A RELAXATION APPROACH TO HENCKY'S PLASTICITY

ANDREA BRAIDES[1], ANNELIESE DEFRANCESCHI[2] AND ENRICO VITALI[3]

Abstract. Given μ, κ, $c > 0$, we consider the functional

$$F(u) = \int_{\Omega\setminus S_u} (\mu|E^D u|^2 + \frac{\kappa}{2}(\mathrm{div} u)^2)dx + c\int_{S_u} |u^+ - u^-| d\mathcal{H}^{n-1},$$

defined on all $\mathbf{R}^n$-valued functions u on the open subset Ω of $\mathbf{R}^n$ which are smooth outside a free discontinuity set S_u, on which the traces u^+, u^- on both sides have equal normal component (i.e., u has a tangential jump along S_u). $E^D u = Eu - \frac{1}{3}(\mathrm{div} u)I$, with Eu denoting the linearized strain tensor.

The functional F is obtained from the usual strain energy of linearized elasticity by addition of a term (the second integral) which penalizes the jump discontinuities of the displacement.

The lower semicontinuous envelope $\overline{F}$ is studied, with respect to the $L^1(\Omega;\mathbf{R}^n)$-topology, on the space $P(\Omega)$ of the functions of bounded deformation with distributional divergence in $L^2(\Omega)$ (F is extended with value $+\infty$ on the whole $P(\Omega)$). The following integral representation is proved:

$$\overline{F}(u) = \int_\Omega (\varphi(\mathcal{E}^D u) + \frac{\kappa}{2}(\mathrm{div} u)^2)dx + \int_\Omega \varphi^\infty(\frac{E^D_s u}{|E^D_s u|})\,|E^D_s u|, \qquad u \in P(\Omega),$$

where φ is a convex function with linear growth at infinity. Now Eu is a measure, $\mathcal{E}^D u$ represents the density of the absolutely continuous part of $E^D u$, while $E^D_s u$ denotes the singular part and φ^∞ the recession function of φ.

Finally, we show that $\overline{F}$ coincides with the functional which intervenes in the minimum problem for the displacement in the theory of Hencky's plasticity with Tresca's yield conditions.

Key words. integral functionals, lower semicontinuity, relaxation, functions of bounded deformation, plasticity

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0. Introduction

In this paper we obtain a relaxation result which leads to a classical functional in the theory of Hencky's plasticity.

Let Ω be a bounded domain in $\mathbf{R}^3$ and let $u\colon\Omega \to \mathbf{R}^3$ represent the displacement field of an elastic perfectly plastic body occupying the domain Ω in unstrained position. Denote by $\sigma\colon\Omega \to \mathrm{M}^{\mathrm{sym}}$ the corresponding stress tensor field referred to the configuration Ω ($\mathrm{M}^{\mathrm{sym}}$ stands for the space of 3×3 symmetric real matrices). Let $Eu = \frac{1}{2}(Du + Du^T)$ be the linearized strain tensor and $E^D u$ its deviator: $E^D u = Eu - \frac{1}{3}(\mathrm{div} u)I$, where I is the identity matrix. Then the variational

[1] Dipartimento di Elettronica per l'Automazione, Università di Brescia, via Valotti 9, I-25060 Brescia, Italy.

[2] Dipartimento di Matematica, Università di Parma, via M.D'Azeglio 85/A, I-43100 Parma, Italy.

[3] Dipartimento di Matematica, Università di Pavia, via Abbiategrasso 215, I-27100 Pavia, Italy.

formulation of the displacement and stress problems in the theory of Hencky's plasticity (see [24], [12], [20]) involves respectively the functionals

$$\int_\Omega \psi(Eu(x))dx\,, \qquad \int_\Omega \psi^*(\sigma(x))dx\,,$$

where ψ and ψ^* are non-negative convex functions conjugate each other (in the duality theory of Fenchel-Moreau), and

$$\psi^*(\sigma) \;=\; \begin{cases} \frac{1}{4\mu}|\sigma^D|^2 + \frac{1}{18\kappa}(\operatorname{tr}\sigma)^2\,, & \text{if } \sigma\in K \\ +\infty\,, & \text{otherwise}\,. \end{cases}$$

Here $\operatorname{tr}\sigma = \sum_i \sigma_{ii}$, $\sigma^D = \sigma - \frac{1}{3}(\operatorname{tr}\sigma)\,I$, $\kappa = \lambda + \frac{2}{3}\mu$ is the bulk modulus of the material, λ and μ are the Lamè coefficients and K is a closed convex subset of $\mathrm{M}^{\mathrm{sym}}$ describing the elastic zone for the stress. We mention two classical examples of the set K:

$$\text{(0.1)}\qquad \begin{aligned} K &= \{\sigma\in \mathrm{M}^{\mathrm{sym}} : |\sigma^D| \le \sqrt{2}k\} \qquad \text{(von Mises' model)}\\ K &= \{\sigma\in \mathrm{M}^{\mathrm{sym}} : \lambda_M(\sigma)-\lambda_m(\sigma) \le C\} \qquad \text{(Tresca's model)}, \end{aligned}$$

where k and C are fixed positive constants and $\lambda_M(\sigma)$ and $\lambda_m(\sigma)$ denote the maximum and the minimum eigenvalue of σ. Both these convex sets are of the form $K^D \oplus \mathbf{R}I$, where K^D is the orthogonal projection of K onto the space $\mathrm{M}^{\mathrm{sym}}_0$ of the matrices with null trace. This decomposition allows us to write the functional of the displacement problem as

$$\text{(0.2)}\qquad \int_\Omega (\varphi(E^D u) + \frac{\kappa}{2}(\mathrm{div} u)^2)dx\,,$$

where $\varphi\colon \mathrm{M}^{\mathrm{sym}}_0 \to [0,+\infty[$ is the convex function in duality with

$$\text{(0.3)}\qquad \varphi^*(\sigma) \;=\; \begin{cases} \frac{1}{4\mu}|\sigma|^2\,, & \text{if } \sigma\in K\cap \mathrm{M}^{\mathrm{sym}}_0\,, \\ +\infty\,, & \text{otherwise} \end{cases} \qquad (\sigma\in \mathrm{M}^{\mathrm{sym}}_0)\,.$$

Let us notice that the application of the direct method of the Calculus of Variations to the minimum problem for the displacement, i.e., involving the functional (0.2), requires a space where the functional is coercive and lower semicontinuous. Since φ grows only linearly as $|\xi^D|\to+\infty$ (note that $\varphi^*=+\infty$ outside $K\cap\mathrm{M}^{\mathrm{sym}}_0$) and no Korn's inequality holds on $W^{1,1}(\Omega;\mathbf{R}^3)$ (see [19], [18]), to gain coerciveness the definition of the functional must be extended to account for displacements whose strains are merely measures. This has led, in the general n-dimensional case, to the introduction of the spaces ([19], [23], [25], [26], [18])

$$\begin{aligned} BD(\Omega;\mathbf{R}^n) &= \{u\in L^1(\Omega;\mathbf{R}^n) : Eu \text{ bounded (matrix-valued) Radon measure}\}\,,\\ P(\Omega) &= \{u\in BD(\Omega;\mathbf{R}^n) : \mathrm{div} u\in L^2(\Omega)\}\,. \end{aligned}$$

The natural $L^1(\Omega;\mathbf{R}^3)$-lower semicontinuous extension to $P(\Omega)$ of (0.2) turns out to be ([3])

$$\text{(0.4)}\qquad \int_\Omega (\varphi(\mathcal{E}^D u) + \frac{\kappa}{2}(\mathrm{div} u)^2)dx + \int_\Omega \varphi^\infty(\frac{E^D_s u}{|E^D_s u|})\,|E^D_s u|\,,$$

where $E^D u = \mathcal{E}^D u \mathcal{L}^n + E^D_s u$ is the Lebesgue decomposition of $E^D u$, $|E^D_s u|$ denotes the total variation of $E^D_s u$, $\frac{E^D_s u}{|E^D_s u|}$ stands for the Radon-Nikodym derivative, and $\varphi^\infty(\xi) = \lim_{t\to+\infty} \dfrac{\varphi(t\xi)}{t}$ is the recession function of φ.

We shall propose a different approach to problems involving the functional (0.4) introducing a somehow simpler model. Consider a subspace $S(\Omega)$ of $P(\Omega)$ whose elements are allowed to have "singularities" in the form of tangential jump discontinuities along surfaces (the possible slippage surfaces of the material). It may be natural the attempt to obtain a functional of type (0.4) by relaxation of the following

$$F(u) = \int_\Omega (\mu|\mathcal{E}^D u|^2 + \frac{\kappa}{2}(\mathrm{div} u)^2)dx + c\int_{S_u} |u^+ - u^-| d\mathcal{H}^2 , \qquad u \in S(\Omega), \tag{0.5}$$

where $\mathcal{H}^2$ is the 2-dimensional Hausdorff measure, S_u is the jump set of u, u^+ and u^- are the traces of u on both sides of S_u, and c is a positive constant. Indeed, the volume integral in the definition of F represents the (linearized) strain energy of an elastic material corresponding to a displacement u from the reference configuration Ω. The second term is a surface integral which takes into account the possible sliding of the material. While F models the microscopic form of the energy, from a macroscopic point of view, i.e., as far as minimum problems are concerned, we can equivalently consider its lower semicontinuous envelope $\overline{F}$ with respect to a suitable topology, which turns out to be the $L^1(\Omega;\mathbf{R}^3)$ topology.

In this paper we show, in the general n-dimensional case (Theorem 2.1), how in the relaxed functional $\overline{F}$ the effect of the volume and surface terms in (0.5) combine giving rise to a typical plastic behaviour: slippage is microscopically preferred to large strains $|\mathcal{E}^D u|$ and $\overline{F}$ can be written as

$$\overline{F}(u) = \int_\Omega (\phi(\mathcal{E}^D u) + \frac{\kappa}{2}(\mathrm{div} u)^2)dx + \int_\Omega \phi^\infty(\frac{E^D_s u}{|E^D_s u|})\,|E^D_s u| , \qquad u \in P(\Omega) ,$$

where ϕ is a convex function with linear growth at infinity. We observe (Theorem 2.9) that $\overline{F}$ can also be considered as the limit (in the sense of Γ-convergence) of the following sequence of functionals:

$$F_h(u) = F(u) + \varepsilon_h \mathcal{H}^{n-1}(S_u) ,$$

where (ε_h) is a sequence of positive numbers converging to 0. Such a singular perturbation approach may lead to a choice criterion among minima of boundary value problems involving the functional $\overline{F}$ (see [5]).

