COMPACTNESS AND LOWER SEMICONTINUITY IN *GSBD*

ANTONIN CHAMBOLLE AND VITO CRISMALE

Abstract. In this paper, we prove a compactness and semicontinuity result in $GSBD$ for sequences with bounded Griffith energy. This generalises classical results in $(G)SBV$ by Ambrosio [1, 2, 3] and SBD by Bellettini-Coscia-Dal Maso [9]. As a result, the static problem in Francfort-Marigo's variational approach to crack growth [27] admits (weak) solutions. Moreover, we obtain a compactness property for minimisers of suitable Ambrosio-Tortorelli's type energies [6], which have been shown to Γ-converge to Griffith energy in [16].

1. Introduction

The variational approach to fracture was introduced by Francfort and Marigo in [27] in order to build crack evolutions in brittle materials, following Griffith's laws [32], without *a priori* knowledge of the crack path (or surface in higher dimension). It relies on successive minimisations of the *Griffith energy*:

$$u \mapsto \int_{\Omega\setminus K} \mathbb{C}e(u) : e(u)dx + \gamma\,\mathcal{H}^{n-1}(K)$$

where $\Omega \subset \mathbb{R}^n$ is a bounded open set, the *reference configuration*, $u : \Omega \to \mathbb{R}^n$ is an (infinitesimal) displacement, $e(u)$ its symmetrised gradient (the *infinitesimal elastic strain*) and $\mathbb{C}$ the *Cauchy stress tensor* defining the *Hooke's law* (in particular, $\mathbb{C}a : a$ defines a positive definite quadratic form of the $n \times n$ symmetric tensor a). The symmetrised gradient $e(u)$ is defined out of a crack set K, which is in the theory a compact $(n-1)$-dimensional set and is penalised by its surface (multiplied by a coefficient γ called the *toughness*).

The minimisation of the energy is under the constraint that K should contain a previously computed crack, and that u should satisfy a Dirichlet condition $u = u_0$ on a subset $\partial_D\Omega \setminus K$ of $\partial\Omega$, where $\partial_D\Omega$ is a regular part of the boundary and u_0 a sufficiently regular displacement. Hence an important question in the theory is whether the problem

$$\min_{u=u_0 \text{ on } \partial_D\Omega\setminus K} \int_{\Omega\setminus K} \mathbb{C}e(u) : e(u)dx + \gamma\,\mathcal{H}^{n-1}(K) \tag{1.1}$$

has a solution.

This problem however is not easy to analyse, since the energy controls very little of the function u: for instance if K almost cuts out a connected component of Ω, the function u may have any (arbitrarily large) value in this component at small cost.

For this reason, most of the "sound" approaches to problem (1.1) consider additional assumptions. In particular, a global L^∞ bound on the displacements ensures one may work in the class SBD of *Special functions with Bounded Deformation* [4], provided one considers a *weak formulation* of the problem where K is replaced with the intrinsic jump set J_u of u (which needs not to be closed anymore): in this space minimising sequences are shown to be compact [9], and the energy to be lower semicontinuous. Another possible assumption is, in $2d$, that the crack set K is connected [23, 12].

The natural space for studying (1.1), in fact, is not $SBD(\Omega)$ (which assumes that the symmetrised gradient of u is a measure and hence u is in $L^{n/(n-1)}_{\rm loc}(\Omega;\mathbb{R}^n)$) but the space $GSBD(\Omega)$, introduced by G. Dal Maso in [21]. This space, defined by the slicing properties of the functions, is designed in order to contain "all" displacements u for which the energy is finite. No compactness result was available in $GSBD$ for minimizing sequences until very recently.

The first existence result for (1.1) without further constraint has been proven indeed in [31], in *dimension two*. It relies on a delicate construction showing a *piecewise Korn inequality*, in [28] (for approximated Korn and Korn-Poincaré inequalities see also e.g. [17, 14, 30]). In the *antiplane case*, namely when the displacement u is assumed vertical and depending only on the horizontal components (this provides a control on the absolutely continuous part of the *full* gradient of u) the existence of minimisers has been proven in [1, 2, 3] in combination with [25], passing through the corresponding weak formulation, and in [22, 34], taking the discontinuity set as main variable (these results consider indeed the minimisation of the *Mumford-Shah functional* [35], closely related to antiplane Griffith energy and which inspired this variational theory).

We remark that [31] also proves existence of quasistatic evolutions in dimension two, extending in that case the result in [26], obtained in the *antiplane case* (see [8] for the existence of *strong* quasistatic evolutions in dimension two). Moreover, we mention the works [33, 29, 19, 18, 16, 15] that employ or give further insight on the space $GSBD$.

In this paper, we prove the following general compactness result for sequences bounded in energy, in the space $GSBD(\Omega)$, in any dimension.

Theorem 1.1. *Let $\phi\colon \mathbb{R}^+ \to \mathbb{R}^+$ be a non-decreasing function with*

$$\lim_{t\to+\infty} \frac{\phi(t)}{t} = +\infty\,, \tag{1.2}$$

and let u_h be a sequence in $GSBD(\Omega)$ such that

$$\int_\Omega \phi\big(|e(u_h)|\big)\,\mathrm{d}x + \mathcal{H}^{n-1}(J_{u_h}) < M\,, \tag{1.3}$$

for some constant M independent of h. Then there exists a subsequence, still denoted by u_h, such that $A := \{x\in\Omega\colon |u_h(x)|\to+\infty\}$ has finite perimeter, and $u\in GSBD(\Omega)$ with $u=0$ on A for which

$$u_h \to u \qquad \text{in } L^0(\Omega\setminus A;\mathbb{R}^n)\,, \tag{1.4a}$$

$$e(u_h) \rightharpoonup e(u) \quad \text{in } L^1(\Omega\setminus A;\mathbb{M}^{n\times n}_{sym})\,, \tag{1.4b}$$

$$\mathcal{H}^{n-1}(J_u\cup\partial^*A) \le \liminf_{h\to\infty} \mathcal{H}^{n-1}(J_{u_h})\,. \tag{1.4c}$$

The proof of this theorem is in our opinion simpler than [31], even if a fundamental tool is a quite technical Korn-Poincaré inequality for functions with small jump set, proved in [14]. We combine this inequality with arguments in the spirit of Rellich's type compactness theorems.

Theorem 1.1 gives the existence of minimisers for Griffith energy with Dirichlet boundary conditions in the *weak formulation* (see Theorem 4.1), which by results in [19, 15] satisfy the properties of strong solutions in the interior of Ω. We believe it is possible to prove existence of solutions for the strong formulation of (1.1) by extending the regularity theorems in [19, 15] up to the boundary (which has to be sufficiently regular), this is the subject for future study.

We deduce also a suitable compactness property (Theorem 5.2) for sequences of minimisers for some Ambrosio-Tortorelli's type energies [6], which have been shown to Γ-converge to Griffith energy (see [16] for a proof in $GSBD$, *cf.* also Theorem 5.1 below). This provides a theoretical basis to the numerical simulations in [10] and many subsequent works.

Our paper is organised as follows: we first fix the notation and recall basic properties of the functional spaces employed (Section 2), then we prove, in Section 3, Theorem 1.1. Section 4 is devoted to the existence of minimisers, while in Section 5 we consider the problem of compactness for minimisers of the approximating energies.

2. Notation and preliminaries

For every $x\in\mathbb{R}^n$ and $\varrho>0$ let $B_\varrho(x)$ be the open ball with center x and radius ϱ. For x, $y\in\mathbb{R}^n$, we use the notation $x\cdot y$ for the scalar product and $|x|$ for the norm. We denote by $\mathcal{L}^n$ and $\mathcal{H}^k$ the n-dimensional Lebesgue measure and the k-dimensional Hausdorff measure. For any locally compact subset B of $\mathbb{R}^n$, the space of bounded $\mathbb{R}^m$-valued Radon measures on B is denoted by $\mathcal{M}_b(B;\mathbb{R}^m)$. For $m=1$ we write $\mathcal{M}_b(B)$ for $\mathcal{M}_b(B;\mathbb{R})$ and $\mathcal{M}_b^+(B)$ for the subspace of positive measures of $\mathcal{M}_b(B)$. For every $\mu\in\mathcal{M}_b(B;\mathbb{R}^m)$, its total variation is denoted by $|\mu|(B)$. We write χ_E for the indicator function of any $E\subset\mathbb{R}^n$, which is 1 on E and 0 otherwise. We use also the symbol $L^0(B;\mathbb{R}^m)$ for the space of measurable functions from B to $\mathbb{R}^m$ with the topology of the convergence in measure, while $L^p(B;\mathbb{R}^m)$, with $p\geq 1$ is as usual the space of p-integrable functions with respect to $\mathcal{L}^n$.

