FLATNESS RESULTS FOR NONLOCAL PHASE TRANSITIONS

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Abstract. We consider a nonlocal version of the Allen-Cahn equation, which models phase transitions problems. In the classical setting, the connection between the Allen-Cahn equation and the classification of entire minimal surfaces is well known and motivates a celebrated conjecture by E. De Giorgi on the one-dimensional symmetry of bounded monotone solutions to the (classical) Allen-Cahn equation up to dimension 8. In this note, we present some recent results in the study of the nonlocal analogue of this phase transition problem. In particular we describe the results obtained in several contributions [8, 9, 13, 14, 25, 41, 44, 46] where the classification of certain entire bounded solutions to the fractional Allen-Cahn equation has been obtained. Moreover we describe the connection between the fractional Allen-Cahn equation and the fractional perimeter functional, and we present also some results in the classifications of nonlocal minimal surfaces obtained in [16, 42, 10, 21].

1. Introduction

In this note we present some recent results concerning the classification of certain solutions to the fractional Allen-Cahn equation

$$(-\Delta)^s u = u - u^3 \quad \text{in } \mathbb{R}^n, \tag{1.1}$$

where s is a real parameter in $(0, 1)$. More precisely, we are interested in the analogue, for problem (1.1), of a well known conjecture by E. De Giorgi for solutions of the classical Allen-Cahn equation.

In 1978, De Giorgi conjectured that the level sets of every bounded solution of

$$-\Delta u = u - u^3 \quad \text{in } \mathbb{R}^n, \tag{1.2}$$

which is monotone in one direction, must be hyperplanes at least if $n \leqslant 8$. That is, such solutions depend only on one Euclidean variable.

The original motivation for this conjecture was given by a classical result in the Calculus of Variations due to Modica and Mortola [36], who proved that, after a suitable rescaling, the energy functional associated to (1.2), Γ-converges to the perimeter functional (see Section 2 for more details). Moreover, the classification of area-minimizing surfaces was known: any area-minimizing set in the whole $\mathbb{R}^n$ is necessarily flat if $n \leqslant 7$. The dimension 7 is optimal, indeed in $\mathbb{R}^8$ there exists an area-minimizing singular cone, the Simons cone, defined in the following way:

$$\mathcal{C} := \{(x_1, \dots, x_8) \in \mathbb{R}^8 \,|\, x_1^2 + \cdots + x_4^2 = x_5^2 + \cdots + x_8^2\}.$$

A related result concerns the classification of minimal graphs (the so-called Bernstein problem): any area-minimizing graph in $\mathbb{R}^n$ is necessarily a hyperplane if $n \leqslant 8$.

Coming back to the Allen-Cahn equation, by the Modica-Mortola result, one knows that the level sets of solutions to $-\Delta u = u - u^3$ are asymptotically area-minimizing surfaces. Moreover, if we assume the solution to be monotone in some direction, we have that the level sets are graphs. Hence, by the previous result on the classification of entire minimal graph, we know that the level sets of bounded monotone solutions to the Allen-Cahn equation are asymptotically flat. The De Giorgi conjecture asserts that they are indeed flat, not only asymptotically. The fact that a function u has level sets which are parallel hyperplanes, means that u depends only on one Euclidean direction (the direction perpendicular to all these hyperplanes). When this happens, we say that u has one-dimensional symmetry, or is 1-D, for short.

We consider now the fractional version of the Allen-Cahn equation (1.1) and we are interested in the validity of the analogue of the De Giorgi conjecture. First of all, a natural question is whether a Modica-Mortola type result is valid for the energy functional associated to (1.1) and whether there is a natural connection with an area-minimizing problem. The answer to this question was given by Savin and Valdinoci in [45]: they proved that, after a suitable rescaling the energy associated to (1.1) Γ-converges to the classical perimeter functional if $1/2 \leqslant s < 1$ and to the so-called *nonlocal perimeter* if $0 < s < 1/2$. Hence, when $1/2 \leqslant s < 1$, one expects the analogue of the De Giorgi conjecture to be true up to dimension $n = 8$ as in the classical setting. While, when $0 < s < 1/2$, the level sets of solutions to (1.1) looks, at large scales, like *nonlocal minimal surfaces.*

The nonlocal (or fractional) s-perimeter functional was introduced by Caffarelli, Roquejoffre, and Savin in [16] (see formula (4.1) in Section 4) and the classification of minimizers for this functional is still widely open. In [42], Savin and Valdinoci proved that any s-minimal set in $\mathbb{R}^2$ is necessarily an half-plane. Moreover in [32], Figalli and Valdinoci addressed the nonlocal analogue of the Bernstein problem and they obtained flatness of s-minimal graphs in $\mathbb{R}^3$. These are the only known results about the classification for s-minimal surfaces, except for some asymptotic results that are valid only for s sufficiently close to $1/2$ (see Section 4 for all the precise results).

This lack of information in large dimensions for the geometric problem, is reflected on the PDE side, where the De Giorgi conjecture for s below $1/2$ is still open in dimensions $n > 3$. We recall here the main references for the fractional De Giorgi conjecture: it has been proven in dimension $n = 2$ and for any $s \in (0, 1)$ in [13, 42], in dimension $n = 3$ for $s \in [1/2, 1)$ in [8, 9], in dimension $n = 3$ for $s \in (0, 1/2)$ in [25] and in the forthcoming paper [11], in dimensions $4 \leqslant n \leqslant 8$ for $1/2 < s < 1$ (under an additional assumption on the limits at infinity of the solution) in [41]. In this last reference [41], the author also announces that in a forthcoming paper he will prove the same result for $s = 1/2$.

We comment now on the proof of the De Giorgi conjecture for the fractional problem (1.1). As in the classical setting, two different approaches have been used to deal with the low or high dimensional case. Indeed, for the classical Allen-Cahn equation, the proof of the conjecture in dimensions $n = 2, 3$ is a purely PDE proof, which relies on some energy estimates and a Liouville-type argument, but never uses the classification for area-minimizing surfaces (see [2, 4, 34]). Instead, for $4 \leqslant n \leqslant 8$, the fact that the only area-minimizing surfaces in the whole $\mathbb{R}^n$ are hyperplanes if $n \leqslant 7$ plays a crucial role. The proof of the conjecture in dimensions larger than 3 was given by Savin in [40] who, using the so-called improvement of flatness, proved that if the level sets of certain solutions are asymptotically flat, then the solution needs to be one-dimensional. In [40] all the ingredients needed in the proofs (energy estimates, density estimates, improvement of flatness) require the solution to be a minimizer for the associated energy functional. A first result in Savin's paper is, in fact, the validity of the De Giorgi conjecture for minimizers in dimensions $n \leqslant 7$. On the geometric side, this statement corresponds to the fact that any area-minimizing surface in the whole $\mathbb{R}^n$ is flat if $n \leqslant 7$. The original conjecture by E. De Giorgi was for bounded monotone solutions (which in general are not minimizers without further assumption). A second result in [40] asserts that if u is a bounded monotone solution for the classical Allen-Cahn equation in $\mathbb{R}^n$ (e.g. $u_{x_n} > 0$), such that $\lim_{x_n \to \pm\infty} u(x) = \pm 1$, then u is 1-D for $n \leqslant 8$. This statement corresponds, on the geometric side, to the fact that any area-minimizing graph is flat up to dimension $n = 8$. We stress that the additional assumption on the limits at infinity are needed to ensure that the solution is a minimizer. The conjecture in dimension $4 \leqslant n \leqslant 8$ for monotone solutions without the limits assumption is still open.

Concerning the fractional case, when $n = 2$ for any $0 < s < 1$, and when $n = 3$ for $1/2 \leqslant s < 1$ the pure PDE proof, which uses the ideas developed in [4] for the classical conjecture in the low-dimensional case, still works (see [8, 9, 13, 42]). While for treating the case $n = 3$ and $0 < s < 1/2$ (see [25, 27]), and the case $4 \leqslant n \leqslant 8$ with $1/2 < s < 1$ (see [41]) one needs to use the idea of Savin based on an improvement of flatness result. As said above, in this approach, the classification for nonlocal minimal surfaces is crucial, that is why when $0 < s < 1/2$ and $n > 3$ the conjecture is still open.

