]$ we deduce that

$$\|\dot v_\varepsilon(t)\|_2^2 = \frac{|\dot s(t)|^2}{\varepsilon^2}\int_{\mathbb{R}^2}\left|\partial_1\Phi\left(\frac{x-(s(t),0)}{\varepsilon}\right)\right|^2 dx = |\dot s(t)|^2\int_{\mathbb{R}^2}|\partial_1\Phi(y)|^2\,dy = C_\Phi|\dot s(t)|^2.$$

Hence this term detects the dissipative effects due to the velocity of the moving crack. Notice that, with similar computations, if the crack has a finite number of tips with velocities $\dot s_i(t)$, $i=1,\dots,n$, then $\|\dot v_\varepsilon(t)\|_2^2$ corresponds to a dissipation of the form $\sum_{i=1}^n C_i\dot s_i^2(t)$, with $C_i>0$.

To make precise the notion of solution of problem (2.11)–(2.14), we consider a pair of functions (u,v) satisfying the following regularity assumptions:

$$u\in C^0([0,T];H^1(\Omega;\mathbb{R}^d))\cap C^1([0,T];L^2(\Omega;\mathbb{R}^d))\cap H^2((0,T);H^{-1}_{D_1}(\Omega;\mathbb{R}^d)), \tag{2.18}$$
$$u(t)-w_1(t)\in H^1_{D_1}(\Omega;\mathbb{R}^d) \text{ for every } t\in[0,T], \tag{2.19}$$
$$v\in H^1((0,T);H^k(\Omega)), \tag{2.20}$$
$$v(t)-w_2\in H^1_{D_2}(\Omega) \text{ and } v(t)\le 1 \text{ in } \Omega \text{ for every } t\in[0,T]. \tag{2.21}$$

Definition 2.2. Let w_1, w_2, f, and g be as in (2.8) and (2.9). We say that (u,v) is a *weak solution* of the elastodynamics system (2.11) with boundary conditions (2.12)–(2.14), if (u,v) satisfies (2.18)–(2.21), and for a.e. $t\in(0,T)$ the following equation holds

$$\langle\ddot u(t),\psi\rangle_* + \langle b(v(t))\mathbb{A}eu(t),e\psi\rangle_2 = \langle f(t),\psi\rangle_2 + \langle g(t),\psi\rangle_* \quad \text{for every } \psi\in H^1_{D_1}(\Omega;\mathbb{R}^d). \tag{2.22}$$

Remark 2.3. Since b is a non negative increasing function and $v(t)\le 1$ for every $t\in[0,T]$, the function $b(v(t))$ is uniformly bounded in Ω for every $t\in[0,T]$. Hence the equation (2.22) is well defined for every $\psi\in H^1_{D_1}(\Omega;\mathbb{R}^d)$.

The meaning of the stability condition (2.17) is explained in the following lemma, where it is shown that it is related to the energy dissipation balance. For every (u,v) satisfying (2.18)–(2.21), we define the *total work* over the time interval $[t_1,t_2]\subset[0,T]$ as

$$\begin{aligned}\mathcal{W}(t_1,t_2;u,v) := &\int_{t_1}^{t_2}[\langle f(s),\dot u(s)-\dot w_1(s)\rangle_2 + \langle b(v(s))\mathbb{A}eu(s),e\dot w_1(s)\rangle_2]\,ds\\ &+\langle g(t_2),u(t_2)-w_1(t_2)\rangle_* - \langle g(t_1),u(t_1)-w_1(t_1)\rangle_* - \int_{t_1}^{t_2}\langle\dot g(s),u(s)-w_1(s)\rangle_*\,ds\\ &+\langle\dot u(t_2),\dot w_1(t_2)\rangle_2 - \langle\dot u(t_1),\dot w_1(t_1)\rangle_2 - \int_{t_1}^{t_2}\langle\ddot w_1(s),\dot u(s)\rangle_2\,ds.\end{aligned}$$

Notice that $\mathcal{W}(t_1,t_2;u,v)$ is a continuous function of t_1 and t_2, thanks to the previous assumptions on $\mathbb{A}$, b, w_1, f, and g.

Lemma 2.4. *Let $k>d/2$ and let w_1, w_2, f, g, u_0, u_1, and v_0 be as in (2.8)–(2.10). Assume that (u,v) is a weak solution of the problem (2.11)–(2.14) satisfying the initial conditions (2.15). Then the identity (2.17) is equivalent to the following energy dissipation balance*

$$\mathcal{F}(u(t),\dot u(t),v(t))+2\int_0^t \mathcal{G}(\dot v(s))\,ds=\mathcal{F}(u_0,u_1,v_0)+\mathcal{W}(0,t;u,v)\quad \textit{for every } t\in[0,T]. \tag{2.23}$$

Proof. We follow the main ideas contained in the proof of [7, Lemma 2.6]. Define for $0<h<T$ the function

$$\psi_h(t):=\frac{u(t+h)-u(t)}{h}-\frac{w_1(t+h)-w_1(t)}{h},\quad t\in[0,T-h].$$

We use $\psi_h(t)$ as test function in (2.22) first at time t and then at time $t+h$. Summing the two expressions and integrating in $[t_1,t_2]\subset[0,T-h]$, we obtain the identity

$$\begin{aligned}&\int_{t_1}^{t_2}\langle \ddot u(t+h)+\ddot u(t),\psi_h(t)\rangle_*\,dt+\int_{t_1}^{t_2}\langle b(v(t+h))\mathbb{A}eu(t+h)+b(v(t))\mathbb{A}eu(t),e\psi_h(t)\rangle_2\,dt\\&=\int_{t_1}^{t_2}\langle f(t+h)+f(t),\psi_h(t)\rangle_2\,dt+\int_{t_1}^{t_2}\langle g(t+h)+g(t),\psi_h(t)\rangle_*\,dt.\end{aligned}\tag{2.24}$$

We study these four terms separately. By performing an integration by parts, the first one becomes

$$\begin{aligned}&\int_{t_1}^{t_2}\langle \ddot u(t+h)+\ddot u(t),\psi_h(t)\rangle_*\,dt\\&=-\int_{t_1}^{t_2}\langle \dot u(t+h)+\dot u(t),\dot\psi_h(t)\rangle_2\,dt+\langle \dot u(t_2+h)+\dot u(t_2),\psi_h(t_2)\rangle_2-\langle \dot u(t_1+h)+\dot u(t_1),\psi_h(t_1)\rangle_2\\&=-\frac1h\int_{t_2}^{t_2+h}\|\dot u(t)\|_2^2\,dt+\frac1h\int_{t_1}^{t_1+h}\|\dot u(t)\|_2^2\,dt+\int_{t_1}^{t_2}\left\langle \dot u(t+h)+\dot u(t),\frac{\dot w_1(t+h)-\dot w_1(t)}{h}\right\rangle_2 dt\\&+\langle \dot u(t_2+h)+\dot u(t_2),\psi_h(t_2)\rangle_2-\langle \dot u(t_1+h)+\dot u(t_1),\psi_h(t_1)\rangle_2.\end{aligned}$$

Sending $h\to0$ and using the fact that $u,w_1\in C^1([0,T];L^2(\Omega;\mathbb{R}^d))$ we deduce that

$$\begin{aligned}&-\frac1h\int_{t_2}^{t_2+h}\|\dot u(t)\|_2^2\,dt+\frac1h\int_{t_1}^{t_1+h}\|\dot u(t)\|_2^2\,dt+\langle \dot u(t_2+h)+\dot u(t_2),\psi_h(t_2)\rangle_2-\langle \dot u(t_1+h)+\dot u(t_1),\psi_h(t_1)\rangle_2\\&\to\|\dot u(t_2)\|_2^2-2\langle \dot u(t_2),\dot w_1(t_2)\rangle-\|\dot u(t_1)\|_2^2+2\langle \dot u(t_1),\dot w_1(t_1)\rangle_2.\end{aligned}\tag{2.25}$$

Notice that $[\dot w_1(\,\cdot\,+h)-\dot w_1]/h$ converges strongly in $L^2((t_1,t_2);L^2(\Omega;\mathbb{R}^d))$ to $\ddot w_1$, since $\dot w_1$ belongs to $H^1((0,T);L^2(\Omega;\mathbb{R}^d))$. In particular there exists a subsequence $(h_i)_i$ and a function $\kappa\in L^2(t_1,t_2)$ such that for a.e. $t\in(t_1,t_2)$

$$\begin{gathered}\left\langle \dot u(t+h_i)+\dot u(t),\frac{\dot w_1(t+h_i)-\dot w_1(t)}{h_i}\right\rangle_2\to2\langle \dot u(t),\ddot w_1(t)\rangle_2,\\ \left|\left\langle \dot u(t+h_i)+\dot u(t),\frac{\dot w_1(t+h_i)-\dot w_1(t)}{h_i}\right\rangle_2\right|\le2\|\dot u\|_{L^\infty(L^2)}k(t).\end{gathered}$$

The Dominated Convergence Theorem and the fact that the limit does not depend on the subsequence allow us to conclude that

$$\lim_{h\to0}\int_{t_1}^{t_2}\left\langle \dot u(t+h)+\dot u(t),\frac{\dot w_1(t+h)-\dot w_1(t)}{h}\right\rangle_2 dt=2\int_{t_1}^{t_2}\langle \dot u(t),\ddot w_1(t)\rangle_2\,dt.\tag{2.26}$$

For the term involving f, observe that $f(\,\cdot\,+h)\to f$ and $\psi_h\to\dot u-\dot w_1$ in $L^2((t_1,t_2);L^2(\Omega;\mathbb{R}^d))$. Hence

$$\lim_{h\to0}\int_{t_1}^{t_2}\langle f(t+h)+f(t),\psi_h(t)\rangle_2\,dt=2\int_{t_1}^{t_2}\langle f(t),\dot u(t)-\dot w_1(t)\rangle_2\,dt.\tag{2.27}$$

Moreover, using the identity

$$\int_{t_1}^{t_2}\langle g(t+h)+g(t),\psi_h(t)\rangle_*\,dt=\frac2h\int_{t_2}^{t_2+h}\langle g(t),u(t)-w_1(t)\rangle_*\,dt-\frac2h\int_{t_1}^{t_1+h}\langle g(t),u(t)-w_1(t)\rangle_*\,dt$$

$$- \int_{t_1}^{t_2} \left\langle \frac{g(t+h)-g(t)}{h}, u(t+h)+u(t)-w_1(t+h)-w_1(t) \right\rangle_* dt,$$

and proceeding as for (2.25) and (2.26), it is possible to deduce that

$$\begin{aligned} &\lim_{h\to 0} \int_{t_1}^{t_2} \langle g(t+h)+g(t), \psi_h(t)\rangle_* \, dt \\ &= 2\langle g(t_2), u(t_2)-w_1(t_2)\rangle_* - 2\langle g(t_1), u(t_1)-w_1(t_1)\rangle_* - 2\int_{t_1}^{t_2} \langle \dot g(t), u(t)-w_1(t)\rangle_* \, dt. \end{aligned} \tag{2.28}$$

It remains to study the last term, that we can write in the following form

$$\begin{aligned} &\int_{t_1}^{t_2} \langle b(v(t+h))\mathbb{A}eu(t+h) + b(v(t))\mathbb{A}eu(t), e\psi_h(t)\rangle_2 \, dt \\ =&\frac{1}{h}\int_{t_2}^{t_2+h} \langle b(v(t))\mathbb{A}eu(t), eu(t)\rangle_2 \, dt - \frac{1}{h}\int_{t_1}^{t_1+h} \langle b(v(t))\mathbb{A}eu(t), eu(t)\rangle_2 \, dt \\ &- \int_{t_1}^{t_2} \left\langle \frac{b(v(t+h))-b(v(t))}{h}\mathbb{A}eu(t), eu(t+h)\right\rangle_2 dt \\ &- \int_{t_1}^{t_2} \left\langle b(v(t+h))\mathbb{A}eu(t+h) + b(v(t))\mathbb{A}eu(t), \frac{ew_1(t+h)-ew_1(t)}{h}\right\rangle_2 dt. \end{aligned}$$

For $k > d/2$ the injection $H^k(\Omega) \hookrightarrow C^0(\overline{\Omega})$ is continuous and compact (see, e.g., [1]). Hence the function $v \in H^1((0,T);C^0(\overline{\Omega}))$ and this, together with $u \in C^0([0,T];H^1(\Omega;\mathbb{R}^d))$, gives us that as $h \to 0$

$$\begin{aligned} &\frac{1}{h}\int_{t_2}^{t_2+h} \langle b(v(t))\mathbb{A}eu(t), eu(t)\rangle_2 \, dt - \frac{1}{h}\int_{t_1}^{t_1+h} \langle b(v(t))\mathbb{A}eu(t), eu(t)\rangle_2 \, dt \\ &\to \langle b(v(t_2))\mathbb{A}eu(t_2), eu(t_2)\rangle_2 - \langle b(v(t_1))\mathbb{A}eu(t_1), eu(t_1)\rangle_2. \end{aligned} \tag{2.29}$$

Since $\dot v \in L^2((0,T);C^0(\overline{\Omega}))$, we deduce that $[v(t+h)-v(t)]/h \to \dot v(t)$ in $C^0(\overline{\Omega})$ for a.e. $t \in (0,T)$. In particular, using also the differentiability of b and the fact that $u \in C^0([0,T];H^1(\Omega;\mathbb{R}^d))$, we obtain that

$$\left\langle \frac{b(v(t+h))-b(v(t))}{h}\mathbb{A}eu(t), eu(t+h)\right\rangle_2 \to \langle b'(v(t))\dot v(t)\mathbb{A}eu(t), eu(t)\rangle_2 \quad \text{for a.e. } t \in (0,T).$$

Moreover $[v(\cdot+h)-v]/h \to \dot v$ in $L^2((t_1,t_2);C^0(\overline{\Omega}))$, and so there exists a subsequence $(h_i)_i$ and a function $\kappa \in L^2(t_1,t_2)$ such that

$$\left\| \frac{v(\cdot+h_i)-v(t)}{h_i}\right\|_\infty \le \kappa(t) \quad \text{for a.e. } t \in [t_1,t_2].$$

Thus, the following estimate holds

$$\left| \left\langle \frac{b(v(t+h_i))-b(v(t))}{h_i}\mathbb{A}eu(t), eu(t+h_i)\right\rangle_2 \right| \le b'(\|v\|_\infty)C_{\mathbb{A}}\|eu\|^2_{L^\infty(L^2)}\kappa(t) \quad \text{for a.e. } t \in (0,T).$$

By the Dominated Convergence Theorem and the fact that the limit does not depend on the subsequence, we conclude that

$$\lim_{h\to 0}\int_{t_1}^{t_2} \left\langle \frac{b(v(t+h))-b(v(t))}{h}\mathbb{A}eu(t), eu(t+h)\right\rangle_2 dt = \int_{t_1}^{t_2} \langle b'(v(t))\dot v(t)\mathbb{A}eu(t), eu(t)\rangle_2 \, dt. \tag{2.30}$$

Finally, notice that $[ew_1(\cdot+h)-ew_1]/h$ converges strongly to $e\dot w_1$ in $L^2((t_1,t_2);L^2(\Omega;\mathbb{R}^{d\times d}))$. Arguing again as in (2.26), this implies the following fact

$$\begin{aligned} &\lim_{h\to 0}\int_{t_1}^{t_2} \left\langle b(v(t+h))\mathbb{A}eu(t+h) + b(v(t))\mathbb{A}eu(t), \frac{ew_1(t+h)-ew_1(t)}{h}\right\rangle_2 dt \\ &= 2\int_{t_1}^{t_2} \langle b(v(t))\mathbb{A}eu(t), e\dot w_1(t)\rangle \, dt. \end{aligned} \tag{2.31}$$

Combining together (2.24) with (2.25)–(2.31) we get that (u,v) satisfies

$$\begin{aligned}&\mathcal{K}(\dot u(t_2))+\mathcal{E}(u(t_2),v(t_2))-\frac{1}{2}\int_{t_1}^{t_2}\langle b'(v(t))\dot v(t)\mathbb{A}eu(t),eu(t)\rangle_2\,dt\\&=\mathcal{K}(\dot u(t_1))+\mathcal{E}(u(t_1),v(t_1))+\mathcal{W}(t_1,t_2;u,v)\end{aligned}\tag{2.32}$$

for every $t_1,t_2\in[0,T)$ with $t_1<t_2$. In particular, since all the terms inside (2.32) are continuous with respect to t_2, we deduce that a weak solution of (2.11)–(2.14) satisfies the identity

$$\mathcal{K}(\dot u(t))+\mathcal{E}(u(t),v(t))-\frac{1}{2}\int_0^t\langle b'(v(t))\dot v(t)\mathbb{A}eu(t),eu(t)\rangle_2\,dt=\mathcal{K}(u_1)+\mathcal{E}(u_0,v_0)+\mathcal{W}(0,t;u,v)\tag{2.33}$$

for every $t\in[0,T]$.