Definition 2.1. Let $A\subset\mathbb{R}^n$, $v\colon A\to\mathbb{R}^m$ an $\mathcal{L}^n$-measurable function, $x\in\mathbb{R}^n$ such that

$$\limsup_{\varrho\to 0^+}\frac{\mathcal{L}^n(A\cap B_\varrho(x))}{\varrho^n}>0\,.$$

A vector $a\in\mathbb{R}^n$ is the *approximate limit* of v as y tends to x if for every $\varepsilon>0$

$$\lim_{\varrho\to 0^+}\frac{\mathcal{L}^n(A\cap B_\varrho(x)\cap\{|v-a|>\varepsilon\})}{\varrho^n}=0\,,$$

and then we write

$$\operatorname*{ap\,lim}_{y\to x} v(y)=a\,. \tag{2.1}$$

Remark 2.2. Let A, v, x, and a be as in Definition 2.1 and let ψ be a homeomorphism between $\mathbb{R}^m$ and a bounded open subset of $\mathbb{R}^m$. Then (2.1) holds if and only if

$$\lim_{\varrho\to 0^+}\frac{1}{\varrho^n}\int\limits_{A\cap B_\varrho(x)}|\psi(v(y))-\psi(a)|\,\mathrm{d}y=0\,.$$

Definition 2.3. Let $U\subset\mathbb{R}^n$ open, and $v\colon U\to\mathbb{R}^m$ be $\mathcal{L}^n$-measurable. The *approximate jump set* J_v is the set of points $x\in U$ for which there exist a, $b\in\mathbb{R}^m$, with $a\neq b$, and $\nu\in\mathbb{S}^{n-1}$ such that

$$\operatorname*{ap\,lim}_{(y-x)\cdot\nu>0,\,y\to x} v(y)=a \quad\text{and}\quad \operatorname*{ap\,lim}_{(y-x)\cdot\nu<0,\,y\to x} v(y)=b\,.$$

The triplet (a,b,ν) is uniquely determined up to a permutation of (a,b) and a change of sign of ν, and is denoted by $(v^+(x),v^-(x),\nu_v(x))$. The jump of v is the function defined by $[v](x):=v^+(x)-v^-(x)$ for every $x\in J_v$. Moreover, we define

$$J_v^1:=\{x\in J_v\colon |[v](x)|\geq 1\}\,. \tag{2.2}$$

Remark 2.4. By Remark 2.2, J_v and J_v^1 are Borel sets and $[v]$ is a Borel function. By Lebesgue's differentiation theorem, it follows that $\mathcal{L}^n(J_v)=0$.

***BV* and *BD* functions.** If $U\subset\mathbb{R}^n$ open, a function $v\in L^1(U)$ is a *function of bounded variation* on U, and we write $v\in BV(U)$, if $\mathrm{D}_i v\in\mathcal{M}_b(U)$ for $i=1,\dots,n$, where $\mathrm{D}v=(\mathrm{D}_1 v,\dots,\mathrm{D}_n v)$ is its distributional gradient. A vector-valued function $v\colon U\to\mathbb{R}^m$ is in $BV(U;\mathbb{R}^m)$ if $v_j\in BV(U)$ for every $j=1,\dots,m$. The space $BV_{\mathrm{loc}}(U)$ is the space of $v\in L^1_{\mathrm{loc}}(U)$ such that $\mathrm{D}_i v\in\mathcal{M}_b(U)$ for $i=1,\dots,n$.

A $\mathcal{L}^n$-measurable bounded set $E\subset\mathbb{R}^n$ is a set of *finite perimeter* if χ_E is a function of bounded variation. The *reduced boundary* of E, denoted by ∂^*E, is the set of points $x\in\operatorname{supp}|\mathrm{D}\chi_E|$ such that the limit $\nu_E(x):=\lim_{\varrho\to0^+}\frac{\mathrm{D}\chi_E(B_\varrho(x))}{|\mathrm{D}\chi_E|(B_\varrho(x))}$ exists and satisfies $|\nu_E(x)|=1$. The reduced boundary is countably $(\mathcal{H}^{n-1},n-1)$ rectifiable, and the function ν_E is called *generalised inner normal* to E.

A function $v\in L^1(U;\mathbb{R}^n)$ belongs to the space of *functions of bounded deformation* if its distributional symmetric gradient $\mathrm{E}v$ belongs to $\mathcal{M}_b(U;\mathbb{R}^n)$. It is well known (see [4, 36]) that for $v\in BD(U)$, J_v is countably $(\mathcal{H}^{n-1},n-1)$ rectifiable, and that

$$\mathrm{E}v=\mathrm{E}^a v+\mathrm{E}^c v+\mathrm{E}^j v\,, \tag{2.3}$$

where $\mathrm{E}^a v$ is absolutely continuous with respect to $\mathcal{L}^n$, $\mathrm{E}^c v$ is singular with respect to $\mathcal{L}^n$ and such that $|\mathrm{E}^c v|(B)=0$ if $\mathcal{H}^{n-1}(B)<\infty$, while $\mathrm{E}^j v$ is concentrated on J_v. The density of $\mathrm{E}^a v$

with respect to $\mathcal{L}^n$ is denoted by $e(v)$, and we have that (see [4, Theorem 4.3] and recall (2.1)) for $\mathcal{L}^n$-a.e. $x \in U$

$$\operatorname*{ap\,lim}_{y\to x} \frac{\big(v(y)-v(x)-e(v)(x)(y-x)\big)\cdot(y-x)}{|y-x|^2} = 0\,. \tag{2.4}$$

The space $SBD(U)$ is the subspace of all functions $v \in BD(U)$ such that $\mathrm{E}^c v = 0$, while for $p \in (1,\infty)$

$$SBD^p(U) := \{v \in SBD(U)\colon e(v) \in L^p(\Omega;\mathbb{M}^{n\times n}_{sym}),\ \mathcal{H}^{n-1}(J_v) < \infty\}\,.$$

Analogous properties hold for BV, as the countable rectifiability of the jump set and the decomposition of $\mathrm{D}v$, and the spaces $SBV(U;\mathbb{R}^m)$ and $SBV^p(U;\mathbb{R}^m)$ are defined similarly, with ∇v, the density of $\mathrm{D}^a v$, in place of $e(v)$. For a complete treatment of BV, SBV functions and BD, SBD functions, we refer to [5] and to [4, 9, 7, 36], respectively.

GBD **functions.** We now recall the definition and the main properties of the space GBD of *generalised functions of bounded deformation*, introduced in [21], referring to that paper for a general treatment and more details. Since the definition of GBD is given by slicing (differently from the definition of GBV, *cf.* [24, 2]), we introduce before some notation.

Fixed $\xi \in \mathbb{S}^{n-1} := \{\xi \in \mathbb{R}^n\colon |\xi| = 1\}$, for any $y \in \mathbb{R}^n$ and $B \subset \mathbb{R}^n$ let

$$\Pi^\xi := \{y \in \mathbb{R}^n\colon y\cdot\xi = 0\}, \qquad B^\xi_y := \{t \in \mathbb{R}\colon y + t\xi \in B\}\,,$$

and for every function $v\colon B \to \mathbb{R}^n$ and $t \in B^\xi_y$ let

$$v^\xi_y(t) := v(y+t\xi), \qquad \widehat{v}^\xi_y(t) := v^\xi_y(t)\cdot\xi\,.$$

Definition 2.5 ([21])**.** Let $\Omega \subset \mathbb{R}^n$ be bounded and open, and $v\colon \Omega \to \mathbb{R}^n$ be $\mathcal{L}^n$-measurable. Then $v \in GBD(\Omega)$ if there exists $\lambda_v \in \mathcal{M}^+_b(\Omega)$ such that the following equivalent conditions hold for every $\xi \in \mathbb{S}^{n-1}$:

(a) for every $\tau \in C^1(\mathbb{R})$ with $-\frac{1}{2} \le \tau \le \frac{1}{2}$ and $0 \le \tau' \le 1$, the partial derivative $\mathrm{D}_\xi\big(\tau(v\cdot\xi)\big) = \mathrm{D}\big(\tau(v\cdot\xi)\big)\cdot\xi$ belongs to $\mathcal{M}_b(\Omega)$, and for every Borel set $B \subset \Omega$

$$\big|\mathrm{D}_\xi\big(\tau(v\cdot\xi)\big)\big|(B) \le \lambda_v(B);$$

(b) $\widehat{v}^\xi_y \in BV_{\mathrm{loc}}(\Omega^\xi_y)$ for $\mathcal{H}^{n-1}$-a.e. $y \in \Pi^\xi$, and for every Borel set $B \subset \Omega$

$$\int_{\Pi_\xi} \Big(\big|\mathrm{D}\widehat{v}^\xi_y\big|\big(B^\xi_y \setminus J^1_{\widehat{v}^\xi_y}\big) + \mathcal{H}^0\big(B^\xi_y \cap J^1_{\widehat{v}^\xi_y}\big)\Big)\,\mathrm{d}\mathcal{H}^{n-1}(y) \le \lambda_v(B)\,, \tag{2.5}$$

where $J^1_{\widehat{u}^\xi_y} := \Big\{t \in J_{\widehat{u}^\xi_y} : |[\widehat{u}^\xi_y]|(t) \ge 1\Big\}$.