We conclude this Introduction, commenting on the class of solutions for which one expects 1-D symmetry to hold true. As already mentioned, the original conjecture was for monotone solutions, which corresponds to having area-minimizing graphs on the geometric side. For these solutions the conjecture is true up to dimension 8 for $s \in (1/2, 1]$ with the additional assumptions on the limits at infinity (as we will see in Section 3, when $n = 3$ this additional assumption is not needed). On the other hand the problem has a variational structure and it is natural to ask the same question for minimizers of the energy: in this version the conjecture is true up to dimension 7 for $s \in (1/2, 1]$. Another class of solutions for which one expects the conjecture to hold true is the one of *stable* solutions (here stability is in the variational sense, that is one requires the second variation of the energy functional to be nonnegative). For stable solutions, even the conjecture for the classical Allen-Cahn equation is still open in all dimensions $n > 2$. This lack of information for the PDE is reflected at the geometric level: it is still an open question whether *stable* minimal surfaces are necessarily hyperplanes in dimension $3 < n \leqslant 7$ (see Section 4 for the precise references). We stress that, instead, stable minimal *cones* are completely classified: they are hyperplanes in dimensions $n \leqslant 7$. In $\mathbb{R}^8$ the Simons cone is an example of stable singular cone. One would expect that this classification of stable cones would imply an analogue classification for any stable surfaces and, on the PDE side, the 1-D symmetry of stable solutions. One of the main obstruction in classifying stable objects is given by the lack of energy estimates. Surprisingly, in the nonlocal setting, some techniques have been recently developed to attack the study of stable objects and some results (such as energy and BV estimates) have been obtained. The analogue results in the local setting are still unknown and the study of stable solutions to the Allen-Cahn equation (and of stable classical minimal surfaces) is still widely open. We will address the classification of stable objects for both the Allen-Cahn equation and the theory of minimal surfaces, in the last section.

The paper is organized as follows:

- In Section 2, we describe the connections between the fractional Allen-Cahn equation and the theory of local/nonlocal minimal surfaces. The main results of this Section have been obtained in [43, 45];
- Section 3 deals with the De Giorgi conjecture for the fractional Allen-Cahn equation. In particular we address the low dimensional case, giving a sketch of the proofs of the results contained in [8, 9].
- In Section 4, we describe some recent results concerning the classification of nonlocal minimal surfaces contained in [21, 10, 42];
- In Section 5 we present some very recent results on the classification of stable objects, both for the fractional Allen-Cahn equation and for the fractional perimeter and we conclude with some related open questions.

2. Γ-convergence results for nonlocal phase transitions

The classical setting. We start by describing a classical model for phase transitions and the rigorous mathematical results which explained the connection between the Allen-Cahn equation and the theory of minimal surfaces. For this part, we refer to [1] and references therein.

Let us consider a container, represented by a bounded and regular subset Ω of $\mathbb{R}^3$, which is filled with two phases of the same fluid. The configuration of the system is described by a function u. There are two different models for the phase transition phenomenon, depending whether the transition between the two phases is given by a separating interface or is a continuous transition which occurs in a thin layer.

In the first model, usually called *sharp-interface model*, the configuration function u only takes two values, e.g. $+1$ and -1, which corresponds to the two pure phases. The classical theory of phase transitions, assume that at equilibrium the two fluids arrange themselves in order to minimize the area of the separating interface, that is the measure of the jump set of u. Hence, in this model, the energy of the system is a pure interface energy given by

$$F(u) = \sigma \mathcal{H}^2(S_u), \tag{2.1}$$

where S_u denotes the jump set of u, $\mathcal{H}^2$ the 2-dimensional Hausdorff measure, and σ is a parameter representing the surface tension between the two phases.

Imposing a volume constraint, the space of all admissible configurations is given by $A = \{u : \Omega \to \{-1, 1\} \,:\, \int_\Omega u = V\}$, where $-|\Omega| < V < |\Omega|$ and the configuration of the system at equilibrium is obtained by minimizing F over A.

The second model, often called the *diffusive model*, was proposed by J.W. Cahn and J.E. Hilliard and allows a fine mixture of the two phases, which corresponds to the fact that the configuration function u can take values in the whole interval $[-1, 1]$. Now, the space of all admissible configurations is given by $A = \{u : \Omega \to [-1, 1] \,:\, \int_\Omega u = V\}$ and the energy has the following form:

$$\mathcal{E}_\varepsilon(u, \Omega) = \int_\Omega \left(\frac{\varepsilon^2}{2} |\nabla u|^2 + W(u) \right) dx, \tag{2.2}$$

where $\varepsilon > 0$ is a small parameter and W is a continuous function which vanishes only at -1 and 1 and is positive elsewhere (usually called a double-well potential). We observe that the Dirichlet term and the potential one are in competition, indeed $W(u)$ forces the configurations to take values close to -1 and 1 and hence favourites the separation of the two phases, while the first term in the energy penalizes the spatial inhomogeneity of u. For small ε the potential term prevails, and the minimum of $\mathcal{E}_\varepsilon$ is attained by a function u_ε which takes values close to -1 and 1 and the transition between these two phases happens in a thin layer of thickness ε.

The Euler-Lagrange equation for the energy (2.2) is given by the (rescaled) Allen-Cahn equation

$$\varepsilon^2 \Delta u = W'(u).$$

A rigorous mathematical justification of the connection between the sharp-interface and the diffuse models was given by Modica and Mortola in [36]. They proved that, when $\varepsilon \to 0$, the rescaled functional $\varepsilon^{-1}\mathcal{E}_\varepsilon$ Γ-converges to F defined by (2.1), and hence minimizers of $\mathcal{E}^\varepsilon$ converges to minimizers of F (this Γ-convergence result holds in any dimension n). The right setting for functions u_0 obtained as limits of minimizers u_ε of $\mathcal{E}_\varepsilon$ is the one of BV functions and the limit functional is the perimeter in the sense of De Giorgi of the sublevelsets of u_0 (which agrees with the $(n-1)$-dimensional Hausdorff measure for smooth objects). The Modica-Mortola theorem establishes convergence, in the L^1-sense, for sequences of minimizers u_ε to a BV function taking values in $\{-1, 1\}$, whose jump set is an area-minimizing surface. Later, in [15] Caffarelli and Cordoba proved that actually the convergence of minimizers is not only in L^1 but in the Hausdorff distance sense.

The nonlocal setting. We pass now to describe what happens when one replaces the standard Dirichlet energy with a nonlocal term which takes into account long range interactions. For a bounded subset Ω of $\mathbb{R}^n$, we consider an energy functional of the form

$$\mathcal{E}^s(u, \Omega) = \frac{1}{2} \iint_{(\mathbb{R}^n\times\mathbb{R}^n)\setminus(\Omega^c\times\Omega^c)} \frac{|u(x) - u(\bar{x})|^2}{|x - \bar{x}|^{n+2s}} \, dx \, d\bar{x} + \int_\Omega W(u) \, dx, \tag{2.3}$$

where Ω^c denotes the complement of Ω.

Observe that the set of integration in the Dirichlet term is given by $(\mathbb{R}^n \times \mathbb{R}^n) \setminus (\Omega^c \times \Omega^c)$. This term represents to contribution of the H^s-seminorm of u in Ω and takes into account the interactions between all possible couple of points x, $\bar{x}$ except the ones for which both x and $\bar{x}$ do not belong to Ω. The reason for this choice is that the energy in the whole space $\mathbb{R}^n \times \mathbb{R}^n$ could not be finite, and the notion of minimality that we consider is the one with fixed "boundary" data, that is competitors must agree with the minimizer u in the complement of Ω, according to the following definition.

Definition 2.1. We say that a function u is a *minimizer* for the energy $\mathcal{E}^s$ if

$$\mathcal{E}^s(u, \Omega) \leqslant \mathcal{E}^s(w, \Omega), \quad \text{for any } w \text{ such that } u \equiv w \ \text{ in } \Omega^c.$$

In [45], Savin and Valdinoci proved a Γ-convergence result for the (suitably rescaled) functional $\mathcal{E}^s$, that is the analogue of the Modica-Mortola theorem in the nonlocal setting. Interestingly, the Γ-limit of $\mathcal{E}^s$ is different depending whether s is below or above $1/2$. Before stating the main result in [45], we introduce all the necessary ingredients.

In the sequel, W will denote a potential with a double-well structure, i.e. we assume that

$$W : [-1, 1] \to [0, +\infty), \;\; W \in C^2([-1, 1]), \;\; W > W(\pm 1) = 0 \text{ in } (-1, 1)$$
$$W'(\pm 1) = 0 \quad \text{and} \quad W''(\pm 1) > 0.$$

The class of admissible functions is given by
$$X = \{u \in L^\infty(\mathbb{R}^n) \,|\, ||u||_\infty \leqslant 1\}.$$
We set
$$\mathcal{E}^s_\varepsilon(u,\Omega) = \frac{1}{2}\varepsilon^{2s} \iint_{\mathbb{R}^n\times\mathbb{R}^n\setminus(\Omega^c\times\Omega^c)} \frac{|u(x)-u(\bar{x})|^2}{|x-\bar{x}|^{n+2s}}\,dx\,d\bar{x} + \int_\Omega W(u)\,dx,$$
and we consider the functional
$$\mathcal{F}^s_\varepsilon(u,\Omega) = \begin{cases} \varepsilon^{-2s}\mathcal{E}^s_\varepsilon(u,\Omega) & \text{if } 0<s<1/2 \\ |\varepsilon\log\varepsilon|^{-1}\mathcal{E}^s_\varepsilon(u,\Omega) & \text{if } s=1/2 \\ \varepsilon^{-1}\mathcal{E}^s_\varepsilon(u,\Omega) & \text{if } 1/2<s<1. \end{cases}$$
We can now state the main result in [45].