The relationship of the functional $\overline{F}$ with the classical functionals (0.4) is further studied in Section 3. It turns out that this relaxation (or limiting) procedure recovers exactly the functional (0.4) when adopting Tresca's yielding model (see Corollary 3.2). Indeed, the integrand function ϕ coincides with the function φ of (0.4) when the convex set K is the second one displayed in (0.1), with $C = 2c$. Let us remark that von Mises' model cannot be recovered following a similar approach, since the corresponding energy density φ is not determined by its behaviour on rank-one matrices (see Remark 2.3(ii)).

For similar results about functionals defined on spaces of functions of bounded variation, also in a non-linear setting, we refer to [5] and [6]. The functional in (0.4) differs from these also in that it satisfies different growth conditions in different directions.

1. Notation and preliminaries

Notation. Throughout this paper n is a fixed positive integer, $n \geq 2$. Given $m \in \mathbf{N}$, the elements of $\mathbf{R}^m$ are usually considered as column vectors, $\langle \cdot, \cdot \rangle$ denotes the scalar product on $\mathbf{R}^m$; and $|\cdot|$ will be the usual euclidean norm. We shall denote by $\mathrm{M}^{n\times n}$ the vector space of the $n \times n$ real matrices; given $\xi \in \mathrm{M}^{n\times n}$ and $x \in \mathbf{R}^n$, $\xi \cdot x$ is the vector of $\mathbf{R}^n$ with components $(\xi \cdot x)_i = \sum_{k=1}^n \xi_{ik} x_k$, $i \in \{1, \dots, n\}$. We shall identify $\mathrm{M}^{n\times n}$ with $\mathbf{R}^{n^2}$. The set of the symmetric $n \times n$ real matrices will be denoted by $\mathrm{M}^{\mathrm{sym}}$, while $\mathrm{M}^{\mathrm{sym}}_0$ is the subspace of $\mathrm{M}^{\mathrm{sym}}$ consisting of the matrices ξ with null trace; i.e., $\mathrm{tr}\,\xi = \sum_i \xi_{ii} = 0$. If a and b are in $\mathbf{R}^n$, the *tensor product* $a \otimes b \in \mathrm{M}^{n\times n}$ is the matrix whose entries are $a_i b_j$ with $i, j = 1, \dots, n$. The *symmetric tensor product* is defined by $a \odot b = \frac{1}{2}(a \otimes b + b \otimes a)$. Remark that $|a \otimes b| = |a||b|$, $\mathrm{tr}\,(a \otimes b) = \mathrm{tr}\,(a \odot b) = \langle a, b \rangle$, and $|a \odot b| = \frac{1}{\sqrt{2}}|a||b|$ if $\langle a, b \rangle = 0$. For any matrix $\xi \in \mathrm{M}^{n\times n}$ we define the *deviator* of ξ by

$$\xi^D = \xi - \frac{1}{n}(\mathrm{tr}\,\xi)\, I\,,$$

where I denotes the identity matrix.

Let Ω be a bounded open subset of $\mathbf{R}^n$ with Lipschitz boundary. We shall denote by $\mathcal{A}(\Omega)$ the family of the open subsets of Ω and by $\mathcal{B}(\Omega)$ the family of its Borel subsets. The Lebesgue measure on $\mathbf{R}^n$ will be denoted by $\mathcal{L}^n$, and the $(n-1)$-dimensional Hausdorff measure in $\mathbf{R}^n$ by $\mathcal{H}^{n-1}$. If $B \subseteq \mathbf{R}^n$ is a Borel set, we shall also use the notation $|B|$ for $\mathcal{L}^n(B)$. The notation *a.e.* stands for almost everywhere with respect to the Lebesgue measure, unless otherwise specified.

Given $p \in \mathbf{R}$, $1 \leq p < +\infty$, we shall use standard notations for the Lebesgue and Sobolev spaces $L^p(\Omega; \mathbf{R}^n)$ and $W^{1,p}(\Omega; \mathbf{R}^n)$, respectively. When dealing with scalar functions defined on Ω, we shall drop the target space $\mathbf{R}^n$ in the notation, and write just $L^p(\Omega)$, $W^{1,p}(\Omega)$ and the like.

If $m \geq 1$ is an integer and $f\colon \mathbf{R}^m \to [0, +\infty]$ is a convex function, we define $f^\infty\colon \mathbf{R}^m \to [0, +\infty]$, the *recession function* of f, by setting

$$f^\infty(\xi) = \lim_{t \to +\infty} \frac{f(t\xi)}{t}\,.$$

One can prove easily that this limit exists for all $\xi \in \mathbf{R}^m$; moreover, f^∞ is a Borel function which is convex and positively homogeneous of degree one.

Given a vector-valued measure μ on Ω, we adopt the notation $|\mu|$ for its total variation. If E is a μ-measurable set, we define the measure $\mu \llcorner E$ by setting $(\mu \llcorner E)(A) = \mu(E \cap A)$. If μ is a (vector-valued) measure on Ω, and f is a positively homogeneous function of degree one we shall write $\int_\Omega f(\mu)$ in place of $\int_\Omega f(\frac{d\mu}{d|\mu|})d|\mu|$, where $\frac{d\mu}{d|\mu|}$ stands for the Radon-Nikodym derivative (sometimes also denoted by $\frac{\mu}{|\mu|}$).

Functions of bounded deformation. We say that $u \in L^1(\Omega; \mathbf{R}^n)$ is a *function of bounded deformation*, and we write $u \in BD(\Omega; \mathbf{R}^n)$, if for any $i, j = 1, \dots, n$

$$E_{ij} u = \frac{1}{2}\big(D_j u^i + D_i u^j\big)$$

is a (Radon) measure with finite total variation in Ω. We denote by Eu the $\mathrm{M}^{n\times n}$-valued measure whose entries are $E_{ij}u$. $BD(\Omega;\mathbf{R}^n)$ is a Banach space under the norm

$$\|u\|_{BD(\Omega;\mathbf{R}^n)} = \|u\|_{L^1(\Omega;\mathbf{R}^n)} + |Eu|(\Omega)\,.$$

For $u \in BD(\Omega;\mathbf{R}^n)$ we shall consider the decomposition

$$Eu = \mathcal{E}u\,\mathcal{L}^n + E_s u$$

of the measure Eu in its absolutely continuous and singular parts with respect to the Lebesgue measure. We shall also use the space $P(\Omega)$ of functions of bounded deformation whose distributional divergence is absolutely continuous with respect to the Lebesgue measure with square summable density:

$$P(\Omega) = \{u \in BD(\Omega;\mathbf{R}^n) : \mathrm{div}u \in L^2(\Omega)\}.$$

Note that if $u \in P(\Omega)$ then $E^D_s u = E_s u$; hence $E^D u = \mathcal{E}^D u\,\mathcal{L}^n + E_s u$.

Relaxation. Let $F : X \to \overline{\mathbf{R}}$ be a functional on a topological space (X,τ). The *relaxed functional* $\overline{F}$ of F, or *lower semicontinuous envelope* of F, with respect to the τ-topology is the greatest τ-lower semicontinuous functional less than or equal to F. If τ is first countable, then

$$\overline{F}(x) = \inf\bigl\{\liminf_{h\to\infty} F(x_h) : x_h \to x \text{ in } X\bigr\}$$

and the infimum is attained. For a general treatment of this subject we refer to the books by Buttazzo [7] and by Dal Maso [9].

Γ-convergence. Let (X,d) be a metric space and let $F_h\colon X \to \overline{\mathbf{R}}$ be a sequence of functionals on X.

We say that (F_h) Γ-converges to F, if the following conditions are satisfied:

(i) for every $x \in X$ and for every sequence (x_h) in X such that $x_h \to x$, we have $F(x) \le \liminf_{h\to\infty} F_h(x_h)$;

(ii) for every $x \in X$ there exists a sequence (x_h) in X such that $x_h \to x$ and $F(x) = \lim_{h\to\infty} F_h(x_h)$.

For a complete treatment of the subject we refer to [9]. Here we only recall that under suitable coercivity conditions this kind of variational convergence guarantees the convergence of the minimum values of the functionals.

Some additional results.

Proposition 1.1. *The set $\mathcal{S} = \{a \odot b : a, b \in \mathbf{R}^n, \langle a,b\rangle = 0\}$ is a closed subset of $\mathrm{M}^{\mathrm{sym}}_0$ which spans $\mathrm{M}^{\mathrm{sym}}_0$. Moreover, $\mathcal{S}$ is a proper subset of $\mathrm{M}^{\mathrm{sym}}_0$ unless $n = 2$.*

Proof. Take a sequence (ξ_h) in $\mathcal{S}$ converging to a matrix $\xi \neq 0$. Let $\xi_h = a_h \odot b_h$, a_h, $b_h \in \mathbf{R}^n$. We can assume that a_h, $b_h \neq 0$ and that the sequences $(\frac{a_h}{|a_h|})$, $(\frac{b_h}{|b_h|})$ converge to some vectors a and b, respectively. Hence

$$\xi_h = |\xi_h|\frac{a_h \odot b_h}{|a_h \odot b_h|} = \sqrt{2}|\xi_h|\frac{a_h}{|a_h|} \odot \frac{b_h}{|b_h|} \quad \to \quad \sqrt{2}|\xi|a \odot b\,.$$

Therefore ξ must be equal to $\sqrt{2}|\xi| a \odot b \in \mathcal{S}$. This proves that $\mathcal{S}$ is closed.

A basis of $\mathrm{M}^{\mathrm{sym}}_0$ is given by the elements $e_i \odot e_j$ for $i \neq j$ and $(e_i + e_{i+1}) \odot (e_i - e_{i+1})$ for $i = 1, \dots, n-1$.