The function v belongs to $GSBD(\Omega)$ if $v \in GBD(\Omega)$ and $\widehat{v}^\xi_y \in SBV_{\mathrm{loc}}(\Omega^\xi_y)$ for every $\xi \in \mathbb{S}^{n-1}$ and for $\mathcal{H}^{n-1}$-a.e. $y \in \Pi^\xi$.

$GBD(\Omega)$ and $GSBD(\Omega)$ are vector spaces, as stated in [21, Remark 4.6], and one has the inclusions $BD(\Omega) \subset GBD(\Omega)$, $SBD(\Omega) \subset GSBD(\Omega)$, which are in general strict (see [21, Remark 4.5 and Example 12.3]). For every $v \in GBD(\Omega)$ the *approximate jump set* J_v is still countably $(\mathcal{H}^{n-1}, n-1)$-rectifiable (*cf.* [21, Theorem 6.2]) and can be reconstructed from the jump of the slices $\widehat{v}^\xi_y$ ([21, Theorem 8.1]). Indeed, for every C^1 manifold $M \subset \Omega$ with unit normal ν, it holds that for $\mathcal{H}^{n-1}$-a.e. $x \in M$ there exist the *traces* $v^+_M(x)$, $v^-_M(x) \in \mathbb{R}^n$ such that

$$\operatorname*{ap\,lim}_{\pm(y-x)\cdot\nu(x)>0,\, y\to x} v(y) = v^\pm_M(x) \tag{2.6}$$

and they can be reconstructed from the traces of the one-dimensional slices (see [21, Theorem 5.2]). Every $v \in GBD(\Omega)$ has an *approximate symmetric gradient* $e(v) \in L^1(\Omega;\mathbb{M}^{n\times n}_{sym})$, characterised by (2.4) and such that for every $\xi \in \mathbb{S}^{n-1}$ and $\mathcal{H}^{n-1}$-a.e. $y \in \Pi^\xi$

$$e(v)^\xi_y\xi\cdot\xi = \nabla\widehat{v}^\xi_y \quad \mathcal{L}^1\text{-a.e. on } \Omega^\xi_y\,. \tag{2.7}$$

By these properties of slices it follows that, if $v \in GSBD(\Omega)$ with $e(v) \in L^1(\Omega;\mathbb{M}^{n\times n}_{sym})$ and $\mathcal{H}^{n-1}(J_v) < +\infty$, then for every Borel set $B \subset \Omega$

$$\mathcal{H}^{n-1}(J_v \cap B) = (2\omega_{n-1})^{-1} \int\limits_{\mathbb{S}^{n-1}} \bigg(\int\limits_{\Pi_\xi} \mathcal{H}^0(J_{v^\xi_y} \cap B^\xi_y) \,\mathrm{d}\mathcal{H}^{n-1}(y) \bigg) \,\mathrm{d}\mathcal{H}^{n-1}(\xi) \tag{2.8}$$

and the two conditions in the definition of $GSBD$ for v hold for $\lambda_v \in \mathcal{M}^+_b(\Omega)$ such that

$$\lambda_v(B) \le \int_B |e(v)| \,\mathrm{d}x + \mathcal{H}^{n-1}(J_v \cap B)\,, \tag{2.9}$$

for every Borel set $B \subset \Omega$ (*cf.* also [29, Theorem 1] and [33, Remark 2]).

We now recall the following result, proven in [14, Proposition 2]. Notice that the proposition is therein stated in SBD, but the proof, which is based on the Fondamental Theorem of Calculus along lines, still holds for $GSBD$, with small adaptations.

Proposition 2.6 ([14])**.** *Let $Q_r = (-r,r)^n$, $v \in GSBD(Q)$, $p \in [1,\infty)$. Then there exist a Borel set $\omega \subset Q_r$ and an affine function $a\colon \mathbb{R}^n \to \mathbb{R}^n$ with $e(a) = 0$ such that*

$$\mathcal{L}^n(\omega) \le cr\mathcal{H}^{n-1}(J_v)$$

and

$$\int\limits_{Q_r\setminus\omega} |v - a|^p \,\mathrm{d}x \le cr^p \int\limits_{Q_r} |e(v)|^p \,\mathrm{d}x\,. \tag{2.10}$$

The constant c depends only on p and n.

3. The main compactness and lower semicontinuity result

In this section we prove Theorem 1.1, the main result of the paper.

Proof of Theorem 1.1. For every $k \in \mathbb{N}$ and $z \in (2k^{-1})\mathbb{Z}^n$ we consider the cubes of center z

$$q_{k,z} := z + (-k^{-1}, k^{-1})^n.$$

Then $\Omega_k := \Omega \setminus \bigcup_{q_{k,z}\not\subset\Omega} \overline{q_{k,z}}$ is essentially the union of the cubes which are contained in Ω.

We apply Proposition 2.6 with $p = 1$ in any $q_{k,z} \subset \Omega$, so for $r = k^{-1}$. Then there exist sets $\omega^h_{k,z} \subset q_{k,z}$ with

$$\mathcal{L}^n(\omega^h_{k,z}) \le ck^{-1}\mathcal{H}^{n-1}(J_{u_h} \cap q_{k,z}) \tag{3.1}$$

and affine functions $a^h_{k,z}\colon \mathbb{R}^n \to \mathbb{R}^n$, with $e(a^h_{k,z}) = 0$, such that

$$\int\limits_{q_{k,z}\setminus\omega^h_{k,z}} |u_h - a^h_{k,z}| \,\mathrm{d}x \le c\,k^{-1} \int\limits_{q_{k,z}} |e(u_h)| \,\mathrm{d}x\,. \tag{3.2}$$

The functions $(a^h_{k,z})_{h\ge1}$ belong to the finite dimensional space of affine functions. Consider a component $(a^h_{k,z}\cdot e_i)_h$ $(i = 1,\dots,n)$, of the sequence: one can either extract a subsequence such that it converges to an affine function, otherwise, the sequence is unbounded and either it converges globally, up to a subsequence, to $+\infty$ or $-\infty$, or one can find a hyperplane $\{x\cdot\nu = t\}$ ($\nu \in \mathbb{R}^n$, $t \in \mathbb{R}$) and a subsequence such that $a^h_{k,z}(x)\cdot e_i \to +\infty$ if $x\cdot\nu > t$ and $a^h_{k,z}(x)\cdot e_i \to -\infty$ if $x\cdot\nu < t$. If all components of $a^h_{k,z}$ are bounded the limit, clearly, is also an infinitesimal rigid motion (that is, an affine function with skew-symmetric gradient).

Let τ denote the function tanh (or any smooth, 1-Lipschitz increasing function from -1 to 1). As a consequence, we obtain that up to a subsequence, the function

$$a^h_k(x) := \sum_{q_{z,k}\subset\Omega} a^h_{k,z}(x)\chi_{q_{k,z}}(x)$$

is such that $\tau(a^h_k\cdot e_i)$ converges to some function in $L^1(\Omega_k)$, for any $i = 1,\dots,n$.

Clearly the subsequence could be extracted from a previous subsequence built at the stage $k-1$, hence by a diagonal argument, we may assume that for any k, $(\tau(a^h_k\cdot e_i))_h$ converges for all $i = 1,\dots n$, in $L^1(\Omega_k)$.