Assume now that the pair (u,v) satisfies (2.17). Since $v\in H^1((0,T);H^k(\Omega))$, the function $h(t):=\mathcal{H}(v(t))$ belongs to $AC([0,T])$ with $\dot h(t)=\partial\mathcal{H}(v(t))[\dot v(t)]$ for a.e. $t\in(0,T)$. Hence, by integrating (2.17) in $[0,t]$, with $t\in[0,T]$, we get that

$$-\frac{1}{2}\int_0^t\langle b'(v(t))\dot v(t)\mathbb{A}eu(t),eu(t)\rangle_2\,dt=\mathcal{H}(v(t))-\mathcal{H}(v_0)+2\int_0^t\mathcal{G}(v(t))\,dt.\tag{2.34}$$

The above identity, together with (2.33), implies the energy equality (2.23). Similarly, if (2.23) holds, by combining it with (2.33) we deduce (2.34) for every $t\in[0,T]$, from which (2.17) follows. □

Remark 2.5. In according to Griffith's criterion (see [11]), we expect that the sum of kinetic energy and elastic energy is dissipated during the evolution, while it is balanced when we take into account the surface energy associated to the phase–field function v. In our case this happens if we consider also the energy term $2\int_0^t\mathcal{G}(\dot v(s))\,ds$. This takes in account the rate at which the function v is decreasing and it is a consequence of the dissipation on $\dot v$ that we added to (2.16). In particular to obtain the energy equality (2.23) we need $k>d/2$, since we pass throughout (2.33), which is well defined only when $\dot v(t)\in L^\infty(\Omega)$.

We state now our main result, whose proof will be given at the end of the paper.

Theorem 2.6. *Let $k>d/2$ and let w_1, w_2, f, g, u_0, u_1, and v_0 be as in (2.8)–(2.10). Then there exist a weak solution (u,v) of the problem (2.11)–(2.14) which satisfies the initial conditions (2.15). Moreover $\dot v(t)\le 0$ in Ω for a.e. $t\in(0,T)$. Finally, the crack stability condition (2.16) and the identity (2.17) hold true for a.e. $t\in(0,T)$.*

Remark 2.7. In Theorem 2.6 we consider only the case of zero Neumann boundary data. Anyway, the previous result can be easily adapted to Neumann boundary conditions of the form

$$b(v(t))\mathbb{A}eu(t)\nu=h(t)\quad\text{on }\partial_{N_1}\Omega\quad\text{for a.e. }t\in(0,T),\tag{2.35}$$

provided that $h\in H^1((0,T);L^2(\partial_{N_1}\Omega;\mathbb{R}^d))$. Indeed, in this case a *weak solution* of the problem (2.11)–(2.13) with Neumann boundary condition (2.35) is a pair (u,v) which satisfies (2.18)–(2.21) and is a solution for a.e. $t\in(0,T)$ of the equation

$$\langle\ddot u(t),\psi\rangle_*+\langle b(v(t))\mathbb{A}eu(t),e\psi\rangle_2=\langle f(t),\psi\rangle_2+\langle\tilde g(t),\psi\rangle_*\quad\text{for every }\psi\in H^1_{D_1}(\Omega;\mathbb{R}^d),$$

where the term $\tilde g(t)\in H^{-1}_{D_1}(\Omega;\mathbb{R}^d)$ is defined for $t\in[0,T]$ as

$$\langle\tilde g(t),\psi\rangle_*:=\langle g(t),\psi\rangle_*+\int_{\partial_{N_1}\Omega}h(t)\cdot\psi\,d\mathcal{H}^{d-1}\quad\text{for every }\psi\in H^1_{D_1}(\Omega;\mathbb{R}^d).$$

Since $\tilde g\in H^1((0,T);H^{-1}_{D_1}(\Omega;\mathbb{R}^d))$, we can apply the previous statement with $\tilde g$ instead of g, and we obtain the existence of a weak solution of the problem (2.11)–(2.13) with Neumann boundary conditions (2.35).

3. The time discretization scheme

In this section we show some general results that are true for every $k\in\mathbb{N}$, where k is the greatest order of derivatives that appear in $\mathcal{G}$. In particular we prove that the problem (2.11)–(2.15) admits always a solution, at least in a weaker sense, for which the minimality condition (2.16) hold. From now on, we always assume that w_1, w_2, f, g, u_0, u_1, and v_0 satisfy (2.8)–(2.10).

We start by introducing the following notion of solution, which requires less regularity in u on the time variable.

Definition 3.1. We say that (u, v) is a *generalized solution* of problem (2.11)–(2.14) if

$$u \in L^\infty((0,T); H^1(\Omega;\mathbb{R}^d)) \cap W^{1,\infty}((0,T); L^2(\Omega;\mathbb{R}^d)) \cap H^2((0,T); H^{-1}_{D_1}(\Omega;\mathbb{R}^d)), \tag{3.1}$$

$$u - w_1 \in L^\infty((0,T); H^1_{D_1}(\Omega;\mathbb{R}^d)), \tag{3.2}$$

v satisfies (2.20) and (2.21), and the equation (2.22) holds for a.e. $t \in (0,T)$.

Remark 3.2. Recall that, given a reflexive Banach space X, the space of weakly continuous functions is defined as

$$C_w([0,T]; X) := \{\eta\colon [0,T] \to X : \text{for all } x^* \in X^* \text{ the function } t \mapsto \langle x^*, \eta(t)\rangle_X \text{ is continuous}\}.$$

If Y is another Banach space, with injection $X \hookrightarrow Y$ continuous, then

$$C_w([0,T]; Y) \cap L^\infty((0,T); X) \subset C_w([0,T]; X)$$

(see, e.g., [8]). Hence, if (u, v) is a generalized solution of (2.11)–(2.14), then $u \in C_w([0,T]; H^1(\Omega;\mathbb{R}^d))$ and $\dot u \in C_w([0,T]; L^2(\Omega;\mathbb{R}^d))$. In particular the initial conditions (2.15) make sense, since the functions $u(t)$ and $\dot u(t)$ are uniquely defined for every $t \in [0,T]$ as elements of $H^1(\Omega;\mathbb{R}^d)$ and $L^2(\Omega;\mathbb{R}^d)$, respectively.

To show the existence of a generalized solution for (2.11)–(2.14), we approximate our problem by using a time discretization scheme, as done in [4, 14]. We divide the time interval $[0,T]$ introducing N equispaced nodes, and in each node we solve, in an alternating way, the wave equation (2.22) with v fixed, and the unilateral minimality condition (2.16) with u fixed. Finally, we consider some interpolants of the resulting approximants and, thanks to an a priori estimate, we pass to the limit as $N \to \infty$.

Given $N \in \mathbb{N}$ we define the time–discrete problems in the following way: we set

$$\tau = T/N, \quad u^0_\tau := u_0, \quad \delta u^0_\tau := u_1, \quad v^0_\tau := v_0,$$

and for $n = 1, \dots, N$ we define inductively

$$u^n_\tau \in \operatorname*{argmin}_{u^* - w^n_\tau \in H^1_{D_1}(\Omega;\mathbb{R}^d)} \left\{ \frac12 \left\| \frac{u^* - u^{n-1}_\tau}{\tau} - \delta u^{n-1}_\tau \right\|_2^2 + \mathcal{E}(u^*, v^{n-1}_\tau) - \langle f^n_\tau, u^*\rangle_2 - \langle g^n_\tau, u^* - w^n_\tau\rangle_* \right\}, \tag{3.3}$$

$$v^n_\tau \in \operatorname*{argmin}_{\substack{v^* - w_2 \in H^1_{D_2}(\Omega) \cap H^k(\Omega) \\ v^* \le v^{n-1}_\tau}} \left\{ \mathcal{E}(u^n_\tau, v^*) + \mathcal{H}(v^*) + \frac{1}{\tau}\mathcal{G}(v^* - v^{n-1}_\tau) \right\}, \tag{3.4}$$

where

$$\delta u^n_\tau = \frac{u^n_\tau - u^{n-1}_\tau}{\tau}, \quad \delta^2 u^n_\tau := \frac{\delta u^n_\tau - \delta u^{n-1}_\tau}{\tau}, \quad f^n_\tau := \frac{1}{\tau}\int_{(n-1)\tau}^{n\tau} f(s)ds, \quad n = 1, \dots, N,$$

$$g^n_\tau := g(n\tau), \quad w^n_\tau := w_1(n\tau), \quad n = 0, \dots, N.$$

Since $\mathbb{A}$ and b satisfy (2.1)–(2.3), (2.6), and (2.7), the time discrete problems are well defined. In particular for every $N \in \mathbb{N}$ and $n = 1, \dots, N$ there exists a unique pair of functions $(u^n_\tau, v^n_\tau) \in H^1(\Omega;\mathbb{R}^d) \times H^k(\Omega)$ which solve (3.3) and (3.4).

The minimality of u^n_τ implies that for $n = 1, \dots, N$

$$\langle \delta^2 u^n_\tau, \psi\rangle_2 + \langle b(v^{n-1}_\tau)\mathbb{A} e u^n_\tau, e\psi\rangle_2 = \langle f^n_\tau, \psi\rangle_2 + \langle g^n_\tau, \psi\rangle_* \quad \text{for every } \psi \in H^1_{D_1}(\Omega;\mathbb{R}^d), \tag{3.5}$$

which is the discrete counterpart of (2.22). Moreover, we can characterize v^n_τ in the following way.

Lemma 3.3. *For every $n = 1, \dots, N$ the function $v^n_\tau \in \mathcal{V}^n_\tau := \{\varphi - w_2 \in H^1_{D_2}(\Omega) \cap H^k(\Omega) : \varphi \le v^{n-1}_\tau\}$ is the unique solution in $\mathcal{V}^n_\tau$ of the variational inequality*

$$\mathcal{E}(u^n_\tau, \varphi) - \mathcal{E}(u^n_\tau, v^n_\tau) + \partial\mathcal{H}(v^n_\tau)[\varphi - v^n_\tau] + \partial\mathcal{G}(\delta v^n_\tau)[\varphi - v^n_\tau] \ge 0 \quad \textit{for every } \varphi \in \mathcal{V}^n_\tau. \tag{3.6}$$

In particular the following relation holds

$$\frac{\mathcal{E}(u^n_\tau, v^n_\tau) - \mathcal{E}(u^n_\tau, v^{n-1}_\tau)}{\tau} + \partial\mathcal{H}(v^n_\tau)[\delta v^n_\tau] + 2\mathcal{G}(\delta v^n_\tau) \le 0 \quad \textit{for } n = 1, \dots, N. \tag{3.7}$$

Proof. Let $v_\tau^n \in \mathcal{V}_\tau^n$ be the solution of (3.4) and let φ be an element of $\mathcal{V}_\tau^n$. For every $s \in [0,1]$ the function $\varphi_s := v_\tau^n + s(\varphi - v_\tau^n) \in \mathcal{V}_\tau^n$ is a competitor for (3.4). Hence, by the minimality of v_τ^n it results that

$$\mathcal{E}(u_\tau^n, v_\tau^n) + \mathcal{H}(v_\tau^n) + \frac{1}{\tau}\mathcal{G}(v_\tau^n - v_\tau^{n-1}) \leq \mathcal{E}(u_\tau^n, \varphi_s) + \mathcal{H}(\varphi_s) + \frac{1}{\tau}\mathcal{G}(\varphi_s - v_\tau^{n-1}).$$

After some simplification and dividing by $s \in (0,1]$, we deduce the following inequality

$$\frac{\mathcal{E}(u_\tau^n, \varphi_s) - \mathcal{E}(u_\tau^n, v_\tau^n)}{s} + \partial\mathcal{H}(v_\tau^n)[\varphi - v_\tau^n] + \partial\mathcal{G}(\delta v_\tau^n)[\varphi - v_\tau^n] + s\left[\mathcal{H}(\varphi - v_\tau^n) + \frac{1}{\tau}\mathcal{G}(\varphi - v_\tau^n)\right] \geq 0. \tag{3.8}$$

Since the function b is convex, the difference quotients are increasing. Thus

$$\frac{\mathcal{E}(u_\tau^n, \varphi_s) - \mathcal{E}(u_\tau^n, v_\tau^n)}{s} \leq \mathcal{E}(u_\tau^n, \varphi) - \mathcal{E}(u_\tau^n, v_\tau^n) \quad \text{for every } s \in (0,1]. \tag{3.9}$$

By combining (3.8) with (3.9) and passing to the limit as $s \to 0$, we obtain (3.6). On the other hand, exploiting the convexity of $\mathcal{H}$ and $\mathcal{G}$, it is easy to see that every solution of (3.6) satisfies (3.4).

Finally, the inequality (3.7) is obtained by taking $\varphi = v_\tau^{n-1} \in \mathcal{V}_\tau^n$ in (3.6) and by dividing it by τ. □

As done in [14], we now combine both the equation (3.5) and the inequality (3.7) to derive a discrete energy inequality for the family $(u_\tau^n, v_\tau^n)_{n=1}^N$.