If $n = 2$ and

$$\xi = \begin{pmatrix} a & b \\ b & -a \end{pmatrix} \quad \in \mathrm{M}^{\mathrm{sym}}_0 ,$$

then $\xi = (a, b - \sqrt{a^2 + b^2}) \odot (1, \frac{b + \sqrt{a^2+b^2}}{a})$ if $a \neq 0$, while $\xi = 2b e_1 \odot e_2$ if $a = 0$.

If $n = 3$ it is easy to see that, for example, the matrix $\xi = (a_{ij})$ with $a_{ij} = 1 - \delta_{ij}$ does not belong to $\mathcal{S}$. □

Theorem 1.2. (Integral representation theorem; see [8] Theorem 1.1) *Let us consider a functional* $F\colon W^{1,2}(\Omega; \mathbf{R}^n) \times \mathcal{A}(\Omega) \to [0, +\infty[$ *satisfying the following properties, for every* $A \in \mathcal{A}(\Omega)$ *and* $u, v \in W^{1,2}(\Omega; \mathbf{R}^n)$*:*

(i) *(measure property)* $F(u, \cdot)$ *is the restriction to* $\mathcal{A}(\Omega)$ *of a Borel measure;*
(ii) *(locality property)* $F(u, A) = F(v, A)$ *whenever* $u = v$ *a.e. on* A*;*
(iii) *(growth condition) there exist* $a \in L^1(\Omega)$ *and* $b \in \mathbf{R}$ *such that*

$$F(u, A) \;\le\; \int_A (a(x) + b|Du|^2) dx ;$$

(iv) *(translation invariance)* $F(u + c, A) = F(u, A)$ *for every* $c \in \mathbf{R}^n$*;*
(v) *(semicontinuity)* $F(\cdot, A)$ *is sequentially lower semicontinuous with respect to the weak topology of* $W^{1,2}(\Omega; \mathbf{R}^n)$.

Then there exists a Carathéodory function $\psi\colon \Omega \times \mathrm{M}^{n\times n} \to [0, +\infty[$*, convex in the second variable for a.e.* $x \in \Omega$*, such that the integral representation*

$$F(u, A) = \int_A \psi(x, Du(x))\, dx$$

holds for every $u \in W^{1,2}(\Omega; \mathbf{R}^n)$ *and for every* $A \in \mathcal{A}(\Omega)$.

Theorem 1.3. *Let* $m \ge 1$ *and* $f : \mathbf{R}^m \to \mathbf{R}$ *be a convex function such that* $0 \le f(\xi) \le c(1 + |\xi|)$ *for all* $\xi \in \mathbf{R}^m$ *and a suitable constant* $c \ge 0$*. Let* (μ_h) *be a sequence of* $\mathbf{R}^m$*-valued Radon measures on* Ω *converging weakly to an* $\mathbf{R}^m$*-valued Radon measure* μ *in the sense that*

$$\int_\Omega \psi\, d\mu_h \quad \to \quad \int_\Omega \psi\, d\mu \qquad \text{for every } \psi \in \mathcal{C}^0_0(\Omega) .$$

Assume that the sequence $\big(|\mu_h|(\Omega)\big)$ *is bounded. Then we have*

$$\int_\Omega f\Big(\frac{d\mu}{d\mathcal{L}^n}\Big)\, dx + \int_\Omega f^\infty(\mu^s) \le \liminf_{h\to\infty} \Big(\int_\Omega f\Big(\frac{d\mu_h}{d\mathcal{L}^n}\Big)\, dx + \int_\Omega f^\infty(\mu^s_h) \Big) . \tag{1.1}$$

(here we use the notation λ^s *to denote the singular part of the measure* λ *with respect to* $\mathcal{L}^n$*). Moreover, if* $|\mu|(\Omega) = \lim_{h\to\infty} |\mu_h|(\Omega)$ *then*

$$\int_\Omega f\Big(\frac{d\mu}{d\mathcal{L}^n}\Big)\, dx + \int_\Omega f^\infty(\mu^s) = \lim_{h\to\infty} \Big(\int_\Omega f\Big(\frac{d\mu_h}{d\mathcal{L}^n}\Big)\, dx + \int_\Omega f^\infty(\mu^s_h) \Big) . \tag{1.2}$$

Proof. In the case of f positively homogeneous of degree one, inequality (1.1) is proven in [16] Theorem 3 or [21] Theorem 2, while (1.2) is proven in [21] Theorem 3. In the general case, it is possible to reduce to f positively homogeneous of degree one by introducing the auxiliary function $\overline{f}: \mathbf{R}^m \times [0,+\infty[\to [0,+\infty[$ defined by

$$\overline{f}(\xi,t) = \begin{cases} tf(\xi/t) & \text{if } t>0 \\ f^\infty(\xi) & \text{if } t=0 \end{cases}$$

(see [16] Theorem 2′ or [4] Lemma 1.1). ∎

2. The main result

Our first step will be the definition of a space of "piecewise C^1-functions". We define the space $S(\Omega)$ as the set of all functions $u \in L^\infty(\Omega;\mathbf{R}^n)$ satisfying the following conditions (i), (ii) and (iii):

(i) there exists a closed set K with $\mathcal{H}^{n-1}(K\cap\Omega) < +\infty$ and $K = \bigcup_j \overline{S_j}$, where $(S_j)_{j\in J}$ is a finite family of C^1-hypersurfaces such that $\mathcal{H}^{n-1}((\overline{S_j}\setminus S_j)\cap\Omega) = 0$, and $u \in C^1(\Omega\setminus K)$;

(ii) $\int_{\Omega\setminus K} |Du| dx < +\infty, \quad \int_{\Omega\setminus K} (\mathrm{div} u)^2\, dx < +\infty$.

Before stating condition (iii) let us make some remarks. The set K represents (a set containing) the set of non-differentiability points for u. Let (S_j) be as in (i), and for every $j\in J$ let ν_j be a unit normal field to S_j. It is not difficult to see that the limits

$$u_j^+(x) = \lim_{t\to 0+} u(x+t\nu_j(x)), \qquad u_j^-(x) = \lim_{t\to 0-} u(x+t\nu_j(x)),$$

exist $\mathcal{H}^{n-1}$-a.e. on S_j (here we use the fact that, by the implicit function theorem, $\nu_i = \pm\nu_j$ $\mathcal{H}^{n-1}$-a.e. on $S_i\cap S_j$ whenever $i,j\in J$). The *set of jump points* or discontinuity points of u,

$$S_u = \bigcup_j \Big\{x\in S_j:\ u_j^+(x) \neq u_j^-(x)\Big\},$$

turns out to be well-defined up to sets of $\mathcal{H}^{n-1}$ measure zero, and is independent of the choice and representation of K.

We can now define the *unit normal field* ν_u to S_u and the *traces* u^+ and u^- of u on both sides of S_u as

$$\nu_u = \nu_j, \qquad u^\pm = u_j^\pm \qquad \mathcal{H}^{n-1}\text{-a.e. on } S_j.$$

The triplet $(u^+(x),u^-(x),\nu_u(x))$ is uniquely determined, for $\mathcal{H}^{n-1}$-a.e. $x\in S_u$, up to a change of sign of ν_u and an interchange between u^+ and u^-.

Finally, we can state condition (iii) of the definition of $S(\Omega)$:

(iii) $$\langle u^+(x)-u^-(x),\nu_u(x)\rangle = 0 \qquad \text{for } \mathcal{H}^{n-1}\text{-a.e. } x\in S_u.$$

$S(\Omega)$ turns out to be a vector space; moreover, if $\varphi \in C^1(\overline{\Omega})$ and $u \in S(\Omega)$, then $\varphi u \in S(\Omega)$. It is not difficult to prove (see, for instance, [11] Lemma 2.3) that if $u \in S(\Omega)$, then the singular part of the measure strain Eu is

$$E_s u = E_s^D u = (u^+ - u^-) \odot \nu_u \, \mathcal{H}^{n-1} \llcorner S_u \, ,$$

and $|(u^+ - u^-) \odot \nu_u| = \frac{1}{\sqrt{2}}|u^+ - u^-|$. In particular it follows that $S(\Omega) \subseteq P(\Omega)$.

In this section we will characterize (Theorem 2.1) the lower semicontinuous envelope, with respect to the $L^1(\Omega; \mathbf{R}^n)$-topology, of the functional $F : P(\Omega) \to [0, +\infty]$ defined as follows:

$$F(u) = \begin{cases} \displaystyle\int_\Omega (\frac{1}{2}|\mathcal{E}^D u|^2 + \alpha(\mathrm{div} u)^2)\, dx + \beta \int_{S_u} |(u^+ - u^-) \odot \nu_u| d\mathcal{H}^{n-1} & \text{if } u \in S(\Omega), \\ +\infty \, , & \text{otherwise.} \end{cases} \tag{2.1}$$

Let us remark that the singular set of a function $u \in S(\Omega)$ has finite $(n-1)$-dimensional measure, so that $\mathcal{E}^D u$ and $\mathrm{div} u$ are defined a.e. on Ω, and we can write the volume integral as an integral over the whole Ω. In Theorem 2.9 we shall also provide an approximation of $\overline{F}$ by means of a sequence of functionals obtained by a suitable perturbation of F.

Let $f : \mathrm{M}_0^{\mathrm{sym}} \to [0, +\infty[$ be defined as follows:

$$\begin{aligned} f = \sup\{\psi : \mathrm{M}_0^{\mathrm{sym}} \to [0, +\infty[\text{ convex, } \psi(\xi) \le \tfrac{1}{2}|\xi|^2 \text{ for every } \xi \in \mathrm{M}_0^{\mathrm{sym}}, \\ \psi(a \odot b) \le \beta|a \odot b| \text{ if } a, b \in \mathbf{R}^n,\ \langle a, b\rangle = 0\} \, . \end{aligned} \tag{2.2}$$

Hence f is the convex envelope in $\mathrm{M}_0^{\mathrm{sym}}$ of the function $h : \mathrm{M}_0^{\mathrm{sym}} \to [0, +\infty[$ given by $h(\xi) = (\frac{1}{2}|\xi|^2) \wedge g(\xi)$, where

$$g : \mathrm{M}_0^{\mathrm{sym}} \to [0, +\infty] \, , \qquad g(\xi) = \begin{cases} \beta|\xi| & \text{if } \xi = a \odot b \text{ for some } a, b \in \mathbf{R}^n, \\ +\infty & \text{otherwise.} \end{cases} \tag{2.3}$$

Theorem 2.1. *Let F be the functional defined in* (2.1). *The lower semicontinuous envelope of F with respect to the $L^1(\Omega; \mathbf{R}^n)$-topology on $P(\Omega)$ is given by*

$$\overline{F}(u) = \int_\Omega (f(\mathcal{E}^D u) + \alpha(\mathrm{div} u)^2)\, dx + \int_\Omega f^\infty(E_s^D u) \tag{2.4}$$

for every $u \in P(\Omega)$, where f is the function in (2.2).