We have that for each $i = 1, \dots, n$, $k \geq 1$, and $l, m \geq 1$,

$$\int_\Omega |\tau(u_m \cdot e_i) - \tau(u_l \cdot e_i)|\,\mathrm{d}x \leq 2|\Omega \setminus \Omega_k| + \int_{\Omega_k} |\tau(u_m \cdot e_i) - \tau(a_k^m \cdot e_i)|\,\mathrm{d}x$$
$$+ \int_{\Omega_k} |\tau(a_k^m \cdot e_i) - \tau(a_k^l \cdot e_i)|\,\mathrm{d}x + \int_{\Omega_k} |\tau(u_l \cdot e_i) - \tau(a_k^l \cdot e_i)|\,\mathrm{d}x. \quad (3.3)$$

By construction,

$$\lim_{l,m\to+\infty} \int_{\Omega_k} |\tau(a_k^m \cdot e_i) - \tau(a_k^l \cdot e_i)|\,\mathrm{d}x = 0.$$

On the other hand,

$$\begin{aligned}\int_{\Omega_k} |\tau(u_m \cdot e_i) - \tau(a_k^m \cdot e_i)|\,\mathrm{d}x &= \sum_{q_{k,z}\subset\Omega} \int_{q_{k,z}} |\tau(u_m \cdot e_i) - \tau(a_{k,z}^m \cdot e_i)|\,\mathrm{d}x \\ &\leq \sum_{q_{k,z}\subset\Omega} \Big(2|\omega_{k,z}^m| + \int_{q_{k,z}\setminus\omega_{k,z}^m} |u_m - a_{k,z}^m|\,\mathrm{d}x\Big) \\ &\leq \frac{2c}{k}\Big(\mathcal{H}^{n-1}(J_{u_m}) + \int_{\Omega_k} |e(u_m)|\,\mathrm{d}x\Big) \leq \frac{C}{k}.\end{aligned}$$

Using that $|\Omega \setminus \Omega_k| \to 0$ as $k \to \infty$, we deduce from (3.3) that $(\tau(u_h \cdot e_i))_h$ is a Cauchy sequence (for each i) and therefore converges in $L^1(\Omega)$ to some limit which we denote $\tilde{\tau}_i$. Up to a subsequence, we also assume that the convergence occurs almost everywhere.

We define $\bar{u}\colon \Omega \to (\widetilde{\mathbb{R}})^n$ and $u\colon \Omega \to \mathbb{R}^n$ such that

$$\bar{u} := (u^1, \dots, u^n)\,, \qquad \text{where } u^i = \tau^{-1}(\tilde{\tau}_i)\,; \qquad u := \bar{u}\,\chi_{\Omega\setminus A}\,, \quad (3.4)$$

with the convention that $\tau^{-1}(\pm 1) = \pm\infty$. (We observe that we could in fact assign any constant value to u in A, and even, any infinitesimal rigid motion.)

The set $\{x \in \Omega\colon\ u^i(x) \in \mathbb{R} \text{ for all } i = 1, \dots, n\}$ is measurable, since $u^i(x) \in \mathbb{R}$ if and only if $|\tau(u^i)| < 1$ and the functions $\tilde{\tau}_i\colon \Omega \to [-1, 1]$ are measurable. Moreover $u_h \cdot e_i$ converges in $\mathcal{L}^n$ measure to u^i on this set, for every i, while the norm of u_h is unbounded outside. Therefore,

$$A = \Omega \setminus \{x \in \Omega\colon\ u^i(x) \in \mathbb{R} \text{ for all } i = 1, \dots, n\} \quad (3.5)$$

up to a set of null $\mathcal{L}^n$ measure, where

$$A := \{x \in \Omega\colon\ |u_h(x)| \text{ is unbounded}\}\,. \quad (3.6)$$

Since $u_h \cdot e_i \to u^i$ in $L^0(\Omega \setminus A)$ for every i, we have that

$$u_h \cdot \xi \to u \cdot \xi \quad \text{in } L^0(\Omega \setminus A) \quad \text{for every } \xi \in \mathbb{S}^{n-1}\,. \quad (3.7)$$

Notice that we have not extracted further subsequences depending on ξ, and that the limit function u (equal to $\bar{u}$ since we are in $\Omega \setminus A$) does not depend on ξ.

We claim that

$$|u_h \cdot \xi| \to +\infty \quad \mathcal{L}^n\text{-a.e. in } A \quad \text{for } \mathcal{H}^{n-1}\text{-a.e. } \xi \in \mathbb{S}^{n-1}\,. \quad (3.8)$$

On the sets $A_i := \{|u_h \cdot e_i| \to +\infty\} \cap \bigcap_{j\neq i}\{\limsup_{h\to\infty}(|u_h \cdot e_j|/|u_h \cdot e_i|) < +\infty\}$, we have that (3.8) holds for every ξ in $\{\xi \in \mathbb{S}^{n-1}\colon\ \xi_i \neq 0\}$, which is of full $\mathcal{H}^{n-1}$ measure in $\mathbb{S}^{n-1}$.

Let us thus consider the case when there are m components of u_h, with $1 < m \leq n$, that we may assume up to a permutation $u_h \cdot e_1, \dots, u_h \cdot e_m$, such that $\frac{u_h\cdot e_i}{u_h\cdot e_j} \to \xi_{i,j} \in \mathbb{R}^*$ for $1 \leq i < j \leq m$ and $|\frac{u_h\cdot e_i}{u_h\cdot e_j}| \to +\infty$ for $i \in \{1, \dots, m\}$ and $j \in \{m+1, \dots, n\}$ (if $m < n$). In this case (3.8) does not hold only for $\mathbb{S}^{n-1} \cap (1, \xi_{1,2}^{-1}, \dots, \xi_{1,m}^{-1}, 0 \dots, 0)^\perp$, which has dimension $n - 2$. Notice now than for every m for which m components go faster to infinity than the other ones, there is an at most countable collection of $(\xi_{1,2}, \dots, \xi_{1,m}) \in (\mathbb{R}^*)^{m-1}$ for which

$\frac{u_h\cdot e_1}{u_h\cdot e_j}\to\xi_{1,j}$ for $j\in\{2,\dots,m\}$ on a subset of Ω of positive $\mathcal{L}^n$ measure. Thus (3.8) holds for every ξ except on an at most countable union of $\mathcal{H}^{n-1}$-negligible sets of $\mathbb{S}^{n-1}$.

We now follow the lines of the proof of [9, Theorem 1.1] (see also [21, Theorem 11.3]), introducing

$$I^\xi_y(u_h):=\int\limits_{\Omega^\xi_y}\phi\big(|(\dot u_h)^\xi_y|\big)\,\mathrm{d}t\,,\tag{3.9}$$

where $(\dot u_h)^\xi_y$ is the density of the absolutely continuous part of $\mathrm{D}(\widehat u_h)^\xi_y$, the distributional derivative of $(\widehat u_h)^\xi_y$ ($(\widehat u_h)^\xi_y\in SBV_{\mathrm{loc}}(\Omega^\xi_y)$ for every $\xi\in\mathbb{S}^{n-1}$ and for $\mathcal{H}^{n-1}$-a.e. $y\in\Pi^\xi$, since $u_h\in GSBD(\Omega)$). Thus for any $\xi\in\mathbb{S}^{n-1}$ it holds that

$$\int\limits_{\Pi_\xi}I^\xi_y(u_h)\,\mathrm{d}\mathcal{H}^{n-1}(y)=\int\limits_\Omega\phi\big(|e(u_h)(x)\xi\cdot\xi|\big)\le\int\limits_\Omega\phi\big(|e(u_h)|\big)\,\mathrm{d}x\le M\,,\tag{3.10}$$

by Fubini-Tonelli's theorem and (1.3), recalling that ϕ is non-decreasing. Moreover, since $u_h\in GSBD(\Omega)$, $\mathrm{D}_\xi\big(\tau(u_h\cdot\xi)\big)\in\mathcal{M}^+_b(\Omega)$ for every $\xi\in\mathbb{S}^{n-1}$ and

$$\int\limits_{\Pi_\xi}|\mathrm{D}\big(\tau(u_h\cdot\xi)^\xi_y\big)|(\Omega^\xi_y)\,\mathrm{d}\mathcal{H}^{n-1}(y)=|\mathrm{D}_\xi\big(\tau(u_h\cdot\xi)\big)|(\Omega)\le M\,,\tag{3.11}$$

by (2.9) and (1.3). We denote

$$II^\xi_y(u_h):=|\mathrm{D}\big(\tau(u_h\cdot\xi)^\xi_y\big)|(\Omega^\xi_y)\,.\tag{3.12}$$

Let $u_k=u_{h_k}$ be a subsequence of u_h such that

$$\lim_{k\to\infty}\mathcal{H}^{n-1}(J_{u_k})=\liminf_{h\to\infty}\mathcal{H}^{n-1}(J_{u_h})<+\infty\,,\tag{3.13}$$

so that, by (2.8), (3.10), and Fatou's lemma, we have that for $\mathcal{H}^{n-1}$-a.e. $\xi\in\mathbb{S}^{n-1}$

$$\liminf_{k\to\infty}\int\limits_{\Pi_\xi}\Big[\mathcal{H}^0\big(J_{(\widehat u_k)^\xi_y}\big)+\varepsilon\big(I^\xi_y(u_k)+II^\xi_y(u_k)\big)\Big]\,\mathrm{d}\mathcal{H}^{n-1}(y)<+\infty\,,\tag{3.14}$$

for a fixed $\varepsilon\in(0,1)$. Let us fix $\xi\in\mathbb{S}^{n-1}$ such that (3.8) and (3.14) hold. Then there is a subsequence $u_m=u_{k_m}$ of u_k, depending on ε and ξ, such that

$$\begin{aligned}\lim_{m\to\infty}&\int\limits_{\Pi_\xi}\Big[\mathcal{H}^0\big(J_{(\widehat u_m)^\xi_y}\big)+\varepsilon\big(I^\xi_y(u_m)+II^\xi_y(u_m)\big)\Big]\,\mathrm{d}\mathcal{H}^{n-1}(y)\\&=\liminf_{k\to\infty}\int\limits_{\Pi_\xi}\Big[\mathcal{H}^0\big(J_{(\widehat u_k)^\xi_y}\big)+\varepsilon\big(I^\xi_y(u_k)+II^\xi_y(u_k)\big)\Big]\,\mathrm{d}\mathcal{H}^{n-1}(y)\,.\end{aligned}\tag{3.15}$$

