UNIQUENESS OF THE BLOW-UP AT ISOLATED SINGULARITIES FOR THE ALT-CAFFARELLI FUNCTIONAL

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Abstract. In this paper we prove uniqueness of blow-ups and $C^{1,\log}$-regularity for the free-boundary of minimizers of the Alt-Caffarelli functional at points where one blow-up has an isolated singularity. We do this by establishing a (log-)epiperimetric inequality for the Weiss energy for traces close to that of a cone with isolated singularity, whose free-boundary is graphical and smooth over that of the cone in the sphere. With additional assumptions on the cone, we can prove a classical epiperimetric inequality which can be applied to deduce a $C^{1,\alpha}$ regularity result. We also show that these additional assumptions are satisfied by the De Silva-Jerison-type cones, which are the only known examples of minimizing cones with isolated singularity. Our approach draws a connection between epiperimetric inequalities and the Łojasiewicz inequality, and, to our knowledge, provides the first regularity result at singular points of the Alt-Caffarelli functional.

1. Introduction

In this paper we prove a *uniqueness of blow-up (with logarithmic decay) and regularity result* for the free-boundary of minimizers of the Alt-Caffarelli functional at points where one blow-up has an isolated singularity. In the special case where one of the blowups is *integrable through rotation* (which includes the only known cones with an isolated singularity), we also get a Hölder rate of convergence to the blowup. We do this by establishing a log-epiperimetric inequality for the Weiss-monotonicity formula around such singularities (see below for relevant definitions). One key tool here is the Łojasiewicz inequality. We remark that this is the *first regularity result for singular points of the Alt-Caffarelli functional* that we are aware of. Before stating the theorem, we need to introduce some notation. We shall denote by $\mathcal{E}$ the Alt-Caffarelli functional:

$$\mathcal{E}(u) := \int_D |\nabla u|^2 \, dx + \big|\{u > 0\} \cap D\big|, \qquad u \in H^1(D, \mathbb{R}^+), \tag{1.1}$$

(here and throughout $D \subset \mathbb{R}^d$ is a connected open set with Lipschitz regular boundary).

The functional in (1.1) was first studied systematically by Alt and Caffarelli [1], who showed that minimizers exist, and satisfy the following, overdetermined, boundary value problem in a weak sense:

$$u \geq 0, \qquad \Delta u = 0 \ \text{ in } \ \{u > 0\} \cap D, \qquad |\nabla u| = 1 \ \text{ on } \ \partial\{u > 0\} \cap D. \tag{1.2}$$

Furthermore, it was shown in [1] that the free boundary, $\partial\{u > 0\}$, is a set of locally finite perimeter and that around $\mathcal{H}^{d-1}$-almost every point $x_0 \in \partial\{u > 0\} \cap D$, the free boundary can be written as the graph of an analytic function.

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In [30], Weiss introduced the quantity, $W(u,x_0,r)$, (or $W(u,r)$, when $x_0=0$, and $W(u)$, when $x_0=0$ and $r=1$) which monotonically increases with r for every minimizer u,

$$W(u,x_0,r) := \frac{1}{r^d}\int_{B_r(x_0)} |\nabla u|^2\, dx - \frac{1}{r^{d+1}}\int_{\partial B_r(x_0)} u^2\, d\mathcal{H}^{d-1} + \frac{1}{r^d}\big|\{u>0\}\cap B_r(x_0)\big|\,, \tag{1.3}$$

where $x_0 \in \partial\{u>0\}\cap D$ and $0<r<\mathrm{dist}(x_0,\partial D)$. A consequence of this monotonicity is that every blow-up (see Section 2) is a 1-homogenous globally defined minimizer to (1.1).
Non-flat 1-homogenous minimizers (and thus singular points) do not exist when $d\le 4$ (for $d=2$ this is due to [1], for $d=3$ to [4], and for $d=4$ to [19]). In contrast, De Silva and Jerison [10] constructed a class of cones with isolated singularities at the origin which are minimizers in dimensions $d\ge 7$. We refer to this class of cones as $\mathcal{B}$; specifically,

$$\mathcal{B} := \big\{b_{\nu,\theta_0} \colon \mathbb{R}^d \to \mathbb{R}^+ \text{ 1-homogeneous minimizers of } \mathcal{E} \text{ with } \{b_{\nu,\theta_0}>0\} = \mathcal{C}_{\nu,\theta_0}\big\},$$

where for given $\theta_0\in(0,\pi/2)$ and $\nu\in\mathbb{S}^{d-1}$ we define the cone

$$\mathcal{C}_{\nu,\theta_0} := \left\{ x\in\mathbb{R}^d\setminus\{0\}\ :\ \left|\frac{x}{|x|}\cdot\nu\right| < \sin(\theta_0)\right\}.$$

Notice that W restricted to the class $\mathcal{B}$ depends only on θ_0, and we set $W(b_{\nu,\theta_0}) := \Theta_{\theta_0}$.

Rephrased in this notation, the aforementioned result of De Silva and Jerison [10] says that the class $\mathcal{B}$ is non-empty in dimension $d\ge 7$, while the aforementioned results of [4, 19] show that the class $\mathcal{B}$ is empty up to dimension 6. We note that the existence of (non-rotationally symmetric) non-flat one-homogeneous minimizers in dimension $d=5,6$ is unknown and is an important open problem in the field. These results are related to analogous ones in the context of minimal surfaces: the proof that $\mathcal{B}$ is empty for $d\le 6$ is based on the result by Simons [25] that there are no co-dimension one area-minimizing surfaces in dimension $d\le 7$. On the other hand, the cones in $\mathcal{B}$ are very similar to the Simons-type cones, which are area-minimizing in dimensions $d\ge 8$ (see [2]).

Left open in the works of [30, 4, 10, 19] is the question of whether or not blowups at singular points are unique. *A priori*, it is possible that the free boundary around a singular point asymptotically approaches one singular cone at a certain set of small scales, but approaches a different singular cone at a separate set of scales. The problem of "uniqueness for blowups" is a central one in geometric analysis and free boundary problems (see, e.g. [24, 15]). Our main theorem is that if one blowup at $x_0\in\partial\{u>0\}$ is a cone, b, with isolated singularity, then every blowup at x_0 is equal to b.

Theorem 1 (Regularity for isolated singularities)**.** *Let $u\in H^1(D)$ be a minimizer of the Alt-Caffarelli functional $\mathcal{E}$ on a domain $D\subset\mathbb{R}^d$ and let $x_0\in\partial\{u>0\}\cap D$ be a singular point of the free-boundary such that there exists a blow up b for u at x_0 with* isolated singularity*. Then b is the unique blow up and, furthermore, there exists $r_0>0$ such that $\partial\{u>0\}\cap B_{r_0}(x_0)$ is a $C^{1,\log}$ graph over $\partial\{b>0\}\cap B_{r_0}(x_0)$.*

If we have additional information on the blowup, b (namely that it is integrable through rotations, see Definition 2.4) we can improve the rate of convergence to the minimal cone.

Theorem 2 (Regularity for isolated and integrable singularities)**.** *Let $u\in H^1(D)$ be a minimizer of the Alt-Caffarelli functional $\mathcal{E}$ on a domain $D\subset\mathbb{R}^d$ and let $x_0\in\partial\{u>0\}\cap D$ be a singular point of the free-boundary such that there exists a blow up b, of u at x_0, with* isolated singularity *and which is* integrable through rotation *(see Definition 2.4). Then b is the unique blow up and there exists $r_0>0$ such that $\partial\{u>0\}\cap B_{r_0}(x_0)$ is a $C^{1,\alpha}$ graph over $\partial\{b>0\}\cap B_{r_0}(x_0)$.*

We will prove that all the cones in the class $\mathcal{B}$ satisfy the above integrability condition, thus leading to the following corollary.

Corollary 1.1 (Regularity for isolated singularities of De Silva-Jerison type)**.** *Let $u\in H^1(D)$ be a minimizer of the Alt-Caffarelli functional $\mathcal{E}$ on a domain $D\subset\mathbb{R}^d$ and let $x_0\in\partial\{u>0\}\cap D$ be a singular point of the free-boundary such that for some $\nu\in\mathbb{S}^{d-1}$ and $\theta_0\in(0,\pi/2)$ the function*

b_{ν,θ_0} is a blow-up for u at x_0. Then b_{ν,θ_0} is the unique blow-up and furthermore there exists $r_0 > 0$ such that $\partial\{u > 0\} \cap B_{r_0}(x_0)$ is a $C^{1,\alpha}$ graph over $\partial\mathcal{C}_{\nu,\theta_0} \cap B_{r_0}(x_0)$.

We remark that there are no other known examples of one-homogeneous minimizers with isolated singularity outside the elements of $\mathcal{B}$. Moreover, we know of no other theorem that shows uniqueness of blowups at singular points for the Alt-Caffarelli functional. In fact, surprisingly little is known about the structure and size of the singular set for minimizers to (1.1). Weiss [30] proved dimension bounds via a dimension-reduction type argument. Later, the first author, with Edelen [11], established size estimates and proved that the singular set is rectifiable.

The main ingredient in the proofs of Theorem 1 and 2 is a (log-)epiperimetric inequality for minimizers whose trace on ∂B_1 has a free boundary which can be written as a smooth graph over $\partial\Omega_b$, where $\Omega_b = \{b > 0\} \cap \partial B_1$. We will often consider graphs on the sphere of the following form: given a function $\zeta\colon \partial\Omega_b \to \mathbb{R}$, we define the set on the sphere

$$\mathrm{graph}_{\partial\Omega_b}(\zeta) := \big\{ \exp_x(\zeta(x)\,\nu(x)) \in \mathbb{S}^{d-1} \,:\, x \in \partial\Omega_b \big\}, \tag{1.4}$$

where $\nu(x) \in T_x(\partial\{b > 0\} \cap \mathbb{S}^{d-1})$ is the unit normal in the sphere to $\partial\{b > 0\}$ pointing outside $\{b > 0\}$, and $\exp : T\mathbb{S}^{d-1} \to \mathbb{S}^{d-1}$ is the exponential map

$$\exp_x(v) := \cos(\|v\|)\,x + \sin(\|v\|)\frac{v}{\|v\|}\,, \quad \text{for every} \quad v \in T_x\mathbb{S}^{d-1}\,.$$

Theorem 3 ((Log-)Epiperimetric inequality for traces with smooth free-boundary)**.** *Let $b \in H^1(B_1)$ be a one-homogeneous minimizer of the Alt-Caffarelli functional $\mathcal{E}$ with an isolated singularity at the origin. There exist constants $\varepsilon = \varepsilon(d, b) > 0, \gamma = \gamma(d, b) \in [0, 1)$ and $\delta_0 = \delta_0(d, b) > 0$, depending on b and on the dimension d, such that the following holds.*

If $c \in H^1(\partial B_1, \mathbb{R}_+)$ is such that there exists $\zeta \in C^{2,\alpha}(\partial\Omega_b)$ satisfying

$$\partial\{c > 0\} = \mathrm{graph}_{\partial\Omega_b}(\zeta)\,, \quad \textit{with} \quad \|\zeta\|_{C^{2,\alpha}} \le C_d\|\zeta\|_{L^2} < \delta\,, \quad \textit{and} \quad \|c - b\|_{L^2(\partial B_1)} < \delta\,, \tag{1.5}$$

then there exists a function $h \in H^1(B_1, \mathbb{R}_+)$ such that $h = c$ on ∂B_1 and

$$W(h) - W(b) \le \Big(1 - \varepsilon\big|W(z) - W(b)\big|^\gamma\Big)\big(W(z) - W(b)\big), \tag{1.6}$$

where z is the 1-homogeneous extension of c to B_1.

In the case where b is integrable through rotations (see Definition 2.4), we can take $\gamma = 0$ in (1.6) above.

We wish to remark that using ideas taken from this work, we were also able to prove an analogous (log-)epiperimetric inequality for multiplicity-one stationary cones, which we used to deduce a uniqueness of the blow-up result for almost area minimizing currents. This is the content of [12], where we also explore the relation between an epiperimetric inequality for, and the integrability of, a cone.

1.1. **Epiperimetric and logarithmic epiperimetric inequalities.** Epiperimetric inequalities (i.e. inequalities like, (1.6) when $\gamma = 0$) were first introduced by Reifenberg [23], who used one to prove that minimal surfaces are smooth near flat points. Later Jean Taylor [28, 27] established an epiperimetric inequality around singular points for dimension two size-minimizing surfaces in co-dimension 1. A consequence of this theorem is a precise description of the structure of the singular set of dimension two size-minimizing surfaces. Brian White [31] proved the same results for two-dimensional area-minimizing currents. The proof in [31] is also remarkable in that it is done directly, as opposed to the proofs by contradiction in [28, 27]. Our approach is partially inspired by [31]. However, in the setting of area-minimizing surfaces the previously known epiperimetric inequalities at singular points hold only in dimension two and are not logarithmic.

More recently, epiperimetric inequalities have been applied to free boundary problems. Weiss [29] proved an epiperimetric inequality at regular points for the obstacle problem, thus giving a different proof of the regularity of the free boundary there. Later, Focardi and Spadaro [14] and Garofalo, Petrosyan and Smit Vega-Garcia [16] established similar epiperimetric inequalities for the thin obstacle problem (again at regular points and at singularities in dimension 2). All of these results are proved by contradiction, whereas the proof of our Theorem 3 is done by directly constructing the competitor, h. In some sense this difference is essential; in [29], Weiss remarks that to prove an *epiperimetric inequality for singular points in general dimension*, one must proceed directly. This has been shown to be true for the obstacle and thin-obstacle problems by the second and third author in joint work with Maria Colombo (see [6, 5]).

The works of [6, 5] also introduced the concept of "Log-Epiperimetric" inequalities, where the gain in (1.6) depends on the distance between the 1-homogenous extension and the blow-up. It was shown in [6, 5] that log-epiperimetric inequalities imply uniqueness of blowups and logarithmic rates of convergence. In contrast to [6, 5], however, here we give a completely different and more general proof of the inequality, which we expect to hold in many other situations, by drawing a connection between log-epiperimetric inequalities and the Łojasiewicz inequality approach of L. Simon [24]. Furthermore we only assume one blow-up to be of the right form, thus dealing with the additional difficulty of showing that blow-ups at different scales have the same structure.

In [26] the second and third author proved a direct epiperimetric inequality for the Alt Caffarelli functional at flat points in dimension $d = 2$ (this implies, in dimension two, the classical ε-regularity result of [1] and new regularity results for the two-phase and vectorial analogues of (1.1)). While this paper is inspired by the ideas in [26], we face many additional complications in higher dimensions, where there is no direct relationship between the area of a connected open subset of the sphere and the first eigenvalue of that subset. In work in progress, we use many of the ideas in this paper to extend the results of [26] and prove an epiperimetric inequality near flat points in arbitrary dimensions. We also plan to apply the ideas here to prove a quantitative Faber-Krahn inequality on the sphere (in the vein of [3]).

1.2. **Sketch of the proof and plan of the paper.** Before we introduce some additional terminology and notation, let us briefly sketch the main ideas of the proof. Recall that the epiperimetric inequality asks if, given a trace, c, on the sphere which is sufficiently close to a one-homogeneous minimizer b, can one construct a $h \in H^1(B_1)$, with $h|_{\partial B_1} = c$, whose energy is quantitatively smaller than that of the one-homogenous extension of c.

The first step is to write c as the sum of the Dirichlet eigenfunctions of $\{c > 0\}$. In the first part of Section 3, we show how the epiperimetric inequality reduces to two separate estimates, one on the first eigenfunction of $\{c > 0\}$ and another on the higher modes. We can deal with the higher modes by using a harmonic extension and cutoff argument. To address the first eigenfunction we do an "internal variation" – smoothly changing the first eigenfunction of $\{c > 0\}$ into the first eigenfunction of the "closest" cone, b. This is the most technical part of the argument and it requires a careful analysis of the Taylor-series expansion of the energy around b, to show that the "internal variation" decreases W; we do this by writing $\partial\{c > 0\}$ (considered as a graph over the cone $\{b > 0\} \cap \partial B_1$) in terms of the eigenfunctions of the second variation of the Alt-Caffarelli functional restricted to the sphere (see Lemma 2.3 and associated definitions). It is here that we use the assumption of integrability through rotations to ensure that we can disregard elements in the kernel of the second variation.

When the cone is not integrable through rotations, we must address the kernel elements. In this situation, we adapt the ideas of L. Simon [24], and apply the Łojasiewicz inequality. More precisely, we do a Lyapunov-Schmidt reduction (see Appendix B) and then move the elements in the kernel by a gradient flow. The fact that the gradient flow always gives a quantitative improvement follows from the Łojasiewicz inequality [20].

In Section 4 we apply the (log-)epiperimetric inequality to obtain our main Theorems 1 and 2. Some of this is standard; once one gets the (log-)epiperimetric inequality, the decay of $W(r)$ and thus the power-rate (or logarithmic rate) of convergence to the blowup follows by purely elementary considerations. However, there is an additional difficulty that our (log-)epiperimetric inequality, Theorem 3, applies only to minimizers whose free boundary restricted to the sphere is a smooth graph over the cone, b. To apply the theorem to arbitrary minimizers which are close enough to a cone with isolated singularity, we employ a compactness argument and use the ε-regularity result of Alt-Caffarelli. In this argument we are inspired by Simon [24].

Finally, in Section 5, we finish the proof of Corollary 1.1 by showing that $b_{\nu,\theta_0} \in \mathcal{B}$ is integrable through rotations. First we produce, with spherical harmonics, an orthogonal basis of $H^{1/2}(\partial\{b_{\nu,\theta_0} > 0\} \cap B_1)$ and, using the symmetries of the cone, show that this is an eigenbasis for the second variation, around b_{ν,θ_0}, of the Alt-Caffarelli functional restricted to the sphere. Integrability follows after explicitly computing some of the associated eigenvalues.

2. Notations and preliminary results

2.1. **Notations.** It will be convenient to split the Weiss energy into two pieces, one without the measure term and one with it, so we set

$$W_0(u) := \int_{B_1} |\nabla u|^2 \, dx - \int_{\partial B_1} u^2 \, d\mathcal{H}^{d-1} \qquad \text{and} \qquad W(u) := W_0(u) + \big|B_1 \cap \{u > 0\}\big|.$$

We will often work in spherical coordinates: a point $x \in \mathbb{R}^d$ can be written as (r,θ), where $r = |x|$ and $\theta \in \mathbb{S}^{d-1} = \partial B_1$. Given a function $u\colon \Omega \to \mathbb{R}$ (almost always a minimizer), a point x_0 (almost always in $\partial\{u > 0\}$) and a radius $r > 0$ we can define the rescaled function

$$u_{r,x_0}(x) := \frac{u(rx + x_0)}{r} \qquad \text{for } 0 < r < \operatorname{dist}(x_0, \partial\Omega). \tag{2.1}$$

When x_0 is clear from the context and $r \in \{r_j\}_{j\in\mathbb{N}}$ we will write u_j to mean u_{r_j,x_0}.
Finally, let u be a minimizer to (1.1) and $x_0 \in \partial\{u > 0\}$. Assume there is a sequence $r_j \downarrow 0$ such that $u_{r_j,x_0} \to u_\infty$ uniformly on compacta as $j \to \infty$. We then say that u_∞ is a *blow-up of u at x_0*. It follows by the convergence theorems of [1] and the work of [30] that u_∞ is a one-homogenous, globally-defined minimizer to (1.1).

2.2. **Eigenvalues and eigenfunction on a spherical set S.** Let S be an open subset of the sphere $\partial B_1 \subset \mathbb{R}^d$. On S we consider the family of eigenfunctions $\{\phi_j^S\}_{j\geq 1}$ and the corresponding sequence of eigenvalues

$$0 \leq \lambda_1^S \leq \lambda_2^S \leq \cdots \leq \lambda_j^S \leq \ldots,$$

counted with their multiplicity. Each function ϕ_j is a solution of the the PDE

$$-\Delta_{\partial B_1}\phi_j^S = \lambda_j^S \phi_j^S \quad \text{in} \quad S, \qquad \phi_j^S = 0 \quad \text{on} \quad \partial S, \qquad \int_{\partial B_1} |\phi_j^S|^2 \, d\mathcal{H}^{d-1} = 1, \tag{2.2}$$

where $\mathcal{H}^{d-1}$ is the $(d-1)$-dimensional Hausdorff measure, $\Delta_{\partial B_1}$ is the Laplace-Beltrami operator on the unit sphere ∂B_1. Thus, every function $c \in H_0^1(S)$ can be expressed in a unique way in Fourier series as

$$c(\theta) = \sum_{j=1}^{\infty} c_j \phi_j^S(\theta), \qquad \text{where, for every } j \geq 1, \qquad c_j := \int_S c(\theta)\phi_j^S(\theta) \, d\mathcal{H}^{d-1}(\theta).$$

The harmonic extension h_c of c to the cone

$$\mathcal{C}_S := \big\{(r,\theta) \in \mathbb{R}^+ \times \partial B_1 \ : \ r > 0, \ \theta \in S\big\}, \tag{2.3}$$

is the solution of the equation

$$-\Delta h_c = 0 \quad \text{in} \quad \mathcal{C}_S\,, \qquad h_c = c \quad \text{on} \quad S\,, \qquad h_c = 0 \quad \text{on} \quad B_1 \cap \partial\mathcal{C}_S\,,$$

and can be written explicitly in polar coordinates as

$$h_c(r,\theta) = \sum_{j=1}^{\infty} c_j r^{\alpha_j^S} \phi_j^S(\theta),$$

where for every $j \in \mathbb{N}$ the homogeneity constant $\alpha_j^S > 0$ is uniquely determined by the equation

$$\alpha_j^S(\alpha_j^S + d - 2) = \lambda_j^S\,.$$

In what follows we will often drop the index S and we simply use the notation ϕ_j, λ_j and α_j.

Remark 2.1. [The case of 1-homogeneous minimizers] Let b be a 1-homogeneous minimizer of the Alt-Caffarelli functional (1.1) and let us denote its trace on ∂B_1, with a slight abuse of notation, still by b. In polar coordinates $b(r,\theta) = r\, b(\theta)$. Let $\Omega := \{b > 0\} \cap \partial B_1$ be the positivity set of b on the unit sphere. Then, using (1.2), it is easy to see that Ω is a connected open set and there is a constant $\kappa_0 > 0$ such that

$$b \equiv \kappa_0\, \phi_1^\Omega\,, \qquad \lambda_1^\Omega = d-1 \qquad \text{and} \qquad \lambda_2^\Omega > d-1, \tag{2.4}$$

where ϕ_1^Ω is the normalized eigenfunction given by (2.2) with $S = \Omega$. Moreover, on (the regular part of) the boundary of the spherical set Ω the following extremality condition is satisfied:

$$\kappa_0\, \partial_\nu \phi_1^\Omega(x) = \partial_\nu b(x) = -1 \quad \text{for every} \quad x \in \partial\Omega\,,$$

where ν denotes the outward pointing normal to $\partial\Omega$.

Notice that if two smooth connected open sets are close in C^1, then their spectra are also close. Thus λ_2^S remains strictly greater than $d-1$ for spherical sets S that are close to Ω. We give the precise statement in the following remark.

