NULL LAGRANGIAN MEASURES IN PLANES, COMPENSATED COMPACTNESS AND CONSERVATION LAWS

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ABSTRACT. Compensated compactness is an important method used to solve nonlinear PDEs, in particular in the study of hyperbolic conservation laws. One of the simplest formulations of a compensated compactness problem is to ask for conditions on a compact set $\mathcal{K} \subset M^{m\times n}$ such that

$$\lim_{j\to\infty} \|\mathrm{dist}(Du_j, \mathcal{K})\|_{L^p} = 0 \text{ and } \sup_j \|u_j\|_{W^{1,p}} < \infty \;\Rightarrow \{Du_j\}_j \text{ is precompact in } L^p. \tag{1}$$

Let $M_1, M_2, \dots, M_q$ denote the set of all minors of $M^{m\times n}$. A sufficient condition for (1) is that any probability measure μ supported on $\mathcal{K}$ satisfying

$$\int M_k(X)d\mu(X) = M_k\left(\int X d\mu(X)\right) \text{ for all } k \tag{2}$$

is a Dirac measure. We call measures that satisfy (2) *Null Lagrangian Measures* and following [Mü 99], we denote the set of Null Lagrangian Measures supported on $\mathcal{K}$ by $\mathcal{M}^{pc}(\mathcal{K})$. For general m, n, a necessary and sufficient condition for triviality of $\mathcal{M}^{pc}(\mathcal{K})$ is unknown even in the case where $\mathcal{K}$ is a linear subspace of $M^{m\times n}$. We provide a condition that is sufficient for any linear subspace $\mathcal{K}$ and also necessary in the case where $2 \leq \min\{m, n\} \leq 3$. A corollary of this result is the fact that two dimensional subspaces $\mathcal{K} \subset M^{m\times n}$ support nontrivial Null Lagrangian Measures if and only if $\mathcal{K}$ has Rank-1 connections.

Using the ideas developed for this purpose we are able to answer (up to first order) a question of Kirchheim, Müller and Šverák [Ki-Mü-Sv 03]. Let $P_1(u,v) := \begin{pmatrix} u & v \\ a(v) & u \\ ua(v) & \frac{1}{2}u^2 + F(v) \end{pmatrix}$ and $\mathcal{K}_1 := \{P_1(u,v) : u, v \in \mathbb{R}\}$ for some function a and its primitive F. The set $\mathcal{K}_1$ arises in the study of entropy solutions to the 2×2 system of conservation laws

$$u_t = a(v)_x \quad \text{and} \quad v_t = u_x. \tag{3}$$

In [Ki-Mü-Sv 03], the authors asked what are the conditions on the function a such that $\mathcal{M}^{pc}(\mathcal{K}_1 \cap U)$ consists of Dirac measures, where U is an open neighborhood of an arbitrary matrix in $\mathcal{K}_1$. Given $\alpha = (\alpha_1, \alpha_2) \in \mathbb{R}^2$, if $a'(\alpha_2) > 0$ then we construct nontrivial measures in $\mathcal{M}^{pc}(\mathcal{K}_1 \cap B_\delta(P_1(\alpha)))$ for any $\delta > 0$. On the other hand if $a'(\alpha_2) < 0$ then for sufficiently small $\delta > 0$, we show that $\mathcal{M}^{pc}(\mathcal{K}_1 \cap B_\delta(P_1(\alpha)))$ consists of Dirac measures.

Further exploiting these ideas leads to a strategy of proving triviality of Null Lagrangian Measures supported on a small neighborhood of a smooth submanifold. We use this strategy to provide a more direct proof of DiPerna's well known result [DP 85] on existence of entropy solutions to (3). The strategy could potentially be applied to other systems of conservation laws with few entropies.

1. INTRODUCTION

Compensated compactness (coupled with a-priori L^p bounds) is an important method of solving nonlinear PDEs. Amongst its most celebrated successes are the proofs of the first existence theorems for solutions of systems of hyperbolic conservation laws with large data by Tartar [Ta 79], [Ta 83] and DiPerna [DP 83], [DP 85]. One of the simplest and most

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natural formulations of compensated compactness is to ask for conditions on a compact set of matrices $\mathcal{K} \subset M^{m\times n}$ such that for any sequence $\{u_j\} \subset W^{1,p}(\Omega;\mathbb{R}^m)$, $1 \leq p < \infty$, defined on a bounded domain $\Omega \subset \mathbb{R}^n$, if

$$\lim_{j\to\infty} \|\mathrm{dist}(Du_j,\mathcal{K})\|_{L^p(\Omega)} = 0 \quad \text{and} \quad u_j \rightharpoonup u \text{ in } W^{1,p}(\Omega;\mathbb{R}^m) \text{ as } j \to \infty, \tag{4}$$

then there exists a subsequence such that

$$Du_{j_k} \to Du \text{ in } L^p(\Omega;\mathbb{R}^m) \text{ as } k \to \infty. \tag{5}$$

It turns out that a necessary and sufficient condition on a compact set $\mathcal{K}$ for hypothesis (4) to imply (5) is the following: for any probability measure μ with $\mathrm{Spt}\mu \subset \mathcal{K}$, if

$$\int f(X)d\mu(X) \geq f\left(\int X d\mu(X)\right) \text{ for all Quasiconvex functions } f,$$

then μ is a Dirac measure. Firstly note that by Corollary 3 in [Mü 99′], since $\mathcal{K}$ is compact, we can without loss of generality assume $\{u_j\} \subset W^{1,\infty}(\Omega;\mathbb{R}^m)$. Then this follows from Theorem 4.7 in [Mü 99] (see [Ki-Pe 91, Ki-Pe 94] for the original source) and the fundamental theorem of Young measures (Theorem 3.1 and Corollary 3.2 in [Mü 99]). However Quasiconvex functions are very hard to understand [1], so more commonly a smaller class of functions known as *Polyconvex* functions are considered. These functions were introduced by Ball [Ba 77] in his fundamental work on existence of minimizers of elasticity functionals. Given $X \in M^{m\times n}$, let $\hat{X}$ denote the vector of all minors of X. A polyconvex function is a function $f : M^{m\times n} \to \mathbb{R}$ that can be written as $f(X) = g(\hat{X})$ where g is convex.

Following [Mü 99], given $\mathcal{K} \subset M^{m\times n}$, we denote

$$\mathcal{M}^{pc}(\mathcal{K}) := \left\{ \begin{array}{cc} \nu \in \mathcal{P}(M^{m\times n}) : & \mathrm{Spt}(\nu) \subset \mathcal{K}, \int f(X)d\nu(X) = f\left(\overline{X}\right) \text{ for all} \\ & \text{polyconvex functions } f, \text{where } \overline{X} = \int X d\nu(X) \end{array} \right\}.$$

A function $g : M^{m\times n} \to \mathbb{R}$ is a *Null Lagrangian* if $g(X)$ is an affine combination of the minors of $X \in M^{m\times n}$. Clearly if g is a *Null Lagrangian* then both g and $-g$ are polyconvex. Therefore $\mu \in \mathcal{M}^{pc}(\mathcal{K})$ if and only if

$$\int M(X)d\mu(X) = M\left(\int X d\mu(X)\right) \text{ for all minors } M.$$

For this reason we shall call measures $\mu \in \mathcal{M}^{pc}(\mathcal{K})$ *Null Lagrangian Measures*.

As we will briefly sketch, the heart of a number of well known compensated compactness results is a proof that for some submanifold $\mathcal{K}$ in the space of matrices, $\mathcal{M}^{pc}(\mathcal{K})$ consists of Dirac measures. There is overall little understanding of what general conditions a set $\mathcal{K}$ has to have in order for $\mathcal{M}^{pc}(\mathcal{K})$ to consist of Dirac measures only, i.e., to be trivial. Even in the case when $\mathcal{K}$ is a linear subspace in the space of matrices, it is an open problem to determine necessary and sufficient conditions on $\mathcal{K}$ for $\mathcal{M}^{pc}(\mathcal{K})$ to be trivial [2]. One of our main results provides such sufficient condition for subspaces in $M^{m\times n}$ that is necessary in the case where $2 \leq \min\{m,n\} \leq 3$. This is a corollary to the following Theorem 1.

First we require some notations. Given a set of homogeneous polynomials

$$\mathcal{F} := \{f_k : \mathbb{R}^n \to \mathbb{R} : k = 1,2,\ldots,q\},$$

we define the set of Null Lagrangian Measures with respect to the set of polynomials $\mathcal{F}$ by

$$\mathbb{M}^{pc}_{\mathcal{F}} := \left\{\mu \in \mathcal{P}(\mathbb{R}^n) : f\left(\int z\, d\mu(z)\right) = \int f(z)\, d\mu(z) \text{ for all } f \in \mathcal{F}\right\} \tag{6}$$

[1]Indeed one of the most important problems in Calculus of Variations is the question of whether in 2×2 matrices, Rank-1 convex functions are Quasiconvex [Ba 85], [Ast 98].

[2]This was asked to the first author by V. Šverák during a brief sabbatical visit to Minnesota in 2016.

and further we define $\mathbb{M}^{pc}_{\mathcal{F}}(\varpi) := \left\{\mu \in \mathbb{M}^{pc}_{\mathcal{F}} : \int z\, d\mu(z) = \varpi\right\}$.

Theorem 1. *Let $f_1, f_2, \ldots, f_{M_1} : \mathbb{R}^n \to \mathbb{R}$ be homogeneous polynomials with the property that*

$$2 \leq \deg(f_k) \leq 3 \text{ for } k = 1, 2, \ldots, M_1,$$

where we have ordered them such that $\deg(f_k) = 2$ for $k = 1, 2, \ldots, M_0$ and $\deg(f_k) = 3$ for $k = M_0 + 1, M_0 + 2, \ldots, M_1$. Let $\mathcal{F} := \{f_1, f_2, \ldots, f_{M_1}\}$. Then

$$\mathbb{M}^{pc}_{\mathcal{F}} \text{ consists of Dirac measures}$$

if and only if the following condition holds:

$$\text{For each subspace } V \subset \mathbb{R}^n \text{ there exists } \beta \in S^{M_0-1} \text{ such that}$$

$$\sum_{k=1}^{M_0} \beta_k f_k(z) \geq 0 \text{ for all } z \in V \text{ and } \sum_{k=1}^{M_0} \beta_k f_k \not\equiv 0 \text{ on } V. \tag{7}$$

Further if V is a subspace for which (7) fails, then for any $\varpi \in V$ there exists a nontrivial measure $\mu \in \mathbb{M}^{pc}_{\mathcal{F}}(\varpi)$.

As a consequence, we have

Corollary 2. *Suppose $K \subset M^{m\times n}$ is a linear subspace. Let $M_1, \ldots, M_{q_0} : M^{m\times n} \to \mathbb{R}$ be the set of all 2×2 minors in $M^{m\times n}$. If*

$$\text{for each subspace } L \subset K \text{ there exists } \beta \in S^{q_0-1} \text{ such that}$$

$$\sum_{k=1}^{q_0} \beta_k M_k(X) \geq 0 \text{ for all } X \in L \text{ and } \sum_{k=1}^{q_0} \beta_k M_k \not\equiv 0 \text{ on } L, \tag{8}$$

then

$$\mathcal{M}^{pc}(K) \text{ consists of Dirac measures.} \tag{9}$$

Conversely if $2 \leq \min\{m, n\} \leq 3$ then (9) implies (8).

Remark 1. We say that a set $\Sigma \subset M^{m\times n}$ has *Rank*-1 *connections* if and only if there exist $A, B \in \Sigma$ such that $A \neq B$ and $\text{Rank}(A - B) = 1$. Note that if $K \subset M^{m\times n}$ is a subspace that satisfies (8), then K has no Rank-1 connections. Indeed, were this not the case, there would be a nontrivial subspace $L \subset K$ with the property that $\text{Rank}(A) = 1$ for every $A \in L$, and therefore $M_k(A) = 0$ for every $k = 1, 2, \ldots, q_0$, which contradicts condition (8). It could be that for subspaces, condition (8) is actually equivalent to having no Rank-1 connections. We will present a proof of this for two dimensional subspaces in $M^{m\times n}$ in Lemma 13 of Section 5. As a consequence of Lemma 13 and Corollary 2, we have the following attractive consequence:

Corollary 3. *Let $K \subset M^{m\times n}$ be a two dimensional subspace, then $\mathcal{M}^{pc}(K)$ consists of Diracs if and only if K does not contain Rank-1 connections.*

Note that in [Sv 93], Šverák proved the beautiful result that for connected sets $\mathcal{K} \subset M^{2\times 2}$, $\mathcal{M}^{pc}(\mathcal{K})$ is trivial if and only if $\mathcal{K}$ does not contain Rank-1 connections. It is known that this result is false for general $M^{m\times n}$ (see the methods of Section 2.5 in [Mü 99]), however it might be true for subspaces. This would be a very attractive result, however for the applications that we have developed in this paper, condition (8) is actually more useful and informative.

Note that the set of k dimensional subspaces in $M^{m\times n}$ is essentially the Grassmannian space $G(k, mn)$. As is well known, $G(k, mn)$ forms a $k(mn - k)$ dimensional smooth compact connected manifold, see [Pi-Ta 08] or Section 6, Lemma 15. As such we say that a property holds "generically" for k dimensional subspaces in $M^{m\times n}$ if the set of k dimensional subspaces for which the property does not hold can be covered by the Lipschitz images of a finite collection of submanifolds in $\mathbb{R}^{k(mn-k)}$ of dimension less than $k(mn-k)$, see Definition 16. Using this point of view, we have the following result:

Theorem 4. *Suppose k, m, n are positive integers with $m, n \geq 2$ and $k \leq \frac{1}{2}\min\{m, n\}$. Then for a "generic" k dimensional subspace $K \subset M^{m\times n}$, $\mathcal{M}^{pc}(K)$ consists of Dirac measures and hence K has no Rank-1 connections.*

Contrast this with the interesting result of Bhattacharya, Firoozye, James and Kohn (Proposition 4.4 in [Bh-Fi-Ja-Ko 94]), in which it is shown that $l(m,n) \leq mn - n$, where $l(m,n)$ denotes the maximum possible dimension of a linear subspace in $M^{m\times n}$ that does not have Rank-1 connections. So Theorem 4 is completely false for higher dimensional subspaces. The bound $k \leq \frac{1}{2}\min\{m,n\}$ is surely not sharp, and an interesting and possibly accessible question is to determine the sharp bound on k such that the conclusion of Theorem 4 holds true. Although Theorem 4 is not a consequence of Theorem 1, the ideas of its proof are very closely related to those of the proof of Theorem 1.

One of the motivations for studying Null Lagrangian Measures supported on planes is that such results might rather directly yield insights into how to prove triviality of $\mathcal{M}^{pc}(\mathcal{K} \cap U)$ where $\mathcal{K}$ is any smooth submanifold and U is a small neighborhood around an arbitrary point in $\mathcal{K}$. In the following subsection we apply these insights to study a well known 2×2 system of conservation laws.

1.1. **Connections and applications to conservation laws.** As mentioned above one of the main successes of compensated compactness is the proof of existence theorems for hyperbolic conservation laws. To sketch this briefly, the standard way to solve a scalar equation is to add a viscosity term and obtain a solution to

$$u_t^\epsilon + G(u^\epsilon)_x = \epsilon u_{xx}^\epsilon \text{ in } (0,\infty)\times\mathbb{R}. \tag{10}$$

Assuming $\{u^\epsilon\}_\epsilon$ is bounded in $L^\infty((0,\infty)\times\mathbb{R})$ we can extract a subsequence $u^{\epsilon_k} \overset{*}{\rightharpoonup} u$ in $L^\infty((0,\infty)\times\mathbb{R})$. Letting $\{\nu_{t,x}\}$ be the Young measures associated with the weak* convergence, i.e., $u(t,x) = \int_{\mathbb{R}} y\, d\nu_{t,x}$, we have $G(u^{\epsilon_k}) \overset{*}{\rightharpoonup} \overline{G}$ in $L^\infty((0,\infty)\times\mathbb{R})$ where $\overline{G}(t,x) = \int G(y)\, d\nu_{t,x}$.

Now for any convex function $\Phi : \mathbb{R} \to \mathbb{R}$, define $\Psi(y) := \int_0^y \Phi'(s)G'(s)ds$. The pair (Φ, Ψ) is called an entropy/entropy flux pair. The key point is that by virtue of the Div-Curl lemma we know that

$$\int (G(y)\Phi(y) - y\Psi(y))\; d\nu_{t,x} = \overline{G}(t,x)\overline{\Phi}(t,x) - u(t,x)\overline{\Psi}(t,x), \tag{11}$$

where $\overline{\Phi}(t,x) = \int \Phi(y) d\nu_{t,x}$ and $\overline{\Psi}(t,x) = \int \Psi(y) d\nu_{t,x}$. Define $P_\Phi : \mathbb{R} \to M^{2\times 2}$ by $P_\Phi(z) := \begin{pmatrix} G(z) & z \\ \Psi(z) & \Phi(z) \end{pmatrix}$ and the measure μ_Φ on the set $\mathcal{K}_\Phi := \{P_\Phi(z) : z \in \mathbb{R}\}$ by $\mu_\Phi := (P_\Phi)_\sharp \nu_{t,x}$, the push forward of $\nu_{t,x}$ by the map P_Φ. By (11), $\mu_\Phi \in \mathcal{M}^{pc}(\mathcal{K}_\Phi)$. So to prove triviality of $\nu_{t,x}$ it suffices to prove $\mathcal{M}^{pc}(\mathcal{K}_\Phi)$ is trivial for any choice of convex function Φ. As this is such a wide class, for a lot of scalar conservation laws, one can find an appropriate convex function Φ for which $\mathcal{M}^{pc}(\mathcal{K}_\Phi)$ is trivial, and hence the Young measures are trivial. The fact that u is a weak solution to (10) without viscosity term follows from triviality of Young measures in a standard way.

For systems of conservation laws (other than 2×2 systems or scalar conservation laws) there are only finitely many entropy/entropy flux pairs (Φ_1, Ψ_1), (Φ_2, Ψ_2), $\ldots$, (Φ_m, Ψ_m). By analogous argument to the scalar case, the Young measures can be pushed forward into $\mathcal{M}^{pc}(\mathcal{K})$, where $\mathcal{K}$ is the subset of matrices whose rows consist of the conservation laws and the entropy/entropy flux pairs (Φ_j, Ψ_j). Then triviality of the Young measures and hence (given appropriate a-priori L^p bounds) proof of existence of solutions via compensated compactness comes down to proving triviality of $\mathcal{M}^{pc}(\mathcal{K})$.

One of the best known results in this area is the work of DiPerna [DP 83] on strong convergence of solutions to a class of systems of two genuinely nonlinear conservation laws in one dimension, where the hypotheses are compactness in $W^{-1,2}$ of *every* entropy/entropy flux pair acting on the approximating solutions. As a particular example, the result applies to the system with the form of the Lagrangian equations of elasticity given by

$$\begin{cases} v_t - u_x = 0, \\ u_t - a(v)_x = 0 \end{cases} \tag{12}$$

for some given smooth function $a : \mathbb{R} \to \mathbb{R}$ that is strictly convex and increasing. Possibly motivated by the question of compactness for higher dimensional systems, in another well known work DiPerna [DP 85] proves a local existence result for the system (12) with just two entropy/entropy flux pairs. So following [DP 85] we introduce the natural entropy/entropy flux pair (η_1, q_1), and its dual pair (η_2, q_2), associated to the system (12). More precisely, we define

$$\eta_1(u,v) := \frac{1}{2}u^2 + F(v), \quad q_1(u,v) := -ua(v), \tag{13}$$

and

$$\eta_2(u,v) := uv, \quad q_2(u,v) := -\frac{1}{2}u^2 - \tau(v), \tag{14}$$

where $F(\xi) = \int_0^{\xi} a(s)ds$ and τ is related to the Legendre transform of F. Formally $\tau(p) := \tilde{\tau}(a(p))$ where $\tilde{\tau}$ is the Legendre transform of F. Thus

$$\tau(p) = a(p)p - F(p), \tag{15}$$

see (176) of Lemma 28 in Appendix I. Note that this is not how DiPerna [DP 85] defined q_2. Indeed he defined $q_2(u,v) := \frac{u^2}{2} + \tilde{\tau}(v)$. However with q_2 defined this way, the pair (η_2, q_2) does not form an entropy/entropy flux pair [3].

As in [Ki-Mü-Sv 03], we consider entropy solutions of (12) defined as L^∞ functions (u,v) satisfying either

$$\begin{cases} v_t - u_x = 0, \\ u_t - a(v)_x = 0, \\ (\eta_1)_t + (q_1)_x \le 0 \end{cases} \tag{16}$$

or

$$\begin{cases} v_t - u_x = 0, \\ u_t - a(v)_x = 0, \\ (\eta_1)_t + (q_1)_x \le 0, \\ (\eta_2)_t + (q_2)_x \le 0 \end{cases} \tag{17}$$

in the sense of distributions. Adding a viscosity term to the first two equations of (16), (17), we obtain the pair (u^ϵ, v^ϵ) that solves

$$v_t^\epsilon - u_x^\epsilon = \epsilon v_{xx}^\epsilon, \;\; u_t^\epsilon - a\left(v^\epsilon\right)_x = \epsilon u_{xx}^\epsilon.$$

Assuming appropriate bounds on $u^\epsilon, v^\epsilon, u_x^\epsilon, v_x^\epsilon$ (see (5.38) of [Ev 90]), we obtain the system (16) or (17) with right hand side precompact in $W^{-1,2}_{loc}$. Hence as we have sketched for scalar

[3]Further Equations (9.8) of [DP 85] do not hold true if $q_2(u,v) = \frac{u^2}{2} + \tilde{\tau}(v)$. However with q_2 defined as in (14), (η_2, q_2) does form an entropy/entropy flux pair and Equations (9.8) of [DP 85] do hold true, see Lemma 28 in Appendix I.

equations, we have $(u^\epsilon, v^\epsilon) \overset{*}{\rightharpoonup} (u,v)$ in L^∞ and the Young measures can be pushed forward into the set $\mathcal{K}_1$ or $\mathcal{K}_2$, where $\mathcal{K}_1$ and $\mathcal{K}_2$ are defined to be

$$\mathcal{K}_1 := \left\{ \begin{pmatrix} u & v \\ a(v) & u \\ ua(v) & \frac{1}{2}u^2 + F(v) \end{pmatrix} : u, v \in \mathbb{R} \right\}, \tag{18}$$

and

$$\mathcal{K}_2 := \left\{ \begin{pmatrix} u & v \\ a(v) & u \\ ua(v) & \frac{1}{2}u^2 + F(v) \\ \frac{1}{2}u^2 + \tau(v) & uv \end{pmatrix} : u, v \in \mathbb{R} \right\}, \tag{19}$$

respectively. By use of the Div-Curl lemma we have measures in $\mathcal{M}^{pc}(\mathcal{K}_1)$ or $\mathcal{M}^{pc}(\mathcal{K}_2)$. Thus the existence of solutions of the system (12) would follow from triviality of $\mathcal{M}^{pc}(\mathcal{K}_1)$ or $\mathcal{M}^{pc}(\mathcal{K}_2)$ (clearly triviality of $\mathcal{M}^{pc}(\mathcal{K}_1)$ would imply triviality $\mathcal{M}^{pc}(\mathcal{K}_2)$).

1.1.1. *Nontrivial Null Lagrangian Measures for the system with one entropy.* Understanding the structure of $\mathcal{K}_1$ or $\mathcal{K}_2$ plays a fundamental role in understanding the system (16) or (17). If there is so little rigidity of the structure of $\mathcal{K}_1$ that certain subset $\mathcal{K}_1^{rc}$ of $\mathcal{K}_1^{pc}$ ($\mathcal{K}_1^{pc}$ and $\mathcal{K}_1^{rc}$ are called the Polyconvex hull and Rank-1 convex hull of $\mathcal{K}_1$, respectively, see Section 4.4 in [Mü 99]) is sufficiently nontrivial, then a very different kind of nontrivial solution to (16) can be obtained as a differential inclusion into $\mathcal{K}_1$[4]. There have been enormous interests and specular progresses in reformulating PDEs as differential inclusions and obtaining solutions via *convex integration* [De-Sz 09], [De-Sz 13], [Bu-De-Is-Sz 15], [Is 17]. Some of the initial impetus for these works come from the pioneering work on Calculus of Variations by [Mü-Sv 96], [Mü-Sv 03], [Mü-Sy 01], [Ki 01], [Ki 03] and the initial ideas come from the work on isometric immersions of Nash [Na 54], Kuiper [Ku 55] and Gromov [Gr 86]. Note that these solutions are wildly non-unique and constructed to have many strange and non-physical properties. The solutions obtainable through vanishing viscosity and compensated compactness are good (in the scalar case) and reasonable solutions in the sense that they are unique and have BV regularity. In contrast, the solutions obtained via convex integration are wild solutions whose existence essentially implies that the problem requires more assumptions to be reasonably posed.

As briefly mentioned already, for a set of a matrices $\mathcal{K}$, the notions of *Rank*-1 *convex hull* $\mathcal{K}^{rc}$, *Quasiconvex hull* $\mathcal{K}^{qc}$ and *Polyconvex hull* $\mathcal{K}^{pc}$ can be found, for example, in [Mü 99] (see also [Ki-Pe 91, Sv 95]). Since *Polyconvexity* $\Rightarrow$ *Quasiconvexity* $\Rightarrow$ *Rank*-1 *convexity* , we have $\mathcal{K}^{rc} \subset \mathcal{K}^{qc} \subset \mathcal{K}^{pc}$. As Quasiconvex functions are hard to understand, a first step to proving non-triviality of $\mathcal{K}^{rc}$ is to show non-triviality of $\mathcal{K}^{pc}$. Further as $\mathcal{K}^{pc} = \left\{ \int X d\mu(X) : \mu \in \mathcal{M}^{pc}(\mathcal{K}) \right\}$ (see Exercise 1, Section 4.4 in [Mü 99]), a first step towards this goal is to show non-triviality of $\mathcal{M}^{pc}(\mathcal{K})$. For this reason Kirchheim, Müller and Šverák [Ki-Mü-Sv 03] asked the following question with respect to the system (16) and its associated differential inclusion into the set $\mathcal{K}_1$ under more general conditions on the function a, namely, what are the natural assumptions on the function a such that the following statement is true:

(S1) For each point $\zeta \in \mathcal{K}_1$, there exists a neighborhood $U \subset M^{3\times 2}$ of ζ such that $\mathcal{M}^{pc}(\mathcal{K}_1 \cap \overline{U})$ is trivial.

For the system (12) without implementing any entropy/entropy flux pairs, the statement **(S1)** for the corresponding set $\mathcal{K}_0 := \left\{ \begin{pmatrix} u & v \\ a(v) & u \end{pmatrix} : u, v \in \mathbb{R} \right\}$ is well understood using results in [Sv 93]. On the other hand, it is proved in [DP 85] that a set closely related to $\mathcal{K}_2$ (see

[4]Note that a differential inclusion into set $\mathcal{K}_1$ gives a solution to (16) with the inequality replaced by an equality.

Subsection 1.1.2 for more details) associated to the system (17) with two entropy/entropy flux pairs satisfies statement **(S1)** when the function a satisfies $a' > 0$ and $a'' \neq 0$. However, this question (as well as some other related properties) for the set $\mathcal{K}_1$ defined in (18) (which is associated with the system (16) with just one entropy/entropy flux pair) remained open. (For more details, see [Ki-Mü-Sv 03], Section 7.)