Therefore, by (3.15), (3.7), and (3.8), employing Fatou's lemma, we have that for $\mathcal{H}^{n-1}$-a.e. $y\in\Pi^\xi$

$$\liminf_{m\to\infty}\Big[\mathcal{H}^0\big(J_{(\widehat u_m)^\xi_y}\big)+\varepsilon\big(I^\xi_y(u_m)+II^\xi_y(u_m)\big)\Big]<+\infty\,,\tag{3.16}$$

$$(\widehat u_m)^\xi_y\to\widehat u^\xi_y\quad\text{in }L^0\big((\Omega\setminus A)^\xi_y\big)\qquad|(\widehat u_m)^\xi_y|\to+\infty\,,\quad\mathcal{L}^1\text{-a.e. in }A^\xi_y\,,\tag{3.17}$$

and

$$\tau(u_m\cdot\xi)^\xi_y\to\tilde\tau^\xi_y\quad\text{in }L^1(\Omega^\xi_y)\,,\tag{3.18}$$

for a suitable $\tilde\tau^\xi_y\in L^1(\Omega^\xi_y)$. Now we employ (3.7), (3.8), and (3.17), (3.18) to get

$$\begin{cases}\tilde\tau^\xi_y=\tau(u\cdot\xi)^\xi_y\quad\mathcal{L}^1\text{-a.e. in }(\Omega\setminus A)^\xi_y\\|\tilde\tau^\xi_y|=1\quad\mathcal{L}^1\text{-a.e. in in }A^\xi_y\,.\end{cases}\tag{3.19}$$

Fixed $y\in\Pi^\xi$ satisfying (3.16) and (3.17), and such that $(\widehat u_m)^\xi_y\in SBV_{\mathrm{loc}}(\Omega^\xi_y)$ for every m, we extract a subsequence $u_j=u_{m_j}$ from u_m, depending also on y, for which

$$\lim_{j\to\infty}\Big[\mathcal{H}^0\big(J_{(\widehat u_j)^\xi_y}\big)+\varepsilon\big(I^\xi_y(u_j)+II^\xi_y(u_j)\big)\Big]=\liminf_{m\to\infty}\Big[\mathcal{H}^0\big(J_{(\widehat u_m)^\xi_y}\big)+\varepsilon\big(I^\xi_y(u_m)+II^\xi_y(u_m)\big)\Big]\,.\tag{3.20}$$

By (3.18) we have that

$$\tau(u_j\cdot\xi)^\xi_y \overset{*}{\rightharpoonup} \tilde\tau^\xi_y \quad \text{in } SBV(\Omega^\xi_y)\,. \tag{3.21}$$

We claim that

$$\partial A^\xi_y \subset J_{\tilde\tau^\xi_y}\,. \tag{3.22}$$

Indeed, up to consider a subsequence of $(\widehat{u}_j)^\xi_y$, we may assume that for every j there is a fixed number N_y of jump points that tends to $M_y \leq N_y$ points $t_1,\dots t_{M_y}$. Then (recall that $I^\xi_y(u_j)$ is equibounded in j) for every $l=1,\dots,M_y-1$

$$\tau(u_j\cdot\xi)^\xi_y \rightharpoonup \tilde\tau^\xi_y \quad \text{in } W^{1,1}_{\mathrm{loc}}(t_l,t_{l+1})\,,$$

and the convergence above is locally uniform for the precise representatives. Moreover, employing the Fundamental Theorem of Calculus and the bound for $I^\xi_y(u_j)$, which is uniform in j, for each interval (t_l,t_{l+1}) either $(\widehat{u}_j)^\xi_y$ are pointwise bounded (in j) and then they converge locally uniformly to $\widehat{u}^\xi_y \in W^{1,1}(t_l,t_{l+1})$, or $(\widehat{u}_j)^\xi_y$ are unbounded from above (from below) in a.e. $x\in(t_l,t_{l+1})$, and then $\tau(\widehat{u}_j)^\xi_y = \tau(u_j\cdot\xi)^\xi_y$ converge to $\tilde\tau^\xi_y = 1$ ($\tilde\tau^\xi_y=-1$, respectively). Therefore, in view of (3.19), the inclusion (3.22) is proven and A^ξ_y is a finite union of intervals where $\tilde\tau^\xi_y$ is 1 or -1.

By (3.20), (3.21), (3.22), and since the jump sets of $\tau(u_j\cdot\xi)^\xi_y$ and $(\widehat{u}_j)^\xi_y$ coincide, we deduce that

$$\begin{aligned}\mathcal{H}^0\big(J_{\widehat{u}^\xi_y}\cap(\Omega\setminus A)^\xi_y\big)+\mathcal{H}^0\big(\partial A^\xi_y\big) &\leq \mathcal{H}^0(J_{\tilde\tau^\xi_y})\\ &\leq \liminf_{m\to\infty}\Big[\mathcal{H}^0\big(J_{(\widehat{u}_m)^\xi_y}\big)+\varepsilon\big(I^\xi_y(u_m)+II^\xi_y(u_m)\big)\Big]\,.\end{aligned}\tag{3.23}$$

We now integrate over $y\in\Pi_\xi$ and use Fatou's lemma with (3.15) to get

$$\begin{aligned}&\int_{\Pi_\xi}\Big[\mathcal{H}^0\big(J_{\widehat{u}^\xi_y}\cap(\Omega\setminus A)^\xi_y\big)+\mathcal{H}^0\big(\partial A^\xi_y\big)\Big]\,\mathrm{d}\mathcal{H}^{n-1}(y)\\ &\qquad\leq \liminf_{k\to\infty}\int_{\Pi_\xi}\Big[\mathcal{H}^0\big(J_{(\widehat{u}_k)^\xi_y}\big)+\varepsilon\big(I^\xi_y(u_k)+II^\xi_y(u_k)\big)\Big]\,\mathrm{d}\mathcal{H}^{n-1}(y)\end{aligned}\tag{3.24}$$

for $\mathcal{H}^{n-1}$-a.e. $\xi\in\mathbb{S}^{n-1}$. In particular we deduce that A has finite perimeter (*cf.* [5, Remark 3.104]).

We integrate (3.24) over $\xi\in\mathbb{S}^{n-1}$; by (2.8), (3.10), (3.11), and (3.13) we get

$$\mathcal{H}^{n-1}(J_u\cap(\Omega\setminus A))+\mathcal{H}^{n-1}(\partial^*A)\leq C\,M\varepsilon+\liminf_{h\to\infty}\mathcal{H}^{n-1}(J_{u_h})\,, \tag{3.25}$$

for a universal constant C. By the arbitrariness of ε we obtain (1.4c) (the property follows immediately also for the extension of u with the value 0 in A).

Employing (2.9) and recalling (1.3), we have that there exist $\lambda_{u_h}\in\mathcal{M}^+_b(\Omega)$ such that for every $\xi\in\mathbb{S}^{n-1}$

$$|\mathrm{D}_\xi\big(\tau(u_h\cdot\xi)\big)|(B)\leq\lambda_{u_h}(B)\,,$$

and

$$\lambda_{u_h}(\Omega)\leq M\,.$$

Let $\lambda_u\in\mathcal{M}^+_b(\Omega)$ be a weak* limit of a subsequence of λ_{u_h}, so that $\lambda_u(\Omega)\leq M$. Notice that

$$\mathrm{D}_\xi\tau(u\cdot\xi)\in\mathcal{M}_b(\Omega)\quad\text{for every }\xi\in\mathbb{S}^{n-1} \tag{3.26}$$

and

$$|\mathrm{D}_\xi\tau(\tilde u\cdot\xi)|(B)\leq\lambda_u(B) \tag{3.27}$$

for every open set $B\subset\Omega$, where λ_u has been defined above. This follows by a slicing procedure and the use of Fatou's lemma for every ξ, to reconstruct at the end $|\mathrm{D}_\xi(\tau(u\cdot\xi))|(\Omega)$ from $II^\xi_y(u):=|\mathrm{D}\big(\tau(u\cdot\xi)^\xi_y\big)|(\Omega^\xi_y)$ (see (3.12)), as in (3.11). The important point here is to get the semicontinuity

$$II^\xi_y(u)\leq\liminf_{j\to\infty}II^\xi_y(u_j)=\liminf_{j\to\infty}|\mathrm{D}\big(\tau(u_j\cdot\xi)^\xi_y\big)|(\Omega^\xi_y)\,,$$

for the slices, which follows from (3.21). Indeed $II^\xi_y(u) \leq |\mathrm{D}(\tilde\tau^\xi_y))|(\Omega^\xi_y)$ because $\tau(u\cdot\xi)^\xi_y = \tilde\tau^\xi_y$ in $(\Omega\setminus A)^\xi_y$ by (3.19) and $\tau(u\cdot\xi)=0$ in A^ξ_y, so we employ (3.22). Moreover, it is immediate that $\widehat{u}^\xi_y \in SBV_{\mathrm{loc}}(\Omega^\xi_y)$. Therefore $\tilde u \in GSBD(\Omega)$.