Remark 2.2. Suppose that $\partial\Omega$ is smooth and that ∂S is the graph of the function $\zeta : \partial\Omega \to \mathbb{R}$ with $\|\zeta\|_{C^{1,\alpha}} \le \delta$, where $\delta > 0$ is small enough depending only on the dimension and b. Then

$$\lambda_2^S - (d-1) > (\lambda_2^S - \lambda_2^\Omega) + (\lambda_2^\Omega - (d-1)) \ge \gamma(d,b)/2, \tag{2.5}$$

where $\gamma(d,b) := \lambda_2^\Omega - (d-1) = \lambda_2^\Omega - \lambda_1^\Omega > 0$ depends only on b and the dimension d.

2.3. **Internal variations on spherical domains and integrability.** Given a smooth domain $\Omega \subset \partial B_1$, determined by a 1-homogeneous minimizer b as in Remark 2.1, and a function, $\zeta \in C^{2,\alpha}(\partial\Omega)$, we consider the functional

$$\mathcal{F}(\zeta) := \kappa_0^2\, \lambda(\zeta) + m(\zeta) - \big(\kappa_0^2\,(d-1) + m(0)\big)\,,$$

where the various terms are defined in the following way:

- κ_0 is the constant given by (2.4);
- Ω_ζ is the domain in ∂B_1 whose boundary is the graph of ζ over $\partial\Omega$ in the sense of (1.4);
- $\lambda(\zeta)$ is the first Dirichlet-Laplacian eigenvalue of Ω_ζ and ϕ_ζ is the corresponding eigenfunction given by (2.2) with $S = \Omega_\zeta$;
- $m(\zeta) := \mathcal{H}^{d-1}(\Omega_\zeta)$ and in particular $m_0 := \mathcal{H}^{d-1}(\Omega)$.

Given two functions $g, \zeta \in C^{2,\alpha}(\partial\Omega)$, the first and the second variations of $\mathcal{F}$ are given by

$$\delta\mathcal{F}(g)[\zeta] = \frac{d}{dt}\Big|_{t=0} \mathcal{F}(g + t\zeta) \qquad \text{and} \qquad \delta^2\mathcal{F}(g)[\zeta,\zeta] = \frac{d^2}{dt^2}\Big|_{t=0} \mathcal{F}(g + t\zeta).$$

The most important properties of the functional $\mathcal{F}$ are listed in the following lemma, which is essential for the proof of Theorem 3.

Lemma 2.3. *Let $g, \zeta \in C^{2,\alpha}(\partial\Omega)$. With the above definitions the following holds.*

(i) There is a constant C, depending only on d and b, such that if $\zeta_i \in C^{2,\alpha}(\partial\Omega)$, for $i = 1, \dots, n$, are such that $\|g\|_{C^{2,\alpha}(\partial\Omega)} \le C$ and $\|\zeta_i\|_{C^{2,\alpha}(\partial\Omega)} \le C$, then the function $(t_1, \dots, t_n) \mapsto \mathcal{F}(g + \sum_{i=1}^n t_i \zeta_i)$ is analytic on $]-2,2[$.

(ii) The first variation vanishes at zero, that is $\delta\mathcal{F}(0)[\zeta] = 0$, for every $\zeta \in C^{2,\alpha}(\partial\Omega)$.

(iii) The function $t \mapsto \phi_{g+t\zeta}$ is differentiable in $t \in]-2,2[$ and the derivative $\frac{d}{dt}\Big|_{t=0} \phi_{g+t\zeta} =: \delta\phi_g[\zeta] \in L^2(\partial B_1)$ satisfies the estimate

$$\int_{\partial B_1} |\delta\phi_g[\zeta]|^2\, d\mathcal{H}^{d-1} < C\, \|\zeta\|^2_{L^2(\partial\Omega)}\,, \tag{2.6}$$

where C depends only on d, b and $\|g\|_{C^{2,\alpha}(\partial\Omega)}$.

(iv) $\delta^2\mathcal{F}(g)$ is a quadratic form on $H^{1/2}(\partial\Omega) \times H^{1/2}(\partial\Omega)$.

(v) There is a function $\omega(t) : [0,1] \to \mathbb{R}^+$, satisfying $\lim_{t\downarrow 0} \omega(t) = 0$, such that

$$\left|\delta^2\mathcal{F}(g)[\zeta,\zeta] - \delta^2\mathcal{F}(0)[\zeta,\zeta]\right| < \omega\big(\|g\|_{C^{2,\alpha}(\partial\Omega)}\big)\|\zeta\|^2_{H^{1/2}(\partial\Omega)}. \tag{2.7}$$

(vi) There exists an orthonormal basis $(\xi_i)_i$ of $H^{1/2}(\partial\Omega)$ such that $\xi_i \in C^{2,\alpha}(\partial\Omega)$, for $i \in \mathbb{N}$, and

$$\delta^2\mathcal{F}(0)[\xi_i,\xi_j] := \lambda_i\delta_{ij} \qquad \text{for every } i,j \in \mathbb{N}\,, \tag{2.8}$$

where $\lambda_i \to \infty$ is a non-decreasing sequence of eigenvalues. In particular, there is a constant $C = C(d,b) > 0$ such that

$$(d-1) \le \dim\ker(\delta^2\mathcal{F}(0)) \le C < \infty, \tag{2.9}$$

where the lower bound depends on the fact that the infinitesimal generators of rotations of $\partial\Omega$ are in the kernel of $\delta^2\mathcal{F}(0)$.

The proof of this Lemma is rather technical so we give it in Appendix A in order not to disrupt the flow of the proof. We only remark here that (i) follows from [21, 22], (ii) and (iv) follow by the general representation of the first two variations given in Subsection A.1, (iii) is proved in Subsection A.4, (v) in Subsection A.5 and (vi) in Subsection A.2. Notice that (2.9) leads naturally to the definition of integrability through rotations.

Definition 2.4 (Integrability)**.** *Let $b \in H^1(B_1)$ be a 1-homogeneous minimizer of $\mathcal{E}$. We say that b is integrable through rotation if*

$$\dim\ker(\delta^2\mathcal{F}(0)) = d-1 \quad \text{and} \quad \int_{\partial\Omega}\xi\, d\mathcal{H}^{d-2} = 0, \quad \text{for every negative eigenfunction } \xi \text{ of } \delta^2\mathcal{F}(0),$$

that is, if all the eigenfunctions corresponding to the eigenvalue 0 are infinitesimal generators of rotations of $\Omega = \{b > 0\}$ and if the constant perturbation is orthogonal in $L^2(\partial\Omega)$ to all the negative eigenfunctions.

This definition differs slightly from the usual one given in the context of minimal surfaces, in that it requires orthogonality for the *negative eigenfunctions* that is, the eigenfunctions associated to negative eigenvalues. This is due to the fact that $\mathcal{F}$ can be perturbed not only geometrically (i.e. by changing the domain) but also by changing κ_0 (the coefficient in front of the first eigenfunction of the domain). We also notice that, by Subsection A.1, $\int_{\partial\Omega}\xi\, d\mathcal{H}^{d-2} = \delta\lambda(0)[\xi]$.

We shall prove in Section 5 that the De Silva-Jerison cone is integrable through rotations by explicitly finding the basis $\{\xi_i\}$ and computing the associated eigenvalues.

3. Proof of Theorem 3

This section is the core of the paper and deals with the proof of Theorem 3. We split it into several parts. We first define the competitor and list several of its properties, from which Theorem 3 will follow. Then, in the subsequent subsections, we prove that these properties hold through external and internal variations.

3.1. **Definition of the competitor and its main properties.** Given a function $c \in H^1(S;\mathbb{R}_+)$, on the set $S = \{c > 0\} \subset \partial B_1$, we consider its decomposition in Fourier series over S

$$c(\theta) = c_1\phi_1(\theta) + g(\theta)\,, \quad \text{where} \quad g(\theta) := \sum_{j=2}^{\infty} c_j\phi_j(\theta). \tag{3.1}$$

In this notation, the one-homogeneous extension of c in B_1 is given by

$$z(r,\theta) = c_1 r\phi_1(\theta) + rg(\theta). \tag{3.2}$$

Our *competitor* $h \in H^1(B_1;\mathbb{R}_+)$ will be defined as

$$h(r,\theta) = h_1(r,\theta) + \psi_\rho(r)h_g(r,\theta), \tag{3.3}$$

where

- $\psi_\rho : B_1 \to \mathbb{R}_+$ is the function defined by
$$\Delta\psi_\rho = 0 \quad \text{in} \quad B_{2\rho} \setminus B_\rho\,, \qquad \psi_\rho = 1 \quad \text{on} \quad B_1 \setminus B_{2\rho}\,, \qquad \psi_\rho = 0 \quad \text{on} \quad B_\rho\,. \tag{3.4}$$
In particular, ψ_ρ depends only on the variable r, $\psi_\rho = \psi_\rho(r)$.
- The radius $\rho \in (0,1)$ depends only on the dimension d and is chosen in Proposition 3.1.
- $h_g : B_1 \to \mathbb{R}$ is the harmonic extension of g to the cone $\mathcal{C}_S := \{(r,\sigma) \ : \ r > 0,\ \sigma \in S\}$,
$$\Delta h_g = 0 \quad \text{in} \quad \mathcal{C}_S\,, \qquad h_g = g \quad \text{on} \quad \partial B_1\,, \qquad h_g = 0 \quad \text{on} \quad B_1 \setminus \mathcal{C}_S\,. \tag{3.5}$$
- $h_1 : B_1 \to \mathbb{R}$ is defined by
$$h_1(r,\theta) := \begin{cases} c_1\, r\, \phi_1(\theta) & \text{if } r \in [\rho, 1]\,, \\ \rho\, \bar{h}(r/\rho, \theta) & \text{if } r \in [0, \rho]\,, \end{cases} \tag{3.6}$$
where $\bar{h}$ is the competitor from Proposition 3.2 corresponding to the trace $c_1\phi_1$.

The epiperimetric inequality (1.6) is then a consequence of the following two propositions. The first deals with the terms corresponding to the higher eigenvalues, for which the energy can easily be improved by taking the harmonic extension.

Proposition 3.1 (Homogeneity improvement of the higher modes: the external variation)**.** *Let $S \subset \partial B_1$ be an open set and $g \in H^1_0(S)$ be a function, expressed in Fourier series over S as*
$$g(\theta) = \sum_{j=k}^{\infty} c_j \phi_j^S(\theta)\,, \quad \text{where} \quad k \geq 1 \quad \text{is such that} \quad \lambda_k^S > d - 1.$$
Then, there are constants $\varepsilon_0 > 0$ and $\rho_0 > 0$, depending only on the dimension and the gap $\lambda_k^S - (d-1)$, such that for every $0 < \varepsilon \leq \varepsilon_0$ and $0 < \rho \leq \rho_0$ we have
$$W_0(\psi_\rho h_g) - (1-\varepsilon) W_0(z_g) \leq 0, \tag{3.7}$$
where $z_g(r,\theta) = r g(\theta)$, and ψ_ρ and h_g satisfy (3.4) *and* (3.5)*, respectively.*

The second proposition deals with the projection of c onto the first eigenfunction and is more difficult. In this case, the energy term no longer dominates the measure term and the construction of the competitor is more complicated.

Proposition 3.2 (Epiperimetric inequality for the first mode: the internal variation)**.** *Let $b \in H^1(B_1, \mathbb{R}_+)$ be a one-homogeneous minimizer of the Alt-Caffarelli functional. There exists $\varepsilon = \varepsilon(b,d) > 0, \gamma = \gamma(b,d) \in [0,1)$ and $\delta_0 = \delta_0(b,d) > 0$ such that the following holds. If*

- *$\{c > 0\} = S \subset \partial B_1$ is a connected spherical set whose boundary is given as the (spherical) graph of the function $\zeta \in C^{2,\alpha}(\partial\Omega)$ satisfying* (1.5)*,*
- *$c \equiv \kappa\, \phi_1^S : \partial B_1 \to \mathbb{R}$ (a multiple of the first Dirichlet eigenfunction of S), extended by zero on $\partial B_1 \setminus S$, where $\kappa \in \mathbb{R}$ satisfies $0 < \kappa < 2\kappa_0$ (defined in* (2.4)*),*

then there exists a function $\bar{h} \in H^1(B_1, \mathbb{R}_+)$ such that $\bar{h} = c$ on ∂B_1 and
$$W(\bar{h}) - W(b) \leq \Big(1 - \varepsilon_1 \big|W(z) - W(b)\big|^\gamma\Big)\big(W(z) - W(b)\big) \tag{3.8}$$
where z is the 1-homogeneous extension of c to B_1. Moreover, for every $r \in (0,1)$, $\bar{h}(r,\cdot)$ is a positive multiple of the first Dirichlet eigenfunction on the spherical set $S_r := \{\bar{h}(r,\cdot) > 0\} \subset \partial B_r$. Furthermore, if b is integrable through rotations, then we can take $\gamma \equiv 0$ in (3.8)*.*

3.2. **Proof of Theorem 3.** We set for simplicity, in the notation of (3.1)-(3.2),
$$z_1(r,\theta) := c_1 r \phi_1(\theta) \qquad \text{and} \qquad z_g(r,\theta) := r g(\theta).$$
Since the eigenfunctions $\{\phi_j\}_{j\geq 1}$ are orthogonal in $L^2(\partial B_1)$ and $H^1(\partial B_1)$, we get that
$$W_0(z_1 + z_g) = W_0(z_1) + W_0(z_g).$$
Moreover, since the set $S = \{c > 0\}$ is connected, we have that $\{\phi_1 > 0\} = \{c > 0\}$. Thus
$$W(z) = W(z_1 + z_g) = W(z_1) + W_0(z_g).$$

We notice that, for every $r \in (0,1)$, the functions $h_1(r,\cdot)$ and $h_g(r,\cdot)$ are orthogonal in $L^2(\partial B_1)$, as well as in $H^1(\partial B_1)$. Indeed, $h_g(r,\cdot) \equiv 0$ if $r \le \rho_0$, while for $r \in [\rho_0, 1]$ the claim follows by the Fourier decomposition of h_g (see Subsection 3.3) and the orthogonality of the eigenfunctions. Thus,

$$W_0(h_1 + \psi_\rho h_g) = W_0(h_1) + W_0(\psi_\rho h_g).$$

Moreover, we have that

$$\{h(r,\cdot) > 0\} \subset S = \{h_1(r,\cdot) > 0\} \quad \text{for every} \quad r \in [\rho_0, 1],$$

$$\{h(r,\cdot) > 0\} = \{h_1(r,\cdot) > 0\} \quad \text{for every} \quad r \in [0, \rho_0].$$

Thus,

$$W(h) = W(h_1 + \psi_\rho h_g) \le W(h_1) + W_0(\psi_\rho h_g)\,,$$

and, for every $0 < \varepsilon \le \varepsilon_0$ (ε_0 being the constant from Proposition 3.1), we have

$$\begin{aligned}
\big(W(h) - W(b)\big) &- (1-\varepsilon)\big(W(z) - W(b)\big) \\
&\le \big(W(h_1) + W_0(\psi_\rho h_g) - W(b)\big) - (1-\varepsilon)\big(W(z_1) + W_0(z_g) - W(b)\big) \\
&= W(h_1) - W(b) - (1-\varepsilon)\big(W(z_1) - W(b)\big) + W_0(\psi_\rho h_g) - (1-\varepsilon) W_0(z_g) \\
&\le W(h_1) - W(b) - (1-\varepsilon)\big(W(z_1) - W(b)\big) + (\varepsilon - \varepsilon_0) W_0(z_g), \qquad (3.9)
\end{aligned}$$

where the last inequality follows by Proposition 3.1. Note that, since by (2.5) the gap $\lambda_2^S - (d-1) > \gamma(d,b)$, the constant $\varepsilon_0 > 0$ depends only on the dimension and the cone b. Also note that if $W_0(z_g) > (W(z_1) - W(b))$, then we can let $h_1 = z_1$ and $\varepsilon = \varepsilon_0/2$ and get the epiperimetric inequality. Thus we can assume that $W_0(z_g) \le \big(W(z_1) - W(b)\big)$, so that

$$W(z) - W(b) \le 2\big(W(z_1) - W(b)\big). \qquad (3.10)$$

In order to conclude the proof, we notice that

$$\begin{aligned}
W(h_1) &= \int_{B_1} |\nabla h_1|^2 - \int_{\partial B_1} h_1^2 + \big|\{h_1 > 0\} \cap B_1\big| \\
&= \int_{B_1 \setminus B_\rho} |\nabla z_1|^2 - \int_{\partial B_1} z_1^2 + \big|\{z_1 > 0\} \cap (B_1 \setminus B_\rho)\big| + \int_{B_\rho} |\nabla h_1|^2 + \big|\{h_1 > 0\} \cap B_\rho\big| \\
&= \int_\rho^1 \left[\int_{\partial B_1} \big(|\nabla_\theta z_1|^2 - (d-1) z_1^2 + \mathbb{1}_{\{z_1 > 0\}}\big)\right] r^{d-1}\, dr \\
&\qquad + \int_{B_\rho} |\nabla h_1|^2 - d \int_0^\rho \left[\int_{\partial B_1} z_1^2\right] r^{d-1}\, dr + \big|\{h_1 > 0\} \cap B_\rho\big| \\
&= (1-\rho^d) \int_0^1 \left[\int_{\partial B_1} \big(|\nabla_\theta z_1|^2 - (d-1) z_1^2 + \mathbb{1}_{\{z_1 > 0\}}\big)\right] r^{d-1}\, dr \\
&\qquad + \int_{B_\rho} |\nabla h_1|^2 - \rho^d \int_{\partial B_1} z_1^2 + \big|\{h_1 > 0\} \cap B_\rho\big| \\
&= (1-\rho^d) W(z_1) + \rho^d W(h_1, \rho) = (1-\rho^d) W(z_1) + \rho^d W(\bar h).
\end{aligned}$$

Let $\bar\varepsilon > 0$. By Proposition 3.2, we then have

$$\begin{aligned}
\big(W(h_1) - W(b)\big) &- (1-\bar\varepsilon)\big(W(z_1) - W(b)\big) \\
&= (1-\rho^d)\big(W(z_1) - W(b)\big) + \rho^d\big(W(\bar h) - W(b)\big) - (1-\bar\varepsilon)\big(W(z_1) - W(b)\big) \\
&\le (\bar\varepsilon - \rho^d)\big(W(z_1) - W(b)\big) + \rho^d\Big(1 - \varepsilon_1 |W(z_1) - W(b)|^\gamma\Big)\big(W(z_1) - W(b)\big) \\
&= \Big(\bar\varepsilon - \rho^d \varepsilon_1 |W(z_1) - W(b)|^\gamma\Big)\big(W(z_1) - W(b)\big). \qquad (3.11)
\end{aligned}$$

Note, all the assumptions of Proposition 3.2 are obviously satisfied except for the condition that $0 < c_1 < 2\kappa_0$. This follows from the hypothesis in Theorem 3 that $\|c - b\|_{L^2} < \delta$ and the fact that

$b=\kappa_0\,\phi_1$ on ∂B_1 (see (2.4)). Note that by (3.10) we can replace the term $\varepsilon_1|W(z_1)-W(b)|^\gamma$ by the smaller $2^{-\gamma}\varepsilon_1|W(z)-W(b)|^\gamma$. Setting $\rho=\rho_0$ and

$$\bar\varepsilon=\min\big\{\varepsilon_0,\rho_0^d 2^{-\gamma}\varepsilon_1|W(z)-W(b)|^\gamma\big\},$$

and using the estimates (3.9) and (3.11), we obtain the epiperimetric inequality (1.6) with $\varepsilon=\rho_0^d2^{-\gamma}\varepsilon_1$, where ε_0, ρ_0,γ and ε_1 are the constants from Proposition 3.1 and Proposition 3.2, depending only on b and the dimension d. □

We prove Proposition 3.1 in Subsection 3.3 using a general argument from [26]. The proof of Proposition 3.2 is more involved and is contained in Subsections 3.4, 3.6 and 3.5.

3.3. **Homogeneity improvement for the higher modes: proof of Proposition 3.1.** In this subsection we prove Proposition 3.1. The proof will be a consequence of the following two Lemmas from [26] of which we recall the statements below. The first lemma shows how the harmonic extension of the high modes has smaller energy than the 1-homogeneous extension.

Lemma 3.3 (Harmonic extension [26, Lemma 2.5])**.** *Let $S\subset\partial B_1$ be an open subset of the unit sphere and g, h_g and z_g be as in Proposition 3.1. Then $W_0(z_g)>0$ and*

$$W_0(h_g)-(1-\varepsilon)W_0(z_g)\le 0\quad\textit{for every}\quad 0<\varepsilon\le\frac{\alpha_k-1}{d+\alpha_k-1}\,,\tag{3.12}$$

where $\alpha_k=\alpha_k^S$ is the exponent from Subsection 2.2.

The second lemma shows that properly cutting-off the harmonic competitor near the origin preserves the energy improvement. This is needed to preserve the orthogonality between h_g and h_1.

Lemma 3.4 (Truncation of the harmonic extension [26, Lemma 2.6])**.** *Let $S\subset\partial B_1$ be an open subset of the unit sphere and g, h_g and z_g be as in Proposition 3.1. Let $\rho>0$ and $\psi:=\psi_\rho$ be given by (3.4). Then, there is a dimensional constant $C_d>0$ such that*

$$W_0(\psi_\rho h_g)\le\left(1+\frac{C_d\,\alpha_k\,(2\rho)^{2(\alpha_k-1)+d}}{\alpha_k-1}\right)W_0(h_g),\tag{3.13}$$

where $\alpha_k=\alpha_k^S$ is the homogeneity exponent corresponding to the eigenvalue λ_k^S (see Subsection 2.2).

Proof of Proposition 3.1. We only consider the case $W_0(z_g)>0$ since otherwise the statement is trivial. Thus, it is sufficient to prove (3.7) for $\varepsilon=\varepsilon_0$. By Lemma 3.3 and Lemma 3.4 we have

$$\begin{aligned}W_0(\psi_\rho h_g)&\le\left(1+\frac{C_d\,\alpha_k\,(2\rho)^{2(\alpha_k-1)+d}}{\alpha_k-1}\right)W_0(h_g)\\&\le\left(1+\frac{C_d\,\alpha_k\,(2\rho_0)^{2(\alpha_k-1)+d}}{\alpha_k-1}\right)\left(1-\frac{\alpha_k-1}{d+\alpha_k-1}\right)W_0(z_g)\le(1-\varepsilon_0)W_0(z_g),\end{aligned}$$

for some constants $\varepsilon_0>0$ and $\rho_0>0$ depending only on the dimension and the gap $\alpha_k-1>0$. □

3.4. **Internal variation and slicing.** This subsection contains a preliminary result for the proof of Proposition 3.2. We treat it separately since it offers a new perspective on the epiperimetric inequality. Suppose that $u:B_1\to\mathbb{R}^+$ is a minimizer of the Alt-Caffarelli functional $\mathcal{E}$ in B_1 such that $0\in\partial\{u>0\}$ and b is a blow-up of u at 0. By definition, u is the best choice for a test function in the left-hand side in the epiperimetric inequality

$$W(u)-W(b)\le(1-\varepsilon)\big(W(z)-W(b)\big).$$

The Lipschitz continuity of u implies that using the coordinates $(r,\theta)\in\mathbb{R}^+\times\mathbb{S}^{d-1}$, the function u can be written in the form $u(r,\theta)=r\phi_r(\theta)$, for some $\phi_r\in L^2(\partial B_1)$. Thus u can be identified with the flow $r \mapsto \phi_r \in L^2(\partial B_1)$ and so, given a trace $c : \partial B_1 \to \mathbb{R}^+$, it is not restrictive to search for a competitor h, written directly in the form $h(r,\theta) = r\phi_r(\theta)$.