Remark 2.2. It is possible to give a weak definition of the functional F in (2.1), by replacing $S(\Omega)$ with a larger space of weakly differentiable functions, such as the space $P(\Omega) \cap SBV(\Omega; \mathbf{R}^n)$, where $SBV(\Omega; \mathbf{R}^n)$ is the space of *special functions of bounded variation* defined by De Giorgi and Ambrosio (see [10]). With this change Theorem 2.1

remains valid, and it can be proven following exactly the same proof given in the sequel, remarking that ([11] Lemma 2.3) we have $S(\Omega) \subseteq SBV(\Omega; \mathbf{R}^n)$. For the properties of SBV functions we refer to [1], [2]; for the general exposition of the theory of functions of bounded variation we refer to [14], [15], [27], and [28].

Remark 2.3. (i) It is easy to see that f grows linearly at infinity: apply Lemma 2.4 below and observe that f is majorized by the convex envelope of g, which turns out to be finite everywhere in view of Proposition 1.1, and positively homogeneous of degree one. In particular it follows that the relaxed functional $\overline{F}$ is finite on $P(\Omega)$. A more accurate estimate of f will be given in Proposition 3.3.

(ii) The proof of Theorem 2.1 fits also functionals of the form

$$F(u) = \begin{cases} \displaystyle\int_\Omega \left(\frac{1}{2}|\mathcal{E}^D u|^2 + \alpha(\mathrm{div} u)^2\right) dx + \int_{S_u} g\big((u^+ - u^-) \odot \nu_u\big) d\mathcal{H}^{n-1} & \\ & \text{if } u \in S(\Omega), \\ +\infty\,, & \text{otherwise,} \end{cases}$$

where g is a non-negative finite convex function positively homogeneous of degree one and verifying $\lambda|a||b| \le g(a \odot b)$ (λ strictly positive constant). In this case we have

$$f = \sup\{\psi : \mathrm{M}_0^{\mathrm{sym}} \to [0, +\infty[\text{ convex, } \psi(\xi) \le \tfrac{1}{2}|\xi|^2 \text{ for every } \xi \in \mathrm{M}_0^{\mathrm{sym}}, \\ \psi(a \odot b) \le g(a \odot b) \text{ if } a, b \in \mathbf{R}^n,\ \langle a, b\rangle = 0\}\,.$$

From this formula it is easy to see that no such f can equal the function φ given by von Mises' yielding model.

Theorem 2.1 will be proved in several steps.

<u>*Step 1*</u>: *Let* $F_1 \colon P(\Omega) \to [0, +\infty[$ *be the functional on the right-hand side of* (2.4). *We prove that* $F_1 \le \overline{F}$ *on* $P(\Omega)$.

Let us introduce the function

$$f_\beta : [0, +\infty[\to [0, +\infty[\qquad f_\beta(t) = \begin{cases} \frac{1}{2}t^2 & \text{if } 0 \le t \le \beta \\ \beta t - \frac{1}{2}\beta^2 & \text{if } t \ge \beta. \end{cases} \tag{2.5}$$

It is easy to see that $t \mapsto f_\beta(|t|)$ is the convex envelope of $t \mapsto (\frac{1}{2}t^2) \wedge (\beta|t|)$ on $\mathbf{R}$.

Lemma 2.4. *Let* f *be the function defined in* (2.2). *For every* $\xi \in \mathrm{M}_0^{\mathrm{sym}}$ *we have* $f(\xi) \ge f_\beta(|\xi|)$, *and*

$$f(a \odot b) = f_\beta(|a \odot b|)\,,$$

whenever $a, b \in \mathbf{R}^n$ *and* $\langle a, b \rangle = 0$.

Proof. Since $\xi \mapsto f_\beta(|\xi|)$ is a convex function on $\mathrm{M}_0^{\mathrm{sym}}$, and $f_\beta(|\xi|) \le h(\xi)$ for every $\xi \in \mathrm{M}_0^{\mathrm{sym}}$, then $f_\beta(|\xi|) \le f(\xi)$.

Let now a and b be as in the statement and $T = \{t(a \odot b) : t \in \mathbf{R}\}$. Clearly $f_{|T}$ is convex and $f(\xi) \le (\frac{1}{2}|\xi|^2) \wedge (\beta|\xi|)$ for every $\xi \in T$. Hence $f(\xi) \le f_\beta(|\xi|)$ for every $\xi \in T$. We conclude that $f(\xi) = f_\beta(|\xi|)$ if $\xi \in T$. $\square$

Proposition 2.5. *F_1 is lower semicontinuous on $P(\Omega)$ with respect to the $L^1(\Omega;\mathbf{R}^n)$-topology, and $F_1 \le F$. Hence, $F_1 \le \overline{F}$.*

Proof. Let (u_h) be a sequence in $P(\Omega)$ converging in $L^1(\Omega;\mathbf{R}^n)$ to a function $u \in P(\Omega)$. We have to prove that $F_1(u) \le \liminf_{h\to\infty} F_1(u_h)$. Since

$$f(\xi) \ \ge\ f_\beta(|\xi|) \ \ge\ \beta|\xi| - \frac{1}{2}\beta^2 \qquad \text{for every } \xi \in \mathrm{M}_0^{\mathrm{sym}},$$

it is not restrictive to assume that the sequence $(\int_\Omega |E^D u_h|)$ is bounded. Hence, we easily obtain that

$$\int_\Omega \psi\, E^D u_h \quad \to \quad \int_\Omega \psi\, E^D u \qquad \text{for every } \psi \in \mathcal{C}_0^0(\Omega).$$

By Theorem 1.3 it follows that

$$\begin{aligned}\int_\Omega f(\mathcal{E}^D u)\,dx + \int_\Omega f^\infty(E^D_s u) \\ \le \liminf_{h\to\infty}\Big(\int_\Omega f(\mathcal{E}^D u_h)\,dx + \int_\Omega f^\infty(E^D_s u_h)\Big).\end{aligned} \tag{2.6}$$

Moreover, since it is not restrictive to assume $\int_\Omega (\mathrm{div} u_h)^2\,dx$ bounded independently of h, we can suppose that $(\mathrm{div} u_h)$ converges weakly in $L^2(\Omega)$ to some function v. Hence we have $v = \mathrm{div} u$ and

$$\int_\Omega (\mathrm{div} u)^2 dx \ \le\ \liminf_{h\to\infty} \int_\Omega (\mathrm{div} u_h)^2 dx. \tag{2.7}$$

Inequalities (2.6) and (2.7) prove the lower semicontinuity of F_1.

Let us now prove that $F_1(u) \le F(u)$ for every $u \in P(\Omega)$. Clearly we can assume $u \in S(\Omega)$. Then

$$E^D_s u = (u^+ - u^-) \odot \nu_u \mathcal{H}^{n-1} \llcorner S_u, \qquad \langle (u^+ - u^-), \nu_u\rangle = 0 \quad \mathcal{H}^{n-1}\text{-a.e. on } S_u.$$

It follows, by Lemma 2.4, that

$$f^\infty\Big(\frac{E^D_s u}{|E^D_s u|}\Big) = f^\infty\Big(\frac{(u^+ - u^-)\odot \nu_u}{|(u^+ - u^-)\odot\nu_u|}\Big) = \beta;$$

hence, the desired inequality. ∎

We localize the functional F by defining for every open subset A of Ω

$$F(u,A) = \begin{cases} \displaystyle\int_A \Big(\frac{1}{2}|\mathcal{E}^D u|^2 + \alpha(\mathrm{div} u)^2\Big)\,dx + \beta\int_{A\cap S_u} |(u^+ - u^-)\odot \nu_u|\,d\mathcal{H}^{n-1} & \text{if } u \in S(\Omega), \\ +\infty, & \text{elsewhere on } P(\Omega). \end{cases}$$

Moreover, for every $u \in P(\Omega) \cap L^2(\Omega;\mathbf{R}^n)$ and $A \in \mathcal{A}(\Omega)$ we define

$$F_2(u,A) = \inf\big\{\liminf_{h\to\infty} F(u_h,A) : (u_h) \text{ in } P(\Omega)\cap L^2(\Omega;\mathbf{R}^n),\ u_h \to u \text{ in } L^2(\Omega;\mathbf{R}^n)\big\};$$

in other words, $F_2(\cdot,A)$ is the lower semicontinuous envelope of $F(\cdot,A)$ on $P(\Omega)\cap L^2(\Omega;\mathbf{R}^n)$ with respect to the $L^2(\Omega;\mathbf{R}^n)$-topology.

Step 2 : *In this step we use Theorem* 1.2 *to obtain an integral representation for* F_2 *on* $W^{1,2}(\Omega;\mathbf{R}^n)\times\mathcal{A}(\Omega)$ *(Proposition* 2.7*).*

Lemma 2.6. $F_2(u,\cdot)$ *is the restriction to* $\mathcal{A}(\Omega)$ *of a non-negative Borel measure on* Ω *for every* $u \in W^{1,2}(\Omega;\mathbf{R}^n)$.

Proof. Let $u \in W^{1,2}(\Omega;\mathbf{R}^n)$ be fixed.