Lemma 3.4. *The family $(u_\tau^n, v_\tau^n)_{n=1}^N$ satisfies the discrete energy inequality*

$$\begin{aligned}
&\mathcal{F}(u_\tau^j, \delta u_\tau^j, v_\tau^j) + 2\sum_{n=1}^j \tau\mathcal{G}(\delta v_\tau^n) + \sum_{n=1}^j \tau^2 D_\tau^n \leq \mathcal{F}(u_0, u_1, v_0) + \sum_{n=1}^j \tau\langle f_\tau^n, \delta u_\tau^n - \delta w_\tau^n\rangle_2 \\
&+ \langle g_\tau^j, u_\tau^j - w_\tau^j\rangle_* - \langle g(0), u_0 - w_1(0)\rangle_* - \sum_{n=1}^j \tau\langle \delta g_\tau^n, u_\tau^{n-1} - w_\tau^{n-1}\rangle_* \\
&+ \langle \delta u_\tau^j, \delta w_\tau^j\rangle_2 - \langle u_1, \dot w_1(0)\rangle_2 - \sum_{n=1}^j \tau\langle \delta u_\tau^{n-1}, \delta^2 w_\tau^n\rangle_2 + \sum_{n=1}^j \tau\langle b(v_\tau^{n-1})\mathbb{A}eu_\tau^n, e\delta w_\tau^n\rangle_2
\end{aligned} \tag{3.10}$$

for every $j = 1, \dots, N$, where $\delta w_\tau^0 := \dot w_1(0)$, $\delta w_\tau^n := (w_\tau^n - w_\tau^{n-1})/\tau$, $\delta^2 w_\tau^n := (\delta w_\tau^n - \delta w_\tau^{n-1})/\tau$, $\delta g_\tau^n := (g_\tau^n - g_\tau^{n-1})/\tau$ for $n = 1, \dots, N$, and the dissipation terms D_τ^n are defined as

$$D_\tau^n := \frac{1}{2}\|\delta^2 u_\tau^n\|_2^2 + \frac{1}{2}\langle b(v_\tau^{n-1})\mathbb{A}e\delta u_\tau^n, e\delta u_\tau^n\rangle_2 + \frac{1}{4\varepsilon}\|\delta v_\tau^n\|_2^2 + \varepsilon\|\nabla\delta v_\tau^n\|_2^2 \quad \textit{for } n = 1, \dots, N.$$

Proof. By using $\psi = \tau(\delta u_\tau^n - \delta w_\tau^n) \in H^1_{D_1}(\Omega;\mathbb{R}^d)$ as test function in (3.5), we deduce the following relation for every $n = 1, \dots, N$

$$\begin{aligned}
\tau\langle \delta^2 u_\tau^n, \delta u_\tau^n\rangle_2 + \tau\langle b(v_\tau^{n-1})\mathbb{A}eu_\tau^n, e\delta u_\tau^n\rangle_2 &= \tau\langle f_\tau^n, \delta u_\tau^n - \delta w_\tau^n\rangle_2 + \tau\langle g_\tau^n, \delta u_\tau^n - \delta w_\tau^n\rangle_* \\
&\quad + \tau\langle \delta^2 u_\tau^n, \delta w_\tau^n\rangle_2 + \tau\langle b(v_\tau^{n-1})\mathbb{A}eu_\tau^n, e\delta w_\tau^n\rangle_2.
\end{aligned} \tag{3.11}$$

The identity $|a|^2 - a\cdot b = \frac{1}{2}|a|^2 - \frac{1}{2}|b|^2 + \frac{1}{2}|a-b|^2$ for every $a, b \in \mathbb{R}^d$ allows us to write the first term as

$$\tau\langle \delta^2 u_\tau^n, \delta u_\tau^n\rangle_2 = \|\delta u_\tau^n\|_2^2 - \langle \delta u_\tau^{n-1}, \delta u^n\rangle_2 = \mathcal{K}(\delta u_\tau^n) - \mathcal{K}(\delta u_\tau^{n-1}) + \frac{\tau^2}{2}\|\delta^2 u_\tau^n\|_2^2. \tag{3.12}$$

Similarly, we have that

$$\begin{aligned}
\tau\langle b(v_\tau^{n-1})\mathbb{A}eu_\tau^n, e\delta u_\tau^n\rangle_2 &= \mathcal{E}(u_\tau^n, v_\tau^n) - \mathcal{E}(u_\tau^{n-1}, v_\tau^{n-1}) + \frac{\tau^2}{2}\langle b(v_\tau^{n-1})\mathbb{A}e\delta u_\tau^n, e\delta u_\tau^n\rangle_2 \\
&\quad + \frac{1}{2}\langle [b(v_\tau^{n-1}) - b(v_\tau^n)]\mathbb{A}eu_\tau^n, eu_\tau^n\rangle_2.
\end{aligned} \tag{3.13}$$

Thanks to (3.7) we can estimate from below the last term in (3.13) in the following way

$$\begin{aligned}
\frac{1}{2}\langle [b(v_\tau^{n-1}) - b(v_\tau^n)]\mathbb{A}eu_\tau^n, eu_\tau^n\rangle_2 &\geq \frac{\tau}{2\varepsilon}\langle v_\tau^n - 1, \delta v_\tau^n\rangle_2 + 2h\varepsilon\langle \nabla v_\tau^n, \nabla\delta v_\tau^n\rangle_2 + 2\tau\mathcal{G}(\delta v_\tau^n) \\
&= \mathcal{H}(v_\tau^n) - \mathcal{H}(v_\tau^{n-1}) + 2\tau\mathcal{G}(\delta v_\tau^n) + \frac{\tau^2}{4\varepsilon}\|\delta v_\tau^n\|_2^2 + \varepsilon\tau^2\|\nabla\delta v_\tau^n\|_2^2.
\end{aligned} \tag{3.14}$$

Hence, combining (3.11)–(3.14) we obtain that

$$\begin{aligned}&[\mathcal{K}(\delta u_\tau^n)+\mathcal{E}(u_\tau^n,v_\tau^n)+\mathcal{H}(v_\tau^n)]-[\mathcal{K}(\delta u_\tau^{n-1})+\mathcal{E}(u_\tau^{n-1},v_\tau^{n-1})+\mathcal{H}(v_\tau^{n-1})]+2\tau\mathcal{G}(\delta v_\tau^n)+\tau^2D_\tau^n\\&\le\tau\langle f_\tau^n,\delta u_\tau^n-\delta w_\tau^n\rangle_2+\tau\langle g_\tau^n,\delta u_\tau^n-\delta w_\tau^n\rangle_*+\tau\langle\delta^2u_\tau^n,\delta w_\tau^n\rangle_2+\tau\langle b(v_\tau^{n-1})\mathbb{A}eu_\tau^n,e\delta w_\tau^n\rangle_2\end{aligned}\tag{3.15}$$

for every $n=1,\dots,N$. Finally, we sum over $n=1,\dots,j$ for every $j=1,\dots,N$, and we use the identities

$$\sum_{n=1}^j\tau\langle g_\tau^n,\delta u_\tau^n-\delta w_\tau^n\rangle_*=\langle g_\tau^j,u_\tau^j-w_\tau^j\rangle_*-\langle g(0),u_0-w_1(0)\rangle_*-\sum_{n=1}^j\tau\langle\delta g_\tau^n,u_\tau^{n-1}-w_\tau^{n-1}\rangle_*,\tag{3.16}$$

$$\sum_{n=1}^j\tau\langle\delta^2u_\tau^n,\delta w_\tau^n\rangle_2=\langle\delta u_\tau^j,\delta w_\tau^j\rangle_2-\langle u_1,\dot w_1(0)\rangle_2-\sum_{n=1}^j\tau\langle\delta u_\tau^{n-1},\delta^2w_\tau^n\rangle_2.\tag{3.17}$$

to deduce the discrete energy inequality (3.10) from (3.15). □

The first consequence of (3.10) is the following a priori estimate.

Lemma 3.5. *There exists a constant $C>0$, independent of τ, such that*

$$\max_{n=0,\dots,N}\{\|u_\tau^n\|_{H^1}+\|\delta u_\tau^n\|_2+\|v_\tau^n\|_{H^1}\}+\sum_{n=1}^N\tau\|\delta v_\tau^n\|_{H^k}^2+\sum_{n=1}^N\tau^2D_\tau^n\le C.\tag{3.18}$$

Proof. Thanks to (2.3) and (2.7) we can estimate from below the left–hand side of (3.10) as

$$\mathcal{F}(u_\tau^j,\delta u_\tau^j,v_\tau^j)+2\sum_{n=1}^j\tau\mathcal{G}(\delta v_\tau^n)+\sum_{n=1}^j\tau^2D_\tau^n\ge\frac12\|\delta u_\tau^j\|_2^2+\frac{\eta c_{\mathbb{A}}}{2}\|eu_\tau^j\|_2^2\quad\text{for every }j=1,\dots,N.\tag{3.19}$$

Let us now bound from above the right–hand side of (3.19). We define

$$L_\tau:=\max_{n=0,\dots,N}\|\delta u_\tau^n\|_2,\quad M_\tau:=\max_{n=0,\dots,N}\|eu_\tau^n\|_2,\quad N_\tau:=\max_{n=0,\dots,N}\|u_\tau^n\|_{H^1},$$

and we use (2.8)–(2.10) to derive for every $j=1,\dots,N$ the following upper bounds

$$\langle g_\tau^j,u_\tau^j-w_\tau^j\rangle_*-\langle g(0),u_0-w_1(0)\rangle_*-\sum_{n=1}^j\tau\langle\delta g_\tau^n,u_\tau^{n-1}-w_\tau^{n-1}\rangle_*\le C_1N_\tau+C_2,\tag{3.20}$$

$$\sum_{n=1}^j\tau\langle f_\tau^n,\delta u_\tau^n-\delta w_\tau^n\rangle_2+\langle\delta u_\tau^j,\delta w_\tau^j\rangle_2-\langle u_1,\dot w_1(0)\rangle_2-\sum_{n=1}^j\tau\langle\delta u_\tau^{n-1},\delta^2w_\tau^n\rangle_2\le C_1L_\tau+C_2,\tag{3.21}$$

for some positive constants C_1 and C_2 independent of τ. Moreover, since $\mathbb{A}\in L^\infty(\Omega;\mathscr{L}(\mathbb{R}^{d\times d};\mathbb{R}^{d\times d}))$, b is non decreasing, and $v_\tau^{n-1}\le1$, we get that

$$\sum_{n=1}^j\tau\langle b(v_\tau^{n-1})\mathbb{A}eu_\tau^n,e\delta w_\tau^n\rangle_2\le b(1)C_{\mathbb{A}}\sqrt{T}\|e\dot w_1\|_{L^2(L^2)}M_\tau\quad\text{for every }j=1,\dots,N,\tag{3.22}$$

where $C_{\mathbb{A}}:=\|\mathbb{A}\|_{L^\infty(\Omega;\mathcal{L}(\mathbb{R}^{d\times d}_{sym};\mathbb{R}^{d\times d}_{sym}))}$. Thanks to the Korn Inequality (2.4) and the fact that

$$\|u_\tau^j\|_2\le\sum_{n=1}^N\tau\|\delta u_\tau^n\|_2+\|u_0\|_2\le L_\tau T+\|u_0\|_2\quad\text{for every }j=0,\dots,N,$$

we obtain the following relation between N_τ, L_τ, and M_τ

$$N_\tau\le K\max_{n=0,\dots,N}(\|u_\tau^n\|_2+\|eu_\tau^n\|_2)\le L_\tau KT+M_\tau K+K\|u_0\|_2.\tag{3.23}$$

Hence, by combining (3.10) with (3.19)–(3.23), we deduce the existence of two positive constants $\tilde C_1$ and $\tilde C_2$, independent of τ, such that $(L_\tau+M_\tau)^2\le\tilde C_1(L_\tau+M_\tau)+\tilde C_2$. This gives that L_τ and M_τ are uniformly bounded in τ, as well as N_τ, thanks to (3.23). In particular there exists a constant $C>0$, independent of τ, such that

$$\mathcal{K}(\delta u_\tau^j)+\mathcal{E}(u_\tau^j,v_\tau^j)+\mathcal{H}(v_\tau^j)+2\sum_{n=1}^N\tau\mathcal{G}(\delta v_\tau^n)+\sum_{n=1}^N\tau^2D_\tau^n\le C\quad\text{for every }j=1,\dots,N.$$

Finally, observe that for $j = 1, \dots, N$

$$\min\left\{\varepsilon, \frac{1}{4\varepsilon}\right\}\|v_\tau^j\|_{H^1} \le \mathcal{H}(v_\tau^j) \le C \quad \text{and} \quad \min_{i=0,\cdots,k} c_i \sum_{n=1}^N \tau\|\delta v_\tau^n\|_{H^k}^2 \le 2\sum_{n=1}^N \tau\mathcal{G}(\delta v_\tau^n) \le C.$$

This implies the remaining estimates. □

Remark 3.6. By combining together (3.5) and (3.18) we obtain also that

$$\sum_{n=1}^N \tau\|\delta^2 u_\tau^n\|_*^2 + \max_{n=0,\dots,N}\|v_\tau^n\|_{H^k} \le C \tag{3.24}$$

for some positive constant C independent of τ. Indeed by (3.5) we have that for every $n = 1, \dots, N$

$$\|\delta^2 u_\tau^n\|_* = \sup_{\substack{\psi\in H^1_{D_1}(\Omega;\mathbb{R}^d)\\ \|\psi\|_{H^1}\le 1}} |\langle\delta^2 u_\tau^n, \psi\rangle_2| \le b(1)C_{\mathbb{A}}\|eu_\tau^n\|_2 + \|f_\tau^n\|_2 + \|g\|_{L^\infty(H^{-1}_{D_1})}.$$

Hence, thanks to (2.9) and (3.18) there exists a constant $C > 0$, independent of τ, such that

$$\sum_{n=1}^N \tau\|\delta^2 u_\tau^n\|_*^2 \le C(1 + \|f\|_{L^2(L^2)} + \|g\|_{L^\infty(L^2)}).$$

Finally, also $\|v_\tau^j\|_{H^k}$ is uniformly bounded with respect to j and τ, since

$$\|v_\tau^j\|_{H^k} \le \sqrt{T}\left(\sum_{n=1}^N \tau\|\delta v_\tau^n\|_{H^k}^2\right)^{1/2} + \|v_0\|_{H^k} \quad \text{for every } j = 0, \dots, N.$$

We now use the functions u_τ^n and v_τ^n to construct a generalized solution for the problem (2.11)–(2.16). First, let us denote by u_τ the piecewise affine interpolant of $(u_\tau^n)_{n=1}^N$, which is defined for $t \in [0, T]$ as

$$u_\tau(t) := u_\tau^n + (t - n\tau)\delta u_\tau^n, \quad t \in [(n-1)\tau, n\tau], \quad n = 1, \dots, N.$$

Furthermore we define the backward interpolant u_τ^+ and the forward interpolant u_τ^- in the following way:

$$u_\tau^+(t) := u_\tau^n, \quad t \in ((n-1)\tau, n\tau], \quad n = 1, \dots, N, \quad u_\tau^+(0) := u_\tau^0,$$
$$u_\tau^-(t) := u_\tau^{n-1}, \quad t \in [(n-1)\tau, n\tau), \quad n = 1, \dots, N, \quad u_\tau^-(T) := u_\tau^N.$$

Similarly, we define the piecewise affine interpolant u'_τ of $(\delta u_\tau^n)_{n=1}^N$, as well as the backward interpolant $u_\tau'^+$ and the forward interpolant $u_\tau'^-$. Notice that $u_\tau \in H^1((0, T); H^1(\Omega; \mathbb{R}^d))$ and $u'_\tau \in H^1((0, T); L^2(\Omega; \mathbb{R}^d))$, with $\dot u_\tau(t) = u_\tau'^+(t) = \delta u_\tau^n$ and $\dot u'_\tau(t) = \delta^2 u_\tau^n$ for $t \in ((n-1)\tau, n\tau)$ and $n = 1, \dots N$.