Theorem 2.2 (Theorem 1.4 in [45]). *Let $s \in (0,1)$ and Ω be a bounded Lipschitz domain of $\mathbb{R}^n$.*

Then,
$$\mathcal{F}^s_\varepsilon(u,\Omega) \xrightarrow{\Gamma} \mathcal{F}(u,\Omega) \quad \text{as} \quad \varepsilon \to 0,$$
where $\mathcal{F}(u,\Omega)$ is defined as follows:
$$\text{if } 0<s<1/2 \qquad \mathcal{F}(u,\Omega) = \begin{cases} \mathrm{Per}_s(E) & \text{if } u_{|\Omega} = \chi_E - \chi_{E^c} \text{ for some } E\subset\Omega \\ +\infty & \text{otherwise,} \end{cases}$$
$$\text{if } 1/2\leqslant s<1 \qquad \mathcal{F}(u,\Omega) = \begin{cases} c_*\mathrm{Per}(E) & \text{if } u_{|\Omega} = \chi_E - \chi_{E^c} \text{ for some } E\subset\Omega \\ +\infty & \text{otherwise,} \end{cases}$$
and c_* is a constant depending on n, s and W.

The s-perimeter Per_s will be precisely defined in Section 4 below. This Γ-convergence theorem, together with a compactness result (see Theorem 1.5 in [45]), implies that if u_ε is a sequence of minimizers for $\mathcal{F}^s_\varepsilon$ such that $\mathcal{F}^s_\varepsilon$ are uniformly bounded as $\varepsilon \to 0$, then there exists a subsequence, that we still call u_ε, which converges in L^1 to a function $u_0 = \chi_E - \chi_{E^c}$ where E is a minimizer for the fractional perimeter Per_s in Ω if $0<s<1/2$ and a minimizer for the classical perimeter Per in Ω if $1/2 \leqslant s < 1$.

As for the case of the classical phase transition model, one can prove that the convergence is not just in L^1 but in a stronger sense. In [43], Savin and Valdinoci proved some density estimates for minimizers of $\mathcal{E}^s$ which gives a bound on the measure of the volume occupied by the level sets of a minimizer in a ball. As a consequence of the density estimates, we have that level sets of minimizers of $\mathcal{E}^s_\varepsilon$ converges locally uniformly as $\varepsilon \to$ to a nonlocal s-minimal surface when $0<s<1/2$, and to a classical minimal surface when $1/2 \leqslant s < 1$.

As already explained in the Introduction, this convergence results motivates the analogue of the De Giorgi conjecture for certain solutions (monotone solutions and minimizers) to the fractional Allen-Cahn equation, which is the Euler-Lagrange equation of the energy functional $\mathcal{E}^s_\varepsilon$. More precisely, since the Γ-limit of $\mathcal{E}^s_\varepsilon$ when $1/2 \leqslant s < 1$ is exactly the same as in the local case, one expects to have one-dimensional symmetry of bounded monotone solutions up to dimension $n=8$. This has been proven in [8, 9, 13, 42] in the low-dimensional case $n=2,3$, and in [41], under the additional assumption on the limits at infinity, for $4 \leqslant n \leqslant 8$. In [41], the author also announces that in a forthcoming paper he will prove the same result for $s = 1/2$. Instead, when $0 < s < 1/2$, the level sets of minimizers for $\mathcal{E}^s$ are asymptotically nonlocal minimal surfaces, and not much is known yet on their classification. For this range of s, the conjecture is known to be true only in dimensions $n = 2, 3$ (see [13, 42, 25]), while it is still open in dimensions $n > 3$.

3. The De Giorgi conjecture for the fractional Laplacian

In this section we describe the main ideas in the proof of the one-dimensional symmetry for minimizers and bounded monotone solutions to

$$(-\Delta)^s u = f(u) \quad \text{in } \mathbb{R}^n, \tag{3.1}$$

in the low-dimensional case, that is for $n = 2$ with $0 < s < 1$, and for $n = 3$ with $1/2 \leqslant s < 1$. These results are contained in [8, 9, 14, 12, 46]. In all these works, in order to prove the De Giorgi conjecture for solutions to the nonlocal equation (3.1), the authors considered the, so-called, Caffarelli-Silvestre extension (that we recall in the next subsection) and work with solutions to a local problem in the half-space $\mathbb{R}^{n+1}_+$. We emphasize that a new proof of the conjecture in dimension $n = 2$ and for any $0 < s < 1$ has been found by Bucur and Valdinoci in [7], without making use of the extension and working "downstairs". This proof is based on some techniques introduced in [21] and it works only in dimension $n = 2$.

Here, we present the proofs in [8, 9] that cover both cases $n = 2$ with $0 < s < 1$ and $n = 3$ with $1/2 \leqslant s < 1$. As already explained in the Introduction, in this setting the same approach used to prove the original De Giorgi conjecture in dimension $n = 3$ in [2, 4] based on some sharp energy estimates and a Liouville-type argument, works. We stress that this approach allows to consider general nonlinearity f, not necessarily associated to a double well potential.

We give now the precise statement of this result.

Theorem 3.1 (see [8, 9, 13, 42])**.** *Let f be any $C^{1,\gamma}$ nonlinearity with $\gamma > \max\{0, 1 - 2s\}$ and u be either a bounded minimizer or a bounded solution which is monotone in some direction, of*

$$(-\Delta)^s u = f(u) \quad \textit{in } \mathbb{R}^n.$$

Then, if $n = 2$ and $0 < s < 1$ or $n = 3$ and $1/2 \leqslant s < 1$, u depends only on one variable, or equivalently, the level sets of u are flat.

Here below, we describe the main steps of the proof of Theorem 3.1, emphasizing which are the main difficulties in the nonlocal setting. The notion of minimizer that we use is given precisely in Definition 3.2 below.

The proof uses the Caffarelli-Silvestre extension and is based on the three following main ingredients:

- stability of solutions;
- a Liouville-type result;
- energy estimates.

In the following subsections we recall briefly all these ingredients.

3.1. **The Caffarelli-Silvestre extension and the notion of minimality.** In [17], Caffarelli and Silvestre gave an equivalent formulation for nonlocal problems involving the fractional Laplacian in $\mathbb{R}^n$, introducing a new local problem in the positive half-space $\mathbb{R}^{n+1}_+$. More precisely, they established that a bounded function u is a solution of (3.1) if and only if v is a solution of

$$\begin{cases} \mathrm{div}(y^{1-2s}\nabla v) = 0 & \text{in } \mathbb{R}^{n+1}_+ \\ -d_s \lim_{y\to 0} y^{1-2s}\partial_y v = f(v(x,0)) & \text{in } \mathbb{R}^n, \end{cases} \tag{3.2}$$

where v is the bounded extension in the positive half-space of u, that is $v(x,0) = 0$ in $\mathbb{R}^n$, and d_s is a positive constant depending only on s. Here, we denote by $(x,y) = (x_1,\dots,x_n,y)$ a point in $\mathbb{R}^{n+1}_+ = \mathbb{R}^n \times \mathbb{R}^+$.

In the sequel we will call the extension v satisfying (3.2), the *s-extension* of u.

We set $a := 1-2s$, so that $-1 < a < 1$ for any $0 < s < 1$. We recall that the first equation in (3.2) is an equation in divergence form with a weight that belongs to the Muckenhoupt class A_2 and hence is a "good" weight, in the sense that we have a Poincaré inequality, Harnack inequality and Hölder regularity of weak solutions for this kind of equations by the theory of Fabes, Kenig and Serapioni, developed in [29]. We observe moreover that, depending whether s is above or below $1/2$, the equation becomes degenerate or singular.

Problem (3.2) has a variational structure and therefore it is natural to consider the associated energy functional and the related notion of minimizer. Let B_R denote the ball in $\mathbb{R}^n$ centered at 0 and of radius R and let $C_R = B_R \times (0,R)$ denote the cylinder of radius R and height R in the positive half-space $\mathbb{R}^{n+1}_+$. We consider a localized energy functional (since the energy in the whole space is not finite in general) in cylinders, which has the following form:

$$E^s(v, C_R) = \frac{1}{2}\int_{C_R} y^a |\nabla v|^2 \, dx\, dy + \int_{B_R} W(v(x,0))\, dx,$$

where the potential W is such that $W' = -f$.

We can now give the notion of minimizer for problem (3.2).

Definition 3.2. We say that a bounded $C^1(\mathbb{R}^{n+1}_+)$ function v is a minimizer for (3.2) if

$$E^s(v, C_R) \leqslant E^s(w, C_R)$$

for every $R > 0$ and for every bounded competitor w such that $v \equiv w$ on $\partial C_R \cap \{y > 0\}$. We say that a bounded $C^1(\mathbb{R}^n)$ function u is a minimizer for (3.1) if its s-extension v is a minimizer for (3.2).

3.2. **Stability of solutions.** We recall the definition of *stable solution* for (3.1).

Definition 3.3. We say that a bounded solution v of (3.2) is *stable* if

$$\int_{\mathbb{R}^{n+1}_+} y^a |\nabla \xi|^2 \, dx\, dy - \int_{\mathbb{R}^n \times \{y=0\}} f'(u)\xi^2\, dx \geqslant 0$$

for every function $\xi \in C^1_0(\overline{\mathbb{R}^{n+1}}_+)$.