Step (i): If A', A'', $B \in \mathcal{A}(\Omega)$ with $A' \subset\subset A''$ then

$$F_2(u, A'\cup B) \le F_2(u,A'') + F_2(u,B).$$

Let (u_h) and (v_h) be two arbitrary sequences in $P(\Omega)\cap L^2(\Omega;\mathbf{R}^n)$ converging to u in $L^2(\Omega;\mathbf{R}^n)$. Suppose that the limits

$$\lim_{h\to\infty} F(u_h,A''), \qquad \lim_{h\to\infty} F(v_h,B)$$

exist and are finite; in particular, we can assume that for every $h \in \mathbf{N}$, u_h, $v_h \in S(\Omega)\cap L^2(\Omega;\mathbf{R}^n)$, and for a suitable constant $M > 0$

$$\int_{A''} (\tfrac{1}{2}|\mathcal{E}^D u_h|^2 + \alpha(\mathrm{div} u_h)^2)dx \le M, \qquad \int_B (\tfrac{1}{2}|\mathcal{E}^D v_h|^2 + \alpha(\mathrm{div} v_h)^2)dx \le M.$$

The lemma will be proved if we show that

$$F_2(u,A'\cup B) \le \lim_{h\to\infty} F(u_h,A'') + \lim_{h\to\infty} F(v_h,B).$$

Fix $k \in \mathbf{N}$. Let $A_0 = A'$ and let $A_1,\dots,A_k$ be open sets in Ω with boundary of measure zero and satisfying the property

$$A' = A_0 \subset\subset A_1 \subset\subset \cdots \subset\subset A_k \subset\subset A''.$$

For every $i \in \{1,\dots,k\}$ let $\varphi_i \in \mathcal{C}_0^\infty(A_i)$ with $\varphi_i = 1$ on A_{i-1} and $0 \le \varphi_i \le 1$. Define

$$w_{h,i} = \varphi_i u_h + (1-\varphi_i)v_h \qquad (h\in\mathbf{N}, 1 \le i \le k).$$

Then $w_{h,i} \in S(\Omega)\cap L^2(\Omega;\mathbf{R}^n)$, and

$$\begin{aligned} E^D w_{h,i} &= \varphi_i E^D u_h + (1-\varphi_i)E^D v_h \\ &\qquad + (D\varphi_i)\odot(u_h - v_h) - \frac{1}{n}\langle D\varphi_i,(u_h-v_h)\rangle I. \end{aligned} \tag{2.8}$$

By setting $S_i = \overline{A}_i \setminus A_{i-1}$ $(1 \le i \le k)$ we have

$$
\begin{aligned}
F(w_{h,i}, A' \cup B) = & \int_{(A'\cup B)\cap A_{i-1}} (\frac{1}{2}|\mathcal{E}^D u_h|^2 + \alpha(\mathrm{div} u_h)^2)\, dx \\
& + \int_{B\setminus \overline{A_i}} (\frac{1}{2}|\mathcal{E}^D v_h|^2 + \alpha(\mathrm{div} v_h)^2)\, dx \\
& + \int_{B\cap S_i} (\frac{1}{2}|\mathcal{E}^D w_{h,i}|^2 + \alpha(\mathrm{div} w_{h,i})^2)\, dx + \beta \int_{A'\cup B} |E_s^D w_{h,i}|\,.
\end{aligned}
\tag{2.9}
$$

Note now that by (2.8) the last term on the right-hand side of (2.9) becomes, up to the factor β,

$$
\begin{aligned}
\int_{A'\cup B} |E_s^D w_{h,i}| = & \int_{A'\cup B} |\varphi_i E_s^D u_h + (1-\varphi_i) E_s^D v_h| \\
\le & \int_{(A'\cup B)\cap A_{i-1}} |E_s^D u_h| + \int_{B\setminus \overline{A_i}} |E_s^D v_h| + \int_{B\cap S_i} (|E_s^D u_h| + |E_s^D v_h|)\,;
\end{aligned}
$$

thus

$$
\int_{A'\cup B} |E_s^D w_{h,i}| \le \int_{A''} |E_s^D u_h| + \int_B |E_s^D v_h|\,. \tag{2.10}
$$

Let us now estimate the integral over $B \cap S_i$ in (2.9). By (2.8) we have

$$
\int_{B\cap S_i} (\frac{1}{2}|\mathcal{E}^D w_{h,i}|^2 + \alpha(\mathrm{div} w_{h,i})^2)\, dx \le c\Big(\int_{B\cap S_i} z_h dx + N_k^2 \int_{B\cap S_i} |u_h - v_h|^2 dx\Big)\,,
$$

where

$$
\begin{aligned}
z_h &= \frac{1}{2}|\mathcal{E}^D u_h|^2 + \frac{1}{2}|\mathcal{E}^D v_h|^2 + \alpha\big((\mathrm{div} u_h)^2 + (\mathrm{div} v_h)^2\big)\,,\\
N_k &= \sup\{|D\varphi_i(x)| : 1 \le i \le k, x \in \Omega\}\,,
\end{aligned}
$$

and $c > 0$ is a constant which depends only on n and α. Note that for every $h \in \mathbf{N}$ there exists $i_h \in \{1, \dots, k\}$ such that

$$
\int_{B\cap S_{i_h}} z_h dx \le \frac{1}{k} \int_{B\cap(\overline{A}_k \setminus A_0)} z_h dx \le 2\frac{M}{k}\,.
$$

Then

$$
\int_{B\cap S_{i_h}} (\frac{1}{2}|\mathcal{E}^D w_{h,i_h}|^2 + \alpha(\mathrm{div} w_{h,i_h})^2)\, dx \le c\Big(\frac{2M}{k} + N_k^2 \int_\Omega |u_h - v_h|^2 dx\Big)\,. \tag{2.11}
$$

From (2.9), (2.10), and (2.11) we obtain

$$
F(w_{h,i_h}, A' \cup B) \le F(u_h, A'') + F(v_h, B) + c\Big(\frac{2M}{k} + N_k^2 \int_\Omega |u_h - v_h|^2 dx\Big)\,;
$$

since the sequence (w_{h,i_h}) converges to u in $L^2(\Omega;\mathbf{R}^n)$, we have

$$F_2(u, A'\cup B) \le \lim_{h\to\infty} F(u_h, A'') + \lim_{h\to\infty} F(v_h, B) + \frac{2cM}{k}\,.$$

The conclusion is now immediate by taking the limit as k tends to ∞.

Step (ii): For any A, A_1, $A_2 \in \mathcal{A}(\Omega)$ we have

$$F_2(u,A_1) \le F_2(u,A_2) \qquad \text{if } A_1 \subseteq A_2 \tag{2.12}$$

$$F_2(u,A_1\cup A_2) \ge F_2(u,A_1)+F_2(u,A_2) \qquad \text{if } A_1\cap A_2=\emptyset \tag{2.13}$$

$$F_2(u,A) = \sup\{F_2(u,A') : A'\in\mathcal{A}(\Omega)\,, A'\subset\subset A\} \tag{2.14}$$

$$F_2(u,A_1\cup A_2) \le F_2(u,A_1)+F_2(u,A_2)\,. \tag{2.15}$$

Properties (2.12) and (2.13) can be easily proved. Let us show (2.14).

An approximation of u in $W^{1,2}(\Omega;\mathbf{R}^n)$ by means of a sequence of $C^1(\Omega;\mathbf{R}^n)$ functions gives that for every compact subset K of A

$$F_2(u, A\setminus K) \le \int_{A\setminus K} (\frac{1}{2}|\mathcal{E}^D u|^2 + \alpha(\mathrm{div} u)^2)\,dx\,.$$

Hence, for every $\varepsilon > 0$ we can choose K such that $F_2(u, A\setminus K) < \varepsilon$. Let A' and A'' be open sets verifying $K\subseteq A'\subset\subset A''\subset\subset A$. By Step (i) with $B = A\setminus K$ it turns out that

$$F_2(u,A) \le F_2(u,A'') + F_2(u,A\setminus K) \le F_2(u,A'')+\varepsilon\,.$$

By the arbitrariness of $\varepsilon > 0$, we get (2.14).

Let us prove (2.15). Given $\varepsilon > 0$, by (2.14) there exists $G \subset\subset A_1\cup A_2$ such that $F_2(u, A_1\cup A_2) - \varepsilon \le F_2(u,G)$. Let $A'\in\mathcal{A}(\Omega)$ with $A'\subset\subset A_1$ and $G\subseteq A'\cup A_2$. Then Step (i) ensures that

$$F_2(u,A_1\cup A_2)-\varepsilon \le F_2(u,A'\cup A_2) \le F_2(u,A_1)+F_2(u,A_2)\,.$$

Since $\varepsilon > 0$ is arbitrary, (2.15) holds.

Properties (2.12) through (2.15), valid for $F_2(u,\cdot)$, guarantee that $F_2(u,\cdot)$ is the restriction to $\mathcal{A}(\Omega)$ of a non-negative Borel measure on Ω (see [9] Theorem 14.23). □

Proposition 2.7. *There exists a convex function* $\psi : \mathrm{M}^{n\times n} \to \mathbf{R}$ *such that*

$$F_2(u,A) = \int_A \psi(Du)\,dx$$

for every $u\in W^{1,2}(\Omega;\mathbf{R}^n)$ *and for every* $A\in\mathcal{A}(\Omega)$.

Proof. Lemma 2.6 guarantees that condition (i) of Theorem 1.2 is satisfied. Let us now verify (ii) through (v).

(ii) Fix $A\in\mathcal{A}(\Omega)$ and $u,v\in W^{1,2}(\Omega;\mathbf{R}^n)$ with $u=v$ a.e. on A. Let (u_h) be a sequence in $P(\Omega)\cap L^2(\Omega)$ converging in $L^2(\Omega;\mathbf{R}^n)$ to u and such that

$$F(u_h, A) \quad\to\quad F_2(u,A)\,.$$

Given $A' \subset\subset A$, consider a function $\varphi \in \mathcal{C}_0^1(A)$ with $\varphi = 1$ on A' and $0 \leq \varphi \leq 1$. Define $w_h = \varphi u_h + (1-\varphi)v$. Then (w_h) converges to v in $L^2(\Omega;\mathbf{R}^n)$ and

$$F_2(v,A') \leq \liminf_{h\to\infty} F(w_h,A') = \liminf_{h\to\infty} F(u_h,A') \leq \liminf_{h\to\infty} F(u_h,A) = F_2(u,A).$$

Taking the supremum for $A' \subset\subset A$ (recall that $F_2(v,\cdot)$ is a measure) we conclude that $F_2(v,A) \leq F_2(u,A)$. The opposite inequality follows by exchanging the roles of u and v.