Now the property (1.4b) follows by an adaptation of the arguments in [9, Theorem 1.1] as in [21, Theorem 11.3] (which follow Ambrosio-Dal Maso's [1, Prop. 4.4]). □

4. Existence for minimisers of Griffith energy

Employing Theorem 1.1, we deduce in this section the existence of weak solutions to the minimisation problem of Griffith energy with Dirichlet boundary conditions.

4.1. **Existence of weak solutions.** Assume $\Omega\subset\mathbb{R}^n$ be an open, bounded domain for which

$$\partial\Omega = \partial_D\Omega \cup \partial_N\Omega \cup N\,,$$

with $\partial_D\Omega$ and $\partial_N\Omega$ relatively open, $\partial_D\Omega\cap\partial_N\Omega=\emptyset$, $\mathcal{H}^{n-1}(N)=0$, $\partial_D\Omega\neq\emptyset$, and $\partial(\partial_D\Omega)=\partial(\partial_N\Omega)$. Let $u_0\in W^{1,p}(\mathbb{R}^n;\mathbb{R}^n)$ and $W\colon \mathbb{R}\times\mathbb{M}^{n\times n}_{sym}\to[0,\infty)$ be convex in the second argument and lower semicontinuous, with

$$c_1\, s\,|\cdot|^p \leq W(s,\cdot) \leq c_2\,(1+s\,|\cdot|^p) \quad \text{for every } s\in\mathbb{R} \tag{4.1}$$

for some $0<c_1<c_2$. Let $K\subset\Omega\cup\partial_D\Omega$ be $(n-1)$-countably rectifiable with $\mathcal{H}^{n-1}(K)<+\infty$, and consider the minimisation problem:

$$\min_{u\in GSBD^p(\Omega)} \left\{ \int_\Omega W(e(u))\,\mathrm{d}x + \mathcal{H}^{n-1}\big(J_u \cup (\partial_D\Omega \cap \{\mathrm{tr}_\Omega\, u \neq \mathrm{tr}_\Omega\, u_0\})\setminus K\big)\right\}. \tag{4.2}$$

Notice that, defining $\widetilde\Omega := \Omega\cup U$, where U is an open bounded set with $U\cap\partial\Omega=\partial_D\Omega$, we can recast the problem as

$$\min_{u\in GSBD^p(\widetilde\Omega)} \left\{ \int_{\widetilde\Omega} W(e(u))\,\mathrm{d}x + \mathcal{H}^{n-1}(J_u\setminus K)\colon u=u_0 \text{ in } \widetilde\Omega\setminus(\Omega\cup\partial_D\Omega)\right\}. \tag{4.3}$$

Then we have the following existence result.

Theorem 4.1. *Problem* (4.3) *admits solutions.*

Proof. Let $u_h\in GSBD^p(\widetilde\Omega)$ with $u=u_0$ in $\widetilde\Omega\setminus(\Omega\cup\partial_D\Omega)$ be a minimising sequence for (4.3). Observe that the infimum of problem (4.3) is finite, since the functional is nonnegative and u_0 is an admissible competitor.

Assume for the moment that K is compact. Then the functions u_h satisfy the hypotheses of Theorem 1.1 with $\Omega=\widetilde\Omega\setminus K$, and $\phi=W$, so that there exist $A\subset\widetilde\Omega\setminus K$ with finite perimeter and a measurable function $u\colon\widetilde\Omega\setminus K\to\mathbb{R}^n$ with $u=0$ in A such that (up to a subsequence)

$$A=\{x\in\widetilde\Omega\setminus K\colon |u_h(x)|\to\infty\}, \qquad u_h\to u \quad \text{in } L^0(\widetilde\Omega\setminus K;\mathbb{R}^n) \tag{4.4}$$

(since $\mathcal{L}^n(K)=0$ we could consider just $\widetilde\Omega$ above, but we keep $\widetilde\Omega\setminus K$ to indicate the set where we apply Theorem 1.1) and

$$\int_{\widetilde\Omega} W(e(u))\,\mathrm{d}x + \mathcal{H}^{n-1}(J_u\setminus K) \leq \liminf_{h\to\infty}\int_{\widetilde\Omega} W(e(u_h))\,\mathrm{d}x + \mathcal{H}^{n-1}(J_{u_h}\setminus K)\,,$$

Moreover, by (4.4) it follows that $u=u_0$ in $\widetilde\Omega\setminus(\Omega\cup\partial_D\Omega)$, and in particular A does not intersect $\big(\widetilde\Omega\setminus(\Omega\cup\partial_D\Omega)\big)$. Then u solves (4.3) (this holds for any other function which coincides with u in $\Omega\setminus A$ and is any fixed infinitesimal rigid motion in A.) This proves the theorem if K is compact.

If K is not compact, for any $\varepsilon>0$ consider $\widehat K\subset K$ with $\mathcal{H}^{n-1}(K\setminus\widehat K)<\varepsilon$. Then, arguing as above for the open set $\widetilde\Omega\setminus\widehat K\supset\widetilde\Omega\setminus K$, we get still

$$\int_{\widetilde\Omega} W(e(u))\,\mathrm{d}x \leq \liminf_{h\to\infty}\int_{\widetilde\Omega} W(e(u_h))\,\mathrm{d}x\,,$$

and

$$\begin{aligned}\mathcal{H}^{n-1}(J_u\setminus K) &\leq \mathcal{H}^{n-1}(J_u\setminus \widehat{K}) \leq \liminf_{h\to\infty}\mathcal{H}^{n-1}(J_{u_h}\setminus \widehat{K})\\ &\leq \liminf_{h\to\infty}\mathcal{H}^{n-1}(J_{u_h}\setminus K)+\mathcal{H}^{n-1}(K\setminus\widehat{K}) < \liminf_{h\to\infty}\mathcal{H}^{n-1}(J_{u_h}\setminus K)+\varepsilon\,,\end{aligned}$$

since $J_u\setminus K\subset J_u\setminus\widehat{K}$ and $J_{u_h}\setminus\widehat{K}\subset(J_{u_h}\setminus K)\cup(K\setminus\widehat{K})$ (*cf.* also [31, Theorem 2.5]). We conclude since $\varepsilon>0$ is arbitrary. □

Remark 4.2. Since, as observed in the proof, a family of minimisers is obtained by adding any fixed infinitesimal rigid motion in A to a given minimiser, we conclude that $\mathcal{H}^{n-1}(\partial^*A\cap\{\mathrm{tr}\,u=a\})=0$ for every infinitesimal rigid motion a ($a(x)=\mathbf{a}\cdot x+b$, $\mathbf{a}+\mathbf{a}^T=0$), where tr denotes here the trace of u on ∂^*A (which is $(n-1)$-countably rectifiable) from $\Omega\setminus A$.

4.2. **Existence of strong solutions.** In recent works, Chambolle, Conti, Focardi, and Iurlano have shown more regularity for the solutions (assuming their existence, which has been proven above) to (4.3) (or (4.2)) if $W(\xi)=\mathbb{C}e(\xi)\colon e(\xi)$ (in [15]), or $n=2$ and

$$W(\xi)=f_\mu(\xi):=\frac{1}{p}\Big((\mathbb{C}\xi\colon\xi+\mu)^{p/2}-\mu^{p/2}\Big) \tag{4.5}$$

(in [19]), requiring that $\mathbb{C}\colon\mathbb{M}^{n\times n}_{sym}\to\mathbb{M}^{n\times n}_{sym}$ is a symmetric linear map with

$$\mathbb{C}(\xi-\xi^T)=0 \quad\text{and}\quad \mathbb{C}\xi\cdot\xi\geq c_0|\xi+\xi^T|^2 \qquad \text{for all } \xi\in\mathbb{M}^{n\times n}_{sym}\,.$$

This corresponds to the following theorem.

