HARNACK INEQUALITIES FOR DOUBLE PHASE FUNCTIONALS

PAOLO BARONI, MARIA COLOMBO, AND GIUSEPPE MINGIONE

Abstract. We prove a Harnack inequality for minimisers of a class of non-autonomous functionals with non-standard growth conditions. They are characterised by the fact that their energy density switches between two types of different degenerate phases.

To Enzo Mitidieri, with friendship

Contents

1. Introduction and results

In this paper we complete the study of the low order regularity properties of minima of a class of functionals with non-standard growth conditions. They are basically characterised by the fact of having the energy density switching between two different types of degenerate behaviours, according to the size of a "modulating coefficient" $a(\cdot)$ that determines the "phase". Specifically, we consider a family of functionals whose model is given by the following one:

$$\mathcal{P}_{p,q}(w,\Omega) := \int_\Omega (|Dw|^p + a(x)|Dw|^q)\, dx \tag{1.1}$$

where $1 < p \leq q$ and $\Omega \subset \mathbb{R}^n$ is a bounded open set with $n \geq 2$. In this paper the function $a(\cdot)$ will always assumed to be bounded and non-negative. In the standard case $p = q$ the functional in question has standard p-polynomial growth and the regularity theory of minimisers is by now well-understood; see for instance [18, 25]. The case $p < q$ falls in the realm of functionals with non-standard growth conditions of (p, q) type, as initially defined and studied by Marcellini [23, 24]. These are general functionals of the type

$$W^{1,1}(\Omega) \ni w \longmapsto \mathcal{F}_{p,q}(w,\Omega) := \int_\Omega F(x, w, Dw)\, dx\ , \qquad \Omega \subset \mathbb{R}^n\ , \tag{1.2}$$

where the integrand $F\colon \Omega \times \mathbb{R} \times \mathbb{R}^n \to [0,\infty)$ is a Carathédory function satisfying bounds of the type

$$|z|^p \lesssim F(x, v, z) \leq |z|^q + 1 \qquad 1 < p < q \tag{1.3}$$

whenever $(x, v, z) \in \Omega \times \mathbb{R} \times \mathbb{R}^n$. Indeed, the energy density

$$H_{p,q}(x, z) := |z|^p + a(x)|z|^q \tag{1.4}$$

of the functional $\mathcal{P}_{p,q}$ is (1.1) exhibits a polynomial growth of order q with respect to the gradient variable z when $a(x) > 0$ (this is the "(p, q)-phase"), while on the phase transition zero set $\{a(x) = 0\}$ the growth is at rate p (this is the "p-phase"). Therefore, from a global viewpoint, also the functional $\mathcal{P}_{p,q}$ satisfies (1.3) and therefore falls in the realm of those with (p, q)-growth conditions. Now, while in the case of an autonomous energy density of the type $F(x, w, Dw) \equiv F(Dw)$ the regularity theory of minima of functionals with (p, q)-growth conditions is by well-understood (see for instance [4, 5, 23, 14, 25]), the case of non-autonomous integrals is still very much open and indeed new phenomena appear, which are directly liked to the specific structure of the functional. In this paper we are interested in functionals whose structure exhibits a phase transition as in (1.1). The functional $\mathcal{P}_{p,q}$ belongs to a family of variational integrals introduced by Zhikov [28, 31] in order to produce models for strongly anisotropic materials. They intervene in Homogenization theory and Elasticity, where the coefficient $a(\cdot)$ for instance dictates the geometry of a composite made by two different materials. They can also be used in order to provide new examples of Lavrentiev phenomenon [29, 30]. For the functional $\mathcal{P}_{p,q}$ a very sharp interaction occurs between the size of the phase transition, measured by the distance between p and q, and the regularity of the coefficient $a(\cdot)$, as initially shown in [13, 14, 16]. There, for every $\varepsilon > 0$, it has been shown the existence of a coefficient function $a(\cdot) \in C^{0,\alpha}$, and of exponents p, q satisfying

$$n - \varepsilon < p < n < n + \alpha < q < n + \alpha + \varepsilon \, , \tag{1.5}$$

such that there exist *bounded* minimisers of $\mathcal{P}_{p,q}$ whose set of essential discontinuity points has Hausdorff dimension larger than $n - p - \varepsilon$. In other words, minimisers can be almost as bad as any other $W^{1,p}$-function. Regularity assertions are instead more recent. In [6] the last two named authors have shown that the conditions

$$0 \leq a(\cdot) \in C^{0,\alpha}(\Omega) \quad \text{and} \quad q \leq p + \alpha \tag{1.6}$$

for some $\alpha \in (0, 1]$, are sufficient in order to prove local Hölder continuity of locally bounded minimisers of the functional $\mathcal{P}_{p,q}$. The numerology displayed in (1.5) shows that conditions in (1.6) are sharp. It is worthwhile to mention that the results of [6] cover more general functionals than $\mathcal{P}_{p,q}$ and that further conditions, this time involving also the ambient dimension n, eventually allow to conclude that any local minimiser is locally bounded. We shall come back on these points in Remark 1.3.

Starting from the Hölder continuity result of [6] and inspired by what happens in the case of functionals with standard polynomial growth ($p = q$), we now wonder if a suitable Harnack inequality holds for non-negative minimisers. We show here that the answer to this question is positive and that Harnack inequality holds in the case of functionals with measurable coefficients, but still encoding the peculiar structure of $\mathcal{P}_{p,q}$, in terms of growth conditions. We indeed consider functionals of the type in display (1.2) where the energy density $F(\cdot)$ is only assumed to be a Carathéodory function satisfying the bounds

$$\nu \leq \frac{F(x, v, z)}{H_{p,q}(x, z)} \leq L \tag{1.7}$$

whenever $z \in \mathbb{R}^n \setminus \{0\}$, $v \in \mathbb{R}$ and $x \in \Omega$, where $0 < \nu \leq 1 \leq L$; $H_{p,q}(\cdot)$ has been defined in (1.4). In this setting we recall that a function $u \in W^{1,1}_{\rm loc}(\Omega)$ is a local minimiser of the functional in (1.2) if and only if $F(x, u, Du) \in L^1_{\rm loc}(\Omega)$ and the minimality condition

$$\int_{\mathrm{supp}\,(u-w)} F(x,u,Du)\,dx \le \int_{\mathrm{supp}\,(u-w)} F(x,w,Dw)\,dx$$

is satisfied whenever $w \in W^{1,1}_{\mathrm{loc}}(\Omega)$ is such that $\mathrm{supp}\,(u-w) \subset \Omega$. Since we are assuming (1.7), and that $F(x,u,Du) \in L^1_{\mathrm{loc}}(\Omega)$, without loss of generality we may assume that all $W^{1,1}_{\mathrm{loc}}$-minimisers will automatically be in $W^{1,p}_{\mathrm{loc}}(\Omega)$ since the lower bound in (1.3) will always be in force for the functionals we are going to consider. Our first result is now the following:

Theorem 1.1. *Let $u \in W^{1,p}_{\mathrm{loc}}(\Omega) \cap L^\infty_{\mathrm{loc}}(\Omega)$ be a non-negative local minimiser of the functional $\mathcal{F}_{p,q}$, defined in* (1.2)*, under the assumptions* (1.7), (1.6) *and with $p<n$. Then for every ball B_R with $B_{9R} \subset \Omega$ there exists a constant, depending on $n,p,q,\nu,L,\alpha,[a]_{C^{0,\alpha}(\Omega)},\|u\|_{L^\infty(B_{9R})}$ and* $\mathrm{diam}\,(\Omega)$*, such that*

$$\sup_{B_R} u \le c \inf_{B_R} u$$

holds.

In the case $p>n$ minimisers are automatically locally bounded by Sobolev embedding theorem, so that assuming $u \in L^\infty_{\mathrm{loc}}(\Omega)$ is superfluous. The same happens when $p=n$ by means of the results of [9], see Remark 1.3 below. On the other hand, as already noticed in [6, 7], when $p>n$ the condition in (1.6) can be relaxed, see also Remark 1.3 below. Indeed, we shall consider

$$0 \le a(\cdot) \in C^{0,\alpha}(\Omega) \quad \text{and} \quad \frac{q}{p} \le 1 + \frac{\alpha}{n}\,. \tag{1.8}$$

Notice that the bounds in (1.6) and (1.8) coincide in the case $p=n$. The result is now as follows, and indeed incorporates the case when $p=n$:

Theorem 1.2. *Let $u \in W^{1,p}_{\mathrm{loc}}(\Omega)$ be a non-negative local minimiser of the functional $\mathcal{F}_{p,q}$, defined in* (1.2)*, under the assumptions* (1.8), (1.7) *and with $p \ge n$. Then for every ball B_R with $B_{9R} \subset \Omega$ there exists a constant, depending on $n,p,q,\nu,L,\alpha,[a]_{C^{0,\alpha}(\Omega)},\|Du\|_{L^p(B_{9R})}$ and* $\mathrm{diam}\,(\Omega)$*, such that*

$$\sup_{B_R} u \le c \inf_{B_R} u$$

holds.

It is worthwhile observing that, from the proofs given, in the statements of Theorems 1.1-1.2 we can dispense from the dependence of the constant c on $\mathrm{diam}\,(\Omega)$, provided $R \le 1$. Moreover, we also remark that Theorems 1.1-1.2 essentially yields another proof of the Hölder continuity result of [6], since it is well-known that the validity of the Harnack inequality for non-negative minimisers implies the Hölder continuity of general minima. A few remarks are now in order.