For the convenience of later discussions, we parametrize the sets $\mathcal{K}_1$ and $\mathcal{K}_2$ by the mappings

$$P_1(u,v) := \begin{pmatrix} u & v \\ a(v) & u \\ ua(v) & \frac{1}{2}u^2 + F(v) \end{pmatrix} \tag{20}$$

and, recalling $\tau(v) \overset{(15)}{=} va(v) - F(v)$,

$$P_2(u,v) := \begin{pmatrix} u & v \\ a(v) & u \\ ua(v) & \frac{1}{2}u^2 + F(v) \\ \frac{1}{2}u^2 + va(v) - F(v) & uv \end{pmatrix}. \tag{21}$$

In this notation, $\mathcal{K}_1 = \{P_1(u,v) : u,v \in \mathbb{R}\}$ and $\mathcal{K}_2 = \{P_2(u,v) : u,v \in \mathbb{R}\}$. In Section 8, given a point $P_1((\tilde{\alpha}_1, \tilde{\alpha}_2)) \in \mathcal{K}_1$, we will show that statement **(S1)** is false if $a'(\tilde{\alpha}_2) > 0$ and true if $a'(\tilde{\alpha}_2) < 0$. Specifically, we have

Theorem 5. *Suppose $a \in C^2(\mathbb{R})$. Given $\tilde{\alpha} = (\tilde{\alpha}_1, \tilde{\alpha}_2) \in \mathbb{R}^2$, if $a'(\tilde{\alpha}_2) > 0$, then there exist nontrivial measures in $\mathcal{M}^{pc}(\mathcal{K}_1 \cap B_\delta(P_1(\tilde{\alpha})))$ for all $\delta > 0$. On the other hand, if $a'(\tilde{\alpha}_2) < 0$, then there exists $\delta_0 > 0$ depending on the function a and $\tilde{\alpha}_2$ such that $\mathcal{M}^{pc}(\mathcal{K}_1 \cap B_\delta(P_1(\tilde{\alpha})))$ is trivial for all $0 < \delta \leq \delta_0$.*

Indeed the second part of Theorem 5 can be made a bit stronger. More precisely, recall that $\mathcal{K}_0 := \left\{ \left(\begin{smallmatrix} u & v \\ a(v) & u \end{smallmatrix} \right) : u,v \in \mathbb{R} \right\}$. Given $\tilde{\alpha} \in \mathbb{R}^2$, if $a'(\tilde{\alpha}_2) < 0$ then $\mathcal{M}^{pc}\left(\mathcal{K}_0 \cap B_\delta\left(\left(\begin{smallmatrix} \tilde{\alpha}_1 & \tilde{\alpha}_2 \\ a(\tilde{\alpha}_2) & \tilde{\alpha}_1 \end{smallmatrix}\right)\right)\right)$ is trivial for sufficiently small $\delta > 0$ depending on a and $\tilde{\alpha}_2$. As $\mathcal{M}^{pc}(\mathcal{K}_1)$ can be naturally embedded into $\mathcal{M}^{pc}(\mathcal{K}_0)$, this implies the second part of the theorem (see the proof of Theorem 5 in Section 8). Theorem 5 is closely related to Theorem 1. Indeed, one can check directly that for the submanifold $\mathcal{K}_1$ given in (18), there does not exist nontrivial linear combination of all three minors that remains nonnegative. Nevertheless, it should be noted that the set $\mathcal{K}_1$ given in (18) is a nonlinear submanifold in the space of 3×2 matrices whose nonlinear structure poses extremely delicate issues. As a result, the arguments needed are significantly beyond those used for linear spaces. The proof of the first part in Theorem 5 uses a construction of discrete probability measures supported at five points with certain symmetry structure. A key ingredient is to place the majority of the mass of the measure at one point in $\mathcal{K}_1$, and think of the other four points as small perturbations. This way one can manage to handle the nonlinear effect of the submanifold $\mathcal{K}_1$ as a small perturbation of the corresponding linear structure, and as a result, the ideas used in the linear cases can be generalized. Moreover, our construction allows to produce infinitely many elements in $\mathcal{M}^{pc}(\mathcal{K}_1)$ (see Theorem 23).

1.1.2. *A direct proof of DiPerna's theorem for the system with two entropies.* As sketched in Subsection 1.1, if there is enough "rigidity of structure" in the set $\mathcal{K}_1$ or $\mathcal{K}_2$ to force the Null Lagrangian Measures on them to be Dirac measures, we obtain solutions of (12) from the vanishing viscosity approximations via compensated compactness. This motivated the work of DiPerna on existence of weak solutions to the system (12). In [DP 85], the following theorem is proved:

Theorem 6 (DiPerna [DP 85]). *Suppose a is a smooth function with $a' > 0$ and $a'' \neq 0$. Let (η_1, q_1) and (η_2, q_2) be defined by (13) and (14). Assume that $\{(u^\epsilon, v^\epsilon)\}$ is a sequence with small oscillations* *such that the sequences*

$$u_t^\epsilon - (a(v^\epsilon))_x, \quad v_t^\epsilon - u_x^\epsilon, \quad (\eta_i(u^\epsilon, v^\epsilon))_t + (q_i(u^\epsilon, v^\epsilon))_x$$

are precompact in $W_{loc}^{-1,2}(\mathbb{R}^2)$*,* $i = 1, 2$*. Then there exists a subsequence* $\{(u^{\epsilon_n}, v^{\epsilon_n})\}$ *that converges to some* (u, v) *in the strong topology of* $L_{loc}^1(\mathbb{R}^2)$*.*

The way that DiPerna proves Theorem 6 relies on using the entropy/entropy flux pairs (η_i, q_i) to show triviality of Young measures associated to the system. This in turn is closely related to triviality of $\mathcal{M}^{pc}(\mathcal{K}_2)$. However, strictly speaking DiPerna did not prove trivially of $\mathcal{M}^{pc}(\mathcal{K}_2)$. Rather he showed trivially of Null Lagrangian Measures on a set $\mathcal{K}_2^\alpha$ (see (92)). The set $\mathcal{K}_2^\alpha$ is the analogue of $\mathcal{K}_2$ with the entropy/entropy flux pairs (η_1, q_1), (η_2, q_2) replaced by simplified pairs $(Q_\alpha \eta_1, Q_\alpha^* q_1)$, $(Q_\alpha \eta_2, Q_\alpha^* q_2)$ given by (169) and (170). Indeed, the sets $\mathcal{M}^{pc}(\mathcal{K}_2)$ and $\mathcal{M}^{pc}(\mathcal{K}_2^\alpha)$ are equivalent. This fact was implicitly stated in [Ki-Mü-Sv 03], see Lemma 20 in Section 7 for the precise statement and for the proof [5]. In this paper, we provide a direct proof of triviality of $\mathcal{M}^{pc}(\mathcal{K}_2)$ (and also $\mathcal{M}^{pc}(\mathcal{K}_2^\alpha)$) motivated by the perspective of searching for linear combinations of minors that are positive. More precisely, we provide a direct proof of

Theorem 7 (DiPerna [DP 85])**.** *Suppose* $a \in C^\infty(\mathbb{R})$*. Given* $\tilde{\alpha} = (\tilde{\alpha}_1, \tilde{\alpha}_2) \in \mathbb{R}^2$ *such that* $a'(\tilde{\alpha}_2) > 0$ *and* $a''(\tilde{\alpha}_2) \neq 0$*, there exists* $\delta_0 > 0$ *depending on the function a and* $\tilde{\alpha}_2$ *such that for all* $0 < \delta \leq \delta_0$*,* $\mathcal{M}^{pc}(\mathcal{K}_2 \cap B_\delta(P_2(\tilde{\alpha})))$ *is trivial.* [6]

As a simple consequence of Theorem 7, we will give a proof of DiPerna's Theorem 6. The perspective we have learned from studying Null Lagrangian Measures supported on planes L is that triviality of Null Lagrangian Measures follows if (and in lower dimensions only if) we can find a linear combination of minors $\sum \lambda_{ij} M_{ij}$ such that $\sum \lambda_{ij} M_{ij}(X)$ is nonnegative on L and hence in low dimensions is convex, where M_{ij} denotes the (i, j) minor of $m \times 2$ matrices. This does not hold true if we have a subset of matrices $\mathcal{K}$ where $\mathcal{K}$ is not a plane, the main issue being that the barycenter of a measure supported on $\mathcal{K}$ needs not to be on $\mathcal{K}$. However if the measure has small support, then $\mathcal{K}$ is locally very close to a plane and the barycenter will be very close to being on $\mathcal{K}$. So if we can find a linear combination $\sum \lambda_{ij} M_{ij}$ such that the Taylor expansion of

$$X \to \sum \lambda_{ij} M_{ij}(X - X_0) \tag{22}$$

(up to some order) around some $X_0 \in \mathcal{K}$ is convex on $\mathcal{K} \cap B_\delta(X_0)$, then there is a very good chance that we can prove triviality of $\mathcal{M}^{pc}(\mathcal{K} \cap B_\delta(X_0))$. Specifically, given $\mu \in \mathcal{M}^{pc}(\mathcal{K})$ with small support, assume for the moment that the Taylor expansion of the mapping in (22) is convex, then letting $\overline{X} = \int X d\mu(X)$ we have that

$$\int \sum \lambda_{ij} M_{ij}(X - X_0)\, d\mu(X) = \sum \lambda_{ij} M_{ij}(\overline{X} - X_0). \tag{23}$$

Now if $\overline{X}$ was actually on $\mathcal{K}$ then this would contradict Jensen's inequality by choosing $X_0 = \overline{X}$ unless μ is a Dirac measure. In general $\overline{X} \notin \mathcal{K}$, but since locally $\mathcal{K}$ is very close to a plane, $\overline{X}$ is very close to $\mathcal{K}$ and so $\sum \lambda_{ij} M_{ij}(\overline{X} - X_0)$ can be estimated. If it can be shown to be sufficiently small, then the error can be absorbed into the left hand side of (23) and hence μ can be shown to be a Dirac measure.

DiPerna's proof in [DP 85] consists of manipulating the equations of the form

$$\int M_{ij}(X - X_0) d\mu(X) = M_{ij}(\overline{X} - X_0),$$

[5] The proof included was provided to us by S. Müller.

[6] The fact that DiPerna's theorem can be formulated this way is due to [Ki-Mü-Sv 03], as such some credit for Theorem 7 is due to these authors.

or more specifically their Taylor expansions up to a certain order. After a sequence of substitutions and linear combinations, DiPerna obtains an inequality that contradicts Jensen's inequality unless μ is a Dirac measure. However the method outlined in the paragraph above is a *strategy*. In Section 9 we provide a proof of DiPerna's theorem by following this strategy.

Remark 2. In the case $a'(\tilde{\alpha}_2) < 0$, the conclusion of Theorem 7 still holds true. This is clear from the proof of Theorem 5 at the end of Section 8.

Remark 3. In principle the strategy outlined above can be applied to other systems of conservation laws as well as other nonlinear PDE problems. One particular potential application involves a classical problem in Calculus of Variations, called the Aviles-Giga functional. This problem shares common features with scalar conservation laws as observed in [De-Mü-Ko-Ot 01], and the tool of entropies has been used intensively in the study of this problem [Ji-Ko 00, Am-De-Ma 99, De-Mü-Ko-Ot 01]. In particular, using compensated compactness arguments involving infinitely many entropies, the authors in [De-Mü-Ko-Ot 01] were able to prove triviality of Young measures, and hence compactness for the Aviles-Giga functional. Interestingly, the same compactness result was proved independently in [Am-De-Ma 99] by using only the two special entropies introduced by Jin and Kohn [Ji-Ko 00]. This is however not surprising from our new perspective outlined above. Indeed, the matrix space formed by the two special entropies used in [Ji-Ko 00, Am-De-Ma 99] exhibits a locally convex structure as shown in the work of the authors of the current paper [Lo-Pe 18], and we believe that this is the underlying reason why the same compactness result can be obtained using only the two special entropies. We plan to pursue this line of research in future works.

2. Proof Sketch

In Subsection 1.1.2 we sketched how the direct proof of Theorem 6 follows from our general approach. In this section we will sketch briefly the main ideas of the proofs of Theorems 1 and 5.

2.1. **Sketch of proof of Theorem 1.**

To illustrate the key ideas, we sketch the proof in the special case where $\deg(f_k) = 2$ for $k = 1, 2, \dots, M_1$ and $\varpi = 0$. Let $\mu \in \mathbb{M}^{pc}_{\mathcal{F}}(0)$. By definition, we have that

$$\int f_k(z) d\mu(z) = f_k\left(\int z \, d\mu(z)\right) = 0 \text{ for } k = 1, 2, \dots, M_1. \tag{24}$$

If the condition (7) is satisfied, then we can find some $\beta_1, \beta_2, \dots, \beta_{M_1}$ such that $g(z) := \sum_{k=1}^{M_1} \beta_k f_k \geq 0$ on $\mathbb{R}^n$. By (24), we have $\int g(z) d\mu(z) = 0$, and therefore $\mathrm{Spt}\mu \subset V := \{z : g(z) = 0\}$. Since g is a nonnegative quadratic function, the set V is a subspace. By assumption, we can find another linear combination that is nontrivial and nonnegative on V. This way we can iteratively reduce the support of μ onto subspaces of smaller and smaller dimensions until the support is reduced to the origin.

The necessity part of the proof is a bit more intricate. Suppose there exists a subspace V such that $\sum_{k=1}^{M_1} \beta_k f_k$ is nontrivial and changes sign on V for every $\beta \in S^{M_1 - 1}$. To construct a nontrivial measure in $\mathbb{M}^{pc}_{\mathcal{F}}(0)$ it suffices to find a finite collection of Dirac measures $\delta_{\tilde{\zeta}_1}, \delta_{\tilde{\zeta}_2}, \dots, \delta_{\tilde{\zeta}_{2m}}$ and weights $\gamma_1, \gamma_2, \dots, \gamma_{2m} \geq 0$ such that

$$\mu := \sum_{k=1}^{2m} \gamma_k \delta_{\tilde{\zeta}_k} \in \mathbb{M}^{pc}_{\mathcal{F}}(0) \text{ and } \sum_{k=1}^{2m} \gamma_k = 1. \tag{25}$$

Now we claim that solving (25) is equivalent to finding points $\zeta_1, \zeta_2, \dots, \zeta_m$ and weights $\gamma_1, \gamma_2, \dots, \gamma_m \geq 0$ satisfying the system

$$\begin{pmatrix} f_1(\zeta_1) & f_1(\zeta_2) & \dots & f_1(\zeta_m) \\ f_2(\zeta_1) & f_2(\zeta_2) & \dots & f_2(\zeta_m) \\ \dots & \dots & & \\ f_{M_1}(\zeta_1) & f_{M_1}(\zeta_2) & \dots & f_{M_1}(\zeta_m) \\ 1 & 1 & \dots & 1 \end{pmatrix} \begin{pmatrix} \gamma_1 \\ \gamma_2 \\ \dots \\ \dots \\ \gamma_m \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \dots \\ 0 \\ \frac{1}{2} \end{pmatrix}. \tag{26}$$

This is because if we find such points $\zeta_1, \zeta_2, \dots, \zeta_m$ and solve the above system, then simply defining $\tilde{\zeta}_i = \zeta_i$ for $i = 1, 2, \dots, m$ and $\tilde{\zeta}_{m+i} = -\zeta_i$ for $i = 1, 2, \dots, m$ and defining $\gamma_{m+i} = \gamma_i$ for $i = 1, 2, \dots, m$, we have that $\sum_{i=1}^{2m} \gamma_i \tilde{\zeta}_i = 0$. So $\int z \; d\mu(z) = 0$. Further because the functions $\{f_k\}$ are quadratic, we have $f_k(\tilde{\zeta}_i) = f_k(\tilde{\zeta}_{i+m})$ for $i = 1, 2, \dots m$ and so (26) implies $\sum_{i=1}^{2m} \gamma_i f_k(\tilde{\zeta}_i) = 0$ for $k = 1, 2, \dots, M_1$ and hence $\mu \in \mathbb{M}^{pc}_{\mathcal{F}}(0)$.

Now letting a_i denote the columns of the matrix in (26), the existence of some $\gamma \in \mathbb{R}^m_+$ such that (26) holds true is equivalent to the statement that the vector $b := \frac{1}{2} e_{M_1+1}$ is in the cone $\mathcal{C} := \{\sum_{i=1}^m \lambda_i a_i : \lambda_1, \lambda_2, \dots, \lambda_m \in \mathbb{R}_+\}$. This in turn is equivalent to the condition that b cannot be separated from $\mathcal{C}$ with a hyperplane. Formally this is the statement that

$$\text{if } y \in \mathbb{R}^{M_1+1} \text{ satisfies } y \cdot a_i \geq 0 \text{ for all } i = 1, 2, \dots, m, \text{ then } y \cdot b \geq 0.$$

This is the content of the Farkas-Minkowski Lemma. Now since by hypothesis $\sum_{k=1}^{M_1} y_k f_k$ changes sign on the subspace V for all nontrivial $y \in \mathbb{R}^{M_1}$, we are able to find a finite collection of points $\{\zeta_j\} \subset V$ that is sufficiently dense in V such that for all nontrivial $y \in \mathbb{R}^{M_1}$ we have that $\sum_{k=1}^{M_1} y_k f_k$ changes sign on $\{\zeta_j\}$. Given a nontrivial $y \in \mathbb{R}^{M_1+1}$ such that $y \cdot a_i \geq 0$ for all i, there exists some j_0 such that $\sum_{k=1}^{M_1} y_k f_k(\zeta_{j_0}) < 0$. However as $a_{j_0} \cdot y = \sum_{k=1}^{M_1} y_k f_k(\zeta_{j_0}) + y_{M_1+1} \geq 0$, we must have that $y_{M_1+1} = y \cdot e_{M_1+1} > 0$ and hence $b \in \mathcal{C}$. Thus the existence of $\gamma \in \mathbb{R}^m_+$ satisfying (26) is established, and we indeed have a nontrivial measure in $\mathbb{M}^{pc}_{\mathcal{F}}(0)$.

For the general case where we allow functions of degree 3 and arbitrary ϖ, we define an auxiliary function $\tilde{f}$ as in (27), which essentially plays the role of translating the barycenter ϖ to the origin. The sufficiency part works the same way as sketched above by use of the function $\tilde{f}$. However, for the construction of nontrivial measures in $\mathbb{M}^{pc}_{\mathcal{F}}(\varpi)$, we cannot simply position points symmetrically around ϖ in order to create a measure whose barycenter is ϖ. Instead we need to incorporate the equation $\sum_{i=1}^m \gamma_i \zeta_i = \varpi$ into the linear system (26). This is what is done in equation (50) of Lemma 12, but essentially the main ideas are as sketched above.

2.2. **Sketch of proof of Theorem 5.** As sketched briefly in the introduction, the case where $\alpha'(\tilde{\alpha}_2) < 0$ follows easily from a well known result of Šverák [Sv 93]. The case where $\alpha'(\tilde{\alpha}_2) > 0$ is the one that requires real work. As in Subsection 2.1, to streamline the sketch, we consider the special case where $\tilde{\alpha} = 0$ and $a(\tilde{\alpha}_2) = 0$. Given s_0, t_0 sufficiently small, let

$$\zeta_0 := \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}, \; \zeta_1 := \begin{pmatrix} s_0 & 0 \\ 0 & s_0 \\ 0 & \frac{1}{2}s_0^2 \end{pmatrix}, \; \zeta_2 := \begin{pmatrix} -s_0 & 0 \\ 0 & -s_0 \\ 0 & \frac{1}{2}s_0^2 \end{pmatrix}$$

and

$$\zeta_3 := \begin{pmatrix} 0 & t_0 \\ a(t_0) & 0 \\ 0 & F(t_0) \end{pmatrix}, \; \zeta_4 := \begin{pmatrix} 0 & -t_0 \\ a(-t_0) & 0 \\ 0 & F(-t_0) \end{pmatrix}.$$

So $\zeta_0, \zeta_1, \ldots, \zeta_4 \in \mathcal{K}_1$. For $0 < \epsilon < 1$ sufficiently small, we construct nontrivial measures supported at the above five points, with weight $1-\epsilon$ at ζ_0, and total weight ϵ at the other four points. Let D_1, D_2, D_3 denote the $(1,2), (2,3), (1,3)$ minors of a 3×2 matrix, respectively. We set the matrix

$$A := \begin{pmatrix} D_1(\zeta_1) & D_1(\zeta_2) & D_1(\zeta_3) & D_1(\zeta_4) \\ D_2(\zeta_1) & D_2(\zeta_2) & D_2(\zeta_3) & D_2(\zeta_4) \\ D_3(\zeta_1) & D_3(\zeta_2) & D_3(\zeta_3) & D_3(\zeta_4) \\ 1 & 1 & 1 & 1 \end{pmatrix} = \begin{pmatrix} s_0^2 & s_0^2 & -t_0 a(t_0) & t_0 a(-t_0) \\ 0 & 0 & a(t_0)F(t_0) & a(-t_0)F(-t_0) \\ \frac{1}{2}s_0^3 & -\frac{1}{2}s_0^3 & 0 & 0 \\ 1 & 1 & 1 & 1 \end{pmatrix}.$$

By a careful analysis using the special structure of the points ζ_j, we have that $\sum_{i=1}^3 y_i D_i$ changes sign on $\{\zeta_1, \zeta_2, \zeta_3, \zeta_4\}$ for all nontrivial $y \in \mathbb{R}^3$. By virtue of the arguments used for the system (26), if we define $L^\epsilon(\gamma) := A\gamma - (0,0,0,\epsilon)^T$, then the Farkas-Minkowski Lemma guarantees existence of $\gamma_0 \in \mathbb{R}^4_+$ such that $L^\epsilon(\gamma_0) = 0$. However what we need to solve for a measure in $\mathcal{M}^{pc}(\mathcal{K}_1)$ is $G^\epsilon(\gamma) := L^\epsilon(\gamma) - Q(\gamma) = 0$, where

$$Q(\gamma) := \begin{pmatrix} D_1\left(\sum_{j=1}^4 \gamma_j \zeta_j\right) \\ D_2\left(\sum_{j=1}^4 \gamma_j \zeta_j\right) \\ D_3\left(\sum_{j=1}^4 \gamma_j \zeta_j\right) \\ 0 \end{pmatrix}.$$

Since G^ϵ is a quadratic perturbation of an invertible function, it should seem reasonable that for small enough ϵ, $G^\epsilon(\gamma) = 0$ will have a solution. But to actually establish that the solution is nonnegative we carry out an iterative argument inspired by the proof of the inverse function theorem. To this end, we start from the nonnegative solution γ_0 of the linear part $L^\epsilon(\gamma) = 0$, and use an iterative argument to solve for γ_k in each step $k > 0$ such that γ_k converges to the actual solution of $G^\epsilon(\gamma) = 0$. The convergence of this scheme is guaranteed by choosing ϵ sufficiently small. These are the contents of Lemmas 24 and 25.

What is slightly surprising is that to prove the general case we need to work instead with the set $\mathcal{K}_1^\alpha$ defined by (91) of Section 7. This set is essentially a stripping away of the quadratic part of $\mathcal{K}_1$ around a point α. An analogous set $\mathcal{K}_2^\alpha$ was introduced by DiPerna [DP 85] in his proof of Theorem 6. It turns out that Null Lagrangian Measures on $\mathcal{K}_i$ can be transformed into Null Lagrangian Measures on $\mathcal{K}_i^\alpha$ and vice versa, and this is essentially the contents of Section 7. In some sense, the sets $\mathcal{K}_i^\alpha$ play the role of simplifying the problem by allowing the assumptions $\tilde{\alpha} = 0$ and $a(\tilde{\alpha}_2) = 0$. Working with $\mathcal{K}_i^\alpha$ turns out to significantly simplify many technical arguments. For this reason we establish the general result by first establishing it for $\mathcal{K}_1^\alpha$ in Section 8 and the argument follows the skeleton outlined above. We will also utilize the set $\mathcal{K}_2^\alpha$ in the proof of Theorem 7 in Section 9.

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3. Notations and preliminary lemmas for Theorem 1

In this section, we gather some notations and preliminary results that will be used in the proof of Theorem 1. Given a subset K of $\mathbb{R}^n$ or $M^{m \times n}$, let $\mathcal{P}(K)$ denote the space of probability measures supported on K. For a fixed point $\varpi \in \mathbb{R}^n$, a function $f : \mathbb{R}^n \to \mathbb{R}$ and a subspace $V \subset \mathbb{R}^n$, we define

$$\tilde{f}(z) := f(z) - \nabla f(\varpi) \cdot (z - \varpi) - f(\varpi) \tag{27}$$

and

$$\widetilde{V} := V + \varpi. \tag{28}$$

Definition 8. *Let $\mathcal{F}$ be a finite collection of homogeneous polynomials on $\mathbb{R}^n$ and V be a subspace in $\mathbb{R}^n$. We define $\mathcal{F}_V$ to be a subset of $\mathcal{F}$ such that $\left\{ f_{\lfloor V} : f \in \mathcal{F}_V \right\}$ forms a basis of the set $\left\{ f_{\lfloor V} : f \in \mathcal{F} \right\}$.*

Lemma 9. *Let V be a subspace in $\mathbb{R}^n$. Then we have*

$$\mu \in \mathbb{M}^{pc}_{\mathcal{F}}(\varpi), \text{ Spt}\mu \subset V \iff \mu \in \mathbb{M}^{pc}_{\mathcal{F}_V}(\varpi), \text{ Spt}\mu \subset V.$$

Proof. First note that as $\mathcal{F}_V \subset \mathcal{F}$ it is immediate that $\mathbb{M}^{pc}_{\mathcal{F}}(\varpi) \subset \mathbb{M}^{pc}_{\mathcal{F}_V}(\varpi)$.