We say that a bounded function u is a *stable* solution for (3.1) if its s-extension v is a stable solution for (3.2).

We observe that if u is a minimizer for (3.1) then, in particular, it is a stable solution. Moreover, as established in Lemma 6.1 in [13], we have a characterization of stability in terms of existence of a positive solution for the linearized problem. More precisely one can prove that a solution u to (3.1) is stable if and only if there exists a positive Hölder continuous function $\varphi \in H^1_{loc}(\overline{\mathbb{R}^{n+1}_+}, y^a)$ with $\varphi > 0$ in $\overline{\mathbb{R}^{n+1}_+}$, satisfying

$$\begin{cases} \operatorname{div}(y^a \nabla\varphi) = 0 & \text{in } \mathbb{R}^{n+1}_+ \\ -y^a \partial_y \varphi = f'(u)\varphi & \text{on } \{y = 0\}. \end{cases} \tag{3.3}$$

Suppose that u is monotone in some direction, e.g. $\partial_{x_n} u > 0$ then, as an application of the maximum principle, one can easily see that its s-extension v satisfies $\partial_{x_n} v > 0$ in the whole $\mathbb{R}^{n+1}_+$. By using the previous characterization of stability, we deduce that v is a stable solution to (3.2) since its derivative in the x_n direction is a positive solution to the linearized problem (3.3).

3.3. **A Liouville-type result.** We introduce the following weighted Sobolev space:

$$H^1_{\text{loc}}(\overline{\mathbb{R}^{n+1}_+}, y^a) = \{\sigma : \overline{\mathbb{R}^{n+1}_+} \to \mathbb{R} \mid y^a(\sigma^2 + |\nabla\sigma|^2) \in L^1_{\text{loc}}(\overline{\mathbb{R}^{n+1}_+})\}.$$

A second ingredient in the proof of the De Giorgi conjecture is the following Liouville-type lemma.

Lemma 3.4 (Theorem 6.1 in [8] and Theorem 4.10 in [13])**.** *Let φ be a positive function in $L^\infty_{loc}(\mathbb{R}^{n+1}_+)$, $\sigma \in H^1_{loc}(\overline{\mathbb{R}^{n+1}}_+, y^a)$ such that:*

$$\begin{cases} -\sigma \operatorname{div}(y^a \varphi^2 \nabla\sigma) \leqslant 0 & \text{in } \mathbb{R}^{n+1}_+ \\ y^a \sigma \frac{\partial\sigma}{\partial\nu} \leqslant 0 & \text{on } \mathbb{R}^n \times \{y = 0\} \end{cases}$$

in the weak sense. If in addition:

$$\int_{C_R} y^a \big(\varphi\sigma\big)^2 \leqslant CR^2 \log R \tag{3.4}$$

holds for every $R > 1$, then σ is constant.

For the proof of this lemma under the stronger assumption that the quantity in (3.4) grows at most like R^2 (instead of $R^2 \log R$) it is enough to multiply the equation by a cutoff function and integrate by parts. For allowing the logarithmic term one needs a refinement of this argument found in [37].

3.4. **Sketch of the proof of the De Giorgi conjecture in low dimensions.** We can now describe the main ideas in the proof of the one-dimensional symmetry of minimizers and monotone solutions for $n = 2$ with $0 < s < 1$ and $n = 3$ with $1/2 \leqslant s < 1$.

In order to prove that the solution u to (3.1) is one-dimensional, we will show that its s-extension v is depend only on y and on one direction in $\mathbb{R}^n$.

Suppose that u is a stable solution to (3.1) and v is its s-extension, that is a stable solution for problem (3.2). By the characterization of stability, we know that there exists some positive function φ satisfying (3.3) (if in particular v is monotone in the x_n direction, one can take $\varphi = v_{x_n}$).

For any $i = 1, \dots, n$, we define the functions

$$\sigma_i := \frac{v_{x_i}}{\varphi}.$$

An easy computation shows that $\varphi^2\nabla\sigma_i = \varphi\nabla v_{x_i} - v_{x_i}\nabla\varphi$ and using that both v_{x_i} and φ satisfy the linearized problem (3.3), we deduce

$$\mathrm{div}(y^a\varphi^2\nabla\sigma_i) = 0 \quad \text{in } \mathbb{R}^{n+1}_+. \tag{3.5}$$

Moreover, using again that v_{x_i} and φ satisfy the same linearized problem (in particular they have the same Neumann condition on $\{y=0\}$), we have

$$y^a\sigma_i\partial_y\sigma_i = y^a\frac{v_{x_i}}{\varphi^2}v_{x_iy} - y^a\frac{v_{x_i}^2}{\varphi^2}\frac{\varphi_y}{\varphi} = 0 \qquad \text{on } \mathbb{R}^n\times\{y=0\}. \tag{3.6}$$

Suppose now that the following estimate for the Dirichlet energy of v holds:

$$\int_{C_R} y^a|\nabla v|^2\,dx\,dy \leqslant CR^2\log R.$$

Then, we can apply Lemma 3.4 with $\sigma=\sigma_i$. Indeed (3.3) is satisfied by (3.5) and (3.6), moreover,

$$\int_{C_R} y^a(\varphi\sigma_i)^2\,dx\,dy = \int_{C_R} y^a|v_{x_i}|^2\,dx\,dy \leqslant \int_{C_R} y^a|\nabla v|^2\,dx\,dy \leqslant CR^2\log R.$$

Hence we deduce that σ_i is constant for any $i=1,\dots,n$. This concludes the proof observing that if $c_1=\cdots=c_n=0$ then v is constant. Otherwise we have $c_iv_{x_j}=c_jv_{x_i}=0$ for every $i\neq j$ and we deduce that v depends only on y and on the variable parallel to the vector $(c_1,\dots,c_n)$.

Hence, the crucial missing ingredient to conclude the proof is the following estimate for the Dirichlet energy

$$\int_{C_R} y^a|\nabla v|^2\,dx\,dy \leqslant CR^2\log R. \tag{3.7}$$

3.5. **Energy estimates.** By the previous discussion, in order to conclude the proof of Theorem 3.1, it remains to prove that both minimizers and bounded monotone solutions (which are in particular stable solutions) satisfy estimate (3.7), in $\mathbb{R}^2$ with $0<s<1$ and in $\mathbb{R}^3$ with $1/2\leqslant s<1$. This is the aim of this subsection.

We start by stating the energy estimate for minimizers contained in [8, 9], which holds in any dimension n.

Theorem 3.5 (Theorem 1.2 in [9]). *Let $f\in C^{1,\gamma}(\mathbb{R})$, with $\gamma>\max\{0,-a\}$, and let v be a bounded minimizer for problem* (3.2)*.*

Then, the following estimates hold

$$E^s(v,C_R) \leqslant \begin{cases} CR^{n-2s} & \text{if } 0<s<1/2 \\ CR^{n-1}\log R & \text{if } s=1/2 \\ CR^{n-1} & \text{if } 1/2<s<1, \end{cases} \tag{3.8}$$

for any $R\geqslant 2$.

In dimension $n=3$, the same energy estimate holds also for bounded monotone solutions, according to the following result

Theorem 3.6 (Theorem 1.4 in [9]). *Let $f \in C^{1,\gamma}(\mathbb{R})$, with $\gamma > \max\{0, -a\}$, and let v be a bounded solution of (3.2) with $n = 3$ such that its trace $u(x) = v(x, 0)$ is monotone in some direction.*

Then, the following estimates hold

$$E^s(v, C_R) \leqslant \begin{cases} CR^{3-2s} & \text{if } 0 < s < 1/2 \\ CR^2 \log R & \text{if } s = 1/2 \\ CR^2 & \text{if } 1/2 < s < 1, \end{cases} \tag{3.9}$$

for any $R \geqslant 2$.

As a consequence of Theorems 3.5 and 3.6, we have that the required estimate (3.7) is satisfied by minimizers and bounded monotone solutions in dimension $n = 2$ for any $s \in (0, 1)$ and in dimension $n = 3$ for $s \in [1/2, 1)$.

We stress that the main difficulty in the proof of the fractional De Giorgi conjecture in low dimensions, relies precisely in establishing the sharp energy estimates for minimizers, since all the other ingredients (the characterization of stability and the Liouville-type result described above are not difficult adaptations of the analogous local results to the nonlocal setting). The energy estimates for minimizers have also been proven by Savin and Valdinoci in [45], without making use of the extension and working with the nonlocal energy functional associated to (3.1). Here, we present the approach via extension of [9], since, as already explained at the beginning of this section, the extension is needed in order to prove the De Giorgi conjecture for the fractional Laplacian in dimension $n = 3$ and for $1/2 \leqslant s < 1$.