We introduce the following functional, which will be helpful in our analysis. For every function $c \in H^1(\partial B_1)$ on the $(d-1)$-dimensional sphere $\partial B_1 \subset \mathbb{R}^d$, we define

$$E(c) := \int_{\partial B_1} \Big(|\nabla_\theta c|^2 - (d-1)c^2\Big)\, d\mathcal{H}^{d-1} + \mathcal{H}^{d-1}\big(\{c>0\}\cap \partial B_1\big). \tag{3.14}$$

Remark 3.5. We notice that for the one-homogeneous function $z(r,\theta) = r\,c(\theta)$ we have

$$W(z) = \int_0^1 \left[\int_{\partial B_1} \Big(|\nabla_\theta c|^2 - (d-1)c^2 + \mathbb{1}_{\{c>0\}}\Big)\, d\mathcal{H}^{d-1}(\theta)\right] r^{d-1}\, dr = \frac{1}{d}E(c).$$

In particular, $E(c) = d\,W(b)$ if c is the trace of the one-homogeneous solution b on the cone $\{b>0\}\cap\partial B_1$. Moreover, if c is the first eigenfunction, $c = \phi_1^{\Omega_\zeta}$, for some domain $\Omega_\zeta \subset \partial B_1$ whose boundary $\partial\Omega_\zeta$ is the graph of ζ over $\partial\Omega$, then in the notation of Subsection 2.3 we have

$$E(c) = \lambda(\zeta) - (d-1) + m(\zeta) = \mathcal{F}(\zeta) + m(0).$$

The following lemma relates the energy, W, of a function, u, to the energy, E, of its slices, ϕ_r.

Lemma 3.6 (Slicing lemma)**.** *Let $v : \mathbb{R}^+ \times \partial B_1 \ni (t,\theta) \mapsto v_t(\theta)$ be such that $v(t,\theta) \in H^1([0,1]\times\partial B_1)$, $v_t \in H^1(\partial B_1)$ for every $t \in \mathbb{R}^+$ and $t \mapsto v_t$ is Lipschitz continuous as a function with values in $L^2(\partial B_1)$. Let $\eta : [0,1] \to \mathbb{R}^+$ be Lipschitz continuous. We set*

$$u(\rho,\theta) = \rho\, v_{\eta(\rho)}(\theta) \qquad \text{and} \qquad z(\rho,\theta) = \rho\, v_{\eta(1)}(\theta).$$

Then, for every $\varepsilon \geq 0$, we have

$$\begin{aligned} W(u) - (1-\varepsilon)W(z) = &\int_0^1 \Big(E(v_{\eta(r)}) - (1-\varepsilon)E(v_{\eta(1)})\Big) r^{d-1}\, dr \\ &+ \int_0^1 r^{d+1}(\eta'(r))^2 \int_{\partial B_1} |\partial_t v_{\eta(r)}|^2\, dr. \end{aligned} \tag{3.15}$$

Moreover, the same inequality holds with W_0 and E_0 in place of W and E.

Proof. We first calculate the energy of u.

$$\begin{aligned} W_0(u) &= \int_{B_1} |\nabla u|^2\, dx - \int_{\partial B_1} u^2\, d\theta \\ &= \int_0^1 r^{d-1} \int_{\partial B_1} \Big(\big(v_{\eta(r)} + r\eta'(r)\partial_t v_{\eta(r)}\big)^2 + \frac{1}{r^2}(r|\nabla_\theta v_{\eta(r)}|)^2\Big)\, d\theta\, dr - \int_{\partial B_1} v_{\eta(1)}^2\, d\theta \\ &= \int_0^1 r^{d-1} \int_{\partial B_1} \Big(v_{\eta(r)}^2 + r\partial_r\Big(v_{\eta(r)}^2\Big) + r^2(\eta'(r))^2|\partial_t v_{\eta(r)}|^2 + |\nabla_\theta v_{\eta(r)}|^2\Big)\, d\theta\, dr - \int_{\partial B_1} v_{\eta(1)}^2\, d\theta \\ &= \int_0^1 r^{d-1} \int_{\partial B_1} \Big((\eta'(r))^2 r^2 |\partial_t v_{\eta(r)}|^2 + |\nabla_\sigma v_{\eta(r)}|^2 - (d-1)v_{\eta(r)}^2\Big)\, d\theta\, dr, \end{aligned}$$

where in the last step we integrated by parts in r. Thus, we get

$$\begin{aligned} W(u) &= W_0(u) + \big|B_1 \cap \{u>0\}\big| \\ &= \int_0^1 \bigg[\int_{\partial B_1} \Big((\eta'(r))^2 r^2 |\partial_t v_{\eta(r)}|^2 + |\nabla_\theta v_{\eta(r)}|^2 - (d-1)v_{\eta(r)}^2\Big)\, d\theta \\ &\qquad\qquad + \mathcal{H}^{d-1}\big(\partial B_1 \cap \{v_{\eta(r)} > 0\}\big)\bigg] r^{d-1}\, dr. \end{aligned}$$

Analogously, the energy of the one-homogeneous extension z is given by

$$W(z) = \int_0^1 \left[\int_{\partial B_1} \Big(|\nabla_\theta v_{\eta(1)}|^2 - (d-1)v_{\eta(1)}^2\Big)\, d\theta + \mathcal{H}^{d-1}\big(\partial B_1 \cap \{v_{\eta(1)} > 0\}\big)\right] r^{d-1}\, dr. \qquad \square$$

The identity (3.15) suggests that in order to obtain the epiperimetric inequality (3.8) we have to construct a flow, $t \mapsto v_t$, that decreases the energy $\frac{1}{d}E(v_t) - W(b)$. To do this we will look at the spectrum of the bilinear form that corresponds to the second variation of $\mathcal{F}$. On the other hand, changing the initial trace $c = v_{\eta(1)}$ has a cost due to the last term in (3.15). The balance between this last term and the energy E will be the main subject of the following subsection.

3.5. **Proof of Proposition 3.2.** Let $K := \ker(\delta^2\mathcal{F}(0))$, $N := \dim K$ and $p_K, p_{K^\perp}$ be the $L^2(\partial\Omega)$-projections on K and $K^\perp$, respectively. Let $\zeta \in C^{2,\alpha}(\partial\Omega)$ be given (as in (1.5)) and

$$\zeta = \zeta^T + \zeta^\perp = \sum_{j=1}^N \mu_j \iota_j + \zeta^\perp\,, \quad \text{where } \zeta^\perp = p_{K^\perp}\zeta \text{ and } \{\iota_1, \dots, \iota_N\} \text{ is an orthonormal basis of } K.$$

Moreover, slightly abusing the notation, we will write $\mu^0 := \sum_{j=1}^N \mu_j \iota_j \in K \subset H^{1/2}(\partial\Omega)$, and also $\mu^0 = (\mu_1, \dots, \mu_N) \in \mathbb{R}^N$, so that $|\mu^0| = \|\zeta^T\|_{H^{1/2}(\partial\Omega)}$.
Let $\Upsilon\colon K \times \mathbb{R} \to K^\perp$ be the map of Lemma B.1 and let $\tilde\zeta := \zeta^\perp - \Upsilon(\mu^0, s^0) \in K^\perp$. Then we can decompose $\tilde\zeta = \zeta_1 + \zeta_2$, where ζ_1, ζ_2 are defined by

$$\zeta_1 := \sum_{\lambda_i > 0} p_i \xi_i \qquad \text{and} \qquad \zeta_2 := \sum_{\lambda_i < 0} p_i\, \xi_i\,,$$

where $(\xi_i)_i$ is the orthonormal basis of $K^\perp$ given by Lemma A.1 (so $(\xi_i)_{i\in\mathbb{N}} \cup (\iota_j)_{j=1}^N$ is an orthonormal basis of $H^{1/2}(\partial\Omega)$) and λ_i is the eigenvalue of $\delta^2\mathcal{F}(0)$ relative to ξ_i.
Finally recall the function $\mathcal{G} : C^{2,\alpha}(\partial\Omega) \oplus \mathbb{R} \to \mathbb{R}$ defined in (B.1)

$$\mathcal{G}(\zeta, s) = (\kappa_0^2 + s^3)(\lambda(\zeta) - (d-1)) + m(\zeta) - m(0).$$

Definition of the competitor. We will define the competitor $\bar h(r,\theta)$ in such a way that, for each $r > 0$, the positivity set $\{\bar h(r,\cdot) > 0\} \subset \partial B_1$ will be given by the spherical set Ω_{g_r}, whose boundary is the graph of $g_r : \partial\Omega \to \mathbb{R}$, where

$$g_r(\theta) = g(r,\theta) := \eta_1(r)\zeta_1(\theta) + \eta_2(r)\,\zeta_2(\theta) + \Big(\mu(\eta_3(r)) + \Upsilon\big(\mu(\eta_3(r)), s(\eta_3(r))\big)\Big)\,, \tag{3.16}$$

where the function $\mathbb{R} \ni t \mapsto (\mu(t), s(t)) \in \mathbb{R}^N \oplus \mathbb{R}$ is defined through the gradient flow

$$\begin{cases} (\mu'(t), s'(t)) = \dfrac{-\nabla G(\mu(t), s(t))}{|\nabla G(\mu(t), s(t))|} \\ (\mu(0), s(0)) = (\mu^0, s^0), \end{cases} \tag{3.17}$$

the function $G\colon \mathbb{R}^N \oplus \mathbb{R} \to \mathbb{R}$ is the analytic function of Lemma B.1 defined as $G(\mu, s) := \mathcal{G}(\mu + \Upsilon(\mu, s), s)$ and s^0 will be defined below. If $|\nabla G(\mu, s)| = 0$, then we set $(\mu', s') = 0$. The functions, η_1, η_2, η_3, will be chosen later, with the properties that $\eta_1(1) = 1 = \eta_2(1)$ and $\eta_3(1) = 0$, so that

$$g(1,\cdot) = \zeta_1 + \zeta_2 + \Big(\mu^0 + \Upsilon(\mu^0)\Big) = \zeta^\perp - \Upsilon(\mu^0) + \Big(\zeta^T + \Upsilon(\mu^0)\Big) = \zeta^\perp + \zeta^T = \zeta\,.$$

For every $r \in (0,1)$, we denote the first Dirichlet eigenfunction, the first eigenvalue and the measure of Ω_{g_r} by

$$\phi_{g_r} := \phi_1^{\Omega_{g_r}}\,, \qquad \lambda(g_r) := \lambda_1^{\Omega_{g_r}} \qquad \text{and} \qquad m(g_r) := \mathcal{H}^{d-1}(\Omega_{g_r})\,.$$

We define the competitor $\bar h$ in polar coordinates as

$$\bar h(r,\theta) = r\kappa_r\phi_{g_r}(\theta), \tag{3.18}$$

where $\kappa_r^2 = \kappa_0^2 + s^3(\eta_3(r))$, and $s(t)$ is given by (3.17), where $s(0) = s^0 \in \mathbb{R}$ is chosen such that $\kappa_0^2 + (s^0)^3 = \kappa^2$. Recall that the one-homogeneous extension z of the trace $c = \kappa\,\phi_\zeta = \kappa\,\phi_{g_1}$ is given by $z(r,\theta)=r\kappa\phi_{g_1}(\theta)$. We notice that, by the Slicing Lemma 3.6, we have

$$\begin{aligned}&\Big|\Big(W\big(\bar h(r,\theta)\big)-(1-\varepsilon)W(z)+\varepsilon W(b)\Big)-\int_0^1\Big[\mathcal G\big(g_r,s(\eta_3(r))\big)-(1-\varepsilon)\mathcal G(\zeta,s^0)\Big]r^{d-1}\,dr\Big|\\&\qquad\le\int_0^1 r^{d+1}\frac{9s(\eta_3(r))^4}{\kappa_r^2}[s'(\eta_3(r))]^2[\eta_3'(r)]^2\,dr+\int_0^1\kappa_r^2r^{d+1}\int_{\partial B_1}\big|\delta\phi_{g_r}[g'(r)]\big|^2\,d\sigma\,dr.\end{aligned}\tag{3.19}$$

Setting for simplicity $s=s(\eta_3(r))$, $\mu=\mu(\eta_3(r))$ and $g=g(r)$, we can write

$$\begin{aligned}\mathcal G(g,s)-(1-\varepsilon)\mathcal G(\zeta,s^0)&=\mathcal G(g,s)-\mathcal G\big(\mu+\Upsilon(\mu,s),s\big)+\mathcal G\big(\mu+\Upsilon(\mu,s),s\big)\\&\quad-(1-\varepsilon)\Big(\mathcal G(\zeta,s^0)-\mathcal G\big(\mu^0+\Upsilon(\mu^0,s^0),s^0\big)+\mathcal G\big(\mu^0+\Upsilon(\mu^0,s^0),s^0\big)\Big)\\&=\underbrace{\mathcal G(g,s)-\mathcal G\big(\mu+\Upsilon(\mu,s),s\big)-(1-\varepsilon)\Big(\mathcal G(\zeta,s^0)-\mathcal G\big(\mu^0+\Upsilon(\mu^0,s^0),s^0\big)\Big)}_{=:E^\perp}\\&\quad+\underbrace{G(\mu,s)-(1-\varepsilon)G\big(\mu^0,s^0\big)}_{=:E^T}\,.\end{aligned}\tag{3.20}$$

We will estimate separately $E^\perp$, E^T and the two error terms in the right-hand side of (3.19).

Estimate of $E^\perp$. For what concerns $E^\perp$, we first notice that, by (3.16),

$$g(r)-\mu(\eta_3(r))-\Upsilon\big(\mu(\eta_3(r)),s(\eta_3(r))\big)=\eta_1(r)\zeta_1+\eta_2(r)\zeta_2=:\hat\zeta(r)\in K^\perp.$$

Using the identity

$$f(1)=f(0)+f'(0)+\frac12A+\int_0^1(1-t)\big(f''(t)-A\big)\,dt\,,\quad\text{for}\quad A\in\mathbb R,$$

for the function $f(t):=G\big(\mu+\Upsilon(\mu,s)+t\hat\zeta,s\big)$ and $A=\delta^2\mathcal F(0)[\hat\zeta,\hat\zeta]$ we get

$$\begin{aligned}&\Big|\Big(\mathcal G(g,s)-\mathcal G\big(\mu+\Upsilon(\mu,s),s\big)\Big)-\frac12\delta^2\mathcal F(0)[\hat\zeta,\hat\zeta]\Big|\\&\qquad\le\sup_{t\in[0,1]}\Big|\delta^2\mathcal G\big(\mu+\Upsilon(\mu,s)+t\hat\zeta,s\big)-\delta^2\mathcal F(0)\Big|\,\|\hat\zeta\|^2_{H^{1/2}}=:S_1\,\|\hat\zeta\|^2_{H^{1/2}},\end{aligned}\tag{3.21}$$

where we used the fact that $\delta^2\mathcal G(0,0)=\delta^2\mathcal F(0)$ and, by (B.4),

$$f'(0)=\delta\mathcal G\big(\mu+\Upsilon(\mu,s),s\big)[(\hat\zeta,0)]=P_{K^\perp}\left(\delta\mathcal G\big(\mu+\Upsilon(\mu,s),s\big)\right)[(\hat\zeta,0)]=0.$$

Moreover, since $\hat\zeta(r)=\eta_1(r)\zeta_1+\eta_2(r)\zeta_2$ and ζ_1 and ζ_2 are sums of orthogonal eigenfunctions of $\delta^2\mathcal F(0)$, we get that

$$\delta^2\mathcal F(0)[\hat\zeta,\hat\zeta]=\delta^2\mathcal F(0)[\eta_1\zeta_1+\eta_2\zeta_2,\eta_1\zeta_1+\eta_2\zeta_2]=\eta_1^2\,\delta^2\mathcal F(0)[\zeta_1,\zeta_1]+\eta_2^2\,\delta^2\mathcal F(0)[\zeta_2,\zeta_2].$$

Analogously, since $\zeta=\mu^0+\Upsilon(\mu^0,s^0)+\tilde\zeta$ we have

$$\begin{aligned}&\Big|\Big(\mathcal G(\zeta,s^0)-\mathcal G\big(\mu^0+\Upsilon(\mu^0,s^0),s^0\big)\Big)-\frac12\Big(\delta^2\mathcal F(0)[\zeta_1,\zeta_1]+\delta^2\mathcal F(0)[\zeta_2,\zeta_2]\Big)\Big|\\&\qquad\le\sup_{t\in[0,1]}\Big|\delta^2\mathcal G\big(\mu^0+\Upsilon(\mu^0,s^0)+t\tilde\zeta,s^0\big)-\delta^2\mathcal F(0)\Big|\,\|\tilde\zeta\|^2_{H^{1/2}}=:S_2\,\|\tilde\zeta\|^2_{H^{1/2}}.\end{aligned}\tag{3.22}$$

Choosing $\eta_1(r)=1-(d+1)\varepsilon(1-r)$ and $\eta_2(r)=1$ we get that $\|\hat\zeta(r)\|^2_{H^{1/2}}\le\|\zeta_1\|^2_{H^{1/2}}+\|\zeta_2\|^2_{H^{1/2}}=\|\tilde\zeta\|^2_{H^{1/2}}$. Thus, combining the inequalities (3.21) and (3.22), we get that there is a dimensional constant $C_d > 0$ such that for every $\varepsilon \le 1/4$ we have

$$\begin{aligned}\int_0^1 E^\perp r^{d-1}\,dr &\le \frac{1}{2}\delta^2\mathcal{F}(0)[\zeta_1,\zeta_1]\int_0^1(\eta_1^2(r)-(1-\varepsilon))r^{d-1}\,dr\\ &\qquad+\frac{1}{2}\delta^2\mathcal{F}(0)[\zeta_2,\zeta_2]\int_0^1(\eta_2^2(r)-(1-\varepsilon))r^{d-1}\,dr+\int_0^1(|S_1|+|S_2|)\,r^{d-1}\,dr\,\|\tilde\zeta\|^2_{H^{1/2}}\\ &\le C_d\,\varepsilon\left(-\left(\min_{\{i|\lambda_i>0\}}\lambda_i\right)\|\zeta_1\|^2+\left(\max_{\{i|\lambda_i<0\}}\lambda_i\right)\|\zeta_2\|^2\right)r+\int_0^1(|S_1|+|S_2|)\,r^{d-1}\,dr\,\|\tilde\zeta\|^2_{H^{1/2}}\\ &\le\left(-C_{d,b}\varepsilon+\int_0^1(|S_1|+|S_2|)\,r^{d-1}\,dr\right)\|\tilde\zeta\|^2_{H^{1/2}}\,.\end{aligned}$$

In order to bound S_1 and S_2, we notice that the second variation of $\mathcal{G}$ is given by

$$\delta^2\mathcal{G}(g,s)[(\zeta,0),(\zeta,0)]=\delta^2\mathcal{F}(g)[\zeta,\zeta]+s^3\delta^2\lambda(g)[\zeta,\zeta]\,,\quad\text{for every}\quad s\in\mathbb{R}\,,\ g,\zeta\in C^{2,\alpha}(\partial\Omega).$$

Applying this formula to the second variation $\delta^2\mathcal{G}\big(\mu+\Upsilon(\mu,s)+t\hat\zeta,s\big)$ in the directions $(\hat\zeta,0)$ and $(\tilde\zeta,0)$, then using (2.7) and the estimate $\delta^2\lambda(g)[\tilde\zeta,\tilde\zeta]\le C\|\tilde\zeta\|^2_{H^{1/2}}$ (see Subsection A.5), we get that there is a modulus of continuity ω such that

$$|S_1|+|S_2|\le C_{d,b}\big(\omega(\mu+\|\hat\zeta\|_{C^{2,\alpha}})+\omega(\mu^0+\|\tilde\zeta\|_{C^{2,\alpha}})+s^3\big).$$

Recall that $\tilde\zeta=\zeta_1+\zeta_2$, where ζ_2 is a finite linear combination of eigenfunctions of the operator T (see Lemma 2.3) with coefficients which are bounded by $\|\zeta\|_{H^{1/2}}$. In particular, there is a constant $C_{d,b}$, depending only on the $C^{2,\alpha}$ norms of the eigenfunctions in the index of the cone and in the kernel of $\delta^2\mathcal{F}(0)$, such that

$$\|\zeta_1\|_{C^{2,\alpha}}+\|\zeta_2\|_{C^{2,\alpha}}\le\|\zeta\|_{C^{2,\alpha}}+\|\mu_0+\Upsilon(\mu_0,s_0)+\zeta_2\|_{C^{2,\alpha}}+\|\zeta_2\|_{C^{2,\alpha}}\le C_{d,b}\|\zeta\|_{C^{2,\alpha}}.$$

Thus, we get

$$|S_1|+|S_2|\le C_{d,b}\left(\omega(\|\zeta\|_{C^{2,\alpha}})+s^3\right).$$

so that, up to choosing the $\|\zeta\|_{C^{2,\alpha}}$ and s small enough, we have

$$\int_0^1 E^\perp r^{d-1}\,dr\le -C_{d,b}\,\varepsilon\,\|\tilde\zeta\|^2_{H^{1/2}}.\tag{3.23}$$

Estimate of E^T. We can assume without loss of generality that $G(\mu^0,s^0)>0$, otherwise set $\mu\equiv\mu^0$ and $s\equiv s^0$, and the conclusion holds for all sufficiently small ε depending only on b and $\delta>0$. By the Łojasievicz inequality for the analytic function G (see [20]), there exist a neighborhood U of $(0,0)\in\mathbb{R}^N\times\mathbb{R}$ and constants $C,\gamma>0$, depending on b and the dimension d, with $\gamma\in(0,1/2]$, such that

$$|G(\mu,s)|^{1-\gamma}\le C\,|\nabla G(\mu,s)|\,,\qquad\text{for every }(\mu,s)\in U\,.\tag{3.24}$$

Therefore, as long as $0<G(\mu(t),s(t))$, we can use the flow (3.17) to estimate,

$$\begin{aligned}G(\mu(t),s(t))-G(\mu^0,s^0)&=\int_0^t\frac{d}{d\tau}G(\mu(\tau),s(\tau))\,d\tau=\int_0^t\nabla G(\mu(\tau),s(\tau))\cdot(\mu'(\tau),s'(\tau))\,d\tau\\ &=-\int_0^t|\nabla G(\mu(\tau),s(\tau))|\,d\tau\le 0,\end{aligned}\tag{3.25}$$

so that the function $t\mapsto G(\mu(t),s(t))$ is non increasing to 0, and therefore there exists a first time $t_1>0$ such that

$$\begin{cases}G(\mu(t),s(t))\ge\frac{1}{2}G(\mu^0,s^0)>0 & \text{if }0\le t\le t_1\\ G(\mu(t),s(t))\le\frac{1}{2}G(\mu^0,s^0) & \text{if }t\ge t_1\,.\end{cases}$$