Remark 1.3 (Boundedness of minimisers and gap bounds)**.** Here we are going to comment both on the occurrence of the two different limitations adopted in (1.6) and (1.8) (accordingly to the occurrence of $p<n$ or of $p \ge n$, respectively) and on the boundedness assumption on minimisers considered in Theorem 1.1. First of all, let us observe that in the case $p \le n$ the boundedness of local minimisers can be obtained in several ways [6, 9], by imposing a new bound, this time depending on n. A recent, interesting criterion given in [9], allows to conclude that local minimisers are locally bounded provided the condition

$$q \le \begin{cases} \frac{np}{n-p} & \text{if} \quad p<n \\ \infty & \text{if} \quad p \ge n \end{cases} \tag{1.9}$$

is satisfied (the case $p > n$ being obviously trivial). We notice that already in the case of non-autonomous functionals a dependence on n of the considered bound on q/p is necessary, as shown by the counterexamples given in [17, 23]. Summarizing, we conclude that starting from a general non a priori bounded, non-negative minimiser, we can conclude with the Harnack inequality provided the following conditions are satisfied:

$$q \leq \begin{cases} \min\left\{p+\alpha, \dfrac{np}{n-p}\right\} & \text{if} \quad p < n \\ p\Big(1+\dfrac{\alpha}{n}\Big) & \text{if} \quad p \geq n\,. \end{cases} \tag{1.10}$$

We notice that this is exactly the conclusion stemming from the results obtained in [6, 7], where the Hölder continuity *of the gradient* of minima of the specific functional $\mathcal{P}_{p,q}$ is proved assuming that (1.10) holds.

Remark 1.4. Since all our estimates are local, with no loss of generality in the following we shall assume that all minimisers u considered will be assumed to belong to $W^{1,p}(\Omega) \cap L^\infty(\Omega)$ in the case $p < n$, while we shall assume that $u \in W^{1,p}(\Omega)$, when $p \geq n$.

Remark 1.5. By a careful inspection of the proofs of Theorems 1.1-1.2, one could see that the dependence of the constants on the ball considered can be avoided. In particular, once chosen $\Omega' \Subset \Omega$, in Theorem 1.1 the constant can be chosen in an universal way, depending on $\|u\|_{L^\infty(\Omega')}$ instead on $\|u\|_{L^\infty(B_{9R})}$; in Theorem 1.2 the dependence can be chosen in terms of $\|Du\|_{L^p(\Omega)}$ instead of $\|Du\|_{L^p(B_{9R})}$.

We finally remark that the results presented here extend in a quite natural way those valid for the standard case $p = q$. Comments are made in Paragraph 1.1 below. We next present a second contribution, that can be regarded as a borderline case of Theorems 1.1-1.2. This deals with functionals of the type

$$\mathcal{P}_{\log}(w,\Omega) := \int_\Omega \big[|Dw|^p + a(x)|Dw|^p \log(1+|Dw|)\big]\,dx$$

where the coefficient function $a(\cdot)$is still assumed to be bounded and non-negative. In this case the phase transition of the energy density

$$H_{\log}(x,z) := |z|^p + a(x)|z|^p \log(1+|z|) \tag{1.11}$$

is milder, since once switches from a growth of the type $z \to |z|^p \log(1+|z|)$ to a p-polynomial growth on the zero set $\{a(x) = 0\}$. In this case the idea, already exploited in [3], is that a less severe phase transition allows to assume less regularity on the coefficient than the one considered in (1.6)-(1.8). In particular, the Hölder continuity of $a(\cdot)$ is no longer necessary for Harnack inequalities. Specifically, denoting by $\omega(\cdot)$ the modulus of continuity of $a(\cdot)$ in the sense that

$$|a(x) - a(y)| \leq \omega(|x-y|) \qquad \text{holds for every } x, y \in \Omega\,, \tag{1.12}$$

we shall assume a decay of logarithmic type, which is in some sense dual to the phase transition width, i.e.

$$\limsup_{r\to 0}\, \omega(r) \log\left(\frac{1}{r}\right) < \infty\,. \tag{1.13}$$

As usual, here we shall assume that $\omega\colon [0,\infty) \to [0,\infty)$ is a non-decreasing, bounded and concave function such that $\omega(0) = 0$. Accordingly to what we have done before, we shall then consider functionals of the type in (1.2) such that

$$\nu \leq \frac{F(x,v,z)}{H_{\log}(x,z)} \leq L \tag{1.14}$$

holds whenever $z \in \mathbb{R}^n \setminus \{0\}$, $v \in \mathbb{R}$ and $x \in \Omega$, where $0 < \nu \leq 1 \leq L$; $H_{\log}(\cdot)$ is now defined as in (1.11). In this case we will denote the functional as $\mathcal{F}_{\log}$. In [3] we have proved that, under the above assumptions, local minimisers of the functional $\mathcal{F}_{\log}$ are locally Hölder continuous; here we prove the validity of Harnack inequalities under the same assumptions.

Theorem 1.6. *Let $u \in W^{1,p}(\Omega)$ be a non-negative local minimiser of the functional $\mathcal{F}_{\log}$, under the assumptions* (1.14)-(1.13). *Then u is locally bounded; moreover, there exists a constant, depending on $n, p, \nu, L, \omega(\cdot)$ and $\|Du\|_{L^p(B_{9r})}$, such that*

$$\sup_{B_R} u \leq c \inf_{B_R} u$$

for every ball B_R such that $B_{9r} \subset \Omega$.

Remark 1.7. The previous theorem, compared to the previous ones concerning the functional $\mathcal{P}_{p,q}$, tells that in order to rebalance the phase transition size of the functional we need a modulus of continuity on $a(\cdot)$ which dictates a transition which is as fast as the change in the growth with respect to the gradient variable. As a matter of fact, this principle is already visible in the case of assumptions (1.6)-(1.8), where the larger is $q-p$ the higher is required to be the degree of regularity of $a(\cdot)$, and therefore the faster $a(\cdot)$ has to approach zero on $\{a(x) = 0\}$. In other words the phase transition must be faster. Another manifestation of the same principle appears when dealing with functionals with variable growth exponent, that is those whose model is given by

$$w \in W^{1,1}(\Omega) \longmapsto \int_\Omega |Dw|^{p(x)}\, dx\,, \qquad 1 < p(x) < \infty\,. \tag{1.15}$$

In this case, the variability of the growth with respect to the gradient variable of the energy density is very modest, provided the exponent $p(x)$ is continuous. As matter of fact, as shown in [1, 8, 15, 19], assuming that the variable exponent $p(x)$ has a logarithmic modulus of continuity exactly as prescribed in (1.13), allows to prove regularity of minima and the validity of Harnack inequalities. Functionals of the type in (1.15) have been again used to create suitable models in Nonlinear Elasticity, Homogeneization [31] and non-Newtonian fluid-dynamics [2, 26].

1.1. **Comparisons with the standard case.** Our results are the exact counterpart of those valid in the standard case, when $p = q$, or, more more precisely, when $a(x) \equiv 0$. Indeed, in thos case Theorems 1.1-1.2 and 1.6 give back the classical result of DiBenedetto & Trudinger [11] on the validity of Harnack inequality for general minima of integral functionals with polynomial growth. We indeed notice that the only assumptions on the continuity with respect to the x-variable made in Theorems 1.1-1.2 and 1.6 concerns the functions $H_{p,q}(\cdot), H_{\log}(\cdot)$ and not the integrand $F(\cdot)$ which is allowed to depend on x in a measurable way. We notice that in [11] additional results are obtained. In particular, so called quasi-minima (or Q-minima) are considered. For this we recall the following definition [17]:

Definition 1. A function $u \in W^{1,1}_{\text{loc}}(\Omega)$ is Q-minimiser of the functional $\mathcal{F}$ in (1.2) with $Q \geq 1$, if and only if $F(x, u, Du) \in L^1_{\text{loc}}(\Omega)$ and moreover

$$\int_K F(x, u, Du)\, dx \leq Q \int_K F(x, w, Dw)\, dx$$

holds for every $w \in W^{1,1}(\Omega)$ and for every compact subset $K \Subset \Omega$ such that $\operatorname{supp}(u - w) \subset K$.

It is not difficult to see that the proofs of this paper extend verbatim to the case of Q-minima. More in general, in [11] the author proved that the Harnack inequality holds for functions belonging to suitable De Giorgi's classes, that is, functions that satisfy suitable Caccioppoli type inequalities on level sets. This is also the case here, see Section 6 below, provided suitable definitions are given.