Before giving an idea of the proof of Theorem 3.5, we recall how one can get the sharp energy estimate for minimizers of the classical Allen-Cahn equation $-\Delta u = u - u^3$, whose associated energy functional is given by

$$\mathcal{E}(u, B_R) = \int_{B_R} \left(\frac{1}{2} |\nabla u|^2 + W(u) \right) dx.$$

To give a bound for the energy of a minimizer u, we argue using a comparison argument, that is we construct a suitable competitor w, which agrees with u on ∂B_R and whose energy is bounded above by CR^{n-1} (we recall that in the classical setting the energy of minimizers grows like R^{n-1}, that is exactly the same growth as the case $1/2 < s < 1$ in Theorem 3.5).

Given a cut-off function $\eta(x) = \eta(|x|)$ compactly supported in B_R and identically equal to 1 in B_{R-1}, we define the competitor $w = \eta + (1 - \eta)u$ in the whole $\mathbb{R}^n$, so that

$$w = \begin{cases} 1 & \text{in } B_{R-1} \\ u & \text{on } \partial B_R. \end{cases}$$

With this choice of w, it is easy to verify that

$$\begin{aligned} \mathcal{E}(u, B_R) \leqslant \mathcal{E}(w, B_R) &= \int_{B_R} \left(\frac{1}{2} |\nabla w|^2 + W(w) \right) dx \\ &= \int_{B_R \setminus B_{R-1}} \left(\frac{1}{2} |\nabla w|^2 + W(w) \right) dx \leqslant CR^{n-1}, \end{aligned} \tag{3.10}$$

where in the first inequality we have used the minimality of u and in the last inequality we have used that $|\nabla u| \in L^\infty(\mathbb{R}^n)$ by standard elliptic estimates, and that the measure of the annulus $B_R \setminus B_{R-1}$ in $\mathbb{R}^n$ is estimated by CR^{n-1}.

We consider now the case of the fractional Laplacian. In this case, we need to construct a suitable competitor w for the minimizers v of E^s in C_R, which fulfills the energy estimates of Theorem 3.5. We recall that, due to the Neumann condition in problem (3.2), now the competitor w has to agree with v on $\partial C_R \cap \{y > 0\}$ but it is free on the bottom of the cylinder $\partial C_R \times \{y = 0\}$. This fact will play a crucial role in the construction of w. On the other hand, let us emphasize that now the cylinder C_R is an $(n + 1)$-dimensional object, and we hope for an estimate for the energy that grows at most like $R^{n-1} \log R$.

Let us describe now how we build the competitor w. We start by defining a function $\bar{w}$ on ∂C_R. Then, we will define w as a suitable extension of $\bar{w}$ inside C_R. First of all, in order to use a comparison argument, $\bar{w}$ needs to agree with v on $\partial C_R \cap \{y > 0\}$. Secondly, since the potential energy appears only as a boundary term on the bottom of the cylinder C_R, and in this part of the boundary w is free, we define $\bar{w}$ as done for the local case, that is in such a way that it agrees with $v(x, 0)$ on $B_R \times \{y = 0\}$ and is identically 1 in $B_{R-1} \times \{y = 0\}$. Resuming, $\bar{w}$ is defined on the whole ∂C_R and satisfies

$$\bar{w} = \begin{cases} 1 & \text{in } B_{R-1} \times \{y = 0\} \\ u & \text{on } \partial C_R \cap \{y > 0\}. \end{cases}$$

It remains now to extend $\bar{w}$ to a function w defined on the whole cylinder C_R. Since we want w to have the least possible Dirichlet energy, we choose w to be the solution of the Dirichlet problem

$$\begin{cases} \operatorname{div}(y^a \nabla w) = 0 & \text{in } C_R \\ w = \bar{w} & \text{on } \partial C_R. \end{cases} \tag{3.11}$$

With this choice of w, we have

$$E^s(v, C_R) \leqslant E^s(w, C_R) = \int_{C_R} y^a |\nabla w|^2 \, dx \, dy + \int_{B_R} W(w) \, dx \leqslant \int_{C_R} y^a |\nabla w|^2 \, dx \, dy + CR^{n-1}.$$

The final step of the proof of the energy estimate for minimizers consists in giving an estimate for the Dirichlet energy of w. This is achieved in [8, 9] using some suitable trace inequalities and optimal gradient bounds for the solutions of (3.2) (for details see Theorems 1.7 and 1.9 in [9]).

To conclude, we comment on the proof of the energy estimate for monotone solutions. In [8, 9], the authors follow the idea of [2] which is based on the following result: bounded monotone solutions are minimizers in the restricted class of functions

$$\mathcal{A}_v := \{ \lim_{x_n \to -\infty} v \leqslant w \leqslant \lim_{x_n \to +\infty} v \}.$$

This result can be proven by a sliding argument and the use of the maximum principle (see proof of Proposition 6.1 in [9]). Once one has this minimality property of monotone solutions, it is enough to show that the competitor w constructed in the proof of Theorem 3.5 belongs to the class $\mathcal{A}_v$. For this last step we refer to Lemma 6.1 and the proof of Theorem 1.4 in [9].

4. Classification for nonlocal minimal surfaces

We start by recalling the notion of fractional perimeter, which was introduced in [16].

Let $s \in (0,1/2)$ and let Ω be a bounded domain in $\mathbb{R}^n$. We define the fractional s-perimeter of a measurable set $E \subset \mathbb{R}^n$ relative to Ω as

$$\operatorname{Per}_s(E,\Omega) := \int_{E\cap\Omega}\int_{E^c} \frac{1}{|x-\bar{x}|^{n+2s}}\,dx\,d\bar{x} + \int_{E\setminus\Omega}\int_{\Omega\setminus E} \frac{1}{|x-\bar{x}|^{n+2s}}\,dx\,d\bar{x}, \tag{4.1}$$

where E^c denotes the complement of E in $\mathbb{R}^n$.

Observe that here we use the notation $\mathrm{Per_s}$ for the perimeter associated with the kernel $|z|^{-n-2s}$, with $0 < s < 1/2$. In many references the order $2s$ is replaced by s, that is one writes Per_s for the perimeter associated to the power $|z|^{-n-s}$ and in this notation s belongs to $(0,1)$. Here, we use the first notations for consistency with the notations used for the fractional Laplacian $(-\Delta)^s$.

The choice of the set of integration in the definition of the fractional perimeter is the natural one, similarly as for the Dirichlet term in the energy $\mathcal{E}^s$ defined in (2.3), in order to avoid infinite contributions coming from the complement of Ω and it does not change the variational structure of the functional once we have fixed the set E outside of Ω. More precisely, similarly to Definition 3.2, we give the following definition.

Definition 4.1. We say that a set E is a *minimizer* for the s-perimeter in Ω if

$$\operatorname{Per}_s(E,\Omega) \leqslant \operatorname{Per}_s(F,\Omega), \qquad \text{for all } F \text{ such that } E\setminus\Omega = F\setminus\Omega.$$

Moreover, we say that E is a minimizer for the s-perimeter in $\mathbb{R}^n$, if E is a minimizer in B_R for all $R > 0$.

The (boundaries of) minimizers of the s-perimeter are usually called *nonlocal minimal* (or s-minimal) *surfaces.*

While the classical perimeter (in the De Giorgi sense) of a set E relative to Ω is the BV-seminorm of the characteristic function χ_E in Ω, the s-perimeter is the H^s (or $W^{2s,1}$) seminorm of the characteristic function χ_E in Ω (we remind that the characteristic function of a set belongs to H^s only if $0 < s < 1/2$). Hence, a nonlocal minimal surface is the boundary of a set E, whose characteristic function minimize the H^s seminorm, among all sets which coincide with E in the complement of Ω.

Another motivation for referring to Per_s as a fractional *perimeter* comes from the asymptotics of this nonlocal functional as $s \to 1/2$. Indeed it is known that the s-perimeter (multiplied by the factor $1/2 - s$) tends to the classical perimeter as $s \to 1/2$, up to a dimensional constant. This fact has been established in several contributions where different notions of convergence are considered (see [23] for the precise limit in the class of BV functions, [18, 19] for a geometric approach to prove regularity and [5] for a Γ-convergence result). The limit as $s \to 0$ was studied in [26], where the authors proved that it is related to the Lebesgue measure of the sets $E \cap \Omega$ and $\Omega \setminus E$.

Making the first variation of the nonlocal perimeter functional, one can introduce the notion of *nonlocal mean curvature.* The nonlocal mean curvature of a set E at a point $x \in \partial E$ is defined as follows

$$H^s_E(x) := \int_{\mathbb{R}^n} \frac{\chi_{E^c}(y) - \chi_E(y)}{|x-y|^{n+2s}}\,dy.$$

Hence, a necessary condition for a set E to be an s-minimal surface is that $H^s_E = 0$ (see Theorem 5.1 in [16]). The first example of a surface with zero nonlocal mean curvature is a half-space. Other examples of sets with vanishing nonlocal mean curvature have been studied in the recent contributions [20, 24]. In [24], the nonlocal analogue of catenoids are constructed, but they differ from the standard catenoids since they approach a singular cone at infinity instead of having a logarithmic growth. These surfaces are constructed using perturbative methods, by performing small perturbation along the normal vector to ∂E. Instead in [20] it is proven that the standard helicoids are surfaces with zero nonlocal mean curvature.