Lemma 3.7. *There exist a function u in $L^\infty((0, T); H^1(\Omega; \mathbb{R}^d))$, with $\dot u$ in $L^\infty((0, T); L^2(\Omega; \mathbb{R}^d))$ and $\ddot u$ in $L^2((0, T); H^{-1}_{D_1}(\Omega; \mathbb{R}^d))$, and a subsequence of τ, not relabeled, such that the following convergences hold*

$$u_\tau \to u \quad \text{in } C^0([0, T]; L^2(\Omega; \mathbb{R}^d)), \qquad u_\tau \xrightharpoonup{*} u \quad \text{in } L^\infty((0, T); H^1(\Omega; \mathbb{R}^d)),$$
$$u_\tau^+ \xrightharpoonup{*} u \quad \text{in } L^\infty((0, T); H^1(\Omega; \mathbb{R}^d)), \qquad u_\tau^- \xrightharpoonup{*} u \quad \text{in } L^\infty((0, T); H^1(\Omega; \mathbb{R}^d)),$$
$$\dot u_\tau \xrightharpoonup{*} \dot u \quad \text{in } L^\infty((0, T); L^2(\Omega; \mathbb{R}^d)), \qquad u_\tau'^- \xrightharpoonup{*} \dot u \quad \text{in } L^\infty((0, T); L^2(\Omega; \mathbb{R}^d)),$$
$$u'_\tau \to \dot u \quad \text{in } C^0([0, T]; H^{-1}_{D_1}(\Omega; \mathbb{R}^d)), \qquad u'_\tau \xrightharpoonup{*} \dot u \quad \text{in } L^\infty((0, T); L^2(\Omega; \mathbb{R}^d)),$$
$$\dot u'_\tau \rightharpoonup \ddot u \quad \text{in } L^2((0, T); H^{-1}_{D_1}(\Omega; \mathbb{R}^d)).$$

In particular we have that

$$u_\tau(t) \rightharpoonup u(t) \quad \text{in } H^1(\Omega; \mathbb{R}^d) \quad \text{for every } t \in [0, T], \qquad \dot u_\tau(t) \rightharpoonup \dot u(t) \quad \text{in } L^2(\Omega; \mathbb{R}^d) \quad \text{for a.e. } t \in (0, T),$$
$$u_\tau^+(t) \rightharpoonup u(t) \quad \text{in } H^1(\Omega; \mathbb{R}^d) \quad \text{for a.e. } t \in (0, T), \qquad u_\tau^-(t) \rightharpoonup u(t) \quad \text{in } H^1(\Omega; \mathbb{R}^d) \quad \text{for a.e. } t \in (0, T).$$

Proof. Thanks to estimates (3.18), the sequence u_τ is uniformly bounded in the space $L^\infty((0, T); H^1(\Omega; \mathbb{R}^d)) \cap W^{1,\infty}((0, T); L^2(\Omega; \mathbb{R}^d))$. Hence, by Aubin–Lions Lemma (see [18]) there exist a function

$$u \in L^\infty((0, T); H^1(\Omega; \mathbb{R}^d)) \cap W^{1,\infty}((0, T); L^2(\Omega; \mathbb{R}^d))$$

and a subsequence of τ, not relabeled, such that

$$u_\tau \overset{*}{\rightharpoonup} u \quad \text{in } L^\infty((0,T);H^1(\Omega;\mathbb{R}^d)), \qquad \dot u_\tau \overset{*}{\rightharpoonup} \dot u \quad \text{in } L^\infty((0,T);L^2(\Omega;\mathbb{R}^d)),$$
$$u_\tau \to u \quad \text{in } C^0([0,T];L^2(\Omega;\mathbb{R}^d)), \qquad u_\tau(t) \to u(t) \quad \text{in } L^2(\Omega;\mathbb{R}^d) \quad \text{for every } t\in[0,T].$$

Moreover the functions $u_\tau(t)$ converge weakly to $u(t)$ in $H^1(\Omega;\mathbb{R}^d)$ for every $t\in[0,T]$, since they are uniformly bounded in $H^1(\Omega;\mathbb{R}^d)$. Observe now that there exists a constant $C>0$, independent of τ, such that $\|u_\tau^+(t)\|_{H^1}\le C$ for every $t\in[0,T]$ and $\|u_\tau-u_\tau^+\|_{L^\infty(L^2)}\le\tau\|\dot u_\tau\|_{L^\infty(L^2)}\le C\tau$. Hence we deduce that

$$u_\tau^+ \to u \quad \text{in } L^\infty((0,T);L^2(\Omega;\mathbb{R}^d)), \qquad u_\tau^+(t)\to u(t) \quad \text{in } L^2(\Omega;\mathbb{R}^d) \quad \text{for a.e. } t\in(0,T),$$
$$u_\tau^+ \overset{*}{\rightharpoonup} u \quad \text{in } L^\infty((0,T);H^1(\Omega;\mathbb{R}^d)), \qquad u_\tau^+(t)\rightharpoonup u(t) \quad \text{in } H^1(\Omega;\mathbb{R}^d) \quad \text{for a.e. } t\in(0,T).$$

In a similar way we get also the following convergences

$$u_\tau^- \to u \quad \text{in } L^\infty((0,T);L^2(\Omega;\mathbb{R}^d)), \qquad u_\tau^-(t)\to u(t) \quad \text{in } L^2(\Omega;\mathbb{R}^d) \quad \text{for a.e. } t\in(0,T),$$
$$u_\tau^- \overset{*}{\rightharpoonup} u \quad \text{in } L^\infty((0,T);H^1(\Omega;\mathbb{R}^d)), \qquad u_\tau^-(t)\rightharpoonup u(t) \quad \text{in } H^1(\Omega;\mathbb{R}^d) \quad \text{for a.e. } t\in(0,T).$$

Let us consider the sequence u_τ'. Since the functions u_τ' are uniformly bounded in $L^\infty((0,T);L^2(\Omega;\mathbb{R}^d))\cap H^1((0,T);H^{-1}_{D_1}(\Omega;\mathbb{R}^d))$ we can apply again Aubin–Lions Lemma. Thus we obtain the existence of a function

$$z\in L^\infty((0,T);L^2(\Omega;\mathbb{R}^d))\cap H^1((0,T);H^{-1}_{D_1}(\Omega;\mathbb{R}^d))$$

such that, up to a further subsequence

$$u_\tau' \overset{*}{\rightharpoonup} z \quad \text{in } L^\infty((0,T);L^2(\Omega;\mathbb{R}^d)), \qquad u_\tau'(t)\rightharpoonup z(t) \quad \text{in } L^2(\Omega;\mathbb{R}^d) \text{ for every } t\in[0,T],$$
$$u_\tau' \to z \quad \text{in } C^0([0,T];H^{-1}_{D_1}(\Omega;\mathbb{R}^d)), \qquad u_\tau'(t)\to z(t) \quad \text{in } H^{-1}_{D_1}(\Omega;\mathbb{R}^d) \text{ for every } t\in[0,T],$$
$$\dot u_\tau' \rightharpoonup \dot z \quad \text{in } L^2((0,T);H^{-1}_{D_1}(\Omega;\mathbb{R}^d)).$$

Furthermore $u_\tau'-\dot u_\tau$ converges strongly to 0 as in $L^2((0,T);H^{-1}_{D_1}(\Omega;\mathbb{R}^d))$ as $\tau\to0$, since

$$\|u_\tau'-\dot u_\tau\|_{L^2(H^{-1}_{D_1})}\le\tau\|\dot u_\tau'\|_{L^2(H^{-1}_{D_1})}\le C\tau\to0.$$

This, together with the previous convergences, gives that $z=\dot u$ and that $\dot u_\tau\to\dot u$ in $L^2((0,T);H^{-1}_{D_1}(\Omega;\mathbb{R}^d))$. In particular, up to a further subsequence, it results that

$$\dot u_\tau(t)\to\dot u(t) \quad \text{in } H^{-1}_{D_1}(\Omega;\mathbb{R}^d) \quad \text{and} \quad \dot u_\tau(t)\rightharpoonup\dot u(t) \quad \text{in } L^2(\Omega;\mathbb{R}^d) \quad \text{for a.e. } t\in(0,T),$$

since $\dot u_\tau'(t)$ are uniformly bounded in $L^2(\Omega;\mathbb{R}^d)$ for a.e. $t\in(0,T)$. Finally, notice that $\dot u_\tau(t)-u_\tau'^-(t)=\tau\ddot u_\tau(t)$ for a.e. $t\in(0,T)$, which gives that $u_\tau'^- \overset{*}{\rightharpoonup} \dot u$ in $L^\infty((0,T);L^2(\Omega;\mathbb{R}^d))$. □

We now define the piecewise affine interpolant of $(v_\tau^n)_{n=1}^N$ as

$$v_\tau(t):=v_\tau^n+(t-n\tau)\delta v_\tau^n, \quad t\in[(n-1)\tau,n\tau], \quad n=1,\dots,N,$$

and, as done before for u_τ^n, the backward interpolant v_τ^+ and the forward interpolant v_τ^-. Notice that $v_\tau\in H^1((0,T);H^k(\Omega))$ and that $\dot v_\tau(t)=\delta v_\tau^n$ for $t\in((n-1)\tau,n\tau)$ and $n=1,\dots,N$.

Lemma 3.8. *There exists a function $v\in H^1((0,T);H^k(\Omega))$ such that, up to a not relabeled further subsequence of τ, the following convergences hold*

$$v_\tau\to v \quad \text{in } C^0([0,T];L^2(\Omega)), \qquad v_\tau \overset{*}{\rightharpoonup} v \quad \text{in } L^\infty((0,T);H^1(\Omega)),$$
$$v_\tau^+\to v \quad \text{in } L^2((0,T);L^2(\Omega)), \qquad v_\tau^+ \overset{*}{\rightharpoonup} v \quad \text{in } L^\infty((0,T);H^1(\Omega)),$$
$$v_\tau^-\to v \quad \text{in } L^2((0,T);L^2(\Omega)), \qquad v_\tau^- \overset{*}{\rightharpoonup} v \quad \text{in } L^\infty((0,T);H^1(\Omega)),$$
$$v_\tau\rightharpoonup v \quad \text{in } H^1((0,T);H^k(\Omega)).$$

In particular we have that

$$v_\tau(t)\to v(t) \quad \text{in } L^2(\Omega) \text{ for every } t\in[0,T], \qquad v_\tau(t)\rightharpoonup v(t) \quad \text{in } H^1(\Omega) \text{ for every } t\in[0,T],$$
$$v_\tau^+(t)\to v(t) \quad \text{in } L^2(\Omega) \text{ for a.e. } t\in(0,T), \qquad v_\tau^+(t)\rightharpoonup v(t) \quad \text{in } H^1(\Omega) \text{ for a.e. } t\in(0,T),$$
$$v_\tau^-(t)\to v(t) \quad \text{in } L^2(\Omega) \text{ for a.e. } t\in(0,T) \qquad v_\tau^-(t)\rightharpoonup v(t) \quad \text{in } H^1(\Omega) \text{ for a.e. } t\in(0,T).$$

Proof. We follow the same procedure adopted in Lemma 3.7, exploiting the bounds on v_τ, v_τ^+, and v_τ^-, and using the compactness of the injection $H^k(\Omega) \hookrightarrow L^2(\Omega)$. □

We are now in position to pass to the limit in the time discrete problem (3.5).

Lemma 3.9. *The limit point (u, v) is a generalized solution of problem (2.11)–(2.14) and satisfies the initial conditions (2.15). Moreover $\dot v(t) \le 0$ in Ω for a.e. $t \in (0, T)$.*

Proof. For every $n = 1, \dots, N$ the pair (u_τ^n, v_τ^n) solves the equation (3.5). In particular, using the previous notation and integrating in the time interval $[t_1, t_2] \subset [0, T]$, we deduce that

$$\int_{t_1}^{t_2} \langle \dot u_\tau'(t), \psi\rangle_* \, dt + \int_{t_1}^{t_2} \langle b(v_\tau^-(t))\mathbb{A} e u_\tau^+(t), e\psi\rangle_2 \, dt = \int_{t_1}^{t_2} \langle f_\tau^+(t), \psi\rangle_2 \, dt + \int_{t_1}^{t_2} \langle g_\tau^+(t), \psi\rangle_* \, dt \tag{3.25}$$

for every $\psi \in H^1_{D_1}(\Omega; \mathbb{R}^d)$, where f_τ^+ and g_τ^+ are the backward interpolant of f_τ^n and g_τ^n, respectively. We now pass to the limit as $\tau \to 0$ in (3.25). We first have that

$$\lim_{\tau\to 0} \int_{t_1}^{t_2} \langle \dot u_\tau'(t), \psi\rangle_* \, dt = \int_{t_1}^{t_2} \langle \ddot u(t), \psi\rangle_* \, dt,$$

by the fact that $\dot u_\tau' \rightharpoonup \ddot u$ in $L^2((0,T); H^{-1}_{D_1}(\Omega; \mathbb{R}^d))$. Moreover, it is easy to check that $f_\tau^+ \to f$ in $L^2((0,T); L^2(\Omega; \mathbb{R}^d))$ and $g_\tau^+ \to g$ in $L^2((0,T); H^{-1}_{D_1}(\Omega; \mathbb{R}^d))$. Hence

$$\lim_{\tau\to 0} \int_{t_1}^{t_2} \langle f_\tau^+(t), \psi\rangle_2 \, dt + \lim_{\tau\to 0} \int_{t_1}^{t_2} \langle g_\tau^+(t), \psi\rangle_* \, dt = \int_{t_1}^{t_2} \langle f(t), \psi\rangle_2 \, dt + \int_{t_1}^{t_2} \langle g(t), \psi\rangle_* \, dt.$$

It remains to analyze the second term of (3.25). The strong convergence of $v_\tau^-(t)$ toward $v(t)$ in $L^2(\Omega)$ and the estimate $|b(v_\tau^-(t))\mathbb{A} e\psi| \le b(1) C_{\mathbb{A}} |e\psi|$ in Ω, imply that $b(v_\tau^-)\mathbb{A} e\psi \to b(v)\mathbb{A} e\psi$ in $L^2((0,T); L^2(\Omega; \mathbb{R}^{d\times d}))$ by the Dominated Convergence Theorem. Hence, using also the weak convergence in $L^2((0,T); L^2(\Omega; \mathbb{R}^{d\times d}))$ of eu_τ^+ toward eu, we conclude that

$$\begin{aligned}\lim_{\tau\to 0} \int_{t_1}^{t_2} \langle b(v_\tau^-(t))\mathbb{A} e u_\tau^+(t), e\psi\rangle_2 \, dt &= \lim_{\tau\to 0} \int_{t_1}^{t_2} \langle b(v_\tau^-(t))\mathbb{A} e\psi, e u_\tau^+(t)\rangle_2 \, dt \\ &= \int_{t_1}^{t_2} \langle b(v(t))\mathbb{A} e\psi, eu(t)\rangle_2 \, dt = \int_{t_1}^{t_2} \langle b(v(t))\mathbb{A} eu(t), e\psi\rangle_2 \, dt.\end{aligned}$$

Therefore, the pair (u, v) solves

$$\int_{t_1}^{t_2} \langle \ddot u(t), \psi\rangle_* \, dt + \int_{t_1}^{t_2} \langle b(v(t))\mathbb{A} eu(t), e\psi\rangle_2 \, dt = \int_{t_1}^{t_2} \langle f(t), \psi\rangle_2 \, dt + \int_{t_1}^{t_2} \langle g(t), \psi\rangle_* \, dt$$

for every $\psi \in H^1_{D_1}(\Omega; \mathbb{R}^d)$ and $[t_1, t_2] \subset [0, T]$. Choosing a countable dense set $\mathcal{D} \subset H^1_{D_1}(\Omega; \mathbb{R}^d)$, by the Lebesgue Differentiation Theorem we obtain that for a.e. $t \in (0, T)$

$$\langle \ddot u(t), \psi\rangle_* + \langle b(v(t))\mathbb{A} eu(t), e\psi\rangle_2 = \langle f(t), \psi\rangle_2 + \langle g(t), \psi\rangle_* \quad \text{for every } \psi \in \mathcal{D}.$$

Finally by density we conclude that (u, v) solves (2.22) for a.e. $t \in (0, T)$ and for every $\psi \in H^1_{D_1}(\Omega; \mathbb{R}^d)$.