2. Notation and preliminary results

2.1. Notation. With $\mathcal{B} \subset \mathbb{R}^n$ being a measurable set with positive, finite measure $|\mathcal{B}| > 0$, and with $g \colon \mathcal{B} \to \mathbb{R}^k$, $k \geq 1$, being a measurable map, we shall denote by

$$(g)_{\mathcal{B}} \equiv \fint_{\mathcal{B}} g(x)\,dx := \frac{1}{|\mathcal{B}|}\int_{\mathcal{B}} g(x)\,dx$$

its integral average. For $\gamma \in (0,1]$, in the following we shall as usual denote

$$[g]_{C^{0,\gamma}(\mathcal{B})} := \sup_{x,y\in\mathcal{B},x\neq y} \frac{|g(x)-g(y)|}{|x-y|^{\gamma}}. \tag{2.1}$$

We denote by c a general constant *always larger or equal than one*, possibly varying from line to line; relevant dependencies on parameters will be emphasised using parentheses, i.e., $c_1 \equiv c_1(n,p,q)$ means that c_1 depends on n,p,q. For the ease of notation, when dealing with the functional in (1.2)-(1.7) and a related local minimiser u, we shall also use the following abbreviation:

$$\texttt{data}(B) := \begin{cases} \{n,p,q,\nu,L,\alpha,[a]_{C^{0,\alpha}(\Omega)},\operatorname{diam}(\Omega),\|u\|_{L^\infty(B)}\} & \text{if } p<n, \\ \{n,p,q,\nu,L,\alpha,[a]_{C^{0,\alpha}(\Omega)},\operatorname{diam}(\Omega),\|u\|_{L^p(B)},\|H_{p,q}(\cdot,Du)\|_{L^1(B)}\} & \\ & \text{if } p=n, \\ \{n,p,q,\nu,L,\alpha,[a]_{C^{0,\alpha}(\Omega)},\operatorname{diam}(\Omega),\|Du\|_{L^p(B)}\} & \text{if } p\geq n\ , \end{cases}$$

where B is a ball on which u is defined. The above distinction between the case $p<n$ and $p \geq n$ is clearly motivated by the different dependences occurring in the statements of Theorems 1.1 and 1.2, see also (1.10). Instead, when dealing with the functional in (1.2)-(1.14), we shall denote

$$\texttt{data}(B) := \{n,p,\nu,L,\tilde{L},\|Du\|_{L^p(B)}\}\ ,$$

where $\tilde{L}$ denotes a constant such that

$$\omega(r)\log\left(\frac{1}{r}\right) \leq \tilde{L} \qquad \text{holds for every positive } r \leq 1\ . \tag{2.2}$$

Such a constant exists since in this paper we shall always assume that (1.13) is in force whenever $\mathcal{F}_{\log}$ will be considered. We denote by

$$B_R(x_0) := \{x \in \mathbb{R}^n \,:\, |x-x_0| < R\}$$

the open ball with center x_0 and radius $R>0$; when not important, or clear from the context, we shall omit denoting the centre as follows: $B_R \equiv B_R(x_0)$. Unless otherwise explicitly stated, different balls in the same context will have the same center. We shall denote, for a magnifying factor $\alpha \geq 1$ by αB_R the enlarged ball $B_{\alpha R}$. As usual, ω_n is the volume of the n-dimensional ball of radius one: $\omega_n = |B_1(0)|$, with $B_1(0) \subset \mathbb{R}^n$. Finally, with f being a function, we shall denote

$$f_+ := \max\{f,0\} \qquad \text{and} \qquad f_- := \max\{-f,0\}\ .$$

2.2. **Technical tools.** The next Sobolev-type Lemma has been proven by De Giorgi in [10].

Lemma 2.1. *Let $u \in W^{1,1}(B)$ for some ball $B \subset \mathbb{R}^n$ and let $m < l$ be two numbers. Then the following inequality holds:*

$$(l-m)|B \cap \{u \le m\}|^{1-1/n} \le \frac{c(n)|B|}{|B \cap \{u \ge l\}|} \int_{B \cap \{m<u\le l\}} |Du|\, dx.$$

The following is a version of the celebrated Krylov & Safonov covering Lemma [21], which usually deals with cubes. In this version it can be found in [20].

Lemma 2.2. *Let $B_{r_0} \equiv B_{r_0}(x_0)$ be a ball in $\mathbb{R}^n$ and let $E \subset B_{r_0}$ be a measurable subset; let moreover $\tilde{\delta} \in (0,1)$. Call*

$$E_{\tilde{\delta}} := \bigcup_{\substack{x \in B_{r_0} \\ \rho > 0}} \left\{ B_{3\rho}(x) \cap B_{r_0} : |B_{3\rho}(x) \cap E| \ge \tilde{\delta} |B_\rho(x)| \right\} . \tag{2.3}$$

Then either $E_{\tilde{\delta}} = B_{r_0}$ or $|E_{\tilde{\delta}}| \ge |E|/(2^n \tilde{\delta})$ holds.

Remark 2.3. In the definition of the set $E_{\tilde{\delta}}$ it is clearly sufficient to consider only the balls $B_{3\rho}(x)$ with $3\rho < 2r$.

The following iteration lemma can be found in [18, Lemma 6.1].

Lemma 2.4. *Let $\varphi \colon [r, 2r] \to [0, \infty)$ be a function such that*

$$\varphi(\sigma r) \le \frac{1}{2}\varphi(\tau r) + \frac{A}{(\tau r - \sigma r)^\kappa} \qquad \text{for every } 1 \le \sigma < \tau \le 2,$$

some $A \ge 0$ and $\kappa > 0$. Then it holds that

$$\varphi(r) \le c(\kappa) \frac{A}{r^\kappa} .$$

2.3. **Frozen functionals.** In the following we are going to deal with frozen functionals of the type

$$W^{1,1}(B) \ni w \longmapsto \int_B \left[|Dw|^p + a_0 |Dw|^q \right] dx \tag{2.4}$$

and

$$W^{1,1}(B) \ni w \longmapsto \int_B \left[|Dw|^p + a_0 |Dw|^p \log(1+|Dw|) \right] dx \tag{2.5}$$

with $B \subset \mathbb{R}^n$ being a ball, where $a_0 \ge 0$ is a constant, and $q > p > 1$. Both these functionals fall in the class of functionals with general growth conditions considered by Lieberman [22], for which regularity results are well known. Indeed we have the following

Theorem 2.5. *Let $u \in W^{1,p}(B)$ be a Q-minimiser of the functional in (2.4) or of the functional in (2.5) in the sense of Definition 1. Then for any $\sigma \in (0,1)$, $\tau \in (\sigma, 1)$ and every $q_+ > 0$ there exists a constant $c_{L,s}$ such that*

$$\sup_{B_{\sigma B}} u \le c_{L,s} \left(\fint_{\tau B} |u|^{q_+}\, dx \right)^{1/q_+} \tag{2.6}$$

holds. The constant $c_{L,s}$ depends on $n, p, q, Q, \tau - \sigma$ and q_+ for the first functional and on $n, p, Q, \tau - \sigma, q_+$ for the second one. Moreover, if u is non-negative, then there exists an exponent $q_- \in (0,1)$ such that for every $\sigma, \tau \in (0,1)$ it holds that

$$\inf_{\sigma B} u \ge \frac{1}{c_{L,i}} \left(\fint_{\tau B} u^{q_-}\, dx \right)^{1/q_-} . \tag{2.7}$$

The exponent q_- depends on n,p,q,Q in the first case and on n,p,Q in the second one; the constant $c_{L,i}$, which is larger than one, is depending on n,p,q,Q,σ,τ in the first case and on n,p,Q,σ,τ in the second one. Both the constants $c_{L,s}$ and $c_{L,i}$ are independent of a_0.

Proof. These estimates are the sup-estimate of Theorem 6.1 and the weak Harnack inequality of Theorem 6.5 of [22] for the choice $\Omega = B$, respectively. Note that both these two estimates hold for quasi-minima of functionals of the type

$$w \longmapsto \int_B H(|Dw|)\,dx\ , \quad H(t) = \begin{cases} t^p + a_0 t^q & \text{for (2.4)}, \\ t^p + a_0 t^p \log(1+t) & \text{for (2.5)}\ , \end{cases} \tag{2.8}$$

defined for $w \in W^{1,1}(B)$. The constants (and exponents) involved, depend, apart from possibly σ,τ, on n, c_H, Q, where

$$\frac{1}{c_H} \leq \frac{H'(t)t}{H(t)} \leq c_H \qquad \text{for all } t>0. \tag{2.9}$$

An easy computation (see [3, Remarks 3.1 & 3.2]) shows that for both the functionals we can take the constant c_H depending on p,q (respectively, on p) and in particular *not depending* on a_0. Finally, note that with the notation in [22] we have $\chi = 0$, since our functionals do not have lower order terms in the sense of the assumptions considered in [22]. □

Remark 2.6. The proof in [22] goes on in two steps. First the estimates (2.6) and (2.7) are proven for elements of the De Giorgi classes associated to the general functions $H(\cdot)$ (actually, $C^2((0,\infty))\cap C^1([0,\infty))$-functions satisfying (2.9) and some more natural properties, which are in particular satisfied by our two model cases in (2.8)). Later on it is proven that Q-minimisers of the functionals in (2.8) belong to appropriate De Giorgi classes. In our (simplified) setting, following [18, 22], the De Giorgi class $DG^+_H(\Omega,\gamma)$, for $\Omega \subset \mathbb{R}^n$ and $\gamma \geq 1$, associated to one of the two functions $H(\cdot)$ in (2.8), is the class of functions $u \in W^{1,1}_{\rm loc}(\Omega)$ such that $H(|Du|) \in L^1_{\rm loc}(\Omega)$ and that satisfy

$$\int_{B_{r_1}} H\big(|D(u-k)_+|\big)\,dx \leq \gamma \int_{B_{r_2}} H\Big(\frac{(u-k)_+}{r_2-r_1}\Big)\,dx \tag{2.10}$$

for every $B_{r_1} \subset B_{r_2} \subset \Omega$ concentric balls and for any $k \in \mathbb{R}$. Similarly, we say that u belongs to the De Giorgi class $DG^-_H(\Omega,\gamma)$ if $-u \in DG^+_H(\Omega,\gamma)$; finally we set $DG_G := DG^+_G \cap DG^-_G$. For what already mentioned, Theorem 2.5 holds not only for Q-minimisers of the functionals in (2.4)-(2.5), but more in general for elements of the De Giorgi classes we have just recalled. In particular, (2.6) holds for functions in $DG^+_H(\Omega,\gamma)$ and (2.7) for elements of $DG^-_H(\Omega,\gamma)$; the only difference is in the dependence of constants and exponents, which depend on γ instead of depending on Q. We shall need this observation in Section 6, when we shall deal with De Giorgi classes associated to $H_{p,q}(\cdot)$ and $H_{\log}(\cdot)$.