Theorem 4.3 (Density lower bound and internal regularity, [19, 15])**.** *Let $u\in GSBD^2(\Omega\setminus K)$ (or $u\in GSBD^p(\Omega\setminus K)$, if $\Omega\subset\mathbb{R}^2$) be a minimiser of*

$$\int_\Omega \mathbb{C}e(u)\colon e(u)\,\mathrm{d}x+\mathcal{H}^{n-1}\big(J_u\cup(\partial_D\Omega\cap\{\mathrm{tr}_\Omega\,u\neq\mathrm{tr}_\Omega\,u_0\})\setminus K\big)$$

(a minimiser of (4.3) *with* (4.5), *respectively). Then there exist θ_0 and R_0, depending only on n and $\mathbb{C}$ (W respectively) such that if $x\in\overline{J}_u$, $\varrho\in(0,R_0)$, and $B_\varrho(x)\subset\Omega\setminus K$, then*

$$\mathcal{H}^{n-1}(J_u\cap B_\varrho(x))\geq\theta_0\varrho^{n-1}\,,$$

and

$$\mathcal{H}^{n-1}\big((\Omega\setminus K)\cap(\overline{J}_u\setminus J_u)\big)=0\,,\qquad u\in C^1\big(\Omega\setminus(K\cup\overline{J}_u)\big)\,.$$

The extension of this result up to the boundary is the subject for future study.

5. An approximation result

In this section we show a compactness property for sequences of minimisers of suitable *phase-field* elliptic energies approximating the Griffith fracture energy *à la* Ambrosio-Tortorelli. The Γ-convergence has been proved in [16, Theorem 5.4] for general energies with p-growth of the bulk energy in $e(u)$. In particular the following "Ambrosio-Tortorelli" [6] type approximation result holds (*cf.* [16, Theorem 1.2]):

Theorem 5.1 ([16])**.** *Let $u_0\in H^1(\mathbb{R}^n;\mathbb{R}^n)$, and $\Omega\subset\mathbb{R}^n$ be an open, bounded, Lipschitz domain for which $\partial\Omega=\partial_D\Omega\cup\partial_N\Omega\cup N$, with $\partial_D\Omega$ and $\partial_N\Omega$ relatively open, $\partial_D\Omega\cap\partial_N\Omega=\emptyset$, $\mathcal{H}^{n-1}(N)=0$, $\partial_D\Omega\neq\emptyset$, and $\partial(\partial_D\Omega)=\partial(\partial_N\Omega)$. Assume that there exist $\overline{\delta}$ and $x_0\in\mathbb{R}^n$ such that*

$$O_{\delta,x_0}(\partial_D\Omega)\subset\Omega$$

for $\delta\in(0,\overline{\delta})$, where $O_{\delta,x_0}(x):=x_0+(1-\delta)(x-x_0)$. Moreover let ε_k, $\eta_k>0$ with $\varepsilon_k\to0$, $\frac{\eta_k}{\varepsilon_k}\to0$ as $k\to\infty$. Then, for $H^1_{u_0}(\Omega;\mathbb{R}^n):=\{u\in H^1(\Omega;\mathbb{R}^n)\colon \mathrm{tr}_\Omega\,u=\mathrm{tr}_\Omega\,u_0$ on $\partial_D\Omega\}$ and $V^1_k:=\{v\in H^1(\Omega)\colon \eta_k\leq v\leq1\,,\ \mathrm{tr}_\Omega\,v=1$ on $\partial_D\Omega\}$, the functionals

$$D^2_k(u,v):=\begin{cases}\displaystyle\int_\Omega\Big(v\,\mathbb{C}e(u)\colon e(u)+\frac{(1-v)^2}{4\varepsilon_k}+\varepsilon_k|\nabla v|^2\Big)\,\mathrm{d}x & \text{in } H^1_{u_0}(\Omega;\mathbb{R}^n)\times V^1_k\,,\\ +\infty & \textit{otherwise,}\end{cases}$$

Γ-converge as $k \to \infty$ to

$$D^2(u,v) := \begin{cases} \displaystyle\int_\Omega \mathbb{C}e(u)\colon e(u)\,\mathrm{d}x + \mathcal{H}^{n-1}\Big(J_u \cup \big(\partial_D\Omega \cap \{\mathrm{tr}_\Omega\, u \neq \mathrm{tr}_\Omega\, u_0\}\big)\Big) & \text{in } GSBD^p(\Omega)\times\{v=1\},\\ +\infty & \text{otherwise,}\end{cases}$$

with respect to the topology of the convergence in measure for u and v.

We now show an important relation between minimisers of D^2_k and of D^2.

Theorem 5.2. *Let $(u_k, v_k) \in H^1_{u_0}(\Omega;\mathbb{R}^n)\times V^1_k$ be minimisers of D^2_k (or "almost" minimisers, up to an error ζ_k with $\zeta_k \to 0$). Then, for a subsequence (u_h, v_h), we have that v_h converges to 1 in $L^1(\Omega)$, the set $A := \{x \in \Omega\colon |u_h(x)| \to +\infty\}$ has finite perimeter, there exists $u \in GSBD(\Omega)$ minimiser of D^2 with $u = 0$ in A, and $u_h \to u$ in $L^0(\Omega \setminus A;\mathbb{R}^n)$. Moreover $\partial^* A \subset J_u$ and*

$$\int_\Omega \mathbb{C}e(u)\colon e(u)\,\mathrm{d}x = \lim_{h\to\infty}\int_\Omega v_h\,\mathbb{C}e(u_h)\colon e(u_h)\,\mathrm{d}x\,, \tag{5.1a}$$

$$\mathcal{H}^{n-1}(J_u) = \lim_{h\to\infty}\int_\Omega \Big(\frac{(1-v_h)^2}{4\varepsilon_h} + \varepsilon_h|\nabla v_h|^2\Big)\,\mathrm{d}x\,. \tag{5.1b}$$

Proof. Since $D^2_k(u_0, 1) = C_0$, where $C_0 := \int_\Omega \mathbb{C}e(u_0)\colon e(u_0)\,\mathrm{d}x$, we have that $v_k \to 1$ in $L^2(\Omega)$ and that (*cf.* [13, Theorem 4])

$$C_0 \geq D^2_k(u_k, v_k) \geq \int_0^1 \bigg(\int_{\{v_k>s\}} 2s\,\mathbb{C}e(u_k)\colon e(u_k)\,\mathrm{d}x + (1-s)\mathcal{H}^{n-1}(\partial^*\{v_k > s\})\bigg)\mathrm{d}s\,,$$

by the coarea formula and the Cauchy inequality:

$$\frac{(1-v_k)^2}{4\varepsilon_k} + \varepsilon_k|\nabla v_k|^2 \geq |1-v_k||\nabla v_k|\,.$$

By Fatou's lemma we have that $\liminf_{k\to\infty}\mathcal{H}^{n-1}(\partial^*\{v_k > s\})$ is bounded for $\mathcal{L}^1$-a.e. $s \in (0,1)$, so we fix s satisfying this property and up to a subsequence $\mathcal{H}^{n-1}(\partial^*\{v_k > s\}) \leq C$. By the minimality of v_k we deduce also

$$\mathcal{L}^n(\{v_k > s\}) \leq 4\,\varepsilon_k\,C_0\,. \tag{5.2}$$

Therefore the sequence $\tilde u_k := u_k\chi_{\Omega\setminus\{v_k>s\}}$ satisfies the hypotheses of Theorem 1.1, and so there are $A = \{x \in \Omega\colon |\tilde u_k(x)| \to \infty\}$, with finite perimeter, and $u \in GSBD(\Omega)$ with u any (fixed) infinitesimal rigid motion on A such that $\tilde u_k \to u$ in $L^0(\Omega\setminus A;\mathbb{R}^n)$, and $e(\tilde u_k) \rightharpoonup e(u)$ in $L^2(\Omega\setminus A;\mathbb{M}^{n\times n}_{sym})$. In particular, employing (5.2), we have that

$$A = \{x \in \Omega\colon |u_k(x)| \to \infty\}\,, \qquad u_k \to u \quad \text{in } L^0(\Omega\setminus A;\mathbb{R}^n)\,. \tag{5.3}$$