We pass now to describe the main results in the study of regularity of nonlocal minimal surfaces. In [16], Caffarelli, Roquejoffre, and Savin established some results that are fundamental tools in the study of regularity, such as density estimates, the improvement of flatness for minimizers, a monotonicity formula, a blow up and a dimension reduction argument. Nevertheless, the study of regularity for minimizers of the fractional perimeter is still widely open. In this section we recall the main results related to the classification of entire s-minimal surfaces and to the study of regularity, and we describe the main open questions in the field.

4.1. **The classical setting.** We start by recalling the main results in the theory of *classical area minimizing surfaces*:

a) Every minimal cone in $\mathbb{R}^n$ is a hyperplane, whenever $n < 8$;
b) In $\mathbb{R}^8$ the Simons cone defined as
$$\mathcal{C} := \{x \in \mathbb{R}^8 \,|\, x_1^2 + \cdots + x_4^2 = x_5^2 + \cdots + x_8^2\}$$
is a minimizer for the perimeter functional;
c) If E is a minimizer of the perimeter functional in the whole $\mathbb{R}^n$, then E is a half-space, whenever $n < 8$;
d) If E is a minimizer of the perimeter functional and ∂E is a graph, then E is a half-space, whenever $n < 9$;
e) Any area-minimizing surface is smooth outside of a singular set Σ of Hausdorff dimension $n - 8$.

The classification of minimal cones (point a) is one of the basic tools in both the classification of entire minimal surfaces (that is surfaces that are minimizer of the perimeter functional in the whole $\mathbb{R}^n$) and in the study of regularity for minimizers of the perimeter in a bounded set Ω. Indeed, the classification of minimal cones leads, on side, to the classification of *any* entire area minimizing surfaces (point c) via a blow-down argument. On the other hand the nonexistence of singular minimal cones in space dimension $n \leqslant 7$ implies, via a blow up and a dimension reduction argument, that any minimal surface is $C^{1,\alpha}$ outside of a singular set of Hausdorff dimension $n-8$ (point e). Moreover, again the classification of minimal cones leads to the classification of entire minimal graphs (the so called Bernstein problem). Note that the critical dimension for a graph to be flat is one more than the one for a general set (point d). The main ingredients in the proof of these results are given by density estimates, perimeter estimates, improvement of flatness for minimizers and a monotonicity formula.

4.2. **The nonlocal setting.** We describe now, more in details, what is known in the nonlocal framework and which are the main open questions in the field.

The study of regularity for nonlocal minimal surfaces was started in [16], where density and perimeter estimates, the improvement of flatness and a monotonicity formula were established. With these tools, the authors could reduce the study of regularity for nonlocal minimal surfaces to the classification of nonlocal minimal cones. More precisely they proved that, if the blow up, around the origin, of an s-minimal set E is flat, then ∂E is $C^{1,\alpha}$ in a neighborhood of the origin (see Theorem 9.4 in [16]). As a consequence of a dimension reduction argument they proved $C^{1,\alpha}$ regularity outside of a singular set of Hausdorff dimension at most $n-2$ (see Theorem 10.4 in [16]). The bound $n-2$ on the dimension of the singular set was not optimal due to the fact that in [16] the classification of nonlocal minimal cones was not known, not even in $\mathbb{R}^2$. Basically, they had all the needed ingredients to pass from a) to e) in the above scheme, but the starting point a) was missing.

Later, in [42] Savin and Valdinoci proved that in $\mathbb{R}^2$ an s-minimal cone is necessarily a half-plane. As a consequence they could improve the bound on the Hausdorff dimension of the singular set from $n-2$ to $n-3$ and via a blow-down argument they obtained the classification of any s-minimal surface in $\mathbb{R}^2$. Moreover, in [6] Barrios, Figalli, and Valdinoci shows that if E is an s-minimal set such that $\partial E \in C^{1,\alpha}$, then ∂E is in fact C^∞. This is a consequence of a more general regularity result for solutions to integro-differential equations via a bootstrap argument. In [32], Figalli and Valdinoci address the fractional version of the Bernstein problems and they prove that, if there are not s-minimal singular cones in $\mathbb{R}^n$, then the only entire s-minimal graphs in $\mathbb{R}^{n+1}$ are the hyperplanes (they show how to pass from point a) to point d) in the nonlocal analogue of the previous scheme).

Resuming all these results, we have the following statement:

Theorem 4.2. (1) *Every s-minimal cone in $\mathbb{R}^2$ is a hyperplane ([42]);*
(2) *If E is a minimizer of the s-perimeter in the whole $\mathbb{R}^2$, then E is a half-plane ([42]);*
(3) *If E is a minimizer of the s-perimeter in $\mathbb{R}^n$ and ∂E is a graph, then E is a half-space, whenever $n \leqslant 3$ ([32]);*
(4) *If E is a minimizer of the s-perimeter, then ∂E is C^∞ outside of a singular set Σ of Hausdorff dimension $n-3$ ([6, 16, 42]).*

In addition, when s is close to $1/2$ Caffarelli and Valdinoci proved that all the regularity results that hold in the classical setting are inherited, by a compactness argument, by s-nonlocal minimal surfaces (see [18, 19]).

Theorem 4.3 ([19])**.** *There exists $\varepsilon_0 \in (0, 1/2)$ such that if $s \geqslant 1/2 - \varepsilon_0$, then any s-minimal surfaces is C^∞ outside of a singular set Σ of Hausdorff dimension $n-8$.*

Finally, in the very recent contribution [10], Cabré, Serra and the author proved flatness for nonlocal s-minimal cones in $\mathbb{R}^3$ for s close to $1/2$ (see Theorem 5.3 of the next Section). We emphasize that in this last result, the proof is not by compactness perturbing from $s = 1/2$ and it gives a quantifiable value for the required closeness of s to $1/2$. This last result holds not only for cones that are minimizers for the s-perimeter, but for the more general class of *stable* cones. We will describe this result more in details in the next Section, were we address the classification for stable objects.

We focus now on the classification of s-minimal cones in $\mathbb{R}^2$ proven in [42] (i.e. point 1) in Theorem 4.2). The idea of the proof of this result relies in considering perturbations of the minimizer E, that are translations of E inside a ball $B_{R/2}$ and that coincide with E outside the double ball B_R. The authors work with the extended problem (the Caffarelli-Silvestre extension but, in this setting, for functions that take only values ± 1 on the boundary of the positive half-space) and compare the energy of (the extension of) these competitors with the energy of (the extension of) E itself. A computation shows that this difference in energy is controlled from above with R^{n-2s-2}. Hence, when $n = 2$, the difference in energy between E and the competitors can be made arbitrarily small as $R \to \infty$. On the other hand, if E was not a half-plane, they showed that it could be modified in order to decrease its energy by a small but fixed amount and this leads to a contradiction. We emphasize here that this argument works only in dimension $n = 2$ since a crucial fact that is needed is that the estimate R^{n-2s-2} goes to 0 as $R \to \infty$, and this holds true only when $n = 2$. We emphasize that the factor R^{n-2s} comes from an optimal bound for the perimeter of minimizers. Indeed, by a comparison argument one can show that if E is a minimizers for the s-perimeter in B_R, then

$$\mathrm{Per}_s(E, B_R) \leqslant CR^{n-2s},$$

and this bound is optimal.

These ideas were recently used in [21] to prove a quantitative version of this 2-dimensional flatness result. By point 1) in Theorem 4.2, we know that if E is a minimizer for the non-local perimeter in the whole $\mathbb{R}^2$ (that is, it is a minimizer in balls B_R of radius R *for any* $R > 1$), then E is a half-plane.
Suppose now that E is a minimizer for Per_s in a ball B_R for *some* R large enough. Can we deduce that E is "close" to be a half-plane in B_1? Moreover, can we give an estimate on this closeness depending on R? The following result, contained in [21] gives an answer to these questions.

Theorem 4.4 (Theorem 1.3 in [21])**.** *Let $n = 2$. Let $R \geqslant 2$ and E be a minimizer for the s-perimeter in the ball $B_R \subset \mathbb{R}^2$.*

Then, there exists a halfl-plane $\mathfrak{h}$ such that

$$|(E \triangle \mathfrak{h}) \cap B_1| \leqslant CR^{-s}. \tag{4.2}$$

Moreover, after a rotation, we have that $E \cap B_1$ is the subgraph of a measurable function $g : (-1, 1) \to (-1, 1)$ with $\operatorname{osc} g \leqslant CR^{-s}$ outside a "bad" set $\mathcal{B} \subset (-1, 1)$ with measure CR^{-s}.

The above result can be seen as a quantitative version of the flatness result of Savin and Valdinoci. It says that if E is a minimizer for Per_s in B_R, with R large but fixed, then E is close, in the L^1-sense, to be a half-plane in B_1. The second part of the statement gives an even more precise information: outside of a bad set $\mathcal{B}$ of small measure, $E \cap B_1$ coincides with the subgraph of a function g which has small oscillation. Again, the smallness of both the bad set and the oscillation of g is given explicitly in terms of R.