3. The weak Harnack inequality

The proof of Theorems 1.1-1.2 will follow classically combining a sup-estimate, which we shall prove in Section 4, see Proposition 4.3, with the weak Harnack inequality for non-negative solution that can be found in Theorem 3.5 below.

In this section we hence start the proof of the theorems by proving the weak Harnack inequality. We therefore consider a *non-negative* local minimiser u of which is locally bounded, i.e. $u \in W^{1,p}(\Omega) \cap L^\infty(\Omega)$. Let us expand a bit on this fact, with special attention to the dependence of the constants on the various parameters. When $p < n$, the boundedness of minimisers comes as an assumption, and therefore the dependence on the constants will involve $\|u\|_{L^\infty}$. In the case $p \geq n$ the local boundedness comes from the results in [9] mentioned in Remark 1.3 when $p = n$ and from Sobolev embedding theorem when $p > n$. In this case the dependence of the constants will be on $\|Du\|_{L^p}$ by the a priori estimates that allow to control $\|u\|_{L^\infty}$ via $\|u\|_{L^p}$ and in turn by $\|Du\|_{L^p}$. Finally, accordingly to the notation introduced in (2.1), we shall denote $[a]_{C^{0,\alpha}} \equiv [a]_{C^{0,\alpha}(\Omega)}$.

3.1. **A bound from below in the p-phase.** We fix a ball $B_{4r} \subset \Omega$ and we suppose that we are in the *p-phase*, i.e. that the following condition holds on the concentric ball B_{3r}. :

$$\sup_{B_{3r}} a \leq 12[a]_{C^{0,\alpha}} r^\alpha \ . \tag{3.1}$$

The terminology is motivated by the fact that we shall show that in the p-phase estimates get back to a form that is very similar to that usually available for minimiser of the p-Dirichlet energy. In the following we shall cite and prove a series of Lemmata with very classical flavour. In all of them $u \in W^{1,p}(\Omega) \cap L^\infty(\Omega)$ will be a fixed *bounded and non-negative* local minimiser of the functional $\mathcal{F}_{p,q}$ under the assumptions (1.7), with (1.6) in the case $p \leq n$ and (1.8) if $p > n$. The following Lemma extends [6, Lemma 10.1] to our situation.

Lemma 3.1. *The Caccioppoli-type estimate*

$$\fint_{B_{r_1}} |D(u-k)_\pm|^p \, dx \leq c \Big(\frac{r}{r_2 - r_1} \Big)^q \fint_{B_{r_2}} \frac{(u-k)_\pm^p}{r^p} \, dx \tag{3.2}$$

holds for every $0 < r_1 < r_2 \leq 3r$ and for any $k \in \mathbb{R}$, such that $|k| \leq \|u\|_{L^\infty(B_{3r})}$. The constant c depends on $\mathtt{data}(B_{4r})$.

Proof. In [6] it can be found the following intrinsic Caccioppoli inequality:

$$\fint_{B_{r_1}} H_{p,q}\big(x, D(u-k)_\pm\big) \, dx \leq c \fint_{B_{r_2}} H_{p,q}\Big(x, \frac{(u-k)_\pm}{r_2 - r_1}\Big) \, dx \ , \tag{3.3}$$

which holds independently on the validity of (3.1) for the range of parameters p, q considered here, for the same radii mentioned in the statement and the constant depending on n, p, q, ν, L. We have

$$\begin{aligned}
\fint_{B_{r_1}} |D(u-k)_\pm|^p \, dx &\leq \fint_{B_{r_1}} H_{p,q}\big(x, D(u-k)_\pm\big) \, dx \\
&\leq c \fint_{B_{r_2}} \left| \frac{(u-k)_\pm}{r_2 - r_1} \right|^p \left(1 + a(x) \left| \frac{(u-k)_\pm}{r_2 - r_1} \right|^{q-p} \right) dx \\
&\leq \frac{c\, r^q}{(r_2 - r_1)^q} \fint_{B_{r_2}} \frac{(u-k)_\pm^p}{r^p} \left(1 + a(x) \left| \frac{(u-k)_\pm}{r} \right|^{q-p} \right) dx \ . \qquad (3.4)
\end{aligned}$$

In the case $p \leq n$ we estimate using the boundedness of u as follows: $(u - k)_\pm \leq 2\|u\|_{L^\infty(B_{3r})}$. Moreover, using that $q \leq p + \alpha$, we estimate $r^{\alpha+p-q} \leq [\operatorname{diam}(\Omega)]^{\alpha+p-q}$. Therefore we have

$$\begin{aligned}
a(x) \left| \frac{(u-k)_\pm}{R} \right|^{q-p} &\leq c(p,q) [a]_{C^{0,\alpha}} r^\alpha \|u\|_{L^\infty(B_{3r})}^{q-p} r^{p-q} \\
&\leq c(p, q, [a]_{C^{0,\alpha}}, \operatorname{diam}(\Omega)) \|u\|_{L^\infty(B_{3r})} \ . \qquad (3.5)
\end{aligned}$$

On the other hand, in the case $p > n$ we have, by Morrey's embedding theorem

$$\operatorname*{osc}_{B_{3r}} u \leq c\, r^{1-n/p} \|Du\|_{L^p(B_{3r})}$$

and the bound in (1.8), that

$$\begin{aligned} a(x)\left|\frac{(u-k)_\pm}{r}\right|^{q-p} &\le c(p,q)[a]_{C^{0,\alpha}} r^\alpha \big[\operatorname*{osc}_{B_{3r}} u\big]^{q-p} r^{p-q} \\ &\le c(p,q,[a]_{C^{0,\alpha}}) r^{\alpha+(1-n/p)(q-p)+p-q} \|Du\|_{L^p(B_{3r})}^{q-p} \\ &= c\, r^{n(1+\alpha/n-q/p)} \|Du\|_{L^p(B_{3r})}^{q-p} \\ &\le c\left(p,q,[a]_{C^{0,\alpha}}, \|Du\|_{L^p(B_{3r})}, \operatorname{diam}(\Omega)\right) . \end{aligned} \tag{3.6}$$

Using the content of the last display and of (3.5) in (3.4) yields (3.2). □

Remark 3.2. Notice that in the arguments above, we always used balls contained in B_{3r}, while on the other hand we start requiring that $B_{4r} \subset \Omega$ and considered $\mathtt{data}(B_{4r})$. This is due to the fact, already observed at the beginning of Section 3, that when in the case $p = n$ the norm $\|u\|_{L^\infty(B_{3r})}$ can be controlled in terms of $\|Du\|_{L^p(B_{4r})}$, and therefore we use $\mathtt{data}(B_{4r})$ to indicate the constant dependence. This follows from the result of [9] already mentioned in Remark 1.3. More in general, with a similar reasoning, instead on considering B_{4r} we could consider balls of the type $B_{(3+\varepsilon)r}$ for $\varepsilon > 0$, and the constants would then depend on $\mathtt{data}(B_{(3+\varepsilon)r})$ and ε. This observation will be useful later.

The following result has a proof which follows the arguments of [11]; we however propose the full proof for completeness.

Lemma 3.3. *Assume that* (3.1) *holds on the ball* $B_{3r} \subset \Omega$*; fix* $\delta \in (0, 1]$ *and suppose that*

$$\frac{|B_r \cap \{u \ge \lambda\}|}{|B_r|} \ge \delta \tag{3.7}$$

holds for some positive level $\lambda > 0$*. Then*

$$\inf_{B_r} u \ge \frac{\lambda}{c_{1,\delta}} \tag{3.8}$$

for a constant $c_{1,\delta}$ *depending on* $\mathtt{data}(B_{4r})$ *and on* δ*.*

Proof. The proof splits into two steps.

Step 1: Density estimate. We show that (3.7) implies

$$\frac{|B_{2r} \cap \{u \le 2^{-\bar{\jmath}}\lambda\}|}{|B_{2r}|} \le \frac{c}{(\delta\, \bar{\jmath}^{1/p'})^{1^*}} \tag{3.9}$$

holds for any $\bar{\jmath} \in \mathbb{N}$ and a constant c depending on $\mathtt{data}(B_{4r})$. Using Lemma 2.1 for $l = 2^{-j}\lambda, m = 2^{-(j+1)}\lambda$, $j \in \mathbb{N}_0$, we infer

$$\begin{aligned} 2^{-(j+1)}\lambda|B_{2r} \cap \{u \le 2^{-(j+1)}\lambda\}|^{1-1/n} \\ \le \frac{c(n)}{\delta} \int_{B_{2r}\cap\{2^{-(j+1)}\lambda<u\le 2^{-j}\lambda\}} |Du|\, dx; \end{aligned} \tag{3.10}$$

note indeed that our assumption (3.7) implies

$$|B_{2r} \cap \{u \ge 2^{-j}\lambda\}| \ge |B_r \cap \{u \ge \lambda\}| \ge \delta\omega_n r^n.$$

Estimating at this point the right-hand side of (3.10) via Hölder's inequality and Caccioppoli's estimate (3.2) between B_{2r} and B_{3r}, we thus get

$$\begin{aligned} &\int_{B_{2r}\cap\{2^{-(j+1)}\lambda<u<2^{-j}\lambda\}} |Du|\, dx \\ &\quad \le |B_{2r} \cap \{2^{-(j+1)}\lambda < u \le 2^{-j}\lambda\}|^{1/p'} \Big(\int_{B_{2r}} |D(u-2^{-j}\lambda)_-|^p\, dx\Big)^{1/p} \end{aligned}$$

$$\leq \frac{c|B_{2r}\cap\{2^{-(j+1)}\lambda<u\leq 2^{-j}\lambda\}|^{1/p'}}{r}\Big(\int_{B_{3r}}(u-2^{-j}\lambda)_-^p\,dx\Big)^{1/p}$$
$$\leq c\,2^{-j}\lambda|B_{3r}|^{1/p}\frac{|B_{3r}\cap\{2^{-(j+1)}\lambda<u\leq 2^{-j}\lambda\}|^{1/p'}}{r}\ ,$$

since u is non-negative. Combining this last estimate with (3.10) yields

$$|B_{2r}\cap\{u\leq 2^{-(j+1)}\lambda\}|^{1-1/n}$$
$$\leq \frac{c(n)|B_{3r}|^{1/p-1/n}}{\delta}|B_{3r}\cap\{2^{-(j+1)}\lambda<u\leq 2^{-j}\lambda\}|^{1/p'}\ .$$