Observe that the initial conditions (2.15) for (u, v) are straightforward, since $u_0 = u_\tau(0) \rightharpoonup u(0)$ in $H^1(\Omega; \mathbb{R}^d)$, and the same happens for $\dot u(0)$ and $v(0)$. Moreover the function u_τ^+ satisfies $u_\tau^+(t) = w_1(n\tau)$ on $\partial_{D_1}\Omega$ for a.e. $t \in (0, T)$, where $n := [t/\tau] + 1$ and $[s]$ denotes the integer part of $s \in [0, T]$. Hence, we deduce that $u(t) = w_1(t)$ for a.e. $t \in (0, T)$ thanks to the previous convergences. Similarly, by combining $v_\tau(t) \le 1$ in Ω and $v_\tau(t) = w_2$ on $\partial_{D_2}\Omega$, with $v_\tau(t) \to v(t)$ in $L^2(\Omega)$ and $v_\tau(t) \rightharpoonup v(t)$ in $H^1(\Omega)$, we get that (2.21) holds. Finally, notice that

$$\dot v_\tau \in \mathcal{C} := \{ z \in L^2((0,T) \times \Omega) : z(t, x) \le 0 \text{ for a.e. } (t, x) \in (0, T) \times \Omega \}.$$

Since $\mathcal{C}$ is weakly closed in $L^2((0,T) \times \Omega)$ and $\dot v_\tau$ weakly converges to $\dot v$ in this space, we conclude that $\dot v \in \mathcal{C}$. In particular $\dot v(t) \le 0$ in Ω for a.e. $t \in (0, T)$. □

In the next lemma we use the inequality (3.6) and the above convergences to prove the validity of the unilateral minimality condition (2.16) for (u, v).

Lemma 3.10. *The pair (u, v) satisfies the unilateral minimality condition (2.16) for a.e. $t \in (0, T)$.*

Proof. For every $n = 1, \dots, N$ the functions u^n_τ and v^n_τ satisfy the inequality (3.6), that can be written as

$$\mathcal{E}(u^+_\tau(t), \varphi) - \mathcal{E}(u^+_\tau(t), v^+_\tau(t)) + \partial\mathcal{H}(v^+_\tau(t))[\varphi - v^+_\tau(t)] + \partial\mathcal{G}(\dot v_\tau(t))[\varphi - v^+_\tau(t)] \geq 0 \tag{3.26}$$

for a.e. $t \in (0,T)$ and for every $\varphi - w_2 \in H^1_{D_2}(\Omega) \cap H^k(\Omega)$ with $\varphi \leq v^-_\tau(t)$. Observe that, given $\chi \in H^1_{D_2}(\Omega) \cap H^k(\Omega)$ with $\chi \leq 0$, the function $\varphi := \chi + v^+_\tau(t)$ is admissible for (3.26). Hence, integrating in $[t_1, t_2] \subset [0,T]$, we deduce the following relation

$$\int_{t_1}^{t_2} [\mathcal{E}(u^+_\tau(t), \chi + v^+_\tau(t)) - \mathcal{E}(u^+_\tau(t), v^+_\tau(t))]\, dt + \int_{t_1}^{t_2} \partial\mathcal{H}(v^+_\tau(t))[\chi]\, dt + \int_{t_1}^{t_2} \partial\mathcal{G}(\dot v_\tau(t))[\chi]\, dt \geq 0. \tag{3.27}$$

Let us see what happens when $\tau \to 0$. First, we notice that

$$\lim_{\tau\to 0} \int_{t_1}^{t_2} \partial\mathcal{G}(\dot v_\tau(t))[\chi]\, dt = \int_{t_1}^{t_2} \partial\mathcal{G}(\dot v_\tau(t))[\chi]\, dt, \tag{3.28}$$

since $\dot v_\tau \rightharpoonup \dot v$ in $L^2((0,T); H^k(\Omega))$. Moreover $v^+_\tau \rightharpoonup v$ in $L^2((0,T); H^1(\Omega))$, which implies that

$$\lim_{\tau\to 0} \int_{t_1}^{t_2} \partial\mathcal{H}(v^+_\tau(t))[\chi]\, dt = \int_{t_1}^{t_2} \partial\mathcal{H}(v(t))[\chi]\, dt. \tag{3.29}$$

Finally, the function $\phi(x,y,\xi) := \frac{1}{2}[b(y) - b(\chi(x)+y)]\mathbb{A}(x)\xi\cdot\xi$, $(x,y,\xi) \in \Omega\times\mathbb{R}\times\mathbb{R}^{d\times d}$, satisfies the assumption of the Ioffe–Olech Theorem (see, e.g., [5]), $v^+_\tau(t) \to v(t)$ in $L^2(\Omega)$ and $eu^+_\tau(t) \rightharpoonup eu(t)$ in $L^2(\Omega;\mathbb{R}^{d\times d})$ for a.e. $t \in (0,T)$. Hence, for a.e. $t \in (0,T)$

$$\begin{aligned}
&\mathcal{E}(u(t), v(t)) - \mathcal{E}(u(t), \chi + v(t)) = \int_\Omega \phi(x, v(t,x), eu(t,x))\, dx \\
&\leq \liminf_{\tau\to 0} \int_\Omega \phi(x, v^+_\tau(t,x), eu^+_\tau(t,x))\, dx = \liminf_{\tau\to 0} [\mathcal{E}(u^+_\tau(t), \chi + v^+_\tau(t)) - \mathcal{E}(u^+_\tau(t), v^+_\tau(t))].
\end{aligned}$$

Thus, by Fatou Lemma we deduce that

$$\begin{aligned}
\int_{t_1}^{t_2} [\mathcal{E}(u(t), v(t)) - \mathcal{E}(u(t), \chi + v(t))]\, dt &\leq \int_{t_1}^{t_2} \liminf_{\tau\to 0} [\mathcal{E}(u^+_\tau(t), v^+_\tau(t)) - \mathcal{E}(u^+_\tau(t), \chi + v^+_\tau(t))]\, dt \\
&\leq \liminf_{\tau\to 0} \int_{t_1}^{t_2} [\mathcal{E}(u^+_\tau(t), v^+_\tau(t)) - \mathcal{E}(u^+_\tau(t), \chi + v^+_\tau(t))]\, dt,
\end{aligned}$$

and so

$$\limsup_{\tau\to 0} \int_{t_1}^{t_2} [\mathcal{E}(u^+_\tau(t), \chi + v^+_\tau(t)) - \mathcal{E}(u^+_\tau(t), v^+_\tau(t))]\, dt \leq \int_{t_1}^{t_2} [\mathcal{E}(u(t), \chi + v(t)) - \mathcal{E}(u(t), v(t))]\, dt. \tag{3.30}$$

By combining (3.27)-(3.30) we obtain the following inequality

$$\int_{t_1}^{t_2} [\mathcal{E}(u(t), \chi + v(t)) - \mathcal{E}(u(t), v(t))]\, dt + \int_{t_1}^{t_2} \partial\mathcal{H}(v(t))[\chi]\, dt + \int_{t_1}^{t_2} \partial\mathcal{G}(\dot v(t))[\chi]\, dt \geq 0.$$

Choose now a countable dense set $\mathcal{D} \subset \{\chi \in H^1_{D_2}(\Omega) \cap H^k(\Omega) : \chi \leq 0\}$. By the Lebesgue Differentiation Theorem we deduce that for a.e. $t \in (0,T)$

$$\mathcal{E}(u(t), \chi + v(t)) - \mathcal{E}(u(t), v(t)) + \partial\mathcal{H}(v(t))[\chi] + \partial\mathcal{G}(\dot v(t))[\chi] \geq 0 \quad \text{for every } \chi \in \mathcal{D}. \tag{3.31}$$

Moreover, by a density argument, the inequality (3.31) holds for a.e. $t \in (0,T)$ and for every $\chi \in H^1_{D_2}(\Omega) \cap H^k(\Omega)$ with $\chi \leq 0$. In particular for a.e. $t \in (0,T)$, by taking $\chi = \varphi - v(t)$ with $\varphi - w_2 \in H^1_{D_2}(\Omega) \cap H^k(\Omega)$ and $\varphi \leq v(t)$, we get that (u,v) solves

$$\mathcal{E}(u(t), \varphi) - \mathcal{E}(u(t), v(t)) + \partial\mathcal{H}(v(t))[\varphi - v(t)] + \partial\mathcal{G}(\dot v(t))[\varphi - v(t)] \geq 0. \tag{3.32}$$

This implies the minimality condition (2.16), since the map $v^* \mapsto \mathcal{H}(v^*)$ is convex. □

We conclude this section by showing that this generalized solution (u,v) satisfies an energy dissipation inequality. First, let us observe that the total work $\mathcal{W}(t_1, t_2; u, v)$ is well defined even for a generalized solution. Indeed, as show in Remark 3.2, $u \in C_w([0,T]; H^1(\Omega;\mathbb{R}^d))$ and $\dot u \in C_w([0,T]; H^1(\Omega;\mathbb{R}^d))$, which gives that $u(t) - w_1(t)$ and $\dot u(t)$ are uniquely defined for every $t \in [0,T]$ as elements of $H^1_{D_1}(\Omega;\mathbb{R}^d)$ and $L^2(\Omega;\mathbb{R}^d)$, respectively. Moreover, by combining the weak continuity and the uniform bounds of u and $\dot u$

with the strong continuity of g, w_1, and $\dot w_1$, it is easy to see that $\mathcal{W}(t_1,t_2,u,v)$ is also a continuous function of t_1 and t_2.

Lemma 3.11 (Energy Inequality)**.** *The pair (u,v) satisfies the energy dissipation inequality*

$$\mathcal{F}(u(t),\dot u(t),v(t))+2\int_0^t\mathcal{G}(\dot v(s))\,ds\le\mathcal{F}(u_0,u_1,v_0)+\mathcal{W}(0,t;u,v)\quad\text{for every } t\in[0,T]. \tag{3.33}$$

Proof. Let g_τ, w_τ, and w'_τ be the affine interpolants of g^n_τ, w^n_τ, and δw^n_τ, respectively, and let w^+_τ and w^-_τ be the backward interpolant and the forward interpolant of w^n_τ, respectively. As done in the previous lemmas, we write (3.10) using these interpolants and, neglecting the dissipation terms D^n_τ, which are non negative, we obtain that

$$\begin{aligned}&\mathcal{F}(u^+_\tau(t),\dot u_\tau(t),v^+_\tau(t))+2\int_0^{jh}\mathcal{G}(\dot v_\tau(s))\,ds\le\mathcal{F}(u_0,u_1,v_0)+\int_0^{jh}\langle f^+_\tau(s),\dot u_\tau(s)-\dot w_\tau(s)\rangle_2\,ds\\&+\langle g^+_\tau(t),u^+_\tau(t)-w^+_\tau(t)\rangle_*-\langle g(0),u_0-w_1(0)\rangle_*-\int_0^{jh}\langle\dot g_\tau(s),u^-_\tau(s)-w^-_\tau(s)\rangle_*\,ds\\&+\langle\dot u_\tau(t),\dot w_\tau(t)\rangle_2-\langle u_1,\dot w_1(0)\rangle_2-\int_0^{jh}\langle u'^-_\tau(s),\dot w'_\tau(s)\rangle_2\,ds+\int_0^{jh}\langle b(v^-_\tau(s))\mathbb{A}eu^+_\tau(s),e\dot w_\tau(s)\rangle_2\,ds\end{aligned} \tag{3.34}$$

for a.e. $t\in(0,T)$, where $j:=[t/\tau]+1$. Notice that as $\tau\to0$ the following convergences hold

$$\begin{aligned}&f^+_\tau\to f\quad\text{in } L^2((0,T);L^2(\Omega;\mathbb{R}^d)),\quad g_\tau\to g\quad\text{in } H^1((0,T);H^{-1}_{D_1}(\Omega;\mathbb{R}^d)),\\&w_\tau\to w_1\quad\text{in } H^1((0,T);H^1(\Omega;\mathbb{R}^d)),\quad w'_\tau\to\dot w_1\quad\text{in } H^1((0,T);L^2(\Omega;\mathbb{R}^d)),\\&w^-_\tau\to w_1\quad\text{in } L^2((0,T);H^1(\Omega;\mathbb{R}^d)).\end{aligned}$$

In particular, by using also those of Lemmas 3.7 and 3.8, and the uniform bounds on $\dot u_\tau$, u^+_τ, u^-_τ, and u'^-_τ we deduce that

$$\lim_{\tau\to0}\int_0^{jh}\langle f^+_\tau(s),\dot u_\tau(s)-\dot w_\tau(s)\rangle_2\,ds=\int_0^t\langle f(s),\dot u(s)-\dot w_1(s)\rangle_2\,ds, \tag{3.35}$$

$$\lim_{\tau\to0}\int_0^{jh}\langle\dot g_\tau(s),u^-_\tau(s)-w^-_\tau(s)\rangle_*\,ds=\int_0^t\langle\dot g(s),u(s)-w_1(s)\rangle_*\,ds, \tag{3.36}$$

$$\lim_{\tau\to0}\int_0^{jh}\langle u'^-_\tau(s),\dot w'_\tau(s)\rangle_2\,ds=\int_0^t\langle\dot u(s),\ddot w_1(s)\rangle_2\,ds, \tag{3.37}$$

for every $t\in[0,T]$. Moreover, the strong continuity of g, w_1, and $\dot w_1$ in $H^{-1}_{D_1}(\Omega;\mathbb{R}^d)$, $H^1(\Omega;\mathbb{R}^d)$, and $L^2(\Omega;\mathbb{R}^d)$, respectively, and the convergences of $u^+_\tau(t)$ and $\dot u_\tau(t)$ seen in Lemma 3.7, imply that

$$\lim_{\tau\to0}[\langle g^+_\tau(t),u^+_\tau(t)-w^+_\tau(t)\rangle_*+\langle\dot u_\tau(t),\dot w_\tau(t)\rangle_2]=\langle g(t),u(t)-w_1(t)\rangle_*+\langle\dot u(t),\dot w_1(t)\rangle_2 \tag{3.38}$$

for a.e. $t\in(0,T)$. Finally, it can be easily checked that $b(v^-_\tau)\mathbb{A}e\dot w_\tau\to b(v)\mathbb{A}e\dot w_1$ in $L^2((0,T);L^2(\Omega;\mathbb{R}^{d\times d}))$ by the Dominated Convergence Theorem. Thus, combining it with the weakly convergence of eu^+_τ toward eu in $L^2((0,T);L^2(\Omega;\mathbb{R}^{d\times d}))$, and with the uniform bound on eu in such space, we conclude that

$$\lim_{\tau\to0}\int_0^{jh}\langle b(v^-_\tau(s))\mathbb{A}eu^+_\tau(s),e\dot w_\tau(s)\rangle_2\,ds=\int_0^t\langle b(v(s))\mathbb{A}eu(s),e\dot w_1(s)\rangle_2\,ds. \tag{3.39}$$