The proof of Theorem 4.4 follows the main ideas contained in the proof of flatness of s-minimal cones in $\mathbb{R}^2$ in [42]. We consider again perturbations of the minimizer E obtained by small translations in some fixed direction and we try to refine the arguments in [42] in order to get some quantitative estimates. Differently from [42], we do not use the Caffarelli-Silvestre extension. This will allow us to obtain a statement analogous to the one of Theorem 4.4 for more general notions of nonlocal perimeter (such as the anisotropic fractional perimeter). Here below, we explain the main steps in the proof of Theorem 4.4.

Sketch of the proof of Theorem 4.4. We start by defining two (small) perturbations of the minimizer E. Let φ_R be a smooth function such that

$$\varphi_R(x) = \begin{cases} 1 & \text{for } |x| < R/2 \\ 0 & \text{for } |x| > R. \end{cases}$$

For $\boldsymbol{v} \in S^{n-1} := \{x \in \mathbb{R}^n \ : \ |x| = 1\}$ and $t \in [0,1]$ we define

$$\Psi_{R,+}(x) := x + t\varphi_R(x)\boldsymbol{v} \quad \text{and} \quad \Psi_{R,-}(x) := x - t\varphi_R(x)\boldsymbol{v}. \tag{4.3}$$

We set $u = \chi_E$ and define the new functions

$$u_R^{\pm}(x) := u\big(\Psi_{R,\pm}^{-1}(x)\big). \tag{4.4}$$

In set notations, we are considering the sets E_R^+ and E_R^- defined as

$$E_R^{\pm} = \{x \ : \ u_R^{\pm}(x) = 1\}. \tag{4.5}$$

We recall the following crucial energy estimate for minimizers, obtained via a comparison argument: if E is a minimizer for the s-perimeter in B_R, then

$$\operatorname{Per}_s(E, B_R) \leqslant CR^{n-2s}. \tag{4.6}$$

We divide the proof in three steps:

- **Step 1**: *Estimating the difference* $\operatorname{Per}_s(E_R^{\pm}, B_R) - \operatorname{Per}_s(E, B_R)$ (see Lemma 2.1 in [21]): using the change of variable formula and after some computations one can prove that

$$\operatorname{Per}_s(E_R^+, B_R) + \operatorname{Per}_s(E_R^-, B_R) - 2\operatorname{Per}_{s,B_R}(E) \leqslant Ct^2 \frac{\operatorname{Per}_s(E, B_R)}{R^2}. \tag{4.7}$$

  Using the estimate (4.6) and the fact that we are in dimension $n = 2$, we get

$$\operatorname{Per}_s(E_R^+, B_R) + \operatorname{Per}_s(E_R^-, B_R) - 2\operatorname{Per}_s(E, B_R) \leqslant Ct^2 R^{-2s}.$$

  Observe that here the fact that the we are working in dimension 2 is crucial in order to get a bound that goes to 0 as $R \to \infty$. As described above, in [42] this fact leads to a contradiction if E was not flat. Here we refine this argument, by keeping the above estimate R^{-2s} in order to get a quantitative estimate (depending on R) on how E differs from being a half-plane.

- **Step 2**: *a purely nonlocal Lemma*:

  Lemma 4.5 (Lemma 2.2 in [21])**.** *Let E, $F \subset \mathbb{R}^2$. Assume that E is a minimizer for Per_s in B_R and that F coincides with E outside B_R, that is, $E \setminus B_R = F \setminus B_R$. Assume moreover that*

$$\operatorname{Per}_s(F, B_R) \leqslant \operatorname{Per}_s(E, B_R) + \delta. \tag{4.8}$$

  Then,

$$2 \int_{F\setminus E} \int_{E\setminus F} \frac{1}{|x - \bar{x}|^{2+2s}} dx d\bar{x} \leqslant \delta.$$

Applying this Lemma to $F = E_R^+$ (and similarly to E_R^-), we deduce that

$$\int_{E_R^+\setminus E}\int_{E\setminus E_R^+} \frac{1}{|x-\bar{x}|^{2+2s}}dxd\bar{x} \leqslant Ct^2R^{-2s}.$$

Therefore, in B_1 we have that for any $\boldsymbol{v}\in S^1$ and any $t\in(0,1)$:

$$\big|\{(E+t\boldsymbol{v})\setminus E\}\cap B_1\big| \cdot \big|\{E\setminus(E+t\boldsymbol{v})\}\cap B_1\big| \leqslant Ct^2R^{-2s}$$

and thus

$$\min\Big\{\big|\{(E+t\boldsymbol{v})\setminus E\}\cap B_1\big|\,,\,\big|\{E\setminus(E+t\boldsymbol{v})\}\cap B_1\big|\Big\} \leqslant CtR^{-s}. \tag{4.9}$$

In this step, the nonlocal character of the s-perimeter is crucial and allows to pass from an estimate in the difference of the s-perimeter between the minimizer E and the competitors $E_R^{\pm}$ to an estimate on the volume of their symmetric difference.

Setting $u := \chi_E$, estimate (4.9) can be written as

$$\min\Big\{\int_{B_1}\big(u(x+t\boldsymbol{v})-u(x)\big)_+\,dx\,,\,\int_{B_1}\big(u(x+t\boldsymbol{v})-u(x)\big)_-\,dx\Big\} \leqslant CtR^{-s}. \tag{4.10}$$

- **Step 3**: *Some geometric lemmas and conclusion.* Dividing (4.10) by t and taking the limit as $t\to 0$, we deduce that for any $\boldsymbol{v}\in S^1$, the following holds:

$$\min\{(\nabla u\cdot\boldsymbol{v})_+(B_1),(\nabla u\cdot\boldsymbol{v})_-(B_1)\} \leqslant CR^{-s}, \tag{4.11}$$

where $\nabla u\cdot\boldsymbol{v}$ denotes the distributional derivative in the direction $\boldsymbol{v}$ of u. This last part of the proof is more technical and needs several geometric lemmas (for the details, we refer to Lemma 2.5 and to all lemmas and propositions of Section 4 in [21]). The main underlying idea is the following: If we set $\Phi_{\pm}(\boldsymbol{v}) := (\nabla u\cdot\boldsymbol{v})_{\pm}(B_1)$, by (4.11) we have that

$$\min\{\Phi_+(\boldsymbol{v}),\Phi_-(\boldsymbol{v})\} \leqslant CR^{-s}, \quad \text{for any } \boldsymbol{v}\in S^1.$$

Moreover, since

$$\Phi_+(\boldsymbol{v}) = \Phi_-(-\boldsymbol{v}),$$

by a continuity argument we have that there exists $\boldsymbol{v}^*\in S^1$ such that

$$\max\{(\nabla u\cdot\boldsymbol{v}^*)_+(B_1),(\nabla u\cdot\boldsymbol{v}^*)_-(B_1)\} \leqslant CR^{-s}.$$

Hence, except for a bad set $\mathfrak{B}$ of measure less than CR^{-s} the function $u=\chi_E$ restricted to all lines parallel to $\boldsymbol{v}^*$ will be at the same time monotone nondecreasing and non-increasing; i.e., constant. Since we also have that u is also monotone along most (for large R) lines perpendicular to $\boldsymbol{v}^*$, the only possibility is that the set $E=\{u=1\}$ is equal to a half plane up to the bad set $\mathfrak{B}$ with $|\mathfrak{B}|\leqslant CR^{-s}$. The rigorous proof for this fact is contained in Section 4 of [21].

□

We emphasize that in [42], the authors first prove a flatness result for minimizing cones, and then they deduce, by a blow-down argument, flatness for any s-minimal set in $\mathbb{R}^2$. In this blow-down procedure the monotonicity formula is needed and unfortunately such a formula is available only for the energy functional of the extended problem (see [16]). Instead, in the proof of Theorem 4.4, we consider E to be *any* set which minimizes the

s-perimeter, not necessarily a cone and, as a consequence of the quantitative estimate (4.2) after letting $R \to \infty$, we deduce that if E is a minimizer in the whole $\mathbb{R}^2$, then E is an half-plane. Hence, we give an alternative proof of the classification result in [42], without using the Caffarelli-Silvestre extension and without needing a monotonicity formula. For this reason, we can generalize our Theorem 4.4 and hence the classification of nonlocal minimal surfaces in $\mathbb{R}^2$ to more general notions of nonlocal perimeter, such as the anisotropic fractional perimeter (see [21]).

The techniques developed in [21] and, more precisely, the estimate (4.11) implies also some estimates for the s-perimeter and the classical perimeter of an s-minimal set E. More interestingly, these estimates holds true in the more general class of *stable sets*. We are going to state and comment on these results on stable sets in the next section.

5. What about stable objects?

In this section we present some very recent results in the study of *stable* solutions to the fractional Allen-Cahn equation and of *stable* nonlocal minimal surfaces. In both cases the notion of stability that we use is the variational one, that is the nonnegativity of the second variation of the associated energy functional. Surprisingly, some results recently established for stable objects in the nonlocal setting, are still unknown in the local setting. The nonlocality of the energy functional (for the Allen-Cahn equation or for the nonlocal perimeter) helps in giving sharp estimates that are crucial for classifying stable solutions. In order to explain which are the main difficulties in this setting and to compare the local and nonlocal framework, we start by recalling what is known for classical stable minimal surfaces.