If $\eta_3(r) \le t_1$ then using (3.25) we get

$$\begin{aligned} G\big(\mu(\eta_3(r)), s(\eta_3(r))\big) - (1-\varepsilon)G(\mu^0, s^0) &= -\int_0^{\eta_3(r)} |\nabla G(\mu(\tau), s(\tau))|\, d\tau + \varepsilon\, G(\mu^0, s^0) \\ &\le -C\int_0^{\eta_3(r)} |G(\mu(\tau), s(\tau))|^{1-\gamma}\, d\tau + \varepsilon\, G(\mu^0, s^0) \\ &\le -C\,|G(\mu(\eta_3(r)), s(\eta_3(r)))|^{1-\gamma}\,\eta_3(r) + \varepsilon\, G(\mu^0, s^0) \\ &\le -C\,|G(\mu^0, s^0)|^{1-\gamma}\,\eta_3(r) + \varepsilon\, G(\mu^0, s^0) \\ &= -\big(C\eta_3(r) - \varepsilon G(\mu^0, s^0)^\gamma\big)\; G(\mu^0, s^0)^{1-\gamma}, \end{aligned} \tag{3.26}$$

where in the first inequality we used the Łojasiewicz inequality (3.24), the second follows by the monotonicity of g and the third by the assumption $\eta_3(r) \le t_1$.
If $\eta_3(r) > t_1$, then choosing $\eta_3(r) := \dfrac{d+2}{C}\varepsilon G(\mu^0, s^0)^\gamma(1-r)$, we have

$$\begin{aligned} G\big(\mu(\eta_3(r)), s(\eta_3(r))\big) - (1-\varepsilon)G(\mu^0, s^0) &\le -\left(\frac{1}{2} - \varepsilon\right) G(\mu^0, s^0) \\ &\le -\big(C\,\eta_3(r) - \varepsilon G(\mu^0, s^0)^\gamma\big)\, G(\mu^0, s^0)^{1-\gamma}. \end{aligned}$$

Therefore, for every $r \in [0,1]$, we have the estimate

$$\begin{aligned} \int_0^1 E^T r^{d-1} dr &= \int_0^1 \Big(G\big(\mu(\eta_3(r)), s(\eta_3(r))\big) - (1-\varepsilon)G(\mu^0, s^0)\Big) r^{d-1} dr \\ &\le -\,G(\mu^0, s^0)^{1-\gamma} \int_0^1 \big(C\eta_3(r) - \varepsilon G(\mu^0, s^0)^\gamma\big)\, r^{d-1}\, dr \\ &= -\,\varepsilon G(\mu^0, s^0) \int_0^1 \big((d+1) - (d+2)r\big) r^{d-1} dr = -\frac{\varepsilon G(\mu^0, s^0)}{d(d+1)}. \end{aligned} \tag{3.27}$$

Estimating the two error terms in the right-hand side of (3.19)**.** For the radial term, we notice that since s is 1-Lipschitz, for $\kappa_0/2 \le s_0 \le 2\kappa_0$ and ε and $G(\mu^0, s^0)$ small enough, we have

$$\frac{\kappa_0}{4} \le s^0 - \eta_3(r) \le s(\eta_3(r)) \le s^0 + \eta_3(r) \le 4\kappa_0.$$

Using this estimate in the κ direction we get

$$\int_0^1 r^{d+1} \frac{9s(\eta_3(r))^4}{\kappa_r^2} [s'(\eta_3(r))]^2 [\eta_3'(r)]^2\, dr \le C_{d,b}\, \varepsilon^2\, G(\mu^0, s^0)^{2\gamma}. \tag{3.28}$$

We now estimate the second term in the right-hand side of (3.19). By (3.16) we have

$$g'(r) = \partial_r g(r,\theta) = \eta_1'(r)\zeta_1(\theta) + \eta_3'(r)\Big(\mu'(\eta_3(r)) + \big(\mu'(\eta_3(r)), s'(\eta_3(r))\big)\cdot \nabla\Upsilon\big(\mu(\eta_3(r)), s(\eta_3(r))\big)\Big)\,.$$

Since $\|\zeta_1\|_{L^\infty(\partial\Omega)} \le C_{d,b}$, $|\mu'| \le 1$ and $|\nabla\Upsilon| \le C_{d,b}$ in a neighborhood of $(0,0)$, we get

$$\|g'(r)\|_{L^2(\partial\Omega)} \le C_{d,b}\big(\|\zeta_1\|_{L^2}|\eta_1'(r)| + |\eta_3'(r)|\big)\,,$$

so, using Lemma 2.3 (iii) and the definition of η_3, we conclude that

$$\int_0^1 \kappa_r^2 r^{d+1} \int_{\partial B_1} \big|\delta\phi_{g_r}[g'(r)]\big|^2\, d\sigma\, dr \le C_{d,b}\Big(\varepsilon^2\|\zeta_1\|_{L^2}^2 + \varepsilon^2 G(\mu^0, s^0)^{2\gamma}\Big). \tag{3.29}$$

Conclusion of the proof. In order to conclude the proof, we consider two cases.
Case 1: $\|\tilde\zeta\|_{H^{1/2}} > C^{-1} G(\mu^0, s^0)^{1/2}$ for some universal constant $C > 0$ depending only on b and d. In this scenario, let $\eta_3 \equiv 0$ and combine (3.19), (3.20), (3.23), (3.28) and (3.29) to get

$$W(\bar h) - W(b) - (1-\varepsilon)(W(z) - W(b)) \le -C_{d,b}\varepsilon\|\tilde\zeta\|_{H^{1/2}}^2 + \varepsilon G(\mu^0, s^0) \le \left(-C_{d,b}\varepsilon + \frac{1}{C}\right)\|\tilde\zeta\|_{H^{1/2}}^2 < 0\,,$$

where we used the fact that $\eta_3' \equiv 0$ so that no error term come from the flow. Therefore, the epiperimetric inequality holds for $\varepsilon > 0$ small but universal.

Case 2: Otherwise, we need to chose $\varepsilon > 0$ depending on $G(\mu^0, s^0)$ so that we can absorb the errors in (3.29), (3.28), into the gain (3.27). Letting $\varepsilon = \varepsilon_1 G(\mu^0, s^0)^{1-2\gamma}$ for some $\varepsilon_1 > 0$ small (but universal) and combining (3.23), (3.27), (3.28) and (3.29), we get

$$\begin{aligned} W\big(\bar{h}(r,\theta)\big) - (1-\varepsilon)W(z) + \varepsilon W(b) &\leq -C_{d,b}\,\varepsilon\,\|\tilde{\zeta}\|^2_{H^{1/2}(\partial\Omega)} - \frac{\varepsilon G(\mu^0, s^0)}{d(d+1)} \\ &\quad + C_{d,b}\Big(\varepsilon^2 \|\tilde{\zeta}\|^2_{H^{1/2}(\partial\Omega)} + \varepsilon^2 G(\mu^0, s^0)^{2\gamma}\Big) \\ &\leq -C_{d,b}\,\varepsilon_1\, G(\mu^0, s^0)^{2-2\gamma}\,. \end{aligned} \tag{3.30}$$

Finally, writing $W(z) - W(b)$ as in (3.20) and using the estimate (3.22), we obtain

$$\begin{aligned} W(z) - W(b) = \int_0^1 \mathcal{G}(\zeta, s^0)\, r^{d-1}\, dr =& \frac{1}{d}\left(\mathcal{G}(\zeta, s^0) - \mathcal{G}\big(\mu^0 + \Upsilon(\mu^0, s^0), s^0\big) + G(\mu^0, s^0)\right) \\ \leq& C_{d,b}\left(\|\tilde{\zeta}\|^2_{H^{1/2}} + G(\mu^0, s^0)\right) \leq C_{d,b} G(\mu^0, s^0)\,, \end{aligned}$$

so that

$$-\varepsilon \leq -C_{d,b}\,\varepsilon_1\,\left(W(z) - W(b)\right)^{1-2\gamma}$$

and the conclusion follows by replacing this in (3.30) with $\gamma' = 1 - 2\gamma \in [0, 1)$.

3.6. **Proof of Proposition 3.2: the integrable case.** Before starting with the proof of Proposition 3.2 in the case when b is integrable, we need a preliminary Lemma, which will allow us to kill rotations, that is to choose a parametrization for which the kernel of $\delta^2\mathcal{F}(0)$ is trivial. Throughout this subsection we assume that b is integrable through rotations (Definition 2.4).

Lemma 3.7 (Killing the linear part)**.** *There exists a dimensional constant $\delta > 0$ such that if $\zeta\colon \partial\Omega \to \mathbb{R}$ satisfies $\|\zeta\|_{C^{2,\alpha}(\partial\Omega)} < \delta$, then there exists a rotation U of the coordinate axes such that* $\mathrm{graph}_{\partial\Omega}(\zeta) = \mathrm{graph}_{U(\partial\Omega)}(\tilde{\zeta})$, *where $\tilde{\zeta}$ satisfies*

(i) $\|\tilde{\zeta}\|_{C^{2,\alpha}(U(\partial\Omega))} \leq C\,\|\zeta\|_{C^{2,\alpha}(\partial\Omega)}$, for a dimensional constant C;

(ii) if ξ_i is such that $\delta^2\mathcal{F}(0)[\xi_i, \cdot] = 0$, then $\displaystyle\int_{U(\partial\Omega)} \tilde{\zeta}\,\xi_i\, d\mathcal{H}^{n-2} = 0$.

Proof. Consider the family of functions defined by

$$\mathcal{R} := \{u\colon \partial\Omega \to \mathbb{R} \,:\, u := a_1\,\xi_1 + \cdots + a_{d-1}\,\xi_{d-1}\,,\ a_1, \ldots, a_{d-1} \in \mathbb{R}\}$$

where ξ_i, $i = 1, \ldots, (d-1)$, are the eigenfunctions of $\delta^2\mathcal{F}(0)$ relative to the eigenvalue 0 with $\|\xi_i\|_{L^2(\partial\Omega)} = 1$. Since by Lemma 2.3 rotations are always in the kernel of $\delta^2\mathcal{F}(0)$, it follows that the infinitesimal generators of rotations of $\partial\Omega_b$ are all and the only elements of $\mathcal{R}$. For any $u \in \mathcal{R}$, let U_u be the rotation generated by u of magnitude $\sum_{i=1}^{d-1} |a_i|$. By the integrability of $\partial\Omega$, there is a neighborhood $\mathcal{U}$ of 0 in $\mathcal{R}$ such that, for any $u \in \mathcal{U}$, the functions $(\xi_i^u)_{i=1,\ldots,d-1}$, defined by $\mathrm{graph}_{\partial\Omega}(\xi_i) = \mathrm{graph}_{U_u(\partial\Omega)}(\xi_i^u)$, are a basis for the kernel of $\delta^2\mathcal{F}(0)$.

We consider the functional $\Psi\colon C^{2,\alpha}(\partial\Omega) \times \mathcal{R} \to \mathbb{R}^{d-1}$, defined in a neighborhood of $(0, 0)$, by

$$\Psi(f, u) := \left(\int_{U_u(\partial\Omega)} f_u\,\xi_1^u\, d\mathcal{H}^{n-2}, \ldots, \int_{U_u(\partial\Omega)} f_u\,\xi_{d-1}^u\, d\mathcal{H}^{n-2}\right)$$

where, as above, f_u satisfies $\mathrm{graph}_{\partial\Omega}(f) = \mathrm{graph}_{U_u(\partial\Omega)}(f_u)$. Notice that, since $\mathrm{graph}_{\partial\Omega}(0) = \mathrm{graph}_{U_{t\xi_i}(\partial\Omega)}(-t\xi_i)$, we have with $u = t\xi_i$ that

$$\Psi(0, t\xi_i) = -\left(t\int_{U_u(\partial\Omega)} \xi_i^u\,\xi_1^u, \ldots, t\int_{U_u(\partial\Omega)} \xi_i^u\,\xi_{d-1}^u\right)$$

so that, identifying $\mathcal{R}$ with $\mathbb{R}^{d-1}$ using the basis $(\xi_i)_i$, we compute $\Psi(0, 0) = 0$ and $\nabla_{\mathcal{R}}\Psi(0, 0) = -\mathrm{Id}$. By the implicit function theorem, the conclusion immediately follows. □

Proof of Proposition 3.2 *for integrable cones.* We first notice that by Lemma 3.7 we can assume that $\mu^0 = 0$. Thus, in the notation of Subsection 3.5, we are left with $\zeta = \zeta^\perp = \tilde{\zeta} = \zeta_1 + \zeta_2$. The positivity set of our competitor will be determined by $g(r,\theta) = \eta_1(r)\zeta_1(\theta) + \zeta_2(\theta)$, while the competitor itself is given by $\overline{h}(r,\theta) = r\kappa_r\phi_{g_r}(\theta)$, where

$$\kappa_r^2 = \kappa_0^2 + \eta_3(r)(s^0)^3\ , \qquad (s^0)^3 = \kappa^2 - \kappa_0^2\ , \qquad \eta_3(r) = 1 - \varepsilon a(1-r),$$

and $a \in \mathbb{R}$ is given by $a = \operatorname{sign}(s_0)\,\delta\lambda(0)[\zeta] = \operatorname{sign}(s_0)\,\delta\lambda(0)[\zeta_1]$, since by the integrability assumption $\delta\lambda(0)[\zeta_2] = 0$. Note that $\kappa_1^2 = \kappa^2$, which means that $\overline{h}(r,\theta)$ satisfies the boundary condition $\overline{h}(1,\theta) = \kappa\phi_\zeta(\theta) = c(\theta)$. As above, the slicing lemma gives

$$\begin{aligned}\Big|\Big(W\big(\bar h(r,\theta)\big) - (1-\varepsilon)W(z) + \varepsilon W(b)\Big) - \int_0^1 \Big[\mathcal{G}\big(g_r, s^0\eta_3^{1/3}(r)\big) - (1-\varepsilon)\mathcal{G}(\zeta,s^0)\Big] r^{d-1}\,dr\Big| \\ \le 3(s^0)^3\varepsilon^2 a^2\int_0^1 r^{d+1}\eta_3(r)^2\,dr + \int_0^1 \kappa_r^2 r^{d+1}\int_{\partial B_1}\big|\delta\phi_{g_r}[g_r']\big|^2\,d\theta\,dr.\end{aligned} \tag{3.31}$$

For simplicity let $g = g_r$ and $s = s^0\eta_3(r)^{1/3}$. As in (3.20), we write:

$$\mathcal{G}(g,s) - (1-\varepsilon)\mathcal{G}(\zeta,s^0) = \underbrace{\mathcal{G}(g,0) - (1-\varepsilon)\mathcal{G}(\zeta,0)}_{=:\widetilde{E}^\perp} + \underbrace{(\mathcal{G}(g,s) - \mathcal{G}(g,0)) - (1-\varepsilon)(\mathcal{G}(\zeta,s^0) - \mathcal{G}(\zeta,0))}_{=:\widetilde{E}^T}.$$

To estimate $\widetilde{E}^\perp$ we proceed as for the term $E^\perp$ above, so that by (3.23) with $\Upsilon \equiv 0$ we get

$$\begin{aligned}\widetilde{E}^\perp &= \mathcal{G}(g,0) - (1-\varepsilon)\mathcal{G}(\zeta,0) = \mathcal{F}(g) - (1-\varepsilon)\mathcal{F}(\zeta) \\ &\le \frac{1}{2}\left(\delta^2\mathcal{F}(0)[g,g] - (1-\varepsilon)\delta^2\mathcal{F}(0)[\zeta,\zeta]\right) + \omega(\|g\|_{C^{2,\alpha}})\|g\|^2_{H^{1/2}} + \omega(\|\zeta\|_{C^{2,\alpha}})\|\zeta\|^2_{H^{1/2}} \\ &\le \frac{1}{2}\left(\eta_1^2(r) - (1-\varepsilon)\right)\delta^2\mathcal{F}(0)[\zeta_1,\zeta_1] + \frac{\varepsilon}{2}\delta^2\mathcal{F}(0)[\zeta_2,\zeta_2] + \omega(\|\zeta_1\|_{C^{2,\alpha}} + \|\zeta_2\|_{C^{2,\alpha}})\|\zeta\|^2_{H^{1/2}}.\end{aligned}$$

Integrating in r, and choosing again $\eta_1(r) = 1 - (d+1)\varepsilon(1-r)$, we get

$$\int_0^1 \widetilde{E}^\perp r^{d-1}\,dr \le -C_{d,b}\,\varepsilon\,\|\zeta\|^2_{H^{1/2}}. \tag{3.32}$$

Moreover, again as in (3.29) without the G term, we have

$$\int_0^1 \kappa_r^2 r^{d+1}\int_{\partial B_1}\big|\delta\phi_{g_r}[g_r']\big|^2\,d\theta\,dr = \int_0^1 \kappa_r^2 r^{d+1}(\eta_1'(r))^2\int_{\partial B_1}\big|\delta\phi_{g_r}[\zeta_1]\big|^2\,d\theta\,dr \le C_{d,b}\varepsilon^2\|\zeta_1\|^2_{L^2}. \tag{3.33}$$

To estimate $\widetilde{E}^T$ we write

$$\begin{aligned}\widetilde{E}^T &= (s^0)^3\Big(\eta_3(r)\big(\lambda(g) - (d-1)\big) - (1-\varepsilon)\big(\lambda(\zeta) - (d-1)\big)\Big) \\ &\le (s^0)^3\Big(\eta_3(r)\delta\lambda(0)[g] - (1-\varepsilon)(\delta\lambda(0)[\zeta]\Big) + |s^0|^3\sup_{t\in[0,1]}\Big(\big|\delta^2\lambda(tg)[g,g]\big| + \big|\delta^2\lambda(t\zeta)[\zeta,\zeta]\big|\Big) \\ &\le (s^0)^3\big(\eta_1(r)\eta_3(r) - (1-\varepsilon)\big)\delta\lambda(0)[\zeta_1] + C|s^0|^3\|\zeta\|^2_{H^{1/2}},\end{aligned} \tag{3.34}$$

where in the last line we used the continuity of $\delta^2\lambda$ and that the eigenfunctions of $\delta^2\mathcal{F}(0)$ with negative eigenvalues do not change λ to first order. Now, by the definition of η_1 and η_3 we have

$$\int_0^1 \big(\eta_1(r)\eta_3(r) - (1-\varepsilon)\big)r^{d-1}\,dr = -\frac{\varepsilon a}{d(d+1)} + \frac{2\varepsilon^2 a}{d(d+2)} = -\frac{a\varepsilon}{d}\left(\frac{1}{d+1} - \frac{2\varepsilon}{d+2}\right).$$

Since $a = \operatorname{sign}(s_0)\,\delta\lambda(0)[\zeta_1]$, choosing $\varepsilon > 0$ small enough depending only on d, we get

$$\int_0^1 \widetilde{E}^T r^{d-1}\,dr \le -C_d|s^0|^3(\delta\lambda(0)[\zeta_1])^2\varepsilon + C|s^0|^3\|\zeta\|^2_{H^{1/2}}. \tag{3.35}$$

Putting (3.31), (3.32), (3.33), (3.35) and (3.23) together, we get that

$$\Big(W\big(\bar h(r,\theta)\big)-(1-\varepsilon)W(z)+\varepsilon W(b)\Big)\le -\,\varepsilon C_{d,b}\left(\|\zeta\|^2_{H^{1/2}}+|s^0|^3(\delta\lambda(0)[\zeta_1])^2\right)$$
$$+\,\varepsilon^2 C_{d,b}\left(\|\zeta_1\|^2_{H^{1/2}}+|s^0|^3(\delta\lambda(0)[\zeta_1])^2\right)+C|s^0|^3\|\zeta\|^2_{H^{1/2}}.$$

Letting $\varepsilon>0$ be small enough and then perhaps shrinking $\delta\ge\|u-b\|^2_{L^2(\partial B_1)}\ge|s^0|^3$ so that $\delta<<\varepsilon$, we get the desired result. □

4. Proof of Theorem 1

Theorem 1 will follow by applying Theorem 3 on dyadic annuli thanks to a suitable parametrization lemma. The Smooth Parametrization Lemma 4.1 uses the strong convergence of minimizers and the fundamental regularity result of Alt and Caffarelli [1], to show that if the trace of a minimizer is sufficiently close to that of a cone with isolated singularity, then we can parametrize the free boundary of the minimizer on an annulus over that of the cone. Then in Lemma 4.2 we show that the condition from Lemma 4.1 remains uniform in the annuli away from the origin. Theorem 3 can then be applied in the annulus to show that the closeness decades and so the procedure can be iterated.

4.1. **Smooth parametrization lemma.** We start by proving our main parametrization lemma.

Lemma 4.1 (Smooth parametrization lemma)**.** *Let b be a 1-homogeneous minimizer of $\mathcal{E}$ with isolated singularity in zero and let $\tau>0$. For every $\varepsilon>0$, there exists $\delta_1=\delta_1(\varepsilon,\tau,b)>0$ such that if $u\in H^1(B_1)$ is a minimizer of $\mathcal{E}$ satisfying* $0\in\partial\{u>0\}$

$$\Theta_u(0):=W(u,0)=W(b)\,,\qquad \|u-b\|_{L^2(\partial B_1)}<\delta_1\qquad\text{and}\qquad W(u,1)-W(b)\le\delta_1\,, \tag{4.1}$$

then there exists a function $\zeta(\theta,r)\in C^{2,\alpha}(\partial\{b>0\})$ (indeed analytic) such that

$$\partial\{u>0\}\cap\partial B_r=\mathrm{graph}_{\mathbb{S}}(\zeta(-,r))\qquad\text{and}\qquad \|\zeta(-,r)\|_{C^{2,\alpha}}\le\varepsilon,\ \ \forall r\in(\tau,1-\tau)\,. \tag{4.2}$$

Proof. Suppose the claim is not true, then there are sequences of minimizers $u_j\in H^1(B_1)$ and of numbers $\delta_j\to0$, such that $0\in\partial\{u_j>0\}$,

$$\Theta_{u_j}(0)=W(b)\,,\qquad \|u_j-b\|_{L^2(\partial B_1)}<\delta_j\qquad\text{and}\qquad W(u_j,1)-W(b)\le\delta_j\,, \tag{4.3}$$

but such that $\partial\{u_j>0\}\cap(B_{1-\tau}\setminus B_\tau)$ does not satisfy (4.2). The condition (4.3) implies that

$$\int_{B_1}|\nabla u_j|^2\,dx\le\delta_j+W(b)+\int_{\partial B_1}u_j^2+2\,\omega_d\le3\delta_j+W(b)+\,2\int_{\partial B_1}b^2+2\,\omega_d\le C(d,b)\,.$$

Therefore the sequence $(u_j)_j$ is uniformly bounded in $H^1(B_1)$ and so up to subsequences it converges weakly in $H^1(B_1)\cap L^2(\partial B_1)$ to a function $v\in H^1(B_1)$. Moreover, the minimality of u_j implies that the convergence is $H^1(B_1)$-strong and $|\{u_j>0\}\cap B_1|\to|\{v>0\}\cap B_1|$ (see for instance [9, Theorem 9.1] or [1]). Then we have

$$\|v-b\|_{L^2(\partial B_1)}=\lim_{j\to\infty}\|u_j-b\|_{L^2(\partial B_1)}=0\ ,\qquad W(v,1)-W(b)=\lim_{j\to\infty}(W(u_j,1)-W(b))=0\,,$$

and $\Theta_v(0)=W(b)$, so that $v\equiv b$ on B_1. Furthermore, by the uniform Lipschitz norm and non-degeneracy of u_j, it follows that $\partial\{u_j>0\}$ converges to $\partial\{b>0\}$ in the Hausdorff sense in $B_{1-\frac{\tau}{2}}$ (see for instance [8]), and so by Alt-Caffarelli improvement of flatness (see for instance [8, Section 7]), we can conclude that for every j sufficiently large there exists an analytic function ζ_j such that $\partial\{u_j>0\}\cap(B_{1-\tau}\setminus B_\tau)=\mathrm{graph}(\zeta_j)$. The same argument proves the smallness of the $C^{2,\alpha}$ norm of the graph. Finally using the smallness of the $C^{1,\alpha}$ norm, a simple reparametrization implies that, for every j big enough, $\partial\{u_j>0\}\cap(B_{1-\tau}\setminus B_\tau)=\mathrm{graph}_{\mathbb{S}}(\zeta_j)$ and $\|\zeta_j\|_{C^{2,\alpha}}\to0\,.$ □

Before proving the main theorem we also need a preliminary lemma about blow-up sequences at comparable scales.