Consider now the left–hand side of (3.34). We have that

$$\mathcal{K}(\dot u(t))\le\liminf_{\tau\to0}\mathcal{K}(\dot u_\tau(t))\quad\text{and}\quad\mathcal{H}(v(t))\le\liminf_{\tau\to0}\mathcal{H}(v^+_\tau(t))\quad\text{for a.e. } t\in(0,T), \tag{3.40}$$

since $\dot u_\tau(t)\rightharpoonup\dot u(t)$ in $L^2(\Omega,\mathbb{R}^d)$ and $v^+_\tau(t)\rightharpoonup v(t)$ in $H^1(\Omega)$. Furthermore $\dot v_\tau\rightharpoonup\dot v$ in $L^2((0,T);H^k(\Omega))$ and $t\le jh$, which gives that

$$\int_0^t\mathcal{G}(\dot v(s))\,ds\le\liminf_{\tau\to0}\int_0^t\mathcal{G}(\dot v_\tau(s))\,ds\le\liminf_{\tau\to0}\int_0^{jh}\mathcal{G}(\dot v_\tau(s))\,ds \tag{3.41}$$

for every $t \in [0,T]$. Finally, consider the function $\phi(x,y,\xi) := \frac{1}{2} b(y)\mathbb{A}(x)\xi\cdot\xi$, $(x,y,\xi) \in \Omega\times\mathbb{R}\times\mathbb{R}^{d\times d}$. Notice that ϕ satisfies the assumption of the Ioffe–Olech Theorem, $v_\tau^+(t) \to v(t)$ in $L^2(\Omega)$, and $eu_\tau^+(t) \rightharpoonup eu(t)$ in $L^2(\Omega;\mathbb{R}^{d\times d})$ for a.e. $t \in (0,T)$. Thus

$$\begin{aligned}\mathcal{E}(u(t),v(t)) &= \int_\Omega \phi(x,v(t,x),eu(t,x))\,dx\\ &\le \liminf_{\tau\to 0}\int_\Omega \phi(x,v_\tau^+(t,x),eu_\tau^+(t,x))\,dx = \liminf_{\tau\to 0}\mathcal{E}(u_\tau^+(t),v_\tau^+(t))\end{aligned} \tag{3.42}$$

for a.e. $t \in (0,T)$. Therefore, by combining (3.34) with (3.35)–(3.42) we deduce that (3.33) holds for a.e. $t \in (0,T)$.

To show that (3.33) is true for every $t \in [0,T]$, we fix $t_0 \in [0,T]$ and we take a sequence of points t_m for which (3.33) holds and converging to t_0 as $m \to +\infty$. Since $u \in C_w([0,T];L^2(\Omega;\mathbb{R}^d))$ and $v \in C^0([0,T];H^k(\Omega))$, we have that

$$\mathcal{K}(\dot u(t_0)) \le \liminf_{m\to+\infty}\mathcal{K}(\dot u(t_m)) \quad\text{and}\quad \mathcal{H}(v(t_0)) = \lim_{m\to+\infty}\mathcal{H}(v(t_m)).$$

Moreover the function $\mathcal{G}(\dot v) \in L^2(0,T)$, so that

$$\int_0^{t_0}\mathcal{G}(\dot v(s))\,ds = \lim_{m\to+\infty}\int_0^{t_m}\mathcal{G}(\dot v(s))\,ds.$$

Thanks to the continuity of $\mathcal{W}(0,t,u,v)$ with respect to the variable t, we deduce that

$$\mathcal{W}(0,t_0;u,v) = \lim_{m\to+\infty}\mathcal{W}(0,t_m;u,v),$$

while the Ioffe–Olech Theorem implies again that

$$\mathcal{E}(u(t_0),v(t_0)) \le \liminf_{m\to+\infty}\mathcal{E}(u(t_m),v(t_m)),$$

since $v \in C^0([0,T];L^2(\Omega))$ and $eu \in C_w([0,T];L^2(\Omega;\mathbb{R}^{d\times d}))$. Hence, we can pass to the limit as $m \to +\infty$ in (3.33) computed in t_m, and we get that the same inequality is true in t_0. This conclude the proof, thanks to the arbitrariness of t_0. ☐

Remark 3.12. With similar proofs all the previous results can be proved in the case $k = 0$, that is when $\mathcal{G}(\sigma) = \frac{1}{2}c_0\|\sigma\|_2^2$ and $H^k(\Omega) = L^2(\Omega)$. In this case the phase–field function v belongs to the space $L^\infty((0,T);H^1(\Omega)) \cap H^1((0,T);L^2(\Omega))$, and is non negative whenever the initial value $v_0 \ge 0$. This is a consequence of the fact that $\max\{v_\tau^n, 0\}$ is always a better competitor for the minimum problem (3.4), and that such minimum point is unique.

4. The main result

In this section we show that for $k > d/2$ the generalized solution (u,v) found before is indeed a weak solution and satisfies the identity (2.17). To this aim we need several lemmas: we first prove that, fixed $v \in H^1((0,T);H^k(\Omega))$ with $\dot v(t) \le 0$ in Ω for a.e. $t \in (0,T)$, there exists a unique solution u of (2.22). As a consequence we deduce the energy dissipation balance (2.33) for *every* $t \in [0,T]$, which guarantees that the function u is more regular in time, namely $u \in C^0([0,T];H^1(\Omega;\mathbb{R}^d)) \cap C^1([0,T];L^2(\Omega;\mathbb{R}^d))$. Finally, we use the minimality condition (2.16) and the energy dissipation inequality (3.33) to obtain (2.23) from (2.33), which implies (2.17), as shown in Lemma 2.4.

We start with the following results, which show that the construction made in the time discretization scheme always implies an energy dissipation inequality.

Lemma 4.1. *Let $k > d/2$ and let w_1, f, g, u_0, and u_1 be as in (2.8)–(2.10). Let $\sigma \in H^1((0,T);H^k(\Omega))$ be a function such that $\sigma(t) \le 0$ for a.e. $t \in (0,T)$. Then there exists a function z satisfying (3.1), (3.2), and for a.e. $t \in (0,T)$ the equation*

$$\langle \ddot z(t),\psi\rangle_* + \langle b(\sigma(t))\mathbb{A}ez(t), e\psi\rangle_2 = \langle f(t),\psi\rangle_2 + \langle g(t),\psi\rangle_* \quad \textit{for every } \psi \in H^1_{D_1}(\Omega;\mathbb{R}^d). \tag{4.1}$$

Moreover, z satisfies the initial conditions $z(0) = u_0$ and $\dot z(0) = u_1$.

Proof. As done in the previous section, we fix $N \in \mathbb{N}$ and we define $\tau := T/N$, $z_\tau^0 := u_0$, $\delta z_\tau^0 := u_1$, and $\sigma_\tau^n := \sigma(n\tau)$, $n = 0, \dots, N$. For every $n = 1, \dots, N$ there exists a unique solution $z_\tau^n - w_\tau^n \in H^1_{D_1}(\Omega; \mathbb{R}^d)$ of

$$\langle \delta^2 z_\tau^n, \psi \rangle_2 + \langle b(\sigma_\tau^{n-1}) \mathbb{A} e z_\tau^n, e\psi \rangle_2 = \langle f_\tau^n, \psi \rangle_2 + \langle g_\tau^n, \psi \rangle_* \quad \text{for all } \psi \in H^1_{D_1}(\Omega; \mathbb{R}^d), \tag{4.2}$$

where $\delta z_\tau^n := (z_\tau^n - z_\tau^{n-1})/\tau$ and $\delta^2 z_\tau^n := (\delta z_\tau^n - \delta z_\tau^{n-1})/\tau$ for $n = 1, \dots, N$. Using $\psi = \tau \delta z_\tau^n - h \delta w_\tau^n$ as test function in (4.2) and proceeding as in the proof of Lemma 3.4, we get that z_τ^n satisfies the identity

$$\begin{aligned}
&[\mathcal{K}(\delta z_\tau^n) + \mathcal{E}(z_\tau^n, \sigma_\tau^n)] - [\mathcal{K}(\delta z_\tau^{n-1}) + \mathcal{E}(z_\tau^{n-1}, \sigma_\tau^{n-1})] - \frac{1}{2} \langle [b(\sigma_\tau^n) - b(\sigma_\tau^{n-1})] \mathbb{A} e z_\tau^n, e z_\tau^n \rangle_2 \\
&+ \frac{\tau^2}{2} \|\delta^2 z_\tau^n\|_2^2 + \frac{\tau^2}{2} \langle b(\sigma_\tau^{n-1}) \mathbb{A} e \delta z_\tau^n, e \delta z_\tau^n \rangle_2 \\
&= \tau \langle f_\tau^n, \delta z_\tau^n - \delta w_\tau^n \rangle_2 + \tau \langle g_\tau^n, \delta z_\tau^n - \delta w_\tau^n \rangle_* + \tau \langle \delta^2 z_\tau^n, \delta w_\tau^n \rangle_2 + \tau \langle b(v_\tau^{n-1}) \mathbb{A} e z_\tau^n, e \delta w_\tau^n \rangle_2
\end{aligned}$$

for every $n = 1, \dots, N$. In particular, we sum over $n = 1, \dots, j$ for every $j = 1, \dots, N$, we use the identities (3.16) and (3.17), and we neglect the terms with τ^2 to obtain the discrete energy inequality

$$\begin{aligned}
&\mathcal{K}(\delta z_\tau^j) + \mathcal{E}(z_\tau^j, \sigma_\tau^j) - \frac{1}{2} \sum_{n=1}^j \langle (b(\sigma_\tau^n) - b(\sigma_\tau^{n-1})) \mathbb{A} e z_\tau^n, e z_\tau^n \rangle_2 \le \mathcal{K}(u_1) + \mathcal{E}(u_0, \sigma(0)) + \sum_{n=1}^j \tau \langle f_\tau^n, \delta z_\tau^n \rangle_2 \\
&+ \langle g_\tau^j, z_\tau^j - w_\tau^j \rangle_* - \langle g(0), u_0 - w_1(0) \rangle_* - \sum_{n=1}^j \tau \langle \delta g_\tau^n, z_\tau^{n-1} - w_\tau^{n-1} \rangle_* \\
&+ \langle \delta z_\tau^j, \delta w_\tau^j \rangle_2 - \langle u_1, \dot w_1(0) \rangle_2 - \sum_{n=1}^j \tau \langle \delta z_\tau^{n-1}, \delta^2 w_\tau^n \rangle_2 + \sum_{n=1}^j \tau \langle b(\sigma_\tau^{n-1}) \mathbb{A} e z_\tau^n, e \delta w_\tau^n \rangle_2.
\end{aligned} \tag{4.3}$$

Since $\sigma_\tau^n \le \sigma_\tau^{n-1}$, all the terms in the left–hand side are non negative. Hence, arguing as in Lemma 3.5 and in Remark 3.6, there exists a positive constant C, independent of τ, such that

$$\max_{n=0,\dots,N} \{\|z_\tau^n\|_{H^1} + \|\delta z_\tau^n\|_2\} + \sum_{n=1}^N \tau \|\delta^2 z_\tau^n\|_*^2 \le C.$$

Let z_τ, z_τ^+, and z_τ^- be the piecewise affine interpolant, the backward interpolant, and the forward interpolant of z_τ^n, respectively. Similarly, let z_τ', $z_\tau'^+$, and $z_\tau'^-$ be the piecewise affine interpolant, the backward interpolant, and the forward interpolant of δz_τ^n, respectively. Proceeding as in Lemma 3.7, the previous estimates imply the existence of a function z satisfying (3.1) and (3.2), the initial conditions $z(0) = u_0$ and $\dot z(0) = u_1$, and the convergences state in the thesis of Lemma 3.7.

Define now the backward interpolant σ_τ^+ and the forward interpolant σ_τ^- of σ_τ^n. Integrating (4.2) in $[t_1, t_2] \subset [0, T]$, we obtain that

$$\int_{t_1}^{t_2} \langle \dot z_\tau'(t), \psi \rangle_* \, dt + \int_{t_1}^{t_2} \langle b(\sigma_\tau^-(t)) \mathbb{A} e z_\tau^+(t), e\psi \rangle_2 \, dt = \int_{t_1}^{t_2} \langle f_\tau^+(t), \psi \rangle_2 \, dt + \int_{t_1}^{t_2} \langle g_\tau^+(t), \psi \rangle_* \, dt \tag{4.4}$$

for every $\psi \in H^1_{D_1}(\Omega; \mathbb{R}^d)$. As done in Lemma 3.9, by using the previous convergences and the continuity of σ, we can pass to the limit as $\tau \to 0$, and we deduce that the function z solves

$$\int_{t_1}^{t_2} \langle \ddot z(t), \psi \rangle_* \, dt + \int_{t_1}^{t_2} \langle b(\sigma(t)) \mathbb{A} e z(t), e\psi \rangle_2 \, dt = \int_{t_1}^{t_2} \langle f(t), \psi \rangle_2 \, dt + \int_{t_1}^{t_2} \langle g(t), \psi \rangle_* \, dt$$

for every $\psi \in H^1_{D_1}(\Omega; \mathbb{R}^d)$. Hence, by the Lebesgue Differentiation Theorem and a density argument we can conclude that for a.e. $t \in (0, T)$

$$\langle \ddot z(t), \psi \rangle_* + \langle b(\sigma(t)) \mathbb{A} e z(t), e\psi \rangle_2 = \langle f(t), \psi \rangle_2 + \langle g(t), \psi \rangle_* \quad \text{for every } \psi \in \mathcal{H}^1_{D_1}(\Omega; \mathbb{R}^d). \tag{4.5}$$

□

Corollary 4.2. *Let $k > d/2$ and let w_1, f, g, u_0, u_1, and σ be as in Lemma 4.1. Then the function z given by Lemma 4.1 satisfies for every $t \in [0, T]$ the energy dissipation inequality*

$$\begin{aligned}
&\mathcal{K}(\dot z(t)) + \mathcal{E}(z(t), \sigma(t)) - \frac{1}{2} \int_0^t \langle b'(\sigma(s)) \dot\sigma(s) \mathbb{A} e z(s), e z(s) \rangle_2 \, ds \\
&\le \mathcal{K}(u_1) + \mathcal{E}(u_0, \sigma(0)) + \mathcal{W}(0, t; z, \sigma)
\end{aligned} \tag{4.6}$$