(iii) Clearly there exists a constant c such that $F(u,A) \leq c\int_A |Du|^2 dx$ for every $u \in C^1(\Omega;\mathbf{R}^n) \cap W^{1,2}(\Omega;\mathbf{R}^n)$ and $A \in \mathcal{A}(\Omega)$. Hence the same inequality holds for F_2.

(iv) The translation invariance of F_2 comes easily from the corresponding property for F.

(v) For every $A \in \mathcal{A}(\Omega)$ the function $F_2(\cdot,A)$ is, by definition, lower semicontinuous on $W^{1,2}(\Omega;\mathbf{R}^n)$ with respect to the $L^2(\Omega;\mathbf{R}^n)$-topology. Since $F_2(\cdot,A)$ is easily verified to be convex, we obtain the sequential lower semicontinuity with respect to the weak topology of $W^{1,2}(\Omega;\mathbf{R}^n)$, too.

By Theorem 1.2 there exists a Carathéodory function $\psi : \Omega \times \mathrm{M}^{n\times n} \to [0,+\infty[$ such that

$$F_2(u,A) = \int_A \psi(x,Du)\,dx$$

for every $u \in W^{1,2}(\Omega;\mathbf{R}^n)$ and $A \in \mathcal{A}(\Omega)$.

Finally, let us show that we can assume ψ to be independent of x. Let $B(x_0,r)$ and $B(y_0,r)$ be any pair of congruent balls contained in Ω. Since the integrand functions defining F are independent of x, the evaluation of F_2 by means of its definition gives $F_2(u_1,B(x_0,r)) = F_2(u_2,B(y_0,r))$, where $u_1{:}\,x \mapsto \xi\cdot x$ and $u_2{:}\,x \mapsto \xi\cdot(x+x_0-y_0)$. Thus, by the integral representation, we have

$$\int_{B(x_0,r)} \psi(x,\xi)\,dx = \int_{B(y_0,r)} \psi(x,\xi)\,dx$$

for every $\xi \in \mathrm{M}^{n\times n}$. This equality implies that $\psi(x_0,\xi) = \psi(y_0,\xi)$ at every pair of Lebesgue points of the function $\psi(\cdot,\xi)$. Letting ξ vary in a countable dense subset of $\mathrm{M}^{n\times n}$, and using the continuity of $\psi(x,\cdot)$ we get the existence of a set $N \subseteq \Omega$ with $|N| = 0$ such that $\psi(x,\xi) = \psi(y,\xi)$ for every $\xi \in \mathrm{M}^{n\times n}$ and for every x, $y \in \Omega \setminus N$. Therefore we can assume that ψ is independent of x. □

<u>Step 3</u>: *Here we prove that* $F_2(\cdot,\Omega) \leq F_1$ *on* $W^{1,2}(\Omega;\mathbf{R}^n)$.

To this aim it is enough to apply the previous step and the following proposition.

Proposition 2.8. *Let* ψ *be the function given by Proposition 2.7. For every* $\xi \in \mathrm{M}^{n\times n}$

$$\psi(\xi) \leq f\Big(\big(\frac{\xi+\xi^T}{2}\big)^D\Big) + \alpha(\mathrm{tr}\,\xi)^2.$$

Proof. Fix $\xi \in \mathrm{M}^{n\times n}$ and let $\widehat{\xi} = \frac{1}{2}(\xi + \xi^T)$. It turns out (see [22] Cor. 17.1.5), that

$$f(\widehat{\xi}^D) = \inf\Big\{\sum_{i=1}^N t_i h(\xi_i) : t_i \geq 0\,, \sum_{i=1}^N t_i = 1\,, \widehat{\xi}^D = \sum_{i=1}^N t_i \xi_i\,, \xi_i \in \mathrm{M}_0^{\mathrm{sym}}\Big\}\,,$$

where $N = \frac{n(n+1)}{2}$. Therefore, for every $k \in \mathbf{N}$ there exist $t_1^k, \cdots, t_N^k \geq 0$ with $\sum_{i=1}^N t_i^k = 1$ and $\xi_1^k, \cdots, \xi_N^k \in \mathrm{M}_0^{\mathrm{sym}}$, such that $\widehat{\xi}^D = \sum_{i=1}^N t_i^k \xi_i^k$ and

$$\sum_{i=1}^N t_i^k h(\xi_i^k) \quad \to \quad f(\widehat{\xi}^D) \qquad \text{as } k \to \infty. \tag{2.16}$$

Fix $k \in \mathbf{N}$; we can suppose that for a suitable integer $l_k \geq 0$

$$h(\xi_i^k) = \beta|\xi_i^k| < \frac{1}{2}|\xi_i^k|^2 \qquad \text{if and only if} \quad l_k + 1 \leq i \leq N\,. \tag{2.17}$$

Let us show that ξ_i^k $(1 \leq i \leq N)$ can be chosen in such a way that l_k is at most 1.

If $l_k > 1$, define

$$\eta^k = \sum_{i=1}^{l_k} (t_i^k/\lambda^k)\xi_i^k\,, \qquad \lambda^k = \sum_{i=1}^{l_k} t_i^k\,.$$

It turns out that $\widehat{\xi}^D = \lambda^k \eta^k + \sum_{i=l_k+1}^N t_i^k \xi_i^k$ and

$$\lambda^k h(\eta^k) \leq \lambda^k \frac{1}{2}|\eta^k|^2 \leq \lambda^k \sum_{i=1}^{l_k} \frac{t_i^k}{\lambda^k}\frac{1}{2}|\xi_i^k|^2 = \sum_{i=1}^{l_k} t_i^k h(\xi_i^k)\,.$$

This yields

$$\lambda^k h(\eta^k) + \sum_{i=l_k+1}^N t_i^k h(\xi_i^k) \leq \sum_{i=1}^N t_i^k h(\xi_i^k)\,,$$

thus allowing us to replace $\xi_1^k, \cdots, \xi_N^k$ with $\eta^k, \xi_{l_k+1}^k, \cdots, \xi_N^k$. Therefore the claim is proved.

Let us agree that $\xi_{l_k}^k = 0$ and $t_{l_k}^k = 0$ if $l_k = 0$. In view of (2.17) for every $i \in \{l_k + 1, \ldots, N\}$ we can write

$$\xi_i^k = a_i^k \odot b_i^k \qquad \text{for suitable } a_i^k,\ b_i^k \in \mathbf{R}^n \setminus \{0\} \text{ with } \langle a_i^k, b_i^k\rangle = 0\,.$$

Let us notice that if $a,\ b \in \mathbf{R}^n \setminus \{0\}$ and $\langle a, b\rangle = 0$ then the functions (here and in the sequel $[s]$ denotes the integer part of s)

$$v_j(x) = \frac{|b|}{j}[j\langle \frac{b}{|b|}, x\rangle]a \qquad (j \in \mathbf{N})$$

are piecewise constant, have zero divergence and converge uniformly on $\mathbf{R}^n$ to $(a\otimes b)\cdot x$. Therefore, there exists a sequence (j_k) of integers such that, setting

$$u_k(x) = t^k_{l_k}\xi^k_{l_k}\cdot x + \frac{1}{2}\sum_{i=l_k+1}^{N} t^k_i\Big(\frac{|a^k_i|}{j_k}[j_k\langle\frac{a^k_i}{|a^k_i|},x\rangle]b^k_i + \frac{|b^k_i|}{j_k}[j_k\langle\frac{b^k_i}{|b^k_i|},x\rangle]a^k_i\Big) + \frac{\xi-\xi^T}{2}\cdot x + \frac{1}{n}(\operatorname{tr}\xi)\,x\,,$$

the sequence (u_k) converges uniformly on $\mathbf{R}^n$ to $x\mapsto \widehat{\xi}^D\cdot x + \frac{\xi-\xi^T}{2}\cdot x + \frac{1}{n}(\operatorname{tr}\xi)x = \xi\cdot x$ and $\operatorname{div}u_k = \operatorname{tr}\xi \in L^\infty(\mathbf{R}^n)$. The discontinuity set of u_k is $S_{u_k}\subseteq \bigcup_{i=l_k+1}^N(S^k_i\cup T^k_i)$, where $S^k_i = \{x\in\mathbf{R}^n : \langle x, \frac{a^k_i}{|a^k_i|}\rangle j_k \in \mathbf{Z}\}$ and $T^k_i = \{x\in\mathbf{R}^n : \langle x, \frac{b^k_i}{|b^k_i|}\rangle j_k \in \mathbf{Z}\}$. Moreover,

$$(u^+_k - u^-_k)\odot\nu_{u_k} = \frac{1}{2}\sum_{i=l_k+1}^{N} 1_{S^k_i\cup T^k_i}\frac{t^k_i}{j_k}a^k_i\odot b^k_i \qquad \mathcal{H}^{n-1}\text{-a.e. on } S_{u_k}\,,$$

where $1_{S^k_i\cup T^k_i}$ denotes the characteristic function of the set $S^k_i\cup T^k_i$.

Let B be an open ball contained in Ω; in the sequel it will not be restrictive to assume that $|B|=1$. Then

$$\begin{aligned} F(u_k,B) &\le \frac{1}{2}(t^k_{l_k})^2|\xi^k_{l_k}|^2 + \alpha(\operatorname{tr}\xi)^2 \\ &\quad+\beta\sum_{i=l_k+1}^{N} t^k_i|a^k_i\odot b^k_i|\frac{\mathcal{H}^{n-1}(S^k_i\cap B)+\mathcal{H}^{n-1}(T^k_i\cap B)}{2j_k} \\ &\le \sum_{i=1}^{N} t^k_i h(\xi^k_i) + \alpha(\operatorname{tr}\xi)^2 + \sigma_k\,, \end{aligned} \tag{2.18}$$

where $\sigma_k = \beta\sum_{i=l_k+1}^{N} t^k_i|a^k_i\odot b^k_i|\Big(\frac{\mathcal{H}^{n-1}(S^k_i\cap B)+\mathcal{H}^{n-1}(T^k_i\cap B)}{2j_k} - 1\Big)$. Take now into account that

$$\lim_{k\to\infty}\frac{\mathcal{H}^{n-1}(S^k_i\cap B)}{j_k} = \lim_{k\to\infty}\frac{\mathcal{H}^{n-1}(T^k_i\cap B)}{j_k} = |B| = 1$$

and $\sum_{i=1}^N t^k_i h(\xi^k_i)$ is bounded independently of k (recall (2.16)). From this we deduce that $\lim\limits_{k\to\infty}\sigma_k = 0$.