Lemma 4.2 (Blow-ups at comparable scales)**.** *Let $u \in H^1(B_1, \mathbb{R}_+)$ be a minimizer of $\mathcal{E}$ such that $0 \in \partial\{u > 0\}$. Then for every $\delta_2 > 0$, there exists $r_0 = r_0(\delta_2) > 0$ such that for every $0 < r < r_0$ the following inequality holds*

$$\|u_\rho - u_r\|_{L^2(\partial B_1)} \le \delta_2 \qquad \textit{for every } \rho \in \left[\frac{r}{8}, r\right]. \tag{4.4}$$

Proof. Suppose, by contradiction, that there are sequences $r_n \downarrow 0$ and $\rho_n \in \left[\frac{r_n}{8}, r_n\right]$ such that

$$\int_{\partial B_1} |u_{\rho_n} - u_{r_n}|^2 > \delta_2^2 \qquad \text{for every } \; n \in \mathbb{N}.$$

In particular, notice that $1 \le \frac{r_n}{\rho_n} \le 8$ for every $n \le \mathbb{N}$, so that $1 \le \liminf_{n\to\infty} \frac{r_n}{\rho_n} \le \limsup_{n\to\infty} \frac{r_n}{\rho_n} \le 8$. Now let $b \in H^1(\mathbb{R}^d)$ be a 1-homogeneous minimizer of $\mathcal{E}$ such that $u_{\rho_n}(x) := \frac{u(\rho_n x)}{\rho_n} \to b$ locally uniformly in $\mathbb{R}^d$. Now since $\lim_{n\to\infty} \int_{\partial B_{r_n/\rho_n}} |u_{\rho_n} - b|^2 = 0$, we can compute

$$\lim_{n\to\infty} \int_{\partial B_1} |u_{r_n} - b|^2 = \lim_{n\to\infty} \left(\frac{\rho_n}{r_n}\right)^{d-1} \int_{\partial B_{r_n/\rho_n}} |u_{\rho_n} - b|^2 = 0\,, \tag{4.5}$$

where we used the 1-homogeneity of b. This means that b is the blow up associated to the sequence r_n, and so by the triangular inequality

$$0 < \delta_2^2 \le \lim_{n\to\infty} \int_{\partial B_1} |u_{\rho_n} - u_{r_n}|^2 \le \lim_{n\to\infty} \left(2\int_{\partial B_1} |u_{\rho_n} - b|^2 + 2\int_{\partial B_1} |b - u_{r_n}|^2\right) = 0\,,$$

which gives the desired contradiction. □

4.2. **Proof of Theorem 1.** We first recall the Weiss' monotonicity formula

$$\frac{d}{d\rho} W(u_\rho) = \frac{d}{\rho}\big[W(z_\rho) - W(u_\rho)\big] + \frac{1}{\rho}\int_{\partial B_1} |x\cdot\nabla u_\rho - u_\rho|^2\, d\mathcal{H}^{d-1}, \tag{4.6}$$

where z_ρ is the 1-homogeneous extension of the trace $c_\rho := u_\rho|_{\partial B_1}$. In particular, $r \mapsto W(u_r)$ is increasing, $\lim_{r\to 0} W(u_r) = W(b)$ and for every $0 < s < t \le 1$, we have the estimate

$$\begin{aligned}
\int_{\partial B_1} |u_t - u_s|^2 \, d\mathcal{H}^{d-1} &\le \int_{\partial B_1} \left(\int_s^t \frac{1}{r}\,|x\cdot\nabla u_r - u_r|\, dr\right)^2 d\mathcal{H}^{d-1} \\
&\le \int_{\partial B_1} \left(\int_s^t r^{-1}\, dr\right)\left(\int_s^t r^{-1}\,|x\cdot\nabla u_r - u_r|^2\, dr\right) d\mathcal{H}^{d-1} \\
&\le (\log(t) - \log(s)) \int_s^t r^{-1} \int_{\partial B_1} |x\cdot\nabla u_r - u_r|^2 \, d\mathcal{H}^{d-1}\, dr \\
&\le \log(t/s) \int_s^t \frac{d}{dr}\,[W(u_r) - W(b)]\, dr \le \log(t/s)\,(W(u_t) - W(b))\,.
\end{aligned} \tag{4.7}$$

Moreover, if the logarithmic epiperimetric inequality

$$W(u_\rho) - W(b) \le \big(1 - \varepsilon\,|W(z_\rho) - W(b)|^\gamma\big)(W(z_\rho) - W(b)), \tag{4.8}$$

holds for every $\rho \in [s,t]$, then we can estimate

$$\begin{aligned}
\frac{d}{d\rho}\big(W(u_\rho) - W(b)\big) &= \frac{d}{\rho}\,[W(z_\rho) - W(b) - W(u_\rho) + W(b)] + \frac{1}{\rho}\int_{\partial B_1} |x\cdot\nabla u_\rho - u_\rho|^2\, d\mathcal{H}^{d-1} \\
&\ge \frac{d\varepsilon}{\rho}\,\frac{(W(u_\rho) - W(b))^{1+\gamma}}{1 - \varepsilon\,(W(u_\rho) - W(b))^\gamma} + \frac{1}{\rho}\int_{\partial B_1} |x\cdot\nabla u_\rho - u_\rho|^2\, d\mathcal{H}^{d-1} \\
&\ge \frac{d\varepsilon}{\rho}\,(W(u_\rho) - W(b))^{1+\gamma} + \frac{1}{\rho}\int_{\partial B_1} |x\cdot\nabla u_\rho - u_\rho|^2\, d\mathcal{H}^{d-1}\,,
\end{aligned} \tag{4.9}$$

which in particular gives that

$$\frac{d}{d\rho}\Big(-(W(u_\rho)-W(b))^{-\gamma}-\gamma d\varepsilon\log(\rho)\Big)\geq 0,\quad\text{for every}\quad \rho\in[s,t]\,. \tag{4.10}$$

Thus, setting $e(r)=W(u_r)-W(b)$ and $f(r)=\|u_r-b\|_{L^2(\partial B_1)}$, and using the triangular inequality $f(s)\leq f(t)+\|u_t-u_s\|_{L^2(\partial B_1)}$, we obtain that if (4.8) holds on the interval $[s,t]$, then

$$f(s)\leq \log(t/s)\,e(t)+f(t)\qquad\text{and}\qquad e(s)\leq\frac{e(t)}{(1+\varepsilon\gamma d\log(t/s)e(t)^\gamma)^{1/\gamma}}. \tag{4.11}$$

We next show that using Lemma 4.1, Lemma 4.2 and Theorem 3, we can iterate the above estimate up to zero. Let $\delta_0>0,\gamma\in(0,1)$ and $\varepsilon>0$ be the constants of Theorem 3, and let $\delta_1=\delta_1(\delta_0,1/8,b)>0$ and $\delta_2>0$ be the constants of Lemma 4.1 and Lemma 4.2. Thanks to the assumption that b is a blow-up of u at 0, we can choose $0<\delta_2<\delta_1/2$ and $r_0=r_0(\delta_1,\delta_2,\gamma)>0$ in such a way that

$$(W(u_{r_0})-W(b))^{\frac{1-\gamma}{2}}+2\delta_2+\|u_{r_0}-b\|_{L^2(\partial B_1)}=e(r_0)^{\frac{1-\gamma}{2}}+2\delta_2+f(r_0)\leq\delta_1\,. \tag{4.12}$$

We will first show that

$$e(r)^{\frac{1-\gamma}{2}}+\delta_2+f(r)\leq\delta_1\quad\text{for all}\quad 0<r\leq r_0, \tag{4.13}$$

which means that we can apply Theorem 3 to all u_r for $0<r\leq r_0$. Vacuously, (4.13) is true for r_0 so the assumption (4.1) is satisfied with $u=u_{r_0}$. Thus, by Lemma 4.1, there exists a function $\zeta\in C^{2,\alpha}(\{b>0\})$ such that

$$\partial\{u_{r_0}>0\}\cap(B_{7/8}\setminus B_{1/8})=\mathrm{graph}_{\mathbb{S}}(\zeta)\qquad\text{and}\qquad\|\zeta\|_{C^{2,\alpha}}\leq\delta_0\,.$$

It follows that the condition (1.5) is satisfied for the trace $c_\rho:=u_\rho|_{\partial B_1}$, for every $\rho\in\left[\frac{r_0}{8},\frac{r_0}{2}\right]$, and so we can apply Theorem 3 and the minimality of u_ρ to deduce that (4.8) holds for $\rho\in\left[\frac{r_0}{8},\frac{r_0}{2}\right]$. Thus, for every $r_0/8\leq s<t\leq r_0/2$, we have $\log(t/s)\leq\log(4)$ and by choosing $e(r_0)$ sufficiently small we get

$$(1+\varepsilon\gamma d\log(t/s)e(t)^\gamma)^{-\frac{1-\gamma}{2\gamma}}\leq 1-C\log(t/s)e(t)^\gamma,\quad\text{where}\quad C=\frac14\varepsilon d\,(1-\gamma)\,.$$

Thus, (4.11) gives that

$$\begin{aligned}f(s)+e(s)^{\frac{1-\gamma}{2}}&\leq\log(t/s)\,e(t)+f(t)+e(t)^{\frac{1-\gamma}{2}}\,(1-C\log(t/s)e(t)^\gamma)\\&\leq f(t)+e(t)^{\frac{1-\gamma}{2}}+\log(t/s)\left(e(t)-Ce(t)^{\frac{1+\gamma}{2}}\right)\leq f(t)+e(t)^{\frac{1-\gamma}{2}},\end{aligned}$$

where for the last inequality we choose r_0 such that $e(t)^{\frac{1-\gamma}{2}}\leq e(r_0)^{\frac{1-\gamma}{2}}\leq C$. This proves (4.13) for $\rho\in[r_0/8,r_0/2]$. But now it is clear that $u_{r_0/4}$ satisfies the conditions (4.1), so we can run the argument again and get that (4.8) applies for all $\rho\in[r_0/32,r_0/2]$. Continuing to repeat the argument we obtain that (4.13) holds for all $0<\rho\leq r_0$. In particular, (4.11) applied to $t=r_0/2$ and $s=\rho<r_0/4$, we get that there is a constant C, depending on $W(u_{r_0})-W(b)$, such that

$$e(\rho)=W(u_\rho)-W(b)\leq C(-\log(\rho))^{-1/\gamma}\quad\text{for every}\quad 0<\rho\leq r_0/2. \tag{4.14}$$

Now we are ready to conclude the proof of the theorem: to prove the uniqueness of blowup we will prove that $(u_r)_r$ is a Cauchy sequence in $L^2(\partial B_1)$. Let $0<s<t<r_0/2$. Let $i\leq j$ be such

that $s \in [2^{-2^{j+1}}, 2^{-2^j})$ and $t \in [2^{-2^{i+1}}, 2^{-2^i})$. Then, using (4.14) and (4.7), we calculate

$$\begin{aligned}\|u_s - u_t\|_{L^2(\partial B_1)} \leq &\|u_s - u_{2^{-2^j}}\|_{L^2(\partial B_1)} + \|u_t - u_{2^{-2^{i+1}}}\|_{L^2(\partial B_1)} + \sum_{k=i+1}^{j-1} \|u_{2^{-2^{k+1}}} - u_{2^{-2^k}}\|_{L^2(\partial B_1)} \\ &\overset{(4.7)}{\leq} 2^j e(s) + 2^i e(t) + \sum_{k=i+1}^{j-1} 2^k e(2^{-2^k}) \\ &\overset{(4.14)}{\leq} C2^j(-\log(s))^{-1/\gamma} + C2^i(-\log(t))^{-1/\gamma} + C\sum_{k=i+1}^{j-1} 2^k 2^{-k/\gamma} \\ &\overset{\gamma\in(0,1)}{\leq} C_\gamma 2^{(i+1)\frac{\gamma-1}{\gamma}} \leq C_\gamma(-\log(t))^{\frac{\gamma-1}{\gamma}}.\end{aligned}$$

Since $(-\log(t))^{\frac{\gamma-1}{\gamma}} \downarrow 0$ as $t \downarrow 0$ we have proven the uniqueness of blowups.

Finally, Lemma 4.1, the decay of the L^2 norm and the Weiss' boundary adjusted energy $W(u_r)$ imply that $\partial\{u > 0\} \cap B_r$ is a graph over $\partial\{b > 0\} \cap B_r$. Moreover, again by Lemma 4.1 we have that $\partial\{u > 0\} \cap B_r$ is a C^1 graph over $\partial\{b > 0\} \cap B_r$, that is the graph over $\partial\{b > 0\} \cap B_1$ associated to $\partial\{u_r > 0\} \cap B_1 \setminus B_{1/8}$ converges to zero in C^1 norm as $r \to 0$. This convergence can be improved to $C^{1,\log}$ by a standard argument that we sketch for the readers' convenience. Indeed, since $\partial\{u_r > 0\} \cap B_1 \setminus B_{1/8}$ is a smooth graph with controlled $C^{2,\alpha}$ norm, the (log-) epiperimetric inequality holds at a uniform scale at every point $x_0 \in \partial\{u_r > 0\} \cap B_{1/2} \setminus B_{1/4}$. Thus, the oscillation of the normals $|\nu_{x_0,r} - \nu_{y_0,s}|$, where $x_0 \in \partial\{u_r > 0\} \cap B_{1/2} \setminus B_{1/4}$ and $y_0 \in \partial\{u_s > 0\} \cap B_{1/2} \setminus B_{1/4}$ is controlled by a power of r_0 (for some $0 < r_0 < 1$) and the L^2-distance $\|(u_r)_{x_0,r_0} - (u_s)_{y_0,r_0}\|_{L^2(\partial B_1)}$. Now this last distance has a logarithmic decay due to the logarithmic decay of $\|u_r - b\|_{L^2(\partial B_1)}$ proved above, which implies the $C^{1,\log}$ convergence of the graphs $\partial\{u_r > 0\} \cap B_{1/2} \setminus B_{1/4}$. □

The proof of Theorem 2, the integrable case, follows similarly, but is simpler and mostly standard so we will omit it. Let us just remark out that we must still take care to show that the "closeness" assumptions of Theorem 3 are satisfied on all scales. However, this argument works in essentially the same way in the integrable and non-integrable setting.

5. The Index of the De Silva-Jerison cone in the sphere

In this section we prove that the De Silva-Jerison cone $\mathcal{C}_{\nu,\theta_0}$ satisfies the conditions of Definition 2.4: namely that the dimension of the kernel of $\delta^2\mathcal{F}(0)$ is $d-1$ and that each perturbation in index$(\delta^2\mathcal{F}(0))$ integrates to zero along $\partial\Omega_{b_{\nu,\theta_0}}$. This completes the proof of Corollary 1.1.

To do so, we will produce an eigenbasis of deformations and show that, except for the deformations infinitesimally generated by rotations, each associated eigenvalue is non-zero. Furthermore, the perturbation generated by a constant will be an element of the eigenbasis and we will check that it is associated to a positive eigenvalue. This implies that all the other elements of the eigenbasis (including those in the index) integrate to zero on $\partial\Omega_{b_{\nu,\theta_0}}$. We note that the positivity of the constant perturbation requires a computational check, which we do explicitly for $d = 7$. A more general argument verifies the inequality for $d \geq 21$ and one can check the numerics for $8 \leq d \leq 20$ via the procedure outlined below for $d = 7$.

Throughout this section we will write the points of the sphere $\partial B_1 = \mathbb{S}^{d-1}$ in the spherical coordinates (θ, ϕ), where $\phi \in \mathbb{S}^{d-2}$ and $\theta \in [0, \pi]$. Thus, the trace of the De Silva-Jerison cone on the sphere is simply given by $\Omega_{b_{\nu,\theta_0}} = \left\{(\theta, \phi) \in \mathbb{S}^{d-1} \ : \ \frac{\pi}{2} - \theta_0 < \theta < \frac{\pi}{2} + \theta_0\right\}$.

5.1. **A basis of eigenfunctions.** Let $\{\psi_j\}$ be the eigenfunctions of the Laplace-Beltrami operator on $\mathbb{S}^{d-2}$ (i.e. the spherical harmonics in dimension $d-1$). Then $\zeta_j^{\pm}$ is an orthogonal basis of $L^2(\partial\Omega_{b_{\nu,\theta_0}})$ and $H^{1/2}(\partial\Omega_{b_{\nu,\theta_0}})$, where

$$\zeta_j^+(\pi/2 \pm \theta_0, \varphi) = -\,\psi_j(\varphi) \qquad \text{and} \qquad \zeta_j^-(\pi/2 \pm \theta_0, \varphi) = \mp\,\psi_j(\varphi). \tag{5.1}$$

For $j > 1$, we define

$$-\,\Delta u_j^\pm = (d-1)u_j^\pm \quad \text{in} \quad \Omega_{b_{\nu,\theta_0}}, \qquad u_j^\pm = -\zeta_j^\pm \quad \text{on} \quad \partial\Omega_{b_{\nu,\theta_0}}, \qquad \int_{\Omega_{b_{\nu,\theta_0}}} b u_j^\pm = 0. \tag{5.2}$$

The case $j = 1$ is slightly more complicated, as the variation in the direction of ζ_j^+ (which can be geometrically interpreted as increasing the opening of the cone), changes the measure and the first eigenvalue of the domain to first order. Therefore, we define

$$-\,\Delta u_1^+ = (d-1)u_1^+ + \eta b \quad \text{in} \quad \Omega_{b_{\nu,\theta_0}}, \qquad u_1^+ = -\zeta_1^+ \quad \text{on} \quad \partial\Omega_{b_{\nu,\theta_0}}, \qquad \int_{\Omega_{b_{\nu,\theta_0}}} b u_1^+ = 0, \tag{5.3}$$

$$-\,\Delta u_1^- = (d-1)u_1^- \quad \text{in} \quad \Omega_{b_{\nu,\theta_0}}, \qquad u_1^- = -\zeta_1^- \quad \text{on} \quad \partial\Omega_{b_{\nu,\theta_0}}, \qquad \int_{\Omega_{b_{\nu,\theta_0}}} b u_1^- = 0, \tag{5.4}$$

where

$$\eta := \frac{1}{\kappa_0^2}\frac{\mathcal{H}^{d-2}(\partial\Omega_{b_{\nu,\theta_0}})}{\sqrt{\mathcal{H}^{d-2}(\mathbb{S}^{d-2})}}.$$

Recall from Appendix A, Subsection A.2, that solutions to (5.2), (5.3), (5.4) exist and are unique. In particular, we can write $u_j^\pm$ explicitly for $j > 1$ (this formula also holds for u_1^-) by separating the variables as $u_j^\pm(\theta,\varphi) = \psi_j(\varphi) f_j^\pm(\theta)$, where $f_j^\pm(\theta)$ are defined by

$$\begin{cases} -\dfrac{1}{\sin^{d-2}\theta}\dfrac{\partial}{\partial\theta}\left(\sin^{d-2}\theta\,\dfrac{\partial f_j^+}{\partial\theta}\right) = \left(-\dfrac{\lambda_j^{d-2}}{\sin^2\theta} + (d-1)\right) f_j^+ \quad \text{for} \quad \dfrac{\pi}{2} - \theta_0 < \theta < \dfrac{\pi}{2} + \theta_0, \\ f_j^+\left(\dfrac{\pi}{2} - \theta_0\right) = f_j^+\left(\dfrac{\pi}{2} + \theta_0\right) = 1, \end{cases} \tag{5.5}$$

$$\begin{cases} -\dfrac{1}{\sin^{d-2}\theta}\dfrac{\partial}{\partial\theta}\left(\sin^{d-2}\theta\,\dfrac{\partial f_j^-}{\partial\theta}\right) = \left(-\dfrac{\lambda_j^{d-2}}{\sin^2\theta} + (d-1)\right) f_j^- \quad \text{for} \quad \dfrac{\pi}{2} - \theta_0 < \theta < \dfrac{\pi}{2} + \theta_0, \\ -f_j^-\left(\dfrac{\pi}{2} - \theta_0\right) = f_j^-\left(\dfrac{\pi}{2} + \theta_0\right) = 1. \end{cases} \tag{5.6}$$

Above, and throughout, λ_j^{d-2} refers to the j-th eigenvalue of the Laplace-Beltrami operator on $\mathbb{S}^{d-2}$, counted with multiplicity. Note that solutions to (5.5) and (5.6) are unique and in fact minimize (under the respective boundary conditions) the functionals

$$J_j(f) = \int_{\pi/2-\theta_0}^{\pi/2+\theta_0} \left(|f'(\theta)|^2 + \frac{\lambda_j^{d-2}}{\sin^2\theta} f^2(\theta) - (d-1)f^2(\theta)\right)\sin^{d-2}\theta\,d\theta.$$

In order to show that the deformations $\zeta_j^\pm$ diagonalize $\delta^2\mathcal{F}(0)$ (and thus satisfy (2.8)) we recall that by (A.14) we have

$$\delta^2\mathcal{F}(0)[\xi_1,\xi_2] = -2(d-2)\tan(\theta_0)\int_{\partial\Omega_{b_{\nu,\theta_0}}} \xi_1\xi_2\; d\mathcal{H}^{d-2} + 2\int_{\partial\Omega_{b_{\nu,\theta_0}}} \xi_1\; T\xi_2\, d\mathcal{H}^{d-2}. \tag{5.7}$$

Applying (5.7) to $\zeta_j^\pm$ and $\zeta_k^\pm$ and integrating by parts we get

$$\begin{aligned} \frac{1}{2}\delta^2\mathcal{F}(0)[\zeta_j^\pm,\zeta_k^\pm] &= \int_{\partial\Omega_{b_{\nu,\theta_0}}} u_j^\pm \partial_\nu u_k^\pm\; d\mathcal{H}^{d-2} - (d-2)\tan(\theta_0)\int_{\partial\Omega_{b_{\nu,\theta_0}}} \zeta_j^\pm\zeta_k^\pm\; d\mathcal{H}^{d-2} \\ &= \langle u_j^\pm, u_k^\pm\rangle - (d-2)\tan(\theta_0)\int_{\partial\Omega_{b_{\nu,\theta_0}}} \zeta_j^\pm\zeta_k^\pm\; d\mathcal{H}^{d-2}, \end{aligned} \tag{5.8}$$

where for simplicity we have set $\left\langle u_j^{\pm}, u_k^{\pm}\right\rangle := \int_{\Omega_{b_{\nu,\theta_0}}} \left(\nabla u_j^{\pm}\cdot\nabla u_k^{\pm} - (d-1)u_j^{\pm}u_k^{\pm}\right)\, d\mathcal{H}^{d-1}$.