Let $\left\{ f_{k_1}, f_{k_2}, \dots, f_{k_{N_1}} \right\} = \mathcal{F}_V$. Suppose $\mu \in \mathbb{M}^{pc}_{\mathcal{F}_V}(\varpi)$ and $\text{Spt}\mu \subset V$. For any $f \in \mathcal{F}$ we have that

$$f_{\lfloor V} = \sum_{i=1}^{N_1} \beta_i f_{k_i \lfloor V}, \text{ where } \beta_i \in \mathbb{R}.$$

It follows that

$$\begin{aligned}\int_{\mathbb{R}^n} f(x) d\mu(x) &= \int_V f(x) d\mu(x) \\ &= \sum_{i=1}^{N_1} \beta_i \int_V f_{k_i}(x) d\mu(x) \\ &= \sum_{i=1}^{N_1} \beta_i f_{k_i}(\varpi) = f(\varpi),\end{aligned}$$

and hence $\mu \in \mathbb{M}^{pc}_{\mathcal{F}}(\varpi)$. □

Lemma 10. *Let $f : \mathbb{R}^n \to \mathbb{R}$ be a homogeneous polynomial of degree 2, then*

$$\tilde{f}(z) = f(z - \varpi). \tag{29}$$

Proof. Recall that $\tilde{f}(z) \overset{(27)}{=} f(z) - \nabla f(\varpi) \cdot (z - \varpi) - f(\varpi)$. Let us define $g(z) := f(z - \varpi)$. First note that $\tilde{f}(\varpi) = g(\varpi) = 0$. Further, we have

$$\nabla \tilde{f}(z) = \nabla f(z) - \nabla f(\varpi) \quad \text{and} \quad \nabla g(z) = \nabla f(z - \varpi).$$

Since f is quadratic, we know ∇f is linear. Therefore, $\nabla g(z) = \nabla f(z) - \nabla f(\varpi) = \nabla \tilde{f}(z)$. This together with $\tilde{f}(\varpi) = g(\varpi) = 0$ implies $\tilde{f} \equiv g$, which is (29). □

Lemma 11. *Let $\mathcal{F} = \{f_1, f_2, \dots, f_{M_1}\}$ be a set of homogeneous polynomials on $\mathbb{R}^n$ with the property that*

$$2 \leq \deg(f_k) \leq 3 \textit{ for } k = 1, 2, \dots, M_1.$$

Let $V \subset \mathbb{R}^n$ be a subspace such that $\mathcal{F}_V = \left\{ f_{k_1}, f_{k_2}, \dots, f_{k_{N_1}} \right\}$ is nonempty, where we have ordered $\mathcal{F}_V$ so that $f_{k_1}, f_{k_2}, \dots, f_{k_{N_0}}$ have degree 2 and $f_{k_{N_0+1}}, f_{k_{N_0+2}}, \dots, f_{k_{N_1}}$ have degree 3. Suppose

$$\sum_{i=1}^{N_0} \beta_i f_{k_i}(z) \textit{ changes sign for } z \in S^{n-1} \cap V \textit{ for every } \beta \in S^{N_0 - 1} \tag{30}$$

or

$$\deg(f_{k_i}) = 3 \textit{ for all } i = 1, 2, \dots, N_1. \tag{31}$$

Then for any $\varpi \in V$, there exist positive constants Θ_0 and Θ_1 with $\Theta_0 < \Theta_1$ such that for every $\beta \in \left(\mathbb{R}^{N_1} \times V\right) \cap S^{N_1+n-1}$, we have that

$$\sum_{i=1}^{N_1} \beta_i \tilde{f}_{k_i}(z) + \left(\beta_{N_1+1}, \beta_{N_1+2}, \ldots, \beta_{N_1+n}\right) \cdot (z-\varpi) \text{ changes sign for } z \in A(\varpi, \Theta_0, \Theta_1) \cap V, \quad (32)$$

where $A(\varpi, \Theta_0, \Theta_1) = \{z \in \mathbb{R}^n : \Theta_0 < |z-\varpi| < \Theta_1\}$.

Proof. Step 1. We establish that

$$\sum_{i=1}^{N_0} \beta_i \tilde{f}_{k_i}(z) + \left(\beta_{N_0+1}, \beta_{N_0+2}, \ldots, \beta_{N_0+n}\right) \cdot (z-\varpi) \text{ changes sign on } V \cap \left(S^{n-1} + \varpi\right)$$

for every $\beta \in \left(\mathbb{R}^{N_0} \times V\right) \cap S^{N_0+n-1}$.

Proof of Step 1. Let $\beta \in \left(\mathbb{R}^{N_0} \times V\right) \cap S^{N_0+n-1}$. If $N_0 = 0$ or $\beta_i = 0$ for $i = 1, \ldots, N_0$, then denoting $\hat{\beta} = \left(\beta_{N_0+1}, \ldots, \beta_{N_0+n}\right)$, we have $\hat{\beta} \in V \cap S^{n-1}$. Note that, in this case, we have

$$\sum_{i=1}^{N_0} \beta_i \tilde{f}_{k_i}(z) + \left(\beta_{N_0+1}, \beta_{N_0+2}, \ldots, \beta_{N_0+n}\right) \cdot (z-\varpi) = \hat{\beta} \cdot (z-\varpi).$$

Therefore, plugging in $z = \varpi + \hat{\beta}$ and $z = \varpi - \hat{\beta}$ in the above, it is clear that $\sum_{i=1}^{N_0} \beta_i \tilde{f}_{k_i}(z) + \hat{\beta} \cdot (z-\varpi)$ changes sign on $V \cap \left(S^{n-1} + \varpi\right)$.

Now assume that $N_0 > 0$ and $(\beta_1, \ldots, \beta_{N_0})$ is nontrivial. Using (29) we have

$$\sum_{i=1}^{N_0} \beta_i \tilde{f}_{k_i}(z) = \sum_{i=1}^{N_0} \beta_i f_{k_i}(z-\varpi). \quad (33)$$

By (30), there exist $z_1, z_2 \in S^{n-1} \cap V$ such that $\sum_{i=1}^{N_0} \beta_i f_{k_i}(z_1) > 0$ and $\sum_{i=1}^{N_0} \beta_i f_{k_i}(z_2) < 0$. As f_{k_i} are all quadratic functions, it follows that $\sum_{i=1}^{N_0} \beta_i f_{k_i}(-z_1) = \sum_{i=1}^{N_0} \beta_i f_{k_i}(z_1) > 0$, and hence $z_2 \neq -z_1$. If $z_1 \cdot \hat{\beta} \geq 0$, then letting $v_1 := \varpi + z_1 \in V \cap \left(S^{n-1} + \varpi\right)$, we have

$$\sum_{i=1}^{N_0} \beta_i \tilde{f}_{k_i}(v_1) + \hat{\beta} \cdot (v_1 - \varpi) \overset{(33)}{=} \sum_{i=1}^{N_0} \beta_i f_{k_i}(z_1) + \hat{\beta} \cdot z_1 > 0.$$

If $z_1 \cdot \hat{\beta} < 0$, then letting $v_1 = \varpi - z_1$, we have

$$\sum_{i=1}^{N_0} \beta_i \tilde{f}_{k_i}(v_1) + \hat{\beta} \cdot (v_1 - \varpi) \overset{(33)}{=} \sum_{i=1}^{N_0} \beta_i f_{k_i}(-z_1) + \hat{\beta} \cdot (-z_1) = \sum_{i=1}^{N_0} \beta_i f_{k_i}(z_1) + \hat{\beta} \cdot (-z_1) > 0.$$

Hence, we can always find $v_1 \in V \cap \left(S^{n-1} + \varpi\right)$ such that $\sum_{i=1}^{N_0} \beta_i \tilde{f}_{k_i}(v_1) + \hat{\beta} \cdot (v_1 - \varpi) > 0$. Exactly the same arguments using $v_2 = \varpi + z_2$ if $z_2 \cdot \hat{\beta} \leq 0$ or $v_2 = \varpi - z_2$ if $z_2 \cdot \hat{\beta} > 0$, we obtain that $\sum_{i=1}^{N_0} \beta_i \tilde{f}_{k_i}(v_2) + \hat{\beta} \cdot (v_2 - \varpi) < 0$ for $v_2 \in V \cap \left(S^{n-1} + \varpi\right)$. Hence, we conclude that $\sum_{i=1}^{N_0} \beta_i \tilde{f}_{k_i}(z) + \hat{\beta} \cdot (z-\varpi)$ changes sign on $V \cap \left(S^{n-1} + \varpi\right)$.

Step 2. Assume $N_0 < N_1$. For any $\alpha > 0$, define

$$\mathcal{B}_\alpha := \left\{\beta \in (\mathbb{R}^{N_1} \times V) \cap S^{N_1+n-1} : \left|\left(\beta_{N_0+1}, \beta_{N_0+2}, \ldots, \beta_{N_1}\right)\right| < \alpha\right\}.$$

Then there exists $\delta_0 > 0$ such that for all $\beta \in \mathcal{B}_{\delta_0}$,

$$\inf\left\{\sum_{i=1}^{N_1} \beta_i \tilde{f}_{k_i}(z) + \left(\beta_{N_1+1}, \beta_{N_1+2}, \ldots, \beta_{N_1+n}\right) \cdot (z-\varpi) : z \in V \cap \left(S^{n-1} + \varpi\right)\right\} < -\delta_0 \quad (34)$$

and

$$\sup\left\{\sum_{i=1}^{N_1}\beta_i\tilde{f}_{k_i}(z)+\left(\beta_{N_1+1},\beta_{N_1+2},\ldots,\beta_{N_1+n}\right)\cdot(z-\varpi):z\in V\cap\left(S^{n-1}+\varpi\right)\right\}>\delta_0, \tag{35}$$

and subsequently (32) holds true for all $\beta\in\mathcal{B}_{\delta_0}$.

Proof of Step 2. We argue by contradiction. Then there exists a sequence $\{\delta_k\}$ converging to zero such that for all k, there exists $\beta^k\in\mathcal{B}_{\delta_k}$ such that either (34) or (35) fails. Without loss of generality, we may assume that there exists a subsequence (not relabeled) such that (34) is false for all k. Upon passing to a further subsequence (again not relabeled), we may assume that $\beta^k\to\tilde{\beta}\in\left(\mathbb{R}^{N_1}\times V\right)\cap S^{N_1+n-1}$. It is clear that $\tilde{\beta}_i=0$ for $i=N_0+1,\ldots,N_1$ and

$$\inf\left\{\sum_{i=1}^{N_1}\tilde{\beta}_i\tilde{f}_{k_i}(z)+\left(\tilde{\beta}_{N_1+1},\tilde{\beta}_{N_1+2},\ldots,\tilde{\beta}_{N_1+n}\right)\cdot(z-\varpi):z\in V\cap\left(S^{n-1}+\varpi\right)\right\}\geq 0,$$

and this contradicts Step 1.

Proof of Lemma 11 completed. Fix $\varpi\in V$. If $N_0=N_1$ then by Step 1 (32) is established. Now assume $N_0<N_1$ and therefore $\left\{f_{k_{N_0+1}},f_{k_{N_0+2}},\ldots,f_{k_{N_1}}\right\}\neq\emptyset$. It suffices to establish (32) for all $\beta\in\left(\mathbb{R}^{N_1}\times V\right)\cap S^{N_1+n-1}\backslash\mathcal{B}_{\delta_0}$. Since $\left\{f_{k_{N_0+1}}\lfloor_V,f_{k_{N_0+2}}\lfloor_V,\ldots,f_{k_{N_1}}\lfloor_V\right\}$ is a linearly independent set it follows that $\sum_{i=N_0+1}^{N_1}\beta_i f_{k_i}\lfloor_V\not\equiv 0$ for any $\left(\beta_{N_0+1},\beta_{N_0+2},\ldots,\beta_{N_1}\right)\neq 0$. Hence for each $\beta\in\left(\mathbb{R}^{N_1}\times V\right)\cap S^{N_1+n-1}\backslash\mathcal{B}_{\delta_0}$ there exists some $z_\beta\in V\cap S^{n-1}$ such that

$$\Upsilon_\beta:=\sum_{i=N_0+1}^{N_1}\beta_i f_{k_i}(z_\beta)=\sup_{z\in V\cap S^{n-1}}\left\{\sum_{i=N_0+1}^{N_1}\beta_i f_{k_i}(z)\right\}>0, \tag{36}$$

as otherwise we would have $\sum_{i=N_0+1}^{N_1}\beta_i f_{k_i}\equiv 0$ (note that $\sum_{i=N_0+1}^{N_1}\beta_i f_{k_i}(z)$ is a homogeneous cubic function). Now by a straightforward compactness argument we have

$$\Upsilon:=\inf\left\{\Upsilon_\beta:\beta\in\left(\mathbb{R}^{N_1}\times V\right)\cap S^{N_1+n-1}\backslash\mathcal{B}_{\delta_0}\right\}>0. \tag{37}$$

Let $\beta\in\left(\mathbb{R}^{N_1}\times V\right)\cap S^{N_1+n-1}\backslash\mathcal{B}_{\delta_0}$. For all $\lambda\in\mathbb{R}$, using (29) of Lemma 10 we have

$$\begin{aligned}\left[\sum_{i=1}^{N_1}\beta_i\tilde{f}_{k_i}\right](\lambda z_\beta+\varpi)&\overset{(29),(27)}{=}\sum_{i=1}^{N_0}\beta_i f_{k_i}(\lambda z_\beta)+\sum_{i=N_0+1}^{N_1}\beta_i f_{k_i}(\lambda z_\beta+\varpi)\\&\quad-\left(\sum_{i=N_0+1}^{N_1}\beta_i\nabla f_{k_i}(\varpi)\right)\cdot(\lambda z_\beta)-\left(\sum_{i=N_0+1}^{N_1}\beta_i f_{k_i}(\varpi)\right).\end{aligned} \tag{38}$$

Since all the terms are quadratic or linear in λ or constants, there exists some constant $\mathcal{C}_1>0$ depending only on $|\beta|$, $|z_\beta|$, $|\varpi|$ and the polynomials f_{k_i} such that

$$\begin{aligned}&\left|\sum_{i=1}^{N_0}\beta_i f_{k_i}(\lambda z_\beta)\right|+\left|\left(\sum_{i=N_0+1}^{N_1}\beta_i\nabla f_{k_i}(\varpi)\right)\cdot(\lambda z_\beta)\right|\\&+\left|\left(\sum_{i=N_0+1}^{N_1}\beta_i f_{k_i}(\varpi)\right)\right|+\left|\left(\beta_{N_1+1},\beta_{N_1+2},\ldots,\beta_{N_1+n}\right)\cdot(\lambda z_\beta)\right|\leq\mathcal{C}_1\lambda^2\text{ for all }|\lambda|\geq 1.\end{aligned} \tag{39}$$

Indeed if we look closer, since $|\beta|=1$ and $|z_\beta|=1$, we see that the above constant $\mathcal{C}_1$ can be made independent of β and z_β. We also know that there exists some constant $\mathcal{C}_2>0$

independent of β such that

$$\left| D\left[\sum_{i=N_0+1}^{N_1} \beta_i f_{k_i}(z)\right]\right| \leq \mathcal{C}_2 \, |z|^2 \text{ for all } z \in \mathbb{R}^n.$$

So for all $\lambda \in \mathbb{R}$ with $|\lambda|$ sufficiently large, we have

$$\left|\left[\sum_{i=N_0+1}^{N_1} \beta_i f_{k_i}(\varpi+\lambda z_\beta)\right] - \left[\sum_{i=N_0+1}^{N_1} \beta_i f_{k_i}(\lambda z_\beta)\right]\right| \leq 4|\varpi|\mathcal{C}_2 \, |\lambda|^2. \tag{40}$$

Clearly there exists $L > 1$ such that $\frac{\Upsilon}{2}\lambda^3 > (\mathcal{C}_1 + 4|\varpi|\mathcal{C}_2)\lambda^2$ for all $\lambda > L$. It follows from this, (38), (39), (40) and (36), (37) that, for all $\lambda > L$ we have

$$\begin{aligned} &\left[\sum_{i=1}^{N_1} \beta_i \tilde{f}_{k_i}\right](\lambda z_\beta + \varpi) + \left(\beta_{N_1+1}, \beta_{N_1+2}, \ldots, \beta_{N_1+n}\right) \cdot (\lambda z_\beta) \\ &\qquad \overset{(38),(39),(40)}{\geq} \lambda^3 \sum_{i=N_0+1}^{N_1} \beta_i f_{k_i}(z_\beta) - (\mathcal{C}_1 + 4|\varpi|\mathcal{C}_2)\,\lambda^2 \overset{(36),(37)}{\geq} \frac{\Upsilon}{2}\lambda^3. \end{aligned} \tag{41}$$

In the same way, for all $\lambda < -L$ we have

$$\left[\sum_{i=1}^{N_1} \beta_i \tilde{f}_{k_i}\right](\lambda z_\beta + \varpi) + \left(\beta_{N_1+1}, \beta_{N_1+2}, \ldots, \beta_{N_1+n}\right) \cdot (\lambda z_\beta) \leq -\frac{\Upsilon}{2}\lambda^3. \tag{42}$$

We deduce from (41)-(42) that there exist positive constants $\tilde{\Theta}_0, \tilde{\Theta}_1$ (independent of β) with $L < \tilde{\Theta}_1 < \infty$ such that

$$\inf\left\{\sum_{i=1}^{N_1} \beta_i \tilde{f}_{k_i}(z) + \left(\beta_{N_1+1}, \ldots, \beta_{N_1+n}\right) \cdot (z - \varpi) : z \in A(\varpi, \tilde{\Theta}_0, \tilde{\Theta}_1) \cap V\right\} < -L^3 \frac{\Upsilon}{2}$$

and

$$\sup\left\{\sum_{i=1}^{N_1} \beta_i \tilde{f}_{k_i}(z) + \left(\beta_{N_1+1}, \ldots, \beta_{N_1+n}\right) \cdot (z - \varpi) : z \in A(\varpi, \tilde{\Theta}_0, \tilde{\Theta}_1) \cap V\right\} > L^3 \frac{\Upsilon}{2}.$$

Finally, define $\Theta_0 := \min\{\tilde{\Theta}_0, \frac{1}{2}\}$ and $\Theta_1 := \max\{\tilde{\Theta}_1, 2\}$. In combination with Step 2, we obtain that $\sum_{i=1}^{N_1} \beta_i \tilde{f}_{k_i}(z) + \left(\beta_{N_1+1}, \ldots, \beta_{N_1+n}\right) \cdot (z - \varpi)$ changes sign on $A(\varpi, \Theta_0, \Theta_1) \cap V$ for all $\beta \in \left(\mathbb{R}^{N_1} \times V\right) \cap S^{N_1+n-1}$. □

Lemma 12. *Under the assumptions of Lemma 11, we can construct a nontrivial measure* $\mu \in \mathbb{M}^{pc}_{\mathcal{F}_V}(\varpi)$ *for any* $\varpi \in V$.

Proof. Let $\varpi \in V$ be fixed. By Lemma 11 there exist $0 < \Theta_0 < \Theta_1$ such that (32) holds true. For each $m \in \mathbb{N}$ there exists a finite collection of points $\Pi_m = \left\{\zeta_j^m : j = 1, 2, \ldots, P_m\right\} \subset V \cap A(\varpi, \Theta_0, \Theta_1)$ such that $V \cap A(\varpi, \Theta_0, \Theta_1) \subset \bigcup_{j=1}^{P_m} B_{2^{-m}}(\zeta_j^m)$.

Step 1. There exists $m \in \mathbb{N}$ such that for every $\beta \in \left(\mathbb{R}^{N_1} \times V\right) \cap S^{N_1+n-1}$ we have

$$\sum_{i=1}^{N_1} \beta_i \tilde{f}_{k_i}(z) + \left(\beta_{N_1+1}, \beta_{N_1+2}, \ldots, \beta_{N_1+n}\right) \cdot (z - \varpi) \text{ changes sign on } \Pi_m. \tag{43}$$

Proof of Step 1. Suppose this is false, so for all m we can find $\beta^m \in \left(\mathbb{R}^{N_1} \times V\right) \cap S^{N_1+n-1}$ such that

$$\sum_{i=1}^{N_1} \beta_i^m \tilde{f}_{k_i}(z) + \left(\beta_{N_1+1}^m, \beta_{N_1+2}^m, \ldots, \beta_{N_1+n}^m\right) \cdot (z - \varpi) \geq 0 \text{ for } z \in \Pi_m. \tag{44}$$

Now passing to a subsequence we have $\beta^m \to \bar{\beta} \in S^{N_1+n-1}$. As $\mathbb{R}^{N_1} \times V$ is a closed subspace of $\mathbb{R}^{N_1+n}$, it is clear that $\bar{\beta} \in \left(\mathbb{R}^{N_1} \times V\right) \cap S^{N_1+n-1}$. Note that there exists a constant $\mathcal{C}_3 > 0$ such that

$$\|\sum_{i=1}^{N_1} \bar{\beta}_i D\tilde{f}_{k_i}\|_{L^\infty(B_{\Theta_1}(\varpi))} \leq \mathcal{C}_3. \tag{45}$$

Now by (32) there exists $\zeta \in A(\varpi, \Theta_0, \Theta_1)$ such that

$$\sum_{i=1}^{N_1} \bar{\beta}_i \tilde{f}_{k_i}(\zeta) + \left(\bar{\beta}_{N_1+1}, \bar{\beta}_{N_1+2}, \ldots, \bar{\beta}_{N_1+n}\right) \cdot (\zeta - \varpi) = \delta < 0.$$

Also by (45) for all large enough m we can find $j_m \in \{1, 2, \ldots, P_m\}$ such that $\zeta_{j_m}^m$ is close enough to ζ so that

$$\sum_{i=1}^{N_1} \bar{\beta}_i \tilde{f}_{k_i}(\zeta_{j_m}^m) + \left(\bar{\beta}_{N_1+1}, \bar{\beta}_{N_1+2}, \ldots, \bar{\beta}_{N_1+n}\right) \cdot \left(\zeta_{j_m}^m - \varpi\right) < \frac{\delta}{2}.$$

Since $\beta^m \to \bar{\beta}$, this implies

$$\sum_{i=1}^{N_1} \beta_i^m \tilde{f}_{k_i}(\zeta_{j_m}^m) + \left(\beta_{N_1+1}^m, \beta_{N_1+2}^m, \ldots, \beta_{N_1+n}^m\right) \cdot \left(\zeta_{j_m}^m - \varpi\right) < \frac{\delta}{4}$$

for all large enough m, which contradicts (44). This completes the proof of Step 1.

Proof of Lemma 12 completed. Define $\Pi := \left\{\zeta_j^m : j = 1, 2, \ldots, P_m\right\}$. Now we will show that we can find coefficients $\gamma_1, \gamma_2, \ldots, \gamma_{P_m} \in \mathbb{R}_+$ for which

$$\sum_{j=1}^{P_m} \gamma_j = 1 \tag{46}$$

and for measure μ defined by $\mu := \sum_{j=1}^{P_m} \gamma_j \delta_{\zeta_j^m}$ we have $\mu \in \mathbb{M}^{pc}_{\mathcal{F}_V}(\varpi)$.

To establish $\mu \in \mathbb{M}^{pc}_{\mathcal{F}_V}(\varpi)$ it suffices to show

$$\int z d\mu(z) = \varpi \tag{47}$$

and

$$\int \tilde{f}_{k_i}(z) d\mu(z) = 0 \text{ for } i = 1, 2, \ldots, N_1, \tag{48}$$

because $\int \tilde{f}_{k_i}(z) d\mu(z) \overset{(27)}{=} \int f_{k_i}(z) d\mu(z) - f_{k_i}(\varpi)$ if (47) holds. So to establish (48) for the measure μ we need

$$\sum_{j=1}^{P_m} \gamma_j \tilde{f}_{k_i}(\zeta_j^m) = 0 \text{ for } i = 1, 2, \ldots, N_1. \tag{49}$$

Now (46), (47), (49) are equivalent to the system

$$\begin{pmatrix} \tilde{f}_{k_1}(\zeta_1^m) & \tilde{f}_{k_1}(\zeta_2^m) & \dots & \tilde{f}_{k_1}(\zeta_{P_m}^m) \\ \tilde{f}_{k_2}(\zeta_1^m) & \tilde{f}_{k_2}(\zeta_2^m) & \dots & \tilde{f}_{k_2}(\zeta_{P_m}^m) \\ \dots \\ \dots \\ \tilde{f}_{k_{N_1}}(\zeta_1^m) & \tilde{f}_{k_{N_1}}(\zeta_2^m) & \dots & \tilde{f}_{k_{N_1}}(\zeta_{P_m}^m) \\ [\zeta_1^m]_1 & [\zeta_2^m]_1 & \dots & \left[\zeta_{P_m}^m\right]_1 \\ [\zeta_1^m]_2 & [\zeta_2^m]_2 & \dots & \left[\zeta_{P_m}^m\right]_2 \\ \dots \\ [\zeta_1^m]_n & [\zeta_2^m]_n & \dots & \left[\zeta_{P_m}^m\right]_n \\ 1 & 1 & \dots & 1 \end{pmatrix} \begin{pmatrix} \gamma_1 \\ \gamma_2 \\ \dots \\ \dots \\ \dots \\ \dots \\ \dots \\ \gamma_{P_m} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \dots \\ \dots \\ 0 \\ [\varpi]_1 \\ [\varpi]_2 \\ \dots \\ [\varpi]_n \\ 1 \end{pmatrix}, \tag{50}$$

where for a vector v, we denote by $[v]_p$ the p-th component of v. Let

$$a_j := \begin{pmatrix} \tilde{f}_{k_1}(\zeta_j^m) \\ \tilde{f}_{k_2}(\zeta_j^m) \\ \dots \\ \dots \\ \tilde{f}_{k_{N_1}}(\zeta_j^m) \\ \left[\zeta_j^m\right]_1 \\ \left[\zeta_j^m\right]_2 \\ \dots \\ \left[\zeta_j^m\right]_n \\ 1 \end{pmatrix} \text{ for } j = 1, 2, \dots, P_m \text{ and } b := \begin{pmatrix} 0 \\ 0 \\ \dots \\ \dots \\ 0 \\ [\varpi]_1 \\ [\varpi]_2 \\ \dots \\ [\varpi]_n \\ 1 \end{pmatrix}.$$

Note that we have $a_j, b \in \mathbb{R}^{N_1} \times V \times \mathbb{R}$ for $j = 1, 2, \dots, P_m$. By applying the Farkas-Minkowski Lemma in the subspace $\mathbb{R}^{N_1} \times V \times \mathbb{R}$ of $\mathbb{R}^{N_1+n+1}$, the above system (50) has a nonnegative solution (componentwise) if and only if

$$\text{whenever } y \in \mathbb{R}^{N_1} \times V \times \mathbb{R} \text{ and } y \cdot a_j \geq 0 \text{ for } j = 1, 2, \dots, P_m \text{ then } y \cdot b \geq 0. \tag{51}$$

The hypothesis of (51) is equivalent to

$$\begin{pmatrix} \sum_{i=1}^{N_1} y_i \tilde{f}_{k_i}(\zeta_1^m) + \left(y_{N_1+1}, y_{N_1+2}, \dots, y_{N_1+n}\right) \cdot \zeta_1^m + y_{N_1+n+1} \\ \sum_{i=1}^{N_1} y_i \tilde{f}_{k_i}(\zeta_2^m) + \left(y_{N_1+1}, y_{N_1+2}, \dots, y_{N_1+n}\right) \cdot \zeta_2^m + y_{N_1+n+1} \\ \dots \\ \dots \\ \sum_{i=1}^{N_1} y_i \tilde{f}_{k_i}(\zeta_{P_m}^m) + \left(y_{N_1+1}, y_{N_1+2}, \dots, y_{N_1+n}\right) \cdot \zeta_{P_m}^m + y_{N_1+n+1} \end{pmatrix} = \begin{pmatrix} \geq 0 \\ \geq 0 \\ \dots \\ \dots \\ \geq 0 \end{pmatrix}. \tag{52}$$

So pick $y \in \mathbb{R}^{N_1} \times V \times \mathbb{R}$ such that (52) holds true. By (43) for some $j_0 \in \{1, 2, \dots, P_m\}$ we have that

$$\sum_{i=1}^{N_1} y_i \tilde{f}_{k_i}(\zeta_{j_0}^m) + \left(y_{N_1+1}, y_{N_1+2}, \dots, y_{N_1+n}\right) \cdot \left(\zeta_{j_0}^m - \varpi\right) \leq 0, \tag{53}$$

and from (52) we know

$$\sum_{i=1}^{N_1} y_i \tilde{f}_{k_i}(\zeta_{j_0}^m) + \left(y_{N_1+1}, y_{N_1+2}, \dots, y_{N_1+n}\right) \cdot \zeta_{j_0}^m + y_{N_1+n+1} \geq 0. \tag{54}$$

Now applying (53) to (54) we have that

$$b \cdot y = \left(y_{N_1+1}, y_{N_1+2}, \ldots, y_{N_1+n}\right) \cdot \varpi + y_{N_1+n+1} \geq 0.$$

Thus (51) is satisfied and so we can solve system (50) for a nonnegative solution γ. Hence we have a nontrivial measure $\mu \in \mathbb{M}^{pc}_{\mathcal{F}_V}(\varpi)$. □

4. Proof of Theorem 1 and Corollary 2

In this section, we first give the proof of Theorem 1 using the preliminary results given in the previous section, and then we will prove Corollary 2.