Raising to the power $p'=p/(p-1)$ both sides of the above inequality and then summing up for $j=0,\dots,\bar{\jmath}-1$, $\bar{\jmath}\in\mathbb{N}$, gives

$$\bar{\jmath}|B_{2r}\cap\{u\leq 2^{-\bar{\jmath}}\lambda\}|^{p'/1^*}\leq \frac{c}{\delta^{p'}}|B_{3r}|^{p'/p-p'/n}\sum_{j=0}^{\bar{\jmath}-1}|B_{3r}\cap\{2^{-(j+1)}\lambda<u\leq 2^{-j}\lambda\}|$$
$$\leq \frac{c}{\delta^{p'}}|B_{2r}|^{(n-p)/[n(p-1)]+1},$$

that is (3.9), once we have verified that $p'/1^*=(n-p)/[n(p-1)]+1$, recalling $1^*=n/(n-1)$.

Step 2: (Almost everywhere) pointwise strict positivity. Being $\bar{\jmath}\in\mathbb{N}$ fixed and for $i\in\mathbb{N}_0$, consider the radii and the levels

$$r_i:=2r\Big(\frac{1}{2}+\frac{1}{2^{i+1}}\Big),\qquad \tilde{r}_i:=\frac{r_{i+1}+r_i}{2},\qquad k_i:=\Big(\frac{1}{2}+\frac{1}{2^{i+1}}\Big)2^{-\bar{\jmath}}\lambda.$$

Using Caccioppoli's inequality (3.2) with $r_1=\tilde{r}_i$, $r_2=r_i$ and $k=k_i$ yields, since $r\leq r_i\leq 2r$

$$\fint_{B_{r_{i+1}}}\big|D(u-k_i)_-\big|^p\,dx\leq c\,2^{iq}\fint_{B_{r_i}}\frac{(u-k_i)_-^p}{r_i^p}\,dx\ .$$

Now we take a cut-off function $\eta_i\in C_c^\infty(B_{\tilde{r}_i})$, $\eta_i\in[0,1]$, such that $\eta_i\equiv 1$ on $B_{r_{i+1}}$ and $|D\eta_i|\leq c\,2^i/r$; using also Sobolev's inequality, with the convention that

$$p^*:=\begin{cases}\frac{np}{n-p} & \text{if}\quad p<n\\ 2p & \text{if}\quad p\geq n\ ,\end{cases}\tag{3.11}$$

we have

$$\Big(\fint_{B_{r_{i+1}}}(u-k_i)_-^{p^*}\,dx\Big)^{p/p^*}\leq c(n,p)\Big(\fint_{B_{\tilde{r}_i}}\big[(u-k_i)_-\eta_i\big]^{p^*}\,dx\Big)^{p/p^*}$$
$$\leq c\,r_i^p\fint_{B_{\tilde{r}_i}}\big|D\big[(u-k_i)_-\eta_i\big]\big|^p\,dx$$
$$\leq c\,2^{iq}\fint_{B_{r_i}}(u-k_i)_-^p\,dx\ .$$

To estimate from above the integral in the right hand side, we use the fact that u is non-negative to have

$$(u-k_i)_-\leq \chi_{\{u\leq k_i\}}2^{-\bar{\jmath}}\lambda$$

in B_{3r}, while, in order to estimate from below the integral appearing in the left hand side, we use

$$\chi_{\{u\leq k_{i+1}\}}2^{-\bar{\jmath}-i-2}\lambda=\chi_{\{u\leq k_{i+1}\}}(k_i-k_{i+1})\leq \chi_{\{u\leq k_{i+1}\}}(u-k_i)_-\leq (u-k_i)_-.$$

Merging the content of the last three displays yields

$$A_{i+1}\leq c\,2^{\tilde{c}(n,p,q)\,i}A_i^{1+\frac{p^*-p}{p}}\quad\text{where}\quad A_i:=\fint_{B_{r_i}}\chi_{\{u\leq k_i\}}\,dx$$

and where c depends on $\texttt{data}(B_{4r})$. We apply now a standard hypergeometric lemma from [10] that ensures that $A_i \to 0$, that is (3.8) with $c_{1,\delta} = 2^{\bar{\jmath}+1}$, provided

$$A_0 = \frac{|B_{2r} \cap \{u \leq 2^{-\bar{\jmath}}\lambda\}|}{|B_{2r}|} \leq \frac{1}{\tilde{c}} \tag{3.12}$$

holds with $\tilde{c}$ depending on $\texttt{data}(B_{4r})$ but *not on* $\bar{\jmath}$. In turn, in view of the result of Step 1, (3.12) can be guaranteed by choosing $\bar{\jmath}$ large enough, depending on the quantities in the line above and on δ. This yields the dependence of c on the quantities mentioned in the statement. □

3.2. **A bound from below in the** (p, q)**-phase.** In this section we study what happens in the case (3.1) does not hold, that is in the case where

$$\sup_{B_{3r}} a > 12[a]_{C^{0,\alpha}} r^{\alpha}. \tag{3.13}$$

We call this one the (p, q)*-phase*; we now show that in this case we have regularity since our local minimiser is a quasi-minimum of the frozen functional (2.4) for a_0 being an appropriate positive number. This will ensure directly the weak Harnack inequality on B_r in this case. However, since we have still to improve the estimate (3.8), which holds true in the p-phase, using the Krylov-Safonov-type covering argument of Lemma 2.2, we shall in some sense "worsen" the estimate we have from [22] to make the covering argument work. Indeed we have the following analog go Lemma 3.3:

Lemma 3.4. *Assume that* (3.13) *holds on the ball* $B_{3r} \subset \Omega$*; fix* $\delta \in (0, 1]$ *and suppose that*

$$\frac{|B_r \cap \{u \geq \lambda\}|}{|B_r|} \geq \delta$$

holds for some positive level $\lambda > 0$*. Then*

$$\inf_{B_r} u \geq \frac{\lambda}{c_{2,\delta}} \tag{3.14}$$

for a constant $c_{2,\delta}$ *depending on* $\texttt{data}(B_{3r})$ *and on* δ*.*

Proof. By (3.13) there exists $\bar{x} \in B_{3r}$ such that $a(\bar{x}) > 12[a]_{C^{0,\alpha}} r^{\alpha}$. Moreover, for every $x \in B_{3r}$, we have $a(\bar{x}) - a(x) \leq [a]_{C^{0,\alpha}}(9r)^{\alpha} \leq 6[a]_{C^{0,\alpha}} r^{\alpha}$ and therefore we have

$$\frac{1}{2}a(\bar{x}) \leq a(\bar{x}) - 6[a]_{C^{0,\alpha}} r^{\alpha} \leq a(x) \leq a(\bar{x}) + 6[a]_{C^{0,\alpha}} r^{\alpha} \leq 2a(\bar{x}) \,. \tag{3.15}$$

We hence have that

$$F(x, u, Du) \approx H_{p,q}(x, Du) \approx |Du|^p + a(\bar{x})|Du|^q$$

holds for every $x \in B_{3r}$ up to a constant depending only on ν, L, and thus

$$|Du|^p + a(\bar{x})|Du|^q \in L^1(B_{3r}).$$

Moreover, for every $v \in W^{1,1}(B_{3r})$ such that $|Dv|^p + a(\bar{x})|Dv|^q \in L^1(B_{3r})$ and $K := \operatorname{supp}(u - v) \Subset B_{3r}$, by (3.15) we have

$$\begin{aligned}
\int_K \Big(|Du|^p + a(\bar{x})|Du|^q\Big)\, dx &\leq 2\int_K H_{p,q}(x, Du)\, dx \\
&\leq \frac{2}{\nu}\int_K F(x, u, Du)\, dx \\
&\leq \frac{2}{\nu}\int_K F(x, v, Dv)\, dx \\
&\leq \frac{2L}{\nu}\int_K H_{p,q}(x, Dv)\, dx
\end{aligned}$$

$$\leq \frac{4L}{\nu}\int_K \Big(|Dv|^p + a(\bar{x})|Dv|^q\Big)\,dx.$$

Note that we used (1.7) and the minimality of u too. Hence u is a Q-minimiser of the functional in (2.4) with $a_0 := a(\bar{x})$ in B_{3r} and $Q = 4L/\nu$ in the sense of Definition 1 with $F(x,v,z) \equiv H_{p,q}(x,z)$. At this point we are allowed to use (2.7) from Theorem 2.5 in B_{3R} with $\tau = \sigma = 1/3$ that implies

$$\frac{|B_r \cap \{u > \lambda\}|}{|B_r|} \geq \delta \qquad \Longrightarrow \qquad \inf_{B_r} u \geq \frac{\delta^{1/q_-}\lambda}{c_{L,i}} =: \frac{\lambda}{c_{2,\delta}} \tag{3.16}$$

with $c_{2,\delta}$ depending only on n, p, q, ν, L and on δ. □

3.3. **The weak Harnack estimate: merging the alternatives.** Finally we come to the proof of the weak Harnack inequality (3.17) below; we follow the approach of Trudinger [27] and DiBenedetto & Trudinger [11]. This method is indeed quite flexible, see the recent fractional approach in [12].