Since now we have determined the pointwise limit of u_k, we can follow standard arguments, employing a slicing technique as in [16, Theorem 5.1] or [33, Theorem 8] (*cf.* also [11]) to obtain that

$$\int_\Omega \mathbb{C}e(u)\colon e(u)\,\mathrm{d}x \leq \liminf_{k\to\infty}\int_\Omega v_k\,\mathbb{C}e(u_k)\colon e(u_k)\,\mathrm{d}x\,, \tag{5.4a}$$

$$\mathcal{H}^{n-1}\big(J_u\cap(\Omega\setminus A)\big) + \mathcal{H}^{n-1}(\partial^*A) \leq \liminf_{k\to\infty}\int_\Omega \Big(\frac{(1-v_k)^2}{4\varepsilon_k} + \varepsilon_k|\nabla v_k|^2\Big)\,\mathrm{d}x\,. \tag{5.4b}$$

In particular, observing $J_u \subset J_u \cap (\Omega\setminus A) \cup \partial^*A$, we have

$$D^2(u,1) \leq \liminf_{k\to\infty} D^2_k(u_k, v_k) = \liminf_{k\to\infty}\min D^2_k\,. \tag{5.5}$$

Since D^2_k Γ-converges to D with respect to the topology of the convergence in measure we obtain (*cf.* [20, Proposition 7.1]) that

$$\inf_{GSBD^2(\Omega)} D^2 \geq \liminf_{k\to\infty}\min D^2_k = \liminf_{k\to\infty} D^2_k(u_k, v_k)\,.$$

Therefore we have that u is a minimiser for D^2 (independently of the rigid motion assigned to u in A, see Remark 4.2) and that, up to considering a subsequence $u_h = u_{h_k}$ of u_k,

$$D^2(u, 1) = \lim_{h\to\infty} D^2_h(u_h, v_h)\,.$$

In particular the conditions (5.4) hold as equalities on (u_h, v_h), so we get that $\partial^* A \subset J_u$ and deduce (5.1). □

Remark 5.3. Theorem 5.1 holds under more general assumptions on the growth of the bulk energy with respect to $e(u)$ and on the Modica-Mortola term in the approximating functionals (in particular for W as in (4.1), see [16, Theorem 5.4]). It is not difficult to prove the version of Theorem 5.2 corresponding to these assumptions.

Acknowledgements. Vito Crismale has been supported by a public grant as part of the *Investissement d'avenir* project, reference ANR-11-LABX-0056-LMH, LabEx LMH.

References

[1] L. Ambrosio, *A compactness theorem for a new class of functions of bounded variation*, Boll. Un. Mat. Ital. B (7), 3 (1989), pp. 857–881.

[2] ———, *Existence theory for a new class of variational problems*, Arch. Rational Mech. Anal., 111 (1990), pp. 291–322.

[3] L. Ambrosio, *A new proof of the SBV compactness theorem*, Calc. Var. Partial Differential Equations, 3 (1995), pp. 127–137.

[4] L. Ambrosio, A. Coscia, and G. Dal Maso, *Fine properties of functions with bounded deformation*, Arch. Rational Mech. Anal., 139 (1997), pp. 201–238.

[5] L. Ambrosio, N. Fusco, and D. Pallara, *Functions of bounded variation and free discontinuity problems*, Oxford Mathematical Monographs, The Clarendon Press, Oxford University Press, New York, 2000.

[6] L. Ambrosio and V. M. Tortorelli, *On the approximation of free discontinuity problems*, Boll. Un. Mat. Ital. B (7), 6 (1992), pp. 105–123.

[7] J.-F. Babadjian, *Traces of functions of bounded deformation*, Indiana Univ. Math. J., 64 (2015), pp. 1271–1290.

[8] J.-F. Babadjian and A. Giacomini, *Existence of strong solutions for quasi-static evolution in brittle fracture*, Ann. Sc. Norm. Super. Pisa Cl. Sci. (5), 13 (2014), pp. 925–974.

[9] G. Bellettini, A. Coscia, and G. Dal Maso, *Compactness and lower semicontinuity properties in* SBD(Ω), Math. Z., 228 (1998), pp. 337–351.

[10] B. Bourdin, G. A. Francfort, and J.-J. Marigo, *Numerical experiments in revisited brittle fracture*, J. Mech. Phys. Solids, 48 (2000), pp. 797–826.

[11] A. Braides, A. Chambolle, and M. Solci, *A relaxation result for energies defined on pairs set-function and applications*, ESAIM Control Optim. Calc. Var., 13 (2007), pp. 717–734.

[12] A. Chambolle, *A density result in two-dimensional linearized elasticity, and applications*, Arch. Ration. Mech. Anal., 167 (2003), pp. 211–233.

[13] ———, *An approximation result for special functions with bounded deformation*, J. Math. Pures Appl. (9), 83 (2004), pp. 929–954.

[14] A. Chambolle, S. Conti, and G. Francfort, *Korn-Poincaré inequalities for functions with a small jump set*, Indiana Univ. Math. J., 65 (2016), pp. 1373–1399.

[15] A. Chambolle, S. Conti, and F. Iurlano, *Approximation of functions with small jump sets and existence of strong minimizers of Griffith's energy*, 2017, In preparation.

[16] A. Chambolle and V. Crismale, *A Density Result In $GSBD^p$ With Applications To The Approximation Of Brittle Fracture Energies*. Preprint arXiv:1708.03281, Aug. 2017.

[17] S. Conti, M. Focardi, and F. Iurlano, *Integral representation for functionals defined on SBD^p in dimension two*, Arch. Ration. Mech. Anal., 223 (2017), pp. 1337–1374.

[18] ———, *Approximation of fracture energies with p-growth via piecewise affine finite elements*, 2017, Preprint.

[19] ———, *Existence of strong minimizers for the Griffith static fracture model in dimension two*, 2017, Preprint.

[20] G. Dal Maso, *An introduction to Γ-convergence*, vol. 8 of Progress in Nonlinear Differential Equations and their Applications, Birkhäuser Boston, Inc., Boston, MA, 1993.

[21] ———, *Generalised functions of bounded deformation*, J. Eur. Math. Soc. (JEMS), 15 (2013), pp. 1943–1997.

[22] G. Dal Maso, J.-M. Morel, and S. Solimini, *A variational method in image segmentation: existence and approximation results*, Acta Math., 168 (1992), pp. 89–151.

[23] G. Dal Maso and R. Toader, *A model for the quasi-static growth of brittle fractures: existence and approximation results*, Arch. Ration. Mech. Anal., 162 (2002), pp. 101–135.
[24] E. De Giorgi and L. Ambrosio, *New functionals in the calculus of variations*, Atti Accad. Naz. Lincei Rend. Cl. Sci. Fis. Mat. Natur. (8), 82 (1988), pp. 199–210 (1989).
[25] E. De Giorgi, M. Carriero, and A. Leaci, *Existence theorem for a minimum problem with free discontinuity set*, Arch. Rational Mech. Anal., 108 (1989), pp. 195–218.
[26] G. A. Francfort and C. J. Larsen, *Existence and convergence for quasi-static evolution in brittle fracture*, Comm. Pure Appl. Math., 56 (2003), pp. 1465–1500.
[27] G. A. Francfort and J.-J. Marigo, *Revisiting brittle fracture as an energy minimization problem*, J. Mech. Phys. Solids, 46 (1998), pp. 1319–1342.
[28] M. Friedrich, *A piecewise Korn inequality in SBD and applications to embedding and density results*, 2016, Preprint.
[29] ———, *A derivation of linearized Griffith energies from nonlinear models*, Arch. Ration. Mech. Anal., 225 (2017), pp. 425–467.
[30] ———, *A Korn-type inequality in SBD for functions with small jump sets*, Math. Models Methods Appl. Sci., 27 (2017), pp. 2461–2484.
[31] M. Friedrich and F. Solombrino, *Quasistatic crack growth in 2d-linearized elasticity*, Ann. Inst. H. Poincaré Anal. Non Linéaire, 35 (2018), pp. 27–64.
[32] A. A. Griffith, *The phenomena of rupture and flow in solids*, Philos. Trans. Roy. Soc. London Ser. A, 221 (1920), pp. 163–198.
[33] F. Iurlano, *A density result for GSBD and its application to the approximation of brittle fracture energies*, Calc. Var. Partial Differential Equations, 51 (2014), pp. 315–342.
[34] F. Maddalena and S. Solimini, *Lower semicontinuity properties of functionals with free discontinuities*, Arch. Ration. Mech. Anal., 159 (2001), pp. 273–294.
[35] D. Mumford and J. Shah, *Boundary detection by minimizing functionals.* Proc. IEEE Conf. on Computer Vision and Pattern Recognition, San Francisco, 1985.
[36] R. Temam, *Mathematical problems in plasticity*, Gauthier-Villars, Paris, 1985. Translation of Problèmes mathématiques en plasticité. Gauthier-Villars, Paris, 1983.

CMAP, École Polytechnique, 91128 Palaiseau Cedex, France
E-mail address, Antonin Chambolle: antonin.chambolle@cmap.polytechnique.fr
E-mail address, Vito Crismale: vito.crismale@polytechnique.edu