We now claim that the family of functions $\{\zeta_j^{\pm}\}_{j\in\mathbb{N}}$ is orthogonal with respect to $\delta^2\mathcal{F}(0)$. Indeed, as $\zeta_j^{\pm}$ are orthogonal in $L^2(\partial\Omega_{b_{\nu,\theta_0}})$, by (5.8) it suffices to establish that the $u_j^{\pm}$ are orthogonal in $H^1(\Omega_{b_{\nu,\theta_0}})$. Indeed, that $\left\langle u_j^+, u_k^-\right\rangle = 0$ is trivial; the u_j^+ are even functions of θ across the equator, whereas the u_k^- are odd functions of θ across the equator (this can be seen in (5.2), (5.3),(5.4)). Then $\left\langle u_j^+, u_k^+\right\rangle = 0$ and $\left\langle u_j^-, u_k^-\right\rangle = 0$ follow from the separation of variables; each is equal to a function in θ times a function in φ and for $k \neq j$ the functions in φ are orthogonal in $\mathcal{H}^1(\mathbb{S}^{d-2})$. Since the integrals split, we get the desired orthogonality. Moreover, a standard density argument gives that the family $\{\zeta_j^{\pm}\}_{j\in\mathbb{N}}$ is complete that is, it generates $H^{1/2}(\partial\Omega_{b_{\nu,\theta_0}})$.

5.2. **Integrability through rotations: proof of Corollary 1.1.** In order to prove the integrability trough rotations of the De Silva-Jerison type cones, and so Corollary 1.1, it suffices to estimate $\delta^2\mathcal{F}(0)[\zeta_j^{\pm},\zeta_j^{\pm}]$ and show that ζ_1^+ is a positive direction, since it is the only one that changes λ at first order.

We start by proving that there are d perturbations which correspond to negative eigenvalues (i.e. that the De Silva-Jerison cone has index d in the sphere).

Proposition 5.1 (Index of De Silva-Jerison cone)**.** *The eigenvalues associated to the eigenfunctions ζ_j^+, for $2 \le j \le d$, and ζ_1^- are strictly negative.*

Proof. To compute the energy of u_j^+, for $2 \le j \le d$, note that

$$u_j^+(\theta,\varphi) \equiv \psi_j(\varphi)\frac{\sin(\theta)}{\cos(\theta_0)}, \quad 2 \le j \le d \tag{5.9}$$

satisfies (5.2) (to see this, recall that $\lambda_j^{\mathbb{S}^{d-2}} = (d-2)$, for $2 \le j \le d$, and that $-\Delta\sin(\theta) = (d-1)\sin(\theta) - \frac{d-2}{\sin^2(\theta)}\sin(\theta)$). Plugging this into (5.8), we get

$$\begin{aligned}\delta^2\mathcal{F}(0)[\zeta_j^+,\zeta_j^+] =& 4\cos^{d-2}(\theta_0)\left(\int_{\mathbb{S}^{d-2}} u_j^+\partial_\nu u_j^+ \, d\sigma - (d-2)\tan(\theta_0)\right)\\ =& 4\cos^{d-2}(\theta_0)\left(-\tan(\theta_0) - (d-2)\tan(\theta_0)\right) < 0, \quad \text{for } \; 2 \le j \le d.\end{aligned} \tag{5.10}$$

To see that $\delta^2\mathcal{F}(0)[\zeta_1^-,\zeta_1^-] < 0$, note that $u_1^- = -\frac{1}{\sqrt{\mathcal{H}^{d-2}(\mathbb{S}^{d-2})}}\frac{\cos(\theta)}{\sin(\theta_0)}$. Then a simple computation gives the result. □

To show that ζ_1^- is a negative direction one could also use a more general principle, which will simplify the rest of the proof. Note that (except for ζ_1^+)

$$\delta^2\mathcal{F}(0)[\zeta_j^{\pm},\zeta_j^{\pm}] = 2J_j(f_j^{\pm}) - 4(d-2)\cos^{d-2}(\theta_0)\tan(\theta_0), \tag{5.11}$$

By the definition of J_j (and the fact that $\lambda_j^{\mathbb{S}^{d-2}}$ increases as $j\to\infty$) for any function $w \in H^1$ we have that $J_{j+1}(w) \ge J_j(w)$ (with strict inequality if $\lambda_{j+1}^{\mathbb{S}^{d-2}} > \lambda_j^{\mathbb{S}^{d-2}}$). Since $f_j^{\pm}$ minimizes J_j with the respective boundary conditions, we get $J_{j+1}(f_{j+1}^{\pm}) \ge J_j(f_{j+1}^{\pm}) \ge J_j(f_j^{\pm})$, which gives

$$\begin{aligned}\delta^2\mathcal{F}(0)[\zeta_{j+1}^+,\zeta_{j+1}^+] &\ge \delta^2\mathcal{F}(0)[\zeta_j^+,\zeta_j^+], \quad \text{for all } \; j \ge 2\,,\\ \delta^2\mathcal{F}(0)[\zeta_{j+1}^-,\zeta_{j+1}^-] &\ge \delta^2\mathcal{F}(0)[\zeta_j^-,\zeta_j^-], \quad \text{for all } \; j \ge 1\,,\end{aligned} \tag{5.12}$$

where the strict inequalities hold if and only if $\lambda_{j+1}^{\mathbb{S}^{d-2}} > \lambda_j^{\mathbb{S}^{d-2}}$.

As a consequence of (5.12), the following proposition will conclude the proof of Corollary 1.1.

Proposition 5.2 (Kernel of De Silva-Jerison cone)**.** *Let $\xi \in \{\zeta_j^{\pm}\}_{\pm,j\in\mathbb{N}}$. Then $\delta^2\mathcal{F}(0)[\xi,\xi] = 0$ if and only if ξ is a linear combination of ζ_j^- for $2 \le j \le d$. Furthermore, $\delta^2\mathcal{F}(0)[\zeta_1^+,\zeta_1^+] > 0$.*

Proof. We divide the proof of the Theorem in two steps dealing respectively with the dilation ζ_1^+ and the rotations ζ_j^-, for $2 \le j \le d$.

Step 1. The dilation ζ_1^+. Our first claim is that u_1^+ minimizes $\int_{\Omega_{b_{\nu,\theta_0}}} |\nabla f|^2 - (d-1)f^2 d\mathcal{H}^{d-1}$ amongst all f equal to $-\zeta_1^+ = \frac{1}{\sqrt{\mathcal{H}^{d-2}(\mathbb{S}^{d-2})}}$ on $\partial\Omega_{b_{\nu,\theta_0}}$ and which are L^2 orthogonal to b in $\Omega_{b_{\nu,\theta_0}}$.

Taking the first variation of the energy, subject to the orthogonality constraint, such a minimizer must satisfy an equation of the form $-\Delta f = (d-1)f + kb$, where k is some constant. However, if there exists an f with the same boundary values as u_1^+ and which solves the above equation for some $k \neq \eta$, then it must be the case that f is either a super or sub solution of the equation that u_1^+ solves, and thus must lie either below or above u_1^+ on all of $\Omega_{b_{\nu,\theta_0}}$ (we have a maximum principle because both functions are orthogonal to b). However, it is not possible that both $f > u_1^+$ and $f, u_1^+ \perp b$, a positive function. Similarly if $f < u_1^+$. Thus, given the boundary values ζ_1^+, the only solution is u_1^+.

Consider the function

$$f = \frac{1}{\sqrt{\mathcal{H}^{d-2}(\mathbb{S}^{d-2})}}\left(1 + c\phi_0\right),$$

where $\phi_0 = b/\kappa_0$, $c = -\int_{\Omega_{b_{\nu,\theta_0}}} \phi_0$ so that f is L^2 orthogonal to ϕ_0 on $\Omega_{b_{\nu,\theta_0}}$. We have that

$$\begin{aligned}
\delta^2\mathcal{F}(0)[\zeta_1^+,\zeta_1^+] + E =& 2\int_{\Omega_{b_{\nu,\theta_0}}} |\nabla u_1^+|^2 - (d-1)(u_1^+)^2\, d\mathcal{H}^{d-2} - 4(d-2)\tan(\theta_0)\cos^{d-2}(\theta_0) + E \\
=& 2\int_{\Omega_{b_{\nu,\theta_0}}} |\nabla f|^2 - (d-1)(f)^2\, d\mathcal{H}^{d-2} - 4(d-2)\tan(\theta_0)\cos^{d-2}(\theta_0) \\
=& \frac{2c^2}{\mathcal{H}^{d-2}(\mathbb{S}^{d-2})}\int_{\Omega_{b_{\nu,\theta_0}}} |\nabla\phi_0|^2 - (d-1)(\phi_0)^2\, d\mathcal{H}^{d-1} \\
&+ \frac{2}{\mathcal{H}^{d-2}(\mathbb{S}^{d-2})}\Big(2(d-1)c^2 - (d-1)\mathcal{H}^{d-1}(\Omega_{b_{\nu,\theta_0}})\Big) - 4(d-2)\tan(\theta_0)\cos^{d-2}(\theta_0) \\
=& \frac{2(d-1)}{\mathcal{H}^{d-2}(\mathbb{S}^{d-2})}\Big(2c^2 - \mathcal{H}^{d-1}(\Omega_{b_{\nu,\theta_0}})\Big) - 4(d-2)\tan(\theta_0)\cos^{d-2}(\theta_0),
\end{aligned} \tag{5.13}$$

where $E > 0$ is some error which reflects how far f is from minimizing.

To estimate E we first use that u_1^+ minimizes to get

$$\begin{aligned}
E =& 2\int_{\Omega_{b_{\nu,\theta_0}}} |\nabla f|^2 - (d-1)(f)^2\, d\mathcal{H}^{d-2} - 2\int_{\Omega_{b_{\nu,\theta_0}}} |\nabla u_1^+|^2 - (d-1)(f)^2\, d\mathcal{H}^{d-2} \\
=& 2\int_{\Omega_{b_{\nu,\theta_0}}} |\nabla(u_1^+ - f)|^2 - (d-1)((u_1^+ - f))^2\, d\mathcal{H}^{d-2} = 2\sum_j \tilde{a}_j^2(\lambda_j - (d-1)),
\end{aligned}$$

where $\tilde{a}_j = \langle u_1^+ - f, \phi_j\rangle$ and ϕ_j is the j Dirichlet eigenfunction of $\Omega_{b_{\nu,\theta}}$ with eigenvalue λ_j.

Let us consider the Dirichlet eigenfunctions of $\Omega_{b_{\nu,\theta_0}}$; by separation of variables we can write any eigenfunction $\phi_j(\theta,\phi) = g_j(\theta)\psi_j(\varphi)$ where ψ_j is a spherical harmonic on $\mathbb{S}^{d-2}$ and $g_j(\pi/2-\theta_0) = g_j(\pi/2+\theta_0) = 0$. In order for $\phi_j \not\perp 1, u_1^+$ but for $\phi_j \perp \phi_0$ it must be the case that ψ_j is constant and g_j is an even function with at least two interior zeroes. As such g_j as two local critical points, at $\pi/2-\eta, \pi/2+\eta$ (for $\eta < \theta_0$) and $\partial_\theta g_j$ satisfies

$$\begin{aligned}
-\partial^2_{\theta\theta}(\partial_\theta g_j(\theta)) - (d-2)\cot(\theta)\partial_\theta(\partial_\theta g_j(\theta)) = \left(\lambda_j - \frac{d-2}{\sin^2(\theta)}\right)(\partial_\theta g_j(\theta)), \\
\partial_\theta g_j(\pi/2-\eta) = \partial_\theta g_j(\pi/2+\eta) = 0,
\end{aligned}$$

where λ_j is the eigenvalue associated to $\phi_j = g_j$. But note that if $h(\theta)\psi_2^{\mathbb{S}^{d-2}}$ is an eigenfunction (and it is for some h), then h minimizes the energy associated to the equation $L = -\Delta + \frac{(d-2)}{\sin^2(\theta)}$ but over Dirichlet boundary conditions on a larger integral. Thus the eigenvalue associated to $h(\theta)\psi_2^{\mathbb{S}^{d-2}}$ is smaller than λ_j (as λ_j is a Dirichlet eigenvalue, but not necessarily the first, associated to the same L on a smaller domain). The Rayleigh quotient of $h(\theta)\psi_2^{\mathbb{S}^{d-2}}$ is simply the Rayleigh quotient of $h(\theta)$ times the Raleigh quotient of $\psi_2^{\mathbb{S}^{d-2}}$ (the latter being equal to $d-2$). The former is bounded below by $\lambda_1^{\Omega_{b_{\nu,\theta_0}}} = d-1$. Therefore, $\tilde{a}_j \neq 0 \Rightarrow \lambda_j \geq (d-2)(d-1)$. We claim that

$$\tilde{a}_j = \frac{\int \phi_j}{\sqrt{\mathcal{H}^{d-2}(\mathbb{S}^{d-2})}} \frac{(d-1)}{\lambda_j - (d-1)}, \quad \text{for every} \quad j \quad \text{such that} \quad \tilde{a}_j \neq 0.$$

Indeed $\langle u_1^+ - f, \phi_j\rangle = \langle u_1^+, \phi_j\rangle - \frac{1}{\sqrt{\mathcal{H}^{d-2}(\mathbb{S}^{d-2})}}\int \phi_j$. We also know that $\langle u_1^+, \phi_j\rangle = 0$, for all $j > 0$ as ϕ_j has zero Dirichlet data and is orthogonal to ϕ_0. Integrating by parts we get

$$\begin{aligned}(\lambda_j - (d-1))\langle u_1^+, \phi_j\rangle &= -\frac{1}{\sqrt{\mathcal{H}^{d-2}(\mathbb{S}^{d-2})}}\int_{\partial\Omega_{b_{\nu,\theta_0}}} \partial_\nu \phi_j \\ &= \frac{1}{\sqrt{\mathcal{H}^{d-2}(\mathbb{S}^{d-2})}}\int_{\Omega_{b_{\nu,\theta_0}}} -\Delta\phi_j = \frac{\lambda_j}{\sqrt{\mathcal{H}^{d-2}(\mathbb{S}^{d-2})}}\int_{\Omega_{b_{\nu,\theta_0}}} \phi_j.\end{aligned}$$

Putting everything together we get the claim. Also note that the above argument implies that $u_1^+ \perp \phi_j$ ($\Leftrightarrow 1 \perp \phi_j$) as long as $j > 0$. So

$$\begin{aligned}E &= \frac{2}{\mathcal{H}^{d-2}(\mathbb{S}^{d-2})}\sum_{\{j \mid \tilde{a}_j \neq 0\}} \left(\int \phi_j\right)^2 \frac{(d-1)^2}{\lambda_j - (d-1)} \\ &\overset{\lambda_j \geq (d-2)(d-1)}{\leq} \frac{3}{\mathcal{H}^{d-2}(\mathbb{S}^{d-2})}\sum_{\{j \mid \tilde{a}_j \neq 0\}} \left(\int \phi_j\right)^2 \leq \frac{3}{\mathcal{H}^{d-2}(\mathbb{S}^{d-2})}\left(\mathcal{H}^{d-1}(\Omega_{b_{\nu,\theta_0}}) - c^2\right),\end{aligned} \tag{5.14}$$

where we assume that $d \geq 7$ so that $\frac{d-1}{d-3} \leq \frac{3}{2}$. Putting (5.13) together with (5.14) yields

$$\begin{aligned}\delta^2\mathcal{F}(0)[\zeta_1^+, \zeta_1^+] > &\frac{2(d-1)}{\mathcal{H}^{d-2}(\mathbb{S}^{d-2})}\left(2c^2 - \mathcal{H}^{d-1}(\Omega_{b_{\nu,\theta_0}})\right) - 4(d-2)\tan(\theta_0)\cos^{d-2}(\theta_0) \\ &- \frac{3}{\mathcal{H}^{d-2}(\mathbb{S}^{d-2})}\left(\mathcal{H}^{d-1}(\Omega_{b_{\nu,\theta_0}}) - c^2\right) \\ = &\frac{1}{\mathcal{H}^{d-2}(\mathbb{S}^{d-2})}\left((4d-1)c^2 - (2d+1)\mathcal{H}^{d-1}(\Omega_{b_{\nu,\theta_0}})\right) - 4(d-2)\tan(\theta_0)\cos^{d-2}(\theta_0).\end{aligned} \tag{5.15}$$

The case $d = 7$. We will now verify that the right hand side of (5.15) is positive when $d = 7$ via a numerical calculation. We use Mathematica, though, since these special functions are well known, one could do this by hand. In order to minimize the effect of rounding errors by the computer, we will round up negative terms and round down positive terms. As the calculation is delicate, we will have to go to four places right of the decimal.

When $d = 7$, $\theta_0 \approx \sin^{-1}(.517331) \approx .54372$ (see [10]), and we have

$$\begin{aligned}&\frac{\mathcal{H}^{d-1}(\Omega_{b_{\nu,\theta_0}})}{\mathcal{H}^{d-2}(\mathbb{S}^{d-2})} < \int_{-.5438}^{.5438} \cos(\theta)^5 d\theta < .8650 \\ &4(d-2)\cos^{d-2}(\theta_0)\tan(\theta_0) < 20\cos^5(.5437)\tan(.5438) < 5.5509 \\ &\Rightarrow \frac{(2d+1)\mathcal{H}^{d-1}(\Omega_{b_{\nu,\theta_0}})}{\mathcal{H}^{d-2}(\mathbb{S}^{d-2})} + 4(d-2)\tan(\theta_0)\cos^{d-2}(\theta_0) < 15 * .8650 + 5.5609 = 18.5359.\end{aligned}$$

Now we calculate $\int \phi_0$. Following [10],

$$\phi_0 = c_\theta(1-\cos^2(\theta))^{-\frac{d-3}{4}} Q_{\frac{d-1}{2}}^{\frac{d-3}{2}}(\cos(\theta)),$$

where c_θ is the L^2 normalizing constant and Q^μ_ν is the associated Legendre function of the second kind (when d is even we work with P^μ_ν the associated Legendre function of the first kind). We define these functions following the convention of Mathematica (which is the same convention used in [10] and [13]), namely that,

$$(1-t^2)\frac{d^2Q^\mu_\nu(t)}{dt^2} - 2(\mu+1)t\frac{dQ^\mu_\nu(t)}{dt} + (\nu-\mu)(\nu+\mu+1)Q^\mu_\nu(t) = 0.$$

In the case $d = 7$, we can use Mathematica to calculate

$$\|(1-\cos^2(\theta))^{-1}Q^2_3(\cos(\theta))\|^2_{L^2} < \mathcal{H}^{d-2}(\mathbb{S}^{d-2})\int_{-.5174}^{.5174}\left(Q^2_3(t)\right)^2 dt < 34.6188 \Rightarrow c_\theta > \frac{.1699}{\sqrt{\mathcal{H}^{d-2}(\mathbb{S}^{d-2})}}$$

Then we have (using Mathematica again)

$$\begin{aligned}\int \phi_0 >& \frac{.1699 * \mathcal{H}^{d-2}(\mathbb{S}^{d-2})}{\sqrt{\mathcal{H}^{d-2}(\mathbb{S}^{d-2})}}\int_{\pi/2-\theta_0}^{\pi/2+\theta_0}\sin(\theta)^5 Q^2_3(\cos(\theta))d\theta\\ >& .1699 * \sqrt{\mathcal{H}^{d-2}(\mathbb{S}^{d-2})}\int_{-.5173}^{.5173}(1-t^2)^2Q^2_3(t)dt > .8326 * \sqrt{\mathcal{H}^{d-2}(\mathbb{S}^{d-2})}.\end{aligned}$$

Putting everything together we finally get

$$\frac{4d-1}{\mathcal{H}^{d-2}(\mathbb{S}^{d-2})}\left(\int\phi_0\right)^2 > 27(.685) > 18.7170 \qquad \text{and} \qquad \delta^2\mathcal{F}(0)[\zeta_1^+,\zeta_1^+] > 18.7170 - 18.5359 > 0.$$

An asymptotic argument in higher dimension. Let us briefly sketch an asymptotic argument, which proves that the deformation corresponding to ζ_1^+ is positive for all $d \geq 21$. This completes the proof as $7 < d \leq 20$ can be verified by hand in the manner outlined above. The first key observation is that

$$.62 < \theta_0\sqrt{d} \leq .65, \quad \forall d > 20. \tag{5.16}$$

The proof of (5.16) is relatively straightforward: the lower bound follows from the fact that

$$2\theta_0\mathcal{H}^{d-2}(\mathbb{S}^{d-2}) \geq \mathcal{H}^{d-1}(\Omega_{b_{\nu,\theta_0}}) \geq \frac{1}{2}\mathcal{H}^{d-1}(\mathbb{S}^{d-1}),$$

the Gamma function formulas for the surface area of the sphere and Sterling approximation. The upper bound of (5.16) is a bit harder; the sharp isoperimetric inequality tells us that the perimeter of $\Omega_{b_{\nu,\theta_0}}$ is larger than the half sphere's perimeter. So we have

$$2\cos^{d-2}(\theta_0)\mathcal{H}^{d-2}(\mathbb{S}^{d-2}) > \mathcal{H}^{d-2}(\mathbb{S}^{d-2}).$$

Approximating cos by its Taylor series and doing some elementary estimates yields the result. Now, from (5.16) it is easy to get

$$1.3 > 2\sqrt{d}\theta_0 > \frac{\sqrt{d}\mathcal{H}^{d-1}(\Omega_{b_{\nu,\theta_0}})}{\mathcal{H}^{d-2}(\mathbb{S}^{d-2})} > \frac{\sqrt{d}}{2}\frac{\mathcal{H}^{d-1}(\mathbb{S}^{d-1})}{\mathcal{H}^{d-2}(\mathbb{S}^{d-2})} > \frac{\sqrt{2\pi}}{2} > 1.24. \tag{5.17}$$

Furthermore, we can also estimate

$$\frac{\sqrt{d}c^2}{\mathcal{H}^{d-2}(\mathbb{S}^{d-2})} \geq \frac{4\cos^{2d-4}(\theta_0)}{1.3}, \ \forall d > 20. \tag{5.18}$$

This is a little bit trickier, but it follows once one observes that

$$c \equiv \int\phi_0 = \frac{1}{d-1}\int -\Delta\phi_0 = \frac{\mathcal{H}^{d-2}(\partial\Omega_{b_{\nu,\theta_0}})}{(d-1)\kappa_0} \qquad \text{and} \qquad \frac{\mathcal{H}^{d-1}(\Omega_{b_{\nu,\theta_0}})}{\kappa_0^2} \geq \int|\nabla\phi_0|^2 = d-1.$$

Plugging (5.17) and (5.16) into (5.15) we find that $\delta^2\mathcal{F}(0)[\zeta_1^+,\zeta_1^+] > 0$ for all $d > 20$.

Step 2. Rotations and the kernel of $\delta^2\mathcal{F}(0)$. In order to conclude the proof of Proposition 5.2, and to prove that $\delta^2\mathcal{F}(0)[\zeta_j^-,\zeta_j^-] > 0$ for $j > d$, it suffices to show that

$$\delta^2\mathcal{F}(0)[\zeta_j^-,\zeta_j^-] = 0, \ 2 \leq j \leq d.$$

There are two ways to see this; the first is to notice that u_j^- for $2 \leq j \leq d$ corresponds to deforming the cone by rotation and thus has zero eigenvalue. However, due to the symmetries of the De Silva-Jerison cone, we can verify this fact explicitly in this scenario.