By the definition of F_2, Proposition 2.7, (2.18) and (2.16) we conclude that

$$\psi(\xi) \le \liminf_{k\to\infty} F(u_k,B) \le f(\widehat{\xi}^D) + \alpha(\operatorname{tr}\xi)^2\,. \qquad \square$$

<u>Step 4</u>: *In this step, which concludes the proof of Theorem* 2.1, *we prove the equality* $\overline{F}(u) = F_1(u)$ *for an arbitrary* $u\in P(\Omega)$.

Fix $u \in P(\Omega)$. By Theorem A.2 in [17] there exists a sequence (u_h) in $\mathcal{C}^\infty(\overline{\Omega};\mathbf{R}^n)$ such that

$$\begin{aligned} u_h &\to u \quad \text{in } L^1(\Omega;\mathbf{R}^n), \\ \int_\Omega |\mathcal{E}^D u_h|\,dx &\to \int_\Omega |E^D u| \\ \int_\Omega (\mathrm{div} u_h)^2 dx &\to \int_\Omega (\mathrm{div} u)^2 dx \end{aligned}$$

as h tends to ∞. By Theorem 1.3 (recall Remark 2.3) we get

$$\lim_{h\to\infty} \int_\Omega f(\mathcal{E}^D u_h)\,dx = \int_\Omega f(\mathcal{E}^D u)\,dx + \int_\Omega f^\infty(E^D_s u).$$

Therefore, taking into account the definition of F_2 and Step 3, we have

$$\begin{aligned} \overline{F}(u) &\le \liminf_{h\to\infty} \overline{F}(u_h) \le \liminf_{h\to\infty} F_2(u_h,\Omega) \le \liminf_{h\to\infty} F_1(u_h) = \\ &= \int_\Omega f(\mathcal{E}^D u)\,dx + \int_\Omega f^\infty(E^D_s u) + \alpha \int_\Omega (\mathrm{div} u)^2\,dx = F_1(u). \end{aligned}$$

So far, we have obtained that $\overline{F}(u) \le F_1(u)$ for every $u \in P(\Omega)$. The other inequality is the content of Step 1. The proof of Theorem 2.1 is thus accomplished. ∎

We conclude this section by proving that the functional $\overline{F}$ in Theorem 2.1 can also be seen as the Γ-limit of a sequence of functionals obtained by perturbing F with the term $u \mapsto \varepsilon\mathcal{H}^{n-1}(S_u)$.

Theorem 2.9. *Let (ε_h) be a sequence of positive numbers converging to 0. Let $F_h\colon P(\Omega) \to [0,+\infty]$ be defined, for $h \in \mathbf{N}$, by*

$$F_h(u) = \begin{cases} \displaystyle\int_\Omega \Big(\frac{1}{2}|\mathcal{E}^D u|^2 + \alpha(\mathrm{div} u)^2\Big)\,dx + \beta\int_{S_u} (\varepsilon_h + |(u^+ - u^-)\odot \nu_u|)\,d\mathcal{H}^{n-1} & \text{if } u \in S(\Omega), \\ +\infty, & \text{otherwise.} \end{cases}$$

Then (F_h) Γ-converges to $\overline{F}$ in the $L^1(\Omega;\mathbf{R}^n)$-topology.

Proof. Fix $u \in P(\Omega)$. Let (u_h) be a sequence in $P(\Omega)$ converging to u in $L^1(\Omega;\mathbf{R}^n)$. Then

$$\liminf_{h\to\infty} F_h(u_h) \ge \liminf_{h\to\infty} F(u_h) \ge \overline{F}(u).$$

We have now to construct a sequence (u_h) in $P(\Omega)$ which converges to u in $L^1(\Omega;\mathbf{R}^n)$ and with the property

$$\lim_{h\to\infty} F_h(u_h) = \overline{F}(u).$$

By the definition of $\overline{F}$ there exists a sequence (v_h) in $P(\Omega)$ converging to u in $L^1(\Omega;\mathbf{R}^n)$ and such that $F(v_h) \to \overline{F}(u)$. We can assume that $v_h \in S(\Omega)$, hence

$\mathcal{H}^{n-1}(S_{v_h}) < +\infty$ for every $h \in \mathbf{N}$. Let $(\sigma(k))_{k\in\mathbf{N}}$ be the sequence of integers defined by induction as follows:

$$\sigma(1) = 1$$
$$\sigma(k+1) = \min\{j \in \mathbf{N} : j > \sigma(k), \forall h \geq j\,, \varepsilon_h^{1/2}\mathcal{H}^{n-1}(S_{v_{k+1}}) \leq 1\}\,.$$

The sequence (u_h) we are looking for is given by

$$u_h = v_k \qquad \text{if} \quad \sigma(k) \leq h < \sigma(k+1)\,.$$

Indeed, for every h, $k \in \mathbf{N}$ with $k > 1$, and $\sigma(k) \leq h < \sigma(k+1)$

$$\varepsilon_h \mathcal{H}^{n-1}(S_{u_h}) = \varepsilon_h \mathcal{H}^{n-1}(S_{v_k}) \leq \varepsilon_h^{1/2}\,.$$

Therefore, $\big(\varepsilon_h \mathcal{H}^{n-1}(S_{u_h})\big)$ converges to 0 as $h \to \infty$ and

$$F_h(u_h) = F(u_h) + \beta\varepsilon_h \mathcal{H}^{n-1}(S_{u_h}) \quad \to \quad \overline{F}(u)\,.$$ ◻

3. Mechanical interpretation

In this section we study the functional obtained in Theorem 2.1, computing the conjugate of the integrand function f on $\mathrm{M}_0^{\mathrm{sym}}$ (Proposition 3.1). As an immediate consequence (Corollary 3.2) it follows that relaxation of (0.5) yields exactly, as stated in the introduction, the functional of the displacement problem in the theory of Hencky's plasticity with Tresca's yield conditions.

We recall that if $h: V \to \overline{\mathbf{R}}$ is a function on a vector space V, the *conjugate* (*or polar*) of h is the function h^* from the dual space V^* to $\overline{\mathbf{R}}$ defined by

$$h^*(\sigma) = \sup_{\xi\in V}(\langle\sigma,\xi\rangle_{V^*,V} - h(\xi))\,.$$

(see for instance [13] for further details).

In the sequel, given $\sigma \in \mathrm{M}^{\mathrm{sym}}$ we shall denote by $\lambda_M(\sigma)$ and $\lambda_m(\sigma)$ the maximum and the minimum eigenvalues of σ, respectively.

Proposition 3.1. *Let $f: \mathrm{M}_0^{\mathrm{sym}} \to [0, +\infty[$ be the function introduced in Theorem 2.1 (i.e., defined in (2.2)). Then for every $\sigma \in \mathrm{M}_0^{\mathrm{sym}}$ the conjugate function f^* is given by*

$$f^*(\sigma) = \begin{cases} \frac{1}{2}|\sigma|^2\,, & \textit{if } \lambda_M(\sigma) - \lambda_m(\sigma) \leq \sqrt{2}\beta, \\ +\infty\,, & \textit{otherwise.} \end{cases}$$

Proof. Recall that f is the convex envelope, in $\mathrm{M}_0^{\mathrm{sym}}$, of the function $h(\xi) = (\frac{1}{2}|\xi|^2) \wedge g(\xi)$, where g is defined in (2.3). Thus $f = h^{**}$ and $f^* = h^*$; we now compute h^*.

Let $\sigma \in \mathrm{M}_0^{\mathrm{sym}}$ be fixed. Then

$$g^*(\sigma) = \sup_{\xi\in\mathcal{S}}(\sigma\cdot\xi - \beta|\xi|),$$

where $\mathcal{S} = \{a \odot b : a, b \in \mathbf{R}^n, \langle a, b\rangle = 0\}$.

i) Assume $g^*(\sigma) > 0$. Then there exists $\xi \in \mathcal{S}$ such that $\sigma \cdot \xi - \beta|\xi| > 0$; it follows that for every $t > 0$

$$h^*(\sigma) \ \geq \ \sigma \cdot (t\xi) - g(t\xi) = t(\sigma \cdot \xi - \beta|\xi|) \to +\infty \qquad \text{as } t \to +\infty .$$

Hence $h^*(\sigma) = +\infty$.

ii) Assume $g^*(\sigma) \ \leq \ 0$. Since $h^*(\sigma)$ is the maximum between

$$\sup_{\frac{1}{2}|\xi|^2 \leq g(\xi)} (\sigma \cdot \xi - \frac{1}{2}|\xi|^2) \qquad \text{and} \qquad \sup_{\substack{\frac{1}{2}|\xi|^2 > g(\xi) \\ (\xi \in \mathcal{S})}} (\sigma \cdot \xi - \beta|\xi|) ,$$

the assumption on $g^*(\sigma)$ implies that $h^*(\sigma)$ equals the first of these two values. On the other hand, if $\frac{1}{2}|\xi|^2 > g(\xi)$ $(= \beta|\xi|)$ then

$$\sigma \cdot \xi - \frac{1}{2}|\xi|^2 < \sigma \cdot \xi - \beta|\xi| \ \leq \ 0 .$$

It follows that

$$h^*(\sigma) = \sup_{\xi \in \mathrm{M}_0^{\mathrm{sym}}} (\sigma \cdot \xi - \frac{1}{2}|\xi|^2) ;$$

therefore

$$h^*(\sigma) = \sup_{t \geq 0} \sup_{|\xi|=1} (t\sigma \cdot \xi - \frac{1}{2}t^2) = \sup_{t \geq 0} (-\frac{1}{2}t^2 + |\sigma|t) = \frac{1}{2}|\sigma|^2 .$$