Proof. Notice that the inequality (4.3) can be written as

$$\begin{aligned}
&\mathcal{K}(\dot z_\tau(t))+\mathcal{E}(z_\tau^+(t),\sigma_\tau^+(t))-\frac{1}{2}\int_0^t\Big\langle\frac{b(\sigma_\tau^+(s))-b(\sigma_\tau^-(s))}{\tau}\mathbb{A}ez_\tau^+(s),ez_\tau^+(s)\Big\rangle_2\,ds\\
&\le\mathcal{K}(u_1)+\mathcal{E}(u_0,\sigma(0))+\int_0^{jh}\langle f_\tau^+(s),\dot z_\tau(s)-\dot w_\tau(s)\rangle_2\,ds\\
&+\langle g_\tau^+(t),z_\tau^+(t)-w_\tau^+(t)\rangle_*-\langle g(0),u_0-w_1(0)\rangle_*-\int_0^{jh}\langle\dot g_\tau(s),z_\tau^-(s)-w_\tau^-(s)\rangle_*\,ds\\
&+\langle\dot z_\tau(t),\dot w_\tau(t)\rangle_2-\langle u_1,\dot w_1(0)\rangle_2-\int_0^{jh}\langle z_\tau'^-(s),\dot w_\tau'(s)\rangle_2\,ds+\int_0^{jh}\langle b(\sigma_\tau^-(s))\mathbb{A}ez_\tau^+(s),e\dot w_\tau(s)\rangle_2\,ds,
\end{aligned}\tag{4.7}$$

for a.e. $t\in(0,T)$, where $j:=\big[t/\tau\big]+1$. In particular, in (4.7) we have used the fact that $b(\sigma_\tau^+(s))\le b(\sigma_\tau^-(s))$ for every $s\in[0,T]$ and that $t\le jh$. We pass to the limit as $\tau\to0$ following the same procedure adopted in Lemma 3.11. Thanks to the weak convergences of $z(t)$ and $\dot z(t)$ and the strong convergence of $\sigma_\tau^+(t)$ toward $\sigma(t)$ in $C^0(\overline\Omega)$, we get that

$$\mathcal{K}(\dot z(t))\le\liminf_{\tau\to0}\mathcal{K}(\dot z_\tau(t))\quad\text{and}\quad\mathcal{E}(z(t),\sigma(t))\le\liminf_{\tau\to0}\mathcal{E}(z_\tau^+(t),\sigma_\tau^+(t))\quad\text{for a.e. } t\in(0,T).\tag{4.8}$$

Similarly, we combine the convergences seen in the previous lemmas with the fact that $\sigma_\tau^-(t)\to\sigma(t)$ in $C^0(\overline\Omega)$ and that $\|\sigma_\tau^-(t)\|_\infty$ is uniformly bounded with respect to t, to deduce that

$$\begin{aligned}
\mathcal{W}(0,t;z,\sigma)=\lim_{\tau\to0}\Big\{&\int_0^{jh}\langle f_\tau^+(s),\dot z_\tau(s)-\dot w_\tau(s)\rangle_2\,ds+\langle g_\tau^+(t),z_\tau^+(t)-w_\tau^+(t)\rangle_*-\langle g(0),u_0-w_1(0)\rangle_*\\
&-\int_0^{jh}\langle\dot g_\tau(s),z_\tau^-(s)-w_\tau^-(s)\rangle_*\,ds+\langle\dot z_\tau(t),\dot w_\tau(t)\rangle_2-\langle u_1,\dot w_1(0)\rangle_2\\
&-\int_0^{jh}\langle z_\tau'^-(s),\dot w_\tau'(s)\rangle_2\,ds+\int_0^{jh}\langle b(\sigma_\tau^-(s))\mathbb{A}ez_\tau^+(s),e\dot w_\tau(s)\rangle_2\,ds\Big\}
\end{aligned}\tag{4.9}$$

for a.e. $t\in(0,T)$. Finally, fixed $s\in(0,T)$, for every $x\in\Omega$ there exists $s_\tau^x\in[\sigma_\tau^+(s,x),\sigma_\tau^-(s,x)]$ such that

$$\frac{b(\sigma_\tau^+(s,x))-b(\sigma_\tau^-(s,x))}{\tau}=b'(s_\tau^x)\frac{\sigma_\tau^+(s,x)-\sigma_\tau^-(s,x)}{\tau}$$

by the Lagrange Theorem. Since $\dot\sigma\in L^2((0,T);H^k(\Omega))$, for a.e. $s\in(0,T)$ as $\tau\to0$

$$\frac{\sigma_\tau^+(s)-\sigma_\tau^-(s)}{\tau}=\frac{\sigma(n\tau)-\sigma((n-1)\tau)}{\tau}=\frac{1}{\tau}\int_{(n-1)\tau}^{n\tau}\dot\sigma(r)\,dr\to\dot\sigma(s)\quad\text{in } H^k(\Omega),$$

where we set $n:=[s/\tau]+1$. Moreover $s_\tau^x\to\sigma(s,x)$ as $\tau\to0$ for every $s\in(0,T)$ and $x\in\Omega$, by the fact that $\sigma\in C^0([0,T];H^k(\Omega))$ and $H^k(\Omega)\hookrightarrow C^0(\overline\Omega)$. Hence, for a.e. $s\in(0,T)$ as $\tau\to0$

$$b'(s_\tau^x)\frac{\sigma_\tau^+(s,x)-\sigma_\tau^-(s,x)}{\tau}\to b'(\sigma(s,x))\dot\sigma(s,x)\quad\text{for every } x\in\Omega.$$

Furthermore, there exists a positive constant $C(s)$ depending on s, but independent of τ, such that

$$\left|\frac{b(\sigma_\tau^+(s))-b(\sigma_\tau^-(s))}{\tau}\right|\le b'(\|\sigma\|_\infty)\left\|\frac{\sigma_\tau^+(s)-\sigma_\tau^-(s)}{\tau}\right\|_\infty\le b'(\|\sigma\|_\infty)C(s).$$

Therefore, by using the Dominated Convergence Theorem we deduce that as $\tau\to0$

$$\frac{b(\sigma_\tau^+(s))-b(\sigma_\tau^-(s))}{\tau}\to b'(\sigma(s))\dot\sigma(s)\quad\text{in } L^2(\Omega),\quad\text{for a.e. } s\in(0,T).\tag{4.10}$$

Now the function $\phi(x,y,\xi):=|y|\mathbb{A}(x)\xi\cdot\xi$, $(x,y,\xi)\in\Omega\times\mathbb{R}\times\mathbb{R}^{d\times d}$, satisfies the assumptions of the Ioffe–Olech Theorem and $ez_\tau^+(s)\rightharpoonup ez(s)$ in $L^2(\Omega;\mathbb{R}^{d\times d})$ for a.e. $s\in(0,T)$. Then (4.10) gives that

$$\begin{aligned}
-\langle b'(\sigma(s))\dot\sigma(s)\mathbb{A}ez(s),ez(s)\rangle_2&=\int_\Omega\phi(x,b'(\sigma(s))\dot\sigma(s,x),ez(s,x))\,dx\\
&\le\liminf_{\tau\to0}\int_\Omega\phi\Big(x,\frac{b(\sigma_\tau^+(s,x))-b(\sigma_\tau^-(s,x))}{\tau},ez_\tau^+(s,x)\Big)\,dx
\end{aligned}$$

$$= \liminf_{\tau\to 0} \Big\{ - \Big\langle \frac{b(\sigma_\tau^+(s)) - b(\sigma_\tau^-(s))}{\tau} \mathbb{A} e z_\tau^+(s), e z_\tau^+(s) \Big\rangle_2 \Big\}$$

for a.e.$s \in (0,T)$. In particular, using also Fatou Lemma

$$\begin{aligned} -\int_0^t \langle b'(\sigma(s))\dot\sigma(s)\mathbb{A}ez(s), ez(s)\rangle_2 \, ds &\le \int_0^t \liminf_{\tau\to 0} \Big\{ - \Big\langle \frac{b(\sigma_\tau^+(s)) - b(\sigma_\tau^-(s))}{\tau} \mathbb{A} e z_\tau^+(s), e z_\tau^+(s) \Big\rangle_2 \Big\} \, ds \\ &\le \liminf_{\tau\to 0} \Big\{ - \int_0^t \Big\langle \frac{b(\sigma_\tau^+(s)) - b(\sigma_\tau^-(s))}{\tau} \mathbb{A} e z_\tau^+(s), e z_\tau^+(s) \Big\rangle_2 \, ds \Big\}. \end{aligned} \tag{4.11}$$

By combining (4.8)–(4.11) with (4.7) we obtain that for a.e. $t \in (0,T)$

$$\begin{aligned} &\mathcal{K}(\dot z(t)) + \mathcal{E}(z(t),\sigma(t)) - \frac{1}{2}\int_0^t \langle b'(\sigma(s))\dot\sigma(s)\mathbb{A}ez(s), ez(s)\rangle_2 \, ds \\ &\le \liminf_{\tau\to 0} \Big\{ \mathcal{K}(\dot z_\tau(t)) + \mathcal{E}(z_\tau^+(t),\sigma_\tau^+(t)) - \frac{1}{2}\int_0^t \Big\langle \frac{b(\sigma_\tau^+(s)) - b(\sigma_\tau^-(s))}{\tau} \mathbb{A} e z_\tau^+(s), e z_\tau^+(s) \Big\rangle_2 \, ds \Big\} \\ &\le \mathcal{K}(u_1) + \mathcal{E}(z_0,\sigma(0)) + \mathcal{W}(0,t;z,\sigma). \end{aligned}$$

Finally, the semicontinuity of the left–hand side and the continuity of the right–hand side with respect to t imply that (4.6) holds for every $t \in [0,T]$. □

The other inequality, at least for a.e. $t \in (0,T)$, is a consequence of (4.1), as shown in the next lemma.

Lemma 4.3. *Let $k > d/2$ and let w_1, f, g, u_0, u_1, and σ be as in Lemma 4.1. Assume that z satisfies* (3.1) *and* (3.2) *and that is a solution of* (4.1) *for a.e. $t \in (0,T)$. Then for a.e. $t \in (0,T)$ it holds*

$$\begin{aligned} &\mathcal{K}(\dot z(t)) + \mathcal{E}(z(t),\sigma(t)) - \frac{1}{2}\int_0^t \langle b'(\sigma(s))\dot\sigma(s)\mathbb{A}ez(s), ez(s)\rangle_2 \, ds \\ &\ge \mathcal{K}(u_1) + \mathcal{E}(u_0,\sigma(0)) + \mathcal{W}(0,t;z,\sigma). \end{aligned} \tag{4.12}$$

Proof. It is enough to proceed as in Lemma 2.4, substituting the assumption $z \in C^0([0,T];H^1(\Omega;\mathbb{R}^d)) \cap C^1([0,T];L^2(\Omega;\mathbb{R}^d))$ with $z \in C_w([0,T];H^1(\Omega;\mathbb{R}^d)) \cap L^\infty((0,T);H^1(\Omega;\mathbb{R}^d))$ and $\dot z \in C_w([0,T];L^2(\Omega;\mathbb{R}^d)) \cap L^\infty((0,T);L^\infty(\Omega;\mathbb{R}^d))$, and using the Lebesgue Differentiation Theorem. This guarantees that the function z satisfies

$$\begin{aligned} &\mathcal{K}(\dot z(t_2)) + \mathcal{E}(z(t_2),\sigma(t_2)) - \frac{1}{2}\int_{t_1}^{t_2} \langle b'(\sigma(s))\dot\sigma(s)\mathbb{A}ez(s), ez(s)\rangle_2 \, ds \\ &= \mathcal{K}(\dot z(t_1)) + \mathcal{E}(z(t_1),\sigma(t_1)) + \mathcal{W}(t_1,t_2;z,\sigma) \end{aligned}$$

for a.e. $t_1, t_2 \in (0,T)$ with $t_1 < t_2$. Since the right–hand side is lower semicontinuous with respect to t_1, while the left–hand side is continuous, sending $t_1 \to 0$ we deduce (4.12). □

Combining the previous results we deduce that the solution z, constructed using the time discretization scheme (4.2), satisfies the energy dissipation balance (2.33) with v replaced by σ for a.e. $t \in (0,T)$. To prove that this identity holds for every time, we first show that the solution of (4.1) is unique. This is done in the following lemma, adapting a standard technique due to Ladyzenskaya [12].

Lemma 4.4 (Uniqueness)**.** *Let $k > d/2$ and let w_1, f, g, u_0, u_1, and σ be as in Lemma 4.1. Then there exist at most one solution z of the equation* (4.1) *satisfying* (3.1), (3.2), *and the initial conditions $z(0) = u_0$ and $\dot z(0) = u_1$.*

Proof. Let z_1 and z_2 be two solutions of (4.1) satisfying the assumptions of the lemma, and consider $z = z_1 - z_2$. Then $z \in L^\infty((0,T);H^1_{D_1}(\Omega;\mathbb{R}^d)) \cap W^{1,\infty}((0,T);L^2(\Omega;\mathbb{R}^d)) \cap H^2((0,T);H^{-1}_{D_1}(\Omega;\mathbb{R}^d))$ solves

$$\langle \ddot z(t), \psi\rangle_* + \langle b(\sigma(t))\mathbb{A}ez(t), e\psi\rangle_2 = 0 \quad \text{for every } \psi \in H^1_{D_1}(\Omega;\mathbb{R}^d)$$

for a.e. $t \in (0,T)$, with initial conditions $z(0) = \dot z(0) = 0$. For every $s \in (0,T)$ fixed, we define the function

$$\zeta(t) = \begin{cases} -\int_t^s z(\tau)\, d\tau & \text{if } t \in [0,s], \\ 0 & \text{if } t \in [s,T]. \end{cases}$$

Clearly $\zeta \in C^0([0,T]; H^1_{D_1}(\Omega;\mathbb{R}^d))$ and $\zeta(s) = \zeta(t) = 0$. Moreover

$$\dot\zeta(t) = \begin{cases} z(t) & \text{if } t \in [0,s),\\ 0 & \text{if } t \in (s,T], \end{cases}$$

so that $\dot\zeta \in L^\infty((0,T); H^1_{D_1}(\Omega;\mathbb{R}^d))$. Hence $\zeta(t)$ is an admissible test function in (4.1). In particular, by integrating (4.1) in $(0,s)$, we deduce that

$$\int_0^s \langle \ddot z(t), \zeta(t)\rangle_* \,dt + \int_0^s \langle b(\sigma(t))\mathbb{A}ez(t), e\zeta(t)\rangle_2 \,dt = 0 \quad \text{for a.e. } t \in (0,T). \tag{4.13}$$