Using the commutator relation $[\partial_\theta, -\Delta] = \dfrac{d-2}{\sin^2(\theta)}\partial_\theta$, we get that

$$u_j^-(\theta,\varphi) = -\kappa_0\psi_j(\varphi)\partial_\theta\phi_0,\ 2 \leq j \leq d.$$

It is then easy to compute (after observing $-\partial_{\theta\theta}\phi_0(\pi/2+\theta_0) = -(d-2)\tan(\theta_0)\partial_\theta\phi_0(\pi/2+\theta_0)$)

$$\delta^2\mathcal{F}(0)[\zeta_j^-,\zeta_j^-] = 4(d-2)\cos^{d-2}(\theta_0)(\tan(\theta_0)-\tan(\theta_0)) = 0,\ \forall 2 \leq j \leq d.$$

As we stated above, invoking (5.12), this proves that $\delta^2\mathcal{F}(0)[\zeta_j^-,\zeta_j^-] > 0$ for all $j > d$ and that $\delta^2\mathcal{F}(0)[\zeta_1^-,\zeta_1^-] < 0$.

In the same manner, to prove that $\delta^2\mathcal{F}(0)[\zeta_j^+,\zeta_j^+] > 0$ for all $j > d$ it suffices to prove that

$$\delta^2\mathcal{F}(0)[\zeta_{d+1}^+,\zeta_{d+1}^+] > 0.$$

To do so, we observe that the function

$$u_{d+1}^+(\theta,\varphi) = -\frac{\kappa_0}{(d-2)\tan(\theta_0)}\psi_{d+1}(\varphi)\left(\partial_{\theta\theta}^2\phi_0(\theta)+\phi_0(\theta)\right) \tag{5.19}$$

satisfies (5.2), for $j = d+1$. To see this, note that $\lambda_{d+1}^{\mathbb{S}^{d-2}} = 2(d-1)$ and that

$$-\Delta\partial_{\theta\theta}^2\phi_0 = \left((d-1) - \frac{2(d-1)}{\sin^2(\theta)}\right)\partial_{\theta\theta}^2\phi_0 - \frac{2(d-1)}{\sin^2(\theta)}\phi_0.$$

We can then compute that

$$\begin{aligned}\delta^2\mathcal{F}(0)[\zeta_{d+1}^+,\zeta_{d+1}^+] &= 4\cos^{d-2}(\theta_0)\left(\int_{\mathbb{S}^{d-2}} u_{d+1}^+\partial_\nu u_{d+1}^+\,d\sigma - (d-2)\tan(\theta_0)\right)\\ &= 4\cos^{d-2}(\theta_0)\left((d-1)\tan(\theta_0)-(d-2)\tan(\theta_0)\right) > 0.\end{aligned} \tag{5.20}$$

Appendix A. Proof of Lemma 2.3

We derive the first and second variation of the Alt-Caffarelli functional restricted to the sphere. We show that the second variation is continuous at zero and that at zero it is diagonalizable. The notation is the same as in Subsection 2.3. □

A.1. **First and second variation of $\mathcal{F}$.** The following Lemma contains the explicit formulas for the variations of $\mathcal{F}$ around a domain $\Omega := \{b > 0\}\cap\partial B_1 \subset \partial B_1$, where b is a 1-homogeneous minimizer of $\mathcal{E}$ with isolated singularity, so that, by [1], $\partial\Omega := \partial\{b > 0\}\cap\partial B_1$ is smooth; we denote by ν the exterior normal to $\partial\Omega$ in the sphere. Keeping the notation from Subsection 2.3, b is given by $\kappa_0\phi_0 := b$, where ϕ_0 is the first eigenfunction on the spherical set Ω

$$-\Delta\phi_0 = (d-1)\,\phi_0 \quad\text{in}\quad \Omega\,, \qquad \int_\Omega \phi_0^2\,d\mathcal{H}^{d-1} = 1\,, \qquad \phi_0 = 0 \quad\text{and}\quad \kappa_0\,\partial_\nu\phi_0 = -1 \quad\text{on}\quad \partial\Omega\,.$$

For every $g \in C^{2,\alpha}(\partial\Omega)$, $\zeta \in C^{2,\alpha}(\partial\Omega)$ and $t \in \mathbb{R}$ we define the function $\Psi_{g,t} : \partial\Omega \to \mathbb{S}^{d-1}$ given through the spherical exponential map

$$\Psi_{g,t}(x) := \exp_x\Big(\big(g(x)+t\,\zeta(x)\big)\,\nu(x)\Big) = \cos\big(g(x)+t\,\zeta(x)\big)x + \sin\big(g(x)+t\,\zeta(x)\big)\nu_{\partial\Omega}(x).$$

We notice that the exponential map $\mathbb{R} \mapsto \partial\Omega \ni (s,x) \mapsto \exp_x\big(s\nu(x)\big)$ is a diffeomorphism in a neighbourhood of $\partial\Omega$. Then, for $\|g\|_{C^{2,\alpha}(\partial\Omega)}$ and $\|\zeta\|_{C^{2,\alpha}(\partial\Omega)}$ small enough, the map

$$[-2,2]\times\partial\Omega \ni (t,x) \mapsto \Psi_{g,t}(x),$$

is injective and smooth (and also a diffeomorphism from $]-2,2[\times\{\zeta \neq 0\}$ onto the image of Ψ_g) and so, the derivative

$$\partial_t\Psi_{g,t}(x) = \zeta(x)\xi_t(x), \quad\text{where}\quad \xi_t(x) := \cos\big(g(x)+t\,\zeta(x)\big)\,\nu_{\partial\Omega}(x) - \sin\big(g(x)+t\,\zeta(x)\big)\,x\,,$$

defines a vector field X on the closed set $\Psi_g([-2,2]\times\partial\Omega)$ such that

$$X(\Psi_{g,t}(x)) = \zeta(x)\,\xi_t(x)\,, \quad \text{for every} \quad x\in\partial\Omega, \tag{A.1}$$

which can be extended to the entire sphere by the Whitney's extension Theorem. Moreover, the map $\Psi_{g,t}:\partial\Omega\to\partial\Omega_{g,t}$ is a diffeomorphism, where $\partial\Omega_{g,t}$ is the boundary of a spherical set $\Omega_{g,t}$. Setting, $\Omega_g := \Omega_{g,0}$, we notice that the flow associated to this vector field X is an extension of the map $\Psi_{g,t}\circ\Psi_g^{-1}:\partial\Omega_g\to\partial B_1$, that is we have

$$\partial_t\Psi_{g,t}(\Psi_g^{-1}(x)) := X(\Psi_{g,t}(\Psi_g^{-1}(x))) \qquad \text{and} \qquad \Psi_{g,0}(\Psi_g^{-1}(x)) = x \quad \text{for} \quad x\in\partial\Omega_g.$$

In the next lemma we calculate the first and the second variations of $\mathcal{F}$.

Lemma A.1. *Let Ω, k_0 and X be as above. Then, for every $\zeta, g\in C^{2,\alpha}(\partial\Omega)$, we have*

$$\delta\mathcal{F}(g)[\zeta] = \int_{\partial\Omega_g}(X\cdot\nu_g)\,d\mathcal{H}^{d-2} - \kappa_0^2\int_{\partial\Omega_g}\partial_X\phi_g\,\partial_{\nu_g}\phi_g\,d\mathcal{H}^{d-2},$$

$$\delta^2\mathcal{F}(g)[\zeta,\zeta] = \int_{\partial\Omega_g}\mathrm{div}X\,(X\cdot\nu_g)\,d\mathcal{H}^{d-2} + \kappa_0^2\int_{\partial\Omega_g}\big(-\partial_{XX}\phi_g - 2\partial_X\phi_g'\big)\partial_{\nu_g}\phi_g\,d\mathcal{H}^{d-2},$$

where div *denotes the divergence in $\mathbb{S}^{d-1}$, ν_g is the outward unit normal to $\partial\Omega_g$, $H_{\partial\Omega_g}$ is the scalar mean curvature of $\partial\Omega_g$, ϕ_g is the first eigenfunction of Ω_g and ϕ_g' is the solution of*

$$-\Delta\phi_g' = \lambda_g\phi_g' - \phi_g\int_{\partial\Omega_g}\partial_X\phi_g\,\partial_{\nu_g}\phi_g \quad in \quad \Omega_g, \qquad \phi_g = 0 \quad on \quad \partial\Omega_g, \qquad \int_{\Omega_g}\phi_g'\phi_g = 0.$$

Remark A.2. We notice that the function ϕ_g' depends linearly on ζ. A more precise (but heavier) notation would be $\phi_g' = \delta\phi(g)[\zeta]$.

Proof. Recall the following Hadamard formula, whose proof can be found in [18, Section 5.2]

$$\frac{d}{dt}\int_{\Omega_{g,t}}f(t,x)\,d\mathcal{H}^{d-1}(x) = \int_{\Omega_{g,t}}\partial_t f(t,x)\,d\mathcal{H}^{d-1}(x) + \int_{\partial\Omega_{g,t}}f(t,x)(X(x)\cdot\nu_t(x))\,d\mathcal{H}^{d-2}(x)\,, \tag{A.2}$$

where ν_t denotes the outward pointing normal to a domain $\Omega_{g,t}$ in the sphere. Applying this law we see immediately that

$$\frac{d}{dt}\mathcal{H}^{d-1}(\Omega_{g,t}) = \delta m(g+t\zeta)[\zeta] = \int_{\partial\Omega_{g,t}}(X\cdot\nu_t)\,d\mathcal{H}^{d-2} = \int_{\Omega_{g,t}}\mathrm{div}X\,d\mathcal{H}^{d-1}. \tag{A.3}$$

Then we can apply (A.2) again to get

$$\delta^2 m(g)[\zeta,\zeta] = \int_{\Omega_g}\partial_t(\mathrm{div}X)\,d\mathcal{H}^{d-1} + \int_{\partial\Omega_g}\mathrm{div}X\,(X\cdot\nu_g)\,d\mathcal{H}^{d-2} = \int_{\partial\Omega_g}\mathrm{div}X\,(X\cdot\nu_g)\,d\mathcal{H}^{d-2} \tag{A.4}$$

where we used the fact that X is autonomous to conclude that $\partial_t(\mathrm{div}X) = \mathrm{div}(\partial_t X) = 0$. In particular, when $g=0$, we have that $X = \zeta\,\nu$ on $\partial\Omega$ and so

$$\delta m(0)[\zeta] = \int_{\partial\Omega}\zeta\;d\mathcal{H}^{d-2} \qquad \text{and} \qquad \delta^2 m(0)[\zeta,\zeta] = \int_{\partial\Omega}H_{\partial\Omega}\,\zeta^2\,d\mathcal{H}^{d-2}. \tag{A.5}$$

We now calculate the first and the second variation of the functional $\lambda : C^{2,\alpha}(\partial\Omega)\to\mathbb{R}$. We first notice that by [18, Theorem 5.7.4], the map $t\mapsto\lambda(g+t\zeta)$ is C^∞ in a neighborhood of zero and the first and the second derivatives have been computed in [18, Theorem 5.7.1] and [18, Section 5.9.6] for sets in $\mathbb{R}^d$. We also notice that the C^3 regularity condition from [18, Section 5.9.6] can be replaced by $C^{2,\alpha}$ as it was shown in [7] and [3]. Below, we formally derive the exact expressions of $\delta\lambda$ and $\delta^2\lambda$ on the sphere. For the sake of simplicity we set

$$\phi_t := \phi_{g+t\zeta}\ , \quad \Omega_t := \Omega_{g,t}\ , \quad \lambda_t = \lambda(g+t\zeta)\ , \quad \Psi_t := \Psi_{g,t}.$$

Using (A.2) once again and the fact that $\phi_t = 0$ on $\partial\Omega_t$, we obtain

$$0 = \partial_t\int_{\Omega_t}\phi_t^2 = 2\int_{\Omega_t}\phi_t\,\phi_t' + \int_{\partial\Omega_t}\phi_t^2\,(X\cdot\nu_t) = 2\int_{\Omega_t}\phi_t\,\phi_t'\,. \tag{A.6}$$

By definition of ϕ_t and Ψ_t we have

$$0 = \phi_t(\Psi_t(x)) = \phi_t\big(\cos(g(x) + t\zeta(x))\, x + \sin(g(x) + t\,\zeta(x))\,\nu(x)\big) \qquad \text{for every } x \in \partial\Omega$$

so that differentiating we get for every $x \in \partial\Omega$

$$\phi_t'(\Psi_t) = \zeta\big(\sin(g + t\zeta)\,x - \cos(g + t\,\zeta)\,\nu\big) \cdot D\phi_t(\Psi_t) = -\zeta\xi_t \cdot D\phi_t(\Psi_t)\,, \tag{A.7}$$

$$\begin{aligned}\phi_t''(\Psi_t) = &-\zeta^2\big(\sin(g + t\zeta)\,x - \cos(g + t\,\zeta)\,\nu\big) \cdot D^2\phi_t(\Psi_t)\,\big[\sin(g + t\zeta)\,x - \cos(g + t\,\zeta)\,\nu\big]\\ &+ 2\,\zeta\big(\sin(g + t\zeta)\,x - \cos(g + t\,\zeta)\,\nu\big) \cdot D\phi_t'(\Psi_t)\\ &- \zeta^2\big(\cos(g + t\zeta)\,x + \sin(g + t\,\zeta)\,\nu\big) \cdot D\phi_t(\Psi_t)\\ = &-\zeta^2\,\xi_t \cdot D^2\phi_t(\Psi_t)\,\xi_t - 2\,\zeta\,\xi_t \cdot D\phi_t'(\Psi_t),\end{aligned} \tag{A.8}$$

where, we used that $\Psi_t \cdot D\phi_t(\Psi_t) = 0$. Differentiating formally the equation for ϕ_t, we obtain

$$\begin{aligned}-\Delta\phi_t' &= \lambda_t\,\phi_t' + \lambda_t'\,\phi_t \quad \text{in} \quad \Omega_t\,,\\ -\Delta\phi_t'' &= \lambda_t\,\phi_t'' + 2\,\lambda_t'\,\phi_t' + \lambda_t''\,\phi_t \quad \text{in} \quad \Omega_t\,,\end{aligned}$$

where $\lambda_t' = \delta\lambda(g + t\zeta)[\zeta]$ and $\lambda_t'' = \delta^2\lambda(g + t\zeta)[\zeta, \zeta]$. Multiplying the first equation by ϕ_t, integrating by parts and using (A.6) and (A.7), we get

$$\begin{aligned}\kappa_0^2\,\delta\lambda(g + t\zeta)[\zeta] &= \kappa_0^2 \int_{\partial\Omega_t} \phi_t'\,\partial_{\nu_t}\phi_t\,d\mathcal{H}^{d-2} = \kappa_0^2 \int_{\partial\Omega} \phi_t'(\Psi_t)\,\partial_{\nu_t}\phi_t(\Psi_t)\,J\Psi_t\,d\mathcal{H}^{d-2}\\ &= -\kappa_0^2 \int_{\partial\Omega} \zeta\,\xi_t \cdot D\phi_t(\Psi_t)\,\partial_{\nu_t}\phi_t(\Psi_t)\,J\Psi_t\,d\mathcal{H}^{d-2}\\ &= -\kappa_0^2 \int_{\partial\Omega_t} \partial_X\phi_t\,\partial_{\nu_t}\phi_t\,d\mathcal{H}^{d-2},\end{aligned} \tag{A.9}$$

where ν_t is the normal to $\partial\Omega_t$. Using the definition of κ_0, for $g = 0$ and $t = 0$ yields

$$\kappa_0^2\,\delta\lambda(0)[\zeta] = -\kappa_0^2 \int_{\partial\Omega} \zeta\,|\partial_\nu\phi_0|^2\,d\mathcal{H}^{d-2} = -\int_{\partial\Omega} \zeta\,d\mathcal{H}^{d-2}\,. \tag{A.10}$$

In a similar way, multiplying the equation for ϕ_t'' by ϕ_t, integrating by parts and using (A.6) and (A.7), we get

$$\kappa_0^2\,\delta^2\lambda(g + t\zeta)[\zeta, \zeta] = \kappa_0^2 \int_{\partial\Omega_t} \phi_t''\,\partial_{\nu_t}\phi_t\,d\mathcal{H}^{d-2} = \kappa_0^2 \int_{\partial\Omega} \phi_t''(\Psi_t)\,\partial_{\nu_t}\phi_t(\Psi_t)\,J\Psi_t\,d\mathcal{H}^{d-2}.$$

By (A.8) we get

$$\begin{aligned}\kappa_0^2\,\delta^2\lambda(g + t\zeta)[\zeta, \zeta] &= \kappa_0^2 \int_{\partial\Omega} \Big[-\zeta^2\,\xi_t \cdot D^2\phi_t(\Psi_t)\,\xi_t - 2\,\zeta\,\xi_t \cdot D\phi_t'(\Psi_t)\Big]\,\partial_{\nu_t}\phi_t(\Psi_t)\,J\Psi_t\,d\mathcal{H}^{d-2}\\ &= \kappa_0^2 \int_{\partial\Omega_t} \big(-\partial_{XX}\phi_t - 2\partial_X\phi_t'\big)\,\partial_{\nu_t}\phi_t\,d\mathcal{H}^{d-2}.\end{aligned} \tag{A.11}$$

Evaluating the above expression at $g = 0$ and $t = 0$ and using the definition of κ_0, we conclude

$$\kappa_0^2\,\delta^2\lambda(0)[\zeta, \zeta] = \int_{\partial\Omega} \zeta^2\,H_{\partial\Omega}\,d\mathcal{H}^{d-2} + 2\,\kappa_0^2 \int_{\partial\Omega} \phi_0'\,\partial_\nu\phi_0'\,d\mathcal{H}^{d-2} \tag{A.12}$$

Combining (A.4), (A.5), (A.7), (A.10), (A.11) and (A.12), and setting $u_\zeta := \kappa_0\phi_0'$, we conclude the proof of the lemma. □

A.2. **The first and the second variation in zero.** Let $\zeta \in C^{2,\alpha}(\partial\Omega)$. We first notice that, by Lemma A.1, equations (A.5), (A.10) and (A.12), we have

$$\delta\mathcal{F}(0)[\zeta] = 0\,, \tag{A.13}$$

$$\delta^2\mathcal{F}(0)[\zeta,\zeta] = 2\int_{\partial\Omega} \zeta^2\, H_{\partial\Omega}\, d\mathcal{H}^{d-2} + 2\int_{\partial\Omega} \zeta\; T\zeta\, d\mathcal{H}^{d-2}, \tag{A.14}$$

where $T\colon H^{1/2}(\partial\Omega) \to H^{-1/2}(\partial\Omega)$ is the Dirichlet to Neumann operator defined by $T\zeta = \partial_\nu u_\zeta$, where u_ζ is the solution of

$$\begin{cases} -\Delta u_\zeta &=& (d-1)u_\zeta - \dfrac{1}{\kappa_0^2}\left(\displaystyle\int_{\partial\Omega} \zeta\, d\mathcal{H}^{d-2}\right) b \quad \text{in} \quad \Omega \\ \quad u_\zeta &=& -\zeta \quad \text{on} \quad \partial\Omega \qquad \text{and} \qquad \displaystyle\int_{\Omega} b\, u_\zeta\, d\mathcal{H}^{d-1} = 0\,. \end{cases} \tag{A.15}$$

Remark A.3. In the notation of Lemma A.1, we have $u_\zeta := \kappa_0 \phi_0'$.

In this subsection we prove (2.8) by diagonalizing the bilinear form $\delta^2\mathcal{F}(0)$. To this aim we first notice that the linear operator T is well defined. Indeed, the solution of (A.15) is unique since if there were two solutions u_1 and u_2, then the difference $v := u_1 - u_2$ would be a solution of the eigenvalue problem

$$-\Delta v = (d-1)v \quad \text{in} \quad \Omega, \qquad v = 0 \quad \text{on} \quad \partial\Omega.$$

Thus, $v = C\, b$ for some constant. Now, the orthogonality condition $\displaystyle\int_\Omega v\, b = 0$ implies that the constant is zero. The existence of a solution of (A.15) now follows by the Fredholm alternative.

Therefore we can consider the linear operator

$$H^{1/2}(\partial\Omega) \ni \zeta \mapsto B(\zeta) := T\zeta + (H_{\partial\Omega} + \Lambda)\,\zeta \in H^{-1/2}(\partial\Omega)$$

where $\Lambda := \|H_{\partial\Omega}\|_{L^\infty} + C_\Omega$ ($C_\Omega > 0$ is a constant which depends only on Ω; it is proportional to the sum of the norm of the trace operator and to the bounds given by elliptic regularity for the Laplacian on Ω). B is a self-adjoint, positive linear operator, with compact inverse. Indeed, it is sufficient to notice that

$$\begin{aligned} \int_{\partial\Omega} T(\zeta_1)\,\zeta_2\, d\mathcal{H}^{d-2} &= \int_\Omega (\Delta u_{\zeta_1}\, u_{\zeta_2} + \nabla u_{\zeta_1}\cdot\nabla u_{\zeta_2})\, d\mathcal{H}^{d-2} \\ &= \int_\Omega \left(-(d-1)u_{\zeta_1}\, u_{\zeta_2} + \nabla u_{\zeta_1}\cdot\nabla u_{\zeta_2}\right)\, d\mathcal{H}^{d-2}\,, \end{aligned}$$

where in the last equality we used the orthogonality of u_ζ and b. The theory of compact operators now implies that there exists a sequence of positive eigenvalues $(\tilde\lambda_i)_i$ accumulating to ∞ and of eigenfunctions $(\xi_i)_i$ which form an orthonormal basis of $L^2(\partial\Omega)$ satisfying

$$B\xi_i = \tilde\lambda_i\,\xi_i \qquad \text{for every } i \in \mathbb{N}\,.$$

In particular the sequence $\lambda_i := \tilde\lambda_i - \Lambda$ and the functions $(\xi_i)_i$, for $i \in \mathbb{N}$, satisfy (2.8). We also note that elliptic regularity tells us that the eigenfunctions of $T + H_{\partial\Omega}$ are as regular as $H_{\partial\Omega}$; since $\partial\Omega$ is locally analytic, we can conclude that $\xi_i \in C^{2,\alpha}$.

To conclude the proof of (2.8) it is enough to observe that if ζ is the infinitesimal generator of a rotation of $\partial\Omega$, then it must preserve $\mathcal{F}$ to second order, so that $\zeta \in \ker(\delta^2\mathcal{F}(0))$ (one can see this by taking the Taylor series expansion of $\mathcal{F}$ around zero, and writing the boundary of a rotated copy of Ω as the graph of $t\zeta + O(t^2)$ over $\partial\Omega$). Since $\partial\Omega$ is a smooth codimension 1 submanifold, it is invariant for at most one rotation, so that there are $(d-1)$ nontrivial elements in the kernel. The upper bound follows from the previous spectral analysis.