Proof of Theorem 1. Let $\mathcal{A} := \mathrm{Span}\left\{f_1, f_2, \ldots, f_{M_0}\right\}$. Suppose for every subspace $V \subset \mathbb{R}^n$ the condition (7) holds true. Let $\varpi \in \mathbb{R}^n$ and $\mu \in \mathbb{M}^{pc}_{\mathcal{F}}(\varpi)$. Now by hypothesis there exists a function $g_1 \in \mathcal{A}$ such that $g_1 \geq 0$ on $\mathbb{R}^n$. Let $V_1 := \{z \in \mathbb{R}^n : g_1(z) = 0\}$. Since g_1 is a quadratic and nonnegative function, it is convex, and hence V_1 is a subspace of $\mathbb{R}^n$. Note that $\widetilde{V}_1 \overset{(29)}{=} \{z \in \mathbb{R}^n : \tilde{g}_1(z) = 0\}$. From (6) and (27) we have $\int \tilde{g}_1(z) d\mu = 0$, and therefore $\mathrm{Spt}\mu \subset \widetilde{V}_1$. Now by hypothesis there exists $g_2 \in \mathcal{A}$ that is nonnegative and nontrivial on V_1. Let $V_2 := \{z \in V_1 : g_2(z) = 0\}$. Note that V_2 is a proper subspace of V_1, and thus $\dim V_2 \leq \dim V_1 - 1$. By applying (29) of Lemma 10 we have

$$\begin{aligned} \widetilde{V}_2 &\overset{(28)}{=} V_2 + \varpi \\ &= \{z + \varpi : z \in V_1, g_2(z) = 0\} \\ &\overset{(28)}{=} \left\{\zeta : \zeta \in \widetilde{V}_1, g_2(\zeta - \varpi) = 0\right\} \\ &\overset{(29)}{=} \left\{\zeta : \zeta \in \widetilde{V}_1, \tilde{g}_2(\zeta) = 0\right\}. \end{aligned} \tag{55}$$

As $\int \tilde{g}_2(z) d\mu = 0$, we have $\mathrm{Spt}\mu \subset \widetilde{V}_2$. Repeating this process, we can find a sequence of subspaces $\mathbb{R}^n \supset V_1 \supset V_2 \supset \cdots \supset V_Q$ with strictly decreasing dimensions and V_Q a one dimensional subspace. Also we have $\mathrm{Spt}\mu \subset \widetilde{V}_Q$. Now there exists a function $g_{Q+1} \in \mathcal{A}$ such that $g_{Q+1} \geq 0$ on V_Q and g_{Q+1} is nontrivial on V_Q. Thus g_{Q+1} is convex and quadratic on V_Q. It follows that $\{0\} = \left\{z \in V_Q : g_{Q+1}(z) = 0\right\}$, and hence (as in (55)) by Lemma 10, we have $\{\varpi\} = \left\{z \in \widetilde{V}_Q : \widetilde{g}_{Q+1}(z) = 0\right\}$. Now as $\int \widetilde{g}_{Q+1}(z) d\mu = 0$, we have $\mathrm{Spt}\mu = \{\varpi\}$ and hence μ is a Dirac measure.

Conversely suppose for some subspace V, condition (7) fails to holds. So

$$\text{for all } g \in \mathrm{Span}(\{f_1, f_2, \ldots, f_{M_0}\}), g \text{ changes sign on } V \text{ or is trivial.} \tag{56}$$

We first consider the case when

$$f_{k\lfloor V} \equiv 0 \text{ for } k = 1, 2, \ldots, M_1. \tag{57}$$

Let $q = \dim(V)$. Then for any $\varpi \in V$, let $\nu := \mathcal{H}^q_{\lfloor B_1(\varpi) \cap V} \left(\mathcal{H}^q(B_1(0) \cap V)\right)^{-1}$. Note that

$$\begin{aligned} \int f_k(z) d\nu(z) = 0 &= f_k(\varpi) \\ &= f_k\left(\int z d\nu(z)\right) \text{ for all } k = 1, 2, \ldots, M_1 \end{aligned}$$

and $\overline{\nu} = \varpi$. So $\nu \in \mathbb{M}^{pc}_{\mathcal{F}}(\varpi)$.

Now suppose (57) does not hold true, then $\mathcal{F}_V$ is a nonempty subset of $\mathcal{F}$. Let $\mathcal{F}_V = \left\{f_{k_1}, f_{k_2}, \ldots, f_{k_{N_1}}\right\}$ where we have ordered them so that $\deg(f_{k_i}) = 2$ for $i = 1, 2, \ldots, N_0$ and $\deg(f_{k_i}) = 3$ for $i = N_0 + 1, N_0 + 2, \ldots, N_1$. It is clear that (56) implies that

$$\sum_{i=1}^{N_0} \beta_i f_{k_i} \text{ changes sign or is trivial on } V \text{ for any } \beta \in S^{N_0-1}.$$

Since $\left\{f_{k_i \lfloor V} : i = 1, 2, \ldots, N_0\right\}$ is a linearly independent set, if $N_0 > 0$ and $\sum_{i=1}^{N_0} \beta_i f_{k_i \lfloor V} \equiv 0$ then $\beta_i = 0$ for $i = 1, \ldots, N_0$. Thus (30) of Lemma 11 must hold true if $N_0 > 0$. On the other hand if $\left\{f_{k_i \lfloor V} : i = 1, 2, \ldots, N_0\right\}$ is an empty set, then since (57) is false we must have that (31) of Lemma 11 holds true. By Lemma 12 we can construct a nontrivial measure $\mu \in \mathbb{M}^{pc}_{\mathcal{F}_V}(\varpi)$ for any $\varpi \in V$. Hence by Lemma 9 this gives a nontrivial measure in $\mathbb{M}^{pc}_{\mathcal{F}}(\varpi)$. □

Proof of Corollary 2. Let $M = \dim(K)$. Then there exists a linear isomorphism $\sigma : \mathbb{R}^M \to K$. Let $M_1, M_2, \ldots, M_{q_1}$ be all minors in $M^{m\times n}$ and define $f_k(z) := M_k(\sigma(z))$. Let $\mathcal{F} = \{f_1, f_2, \ldots, f_{q_1}\}$. Without loss of generality, we may assume that all f_k are nontrivial in $\mathbb{R}^M$, as otherwise we can remove the trivial ones.

Step 1. Let $\mu \in \mathcal{P}(K)$ and $\nu \in \mathcal{P}(\mathbb{R}^M)$ be such that $\mu = \sigma_\sharp \nu$, i.e., μ is the push forward of ν under the mapping σ, then $\nu \in \mathbb{M}^{pc}_{\mathcal{F}}$ if and only if $\mu \in \mathcal{M}^{pc}(K)$.

Proof of Step 1. By change of variable formula for push forward measures (see [Am-Fu-Pa 00], P. 32) we have

$$\int f_k(z) d\nu(z) = \int M_k(\sigma(z)) d\nu(z) = \int M_k(X) d\mu(X)$$

and

$$M_k\left(\int X d\mu(X)\right) = M_k\left(\int \sigma(z) d\nu(z)\right) = M_k\left(\sigma\left(\int z d\nu(z)\right)\right) = f_k(\bar{\nu}).$$

Putting the above together it is clear that $\nu \in \mathbb{M}^{pc}_{\mathcal{F}}$ if and only if $\mu \in \mathcal{M}^{pc}(K)$.

Proof of Corollary 2 completed. Suppose (8) holds true. Let $V \subset \mathbb{R}^M$ be a subspace and define $L = \sigma(V)$. By hypothesis there exists $\beta \in S^{q_0-1}$ such that

$$\sum_{k=1}^{q_0} \beta_k f_k(z) = \sum_{k=1}^{q_0} \beta_k M_k(\sigma(z)) \geq 0 \text{ for all } z \in V \text{ and } \sum_{k=1}^{q_0} \beta_k f_k = \sum_{k=1}^{q_0} \beta_k (M_k \circ \sigma) \not\equiv 0.$$

So condition (7) holds true for $\mathbb{R}^M$. For all $\mu \in \mathcal{M}^{pc}(K)$, it follows from Step 1 that $\nu := (\sigma^{-1})_\sharp \mu \in \mathbb{M}^{pc}_{\mathcal{F}}$. It is not hard to see that the proof of the sufficiency part of Theorem 1 does not rely on the assumption that the degrees of the polynomials are less than or equal to three. Hence, by the sufficiency part of Theorem 1, we have $\nu = \delta_{\overline{\nu}}$ and thus μ is also a Dirac measure.

Conversely suppose condition (8) is false and $2 \leq \min\{m, n\} \leq 3$. Note that all the functions f_k are either homogeneous of degree 2 or 3 (when $\min\{m, n\} = 2$, all f_k are simply homogeneous of degree 2). We assume that we have ordered $\{M_k : k = 1, 2 \ldots, q_1\}$ such that $f_1, f_2, \ldots, f_{q_0}$ are quadratics and $f_{q_0+1}, f_{q_0+2}, \ldots, f_{q_1}$ are cubics. There exists some subspace $L \subset K$ such that $\sum_{k=1}^{q_0} \beta_k M_k$ changes sign or is trivial on L for any $\beta \in S^{q_0-1}$. Define $V := \sigma^{-1}(L)$, then $\sum_{k=1}^{q_0} \beta_k f_k$ changes sign or is trivial on V for any $\beta \in S^{q_0-1}$. Thus, by Theorem 1, for any $\varpi \in V$ we can construct a nontrivial measure $\nu \in \mathbb{M}^{pc}_{\mathcal{F}}(\varpi)$. Let $\mu := \sigma_\sharp \nu$. By Step 1 we have that $\mu \in \mathcal{M}^{pc}(K)$. As μ is the push forward by a linear isomorphism of a nontrivial measure, it is clear that μ is nontrivial. □

5. Condition (8) for two dimensional subspaces in $M^{m\times n}$ and proof of Corollary 3

In this section, we first show that for two dimensional subspaces in $M^{m\times n}$, the condition of having no Rank-1 connections implies condition (8). As a consequence, we provide the proof of Corollary 3.

5.1. **Condition (8) for two dimensional subspaces.**

Lemma 13. *Let $M_1, M_2, \dots, M_{q_0}$ denote all 2×2 minors of $M^{m\times n}$. Suppose $K \subset M^{m\times n}$ is a two dimensional subspace without Rank-1 connections, then the following condition holds true:*

$$\text{For each subspace } L \subset K \text{ there exists } \beta \in S^{q_0-1} \text{ such that}$$
$$\sum_{k=1}^{q_0} \beta_k M_k(X) \geq 0 \text{ for all } X \in L \text{ and } \sum_{k=1}^{q_0} \beta_k M_k \not\equiv 0 \text{ on } L. \tag{58}$$

The proof of the above lemma requires some preparation. We begin by introducing some notations. Given $A \in M^{m\times n}$, we denote

$$M^{n_1,n_2}_{m_1,m_2}(A) := \det \begin{pmatrix} a_{m_1n_1} & a_{m_1n_2} \\ a_{m_2n_1} & a_{m_2n_2} \end{pmatrix}$$

for $m_1 \neq m_2 \in \{1,2,\dots,m\}$, $n_1 \neq n_2 \in \{1,2,\dots,n\}$. Let $K \subset M^{m\times n}$ be a two dimensional subspace, then there exist $a_{ij} \in \mathbb{R}^2$ for $i = 1,\dots,m$ and $j = 1,\dots,n$ such that

$$P(z) := \begin{pmatrix} a_{11}\cdot z & a_{12}\cdot z & \dots & a_{1n}\cdot z \\ a_{21}\cdot z & a_{22}\cdot z & \dots & a_{2n}\cdot z \\ \dots & & & \\ \dots & & & \\ a_{m1}\cdot z & a_{m2}\cdot z & \dots & a_{mn}\cdot z \end{pmatrix} \tag{59}$$

is a linear isomorphism of $\mathbb{R}^2$ onto K and hence is a parametrization. Thus we have

$$K = \left\{ P(z) : z \in \mathbb{R}^2 \right\}.$$

For any $z \neq 0$, if we perform an elementary row or column operation on the matrix $P(z)$ we obtain a matrix which we will denote by $T(z)$. Now due to the linearity of the inner product, $T(z)$ can be represented by

$$T(z) := \begin{pmatrix} \tilde{a}_{11}\cdot z & \dots & \tilde{a}_{1n}\cdot z \\ \dots & & \\ \tilde{a}_{m1}\cdot z & \dots & \tilde{a}_{mn}\cdot z \end{pmatrix}$$

where $\tilde{a}_{ij} \in \mathbb{R}^2$. We first show the following

Lemma 14. *Let $M_1, M_2, \dots, M_{q_0}$ denote all 2×2 minors of $M^{m\times n}$. Suppose that $K \subset M^{m\times n}$ is a two dimensional subspace parameterized by (59). For $z \neq 0$, let $T(z) \in M^{m\times n}$ be obtained from $P(z)$ by finitely many elementary row and column operations and define*

$$\tilde{K} := \left\{ T(z) : z \in \mathbb{R}^2 \right\}.$$

Then K has no Rank-1 connections if and only if $\tilde{K}$ has no Rank-1 connections. Further, we have

$$\text{Span}\left\{ M_1(P(z)), \dots, M_{q_0}(P(z)) \right\} = \text{Span}\left\{ M_1(T(z)), \dots, M_{q_0}(T(z)) \right\}. \tag{60}$$

Proof. By induction, it suffices to consider the case where $T(z)$ is obtained from $P(z)$ by an elementary row or column operation. We only show the case where $T(z)$ is obtained from $P(z)$ by an elementary row operation, as the proof for column operation is identical.

Note that

$$\text{Rank}(T(z)) = \text{Rank}(P(z)) \text{ for any } z \neq 0. \tag{61}$$

Therefore

$$\begin{aligned} & K \text{ has no Rank-1 connections} \\ \iff & \operatorname{Rank}(P(z)) \geq 2 \text{ for any } z \neq 0 \\ \overset{(61)}{\iff} & \operatorname{Rank}(T(z)) \geq 2 \text{ for any } z \neq 0 \\ \iff & \tilde{K} \text{ has no Rank-1 connections.} \end{aligned}$$

Next, as $T(z)$ is obtained from $P(z)$ by an elementary tow operation, we have that

$$R_i(T(z)) = \sum_{i'=1}^{m} c_{i,i'} R_{i'}(P(z)), \tag{62}$$

where $R_i(A)$ denotes the i-th row of a matrix A. It follows that

$$\begin{aligned} M_{i_0,i_1}^{j_0,j_1}(T(z)) &= \det \begin{pmatrix} [T(z)]_{i_0,j_0} & [T(z)]_{i_0,j_1} \\ [T(z)]_{i_1,j_0} & [T(z)]_{i_1,j_1} \end{pmatrix} \\ &= \left([T(z)]_{i_0,j_0}, [T(z)]_{i_0,j_1}\right) \wedge \left([T(z)]_{i_1,j_0}, [T(z)]_{i_1,j_1}\right) \\ &= \left(\sum_{i'=1}^{m} c_{i_0,i'} \left([P(z)]_{i',j_0}, [P(z)]_{i',j_1}\right)\right) \wedge \left(\sum_{i''=1}^{m} c_{i_1,i''} \left([P(z)]_{i'',j_0}, [P(z)]_{i'',j_1}\right)\right) \\ &= \sum_{i',i''=1}^{m} c_{i_0,i'} c_{i_1,i''} M_{i',i''}^{j_0,j_1}(P(z)). \end{aligned}$$

This shows that the minors of $T(z)$ are inside the span of the minors of $P(z)$. Conversely, by (62) the rows of $P(z)$ can be represented as linear combinations of the rows of $T(z)$. Therefore, exactly the same argument shows the opposite inclusion in (60). ☐

Proof of Lemma 13. First note that if $m = n = 2$, the lemma is trivial because in this case there is only one minor. If it fails to be positive semidefinite or negative semidefinite then the subspace K has Rank-1 connections. So we can assume $\max\{m, n\} \geq 3$.

We have two cases to consider. The first case is where $\dim(L) = 2$ and the second case is where $\dim(L) = 1$. The former requires much more work and we consider that case first. Note that in this case $L = K$. Assume that K is parameterized by $P(z)$ given in (59).

Step 1. We claim that after a finite number of row and column operations, we can transform $P(z)$ into a diagonal matrix $Q(z) = (q_{ij} \cdot z)$ or a matrix $T(z) = (t_{ij} \cdot z)$ such that there exists $i_0 \in \{1, \ldots, m\}$ with

$$\dim\left(\operatorname{Span}\left\{t_{i_0 1}, \ldots, t_{i_0 n}\right\}\right) = 2 \tag{63}$$

or there exists $j_0 \in \{1, \ldots, n\}$ with

$$\dim\left(\operatorname{Span}\left\{t_{1 j_0}, \ldots, t_{m j_0}\right\}\right) = 2. \tag{64}$$

We start with the matrix $P(z)$. Clearly there exists some $a_{ij} \neq 0 \in \mathbb{R}^2$. Without loss of generality, we may assume that $a_{11} \neq 0 \in \mathbb{R}^2$. If $\dim\left(\operatorname{Span}\left\{a_{11}, \ldots, a_{1n}\right\}\right) = 2$ or $\dim\left(\operatorname{Span}\left\{a_{11}, \ldots, a_{m1}\right\}\right) = 2$, then we are done. Otherwise we can use row and column operations to transform $P(z)$ into $\tilde{P}(z) = \left(\tilde{a}_{ij} \cdot z\right)$ such that $\tilde{a}_{i1} = 0$ for all $i \in \{2, \ldots, m\}$ and $\tilde{a}_{1j} = 0$ for all $j \in \{2, \ldots, n\}$. If $\tilde{a}_{ij} = 0$ for all $i \in \{2, \ldots, m\}$ and $j \in \{2, \ldots, n\}$, then $\tilde{P}(z)$ is diagonal. Otherwise, without loss of generality, we may assume that $\tilde{a}_{22} \neq 0$. Now repeating the above procedure, we either have $\dim\left(\operatorname{Span}\left\{\tilde{a}_{21}, \ldots, \tilde{a}_{2n}\right\}\right) = 2$ or $\dim\left(\operatorname{Span}\left\{\tilde{a}_{12}, \ldots, \tilde{a}_{m2}\right\}\right) = 2$, in which case we are done, or $\dim\left(\operatorname{Span}\left\{\tilde{a}_{21}, \ldots, \tilde{a}_{2n}\right\}\right) = 1$

and $\dim\left(\text{Span}\left\{\tilde{a}_{12},\dots,\tilde{a}_{m2}\right\}\right)=1$, in which case we can further transform $\tilde{P}(z)$ into $\tilde{\tilde{P}}(z)=\left(\tilde{\tilde{a}}_{ij}\cdot z\right)$ with all first two rows and first two columns zero except for $\tilde{\tilde{a}}_{11}\cdot z$ and $\tilde{\tilde{a}}_{22}\cdot z$. Repeating this procedure finitely many times, we establish the claim.

Step 2. Assume that $P(z)$ can be transformed into a diagonal matrix $Q(z)=\left(q_{ij}\cdot z\right)$. Denote the nonzero entries of $Q(z)$ by $q_{i_1i_1}\cdot z, q_{i_2i_2}\cdot z,\dots,q_{i_ri_r}\cdot z$ (where $r\leq\min\{m,n\}$). If $r=2$ or $\dim\left(\text{Span}\left\{q_{i_1i_1},q_{i_2i_2},\dots,q_{i_ri_r}\right\}\right)=1$, then for any $\zeta\in q_{i_1i_1}^{\perp}$ we have that $\text{Rank}\left(Q(\zeta)\right)\leq 1$ which contradicts Lemma 14. Therefore we have $r\geq 3$ and $\dim\left(\text{Span}\left\{q_{i_1i_1},q_{i_2i_2},\dots,q_{i_ri_r}\right\}\right)=2$. Thus we must be able to find distinct $p_1,p_2,p_3\in\{1,2,\dots,r\}$ and $\lambda_1,\lambda_2\in\mathbb{R}$ such that $q_{i_{p_3}i_{p_3}}=\lambda_1 q_{i_{p_1}i_{p_1}}+\lambda_2 q_{i_{p_2}i_{p_2}}$. Now $\lambda_1 M_{i_{p_1},i_{p_3}}^{i_{p_1},i_{p_3}}(Q(z))+\lambda_2 M_{i_{p_2},i_{p_3}}^{i_{p_2},i_{p_3}}(Q(z))=\lambda_1(q_{i_{p_1}i_{p_1}}\cdot z)(q_{i_{p_3}i_{p_3}}\cdot z)+\lambda_2(q_{i_{p_2}i_{p_2}}\cdot z)(q_{i_{p_3}i_{p_3}}\cdot z)=(q_{i_{p_3}i_{p_3}}\cdot z)^2$ which is positive semidefinite and thus we have (58) by Lemma 14.

Step 3. Assume that $P(z)$ can be transformed into a matrix $T(z)=\left(t_{ij}\cdot z\right)$ such that (63) or (64) holds. Without loss of generality assume (64) holds, and thus there exists $j_0\in\{1,2,\dots,n\}$ and $i_0,i_1\in\{1,2,\dots,m\}$ such that $t_{i_0j_0}$ and $t_{i_1j_0}$ are linearly independent. And by performing further row and column operations on $T(z)$ we can assume $i_0=1, j_0=1, i_1=2$. Therefore, all t_{i1} can be expressed as a linear combination of t_{11} and t_{21}. Thus we can perform row operations to further transform $T(z)$ to the matrix

$$\tilde{T}(z)=\begin{pmatrix}\tilde{t}_{11}\cdot z & \tilde{t}_{12}\cdot z & \dots & \tilde{t}_{1n}\cdot z\\ \tilde{t}_{21}\cdot z & \tilde{t}_{22}\cdot z & \dots & \tilde{t}_{2n}\cdot z\\ 0 & \tilde{t}_{32}\cdot z & \dots & \tilde{t}_{3n}\cdot z\\ \dots & & & \\ \dots & & & \\ 0 & \tilde{t}_{m2}\cdot z & \dots & \tilde{t}_{mn}\cdot z\end{pmatrix}.$$

Now we have two subcases to consider. The first subcase is where $\tilde{t}_{i_0j_0}\neq 0$ for some $i_0\in\{3,\dots,m\}$ and $j_0\in\{2,\dots,n\}$, and the second subcase is where $\tilde{t}_{ij}=0$ for all $i\in\{3,\dots,m\}$ and $j\in\{2,\dots,n\}$.

Step 4. We consider the first subcase where $\tilde{t}_{i_0j_0}\neq 0$ for some $i_0\in\{3,\dots,m\}$ and $j_0\in\{2,\dots,n\}$. Now as $\tilde{t}_{11}$ and $\tilde{t}_{21}$ are linearly independent, we may write $\tilde{t}_{i_0j_0}=\tilde{\beta}_1\tilde{t}_{11}+\tilde{\beta}_1\tilde{t}_{21}$ for some $\tilde{\beta}_1,\tilde{\beta}_2\in\mathbb{R}$. It follows that

$$\begin{aligned}&\tilde{\beta}_1 M_{1,i_0}^{1,j_0}\left(\tilde{T}(z)\right)+\tilde{\beta}_2 M_{2,i_0}^{1,j_0}\left(\tilde{T}(z)\right)\\ &\quad=\tilde{\beta}_1\det\begin{pmatrix}\tilde{t}_{11}\cdot z & \tilde{t}_{1j_0}\cdot z\\ 0 & \tilde{t}_{i_0j_0}\cdot z\end{pmatrix}+\tilde{\beta}_2\det\begin{pmatrix}\tilde{t}_{21}\cdot z & \tilde{t}_{2j_0}\cdot z\\ 0 & \tilde{t}_{i_0j_0}\cdot z\end{pmatrix}\\ &\quad=\left((\tilde{\beta}_1\tilde{t}_{11}+\tilde{\beta}_2\tilde{t}_{21})\cdot z\right)\left(\tilde{t}_{i_0j_0}\cdot z\right)\\ &\quad=\left|\tilde{t}_{i_0j_0}\cdot z\right|^2,\end{aligned}\tag{65}$$

which is positive semidefinite. Hence we obtain from (60) that there exists a linear combination of the 2×2 minors of $P(z)$ that is positive semidefinite. This establishes (58) in subcase 1.

Step 5. We consider the second subcase where $\tilde{t}_{ij}=0$ for all $i\in\{3,\dots,m\}$ and $j\in\{2,\dots,n\}$. We know from Lemma 14 that $\text{Rank}\left(\tilde{T}(z)\right)=\text{Rank}\left(P(z)\right)>1$ for all $z\neq 0\in\mathbb{R}^2$. Now we claim that

$$\dim\left(\text{Span}\left\{\tilde{t}_{11},\dots,\tilde{t}_{1n}\right\}\right)=2.$$

Suppose not, then for some $\tilde{t}_{1j_0} \neq 0 \in \mathbb{R}^2$, we have $\tilde{t}_{1j} = \lambda_j \tilde{t}_{1j_0}$ for all $j \in \{1, \ldots, n\}$ and $\lambda_j \in \mathbb{R}$. Then we can choose $z \neq 0 \in \mathbb{R}^2$ such that $z \cdot \tilde{t}_{1j_0} = 0$, and thus this z makes the first row of $\tilde{T}(z)$ zero, which is a contradiction to the fact that $\tilde{T}(z)$ has no Rank-1 connections.

Without loss of generality due to Lemma 14, we may assume that $\tilde{t}_{11}$ and $\tilde{t}_{12}$ are linearly independent. Similar to before, we can perform column operations to reduce the matrix $\tilde{T}(z)$ to

$$\tilde{\tilde{T}}(z) = \begin{pmatrix} \tilde{\tilde{t}}_{11} \cdot z & \tilde{\tilde{t}}_{12} \cdot z & 0 & \ldots & 0 \\ \tilde{\tilde{t}}_{21} \cdot z & \tilde{\tilde{t}}_{22} \cdot z & \tilde{\tilde{t}}_{23} \cdot z & \ldots & \tilde{\tilde{t}}_{2n} \cdot z \\ 0 & 0 & 0 & \ldots & 0 \\ \ldots & & & & \\ \ldots & & & & \\ 0 & 0 & 0 & \ldots & 0 \end{pmatrix}.$$

If $\tilde{\tilde{t}}_{2j_0} \neq 0$ for some $j_0 \in \{3, \ldots, n\}$, then we can write $\tilde{\tilde{t}}_{2j_0} = \tilde{\tilde{\beta}}_1 \tilde{\tilde{t}}_{11} + \tilde{\tilde{\beta}}_2 \tilde{\tilde{t}}_{12}$ for $\tilde{\tilde{\beta}}_1, \tilde{\tilde{\beta}}_2 \in \mathbb{R}$, and similar to (65), we have

$$\tilde{\tilde{\beta}}_1 M_{1,2}^{1,j_0} \left(\tilde{\tilde{T}}(z) \right) + \tilde{\tilde{\beta}}_2 M_{1,2}^{2,j_0} \left(\tilde{\tilde{T}}(z) \right) = \left| \tilde{\tilde{t}}_{2j_0} \cdot z \right|^2.$$

Hence, by (60), there exists a linear combination of the 2×2 minors of $P(z)$ that is positive semidefinite.