We conclude that

$$h^*(\sigma) = \begin{cases} \frac{1}{2}|\sigma|^2 , & \text{if } g^*(\sigma) \ \leq \ 0 , \\ +\infty , & \text{if } g^*(\sigma) > 0. \end{cases} \tag{3.1}$$

Let us now transform the condition $g^*(\sigma) \ \leq \ 0$ (> 0). It is easy to see that

$$g^*(\sigma) \ \leq \ 0 \qquad \text{if and only if} \qquad \sup_{\substack{\xi \in \mathcal{S} \\ |\xi|=1}} \sigma \cdot \xi \ \leq \ \beta .$$

Every $\xi \in \mathcal{S}$ satisfying $|\xi| = 1$ can be written as $a \odot b$ with $|a| = |b| = \sqrt[4]{2}$, and

$$\sigma \cdot \xi = \frac{1}{2} \sum_{i,j} \sigma_{ij}(a_i b_j + a_j b_i) = \frac{1}{2}(a^T \sigma b + a^T \sigma^T b) = a^T \sigma b .$$

We claim that

$$\sup_{\substack{a,b \in \mathbf{R}^n \\ |a|=|b|=1,\, \langle a,b\rangle=0}} a^T \sigma b = \frac{1}{2}(\lambda_M(\sigma) - \lambda_m(\sigma)) . \tag{3.2}$$

Indeed, let $(x,y)\in \mathbf{R}^n\times\mathbf{R}^n$, with $|x|=|y|=1$ and $\langle x,y\rangle=0$, be a point where the supremum is attained. The Lagrange multiplier rule yields that

$$\begin{cases}\sigma y=\lambda y+\mu_1 x\\ \sigma x=\lambda x+\mu_2 y\,.\end{cases}$$

In particular, $\mu_1=\mu_2=x^T\sigma y$, thus

$$\begin{cases}\sigma(x+y)=(\lambda+\mu_1)(x+y)\\ \sigma(x-y)=(\lambda-\mu_1)(x-y)\,.\end{cases}$$

Therefore $\lambda+\mu_1$ and $\lambda-\mu_1$ are eigenvalues of σ, and the supremum in (3.2) is

$$x^T\sigma y=\mu_1=\frac{1}{2}((\lambda+\mu_1)-(\lambda-\mu_1))\ \le\ \frac{1}{2}(\lambda_M(\sigma)-\lambda_m(\sigma))\,.$$

On the other hand, the value $\frac{1}{2}(\lambda_M(\sigma)-\lambda_m(\sigma))$ is attained when $a=v+w$, $b=v-w$, where v and w are orthogonal eigenvectors corresponding to $\lambda_M(\sigma)$ and $\lambda_m(\sigma)$ respectively, and $|v|=|w|=\frac{1}{\sqrt{2}}$. Thus (3.2) is established.

We conclude that

$$g^*(\sigma)\ \le\ 0 \qquad \text{if and only if} \qquad \frac{1}{\sqrt{2}}(\lambda_M(\sigma)-\lambda_m(\sigma))\ \le\ \beta\,;$$

this fact, together with (3.1), proves the proposition. □

Corollary 3.2. *Let $F\colon P(\Omega)\to[0,+\infty]$ be defined as follows (see (0.5))*

$$F(u)=\begin{cases}\int_\Omega(\mu|\mathcal{E}^Du|^2+\frac{\kappa}{2}(\mathrm{div}u)^2)dx+c\int_{S_u}|u^+-u^-|d\mathcal{H}^{n-1}\,, & \text{if } u\in S(\Omega)\,,\\ +\infty\,, & \text{otherwise.}\end{cases}$$

The lower semicontinuous envelope of F with respect to the $L^1(\Omega;\mathbf{R}^n)$-topology on $P(\Omega)$ is given by (compare with (0.4))

$$\overline{F}(u)=\int_\Omega(\varphi(\mathcal{E}^Du)+\frac{\kappa}{2}(\mathrm{div}u)^2)dx+\int_\Omega\varphi^\infty(E^D_su)\,,$$

where $\varphi\colon \mathrm{M}^{\mathrm{sym}}_0\to[0,+\infty[$ is the finite convex function on $\mathrm{M}^{\mathrm{sym}}_0$ whose conjugate is (see (0.3))

$$\varphi^*(\sigma)=\begin{cases}\frac{1}{4\mu}|\sigma|^2\,, & \text{if } \sigma\in K\cap\mathrm{M}^{\mathrm{sym}}_0\,,\\ +\infty\,, & \text{otherwise}\,,\end{cases}$$

and K is Tresca's convex set:

$$K=\{\sigma\in\mathrm{M}^{\mathrm{sym}}:\lambda_M(\sigma)-\lambda_m(\sigma)\ \le\ 2c\}\,.$$

Proof. It is enough to apply Theorem 2.1, with $\alpha=\frac{\kappa}{4\mu}$, $\beta=\frac{c}{\sqrt{2}\mu}$, and Proposition 3.1. □

We conclude by giving an estimate of the function φ in the foregoing Corollary. For every $\gamma > 0$ define

$$\varphi_\gamma : [0, +\infty[\to [0, +\infty[\qquad \varphi_\gamma(t) = \begin{cases} \mu t^2 & \text{if } 0 \le t \le \gamma/2\mu \\ \gamma t - \frac{\gamma^2}{4\mu} & \text{if } t \ge \gamma/2\mu. \end{cases}$$

Proposition 3.3. *(i) For every $\xi \in \mathrm{M}^{\mathrm{sym}}_0$ and R orthogonal $n \times n$ matrix, we have $\varphi(R\xi R^T) = \varphi(\xi)$.*

(ii) For every $\xi \in \mathrm{M}^{\mathrm{sym}}_0$ the following estimates hold:

$$\varphi_\gamma(|\xi|) \le \varphi(\xi) \le \varphi_\delta(|\xi|),$$

where $\gamma = \sqrt{2}c$ and

$$(\frac{\delta}{2c})^2 = \frac{[n/2](n - [n/2])}{n} = \begin{cases} n/4, & \textit{if } n \textit{ is even}, \\ \frac{n^2-1}{4n}, & \textit{if } n \textit{ is odd}. \end{cases}$$

Moreover,

$$\varphi(a \odot b) = \varphi_\gamma(|a \odot b|) \qquad \textit{whenever } \langle a, b\rangle = 0 \tag{3.3}$$

$$\varphi(\xi^D) = \varphi_\delta(|\xi^D|) \qquad \textit{whenever } \xi = t \begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix}, \quad t \in \mathbf{R} \tag{3.4}$$

where I is the identity matrix of order $[n/2]$.

Remark 3.4. We notice that in the physical case $n = 3$, Proposition 3.3 implies that the maximum slope of φ is attained along the directions determined by the matrices of the form $(k \otimes k)^D$.

Proof of Proposition 3.3. (i) Let R be an orthogonal $n \times n$ matrix. Then $R\,(a \odot b)\,R^T = (Ra) \odot (Rb)$, and $\langle Ra, Rb\rangle = 0$ whenever $\langle a, b\rangle = 0$. From this and (2.2) it is not difficult to see that $\varphi(R\xi R^T) = \varphi(\xi)$.

(ii) The inequality $\varphi(\xi) \ge \varphi_\gamma(|\xi|)$ for every $\xi \in \mathrm{M}^{\mathrm{sym}}_0$ and equality (3.3) follow immediately from Lemma 2.4. Let us define now

$$\delta = \max\{|x| : x \in \mathbf{R}^n, \sum_{i=1}^n x_i = 0, |x_i - x_j| \le 2c \text{ for every } i, j\}.$$

Assume that $x \in \mathbf{R}^n$ realizes this maximum, and that, for suitable integers r and s,

$$x_i = m \quad \text{if and only if } 1 \le i \le r, \qquad x_i = M \quad \text{if and only if } r+1 \le i \le r+s,$$

where m and M are the minimum and the maximum of $\{x_1, \ldots, x_n\}$, respectively. We claim that $r + s = n$. Otherwise, assume for simplicity that $r + s = n - 1$, and observe that for every $v \in \mathbf{R}^n \setminus \{0\}$ and $\varepsilon \in \mathbf{R}$ we have $|x| < |x + \varepsilon v|$ whenever $\varepsilon\langle x, v\rangle \ge 0$.

We obtain a contradiction by choosing $|\varepsilon|$ sufficiently small and v satisfying: $v_i = 1$ if $1 \le i \le n-1$, $v_n = 1-n$. We easily deduce then that $r \wedge s = [n/2]$, and δ takes the value given in the statement.

Since for every $\sigma \in \mathrm{M}_0^{\mathrm{sym}}$, $|\sigma|^2$ equals the sum of the squares of the eigenvalues of σ (with their respective multiplicity), we have proved that

$$\max_{\sigma \in K \cap \mathrm{M}_0^{\mathrm{sym}}} |\sigma| = \delta,$$

and the maximum is attained when σ is the deviator of a matrix of the form (3.4), with $t = \pm 2c$.

Let now $\xi \in \mathrm{M}_0^{\mathrm{sym}}$. Then

$$\begin{aligned} \varphi(\xi) &= \sup_{\sigma \in K \cap \mathrm{M}_0^{\mathrm{sym}}} (\xi \cdot \sigma - \frac{1}{4\mu}|\sigma|^2) \\ &= \sup_{0 \le t \le \delta} \sup_{\substack{\sigma \in K \cap \mathrm{M}_0^{\mathrm{sym}} \\ |\sigma| = t}} (\xi \cdot \sigma - \frac{1}{4\mu} t^2) \\ &\le \sup_{0 \le t \le \delta} (t|\xi| - \frac{1}{4\mu} t^2) = \varphi_\delta(|\xi|). \end{aligned}$$

The last inequality is actually an equality in case ξ is the deviator of a matrix of the form (3.4), since $t\frac{\xi}{|\xi|} \in K$ for every $0 \le t \le \delta$. This proves (3.4). □

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