Theorem 3.5 (Weak Harnack inequality). *Let $B_{9r} \equiv B_{9r}(x_0) \subset \Omega$. There exists an exponent $q > 0$, depending on $\mathtt{data}(B_{9r})$ and a constant $c \geq 1$, also depending on $\mathtt{data}(B_{9r})$, such that the following inequality holds:*

$$\inf_{B_r} u \geq \frac{1}{c}\Big(\fint_{B_{2r}} u^{t_-}\,dx\Big)^{1/t_-}. \tag{3.17}$$

Proof. To begin, we fix $\tilde{\delta} \equiv \tilde{\delta}(n) = 1/3^n$ and then we take

$$\mu \equiv \mu(\mathtt{data}(B_{9r})) := \frac{1}{\max\{c_{1,9^{-n}}, c_{2,9^{-n}}, c_{1,(2/3)^n}, c_{2,(2/3)^n}\}} < 1 \tag{3.18}$$

with $c_{1,9^{-n}}$ being the constant $c_{1,\delta}$ appearing in (3.8) corresponding to the choices $\delta = \tilde{\delta}/3^n = 1/9^n$ and $\delta = 1/3^n$, $c_{2,9^{-n}}$ the constant appearing in (3.14) for $\delta = 9^{-n}$ and similarly for $c_{1,(2/3)^n}, c_{2,(2/3)^n}$, with self-explaining notation.

Remark 3.6. By an easy analysis of the monotonicity of $c_{1,\delta}, c_{2,\delta}$ with respect to δ - they are decreasing functions of δ - it can be seen that $c_{1,(2/3)^n} \leq c_{1,9^{-n}}$ and $c_{2,(2/3)^n} \leq c_{2,9^{-n}}$. We preferred however to keep explicit all the arguments of the max for ease of exposition.

Note that all these constants are larger than one; all in all, this fixes μ as a number depending only on $\mathtt{data}(B_{9r})$. We also call

$$\lambda_0 := \inf_{B_{2r}} u\,.$$

Now we consider, for $\lambda > 0$ and $i \in \mathbb{N}$ being fixed, the set

$$E := B_{2r} \cap \{u \geq \mu^{i-1}\lambda\} \subset B_{2r}$$

and also $E_{\tilde{\delta}} = (B_{2r} \cap \{u \geq \mu^{i-1}\lambda\})_{\tilde{\delta}} \subset B_{2r}$, where the subscript $_{\tilde{\delta}}$ indicates the set constructed in Lemma 2.2. We use this Lemma with $B_{r_0} \equiv B_{2r}$, and this implies that either

$$\big(B_{2r} \cap \{u \geq \mu^{i-1}\lambda\}\big)_{\tilde{\delta}} = B_{2r} \tag{3.19}$$

or

$$\big|(B_{2r} \cap \{u \geq \mu^{i-1}\lambda\})_{\tilde{\delta}}\big| \geq (2^n\tilde{\delta})^{-1}\big|B_{2r} \cap \{u \geq \mu^{i-1}\lambda\}\big| \tag{3.20}$$

holds true. We take a point $x \in B_{2r}$ and a sub-ball $B_\rho(x)$ for some $0 < \rho < 2r/3$ (see Remark 2.3); note that $B_{3\rho}(x) \subset B_{4r}(x_0)$. Suppose that the ball $B_{3\rho}(x)$ is one of those contributing to the union in (2.3). Thus, by the given definitions we have

$$\big|B_{3\rho}(x) \cap \{u \geq \mu^{i-1}\lambda\}\big| \geq \big|B_{3\rho}(x) \cap \big(B_{2r} \cap \{u \geq \mu^{i-1}\lambda\}\big)\big|$$

$$\geq \tilde{\delta}|B_\rho(x)| = \frac{\tilde{\delta}}{3^n}|B_{3\rho}(x)| \ .$$

Hence the implication (3.7) or (3.16), depending whether (3.1) holds or not in $B_{9\rho}(x) \subset B_{9r}(x_0)$, and the very definition of μ in (3.18), yield

$$\inf_{B_{3\rho}(x)} u \geq \frac{\mu^{i-1}\lambda}{\max\{c_{1,9^{-n}}, c_{2,9^{-n}}\}} \geq \mu^i\lambda$$

and in turn, this implies

$$\big(B_{2r} \cap \{u \geq \mu^{i-1}\lambda\}\big)_{\tilde{\delta}} \subset B_{2r} \cap \{u \geq \mu^i\lambda\} \ ,$$

since $B_{3\rho}(x)$ was a generic member of the family making the union in (2.3).

Hence, since either (3.19) or (3.20) holds, we have proved that either

$$B_{2r} \cap \{u \geq \mu^i\lambda\} = B_{2r} \quad \Longrightarrow \quad u \geq \mu^i\lambda \quad \text{a.e. in } B_{2r} \tag{3.21}$$

or

$$\big|B_{2r} \cap \{u \geq \mu^i\lambda\}\big| \geq (2^n\tilde{\delta})^{-1}|B_{2r} \cap \{u \geq \mu^{i-1}\lambda\}| \tag{3.22}$$

hold for any $i \in \mathbb{N}$. This said, now, let us take the integer $s \in \mathbb{N}$ such that

$$(2^n\tilde{\delta})^s < \frac{|B_{2r} \cap \{u \geq \lambda\}|}{|B_{2r}|} \leq (2^n\tilde{\delta})^{s-1}$$

holds and let us show that

$$\inf_{B_{2r}} u \geq \mu^s\lambda, \tag{3.23}$$

holds too. Note that such an integer s always exists since $2^n\tilde{\delta} < 1$. To prove (3.23), we consider the dichotomy given by (3.21)-(3.22) and we first separately consider the case $s = 1$. Here we directly have

$$\inf_{B_{2r}} u \geq \frac{\lambda}{\max\{c_{1,(2/3)^n}, c_{2,(2/3)^n}\}} \geq \mu\lambda$$

by (3.7) or (3.16), depending whether (3.1) holds with B_{3r} being $B_{6r} \equiv B_{6r}(x_0)$.

Let us now consider the case where $s \geq 2$ and there exists $i \in \{1, \ldots, s-1\}$ such that (3.21) holds; in this case we conclude that $u \geq \mu^i\lambda$ holds in B_{2r} and therefore (3.23) holds since $\mu \leq 1$ and $\mu^s \leq \mu^i$. We can therefore assume that (3.22) holds whenever $i \in \{1, \ldots, s-1\}$. Then we have

$$|B_{2r} \cap \{u \geq \mu^{s-1}\lambda\}| \geq \cdots \geq (2^n\tilde{\delta})^{-(s-1)}|B_{2r} \cap \{u \geq \lambda\}| \geq 2^n\tilde{\delta}|B_{2r}| = (2/3)^n|B_{2r}|$$

and thus (3.23) follows applying again (3.7) or (3.16), depending whether (3.1) holds with B_{3r} again being B_{6r}; recall Remark 3.6 and note that this holds also if $s - 1 = 1$. Next, we finish the proof by quantifying quantity appearing in the right-hand side of (3.23): for $\beta = \log_{2^n\tilde{\delta}} \mu = \log\mu/\log(2^n\tilde{\delta})$, we clearly have

$$\lambda_0 = \inf_{B_{2r}} u \geq \mu^s\lambda = (2^n\tilde{\delta})^{\beta s}\lambda \geq (2^n\tilde{\delta})^{\beta}\Big(\frac{|B_{2r} \cap \{u \geq \lambda\}|}{|B_{2r}|}\Big)^{\beta}\lambda \ ,$$

that is $|B_{2r} \cap \{u \geq \lambda\}| \leq c\,|B_{2r}|\lambda_0^{1/\beta}\lambda^{-1/\beta}$ with $c \equiv c(\mathtt{data}(B_{9r}))$. We use this inequality to estimate

$$\begin{aligned}
\fint_{B_{2r}} u^{t_-}\,dx &= \frac{t_-}{|B_{2r}|}\int_0^\infty \lambda^{t_- -1}|B_{2r} \cap \{u \geq \lambda\}|\,d\lambda \\
&\leq \lambda_0^{t_-} + \frac{t_-}{|B_{2r}|}\int_{\lambda_0}^\infty \lambda^{t_- -1}|B_{2r} \cap \{u \geq \lambda\}|\,d\lambda \\
&\leq \lambda_0^{t_-} + c\,\lambda_0^{1/\beta}\int_{\lambda_0}^\infty \lambda^{t_- -1/\beta-1}\,d\lambda \leq c\,\lambda_0^{t_-} = c\,\big(\inf_{B_{2r}} u\big)^{t_-},
\end{aligned}$$

provided we choose $t_- < 1/\beta$.

□

4. The sup-estimate & proof of Theorems 1.1-1.2 completed

To conclude the proof of the Harnack inequality, we need to deduce a local sup-estimate; again, this will be done by considering separately the two cases similarly as done with (3.1) and (3.13). Again, $u \in W^{1,p}(\Omega) \cap L^\infty(\Omega)$ is a fixed *bounded* local minimiser of the functional $\mathcal{F}_{p,q}$ under the assumptions (1.7) and (1.6)-(1.8). The setting is precisely the one described at the beginning of Section 3.

We therefore now fix a ball $B_{4r} \equiv B_{4r}(x_0) \subset \Omega$ and we again consider the assumption

$$\sup_{B_{3r}} a \leq 12[a]_{C^{0,\alpha}} r^\alpha \tag{4.1}$$

made on the concentric ball $B_{3r} \equiv B_{3r}(x_0)$. We have the following sup-estimate absolutely analogous to that holding for minimisers of the p-Dirichlet energy.