By integration by parts, the first term becomes

$$\int_0^s \langle \ddot z(t), \zeta(t)\rangle_* \,dt = -\int_0^s \langle \dot z(t), z(t)\rangle_2 \,dt = -\frac12 \|z(s)\|_2^2,$$

since $\zeta(s) = \dot z(0) = z(0) = 0$. Similarly, by integrating again by parts the second terms of (4.13), we obtain that

$$\int_0^s \langle b(\sigma(t))\mathbb{A}e(z(t)), e\zeta(t)\rangle_2 \,dt = -\frac12 \int_0^s \langle b'(\sigma(t))\dot\sigma(t)\mathbb{A}e\zeta(t), e\zeta(t)\rangle_2 \,dt - \frac12 \langle b(\sigma(0))\mathbb{A}e\zeta(0), e\zeta(0)\rangle_2.$$

These two identities imply that z and ζ satisfy

$$\|z(s)\|_2^2 + \langle b(\sigma(0))\mathbb{A}e\zeta(0), e\zeta(0)\rangle_2 = -\int_0^s \langle b'(\sigma(t))\dot\sigma(t)\mathbb{A}e\zeta(t), e\zeta(t)\rangle_2 \,dt.$$

In particular

$$\|z(s)\|_2^2 + \eta c_{\mathbb{A}} \|e\zeta(0)\|_2^2 \le b'(\|\sigma\|_\infty) C_{\mathbb{A}} \int_0^s \|\dot\sigma(t)\|_\infty \|e\zeta(t)\|_2^2 \,dt.$$

Let us define $\xi(s) := \int_0^s z(\tau)\,d\tau$ for every $t \in [0,s]$. Since $\zeta(t) = \xi(t) - \xi(s)$, we get that $\|e\zeta(0)\|_2 = \|e\xi(s)\|_2$ and

$$\begin{aligned}\int_0^s \|\dot\sigma(t)\|_\infty \|e\zeta(t)\|_2^2 \,dt &\le 2\|e\xi(s)\|_2^2 \int_0^s \|\dot\sigma(t)\|_\infty \,dt + 2\int_0^s \|\dot\sigma(t)\|_\infty \|e\xi(t)\|_2^2 \,dt\\ &\le 2\sqrt{s}\|\dot\sigma\|_{L^2(L^\infty)} \|e\xi(s)\|_2^2 + 2\int_0^s \|\dot\sigma(t)\|_\infty \|e\xi(t)\|_2^2 \,dt.\end{aligned}$$

Hence

$$\|z(s)\|_2^2 + (\eta c_{\mathbb{A}} - 2b'(\|\sigma\|_\infty) C_{\mathbb{A}} \|\dot\sigma\|_{L^2(L^\infty)}\sqrt{s}) \|e\xi(s)\|_2^2 \le 2b'(\|\sigma\|_\infty) C_{\mathbb{A}} \int_0^s \|\dot\sigma(t)\|_\infty \|e\xi(t)\|_2^2 \,dt.$$

Define

$$t_0 := \left(\frac{\eta c_{\mathbb{A}}}{4b'(\|\sigma\|_\infty) C_{\mathbb{A}} \|\dot\sigma\|_{L^2(L^\infty)}}\right)^2.$$

Then, for every $s \in [0,t_0]$ we get the estimate

$$\|z(s)\|_2^2 + \frac{\eta c_{\mathbb{A}}}{2} \|e\xi(s)\|_2^2 \le 2b'(\|\sigma\|_\infty) C_{\mathbb{A}} \int_0^s \|\dot\sigma(t)\|_\infty \|e\xi(t)\|_2^2 \,dt.$$

This implies that $z(s) = e\xi(s) = 0$ for every $s \in [0,t_0]$ thanks to Gronwall's Lemma. Finally, notice that t_0 depends only on $\mathbb{A}$, b, and σ. Hence, by repeating this strategy starting from t_0, with a finite number of steps we obtain that $z = 0$ on the whole interval $[0,T]$. □

Corollary 4.5. *Let $k > d/2$ and let w_1, f, g, u_0, u_1, and σ be as in Lemma 4.1. Then the unique solution z of (4.1) satisfies the energy dissipation balance*

$$\begin{aligned}&\mathcal{K}(\dot z(t)) + \mathcal{E}(z(t),\sigma(t)) - \frac12 \int_0^t \langle b'(\sigma(s))\dot\sigma(s)\mathbb{A}ez(s), ez(s)\rangle_2 \,ds\\ &= \mathcal{K}(u_1) + \mathcal{E}(u_0,\sigma(0)) + \mathcal{W}(0,t;z,\sigma)\end{aligned} \tag{4.14}$$

for every $t \in [0,T]$. In particular $\mathcal{K}(\dot z(t)) + \mathcal{E}(z(t),\sigma(t))$ is a continuous function from $[0,T]$ to $\mathbb{R}$ and

$$z \in C^0([0,T]; H^1(\Omega;\mathbb{R}^d)) \cap C^1([0,T]; L^2(\Omega;\mathbb{R}^d)).$$

Proof. We may assume that σ, w_1, f, and g are defined on $[0,2T]$ and satisfy the hypotheses of Lemma 4.1 with T replaced by $2T$. As for w_1 and σ, it is enough to set $w_1(t):=2w_1(T)-w_1(2T-t)$ and $\sigma(t):=\sigma(T)$ for $t\in(T,2T]$, respectively. By Lemma 4.4 the solution z on $[0,T]$ can be extended to a solution on $[0,2T]$ still denoted by z. Moreover, thanks to Corollary 4.2 and Lemma 4.3, z satisfies

$$\begin{aligned}&\mathcal{K}(\dot z(t))+\mathcal{E}(z(t),\sigma(t))-\frac{1}{2}\int_0^t\langle b'(\sigma(s))\dot\sigma(s)\mathbb{A}ez(s),ez(s)\rangle_2\,ds\\&=\mathcal{K}(u_1)+\mathcal{E}(u_0,\sigma(0))+\mathcal{W}(0,t;z,\sigma)\end{aligned}\tag{4.15}$$

for a.e. $t\in(0,T)$. By contradiction assume the existence of a point $t^*\in[0,T]$ such that

$$\begin{aligned}&\mathcal{K}(\dot z(t^*))+\mathcal{E}(z(t^*),\sigma(t^*))-\frac{1}{2}\int_0^{t^*}\langle b'(\sigma(s))\dot\sigma(s)\mathbb{A}ez(s),ez(s)\rangle_2\,ds\\&<\mathcal{K}(u_1)+\mathcal{E}(u_0,\sigma(0))+\mathcal{W}(0,t^*;z,\sigma).\end{aligned}$$

Since $z\in C_w([0,T];H^1(\Omega;\mathbb{R}^d))$ and $\dot z\in C_w([0,T];L^2(\Omega;\mathbb{R}^d))$ we have that $z(t^*)\in H^1_{D_1}(\Omega;\mathbb{R}^d)$ and $\dot z(t^*)\in L^2(\Omega;\mathbb{R}^d)$. Then we can consider the solution z^* of (4.1) in $[t^*,2T]$ with these initial conditions. The function defined by z in $[0,t^*]$ and z^* in $[t^*,2T]$ is still a solution of (4.1) in $[0,2T]$ and so, by Lemma 4.4, $z=z^*$ in $[t^*,2T]$. Furthermore, by (4.6) we deduce that

$$\begin{aligned}&\mathcal{K}(\dot z(t))+\mathcal{E}(z(t),\sigma(t))-\frac{1}{2}\int_{t^*}^t\langle b'(\sigma(s))\dot\sigma(s)\mathbb{A}ez(s),ez(s)\rangle_2\,ds\\&\le\mathcal{K}(z(t^*))+\mathcal{E}(z(t^*),\sigma(t^*))+\mathcal{W}(t^*,t;z,\sigma)\end{aligned}$$

for every $t\in[t^*,2T]$. Hence, combining the last two inequalities, we get that

$$\begin{aligned}&\mathcal{K}(\dot z(t))+\mathcal{E}(z(t),\sigma(t))-\frac{1}{2}\int_0^t\langle b'(\sigma(s))\dot\sigma(s)\mathbb{A}ez(s),ez(s)\rangle_2\,ds\\&\le\mathcal{K}(z(t^*))+\mathcal{E}(z(t^*),\sigma(t^*))+\mathcal{W}(t^*,t;z,\sigma)-\frac{1}{2}\int_0^{t^*}\langle b'(\sigma(s))\dot\sigma(s)\mathbb{A}ez(s),ez(s)\rangle_2\,ds\\&<\mathcal{K}(u_1)+\mathcal{E}(u_0,\sigma(0))+\mathcal{W}(0,t^*;z,\sigma)+\mathcal{W}(t^*,t;z,\sigma)=\mathcal{K}(u_1)+\mathcal{E}(u_0,\sigma(0))+\mathcal{W}(0,t;z,\sigma)\end{aligned}$$

for every $t\in[t^*,2T]$, which contradicts (4.15). Then (4.15) holds for every $t\in[0,T]$, and this implies the continuity of the map $t\mapsto\mathcal{K}(\dot z(t))+\mathcal{E}(z(t),\sigma(t))$.

Let us now prove that z is more regular in time. To this aim we fix $t_0\in[0,T]$ and we consider a sequence of points t_m converging to t_0. Since $z\in C_w([0,T];H^1(\Omega;\mathbb{R}^d))$ and $\dot z\in C_w([0,T];L^2(\Omega;\mathbb{R}^d))$, we have that

$$\|\dot z(t_0)\|_2^2\le\liminf_{m\to+\infty}\|\dot z(t_m)\|_2^2,\quad\langle b(\sigma(t_0))\mathbb{A}ez(t_0),ez(t_0)\rangle_2\le\liminf_{m\to+\infty}\langle b(\sigma(t_0))\mathbb{A}ez(t_m),ez(t_m)\rangle_2,$$

Moreover, the L^∞ bound for $\|ez(t)\|_2$ holds for every $t\in[0,T]$, thanks to the fact that the function $z\in L^\infty((0,T);H^1(\Omega;\mathbb{R}^d))\cap C_w([0,T];H^1(\Omega;\mathbb{R}^d)))$. This, together with $\sigma\in C^0([0,T];C^0(\overline\Omega))$, implies the following result

$$\limsup_{m\to+\infty}|\langle[b(\sigma(t_0))-b(\sigma(t_m))]\mathbb{A}ez(t_m),ez(t_m)\rangle_2|\le b'(\|\sigma\|_\infty)C_{\mathbb{A}}\|\nabla z\|^2_{L^\infty(L^2)}\lim_{m\to+\infty}\|\sigma(t_0)-\sigma(t_m)\|_\infty=0.$$

In particular we deduce that

$$\langle b(\sigma(t_0))\mathbb{A}ez(t_0),ez(t_0)\rangle_2\le\liminf_{m\to+\infty}\langle b(\sigma(t_m))\mathbb{A}ez(t_m),ez(t_m)\rangle_2.$$

The above inequalities and the continuity of $\mathcal{K}(\dot z)+\mathcal{E}(z,\sigma)$ gives that

$$\begin{aligned}\mathcal{K}(\dot z(t_0))+\mathcal{E}(z(t_0),\sigma(t_0))&\le\frac{1}{2}\liminf_{m\to+\infty}\|\dot z(t_m)\|_2^2+\frac{1}{2}\liminf_{m\to+\infty}\langle b(\sigma(t_m))\mathbb{A}ez(t_m),ez(t_m)\rangle_2\\&\le\lim_{m\to+\infty}[\mathcal{K}(\dot z(t_m))+\mathcal{E}(z(t_m),\sigma(t_m))]=\mathcal{K}(\dot z(t_0))+\mathcal{E}(z(t_0),\sigma(t_0)),\end{aligned}$$

which implies that the maps $t\mapsto\|\dot z(t)\|_2^2$ and $t\mapsto\langle b(\sigma(t))\mathbb{A}ez(t),ez(t)\rangle_2$ are continuous in $t_0\in[0,T]$. Hence, by the arbitrariness of t_0 we conclude that $\dot z$ and ∇z are strongly continuous from $[0,T]$ to $L^2(\Omega;\mathbb{R}^d)$ and $L^2(\Omega;\mathbb{R}^{d\times d})$, respectively. ∎

We are now in a position to prove Theorem 2.6.

Proof of Theorem 2.6. By Lemmas 3.9 and 3.10, there exists a generalized solution (u, v) of (2.11)–(2.14) satisfying the initial conditions (2.15) and the unilateral minimality condition (2.16). It remains to prove that u is more regular in time, namely it satisfies (2.18) and (2.19), and the validity of the identity (2.17).

Observe first that $\sigma = v$ is admissible in Lemmas 4.1–4.5. In particular, Corollary 4.5 implies that $u = z$ satisfies (2.18) and (2.19), since $w_1 \in C^0([0,T]; H^1(\Omega;\mathbb{R}^d))$.

As for (2.17), it is sufficient to show that (2.23) holds, thanks to Lemma 2.4. We choose for $h > 0$ small enough $\varphi = v(t+h)$ as test function in (3.32), which is possible since $\dot v(t) \le 0$ for a.e. $t \in (0,T)$. By dividing also by h, we obtain the following inequality

$$\frac{\mathcal{E}(u(t), v(t+h)) - \mathcal{E}(u(t), v(t))}{h} + \partial\mathcal{H}(v(t))\left[\frac{v(t+h)-v(t)}{h}\right] + \partial\mathcal{G}(\dot v(t))\left[\frac{v(t+h)-v(t)}{h}\right] \ge 0 \tag{4.16}$$

for a.e. $t \in (0,T)$. Since $\dot v \in L^2((0,T); H^k(\Omega))$, we know that as $h \to 0$

$$\frac{v(t+h)-v(t)}{h} \to \dot v(t) \quad \text{in } H^k(\Omega) \quad \text{for a.e. } t \in (0,T).$$

Thus, passing to the limit as $h \to 0$ in (4.16) and using the fact that b is differentiable, we conclude that

$$\partial_v \mathcal{E}(u(t), v(t))[\dot v(t)] + \partial\mathcal{H}(v(t))[\dot v(t)] + 2\mathcal{G}(\dot v(t)) \ge 0 \quad \text{for a.e. } t \in (0,T).$$

By integrating the above inequality in $[0,t]$ for every $t \in [0,T]$, we get the following relation

$$\int_0^t \partial_v \mathcal{E}(u(s), v(s))[\dot v(s)]\,ds + \mathcal{H}(v(t)) - \mathcal{H}(v_0) + 2\int_0^t \mathcal{G}(\dot v(s))\,ds \ge 0. \tag{4.17}$$

Notice that for every $t \in [0,T]$ the pair (u, v) satisfies the energy dissipation balance

$$\begin{aligned} &\mathcal{K}(\dot u(t)) + \mathcal{E}(u(t), v(t)) - \frac{1}{2}\int_0^t \langle b'(v(s))\dot v(s)\mathbb{A}eu(s), eu(s)\rangle_2\,ds \\ &= \mathcal{K}(u_1) + \mathcal{E}(u_0, v_0) + \mathcal{W}(0, t; u, v), \end{aligned} \tag{4.18}$$

thanks to Corollary 4.5. Hence, combining (4.17) and (4.18) we deduce that

$$\mathcal{F}(u(t), \dot u(t), v(t)) + 2\int_0^t \mathcal{G}(\dot v(s))\,ds \ge \mathcal{F}(u_0, u_1, v_0) + \mathcal{W}(0, t; u, v)$$

for every $t \in [0,T]$. This inequality, together with (3.33), implies (2.23) and concludes the proof. □

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