On the other hand, if $\tilde{\tilde{t}}_{2j} = 0$ for all $j \in \{3, \ldots, n\}$, then since $\tilde{\tilde{T}}(z)$ has no Rank-1 connections, it is clear that $\det \begin{pmatrix} \tilde{\tilde{t}}_{11} \cdot z & \tilde{\tilde{t}}_{12} \cdot z \\ \tilde{\tilde{t}}_{21} \cdot z & \tilde{\tilde{t}}_{22} \cdot z \end{pmatrix}$ is either positive semidefinite or negative semidefinite. In either case, (58) follows trivially by (60). This establishes subcase 2.

Proof of Lemma 13 completed. Finally, we consider the case where $\dim(L) = 1$. Assume that L is parametrized by $\mathbb{P}(t)$ given by

$$\mathbb{P}(t) := \begin{pmatrix} b_{11}t & b_{12}t & \ldots & b_{1n}t \\ b_{21}t & b_{22}t & \ldots & b_{2n}t \\ \ldots & & & \\ \ldots & & & \\ b_{m1}t & b_{m2}t & \ldots & b_{mn}t \end{pmatrix}$$

for $b_{ij} \in \mathbb{R}$. It is not hard to see that Lemma 14 still holds true for one dimensional subspaces. Using elementary row operations, we reduce the matrix

$$\begin{pmatrix} b_{11} & b_{12} & \ldots & b_{1n} \\ b_{21} & b_{22} & \ldots & b_{2n} \\ \ldots & & & \\ \ldots & & & \\ b_{m1} & b_{m2} & \ldots & b_{mn} \end{pmatrix}$$

to the triangle form

$$\begin{pmatrix} 0 & \ldots & 0 & \tilde{b}_{1j_1} & \ldots & \tilde{b}_{1j_2} & \ldots & \tilde{b}_{1m} \\ 0 & \ldots & 0 & 0 & \ldots & \tilde{b}_{2j_2} & \ldots & \tilde{b}_{2m} \\ \ldots & & & & & & & \\ \ldots & & & & & & & \\ 0 & \ldots & 0 & 0 & \ldots & * & \ldots & * \end{pmatrix}$$

and denote

$$\mathbb{T}(t) := \begin{pmatrix} 0 & \dots & 0 & \tilde{b}_{1j_1}t & \dots & \tilde{b}_{1j_2}t & \dots & \tilde{b}_{1m}t \\ 0 & \dots & 0 & 0 & \dots & \tilde{b}_{2j_2}t & \dots & \tilde{b}_{2m}t \\ \dots \\ \dots \\ 0 & \dots & 0 & 0 & \dots & * & \dots & * \end{pmatrix},$$

where $\tilde{b}_{ij_i}$ is the first nonzero entry in the i-th row. Note that the second row in $\mathbb{T}(t)$ must be nontrivial, as otherwise $\mathbb{P}(t)$ would form a Rank-1 line in the subspace K, which is a contradiction. Now

$$\det \begin{pmatrix} \tilde{b}_{1j_1}t & \tilde{b}_{1j_2}t \\ 0 & \tilde{b}_{2j_2}t \end{pmatrix} = \tilde{b}_{1j_1}\tilde{b}_{2j_2}t^2$$

is nontrivial. Hence it follows from Lemma 14 for one dimensional subspaces that (58) holds in the case where $\dim(L) = 1$. This completes the proof of Lemma 13. □

5.2. **Proof of Corollary 3.** Now we give the proof of Corollary 3.

Proof of Corollary 3. By Lemma 13, if K has no Rank-1 connections then (58) of Lemma 13, and hence (8) of Corollary 2, are satisfied. Thus, by Corollary 2, any $\mu \in \mathcal{M}^{pc}(K)$ is a Dirac measure. Now suppose that K has Rank-1 connections, then there exists a nontrivial subspace $L \subset K$ such that $\text{Rank}(A) = 1$ for any $A \in L$. Pick $A_1, A_2 \in L$, $A_1 \neq A_2$, and let $\mu := \frac{1}{2}\delta_{A_1} + \frac{1}{2}\delta_{A_2}$. Note that for any minor M_q of $M^{m\times n}$, $M_q(A_1) = M_q(A_2) = M_q(\frac{A_1+A_2}{2}) = 0$. It follows that

$$\int_K M_q(X)d\mu(X) = \frac{1}{2}M_q(A_1) + \frac{1}{2}M_q(A_2) = 0 = M_q\left(\frac{A_1+A_2}{2}\right) = M_q\left(\int_K X d\mu(X)\right).$$

Hence $\mu \in \mathcal{M}^{pc}(K)$ and $\mathcal{M}^{pc}(K)$ contains nontrivial measures. □

6. Proof of Theorem 4

We first recall the notion of Grassmannian which is needed in the discussions of this section. Let p, k be fixed integers with $p \geq 0$ and $0 \leq k \leq p$. We denote by $G(k, p)$ the set of all k dimensional subspaces of $\mathbb{R}^p$, and it is called the Grassmannian of k dimensional subspaces of $\mathbb{R}^p$. We have the following property regarding $G(k, p)$, whose proof can be found, for example, in [Pi-Ta 08]:

Lemma 15. *The Grassmannian $G(k, p)$ is a real analytic, compact and connected manifold of dimension $k(p-k)$.*

One can view $G(k, p)$ as a differentiable manifold in the following way. We fix a pair of transversal subspaces (W_0, W_1) of $\mathbb{R}^p$, i.e., $W_0 \cap W_1 = \{0\}$, where $\dim(W_0) = k$ and $\dim(W_1) = p-k$. Then one can view elements in $G^0(k, p, W_1)$ as the graphs of linear maps from W_0 to W_1, where

$$G^0(k, p, W_1) := \{V \in G(k, p) : V \cap W_1 = \{0\}\}.$$

Specifically, we pick a basis $\{a_1, a_2, \dots, a_k\}$ for W_0 and a basis $\left\{b_1, b_2, \dots, b_{p-k}\right\}$ for W_1. We identify $\mathbb{R}^{k(p-k)}$ with $M^{(p-k)\times k}$ in the obvious way, i.e., identify $x \in \mathbb{R}^{k(p-k)}$ with $A_x \in M^{(p-k)\times k}$ where $[A_x]_{ij} = [x]_{(i-1)k+j}$. For each $A \in \mathbb{R}^{k(p-k)} \simeq M^{(p-k)\times k}$, let $T_A : W_0 \to W_1$ be the linear map defined by A and the choices of bases $\{a_1, a_2, \dots, a_k\}$, $\left\{b_1, b_2, \dots, b_{p-k}\right\}$. We define

$$\phi_{W_0,W_1}(A) := \{v + T_A(v) : v \in W_0\}. \quad (66)$$

Note that the mapping ϕ_{W_0,W_1} is one to one from $\mathbb{R}^{k(p-k)}$ onto $G^0(k,p,W_1)$. Hence it defines a chart on $G(k,p)$ that covers $G^0(k,p,W_1)$. As noted in Remark 2.2.4 in [Pi-Ta 08], the charts defined by (66) actually form a real analytic atlas for $G(k,p)$. Further, it is shown in Corollary 2.4.3 in [Pi-Ta 08] that $G(k,p)$ is compact and connected.

Definition 16. *We say that a property holds* **generically** *for k dimensional subspaces of $\mathbb{R}^p$ if there exist finitely many smooth manifolds $\Gamma_1, \Gamma_2, \ldots, \Gamma_r$ in $\mathbb{R}^{k(p-k)}$ of dimension less than $k(p-k)$ and Lipschitz mappings $P_j : \Gamma_j \to G(k,p)$ for $j = 1,2,\ldots,r$ such that the property holds true for every $V \in G(k,p) \setminus \left(\bigcup_{j=1}^r P_j(\Gamma_j)\right)$.*

Remark 4. We let $G(k, M^{m\times n})$ denote the space of k dimensional subspaces in $M^{m\times n}$. In an obvious way we can uniquely identify any $V \in G(k, M^{m\times n})$ with some $W \in G(k,mn)$. For this reason we will not distinguish between $G(k, M^{m\times n})$ and $G(k,mn)$.

Let k,m,n be positive integers with $k \le mn$, and (W_0, W_1) be a pair of transversal subspaces of $\mathbb{R}^{mn}$ with $\dim(W_0) = k$ and $\dim(W_1) = mn - k$. Further let $T : W_0 \to W_1$ be a linear mapping. We fix a basis $\mathcal{B}_0 = \{a_1, a_2, \ldots, a_k\}$ for W_0. Recall that $M_1, M_2, \ldots, M_{q_0}$ denote all 2×2 minors in $M^{m\times n}$. With the linear mapping T and the basis $\mathcal{B}_0$ we can define the set of quadratics $Q_1, Q_2, \ldots, Q_{q_0}$ on $\mathbb{R}^k$ by

$$Q_j(y) := M_j\left(\sum_{l=1}^k y_l\,(a_l + T(a_l))\right) \text{ for } j = 1,2,\ldots,q_0. \tag{67}$$

Then for each Q_j, there exists a unique $X_j \in M^{k\times k}_{sym}$ that represents Q_j. We need the following auxiliary lemma.

Lemma 17. *Let k,m,n be positive integers with $k \le mn$. Suppose that for $\varpi = 1,2$, (W_0^ϖ, W_1^ϖ) is a pair of transversal subspaces of $\mathbb{R}^{mn}$ with* $\dim(W_0^\varpi) = k$, $\dim(W_1^\varpi) = nm - k$, *and $T^\varpi : W_0^\varpi \to W_1^\varpi$ is a linear mapping. Suppose also that for some $V \in G(k,mn)$ we have that*

$$\left\{v + T^1(v) : v \in W_0^1\right\} = V = \left\{v + T^2(v) : v \in W_0^2\right\}. \tag{68}$$

Let $\mathcal{B}_0^\varpi = \left\{a_1^\varpi, a_2^\varpi, \ldots, a_k^\varpi\right\}$ be a basis of W_0^ϖ for $\varpi = 1,2$. We denote by $X_j^\varpi \in M^{k\times k}_{sym}$ the symmetric matrix that represents the quadratic Q_j^ϖ given by (67) with respect to the linear mapping T^ϖ and the basis $\mathcal{B}_0^\varpi$. Then we have

$$\operatorname{Span}\left\{X_1^1, X_2^1, \ldots, X_{q_0}^1\right\} = M^{k\times k}_{sym} \iff \operatorname{Span}\left\{X_1^2, X_2^2, \ldots, X_{q_0}^2\right\} = M^{k\times k}_{sym}. \tag{69}$$

Proof. We begin by showing that for $\varpi = 1,2$,

$$\mathcal{B}^\varpi := \left\{a_l^\varpi + T^\varpi(a_l^\varpi) : l = 1,2,\ldots,k\right\} \text{ forms a basis of } V. \tag{70}$$

From (68) and the fact that $\mathcal{B}_0^\varpi$ is a basis of W_0^ϖ, it is immediate that $\mathcal{B}^\varpi$ spans V. Assume $\sum_{l=1}^k \lambda_l \left(a_l^\varpi + T^\varpi(a_l^\varpi)\right) = 0$. As $\sum_{l=1}^k \lambda_l a_l^\varpi \in W_0^\varpi$, $\sum_{l=1}^k \lambda_l T^\varpi(a_l^\varpi) \in W_1^\varpi$, and W_0^ϖ and W_1^ϖ are transversal, it follows that $\sum_{l=1}^k \lambda_l a_l^\varpi = 0$ and $\sum_{l=1}^k \lambda_l T^\varpi(a_l^\varpi) = 0$. Hence $\lambda_l = 0$ for all $l = 1,2,\ldots,k$ and thus (70) is established.

Let $A \in M^{k\times k}$ denote the change of basis matrix between the two bases $\mathcal{B}^1$ and $\mathcal{B}^2$ of V, i.e., letting $\alpha_{ij} \in \mathbb{R}$ denote the (i,j) entry of A, we have that

$$a_i^1 + T^1(a_i^1) = \sum_{l=1}^k \alpha_{il}(a_l^2 + T^2(a_l^2)) \text{ for } i = 1,2,\ldots,k.$$

So for any $j \in \{1, 2, \dots, q_0\}$, we have

$$\begin{aligned} y^T X_j^1 y &= M_j\left(\sum_{i=1}^k y_i\left(a_i^1 + T^1(a_i^1)\right)\right) \\ &= M_j\left(\sum_{i=1}^k y_i\left(\sum_{l=1}^k \alpha_{il}\left(a_l^2 + T^2(a_l^2)\right)\right)\right) \\ &= M_j\left(\sum_{l=1}^k\left(\sum_{i=1}^k y_i\alpha_{il}\right)\left(a_l^2 + T^2(a_l^2)\right)\right) \\ &= M_j\left(\sum_{l=1}^k\left[A^T y\right]_l\left(a_l^2 + T^2(a_l^2)\right)\right) \\ &= \left[A^T y\right]^T X_j^2\left[A^T y\right] = y^T A X_j^2 A^T y, \end{aligned}$$

and thus

$$X_j^1 = A X_j^2 A^T \text{ for } j = 1, 2, \dots, q_0. \tag{71}$$

Suppose $\text{Span}\left\{X_1^1, X_2^1, \dots, X_{q_0}^1\right\} = M_{sym}^{k\times k}$. For any $S \in M_{sym}^{k\times k}$, note that $ASA^T \in M_{sym}^{k\times k}$. Therefore we have $\sum_{j=1}^k \lambda_j X_j^1 = ASA^T$ for some $\lambda_1, \lambda_2, \dots, \lambda_k \in \mathbb{R}$. It follows from (71) that $\sum_{j=1}^k \lambda_j X_j^2 = \sum_{j=1}^k \lambda_j A^{-1} X_j^1 (A^T)^{-1} = S$, and thus $M_{sym}^{k\times k} = \text{Span}\left\{X_1^2, X_2^2, \dots, X_{q_0}^2\right\}$. Exactly the same argument shows the other implication in (69). □

Lemma 18. *Let k, m, n be positive integers with $m, n \geq 2$ and $k \leq \frac{1}{2}\min\{m, n\}$, and $M_1, M_2, \dots, M_{q_0}$ denote all the 2×2 minors of $M^{m\times n}$. Generically for $V \in G(k, M^{m\times n})$ (in the sense of Definition 16) there exists $\beta \in S^{q_0-1}$ such that*

$$\sum_{j=1}^{q_0} \beta_j M_j(X) > 0 \text{ for all } X \in V\setminus\{0\}. \tag{72}$$

Proof. By Lemma 15, $G(k, M^{m\times n})$ is a $k(mn-k)$ dimensional compact manifold. Therefore we can find finitely many charts of the form (66) whose images cover $G(k, M^{m\times n})$. Formally we can find finitely many pairs of transversal subspaces

$$\left\{(W_0^1, W_1^1), (W_0^2, W_1^2), \dots, (W_0^{p_0}, W_1^{p_0})\right\}$$

with $\dim(W_0^i) = k$ and $\dim(W_1^i) = mn - k$ such that

$$G(k, M^{m\times n}) \subset \bigcup_{i=1}^{p_0} \phi_{W_0^i, W_1^i}(\mathbb{R}^{k(mn-k)}). \tag{73}$$

For each $i \in \{1, 2, \dots, p_0\}$, we fix a basis $\mathcal{B}_0^i = \{a_1^i, \dots, a_k^i\}$ for W_0^i and a basis $\mathcal{B}_1^i = \{b_1^i, \dots, b_{mn-k}^i\}$ for W_1^i. Given $x \in \mathbb{R}^{k(mn-k)}$, define $A_x \in M^{(mn-k)\times k}$ to be the matrix with $[A_x]_{ij} = [x]_{(i-1)k+j}$, and let $T_x^i : W_0^i \to W_1^i$ be the linear mapping defined from A_x given the bases $\mathcal{B}_0^i$ and $\mathcal{B}_1^i$. For each $x \in \mathbb{R}^{k(mn-k)}$, exactly the same arguments used to establish (70) show that the set

$$\mathcal{B}_x^i := \left\{a_l^i + T_x^i(a_l^i) : l = 1, 2, \dots, k\right\} \tag{74}$$

forms a basis of the subspace $\phi_{W_0^i,W_1^i}(x) = \left\{(v, T_x^i(v)) : v \in W_0^i\right\}$. For any $j \in \{1,2,\ldots,q_0\}$, as in (67), the mapping

$$Q_{j,x}^i(y) := M_j\left(\sum_{l=1}^k y_l\left(a_l^i + T_x^i(a_l^i)\right)\right) \tag{75}$$

defines a quadratic mapping on $\mathbb{R}^k$ and hence can be represented by a matrix $X_{j,x}^i \in M_{sym}^{k\times k}$. Given $i \in \{1,2,\ldots,p_0\}$, recall that we have fixed the bases $\mathcal{B}_0^i$ and $\mathcal{B}_1^i$. Thus, each entry of the $m\times n$ matrix $a_l^i + T_x^i(a_l^i)$ is either linear in x or constant. As M_j is a 2×2 minor, it follows that the coefficients in the quadratic $Q_{j,x}^i(y)$ are polynomials (of degree less than or equal to two) of x, and so are all the entries of the matrix $X_{j,x}^i$.

Step 1. For each $i \in \{1,2,\ldots,p_0\}$, we show that there exists a polynomial function $\Lambda^i : \mathbb{R}^{k(mn-k)} \to \mathbb{R}$ such that

$$\text{Span}\left\{X_{1,x}^i, X_{2,x}^i, \ldots, X_{q_0,x}^i\right\} = M_{sym}^{k\times k} \text{ for any } x \in \mathbb{R}^{k(mn-k)} \setminus \left\{x : \Lambda^i(x) = 0\right\}. \tag{76}$$

Proof of Step 1. First note that, since $X_{j,x}^i$ is a symmetric $k\times k$ matrix, it can be uniquely identified with a vector $v_{j,x}^i \in \mathbb{R}^{\frac{k(k+1)}{2}}$. Then it is clear that

$$\text{Span}\left\{X_{1,x}^i, X_{2,x}^i, \ldots, X_{q_0,x}^i\right\} = M_{sym}^{k\times k} \Longleftrightarrow \text{Span}\left\{v_{1,x}^i, v_{2,x}^i, \ldots, v_{q_0,x}^i\right\} = \mathbb{R}^{\frac{k(k+1)}{2}}. \tag{77}$$

Recall that q_0 is the number of 2×2 minors in $M^{m\times n}$, and therefore $q_0 = \frac{m(m-1)}{2}\frac{n(n-1)}{2}$. Without loss of generality, we may assume $m \le n$, and by assumption, we have $k \le \frac{m}{2}$. In particular, $m \ge 2k \ge k+1$. In order to apply Lemma 29 later in the proof we observe the following inequality

$$q_0 = \frac{m(m-1)}{2}\frac{n(n-1)}{2} \ge \frac{m^2(m-1)^2}{4} \ge \frac{(2k)(k+1)}{4}(m-1)^2 \ge \frac{k(k+1)}{2}. \tag{78}$$

Now, viewing $v_{j,x}^i$ as column vectors for all j, we define

$$\Pi^i(x) := \left(v_{1,x}^i, v_{2,x}^i, \ldots, v_{q_0,x}^i\right) \in M^{\frac{k(k+1)}{2}\times q_0},$$

and

$$\Lambda^i(x) := \det\left(\Pi^i(x)\left(\Pi^i(x)\right)^T\right) \text{ for any } x \in \mathbb{R}^{k(mn-k)}.$$

Note that each entry of $\Pi^i(x)$ is an entry of $X_{j,x}^i$ for some $j \in \{1,\ldots,q_0\}$. By previous discussions, we know that each entry of $\Pi^i(x)$ is a polynomial of x, and hence $\Lambda^i(x)$ is also a polynomial of x. Further, by Lemma 29 (note the relation (78)), we have that

$$\dim\left(\text{Span}\left\{v_{1,x}^i, v_{2,x}^i, \ldots, v_{q_0,x}^i\right\}\right) = \frac{k(k+1)}{2} \iff \Lambda^i(x) \neq 0. \tag{79}$$

This together with (77) gives (76).

Step 2. There exists $i_0 \in \{1,2,\ldots,p_0\}$ such that Λ^{i_0} is nontrivial.

Proof of Step 2. We define a subspace $V_0 \in G(k, M^{m\times n})$ to be

$$V_0 := \left\{ \begin{pmatrix} y_1 & 0 & 0 & 0 & \dots & 0 & 0 & 0 & \dots & 0 \\ 0 & y_1 & 0 & 0 & \dots & 0 & 0 & 0 & \dots & 0 \\ 0 & 0 & y_2 & 0 & \dots & 0 & 0 & 0 & \dots & 0 \\ 0 & 0 & 0 & y_2 & \dots & 0 & 0 & 0 & \dots & 0 \\ \dots \\ 0 & 0 & 0 & 0 & \dots & y_k & 0 & 0 & \dots & 0 \\ 0 & 0 & 0 & 0 & \dots & 0 & y_k & 0 & \dots & 0 \\ 0 & 0 & 0 & 0 & \dots & 0 & 0 & 0 & \dots & 0 \\ \dots \\ 0 & 0 & 0 & 0 & \dots & 0 & 0 & 0 & \dots & 0 \end{pmatrix} : (y_1, \dots, y_k) \in \mathbb{R}^k \right\}. \tag{80}$$

Note that the assumption $k \leq \frac{1}{2}\min\{m,n\}$ allows to construct the subspace V_0 in $M^{m\times n}$. Now for some $i_0 \in \{1,2,\dots,p_0\}$ and $x_0 \in \mathbb{R}^{k(mn-k)}$ we have $\phi_{W_0^{i_0}, W_1^{i_0}}(x_0) = V_0$. Thus by (66) we have $V_0 = \left\{ (v, T_{x_0}^{i_0}(v)) : v \in W_0^{i_0} \right\}$. Since $\mathcal{B}_{x_0}^{i_0}$ (recall (74)) is a basis of V_0, the mapping $H : \mathbb{R}^k \to V_0$ defined by

$$H(y) := \sum_{l=1}^{k} y_l \left(a_l^{i_0} + T_{x_0}^{i_0}(a_l^{i_0}) \right) \tag{81}$$

is a linear isomorphism onto V_0. Thus there exist $h_{s,t} \in \mathbb{R}^k$ for $s = 1,2,\dots,m$, $t = 1,2,\dots,n$ such that

$$H(y) = \begin{pmatrix} h_{1,1}\cdot y & h_{1,2}\cdot y & \dots & h_{1,n}\cdot y \\ h_{2,1}\cdot y & h_{2,2}\cdot y & \dots & h_{2,n}\cdot y \\ \dots \\ h_{m,1}\cdot y & h_{m,2}\cdot y & \dots & h_{m,n}\cdot y \end{pmatrix}.$$

By definition of V_0 (recall (80)) we have that

$$h_{s,t} = 0 \text{ for } s \neq t, h_{2s-1,2s-1} = h_{2s,2s} \text{ for } s = 1,2,\dots,k, \text{ and } h_{s,s} = 0 \text{ for } s > 2k. \tag{82}$$

Now we claim that $\{h_{1,1}, h_{3,3}, \dots, h_{2k-1,2k-1}\}$ are linearly independent. Suppose this is false, then pick $y_1 \in \bigcap_{s=1}^{k} (h_{2s-1,2s-1})^{\perp} \setminus \{0\}$. Thus $H(y_1) = 0 \in M^{m\times n}$, contradicting the fact that H is an isomorphism. Thus the claim is established.

Note that $Q_{j,x_0}^{i_0}(y) \overset{(75),(81)}{=} M_j(H(y))$ for $j = 1,2,\dots,q_0$. Using (82), the set $\mathcal{O}_M$ of all the 2×2 minors on V_0 is simply

$$\begin{aligned} \mathcal{O}_M &:= \left\{ Q_{j,x_0}^{i_0}(y) : j = 1,2,\dots,q_0 \right\} \\ &= \left\{ (h_{2s_1-1,2s_1-1}\cdot y)(h_{2s_2-1,2s_2-1}\cdot y) : s_1 < s_2 \in \{1,2,\dots,k\} \right\} \\ &\qquad \bigcup \left\{ (h_{2t-1,2t-1}\cdot y)^2 : t \in \{1,2,\dots,k\} \right\}. \end{aligned}$$

Now we claim that $\mathcal{O}_M$ is linearly independent. So see this, let $\lambda_{s_1,s_2} \in \mathbb{R}$ and $\lambda_t \in \mathbb{R}$ be such that

$$0 = \sum_{s_1 < s_2 \in \{1,2,\dots,k\}} \lambda_{s_1,s_2} (h_{2s_1-1,2s_1-1}\cdot y)(h_{2s_2-1,2s_2-1}\cdot y) + \sum_{t\in\{1,2,\dots,k\}} \lambda_t (h_{2t-1,2t-1}\cdot y)^2. \tag{83}$$

For every $t \in \{1,2,\dots,k\}$, pick $y \in \bigcap_{s\in\{1,2,\dots,k\}\setminus\{t\}} (h_{2s-1,2s-1})^{\perp}$ such that $h_{2t-1,2t-1}\cdot y \neq 0$. Note that such y exists because $\{h_{1,1}, h_{3,3}, \dots, h_{2k-1,2k-1}\}$ are linearly independent. Putting this into (83) we get that $\lambda_t(h_{2t-1,2t-1}\cdot y)^2=0$ and so $\lambda_t=0$. Thus $\lambda_t=0$ for all $t\in\{1,2,\ldots,k\}$. Next, let $s_1<s_2\in\{1,2,\ldots,k\}$. Pick

$$y\in\bigcap_{s\in\{1,2,\ldots,k\}\setminus\{s_1,s_2\}}(h_{2s-1,2s-1})^{\perp}$$

such that $y\cdot h_{2s_1-1,2s_1-1}\neq 0$ and $y\cdot h_{2s_2-1,2s_2-1}\neq 0$. Such y exists for the same reason as above. Putting this into (83) we have that $\lambda_{s_1,s_2}(h_{2s_1-1,2s_1-1}\cdot y)(h_{2s_2-1,2s_2-1}\cdot y)=0$ and so $\lambda_{s_1,s_2}=0$. Thus linear independence of $\mathcal{O}_M$ is established.

It is easy to see that

$$\text{Card}\,(\mathcal{O}_M)=\frac{k!}{2(k-2)!}+k=\frac{k(k+1)}{2}.$$

Thus

$$\dim\left(\text{Span}\left\{X^{i_0}_{j,x_0}:j=1,2,\ldots,q_0\right\}\right)=\dim\left(\text{Span}\left\{Q^{i_0}_{j,x_0}:j=1,2,\ldots,q_0\right\}\right)=\frac{k(k+1)}{2},$$

and so by (77) and (79) we have that $\Lambda^{i_0}(x_0)\neq 0$. This completes the proof Step 2.

Step 3. We show that $\Lambda^i(x)$ is nontrivial on $R^{k(mn-k)}$ for all $i\in\{1,2,\ldots,p_0\}$.

Proof of Step 3. We first denote the zero set of $\Lambda^i(x)$ by

$$Z^i:=\left\{x\in\mathbb{R}^{k(mn-k)}:\Lambda^i(x)=0\right\}.$$

Note that, as $\Lambda^i(x)$ is a polynomial function, the set Z^i is a real algebraic variety. A classical result of Whitney [Wh 57] states that Z^i can be decomposed as a disjoint union of finitely many connected analytic submanifolds of dimension less than $k(mn-k)$, provided that Λ^i is nontrivial on $\mathbb{R}^{k(mn-k)}$.