Lemma 4.1. *Let* (4.1) *holds. Then for any exponent* $t_+ > 0$*, the local estimate*

$$\sup_{B_r} u \leq c\Big(\fint_{B_{2r}} |u|^{t_+}\, dx\Big)^{1/t_+} \tag{4.2}$$

holds for a constant c *depending on the* $\texttt{data}(B_{4r})$ *and on* t_+.

Proof. For $0 < \sigma < \tau \leq 2$, take the radii and the levels, for $d > 0$ to be fixed later,

$$r_j := r(\sigma + 2^{-i}(\tau - \sigma)), \qquad k_i := 2d(1 - 2^{-(i+1)}), \qquad i \in \mathbb{N}_0 \ .$$

Moreover, consider the balls $B_i := B_{r_i}(x_0)$ and the intermediate ones $\tilde{B}_i := B_{(r_i + r_{i+1})/2}$. Now take a test function $\eta_i \in C_c^\infty(\tilde{B}_i)$ with $0 \leq \eta_i \leq 1$, $\eta_i \equiv 1$ on B_{i+1} and $|D\eta_i| \leq c2^i/[r(\tau - \sigma)]$. Using Sobolev's embedding we have (with p^* as in (3.11))

$$\begin{aligned}\Big(\fint_{B_{i+1}} (u - k_i)_+^{p^*}\, dx\Big)^{p/p^*} &\leq c(n,p)\Big(\fint_{\tilde{B}_i} \big[(u - k_i)_+ \eta_i\big]^{p^*}\, dx\Big)^{p/p^*} \\ &\leq c\, r_i^p \fint_{\tilde{B}_i} \big|D\big[(u - k_i)_+ \eta_i\big]\big|^p\, dx \ .\end{aligned}$$

At this point we estimate the right-hand side with the help of the Caccioppoli's inequality (3.2) that, with appropriate choices, here looks like

$$\fint_{\tilde{B}_i} |D(u - k_i)_+|^p\, dx \leq \frac{c\, 2^{iq}}{(\tau - \sigma)^q} \fint_{B_i} \frac{(u - k_i)_+^p}{r^p}\, dx$$

and also using the estimate on $|D\eta_i|$:

$$\begin{aligned}\fint_{\tilde{B}_i} \big|D\big[(u - k_i)_+ \eta_i\big]\big|^p\, dx &\leq c \fint_{\tilde{B}_i} \big|D(u - k_i)_+\big|^p\, dx + \frac{c\, 2^{jp}}{(\tau - \sigma)^p} \fint_{B_i} \frac{(u - k_i)_+^p}{r^p}\, dx \\ &\leq \frac{c\, 2^{iq}}{(\tau - \sigma)^q} \fint_{B_i} \frac{(u - k_i)_+^p}{r^p}\, dx \ ,\end{aligned}$$

with c depending on $\texttt{data}(B_{4r})$. Merging the estimates above and recalling that $r_i \leq r$, and also using Hölder's inequality, we infer

$$\begin{aligned}(2^{-j}d)^{p^* - p} \fint_{B_{i+1}} (u - k_{i+1})_+^p\, dx &\leq \fint_{B_{i+1}} (u - k_i)_+^{p^*}\, dx \\ &\leq c\Big(\frac{c\, 2^{iq}}{(\tau - \sigma)^q} \fint_{B_i} (u - k_i)_+^p\, dx\Big)^{p^*/p}.\end{aligned}$$

To deduce the first inequality we used the estimates

$$(u-k_i)_+^{p^*} \geq (u-k_i)_+^{p^*-p}(u-k_{i+1})_+^p \geq (2^{-j-1}d)^{p^*-p}(u-k_{i+1})_+^p$$

which hold since the levels $\{k_i\}$ are increasing. Now, denoting

$$\Psi_i := d^{-p} \fint_{B_i} (u-k_i)_+^p \, dx \ ,$$

the previous inequality rewrites as

$$\Psi_{i+1} \leq c \, \frac{2^{\tilde{c}(n,p,q)i}}{(\tau-\sigma)^{p^*q/p}} \Psi_i^{1+(p^*-p)/p} \ ,$$

where c depends on $\texttt{data}(B_{4r})$. By the previous estimate, we can now use a standard iteration lemma (see [17, Lemma 7.1]); this implies that the sequence $\{\Psi_i\}$ is infinitesimal if, by a direct computation, the following inequality holds:

$$\Psi_0 = d^{-p} \fint_{B_R} (u-d)_+^p \, dx \leq c(\tau-\sigma)^{\kappa q}$$

where

$$\kappa := \begin{cases} n/p & \text{if } p < n \\ 2 & \text{if } p \geq n \ . \end{cases}$$

This can be provided by choosing

$$d^p = \frac{c}{(\tau-\sigma)^{\kappa q}} \fint_{B_{\tau r}} u^p \, dx$$

that therefore implies

$$\sup_{B_{\sigma r}} u \leq c \Big(\frac{1}{(\tau-\sigma)^{\kappa q}} \fint_{B_{\tau r}} u^p \, dx \Big)^{1/p}$$

with c depending on $\texttt{data}(B_{4r})$. At this point we use a simple interpolation argument to lower the exponent on the right-hand side: indeed, for $0 < t_+ < p$ and considering only the range $1 \leq \sigma < \tau \leq 2$, we have

$$\begin{aligned} \sup_{B_{\sigma r}} u &\leq c \big[\sup_{B_{\tau r}} u \big]^{1-t_+/p} \Big(\frac{1}{(\tau-\sigma)^{\kappa t_+}} \fint_{B_{\tau r}} u^{t_+} \, dx \Big)^{1/p} \\ &\leq \frac{1}{2} \sup_{B_{\tau r}} u + \frac{c \, r^{\kappa}}{(\tau r - \sigma r)^{\kappa}} \Big(\fint_{B_r} u^{t_+} \, dx \Big)^{1/t_+} \ . \end{aligned}$$

We have used Young's inequality with conjugate exponent $(p/(p-t_+), p/t_+)$. Lemma 2.4 with the choice

$$\varphi(s) := \sup_{B_{sr}} u$$

at this point gives (4.2), after a few simple algebraic manipulations. □

We then have the dual version of the previous lemma, that is

Lemma 4.2. *Let the inequality*

$$\sup_{B_{3r}} a > 12[a]_{C^{0,\alpha}} r^{\alpha}$$

hold. Then for any exponent $t > 0$, the local estimate

$$\sup_{B_r} u \leq c \Big(\fint_{B_{2r}} |u|^{t_+} \, dx \Big)^{1/t_+} \tag{4.3}$$

holds for a constant c depending on $\texttt{data}(B_{4r})$ and on t_+.

Proof. Exactly as in Paragraph 3.2, it can be shown that u is a $4L/\nu$-minimiser of the functional (2.4) in B_{3r} and therefore (2.6) with $\sigma = 1/3$, $\tau = 2/3$ directly yields (4.3) for a constant depending only on n, p, q, ν, L and t. □

Combining the two cases, that is Lemma 4.1 and 4.2 above, leads to

Proposition 4.3. *Let $u \in W^{1,p}(\Omega) \cap L^\infty(\Omega)$ be a bounded local minimiser of the functional $\mathcal{F}_{p,q}$ under the assumptions of Theorems 1.1-1.2, and consider a ball $B_{4r} \subset \Omega$. Then for any $t_+ > 0$ the local estimate*

$$\sup_{B_r} u \leq c\Big(\fint_{B_{2r}} |u|^{t_+}\, dx\Big)^{1/t_+}$$

holds for a constant c depending only on $\mathtt{data}(B_{4r})$ and on t_+.

Proof of Theorems 1.1-1.2. Once fixed a ball $B_R \equiv B_R(x_0)$ such that $B_{9R} \subset \Omega$, Theorems 1.1-1.2 follow at once by combining Theorem 3.5 and Proposition 4.3, both for the choice $B_r \equiv B_R$ and for $t_+ = t_-$; we need also to note that in the constant of the latter proposition we can replace the dependence on $\mathtt{data}(B_{4R})$ with that of $\mathtt{data}(B_{9R})$, similarly as described in Remark 1.5. □

5. Proof of Theorem 1.6

For Theorem 1.6 we confine ourselves to give just a sketch of the proof, since this is now quite similar to the one developed for Theorems 1.1-1.2, keeping also in mind the arguments developed in [3]. We start by noting that the local boundedness of local minimisers of the functional (1.2) under the only hypotheses (1.14) and $a \in L^\infty(\Omega)$ follows from the results in [9]. The same paper also features a local estimate in terms of the L^p norm of u on a slightly larger ball, and therefore also in terms of $\|Du\|_{L^p}$.

As for Theorems 1.1-1.2, with $B_{4r} \subset \Omega$, by *p-phase* we mean as usual the occurrence of the inequality

$$\sup_{B_{3r}} a \leq 12\omega(r), \tag{5.1}$$

where $\omega(\cdot)$ is the modulus of continuity defined in (1.12).