5.1. **The classical setting.** Stable minimal cones (for the classical perimeter) are completely classified: they are hyperplanes in space dimensions $n \leqslant 7$. In $\mathbb{R}^8$, the Simons cone is an example of stable cone which is singular. The classification that we have presented in the previous section for classical minimal surfaces holds true for stable cones. In order to pass from the classification of stable cones to the classification of *any* stable surface in the whole $\mathbb{R}^n$, one would like to perform a blow-down procedure using the monotonicity formula. A crucial tool needed for using a blow-down argument would be an optimal estimate for the perimeter of stable sets. It is well known that any minimizer of the classical perimeter in a ball B_R satisfies the estimate

$$\mathrm{Per}(E, B_R) \leqslant C R^{n-1}. \tag{5.1}$$

Unfortunately, an estimate like (5.1) is not known to hold for stable sets, unless we are in dimension $n = 3$ and we require some topological assumption on the set E (see Theorem 5.1 below). While for proving an energy estimate for minimizers it is enough to construct a suitable competitor, which has to agree with E outside B_R but can be modified arbitrarily in B_R, and that satisfies the needed estimate, for proving such an estimate for stable sets we are allowed to consider only competitors which are *small perturbations* of the given set E.

We recall here below the perimeter estimate for classical stable sets, which was proven by Pogorelov [39], and Colding and Minicozzi [22] —see also [35, Theorem 2] and [47, Lemma 34].

Theorem 5.1 ([39, 22]). *Let D be a simply connected, immersed, stable minimal disk of geodesic radius r_0 on a minimal (two-dimensional) surface $\Sigma \subset \mathbb{R}^3$, then*

$$\pi r_0^2 \leqslant \mathrm{Area}\,(D) \leqslant \frac{4}{3}\pi r_0^2.$$

In dimension $n > 3$ the perimeter estimate for stable sets is still completely open. As explained above, having a universal bound for the classical perimeter of embedded minimal surfaces in every dimension $n > 3$ would be a decisive step towards proving the following well-known and long standing conjecture: *The only stable embedded minimal (hyper)surfaces in $\mathbb{R}^n$ are hyperplanes as long as the dimension of the ambient space is less than or equal to 7.* On the other hand, without a universal perimeter bound, the sequence of blow-downs could have perimeters converging to ∞.

In a similar way, one can ask whether the De Giorgi conjecture on one-dimensional symmetry for solutions to the Allen-Cahn equation, holds in the more general class of *stable solutions*. This is known only in dimension $n = 2$ and it is still open in higher dimensions. We have already seen in Section 3 that stability plays a crucial role in the proof of the conjecture, but again another fundamental ingredient was given by the energy estimate (3.10). Also in this case the optimal estimate is known to hold only for minimizers (and for monotone solutions in dimension 3) and it is completely open for stable sets. Nevertheless, when $n = 2$ one can prove the conjecture for stable solutions because, in order to apply the Liouville-type argument described in Section 3, an estimate of the form

$$\mathcal{E}(u, B_R) \leqslant CR^2,$$

is enough. In $\mathbb{R}^2$ this (not sharp) estimate holds true since the measure of B_R is of order R^2 (and $|\nabla u|$ is bounded by standard elliptic estimates).

One important open question in the classification of solutions to the classical Allen-Cahn equation is, then, the following:

Open Question: Is it true that any bounded *stable* solution of $-\Delta u = u - u^2$ in $\mathbb{R}^n$ is one-dimensional for $3 \leqslant n \leqslant 7$?

One would expect a positive answer to this question for all dimensions $3 \leqslant n \leqslant 7$, in the same way one expects a positive answer to the conjecture for stable minimal surfaces stated above. Starting from dimension $n = 8$, instead there are examples of stable solutions to the Allen-Cahn equations which are not one-dimensional. This was established by Pacard and Wei in [38].

5.2. **The nonlocal setting.** Surprisingly, when dealing with stable sets for the nonlocal perimeter (or the nonlocal Allen-Cahn equation) some of the above open problems received a positive answer, at least in some particular cases.

Since the notion of stability that we consider is the one of nonnegativity for the second variation of the s-perimeter functional, we recall here the expression for $\partial^2 \mathrm{Per}_s$, given in [30, 24].

Let $E \subset \mathbb{R}^n$ be such that ∂E is C^2 away from 0. We denote by H^{n-1} the $(n-1)$-dimensional Hasudorff measure. Then, the second variation of the s-perimeter is given by

$$\int_{\partial E} c^2_{s,\partial E}(x)|\zeta(x)|^2\, dH^{n-1}(x) - \iint_{\partial E \times \partial E} \frac{\big|\zeta(x) - \zeta(\bar{x})\big|^2}{|x - \bar{x}|^{n+2s}}\, dH^{n-1}(x)\, dH^{n-1}(\bar{x}),$$

where

$$c_{s,\partial E}^2(x) := \int_{\partial E} \frac{\left|\nu_E(x) - \nu_E(\bar{x})\right|^2}{|x-\bar{x}|^{n+2s}}\, dH^{n-1}(\bar{x}),$$

$\nu_E(x)$ denotes the outward normal vector to ∂E at $x \in \partial E$ and $\zeta \in C_0^2(\mathbb{R}^n \setminus \{0\})$.

In Section 3 of [10], we deal with different possible notions of stability, that, in the case of smooth sets E, are equivalent to require that the expression above is nonnegative for any $\zeta \in C_c^2(\mathbb{R}^n \setminus \{0\})$.

As anticipated in the previous section, the techniques developed in [21] allow to prove some perimeter and energy estimates for nonlocal stable sets.

Theorem 5.2 (Theorem 1.1 in [21])**.** *Let $s \in (0,1)$, $R > 0$ and E be a stable set in the ball B_{2R} for the nonlocal s-perimeter functional. Then,*

$$\mathrm{Per}(E, B_R) \leqslant CR^{n-1},$$

and

$$\mathrm{Per}_{2s}(E, B_R) \leqslant CR^{n-2s}.$$

As a consequence of Theorem 5.2, in [21] we obtained that any nonlocal stable set in the whole $\mathbb{R}^2$ is a half-plane (by using the same argument that we sketch in the proof of Theorem 4.4 in the previous Section).

Analogue estimates for classical stable surfaces are not known when $n > 2$, and even comparing our result with the 2-dimensional result of Theorem 5.1 above, we stress that here we do not need ∂E to be simply connected. In fact, an estimate exactly like ours can not hold for classical stable minimal surfaces since a large number of parallel planes is always a classical stable minimal surface with arbitrarily large perimeter in B_1.

Moreover, we believe that our result in Theorem 5.2 can be used to reduce the classification of stable s-minimal surfaces in the whole $\mathbb{R}^n$ to the classification of stable cones, by means of a blow-down argument and using a monotonicity formula. Somehow the difficulties in the local/nonlocal setting are interchanged: in the local setting we have the complete classifications of stable cones but it is not known yet how to pass from the classification of cones to the classification of any stable surfaces (due to the lack of perimeter estimates). On the other hand, in the nonlocal setting, we have the energy estimates for stable sets, but the classification of stable s-minimal cones is still widely open.

Concerning the classification of stable s-minimal cones, in [10], Cabré, Serra and the author, proved the following Theorem, which is the first result in the 3-dimensional case.

Theorem 5.3 (Theorem 1.2 in [10])**.** *There exists $s_* \in (0,1)$ such that for every $s \in (s_*, 1)$ the following statement holds.*

Let $\Sigma \subset \mathbb{R}^3$ be a cone with nonempty boundary of class C^2 away from 0. Assume that Σ is a stable set for the s-perimeter. Then, Σ is a half-space.

In the proof of Theorem 5.3, the estimate of Theorem 5.2 plays a crucial role, together with several other ingredients, such as the fractional Hardy inequality and some geometric lemmas. We stress that our result is not a perturbative result from $s = 1/2$ which can be obtained by some sort of compactness argument. In fact, a careful inspection of our proof gives an explicit (computable) value for s_*, something impossible when using compactness arguments.

We conclude with some considerations and open problems on the classification of stable solutions for the fractional Allen-Cahn equation. As in the classical setting, when the dimension $n = 2$, one can prove that any bounded stable solution to (3.1) is one-dimensional for any $0 < s < 1$, using the same approach described in Section 3. Indeed, in this case a not optimal energy estimate is enough to apply the Liouville-type argument. What about $n > 3$? In the very recent contribution [31], Figalli and Serra proved that when $n = 3$ and $s = 1/2$, any stable bounded solutions to (3.1) is one-dimensional. Again, surprisingly, in the nonlocal case (even if only for the half-Laplacian) something that is not known for the local case, has been established. To conclude, we announce that the forthcoming paper [11] will contain a careful study of stable solutions to the fractional Allen-Cahn equation in the case $0 < s < 1/2$, including energy estimates, density estimates, convergence of blow-down and some new classification results.

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