As the collection of charts $\left\{\phi_{W_0^1,W_1^1},\phi_{W_0^2,W_1^2},\ldots,\phi_{W_0^{p_0},W_1^{p_0}}\right\}$ satisfy (73) and $G(k,M^{m\times n})$ is connected, it is clear that we can find $i_1\in\{1,2,\ldots,p_0\}$ such that

$$\mathcal{U}_{i_0,i_1}:=\phi_{W_0^{i_0},W_1^{i_0}}\left(\mathbb{R}^{k(mn-k)}\right)\bigcap\phi_{W_0^{i_1},W_1^{i_1}}\left(\mathbb{R}^{k(mn-k)}\right)\neq\emptyset. \tag{84}$$

We begin by showing that Λ^{i_1} is nontrivial. Since $\mathcal{U}_{i_0,i_1}$ is a nonempty open subset of $G(k,M^{m\times n})$ and $\phi_{W_0^{i_0},W_1^{i_0}}$ is a Lipschitz mapping, we know that $(\phi_{W_0^{i_0},W_1^{i_0}})^{-1}(\mathcal{U}_{i_0,i_1})$ is a nonempty open subset of $\mathbb{R}^{k(mn-k)}$. As Λ^{i_0} is nontrivial, we know from Whitney's result [Wh 57] that Z^{i_0} is the disjoint union of finitely many submanifolds of dimension less than $k(mn-k)$. It follows that

$$(\phi_{W_0^{i_0},W_1^{i_0}})^{-1}(\mathcal{U}_{i_0,i_1})\not\subset Z^{i_0}.$$

Thus we must be able to find $\tilde{x}_0\in\mathbb{R}^{k(mn-k)}$ such that $\phi_{W_0^{i_0},W_1^{i_0}}(\tilde{x}_0)\in U_{i_0,i_1}$ and $\Lambda^{i_0}(\tilde{x}_0)\neq 0$. From (84), as $U_{i_0,i_1}\subset\phi_{W_0^{i_1},W_1^{i_1}}\left(\mathbb{R}^{k(mn-k)}\right)$, we can find some $x_1\in\mathbb{R}^{k(mn-k)}$ such that $\phi_{W_0^{i_0},W_1^{i_0}}(\tilde{x}_0)=\phi_{W_0^{i_1},W_1^{i_1}}(x_1)=:V_1$. Thus

$$\left\{v+T^{i_0}_{\tilde{x}_0}(v):v\in W_0^{i_0}\right\}=V_1=\left\{v+T^{i_1}_{x_1}(v):v\in W_0^{i_1}\right\}.$$

Since $\Lambda^{i_0}(\tilde{x}_0)\neq 0$, we know from Step 1 that $\text{Span}\left\{X^{i_0}_{1,\tilde{x}_0},X^{i_0}_{2,\tilde{x}_0},\ldots,X^{i_0}_{q_0,\tilde{x}_0}\right\}=M^{k\times k}_{sym}$. By Lemma 17, we have that $\text{Span}\left\{X^{i_1}_{1,x_1},X^{i_1}_{2,x_1},\ldots,X^{i_1}_{q_0,x_1}\right\}=M^{k\times k}_{sym}$. From Step 1 again, we have $\Lambda^{i_1}(x_1)\neq 0$ and hence Λ^{i_1} is nontrivial. From Lemma 15, $G(k,M^{m\times n})$ is connected. Therefore, because of (73) the above arguments can be repeated to all charts to conclude that $\Lambda^i(x)$

is nontrivial for all $i \in \{1, 2, \ldots, p_0\}$.

Proof of Lemma 18 completed. Let $M^{k\times k}_{sym,+}$ denote the cone of positive definite matrices in $M^{k\times k}_{sym}$, i.e., $A \in M^{k\times k}_{sym,+}$ if and only if $y^T Ay > 0$ for all $y \in \mathbb{R}^k\backslash\{0\}$. If $V \in \phi_{W_0^i,W_1^i}(\mathbb{R}^{k(mn-k)}\backslash Z^i)$, then there exists $x \in \mathbb{R}^{k(mn-k)}$ such that $\phi_{W_0^i,W_1^i}(x) = V$ and $\Lambda^i(x) \neq 0$. Now by (76) we must be able to find some $\beta \in S^{q_0-1}$ such that $\sum_{j=1}^{q_0} \beta_j X^i_{j,x} \in M^{k\times k}_{sym,+}$. By (75) this implies that $\sum_{j=1}^{q_0} \beta_j M_j(X) > 0$ for all $X \in V\backslash\{0\}$. Since Λ^i is nontrivial by Step 3, we know from Whitney's result [Wh 57] that Z^i is the disjoint union of finitely many submanifolds of dimension less than $k(mn-k)$. Thus (72) holds *generically* (recall Definition 16) for $V \in \phi_{W_0^i,W_1^i}(\mathbb{R}^{k(mn-k)})$ for all $i \in \{1, 2, \ldots, p_0\}$. We conclude the proof of the lemma by noting (73). □

Proof of Theorem 4. Given 2×2 matrices A and B, we have that

$$\det(A - B) = \det(A) - A : \mathrm{Cof}(B) + \det(B). \tag{85}$$

Also recall that $M_1, M_2, \ldots, M_{q_0}$ denote all 2×2 minors in $M^{m\times n}$. By Lemma 18, we have that for generic subspace $V \in G(k, M^{m\times n})$ there exists $\beta \in S^{q_0-1}$ such that (72) holds true. Now for any $\mu \in \mathcal{M}^{pc}(V)$, using the expansion (85) we know

$$\int_V \sum_{j=1}^{q_0} \beta_j M_j(X - \overline{X})d\mu = \sum_{j=1}^{q_0} \beta_j M_j(0) = 0. \tag{86}$$

However by (72), unless $\mu = \delta_{\overline{X}}$ the left hand side of (86) is strictly positive, which is a contradiction. □

7. Preliminaries for Theorems 5 and 7

In this section, we gather some preliminary lemmas that will be useful in dealing with $\mathcal{M}^{pc}(\mathcal{K}_1)$ and $\mathcal{M}^{pc}(\mathcal{K}_2)$ in the following two sections. First, we introduce some notations that will be used repeatedly. Given a matrix $A \in M^{m\times n}$, let $R_i(A)$ denote the i-th row of A and $[A]_{ij}$ denote the (i, j) entry of the matrix A. Let $A \in M^{m\times 2}$ with $m \geq 2$ and $1 \leq i < j \leq m$, we define

$$X_{ij}(A) := \begin{pmatrix} [A]_{i1} & [A]_{i2} \\ [A]_{j1} & [A]_{j2} \end{pmatrix} \tag{87}$$

and

$$M_{ij}(A) := R_i(A) \wedge R_j(A) = \det\left(X_{ij}(A)\right). \tag{88}$$

Recall the definitions of $\mathcal{K}_1, \mathcal{K}_2, P_1$ and P_2 in (18), (19), (20) and (21), respectively. Further, given $\alpha \in \mathbb{R}^2$, define

$$P_1^\alpha(u, v) := \begin{pmatrix} u - \alpha_1 & v - \alpha_2 \\ a(v) - a(\alpha_2) & u - \alpha_1 \\ (u - \alpha_1)\,(a(v) - a(\alpha_2)) & \frac{(u-\alpha_1)^2}{2} + F(v) - F(\alpha_2) - a(\alpha_2)(v - \alpha_2) \end{pmatrix}. \tag{89}$$

and

$$P_2^\alpha(u, v) := \begin{pmatrix} u - \alpha_1 & v - \alpha_2 \\ a(v) - a(\alpha_2) & u - \alpha_1 \\ (u - \alpha_1)\,(a(v) - a(\alpha_2)) & \frac{(u-\alpha_1)^2}{2} + F(v) - F(\alpha_2) - a(\alpha_2)(v - \alpha_2) \\ \frac{(u-\alpha_1)^2}{2} + F(\alpha_2) - F(v) + a(v)(v - \alpha_2) & (u - \alpha_1)\,(v - \alpha_2) \end{pmatrix}. \tag{90}$$

Similar to $\mathcal{K}_1$ and $\mathcal{K}_2$, we denote

$$\mathcal{K}_1^\alpha := \{P_1^\alpha(u,v) : u,v \in \mathbb{R}\} \tag{91}$$

and

$$\mathcal{K}_2^\alpha := \{P_2^\alpha(u,v) : u,v \in \mathbb{R}\}. \tag{92}$$

Finally, given a measure μ and a function f which is integrable with respect to the measure μ, define

$$\overline{f} := \int f(z) d\mu(z).$$

We prove a couple of lemmas that will be essential in the following two sections.

Lemma 19. *For all $\alpha \in \mathbb{R}^2$ and $k \in \{1,2\}$, the push forward map $(P_k^\alpha)_\sharp : \mathcal{P}(\mathbb{R}^2) \to \mathcal{P}(\mathcal{K}_k^\alpha)$ defined by $\nu \mapsto \mu := (P_k^\alpha)_\sharp \nu$ forms a bijection. Moreover, $\mu \in \mathcal{M}^{pc}(\mathcal{K}_k^\alpha)$ if and only if*

$$\int_{\mathbb{R}^2} M_{ij}(P_k^\alpha(u,v)) d\nu = M_{ij}\left(\int_{\mathbb{R}^2} P_k^\alpha(u,v) d\nu\right) \text{ for } i<j \in \{1,\ldots,k+2\}. \tag{93}$$

Further, given $\delta > 0$, if $\mathrm{Spt}\mu \subset \mathcal{K}_k^\alpha \cap B_\delta(0)$ then $\mathrm{Spt}\nu \subset B_\delta(\alpha)$, and conversely, if $\mathrm{Spt}\nu \subset B_\delta(\alpha)$ then $\mathrm{Spt}\mu \subset \mathcal{K}_k^\alpha \cap B_{C\delta}(0)$ for some constant C depending on the function a, α_2 and δ.

Proof. First note that, since the first row of $P_k^\alpha(u,v)$ is $(u-\alpha_1, v-\alpha_2)$, it is clear that $P_k^\alpha : \mathbb{R}^2 \to \mathcal{K}_k^\alpha$ is a bijection. Therefore it is straight forward to check that $\left((P_k^\alpha)^{-1}\right)_\sharp$ is the inverse map of $(P_k^\alpha)_\sharp$ and hence $(P_k^\alpha)_\sharp$ is a bijection.

Let $\nu \in \mathcal{P}(\mathbb{R}^2)$ and $\mu \in \mathcal{P}(\mathcal{K}_k^\alpha)$ be related by $\mu = (P_k^\alpha)_\sharp \nu$. By change of variable formula for push forward measures, we have

$$\int_{\mathbb{R}^2} M_{ij}(P_k^\alpha(u,v)) d\nu = \int_{\mathcal{K}_k^\alpha} M_{ij}(\zeta) d\mu(\zeta)$$

and

$$M_{ij}\left(\int_{\mathbb{R}^2} P_k^\alpha(u,v)) d\nu\right) = M_{ij}\left(\int_{\mathcal{K}_k^\alpha} \zeta d\mu(\zeta)\right).$$

It follows that $\mu \in \mathcal{M}^{pc}(K^\alpha)$ if and only if (93) holds.

Next, assume $\mathrm{Spt}\mu \subset \mathcal{K}_k^\alpha \cap B_\delta(0)$. Since the first row of $P_k^\alpha(u,v)$ is $(u-\alpha_1, v-\alpha_2)$, it is clear that $\|(u,v)-\alpha\| \le \|P_k^\alpha(u,v)\|$. Therefore $(P_k^\alpha)^{-1}(\mathcal{K}_k^\alpha \cap B_\delta(0)) \subset B_\delta(\alpha)$. As $\mathrm{Spt}\nu = \left(P_k^\alpha\right)^{-1}\mathrm{Spt}\mu$, it follows that $\mathrm{Spt}\nu \subset B_\delta(\alpha)$. Conversely, assume $\mathrm{Spt}\nu \subset B_\delta(\alpha)$. From the expression of $P_k^\alpha(u,v)$ in (89) or (90), it is clear that the absolute value of each component of $P_k^\alpha(u,v)$ is bounded above by $C\|(u,v)-\alpha\|$ for some constant C depending on the function a, α_2 and δ, provided δ is sufficiently small. Therefore $\|P_k^\alpha(u,v)\| \le \tilde{C}\|(u,v)-\alpha\|$ and hence $P_k^\alpha(B_\delta(\alpha)) \subset \mathcal{K}_k^\alpha \cap B_{\tilde{C}\delta}(0)$. It follows that $\mathrm{Spt}\mu \subset \mathcal{K}_k^\alpha \cap B_{\tilde{C}\delta}(0)$. This completes the proof of the lemma. □

The following lemma is implicitly stated in [Ki-Mü-Sv 03]. We thank S. Müller [Mü 18] for providing us with the elegant proof presented in this section.

Lemma 20 (Kirchheim-Müller-Šverák [Ki-Mü-Sv 03])**.** *Given $\nu \in \mathcal{P}(\mathbb{R}^2)$, for all $\alpha \in \mathbb{R}^2$ and $k \in \{1,2\}$, we have*

$$(P_k)_\sharp \nu \in \mathcal{M}^{pc}(\mathcal{K}_k) \Longleftrightarrow (P_k^\alpha)_\sharp \nu \in \mathcal{M}^{pc}(\mathcal{K}_k^\alpha). \tag{94}$$

We show (94) for $k=2$, and the case for $k=1$ is included. We break the proof into several steps. The first lemma is standard.

Lemma 21. *Given $\nu \in \mathcal{P}(\mathbb{R}^2)$, for all $\alpha \in \mathbb{R}^2$, we have*

$$(P_2^\alpha)_\sharp \nu \in \mathcal{M}^{pc}(\mathcal{K}_2^\alpha) \Longleftrightarrow (\tilde{P}_2^\alpha)_\sharp \nu \in \mathcal{M}^{pc}(\tilde{\mathcal{K}}_2^\alpha), \tag{95}$$

where

$$\tilde{P}_2^\alpha(u,v) := \begin{pmatrix} u & v \\ a(v) & u \\ ua(v) - \alpha_1 a(v) - ua(\alpha_2) & \frac{u^2}{2} - u\alpha_1 + F(v) - va(\alpha_2) \\ \frac{u^2}{2} - u\alpha_1 - F(v) + va(v) - \alpha_2 a(v) & uv - \alpha_1 v - u\alpha_2 \end{pmatrix} \tag{96}$$

and

$$\tilde{\mathcal{K}}_2^\alpha := \left\{ \tilde{P}_2^\alpha(u,v) : u,v \in \mathbb{R} \right\}.$$

Proof. Recall the definition of P_2^α given in (90). Direct calculations show that

$$P_2^\alpha(u,v) = \tilde{P}_2^\alpha(u,v) - \tilde{E}^\alpha \tag{97}$$

for

$$\tilde{E}^\alpha := \begin{pmatrix} \alpha_1 & \alpha_2 \\ a(\alpha_2) & \alpha_1 \\ -\alpha_1 a(\alpha_2) & -\frac{\alpha_1^2}{2} + F(\alpha_2) - \alpha_2 a(\alpha_2) \\ -\frac{\alpha_1^2}{2} - F(\alpha_2) & -\alpha_1\alpha_2 \end{pmatrix}.$$

Note that $\tilde{E}^\alpha$ is the constant part of $P_2^\alpha(u,v)$.

Given $\nu \in \mathcal{P}(\mathbb{R}^2)$, by arguing exactly as in Lemma 19, we have that

$$(P_2)_\sharp \nu \in \mathcal{M}^{pc}(\mathcal{K}_2) \Longleftrightarrow \int_{\mathbb{R}^2} M_{ij}(P_2(u,v))d\nu = M_{ij}\left(\int_{\mathbb{R}^2} P_2(u,v)d\nu\right) \text{ for } i<j \in \{1,2,3,4\}$$

and

$$(\tilde{P}_2^\alpha)_\sharp \nu \in \mathcal{M}^{pc}(\tilde{\mathcal{K}}_2^\alpha) \Longleftrightarrow \int_{\mathbb{R}^2} M_{ij}(\tilde{P}_2^\alpha(u,v))d\nu = M_{ij}\left(\int_{\mathbb{R}^2} \tilde{P}_2^\alpha(u,v)d\nu\right) \text{ for } i<j \in \{1,2,3,4\}. \tag{98}$$

Using the formula (85), the fact that $\tilde{E}^\alpha$ is a constant matrix and the notation $M_{ij}(\cdot) = \det\left(X_{ij}(\cdot)\right)$ (recalling (87)), we have

$$\begin{aligned} \int M_{ij}\left(P_2^\alpha(u,v)\right)d\nu &\overset{(97)}{=} \int M_{ij}\left(\tilde{P}_2^\alpha(u,v) - \tilde{E}^\alpha\right)d\nu \\ &= \int \left[M_{ij}\left(\tilde{P}_2^\alpha(u,v)\right) - X_{ij}\left(\tilde{P}_2^\alpha(u,v)\right) : \mathrm{Cof}\left(X_{ij}\left(\tilde{E}^\alpha\right)\right) + M_{ij}\left(\tilde{E}^\alpha\right)\right]d\nu \\ &= \int M_{ij}\left(\tilde{P}_2^\alpha(u,v)\right)d\nu - \int X_{ij}\left(\tilde{P}_2^\alpha(u,v)\right)d\nu : \mathrm{Cof}\left(X_{ij}\left(\tilde{E}^\alpha\right)\right) + M_{ij}\left(\tilde{E}^\alpha\right). \end{aligned} \tag{99}$$

In a similar way using (85) we have that

$$\begin{aligned} &\det\left(\int X_{ij}\left(P_2^\alpha(u,v)\right)d\nu\right) \\ &= \det\left(\int X_{ij}\left(\tilde{P}_2^\alpha(u,v) - \tilde{E}^\alpha\right)d\nu\right) \\ &= \det\left(\int X_{ij}\left(\tilde{P}_2^\alpha(u,v)\right)d\nu\right) - \int X_{ij}\left(\tilde{P}_2^\alpha(u,v)\right)d\nu : \mathrm{Cof}\left(X_{ij}\left(\tilde{E}^\alpha\right)\right) + M_{ij}\left(\tilde{E}^\alpha\right). \end{aligned} \tag{100}$$

Putting (99) and (100) together and using Lemma 19 we have that

$$
\begin{aligned}
&(P_2^\alpha)_\sharp \nu \in \mathcal{M}^{pc}(\mathcal{K}_2^\alpha)\\
&\overset{(93)}{\Longleftrightarrow} \int M_{ij}\left(P_2^\alpha(u,v)\right) d\nu = \det\left(\int X_{ij}\left(P_2^\alpha(u,v)\right) d\nu\right) \text{ for all } i,j\\
&\overset{(99),(100)}{\Longleftrightarrow} \int M_{ij}\left(\tilde{P}_2^\alpha(u,v)\right) d\nu = \det\left(\int X_{ij}\left(\tilde{P}_2^\alpha(u,v)\right) d\nu\right) \text{ for all } i,j\\
&\overset{(98)}{\Longleftrightarrow} (\tilde{P}_2^\alpha)_\sharp \nu \in \mathcal{M}^{pc}(\tilde{\mathcal{K}}_2^\alpha).
\end{aligned}
$$

This establishes (95). □

Lemma 22 (Müller [Mü 18])**.** *Every row $R_i\left(\tilde{P}_2^\alpha(u,v)\right)$ of the matrix $\tilde{P}_2^\alpha(u,v)$ can be expressed as a linear combination of the rows of $P_2(u,v)$, and conversely every row $R_i\left(P_2(u,v)\right)$ of the matrix $P_2(u,v)$ can be expressed as a linear combination of the rows of $\tilde{P}_2^\alpha(u,v)$, and the coefficients depend only on α, but not on (u,v), i.e.,*

$$R_i\left(\tilde{P}_2^\alpha(u,v)\right) = \sum_{i'=1}^{4} c_{ii'}(\alpha)\, R_{i'}\left(P_2(u,v)\right) \quad \textit{for all } (u,v) \in \mathbb{R}^2 \tag{101}$$

and

$$R_i\left(P_2(u,v)\right) = \sum_{i'=1}^{4} \tilde{c}_{ii'}(\alpha)\, R_{i'}\left(\tilde{P}_2^\alpha(u,v)\right) \quad \textit{for all } (u,v) \in \mathbb{R}^2. \tag{102}$$

Proof. From the definitions of P_2 and $\tilde{P}_2^\alpha$ in (21) and (96), we see that

$$R_1\left(\tilde{P}_2^\alpha(u,v)\right) = R_1\left(P_2(u,v)\right), \quad R_2\left(\tilde{P}_2^\alpha(u,v)\right) = R_2\left(P_2(u,v)\right). \tag{103}$$

Now we calculate

$$
\begin{aligned}
R_3\left(\tilde{P}_2^\alpha(u,v)\right) &= \left(ua(v) - \alpha_1 a(v) - ua(\alpha_2), \frac{u^2}{2} - u\alpha_1 + F(v) - va(\alpha_2)\right)\\
&= R_3\left(P_2(u,v)\right) - \alpha_1 R_2\left(P_2(u,v)\right) - a(\alpha_2) R_1\left(P_2(u,v)\right)
\end{aligned} \tag{104}
$$

and

$$
\begin{aligned}
R_4\left(\tilde{P}_2^\alpha(u,v)\right) &= \left(\frac{u^2}{2} - u\alpha_1 - F(v) + va(v) - \alpha_2 a(v), uv - \alpha_1 v - u\alpha_2\right)\\
&= R_4\left(P_2(u,v)\right) - \alpha_1 R_1\left(P_2(u,v)\right) - \alpha_2 R_2\left(P_2(u,v)\right).
\end{aligned} \tag{105}
$$

This proves (101). Conversely, we have

$$R_3\left(P_2(u,v)\right) \overset{(104),(103)}{=} R_3\left(\tilde{P}_2^\alpha(u,v)\right) + \alpha_1 R_2\left(\tilde{P}_2^\alpha(u,v)\right) + a(\alpha_2) R_1\left(\tilde{P}_2^\alpha(u,v)\right)$$

and

$$R_4\left(P_2(u,v)\right) \overset{(105),(103)}{=} R_4\left(\tilde{P}_2^\alpha(u,v)\right) + \alpha_1 R_1\left(\tilde{P}_2^\alpha(u,v)\right) + \alpha_2 R_2\left(\tilde{P}_2^\alpha(u,v)\right),$$

and therefore we have (102). □

Proof of Lemma 20. By Lemma 21, it suffices to show that

$$(P_2)_\sharp \nu \in \mathcal{M}^{pc}(\mathcal{K}_2) \Longleftrightarrow (\tilde{P}_2^\alpha)_\sharp \nu \in \mathcal{M}^{pc}(\tilde{\mathcal{K}}_2^\alpha). \tag{106}$$

We first show the implication "$\Longrightarrow$". We denote $\mu := (P_2)_\sharp \nu$ and $\mu^\alpha := (\tilde{P}_2^\alpha)_\sharp \nu$, and assume $\mu \in \mathcal{M}^{pc}(\mathcal{K}_2)$. Recall the notation M_{ij} given by (88), and denote $\overline{\zeta} := \int_{\mathcal{K}_2} \zeta \, d\mu(\zeta)$. Then by the change of variable formula for push forward measures, Lemma 22 and bilinearity of the minor we have

$$
\begin{aligned}
\int_{\tilde{\mathcal{K}}_2^\alpha} M_{ij}(\zeta)\, d\mu^\alpha &= \int_{\mathbb{R}^2} M_{ij}\left(\tilde{P}_2^\alpha(u,v)\right)\, d\nu \\
&= \int_{\mathbb{R}^2} \sum_{i',j'=1}^{4} c_{ii'}c_{jj'}\, M_{i'j'}\left(P_2(u,v)\right)\, d\nu \\
&= \sum_{i',j'=1}^{4} c_{ii'}c_{jj'} \int_{\mathcal{K}_2} M_{i'j'}\left(\zeta\right)\, d\mu \\
&\overset{\mu\in\mathcal{M}^{pc}(\mathcal{K}_2)}{=} \sum_{i',j'=1}^{4} c_{ii'}c_{jj'} M_{i'j'}(\overline{\zeta}).
\end{aligned}
\tag{107}
$$

On the other hand bilinearity of the minor implies that

$$
\begin{aligned}
M_{ij}&\left(\int_{\tilde{\mathcal{K}}_2^\alpha} \zeta\, d\mu^\alpha\right) = M_{ij}\left(\int_{\mathbb{R}^2} \tilde{P}_2^\alpha(u,v)\, d\nu\right) \\
&\overset{(88)}{=} \int_{\mathbb{R}^2} R_i\left(\tilde{P}_2^\alpha(u,v)\right)\, d\nu \wedge \int_{\mathbb{R}^2} R_j\left(\tilde{P}_2^\alpha(u,v)\right)\, d\nu \\
&= \sum_{i',j'=1}^{4} c_{ii'}c_{jj'} \int_{\mathbb{R}^2} R_{i'}\left(P_2(u,v)\right)\, d\nu \wedge \int_{\mathbb{R}^2} R_{j'}\left(P_2(u,v)\right)\, d\nu \\
&= \sum_{i',j'=1}^{4} c_{ii'}c_{jj'} M_{i'j'}\left(\int_{\mathbb{R}^2} P_2(u,v)\, d\nu\right) = \sum_{i',j'=1}^{4} c_{ii'}c_{jj'} M_{i'j'}\left(\int_{\mathcal{K}_2} \zeta\, d\mu\right) \\
&= \sum_{i',j'=1}^{4} c_{ii'}c_{jj'} M_{i'j'}(\overline{\zeta}) \overset{(107)}{=} \int_{\tilde{\mathcal{K}}_2^\alpha} M_{ij}(\zeta)\, d\mu^\alpha
\end{aligned}
$$

as desired. The proof of the converse implication is analogous using (102). This completes the proof of (106), and hence Lemma 20. □

8. Existence of nontrivial measure in $\mathcal{M}^{pc}(\mathcal{K}_1)$

In this section, we first construct nontrivial measures in $\mathcal{M}^{pc}(\mathcal{K}_1^{\tilde{\alpha}})$ in the case $a'(\tilde{\alpha}_2) > 0$. Then it follows from Lemma 20 that we also have nontrivial elements in $\mathcal{M}^{pc}(\mathcal{K}_1)$. More precisely, we construct nontrivial measures supported at five points that belong to the space $\mathcal{M}^{pc}(\mathcal{K}_1^{\tilde{\alpha}})$. To begin with, given $s_0, t_0 > 0$, recalling (89), we set

$$
\zeta_0 := P_1^{\tilde{\alpha}}(\tilde{\alpha}_1, \tilde{\alpha}_2) = \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix},\ \zeta_1 := P_1^{\tilde{\alpha}}(\tilde{\alpha}_1 + s_0, \tilde{\alpha}_2) = \begin{pmatrix} s_0 & 0 \\ 0 & s_0 \\ 0 & \frac{1}{2}s_0^2 \end{pmatrix},
$$

$$
\zeta_2 := P_1^{\tilde{\alpha}}(\tilde{\alpha}_1 - s_0, \tilde{\alpha}_2) = \begin{pmatrix} -s_0 & 0 \\ 0 & -s_0 \\ 0 & \frac{1}{2}s_0^2 \end{pmatrix},
$$

$$
\zeta_3 := P_1^{\tilde{\alpha}}(\tilde{\alpha}_1, \tilde{\alpha}_2 + t_0) = \begin{pmatrix} 0 & t_0 \\ a(\tilde{\alpha}_2 + t_0) - a(\tilde{\alpha}_2) & 0 \\ 0 & F(\tilde{\alpha}_2 + t_0) - F(\tilde{\alpha}_2) - a(\tilde{\alpha}_2)t_0 \end{pmatrix},
$$

and

$$\zeta_4 := P_1^{\tilde{\alpha}}(\tilde{\alpha}_1, \tilde{\alpha}_2 - t_0) = \begin{pmatrix} 0 & -t_0 \\ a(\tilde{\alpha}_2 - t_0) - a(\tilde{\alpha}_2) & 0 \\ 0 & F(\tilde{\alpha}_2 - t_0) - F(\tilde{\alpha}_2) + a(\tilde{\alpha}_2)t_0 \end{pmatrix}.$$