The intrinsic Caccioppoli's inequality for the functional in (1.2)-(1.14) has been proven in [3, Lemma 4.1]; the proof is similar to that of [6, Lemma 10.1]. From the intrinsic Caccioppoli's inequality we can infer an almost standard Caccioppoli's inequality, exactly as done to prove Lemma 3.2, and this has been done in [3, Lemma 4.3]: we report the result in the following

Lemma 5.1. *Let $u \in W^{1,p}(\Omega)$ be a local minimiser of the functional in* (1.2)-(1.14) *under the assumptions* (2.2) *and* (5.1). *The following Caccioppoli-type estimate then holds:*

$$\fint_{B_{r_1}} |D(u-k)_\pm|^p\, dx \leq c\Big(\frac{r}{r_2 - r_1}\Big)^{p+1} \fint_{B_{r_2}} \frac{(u-k)_\pm^p}{r^p}\, dx \tag{5.2}$$

for every $0 < r_1 < r_2 \leq 3r$ and for any $k \in \mathbb{R}$, $|k| \leq \|u\|_{L^\infty(B_{3r})}$. The constant c depends on $\mathtt{data}(B_{4r})$.

Note now that in order to prove all the other results in Sections 3 & 4 the restrictions on q (1.6)-(1.8) have not played any role: they have just been used in (3.5)-(3.6). Hence we can now follow the proofs of Lemmata 3.3 & 3.4 and Theorem 3.5 for the choice $q = p+1$ and a different constant in the almost standard Caccioppoli's inequality; the changes needed regard only the quantitative values of the constants in play, and the fact that we have to replace the regularity results of Theorem 2.5 for Q-minimisers to (2.4) with the analogous ones for Q-minimisers of (2.5). One can hence follow verbatim the proofs of the previous Sections, recalling that $\omega(cR) \leq c\omega(R)$ for $c \geq 1$, by the concavity of $\omega(\cdot)$ (this is useful, for instance, in (3.15)).

The sup-estimate of Lemma 4.1 for the logarithmic functional (1.2)-(1.14), in the case where (5.1) holds follows directly by the Caccioppoli's inequality (5.2) as in Section 4. The changes are inessential, and basically concern the values of constants. In the case where (5.1) does not hold, the sup-estimate is (2.6) for the functional in (2.5). Indeed, similarly to the arguments developed in Lemma 3.4, we can prove that u is a Q-minimiser of the functional (2.5) in B_{3r}, for $Q = 4L/\nu$ (see also [3]). Collecting all these results leads to the Harnack inequality of Theorem 1.6 exactly as for Theorems 1.1-1.2.

6. De Giorgi classes

In this paragraph we sketch how Harnack inequalities can be deduced also for functions belonging to properly defined De Giorgi classes, of the type discussed in Remark 2.6. This extends the results presented up to now since it is not difficult to see that minimisers and Q-minimisers of the functionals in (1.2)-(1.7) indeed belong to such classes. We recall that we are extending to our setting similar results known to hold for De Giorgi classes relative to functionals with standard p-polynomial growth, that is those considered in (2.10) with $H(t) = t^p$; see [11]. We restrict to the case of (1.2)-(1.7), just observing that a similar construction can be done replacing $H_{p,q}(\cdot)$ with $H_{\log}(\cdot)$ with slight modifications.

With $H_{p,q}(\cdot)$ as in (1.4), we naturally define the associated De Giorgi class $DG^+_{H_{p,q}}(\Omega, \gamma)$ as the collection of functions $u \in W^{1,p}_{\mathrm{loc}}(\Omega)$ satisfying

$$\fint_{B_{r_1}} H_{p,q}\big(x, D(u-k)_+\big)\,dx \le \gamma \fint_{B_{r_2}} H_{p,q}\Big(x, \frac{(u-k)_+}{r_2 - r_1}\Big)\,dx\ , \tag{6.1}$$

for every couple of concentric balls $B_{r_1} \equiv B_{r_1}(x_0) \subset B_{r_2} \Subset \Omega$; compare (2.10) and (3.3). Accordingly, $u \in DG^-_{H_{p,q}}(\Omega, \gamma)$ iff $-u \in DG^+_{H_{p,q}}(\Omega, \gamma)$ and

$$DG_{H_{p,q}}(\Omega, \gamma) := DG^+_{H_{p,q}}(\Omega, \gamma) \cap DG^-_{H_{p,q}}(\Omega, \gamma)\ .$$

Remark 6.1. Functions belonging to De Giorgi classes $DG_{H_{p,q}}(\Omega, \gamma)$ are locally bounded provided (1.9) holds. This fact is implicit in the proofs from [9], where indeed local boundedness of minima is proved as a consequence of the fact that they belong to De Giorgi classes that in our case coincide with $DG_{H_{p,q}}(\Omega, \gamma)$. Summarizing, exactly as in Theorems 1.1-1.2 and Remark 1.4, we shall assume we are dealing with a locally bounded, non-negative function $u \in DG_{H_{p,q}}(\Omega, \gamma)$ when $p < n$. For these reasons we define

$$\tilde{\mathtt{data}}(B) := \begin{cases} \{n, p, q, \gamma, \alpha, [a]_{C^{0,\alpha}(\Omega)}, \mathrm{diam}\,(\Omega), \|u\|_{L^\infty(B)}\} & \text{if } p < n, \\ \{n, p, q, \gamma, \alpha, [a]_{C^{0,\alpha}(\Omega)}, \mathrm{diam}\,(\Omega), \|Du\|_{L^p(B)}\} & \text{if } p \ge n\ . \end{cases}$$

We then have the following:

Theorem 6.2. *Let $u \in W^{1,p}_{\mathrm{loc}}(\Omega)$ be a non-negative function that belongs to the De Giorgi class $DG_{H_{p,q}}(\Omega, \gamma)$ for some positive number γ. Assume that*

- *$u \in L^\infty_{\mathrm{loc}}(\Omega)$ and (1.6) holds when $p < n$*
- *(1.8) holds when $p \ge n$.*

Then for every ball B_R with $B_{9R} \subset \Omega$ there exists a constant, depending on $\tilde{\mathtt{data}}(B_{9R})$ and γ, such that

$$\sup_{B_R} u \le c \inf_{B_R} u$$

holds.

Proof. The proof follows the one given for minimisers. If on some ball $B_{4r} \subset \Omega$ the inequality

$$\sup_{B_{3r}} a \leq 12[a]_{C^{0,\alpha}} r^{\alpha}, \tag{6.2}$$

is satisfied, then the Caccioppoli's inequality (3.2) holds true for levels $|k| \leq \|u\|_{L^\infty(B_{3r})}$ in the ball B_{3r}: that is,

$$\fint_{B_{r_1}} |D(u-k)_\pm|^p \, dx \leq c \Big(\frac{r}{r_2 - r_1}\Big)^q \fint_{B_{r_2}} \frac{(u-k)_\pm^p}{r^p} \, dx$$

for $0 < r_1 < r_2 \leq 3r$ and concentric balls $B_{r_1} \subset B_{r_2} \subset B_{3r}$. The constant c depends $\mathtt{data}(B_{4r})$; the proof is exactly the same as that of Lemma 3.2. This will allow to treat, analogously as done in Sections 3-4, the p-phase, that is the occurrence of (6.2) on B_{3r}. In particular, Lemma 3.3 holds for u. If on the other hand

$$\sup_{B_{3r}} a > 12[a]_{C^{0,\alpha}} r^{\alpha} \tag{6.3}$$

holds in the ball considered, then we have at hand the estimates of Theorem 2.5. Specifically, similar to what done in Paragraph 3.2, as for (3.15) we have

$$\frac{1}{2} a(\bar{x}) \leq a(x) \leq 2a(\bar{x})$$

for some point $\bar{x} \in B_{3r}$. Next, instead of proving that u is a Q-minimiser of the frozen functional in (2.4) with $a_0 := a(\bar{x})$, we directly prove that u belongs to the De Giorgi class $DG_H(\Omega, 4\gamma)$ as defined in Remark 2.6, for $H(t) = t^p + a(\bar{x})t^q$. This means that (2.10) holds for this particular choice of $H(\cdot)$. Indeed, for $0 < r_1 < r_2 \leq 3r$ we have, also clearly using (6.1)

$$\begin{aligned}
\int_{B_{r_1}} H\big(|D(u-k)_\pm|\big) \, dx &= \int_{B_{r_1}} \Big(|D(u-k)_\pm|^p + a(\bar{x})|D(u-k)_\pm|^q\Big) \, dx \\
&\leq 2 \int_{B_{r_1}} H_{p,q}\big(x, D(u-k)_\pm\big) \, dx \\
&\leq 2\gamma \fint_{B_{r_2}} H_{p,q}\Big(x, \frac{(u-k)_\pm}{r_2 - r_1}\Big) \, dx \\
&\leq 4\gamma \fint_{B_{r_2}} \bigg(\Big|\frac{(u-k)_\pm}{r_2 - r_1}\Big|^p + a(\bar{x})\Big|\frac{(u-k)_\pm}{r_2 - r_1}\Big|^q\bigg) \, dx \\
&\leq 4\gamma \int_{B_{r_2}} H\Big(\frac{(u-k)_\pm}{r_2 - r_1}\Big) \, dx,
\end{aligned}$$

and therefore we are allowed to use (2.6) and (2.7) in the case (6.3) holds. At this point we can conclude with Lemma 3.4. Eventually, combining the results from the two phases, we get Theorem 3.5. In a similar fashion we find back the results in Section 4. With all these ingredients at hand we are now able to conclude the proof as in the case of Theorems 1.1-1.2. □

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Paolo Baroni Department of Mathematics, Uppsala University, Lägerhyddsvägen 1, S-75106 Uppsala, Sweden
E-mail address: paolo.baroni@math.uu.se

Maria Colombo, Scuola Normale Superiore di Pisa, p.za dei Cavalieri 7, I-56126 Pisa, Italy
E-mail address: maria.colombo@sns.it

Giuseppe Mingione, Dipartimento di Matematica e Informatica, Università di Parma, Parco Area delle Scienze 53/a, Campus, 43100 Parma, Italy
E-mail address: giuseppe.mingione@unipr.it.