A.3. **Uniform bounds on $\phi_{g+t\zeta}$.** Recall that $\phi_{g+t\zeta}$ satisfies the elliptic equation

$$(-\Delta-\lambda_{g+t\zeta})\phi_{g+t\zeta}=0 \quad \text{in} \quad \Omega_{g,t}, \qquad \phi_{g+t\zeta}=0 \quad \text{on} \quad \partial\Omega_{g,t}, \qquad \int_{\Omega_{g,t}}\phi_{g+t\zeta}^2=1.$$

The spherical domain Ω_g is $C^{2,\alpha}$ smooth and the $C^{2,\alpha}$ norm depends only on $\|g\|_{C^{2,\alpha}}$. Thus by classical elliptic regularity/Schauder estimates (see [17]), there is a universal constant C such that

$$\|\phi_g\|_{C^{2,\alpha}(\overline{\Omega}_g)}\le C(\|g\|_{C^{2,\alpha}}+1). \tag{A.16}$$

A.4. **Bounds on ϕ'_g.** In this subsection we prove (2.6), which in the notation of this section reads as $\|\phi'_g\|_{L^2(\Omega_g)}\le C\|\zeta\|_{L^2(\partial\Omega)}$. We first prove a bound on $\delta\lambda(g)[\zeta]$. Recall that by (A.9)

$$\lambda'_g=\delta\lambda(g)[\zeta]=-\int_{\partial\Omega_g}\partial_X\phi_g\,\partial_{\nu_g}\phi_g\,d\mathcal{H}^{d-2}.$$

This, together with (A.16), implies that

$$\big|\delta\lambda(g)[\zeta]\big|\le\|\phi_g\|^2_{C^1(\overline{\Omega}_g)}\|\zeta\|_{L^2}\sqrt{\mathcal{H}^{d-2}(\partial\Omega_g)}<C^2\left(\|g\|^2_{C^{2,\alpha}}+1\right)\|\zeta\|_{L^2}, \tag{A.17}$$

where C is the constant from (A.16). Recall that by (A.7) ϕ'_g is a solution of the equation

$$(-\Delta-\lambda_g)\phi'_g=\lambda'_g\phi_g \quad \text{in} \quad \Omega_g, \qquad \phi'_g=-\partial_X\phi_g \quad \text{on} \quad \partial\Omega_g, \qquad \int_{\Omega_g}\phi'_g\phi_g\,d\mathcal{H}^{d-1}=0.$$

Thus, ϕ'_g can be decomposed as

$$\phi'_g=h_g+\psi_g-\phi_g\int_{\Omega_g}h_g\phi_g, \tag{A.18}$$

where h_g and ψ_g are the solutions of the problems

$$\Delta h_g=0 \quad \text{in} \quad \Omega_g, \qquad h_g=-\partial_X\phi_g \quad \text{on} \quad \partial\Omega_g.$$

$$(-\Delta-\lambda_g)\psi_g=\lambda_g h_g+\lambda'_g\phi_g \quad \text{in} \quad \Omega_g, \qquad \psi_g=0 \quad \text{on} \quad \partial\Omega_g, \qquad \int_{\Omega_g}\psi_g\phi_g\,d\mathcal{H}^{d-1}=0. \tag{A.19}$$

Notice that λ_g is the lowest eigenvalue on Ω_g and its eigenspace is one-dimensional and generated by ϕ_g. Thus, the orthogonality $\int_{\Omega_g}\psi_g\phi_g=0$ implies that there is some constant $c_b>0$ such that

$$c_b\int_{\Omega_g}\left(|\nabla\psi_g|^2+\psi_g^2\right)d\mathcal{H}^{d-1}\le\int_{\Omega_g}\left(|\nabla\psi_g|^2-\lambda_g\psi_g^2\right)d\mathcal{H}^{d-1}.$$

Thus, multiplying by ψ_g and integrating by parts in (A.19), we get

$$c_b\|\psi_g\|^2_{H^1(\Omega_g)}\le\int_{\Omega_g}\psi_g(\lambda_g h_g+\lambda'_g\phi_g)\,dx=\lambda_g\int_{\Omega_g}\psi_g h_g\,dx\le\lambda_g\|\psi_g\|_{L^2(\Omega_g)}\|h_g\|_{L^2(\Omega_g)},$$

which in turn gives

$$\|\phi'_g\|_{L^2(\Omega_g)}\le\|\psi_g\|_{L^2(\Omega_g)}+2\|h_g\|_{L^2(\Omega_g)}\le\left(\frac{\lambda_g}{c_b}+2\right)\|h_g\|_{L^2(\Omega_g)}.$$

Now notice that by the maximum principle we have $\|h_g\|_{L^\infty(\Omega_g)}\le C\|\zeta\|_{L^\infty(\partial\Omega)}$ and, by [7, Lemma 10], $\|h_g\|_{L^1(\Omega_g)}\le C\|\zeta\|_{L^1(\partial\Omega)}$. Thus, by interpolation, we get $\|h_g\|_{L^2(\Omega_g)}\le C\|\zeta\|_{L^2(\Omega)}$ and finally we obtain (2.6).

A.5. **Proof of** (2.7)**.** The continuity of $\delta^2\mathcal{F}(g)[\zeta,\zeta]$ at zero follows by the continuity of $\delta^2\lambda(g)[\zeta,\zeta]$ and $\delta^2 m(g)[\zeta,\zeta]$. We give the details for $\delta^2\lambda$ (the more complicated case) but the arguments for $\delta^2 m$ are in the same vein. By (A.11) we have

$$\begin{aligned}\delta^2\lambda(g)[\zeta,\zeta]-\delta^2\lambda(0)[\zeta,\zeta]&=\int_{\partial\Omega}\Big[-\zeta^2\,\xi_g\cdot D^2\phi_g(\Psi_g)[\xi_g]-2\,\zeta\,\xi_g\cdot D\phi_g'(\Psi_g)\Big]\,\partial_{\nu_g}\phi_g(\Psi_g)\,J\Psi_g\,d\mathcal{H}^{d-2}\\&\quad-\int_{\partial\Omega}\Big[-\zeta^2\,\nu\cdot D^2\phi_0[\nu]-2\,\zeta\,\nu\cdot D\phi_0'\Big]\,\partial_\nu\phi_0\,d\mathcal{H}^{d-2},\end{aligned}$$

which in turn can be split into

$$\begin{aligned}I_1+I_2+I_3:=&\int_{\partial\Omega}\zeta^2\Big[-\xi_g\cdot D^2\phi_g(\Psi_g)[\xi_g]\,\partial_{\nu_g}\phi_g(\Psi_g)\,J\Psi_g+\nu\cdot D^2\phi_0[\nu]\,\partial_\nu\phi_0\Big]\,d\mathcal{H}^{d-2}\\&-2\int_{\partial\Omega}\Big[\zeta\,\xi_g\cdot D\phi_g'(\Psi_g)\,\partial_{\nu_g}\phi_g(\Psi_g)-\zeta\,\nu_g(\Psi_g)\cdot D\phi_g'(\Psi_g)\,\partial_{\xi_g}\phi_g(\Psi_g)\Big]\,J\Psi_g\,d\mathcal{H}^{d-2}\\&-2\int_{\partial\Omega}\Big[\zeta\,\nu_g(\Psi_g)\cdot D\phi_g'(\Psi_g)\,\partial_{\xi_g}\phi_g(\Psi_g)\,J\Psi_g-\zeta\,\nu\cdot D\phi_0'\,\partial_\nu\phi_0\Big]\,d\mathcal{H}^{d-2}.\end{aligned}$$

We first notice that there is a universal constant C such that for every $x\in\partial\Omega$ and $g\in C^{2,\alpha}(\partial\Omega)$

$$|\xi_g-\nu|\le C|g|\qquad\text{and}\qquad|\nu_g(\Psi_g)-\nu|\le C\big(|\nabla g|+|g|\big).$$

Moreover, the uniform Schauder estimates on the boundary of Ω_g give that for some universal modulus of continuity ω we have

$$\|D^2\phi_g(\Psi_g)-D^2\phi_0\|_{L^\infty(\Omega)}\le\omega(\|g\|_{C^{2,\alpha}})\qquad\text{and}\qquad\|D\phi_g(\Psi_g)-D\phi_0\|_{L^\infty(\Omega)}\le\omega(\|g\|_{C^{2,\alpha}}).$$

Thus, we get that

$$|I_1|\le\|\zeta\|^2_{L^2(\partial\Omega)}\,\omega(\|g\|_{C^{2,\alpha}}).$$

In order to estimate the third integral I_3, we notice that (A.7), a change of variables and an integration by parts give

$$\begin{aligned}\frac12 I_3&=-\int_{\partial\Omega}\zeta\Big[\nu_g(\Psi_g)\cdot D\phi_g'(\Psi_g)\,\partial_{\xi_g}\phi_g(\Psi_g)\,J\Psi_g-\nu\cdot D\phi_0'\,\partial_\nu\phi_0\Big]\,d\mathcal{H}^{d-2}\\&=\int_{\partial\Omega}\nu_g(\Psi_g)\cdot D\phi_g'(\Psi_g)\,\phi_g'(\Psi_g)\,J\Psi_g\,d\mathcal{H}^{d-2}-\int_{\partial\Omega}\nu\cdot D\phi_0'\,\phi_0'\,d\mathcal{H}^{d-2}\\&=\int_{\partial\Omega_g}\partial_{\nu_g}\phi_g'\,\phi_g'\,d\mathcal{H}^{d-2}-\int_{\partial\Omega}\partial_\nu\phi_0'\,\phi_0'\,d\mathcal{H}^{d-2}=\int_{\Omega_g}|\nabla\phi_g'|^2\,d\mathcal{H}^{d-1}-\int_\Omega|\nabla\phi_0'|^2\,d\mathcal{H}^{d-1}.\end{aligned}$$

Using the decomposition (A.18) and the fact that h_g is harmonic on Ω_g, we get

$$\int_{\Omega_g}|\nabla\phi_g'|^2\,d\mathcal{H}^{d-1}=\int_{\Omega_g}|\nabla h_g|^2\,d\mathcal{H}^{d-1}+\int_{\Omega_g}|\nabla(\psi_g-\phi_g\langle h_g,\phi_g\rangle)|^2\,d\mathcal{H}^{d-1},$$

where $\langle h_g,\phi_g\rangle:=\displaystyle\int_{\Omega_g}h_g\phi_g$. Next we notice that by [3, Lemma A.2, eq. (A.20)] we have

$$\left|\int_{\Omega_g}|\nabla h_g|^2-\int_\Omega|\nabla h_0|^2\right|\le\omega(\|g\|_{C^{2,\alpha}})\|\zeta\|^2_{H^{1/2}}.$$

Thus, it is sufficient to prove that

$$\left|\int_{\Omega_g}|\nabla(\psi_g-\phi_g\langle h_g,\phi_g\rangle)|^2\,d\mathcal{H}^{d-1}-\int_\Omega|\nabla(\psi_0-\phi_0\langle h_0,\phi_0\rangle)|^2\,d\mathcal{H}^{d-1}\right|\le\omega(\|g\|_{C^{2,\alpha}})\|\zeta\|^2_{L^2},$$

which follows by a simple argument by contradiction, which we sketch for completeness. Indeed, we notice that the functions $\|\zeta\|^{-1}_{L^2}\psi_g$ have uniformly bounded H^1-norm and so are converging weakly in H^1 (and since they are solutions of a PDE they converge strongly in H^1) to $\|\zeta\|^{-1}_{L^2}\psi_0$ as $\|g\|_{C^{2,\alpha}}\to0$. On the other hand, $\|\zeta\|^{-1}_{L^2}h_g$ have uniformly bounded L^2 norm and converge weakly in L^2 to $\|\zeta\|^{-1}_{L^2}h_0$. Finally, the strong H^1 convergence of ϕ_g to ϕ_0 gives that

$\|\zeta\|_{L^2}^{-1}\phi_g\langle h_g,\phi_g\rangle \to \|\zeta\|_{L^2}^{-1}\phi_0\langle h_0,\phi_0\rangle$ strongly in H^1. This concludes the estimate of I_3.

We now estimate I_2. Since $D\phi_g$ is parallel to ν_g on $\partial\Omega_g$ we get that

$$\begin{aligned}I_2 &= -2\int_{\partial\Omega}\Big[\zeta\,\xi_g\cdot D\phi_g'(\Psi_g)\,\partial_{\nu_g}\phi_g(\Psi_g) - \zeta\,\nu_g(\Psi_g)\cdot D\phi_g'(\Psi_g)\,\partial_{\xi_g}\phi_g(\Psi_g)\Big]\,J\Psi_g\,d\mathcal{H}^{d-2}\\ &= -2\int_{\partial\Omega}\Big[\zeta\,\xi_g\cdot D\phi_g'(\Psi_g)\,\partial_{\nu_g}\phi_g(\Psi_g) - \zeta\,\nu_g(\Psi_g)\cdot D\phi_g'(\Psi_g)\,\big(\xi_g\cdot\nu_g(\Psi_g)\big)\partial_{\nu_g}\phi_g(\Psi_g)\Big]\,J\Psi_g\,d\mathcal{H}^{d-2}\\ &= -2\int_{\partial\Omega}\zeta\,\big(\xi_g-(\xi_g\cdot\nu_g(\Psi_g))\nu_g(\Psi_g)\big)\cdot D\phi_g'(\Psi_g)\,\partial_{\nu_g}\phi_g(\Psi_g)\,J\Psi_g\,d\mathcal{H}^{d-2}.\end{aligned}$$

We first notice that we have the pointwise estimates

$$\big|\xi_g-(\xi_g\cdot\nu_g(\Psi_g))\nu_g(\Psi_g)\big| \le C\big(|\nabla g|+|g|\big) \qquad\text{and}\qquad |\partial_{\nu_g}\phi_g(\Psi_g)|\le C,$$

and that $\xi_g-(\xi_g\cdot\nu_g(\Psi_g))\nu_g(\Psi_g)$ is parallel to $\partial\Omega_g$. Then, on $\partial\Omega_g$ we define

$$z=\zeta(\Psi_g^{-1})\ ,\quad V=\xi_g(\Phi_g^{-1})\quad\text{and}\quad W=V-(V\cdot\nu_g)\nu_g,$$

so we obtain

$$\begin{aligned}I_2 &= -2\int_{\partial\Omega_g} z\,(W\cdot D\phi_g')\,\partial_{\nu_g}\phi_g\,d\mathcal{H}^{d-2}\\ &= -2\int_{\partial\Omega_g} z\,(W\cdot D(zV\cdot D\phi_g))\,\partial_{\nu_g}\phi_g\,d\mathcal{H}^{d-2}\\ &= -2\int_{\partial\Omega_g} z^2\,(W\cdot D(V\cdot D\phi_g))\,\partial_{\nu_g}\phi_g\,d\mathcal{H}^{d-2} - \int_{\partial\Omega_g} W\cdot D(z^2)\,(V\cdot D\phi_g)\,\partial_{\nu_g}\phi_g\,d\mathcal{H}^{d-2}\\ &= -2\int_{\partial\Omega_g} z^2\,(W\cdot D(V\cdot D\phi_g))\,\partial_{\nu_g}\phi_g\,d\mathcal{H}^{d-2} + \int_{\partial\Omega_g} z^2\,\mathrm{div}\big(W(V\cdot D\phi_g)\,\partial_{\nu_g}\phi_g\big)\,d\mathcal{H}^{d-2}\\ &= -\int_{\partial\Omega_g} z^2\,(W\cdot D(V\cdot D\phi_g))\,\partial_{\nu_g}\phi_g\,d\mathcal{H}^{d-2} + \int_{\partial\Omega_g} z^2\,\mathrm{div}\,W\big((V\cdot D\phi_g)\,\partial_{\nu_g}\phi_g\big)\,d\mathcal{H}^{d-2},\end{aligned}$$

where for the second equality we used the boundary condition $\phi_g'(\Psi_g)=-\zeta\xi_g\cdot D\phi_g(\Psi_g)$ and in the fourth one we integrated by parts. Thus, we get that

$$|I_2|\le\omega\big(\|g\|_{C^{2,\alpha}(\partial\Omega)}\big)\int_{\partial\Omega}|\zeta|^2\,d\mathcal{H}^{d-2},$$

which concludes the proof of Lemma 2.3.

B. Lyapunov-Schmidt reduction for $\mathcal{F}$

We prove the following lemma, inspired by [24]. We shall denote by $K:=\ker(\delta^2(\mathcal{F}(0)))$ and by $N:=\dim K$, its dimension (see Subsection 2.3). We introduce an auxiliary functional

$$\mathcal{G}: C^{2,\alpha}(\partial\Omega)\times\mathbb{R}\to\mathbb{R}\,,\qquad \mathcal{G}(\zeta,s)=(\kappa_0^2+s^3)(\lambda(\zeta)-(d-1))+m(\zeta)-m(0), \tag{B.1}$$

where κ_0 is the constant from Subsection 2.3. Thus, the variable s accounts for the possibility that the coefficient κ, from Proposition 3.2, in front of ϕ_1 may not be equal to κ_0. It is easy to check that the first and the second variation of $\mathcal{G}$ are given by

$$\mathcal{G}(0,0)=0,\quad \delta\mathcal{G}(0,0)[\xi,r]=0,\quad \delta^2\mathcal{G}(0,0)[(\xi,r),(\eta,t)]=\delta^2\mathcal{F}(0)[\xi,\eta]. \tag{B.2}$$

In particular, we have that $\ker\delta^2\mathcal{G}(0,0)=\ker\delta^2\mathcal{F}(0)\oplus\mathbb{R}$. In the lemma below, we let P_K and $P_{K^\perp}$ be the $(L^2(\partial\Omega)\oplus\mathbb{R})$-projections on $K\oplus\mathbb{R}$ and $K^\perp\oplus\{0\}$, respectively.

Lemma B.1 (Lyapunov-Schmidt decomposition)**.** *There exists a neighborhood U of 0 in $C^{1,\alpha}\oplus\mathbb{R}$ and an analytic map $\Upsilon\colon U\cap(K\oplus\mathbb{R})\to K^\perp\subset C^{2,\alpha}(\partial\Omega)$ such that*

$$\Upsilon(0,0)=0\,,\qquad \delta\Upsilon(0,0)=0\,, \tag{B.3}$$

and moreover

$$\begin{cases} P_{K^\perp}\left(\delta\mathcal{G}(\zeta+\Upsilon(\zeta,s),s)\right)=0 & (\zeta,s)\in(K\oplus\mathbb{R})\cap U \\ P_K\left(\delta\mathcal{G}(\zeta+\Upsilon(\zeta,s),s)\right)=-\nabla G(\zeta,s) & (\zeta,s)\in(K\oplus\mathbb{R})\cap U, \end{cases} \tag{B.4}$$

where $G(\zeta,s)=\mathcal{G}(\zeta+\Upsilon(\zeta,s),s)$ *for every* $(\zeta,s)\in(K\oplus\mathbb{R})\cap U$*. Moreover, all the critical points of* $\mathcal{G}$ *in* U *are given by*

$$\mathcal{C}:=\big\{(\zeta+\Upsilon(\zeta,s),s)\,:\,(\zeta,s)\in(K\oplus\mathbb{R})\cap U\quad\text{and}\quad\nabla G(\zeta,s)=0\big\}$$

which is an analytic submanifold of the $N+1$*-dimensional analytic manifold*

$$\mathcal{M}:=\{(\zeta+\Upsilon(\zeta,s),s)\,:\,(\zeta,s)\in(K\oplus\mathbb{R})\cap U\}\,.$$

Proof. Consider the operator

$$\mathcal{N}(\zeta,s):=P_{K^\perp}(\delta\mathcal{G}(\zeta,s))+P_K(\zeta,s):L^2(\partial\Omega)\oplus\mathbb{R}\to L^2(\partial\Omega)\oplus\mathbb{R}$$

and notice that $\mathcal{N}(0,0)=0$, since $\delta\mathcal{G}(0,0)=0$. Moreover,

$$\delta\mathcal{N}(0,0)[(\zeta,s)]=\frac{d}{dt}\Big|_{t=0}\mathcal{N}(t\zeta,ts)=\big(\delta^2\mathcal{F}(0)[\zeta,-],0\big)+P_K(\zeta,s)\,,$$

where we used that P_K and $\delta^2\mathcal{F}(0)$ are linear (and that $\delta^2\mathcal{F}(0)[\zeta,P_{K^\perp}\zeta]=\delta^2\mathcal{F}(0)[\zeta,\zeta]$). In particular $\delta\mathcal{N}(0,0)$ has trivial kernel on $C^{2,\alpha}(\partial\Omega)\oplus\mathbb{R}$ by construction. By standard elliptic theory and Schauder estimates, we conclude that the operator $\delta\mathcal{N}(0,0)=(\delta^2\mathcal{F}(0)+P_K,1)=(T+H_{\partial\Omega}+P_K,1)$ is an isomorphism of $C^{2,\alpha}\oplus\mathbb{R}$ to $C^{1,\alpha}\oplus\mathbb{R}$, for every $\alpha\in(0,1)$, and therefore we can apply the inverse function theorem to the $C^{2,\alpha}$ operator $\mathcal{N}\colon C^{2,\alpha}\oplus\mathbb{R}\to C^{1,\alpha}\oplus\mathbb{R}$, producing $\Psi:=\mathcal{N}^{-1}$ which is a bijection from a neighborhood W of 0 in $C^{1,\alpha}\oplus\mathbb{R}$ to a neighborhood U of 0 in $C^{2,\alpha}\oplus R$.

Now the map we are looking for is simply given by $\Upsilon:=P_{K^\perp}\circ\Psi\colon K\oplus\mathbb{R}\to K^\perp\oplus\{0\}$. Indeed notice that the first conclusion of (B.3) is obvious since $\Psi(0,0)=\Psi(\mathcal{N}(0,0))=(0,0)$, while the second one follows from the more general observation that for every $\zeta\in K,s\in\mathbb{R}$ we have

$$\delta\Upsilon(\zeta,s)[(\eta,r)]=\delta(P_{K^\perp}\Psi(\zeta,s))[(\eta,r)]=P_{K^\perp}(\delta\Psi(\zeta,s))[\eta,r]=0,\ \text{for every}\ \eta\in K,r\in\mathbb{R}\,, \tag{B.5}$$

by the linearity of $P_{K^\perp}$.

For what concerns (B.4) for every $(\zeta,s)\in C^{2,\alpha}\oplus\mathbb{R}$ we have

$$P_K(\zeta,r)+P_{K^\perp}(\zeta,r)=(\zeta,r)=\mathcal{N}(\Psi(\zeta,r))=P_{K^\perp}\delta\mathcal{G}(\Psi(\zeta,r))+P_K(\Psi(\zeta,r))$$

which implies, by applying P_K and $P_{K^\perp}$ respectively on both sides, that

$$P_K(\zeta,r)=P_K(\Psi(\zeta,r))\qquad\text{and}\qquad P_{K^\perp}(\zeta,r)=P_{K^\perp}\delta\mathcal{G}(\Psi(\zeta,r))\,.$$

In particular, using the first identity in the second one we get

$$P_{K^\perp}(\zeta,r)=P_{K^\perp}\delta\mathcal{G}(P_K\Psi(\zeta,r)+P_{K^\perp}\Psi(\zeta,r))=P_{K^\perp}\delta\mathcal{G}(P_K\zeta+\Upsilon(\zeta,r),r)\,,$$

so that, if $\zeta\in K\cap U,r\in\mathbb{R}$, we conclude

$$P_{K^\perp}\delta\mathcal{G}(\zeta+\Upsilon(\zeta,r),r)=0\,.$$

The second conclusion of (B.4) follows by differentiating the function $G(\zeta,r)$ as follows. Let $\eta\in K$, then we have

$$\begin{aligned}\langle\nabla G(\zeta,r),(\eta,s)\rangle&=\delta\mathcal{G}(\zeta+\Upsilon(\zeta,r),r)[\eta+\delta\Upsilon(\zeta,r)[\eta,s],s]\\ &\overset{(B.5)}{=}\delta\mathcal{G}(\zeta+\Upsilon(\zeta,r),r)[\eta,s]=P_K(\delta\mathcal{G}(\zeta+\Upsilon(\zeta,r),r))[\eta,s]\,,\end{aligned}$$

where the last equality follows from the first one in (B.4). □

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