We first prove

Theorem 23. *Suppose $a \in C^2(\mathbb{R})$. Let $\tilde{\alpha} \in \mathbb{R}^2$ be such that $a'(\tilde{\alpha}_2) > 0$. Given $s_0, t_0 > 0$ sufficiently small depending on the function a and $\tilde{\alpha}_2$, there exists $0 < \epsilon_0 < 1$ depending on the function a, $\tilde{\alpha}_2$, s_0 and t_0 such that, for all $\epsilon \leq \epsilon_0$, there exists a collection of weights $\{[\gamma^\epsilon]_j\}_{j=0}^4 \subset \mathbb{R}_+$ with $\sum_{j=1}^4 [\gamma^\epsilon]_j = \epsilon$ and $[\gamma^\epsilon]_0 = 1 - \epsilon$ such that*

$$\mu^\epsilon := \sum_{j=0}^{4} [\gamma^\epsilon]_j \delta_{\zeta_j} \in \mathcal{M}^{pc}(\mathcal{K}_1^{\tilde{\alpha}}).$$

The proof of Theorem 23 will rely on a couple of crucial lemmas. Let us first introduce some notations. We denote by D_1, D_2, D_3 the $(1,2), (2,3), (1,3)$ minors of a 3×2 matrix, respectively. We set the matrix

$$A := \begin{pmatrix} D_1(\zeta_1) & D_1(\zeta_2) & D_1(\zeta_3) & D_1(\zeta_4) \\ D_2(\zeta_1) & D_2(\zeta_2) & D_2(\zeta_3) & D_2(\zeta_4) \\ D_3(\zeta_1) & D_3(\zeta_2) & D_3(\zeta_3) & D_3(\zeta_4) \\ 1 & 1 & 1 & 1 \end{pmatrix}. \tag{108}$$

For any $\epsilon > 0$ and $\gamma \in \mathbb{R}^4$, define

$$L^\epsilon(\gamma) := A\gamma - \begin{pmatrix} 0 \\ 0 \\ 0 \\ \epsilon \end{pmatrix}, \tag{109}$$

$$Q(\gamma) := \begin{pmatrix} D_1\left(\sum_{j=1}^4 [\gamma^\epsilon]_j \zeta_j\right) \\ D_2\left(\sum_{j=1}^4 [\gamma^\epsilon]_j \zeta_j\right) \\ D_3\left(\sum_{j=1}^4 [\gamma^\epsilon]_j \zeta_j\right) \\ 0 \end{pmatrix},$$

and

$$G^\epsilon(\gamma) := L^\epsilon(\gamma) - Q(\gamma). \tag{110}$$

Lemma 24. *Suppose $a \in C^2(\mathbb{R})$. Let $\tilde{\alpha} \in \mathbb{R}^2$ be such that $a'(\tilde{\alpha}_2) > 0$. Given $s_0, t_0 > 0$ sufficiently small depending on the function a and $\tilde{\alpha}_2$, the matrix A defined in (108) is invertible. Moreover, for any $0 < \epsilon < 1$, the unique solution γ_0^ϵ of the system*

$$L^\epsilon(\gamma) = 0 \tag{111}$$

is nonnegative componentwise. Further, there exist constants $0 < \lambda < \Lambda < \infty$ depending on the function a, $\tilde{\alpha}_2$, s_0 and t_0 such that

$$\lambda\epsilon \leq [\gamma_0^\epsilon]_i \leq \Lambda\epsilon \ \text{ for } \ i = 1, 2, 3, 4. \tag{112}$$

Proof. To simplify notation define $a_{\tilde{\alpha}_2}(t) := a(\tilde{\alpha}_2 + t) - a(\tilde{\alpha}_2)$ and $F_{\tilde{\alpha}_2}(t) := F(\tilde{\alpha}_2 + t) - F(\tilde{\alpha}_2) - a(\tilde{\alpha}_2)t$. First, explicit calculations using the formulas for ζ_j, $j = 1, 2, 3, 4$, give

$$A = \begin{pmatrix} s_0^2 & s_0^2 & -t_0 a_{\tilde{\alpha}_2}(t_0) & t_0 a_{\tilde{\alpha}_2}(-t_0) \\ 0 & 0 & a_{\tilde{\alpha}_2}(t_0) F_{\tilde{\alpha}_2}(t_0) & a_{\tilde{\alpha}_2}(-t_0) F_{\tilde{\alpha}_2}(-t_0) \\ \frac{1}{2}s_0^3 & -\frac{1}{2}s_0^3 & 0 & 0 \\ 1 & 1 & 1 & 1 \end{pmatrix}. \tag{113}$$

We claim that for any $(y_1, y_2, y_3) \neq 0 \in \mathbb{R}^3$, we have

$$\min_j \left\{ \sum_{i=1}^{3} y_i D_i(\zeta_j) \right\} < 0 \tag{114}$$

and

$$\max_j \left\{ \sum_{i=1}^{3} y_i D_i(\zeta_j) \right\} > 0. \tag{115}$$

We check (114) by an enumerative argument. Note that since $a'(\tilde{\alpha}_2) > 0$, assuming $t_0 > 0$ is small enough, we have that

$$a_{\tilde{\alpha}_2}(t_0) > 0 \text{ and } a_{\tilde{\alpha}_2}(-t_0) < 0. \tag{116}$$

Recall that $F' = a$. We also know that F is strictly convex in small neighborhood of $\tilde{\alpha}_2$ and so

$$F_{\tilde{\alpha}_2}(t_0) = F(\tilde{\alpha}_2 + t_0) - F(\tilde{\alpha}_2) - a(\tilde{\alpha}_2)t_0 > 0 \text{ and } F_{\tilde{\alpha}_2}(-t_0) = F(\tilde{\alpha}_2 - t_0) - F(\tilde{\alpha}_2) + a(\tilde{\alpha}_2)t_0 > 0. \tag{117}$$

By carefully checking out the columns of A and using (116), (117) we see that

(1) If $y_1 > 0, y_2 \geq 0, y_3 \in \mathbb{R}$, we have $\sum_{i=1}^{3} y_i D_i(\zeta_4) < 0$.
(2) If $y_1 > 0, y_2 \leq 0, y_3 \in \mathbb{R}$, we have $\sum_{i=1}^{3} y_i D_i(\zeta_3) < 0$.
(3) If $y_1 < 0, y_3 \geq 0, y_2 \in \mathbb{R}$, we have $\sum_{i=1}^{3} y_i D_i(\zeta_2) < 0$.
(4) If $y_1 < 0, y_3 \leq 0, y_2 \in \mathbb{R}$, we have $\sum_{i=1}^{3} y_i D_i(\zeta_1) < 0$.
(5) If $y_1 = 0$, and
  (a) $y_2 > 0, y_3 \in \mathbb{R}$, we have $\sum_{i=1}^{3} y_i D_i(\zeta_4) = y_2 D_2(\zeta_4) < 0$;
  (b) $y_2 < 0, y_3 \in \mathbb{R}$, we have $\sum_{i=1}^{3} y_i D_i(\zeta_3) = y_2 D_2(\zeta_3) < 0$;
  (c) $y_2 = 0$, $y_3 > 0$, we have $\sum_{i=1}^{3} y_i D_i(\zeta_2) = y_3 D_3(\zeta_2) < 0$;
  (d) $y_2 = 0$, $y_3 < 0$, we have $\sum_{i=1}^{3} y_i D_i(\zeta_1) = y_3 D_3(\zeta_1) < 0$.

The above (1), (2) cover all cases when $y_1 > 0$, and (3), (4) cover all cases when $y_1 < 0$. For the case $y_1 = 0$, we have either $y_2 \neq 0$ or $y_2 = 0$. The former is covered by (5a), (5b), and the latter is covered by (5c), (5d). Therefore the above enumerative argument shows (114), and (115) is equivalent to (114). As we will briefly sketch, similar to the arguments in Lemma 12 using the Farkas-Minkowski Lemma, condition (114) guarantees that the system (111) has a solution that is nonnegative componentwise. Specifically, let a_1, a_2, a_3, a_4 denote the columns of the matrix A. Given any $y \in \mathbb{R}^4$ such that $y \cdot a_i \geq 0$ for any $i = 1, 2, 3, 4$, we must have that $y_4 > 0$. Hence the vector $\begin{pmatrix} 0 \\ 0 \\ 0 \\ \epsilon \end{pmatrix}$ must be in the cone $\{\sum_{i=1}^{4} \lambda_i a_i : \lambda_i \in \mathbb{R}_+\}$. We deduce from the Farkas-Minkowski Lemma that (111) has a nonnegative solution.

To see the matrix A is invertible, consider the following system

$$A^T y = 0. \tag{118}$$

We claim that the system (118) has only the trivial solution. Indeed, let $y = (y_1, y_2, y_3, y_4) \in \mathbb{R}^4$ be a solution of (118), i.e.,

$$\sum_{i=1}^{3} y_i D_i(\zeta_j) + y_4 = 0 \text{ for } j = 1, 2, 3, 4.$$

It follows from (114) and (115) that $y_4 = 0$, and therefore by (114) and (115) again, we have $y_i = 0$ for $i = 1, 2, 3$. It follows that A^T, and hence A, are invertible.

Finally, we show (112). Let γ_0^ϵ be the unique solution of (111). We already know that γ_0^ϵ is nonnegative componentwise. We will show that all components of γ_0^ϵ are strictly positive.

We argue by contradiction. Suppose $[\gamma_0^\epsilon]_1 = 0$ or $[\gamma_0^\epsilon]_2 = 0$. Then using the third row of (111), we have $\frac{s_0^3}{2}([\gamma_0^\epsilon]_1 - [\gamma_0^\epsilon]_2) \overset{(113)}{=} 0$ and therefore we have $[\gamma_0^\epsilon]_1 = [\gamma_0^\epsilon]_2 = 0$. Now using the first two rows of (111) (see (113)), we have

$$\begin{pmatrix} -t_0 a_{\tilde{\alpha}_2}(t_0) & t_0 a_{\tilde{\alpha}_2}(-t_0) \\ a_{\tilde{\alpha}_2}(t_0) F_{\tilde{\alpha}_2}(t_0) & a_{\tilde{\alpha}_2}(-t_0) F_{\tilde{\alpha}_2}(-t_0) \end{pmatrix} \begin{pmatrix} [\gamma_0^\epsilon]_3 \\ [\gamma_0^\epsilon]_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

It is clear from (116), (117) that the matrix $\begin{pmatrix} -t_0 a_{\tilde{\alpha}_2}(t_0) & t_0 a_{\tilde{\alpha}_2}(-t_0) \\ a_{\tilde{\alpha}_2}(t_0) F_{\tilde{\alpha}_2}(t_0) & a_{\tilde{\alpha}_2}(-t_0) F_{\tilde{\alpha}_2}(-t_0) \end{pmatrix}$ is invertible, and hence $[\gamma_0^\epsilon]_3 = [\gamma_0^\epsilon]_4 = 0$. But now we have $\gamma_0^\epsilon = 0$, which contradicts the fourth row of (111). This contradiction implies $[\gamma_0^\epsilon]_1 > 0$ and $[\gamma_0^\epsilon]_2 > 0$. A similar argument yields $[\gamma_0^\epsilon]_3 > 0$ and $[\gamma_0^\epsilon]_4 > 0$. Since $[\gamma_0^\epsilon]_i = \epsilon [A^{-1}]_{i4}$, $i = 1, 2, 3, 4$, it follows that $[A^{-1}]_{i4} > 0$ for all i. Now we define

$$\lambda := \min_i \{[A^{-1}]_{i4}\} \quad \text{and} \quad \Lambda := \max_i \{[A^{-1}]_{i4}\}.$$

It is clear that $0 < \lambda \le \Lambda < \infty$ and (112) is satisfied. Note that A^{-1} is a fixed matrix independent of ϵ, and so are λ and Λ independent of ϵ. □

Lemma 25. *Suppose $a \in C^2(\mathbb{R})$. Let $\tilde{\alpha} \in \mathbb{R}^2$ be such that $a'(\tilde{\alpha}_2) > 0$. Given $s_0, t_0 > 0$ sufficiently small depending on the function a and $\tilde{\alpha}_2$, there exists $0 < \epsilon_0 < 1$ sufficiently small such that for all $0 < \epsilon \le \epsilon_0$, the system*

$$G^\epsilon(\gamma) = 0 \tag{119}$$

has a nonnegative solution.

Proof. Given a sufficiently small $0 < \epsilon < 1$ whose size will be specified later, by Lemma 24, the linear system $L^\epsilon(\gamma) = 0$ has a unique nonnegative solution γ_0^ϵ that satisfies the estimate (112). We will find a solution to (119) by iteration. For all $k \in \mathbb{N}^+$, define

$$\Delta_k^\epsilon := A^{-1}\left(-G^\epsilon(\gamma_{k-1}^\epsilon)\right) \text{ and } \gamma_k^\epsilon := \gamma_{k-1}^\epsilon + \Delta_k^\epsilon. \tag{120}$$

Then we have

$$\begin{aligned} G^\epsilon(\gamma_k^\epsilon) &\overset{(110)}{=} L^\epsilon(\gamma_k^\epsilon) - Q(\gamma_k^\epsilon) \\ &\overset{(109),(120)}{=} A\left(\gamma_{k-1}^\epsilon + \Delta_k^\epsilon\right) - (0,0,0,\epsilon)^T - Q(\gamma_k^\epsilon) \\ &\overset{(109)}{=} L^\epsilon(\gamma_{k-1}^\epsilon) + A(\Delta_k^\epsilon) - Q(\gamma_k^\epsilon) \\ &\overset{(110)}{=} G^\epsilon\left(\gamma_{k-1}^\epsilon\right) + A\left(\Delta_k^\epsilon\right) + Q(\gamma_{k-1}^\epsilon) - Q(\gamma_k^\epsilon) \\ &\overset{(120)}{=} G^\epsilon\left(\gamma_{k-1}^\epsilon\right) - G^\epsilon\left(\gamma_{k-1}^\epsilon\right) + Q(\gamma_{k-1}^\epsilon) - Q(\gamma_k^\epsilon) \\ &= Q(\gamma_{k-1}^\epsilon) - Q(\gamma_k^\epsilon). \end{aligned} \tag{121}$$

Now let us estimate the sizes of Δ_k^ϵ and γ_k^ϵ. First note that $D_i\left(\sum_{j=1}^4 [\gamma]_j \zeta_j\right)$, $i = 1, 2, 3$, is a fixed quadratic function of γ whose coefficients depend only on the function a, $\tilde{\alpha}_2$, s_0 and t_0. Therefore, for all $r > 0$ and $\gamma, \tilde{\gamma} \in B_r(0) \subset \mathbb{R}^4$, we have

$$\|Q(\gamma)\| \le C_1 \|\gamma\|^2 \tag{122}$$

and

$$\|Q(\gamma) - Q(\tilde{\gamma})\| \le \sup_{z \in B_r(0)} \|DQ(z)\| \cdot \|\gamma - \tilde{\gamma}\| \le C_1 r \|\gamma - \tilde{\gamma}\|, \tag{123}$$

where the above constant C_1 depends only on the coefficients of Q and therefore does not depend on ϵ or $\gamma, \tilde{\gamma}, r$. Let $\theta > 0$ be sufficiently small such that

$$\sum_{p=1}^{\infty} 2^{p-1}\theta^p \leq \frac{\lambda}{4\Lambda} < 1. \tag{124}$$

Clearly such θ exists. We denote

$$C_2 := \left\| A^{-1} \right\| C_1. \tag{125}$$

Let $\epsilon_0 := \frac{\theta}{2C_2\Lambda} > 0$. Now for all $0 < \epsilon \leq \epsilon_0$, it follows from (112) that

$$C_2 \|\gamma_0^\epsilon\| \leq C_2 \cdot 2\Lambda\epsilon_0 = \theta. \tag{126}$$

We claim that

$$\|\Delta_k^\epsilon\| \leq 2^{k-1}\theta^k \|\gamma_0^\epsilon\| \tag{127}$$

and

$$\|\gamma_k^\epsilon\| \leq \left(1 + \sum_{p=1}^{k} 2^{p-1}\theta^p\right) \|\gamma_0^\epsilon\| < 2\|\gamma_0^\epsilon\|. \tag{128}$$

We show this by induction. Recall that $L^\epsilon(\gamma_0^\epsilon) = 0$. We deduce from (122) that

$$\|-G^\epsilon(\gamma_0^\epsilon)\| \overset{(110)}{=} \|Q(\gamma_0^\epsilon)\| \overset{(122)}{\leq} C_1 \|\gamma_0^\epsilon\|^2. \tag{129}$$

It follows from this, (120) and (125), (126) that

$$\|\Delta_1^\epsilon\| \overset{(120),(129)}{\leq} \left\|A^{-1}\right\| \cdot C_1 \|\gamma_0^\epsilon\|^2 \overset{(125)}{=} C_2 \|\gamma_0^\epsilon\|^2 \overset{(126)}{\leq} \theta \|\gamma_0^\epsilon\| \tag{130}$$

and therefore

$$\|\gamma_1^\epsilon\| \overset{(120)}{\leq} (1+\theta)\|\gamma_0^\epsilon\|. \tag{131}$$

So by (130), (131) we have that (127), (128) hold for $k = 1$. Now suppose (127), (128) hold for $k \geq 1$. Using (121), (123) and the induction assumption, we have

$$\begin{aligned} \|G^\epsilon(\gamma_k^\epsilon)\| &\overset{(121)}{=} \|Q(\gamma_{k-1}^\epsilon) - Q(\gamma_k^\epsilon)\| \\ &\overset{(123),(128)}{\leq} C_1 \cdot 2\|\gamma_0^\epsilon\| \|\gamma_{k-1}^\epsilon - \gamma_k^\epsilon\| \\ &\overset{(120)}{\leq} C_1 \cdot 2\|\gamma_0^\epsilon\| \cdot \|\Delta_k^\epsilon\| \overset{(127)}{\leq} C_1 2^k \theta^k \|\gamma_0^\epsilon\|^2. \end{aligned} \tag{132}$$

It follows from (120) and (126) that

$$\left\|\Delta_{k+1}^\epsilon\right\| \overset{(120)}{\leq} \left\|A^{-1}\right\| \cdot \|G^\epsilon(\gamma_k^\epsilon)\| \overset{(132),(125)}{\leq} C_2 2^k \theta^k \|\gamma_0^\epsilon\|^2 \overset{(126)}{\leq} 2^k \theta^{k+1} \|\gamma_0^\epsilon\| \tag{133}$$

and

$$\left\|\gamma_{k+1}^\epsilon\right\| \overset{(120)}{\leq} \|\gamma_0^\epsilon\| + \sum_{p=1}^{k+1} \left\|\Delta_p^\epsilon\right\| \overset{(127),(133)}{\leq} \left(1 + \sum_{p=1}^{k+1} 2^{p-1}\theta^p\right) \|\gamma_0^\epsilon\| \overset{(124)}{\leq} 2\|\gamma_0^\epsilon\|.$$

Thus we have established (127), (128) for general k.

Since $\{\gamma_k^\epsilon\}_k$ forms a bounded sequence, it has a convergent subsequence such that (without relabeling)

$$\lim_{k\to\infty} \gamma_k^\epsilon = \bar{\gamma}^\epsilon$$

for some $\bar{\gamma}^\epsilon$. We claim that $\bar{\gamma}^\epsilon$ is a nonnegative solution to (119). From the estimates (132) and (124), we have

$$\|G^\epsilon(\gamma_k^\epsilon)\| \overset{(132)}{\leq} C_1 2^k \theta^k \|\gamma_0^\epsilon\|^2 \to 0 \quad \text{as} \quad k \to \infty.$$

Since G^ϵ is continuous, we have

$$\|G^\epsilon(\bar{\gamma}^\epsilon)\| = \lim_{k\to\infty} \|G^\epsilon(\gamma_k^\epsilon)\| = 0.$$

It only remains to show that $\bar{\gamma}^\epsilon$ is nonnegative componentwise. We deduce from (127), (112) and (124) that

$$\|\gamma_k^\epsilon - \gamma_0^\epsilon\| \overset{(120)}{\leq} \sum_{p=1}^{k} \left\|\Delta_p^\epsilon\right\| \overset{(127)}{\leq} \sum_{p=1}^{k} 2^{p-1}\theta^p \|\gamma_0^\epsilon\| \overset{(124),(112)}{\leq} \frac{\lambda}{4\Lambda}\cdot 2\Lambda\epsilon = \frac{\lambda}{2}\epsilon. \tag{134}$$

We know from (112) that each component of γ_0^ϵ is bounded below by $\lambda\epsilon$. This together with (134) shows that all components of γ_k^ϵ are bounded below by $\frac{\lambda}{2}\epsilon$ for all k. Therefore, the same holds for $\bar{\gamma}^\epsilon$. In particular, $\bar{\gamma}^\epsilon$ is nonnegative. □

Proof of Theorem 23. Given $0 < \epsilon \leq \epsilon_0 < 1$, let $\bar{\gamma}^\epsilon = ([\bar{\gamma}^\epsilon]_1, [\bar{\gamma}^\epsilon]_2, [\bar{\gamma}^\epsilon]_3, [\bar{\gamma}^\epsilon]_4)$ be the nonnegative solution of (119) found in Lemma 25. Then we have $\sum_{j=1}^4 [\bar{\gamma}^\epsilon]_j = \epsilon$. Define $[\bar{\gamma}^\epsilon]_0 := 1-\epsilon$. Then we have $[\bar{\gamma}^\epsilon]_j \geq 0$ for all $j = 0,1,2,3,4$ and $\sum_{j=0}^4 [\bar{\gamma}^\epsilon]_j = 1$. Now we define

$$\mu^\epsilon := \sum_{j=0}^{4} [\bar{\gamma}^\epsilon]_j \delta_{\zeta_j}.$$

It is clear that μ^ϵ is a probability measure. Since $0 < \epsilon < 1$, μ^ϵ is nontrivial. Since ζ_0 is the trivial matrix and $\bar{\gamma}^\epsilon$ solves the system (119), we have

$$\sum_{j=0}^{4} [\bar{\gamma}^\epsilon]_j D_i(\zeta_j) = \sum_{j=1}^{4} [\bar{\gamma}^\epsilon]_j D_i(\zeta_j) \overset{(119),(110)}{=} D_i\left(\sum_{j=1}^{4} [\bar{\gamma}^\epsilon]_j \zeta_j\right) = D_i\left(\sum_{j=0}^{4} [\bar{\gamma}^\epsilon]_j \zeta_j\right)$$

for all $i = 1,2,3$. This shows that $\mu^\epsilon \in \mathcal{M}^{pc}(\mathcal{K}_1^{\tilde{\alpha}})$. □

Proof of Theorem 5 completed. We first consider the case $a'(\tilde{\alpha}_2) > 0$. Given $0 < \epsilon \leq \epsilon_0 < 1$, let $\mu^\epsilon \in \mathcal{M}^{pc}(\mathcal{K}_1^{\tilde{\alpha}})$ be the measure constructed in Theorem 23. Let $\nu^\epsilon := \left((P_1^{\tilde{\alpha}})^{-1}\right)_\sharp \mu^\epsilon$. Note that since $P_1^{\tilde{\alpha}}$ is a bijection, we have $\mu^\epsilon = (P_1^{\tilde{\alpha}})_\sharp \nu^\epsilon$. Define $\tilde{\mu}^\epsilon := (P_1)_\sharp \nu^\epsilon$. Since $(P_1^{\tilde{\alpha}})_\sharp \nu^\epsilon = \mu^\epsilon \in \mathcal{M}^{pc}(\mathcal{K}_1^{\tilde{\alpha}})$, it follows from Lemma 20 that $\tilde{\mu}^\epsilon = (P_1)_\sharp \nu^\epsilon \in \mathcal{M}^{pc}(\mathcal{K}_1)$. Since P_1 and $P_1^{\tilde{\alpha}}$ are both bijections, it is clear that $\tilde{\mu}^\epsilon$ is also supported at five points, and hence is nontrivial. Further, by choosing s_0, t_0 sufficiently small in Theorem 23, one can make the support of μ^ϵ sufficiently small. It follows from Lemma 19 that the support of $\tilde{\mu}^\epsilon$ can be made sufficiently small. This establishes the case where $a'(\tilde{\alpha}_2) > 0$.

Now suppose $a'(\tilde{\alpha}_2) < 0$, then for some $\delta > 0$ sufficiently small we have that

$$(v_2 - v_1)(a(v_2) - a(v_1)) < 0 \text{ for any } v_1, v_2 \in (a(\tilde{\alpha}_2) - \delta, a(\tilde{\alpha}_2) + \delta). \tag{135}$$

Let $\mathcal{K}_0 := \left\{ \begin{pmatrix} u & v \\ a(v) & u \end{pmatrix} : u, v \in \mathbb{R} \right\}$. Note that if $\det \begin{pmatrix} u_2-u_1 & v_2-v_1 \\ a(v_2)-a(v_1) & u_2-u_1 \end{pmatrix} = 0$ for some (u_1, v_1) and (u_2, v_2) in $B_\delta(\tilde{\alpha})$, then

$$(u_2 - u_1)^2 - (v_2 - v_1)(a(v_2) - a(v_1)) = 0,$$

which by (135) implies $u_1 = u_2$ and $v_1 = v_2$. Thus, for sufficiently small neighborhood $\tilde{U}$ of $\begin{pmatrix} \tilde{\alpha}_1 & \tilde{\alpha}_2 \\ a(\tilde{\alpha}_2) & \tilde{\alpha}_1 \end{pmatrix}$, $\mathcal{K}_0 \cap \tilde{U}$ does not contain Rank-1 connections and therefore $\det(X - Y)$ does not change sign on $(\mathcal{K}_0 \cap \tilde{U}) \times (\mathcal{K}_0 \cap \tilde{U})$ by Lemma 1 in [Sv 93]. By [Sv 93] Lemma 3 we have that $\mathcal{M}^{pc}(\mathcal{K}_0 \cap \tilde{U})$ consists of Dirac measures only. As $\mathcal{M}^{pc}(\mathcal{K}_1 \cap U)$ can be embedded in $\mathcal{M}^{pc}(\mathcal{K}_0 \cap \tilde{U})$, this completes the proof of the case $a'(\tilde{\alpha}_2) < 0$, and hence the proof of Theorem 5. □

9. A more direct proof of DiPerna's theorem

We will present a direct proof of triviality of $\mathcal{M}^{pc}(\mathcal{K}_2)$ (Theorem 7) motivated by the perspective of searching for linear combinations of minors that are nonnegative. As a simple consequence, we give a proof of DiPerna's Theorem 6.

9.1. **Proof of Theorem 7.** The proof of Theorem 7 requires a couple of auxiliary lemmas.

Lemma 26. *Suppose $a \in C^\infty(\mathbb{R})$. Given $\tilde{\alpha} = (\tilde{\alpha}_1, \tilde{\alpha}_2)$ such that $a'(\tilde{\alpha}_2) > 0$ and $a''(\tilde{\alpha}_2) \neq 0$, there exists $\delta_1 > 0$ depending on $\tilde{\alpha}$ and the function a such that for all $\alpha = (\alpha_1, \alpha_2) \in B_{\delta_1}(\tilde{\alpha})$ and $(u, v) \in B_{\delta_1}(\tilde{\alpha})$, we have*

$$
\begin{aligned}
&- M_{34}(P_2^\alpha(u,v)) - \frac{2a'(\alpha_2)}{a''(\alpha_2)} M_{23}(P_2^\alpha(u,v)) - \frac{2(a'(\alpha_2))^2}{a''(\alpha_2)} M_{14}(P_2^\alpha(u,v)) \\
&\geq \frac{1}{8}|u-\alpha_1|^4 + \frac{a'(\tilde{\alpha}_2)^2}{48}|v-\alpha_2|^4.
\end{aligned}
\tag{136}
$$

Proof. Given $\delta_1 > 0$ sufficiently small whose size will be specified later, let $\alpha = (\alpha_1, \alpha_2)$, $(u, v) \in B_{\delta_1}(\tilde{\alpha})$. We denote $s := u - \alpha_1$, $t := v - \alpha_2$. Then using the definition of P_2^α in (90) we have

$$
P_2^\alpha(u,v) = \begin{pmatrix} s & t \\ a(t+\alpha_2) - a(\alpha_2) & s \\ s\,(a(t+\alpha_2) - a(\alpha_2)) & \frac{1}{2}s^2 + F(t+\alpha_2) - F(\alpha_2) - a(\alpha_2)t \\ \frac{1}{2}s^2 + F(\alpha_2) - F(t+\alpha_2) + a(t+\alpha_2)t & st \end{pmatrix}.
$$

Step 1. We write out the Taylor expansion of the entries of $P_2^\alpha(u,v)$ and calculate the Taylor expansion of the minors up to order four. This idea of looking at terms up to order four comes from the work of DiPerna [DP 85]. First we have

$$
a(\alpha_2 + t) - a(\alpha_2) \approx a'(\alpha_2)t + \frac{a''(\alpha_2)}{2}t^2 + \frac{a'''(\alpha_2)}{6}t^3, \tag{137}
$$

$$
s\,(a(\alpha_2+t) - a(\alpha_2)) \approx a'(\alpha_2)st + \frac{a''(\alpha_2)}{2}st^2, \tag{138}
$$

$$
F(\alpha_2+t) - F(\alpha_2) - a(\alpha_2)t \approx \frac{a'(\alpha_2)}{2}t^2 + \frac{a''(\alpha_2)}{6}t^3, \tag{139}
$$

and

$$
\begin{aligned}
F(\alpha_2) - F(\alpha_2+t) + a(\alpha_2+t)t &= -\left(F(\alpha_2+t) - F(\alpha_2) - a(\alpha_2)t\right) + \left(a(\alpha_2+t) - a(\alpha_2)\right)t \\
&\overset{(139)}{\approx} -\frac{a'(\alpha_2)}{2}t^2 - \frac{a''(\alpha_2)}{6}t^3 + a'(\alpha_2)t^2 + \frac{a''(\alpha_2)}{2}t^3 \\
&= \frac{a'(\alpha_2)}{2}t^2 + \frac{a''(\alpha_2)}{3}t^3.
\end{aligned}
\tag{140}
$$

In the above calculations, we have omitted terms that are of order four or higher. Putting the above calculations together, we obtain

$$
P_2^\alpha(u,v) \approx \begin{pmatrix} s & t \\ a'(\alpha_2)t + \frac{a''(\alpha_2)}{2}t^2 + \frac{a'''(\alpha_2)}{6}t^3 & s \\ a'(\alpha_2)st + \frac{a''(\alpha_2)}{2}st^2 & \frac{s^2}{2} + \frac{a'(\alpha_2)}{2}t^2 + \frac{a''(\alpha_2)}{6}t^3 \\ \frac{s^2}{2} + \frac{a'(\alpha_2)}{2}t^2 + \frac{a''(\alpha_2)}{3}t^3 & st \end{pmatrix}.
\tag{141}
$$

Step 2. We denote by $M^p_{jk}\left(P^\alpha_2(u,v)\right)$ the p-th order terms in the minor $M_{jk}\left(P^\alpha_2(u,v)\right)$, and we calculate the $(1,4),(2,3)$ and $(3,4)$ minors of (141) up to order four.

$$\begin{gathered}
M^2_{14}\left(P^\alpha_2(u,v)\right)=0,\\
M^3_{14}\left(P^\alpha_2(u,v)\right)=s^2t-\frac{1}{2}s^2t-\frac{a'(\alpha_2)}{2}t^3=\frac{1}{2}s^2t-\frac{a'(\alpha_2)}{2}t^3,\\
M^4_{14}\left(P^\alpha_2(u,v)\right)=-\frac{a''(\alpha_2)}{3}t^4,
\end{gathered}$$

$$\begin{gathered}
M^2_{23}\left(P^\alpha_2(u,v)\right)=0,\\
M^3_{23}\left(P^\alpha_2(u,v)\right)=\frac{a'(\alpha_2)}{2}s^2t+\frac{a'(\alpha_2)^2}{2}t^3-a'(\alpha_2)s^2t=-\frac{a'(\alpha_2)}{2}s^2t+\frac{a'(\alpha_2)^2}{2}t^3,\\
\begin{aligned}
M^4_{23}\left(P^\alpha_2(u,v)\right)&=\frac{a'(\alpha_2)a''(\alpha_2)}{6}t^4+\frac{a''(\alpha_2)}{2}t^2\left(\frac{s^2}{2}+\frac{a'(\alpha_2)}{2}t^2\right)-\frac{a''(\alpha_2)}{2}s^2t^2\\
&=\frac{5}{12}a'(\alpha_2)a''(\alpha_2)t^4-\frac{a''(\alpha_2)}{4}s^2t^2,
\end{aligned}\\
M^2_{34}\left(P^\alpha_2(u,v)\right)=M^3_{34}\left(P^\alpha_2(u,v)\right)=0,
\end{gathered}$$

$$M^4_{34}\left(P^\alpha_2(u,v)\right)=a'(\alpha_2)s^2t^2-\left(\frac{s^2}{2}+\frac{a'(\alpha_2)}{2}t^2\right)^2=-\frac{1}{4}s^4+\frac{a'(\alpha_2)}{2}s^2t^2-\frac{a'(\alpha_2)^2}{4}t^4.$$

Step 3. We show (136). Without loss of generality, we may assume that δ_1 is sufficiently small such that $a''(\alpha_2)\neq 0$ for all $\alpha\in B_{\delta_1}(\tilde\alpha)$, as otherwise we may shrink the size of δ_1 until this is satisfied. Let us denote by

$$R^p:=-M^p_{34}(P^\alpha_2(u,v))-\frac{2a'(\alpha_2)}{a''(\alpha_2)}M^p_{23}(P^\alpha_2(u,v))-\frac{2a'(\alpha_2)^2}{a''(\alpha_2)}M^p_{14}(P^\alpha_2(u,v))$$

for $p=2,3,4$. From the calculations in Step 2, we have

$$\begin{aligned}
R^2&=0,\\
R^3&=-\frac{2a'(\alpha_2)}{a''(\alpha_2)}\left(-\frac{a'(\alpha_2)}{2}s^2t+\frac{a'(\alpha_2)^2}{2}t^3\right)-\frac{2a'(\alpha_2)^2}{a''(\alpha_2)}\left(\frac{s^2t}{2}-\frac{a'(\alpha_2)}{2}t^3\right)=0,\\
R^4&=\frac{2a'(\alpha_2)^2}{a''(\alpha_2)}\frac{a''(\alpha_2)}{3}t^4-\frac{2a'(\alpha_2)}{a''(\alpha_2)}\left(\frac{5}{12}a'(\alpha_2)a''(\alpha_2)t^4-\frac{a''(\alpha_2)}{4}s^2t^2\right)\\
&\quad-\left(-\frac{1}{4}s^4+\frac{a'(\alpha_2)}{2}s^2t^2-\frac{a'(\alpha_2)^2}{4}t^4\right)=\frac{1}{4}s^4+\frac{a'(\alpha_2)^2}{12}t^4.
\end{aligned}$$

Therefore, we have

$$\begin{aligned}
-M_{34}(P^\alpha_2(u,v))&-\frac{2a'(\alpha_2)}{a''(\alpha_2)}M_{23}(P^\alpha_2(u,v))-\frac{2a'(\alpha_2)^2}{a''(\alpha_2)}M_{14}(P^\alpha_2(u,v))\\
&=\frac{1}{4}s^4+\frac{a'(\alpha_2)^2}{12}t^4+\text{Error terms},
\end{aligned}$$

where

$$|\text{Error terms}|\leq\tilde C\sum_{i+j=5}s^it^j. \tag{142}$$

Note that, by Taylor's theorem, the above constant $\tilde C$ depends only on the functions in the matrix $P^\alpha_2(u,v)$ and their third order derivatives in $B_{\delta_1}(\tilde\alpha)$, and hence can be bounded above by the supremum of the absolute values of these in a sufficiently large ball centered at $\tilde{\alpha}$. In particular, the constant $\tilde{C}$ can be made independent of δ_1, α and (u,v). Now we choose $\delta_1 > 0$ sufficiently small such that $a'(\alpha_2)^2 \geq \frac{a'(\tilde{\alpha}_2)^2}{2}$ for all $\alpha \in B_{\delta_1}(\tilde{\alpha})$ and

$$\tilde{C} \sum_{i+j=5} s^i t^j \leq \tilde{C}\, 2\delta_1 \sum_{i+j=4} s^i t^j \leq \frac{1}{8}s^4 + \frac{a'(\tilde{\alpha}_2)^2}{48}t^4.$$

It follows that

$$\begin{aligned} -M_{34}(P_2^\alpha(u,v)) - \frac{2a'(\alpha_2)}{a''(\alpha_2)}M_{23}(P_2^\alpha(u,v)) - \frac{2a'(\alpha_2)^2}{a''(\alpha_2)}M_{14}(P_2^\alpha(u,v)) \\ \geq \frac{1}{4}s^4 + \frac{a'(\tilde{\alpha}_2)^2}{24}t^4 - \left(\frac{1}{8}s^4 + \frac{a'(\tilde{\alpha}_2)^2}{48}t^4\right) = \frac{1}{8}s^4 + \frac{a'(\tilde{\alpha}_2)^2}{48}t^4 \end{aligned}$$

for all $\alpha, (u,v) \in B_{\delta_1}(\tilde{\alpha})$ and hence we have established (136). □

Lemma 27. *Suppose $a \in C^\infty(\mathbb{R})$. Let $\tilde{\alpha} = (\tilde{\alpha}_1, \tilde{\alpha}_2)$ be such that $a'(\tilde{\alpha}_2) > 0$ and $a''(\tilde{\alpha}_2) \neq 0$. Then there exists $\delta_2 > 0$ depending on $\tilde{\alpha}$ and the function a such that for all $0 < \delta \leq \delta_2$ and all $\nu \in \mathcal{P}(\mathbb{R}^2)$ with $(P_2)_\sharp \nu \in \mathcal{M}^{pc}(\mathcal{K}_2)$ and*

$$\mathrm{Spt}\nu \subset B_\delta(\tilde{\alpha}), \tag{143}$$

we have that

$$-M_{34}(\overline{P}) - \frac{2a'(\overline{v})}{a''(\overline{v})}M_{23}(\overline{P}) - \frac{2a'(\overline{v})^2}{a''(\overline{v})}M_{14}(\overline{P}) \leq C\delta \int |v - \overline{v}|^4\, d\nu \tag{144}$$

for some constant C independent of δ and ν, where $\overline{u} = \int u d\nu$, $\overline{v} = \int v d\nu$ and (recall the definition of P_2^α in (90))

$$\overline{P} = \int P_2^{(\overline{u},\overline{v})}(u,v) d\nu. \tag{145}$$

Proof. Let $\delta_2 > 0$ be sufficiently small such that $a'(\alpha_2) > 0$ and $a''(\alpha_2) \neq 0$ for all $\alpha = (\alpha_1, \alpha_2) \in B_{\delta_2}(\tilde{\alpha})$. Given $0 < \delta \leq \delta_2$, let $\nu \in \mathcal{P}(\mathbb{R}^2)$ be such that $(P_2)_\sharp \nu \in \mathcal{M}^{pc}(\mathcal{K}_2)$ and $\mathrm{Spt}\nu \subset B_\delta(\tilde{\alpha})$. In the following we use $\tilde{C}$ to denote constants that are independent of δ and ν. Note that by Lemma 20 we have $(P_2^{(\overline{u},\overline{v})})_\sharp \nu \in \mathcal{M}^{pc}(\mathcal{K}_2^{(\overline{u},\overline{v})})$. From the expression of $P_2^{(\overline{u},\overline{v})}$ it is clear that

$$\left[\overline{P}\right]_{11} = 0 \text{ and } \left[\overline{P}\right]_{12} = 0. \tag{146}$$

So we have

$$\begin{aligned} \int \left((u-\overline{u})^2 - (a(v) - a(\overline{v}))(v - \overline{v})\right) d\nu &= \int M_{12}\left(P_2^{(\overline{u},\overline{v})}(u,v)\right) d\nu \\ &\overset{(93)}{=} M_{12}\left(\int P_2^{(\overline{u},\overline{v})}(u,v) d\nu\right) \overset{(145)}{=} M_{12}(\overline{P}) \overset{(146)}{=} 0. \end{aligned} \tag{147}$$

Now by Taylor's theorem we have that $a(v) = a(\overline{v}) + a'(\overline{v})(v - \overline{v}) + \frac{a''(d_v)}{2}(v-\overline{v})^2$ for some $d_v \in [\overline{v}, v]$. So we deduce from (147) that

$$\left|\int \left((u-\overline{u})^2 - a'(\overline{v})(v-\overline{v})^2\right) d\nu\right| \leq \tilde{C}\int |v - \overline{v}|^3\, d\nu, \tag{148}$$

where the above constant $\tilde{C}$ can be made independent of δ and ν for similar reasons as in the arguments for the constant in (142). It follows from (137)-(140) that

$$\left|\int (a(v) - a(\overline{v}))\, d\nu - \int \frac{a''(\overline{v})}{2}(v-\overline{v})^2\, d\nu\right| \leq \tilde{C}\int |v-\overline{v}|^3\, d\nu, \tag{149}$$

$$\left|\int (u-\overline{u})(a(v) - a(\overline{v}))\, d\nu - \int a'(\overline{v})(u-\overline{u})(v-\overline{v})\, d\nu\right| \leq \tilde{C}\int \left|(u-\overline{u})(v-\overline{v})^2\right| d\nu, \tag{150}$$

$$\begin{aligned}&\left|\int\left(\frac{(u-\overline{u})^2}{2}+F(v)-F(\overline{v})-a(\overline{v})(v-\overline{v})\right)d\nu-\frac{1}{2}\int\left((u-\overline{u})^2+a'(\overline{v})\,(v-\overline{v})^2\right)d\nu\right|\\&\qquad\leq\tilde{C}\int|v-\overline{v}|^3\,d\nu.\end{aligned}\tag{151}$$

and

$$\begin{aligned}&\left|\int\left(\frac{(u-\overline{u})^2}{2}+F(\overline{v})-F(v)-a(v)(\overline{v}-v)\right)d\nu-\frac{1}{2}\int\left((u-\overline{u})^2+a'(\overline{v})\,(v-\overline{v})^2\right)d\nu\right|\\&\qquad\leq\tilde{C}\int|v-\overline{v}|^3\,d\nu.\end{aligned}\tag{152}$$

It is clear from (148) that

$$\left|\frac{1}{2}\int\left((u-\overline{u})^2+a'(\overline{v})(v-\overline{v})^2\right)d\nu-\int(u-\overline{u})^2\,d\nu\right|\leq\tilde{C}\int|v-\overline{v}|^3\,d\nu.\tag{153}$$

Using (149)-(153) to clean up $\overline{P}$ we obtain

$$\overline{P}=\begin{pmatrix}0&0\\\int\frac{a''(\overline{v})}{2}\,(v-\overline{v})^2\,d\nu+E_{21}&0\\\int a'(\overline{v})\,(u-\overline{u})\,(v-\overline{v})\,d\nu+E_{31}&\int(u-\overline{u})^2\,d\nu+E_{32}\\\int(u-\overline{u})^2\,d\nu+E_{41}&\int(u-\overline{u})\,(v-\overline{v})\,d\nu\end{pmatrix},$$

where

$$|E_{ij}|\leq\tilde{C}\left(\int\left|(u-\overline{u})\,(v-\overline{v})^2\right|d\nu+\int|v-\overline{v}|^3\,d\nu\right).\tag{154}$$

Let

$$\widetilde{P}:=\begin{pmatrix}0&0\\\int\frac{a''(\overline{v})}{2}\,(v-\overline{v})^2\,d\nu&0\\\int a'(\overline{v})\,(u-\overline{u})\,(v-\overline{v})\,d\nu&\int(u-\overline{u})^2\,d\nu\\\int(u-\overline{u})^2\,d\nu&\int(u-\overline{u})\,(v-\overline{v})\,d\nu\end{pmatrix}.\tag{155}$$

Using (143) we have

$$\sup\left\{|(u,v)-(\overline{u},\overline{v})|:(u,v)\in\mathrm{Spt}\nu\right\}\leq2\delta.\tag{156}$$

By Hölder's inequality, we have

$$\begin{aligned}\left|\int a'(\overline{v})\,(u-\overline{u})\,(v-\overline{v})\,d\nu\right|&\leq\tilde{C}\left(\int(u-\overline{u})^2\,d\nu\right)^{\frac{1}{2}}\left(\int|v-\overline{v}|^2\,d\nu\right)^{\frac{1}{2}}\\&\overset{(148)}{\leq}\tilde{C}\left(\int|v-\overline{v}|^2\,d\nu+|v-\overline{v}|^3\,d\nu\right)^{\frac{1}{2}}\left(\int|v-\overline{v}|^2\,d\nu\right)^{\frac{1}{2}}\\&\overset{(156)}{\leq}\tilde{C}\int|v-\overline{v}|^2\,d\nu.\end{aligned}$$

In a similar way using (148) we have that

$$\left|\left[\widetilde{P}\right]_{lk}\right|\leq\tilde{C}\int|v-\overline{v}|^2\,d\nu\text{ for all }l,k.\tag{157}$$

It follows from (154), (157), (156) and Hölder's inequality that for all $1\leq i<j\leq4$ we have

$$\begin{aligned}\left|M_{ij}(\overline{P})-M_{ij}(\widetilde{P})\right|&\leq\tilde{C}\left(\int\left(|v-\overline{v}|^3+|u-\overline{u}|\,|v-\overline{v}|^2\right)d\nu\right)\left(\int|v-\overline{v}|^2\,d\nu\right)\\&\leq\tilde{C}\delta\left(\int|v-\overline{v}|^2\,d\nu\right)^2\leq\tilde{C}\delta\int|v-\overline{v}|^4\,d\nu.\end{aligned}\tag{158}$$

Therefore we have

$$
\begin{aligned}
&- M_{34}(\overline{P}) - \frac{2a'(\overline{v})}{a''(\overline{v})} M_{23}(\overline{P}) - \frac{2a'(\overline{v})^2}{a''(\overline{v})} M_{14}(\overline{P}) \\
&\overset{(155)}{=} - \left(a'(\overline{v}) \left(\int (u-\overline{u})\,(v-\overline{v})\, d\nu \right)^2 - \left(\int (u-\overline{u})^2\, d\nu \right)^2 \right) \\
&\qquad - 2\frac{a'(\overline{v})}{a''(\overline{v})} \left(\frac{a''(\overline{v})}{2} \int (v-\overline{v})^2\, d\nu \right) \left(\int (u-\overline{u})^2\, d\nu \right) + \text{Error terms} \qquad (159) \\
&= -a'(\overline{v}) \left(\int (u-\overline{u})\,(v-\overline{v})\, d\nu \right)^2 + \left(\int (u-\overline{u})^2\ d\nu \right)^2 \\
&\qquad - a'(\overline{v}) \left(\int (v-\overline{v})^2\, d\nu \right) \left(\int (u-\overline{u})^2\, d\nu \right) + \text{Error terms},
\end{aligned}
$$

where we know by (158) that

$$
|\text{Error terms}| \le \tilde{C}\delta \int |v-\overline{v}|^4\, d\nu. \tag{160}
$$

Since $a'(\overline{v}) > 0$, we have $-a'(\overline{v}) \left(\int (u-\overline{u})\,(v-\overline{v})\, d\nu \right)^2 \le 0$. Also note that from (148), (156) and Hölder's inequality, we have

$$
\begin{aligned}
&\left(\int (u-\overline{u})^2\ d\nu \right)^2 - a'(\overline{v}) \left(\int (v-\overline{v})^2\, d\nu \right) \left(\int (u-\overline{u})^2\, d\nu \right) \\
&= \left(\int (u-\overline{u})^2\, d\nu \right) \left(\int (u-\overline{u})^2\ d\nu - a'(\overline{v}) \int (v-\overline{v})^2\, d\nu \right) \\
&\le \tilde{C} \left(\int (v-\overline{v})^2\, d\nu \right) \left(\int |v-\overline{v}|^3\, d\nu \right) \le \tilde{C}\delta \int |v-\overline{v}|^4\, d\nu.
\end{aligned}
$$

The above together with (159) and (160) gives (144). □

Proof of Theorem 7. Let $\tilde{\alpha} = (\tilde{\alpha}_1, \tilde{\alpha}_2)$ be such that $a'(\tilde{\alpha}_2) > 0$ and $a''(\tilde{\alpha}_2) \neq 0$. Define $\tilde{\delta}_0 := \min\{\delta_1, \delta_2\}$, where δ_1 and δ_2 are as in Lemmas 26 and 27, respectively. Note that by the choice of δ_1 and δ_2, we have that $a'(\alpha_2) > 0$ and $a''(\alpha_2) \neq 0$ for all $\alpha = (\alpha_1, \alpha_2) \in B_{\tilde{\delta}_0}(\tilde{\alpha})$. For any $0 < \delta \le \tilde{\delta}_0$, let $\mu \in \mathcal{M}^{pc}(\mathcal{K}_2 \cap B_\delta(P_2(\tilde{\alpha}))$. Define $\nu := \left((P_2)^{-1}\right)_\sharp \mu$ and let $\alpha = (\overline{u}, \overline{v})$. Using arguments similar to those in Lemma 19, we have $\mathrm{Spt}\nu \subset B_\delta(\tilde{\alpha})$.

Let $\lambda_{34} = -1$, $\lambda_{23} = \frac{-2a'(\overline{v})}{a''(\overline{v})}$, $\lambda_{14} = \frac{-2(a'(\overline{v}))^2}{a''(\overline{v})}$. For any $1 \le i < j \le 4$ with $(i,j) \notin \{(3,4),(2,3),(1,4)\}$ define $\lambda_{ij} = 0$. Note that by Lemma 19 we have that

$$
\int \sum_{i<j=1}^{4} \lambda_{ij} M_{ij} \left(P_2^{(\overline{u},\overline{v})}(u,v) \right) d\nu = \sum_{i<j=1}^{4} \lambda_{ij} M_{ij} \left(\int P_2^{(\overline{u},\overline{v})}(u,v) d\nu \right).
$$

By Lemma 26 we have that

$$
\int \sum_{i<j=1}^{4} \lambda_{ij} M_{ij} \left(P_2^{(\overline{u},\overline{v})}(u,v) \right) d\nu \ge \int \left(\frac{|u-\overline{u}|^4}{8} + \frac{(a'(\tilde{\alpha}_2))^2}{48} |v-\overline{v}|^4 \right) d\nu. \tag{161}
$$

On the other hand by Lemma 27 we have that

$$
\sum_{i,j=1}^{4} \lambda_{ij} M_{ij} \left(\int P_2^{(\overline{u},\overline{v})}(u,v) d\nu \right) \le C\delta \int |v-\overline{v}|^4\, d\nu \tag{162}
$$

for some constant C independent of δ and ν. Putting (161) and (162) together we have that

$$\int \left(\frac{|u-\overline{u}|^4}{8} + \frac{(a'(\tilde{\alpha}_2))^2}{48} |v-\overline{v}|^4 \right) d\nu \leq C\delta \int |v-\overline{v}|^4 \, d\nu. \tag{163}$$

Hence there exists $\delta_0 \leq \tilde{\delta}_0$ such that for all $0 < \delta \leq \delta_0$ we can absorb the right hand side of (163) into the left hand side. It follows that $\nu = \delta_{(\overline{u},\overline{v})}$, and hence $\mu = (P_2)_\sharp \nu$ is also a Dirac measure. □

9.2. **Proof of Theorem 6.** Finally we give the proof of Theorem 6 as a consequence of Theorem 7.

Proof of Theorem 6. Let

$$(g_{1,1}(u,v), g_{1,2}(u,v)) := (v, -u) \text{ and } (g_{2,1}(u,v), g_{2,2}(u,v)) := (u, -a(v)). \tag{164}$$

Further, denote

$$(g_{i+2,1}(u,v), g_{i+2,2}(u,v)) := (\eta_i(u,v), q_i(u,v)) \text{ for } i = 1,2. \tag{165}$$

By hypothesis we have that

$$\operatorname{div}(g_{i,1}(u^\epsilon, v^\epsilon), g_{i,2}(u^\epsilon, v^\epsilon)) \text{ is precompact in } W^{-1,2}_{loc} \text{ for } i = 1,2,3,4.$$

Let $\{(u^{\epsilon_n}, v^{\epsilon_n})\}$ be a subsequence that converges weakly* to some $(u,v) \in L^\infty(\mathbb{R}^2)$ and $\nu_{t,x}$ be the Young measures associated with the weak* convergence. Now for any $1 \leq i < j \leq 4$, by the fundamental theorem of Young measures we have that

$$g_{k,1}(u^{\epsilon_n}, v^{\epsilon_n}) \overset{*}{\rightharpoonup} \int g_{k,1}(u,v)\, d\nu_{t,x} \text{ in } L^\infty \text{ for } k \in \{i,j\},$$
$$g_{k,2}(u^{\epsilon_n}, v^{\epsilon_n}) \overset{*}{\rightharpoonup} \int g_{k,2}(u,v)\, d\nu_{t,x} \text{ in } L^\infty \text{ for } k \in \{i,j\}.$$

Since $(u^{\epsilon_n}, v^{\epsilon_n})$ is bounded, it is clear that $g_{k,1}(u^{\epsilon_n}, v^{\epsilon_n})$ and $g_{k,2}(u^{\epsilon_n}, v^{\epsilon_n})$ are in L^2_{loc} and hence it follows that

$$g_{k,1}(u^{\epsilon_n}, v^{\epsilon_n}) \rightharpoonup \int g_{k,1}(u,v)\, d\nu_{t,x} \text{ in } L^2_{loc} \text{ for } k \in \{i,j\},$$
$$g_{k,2}(u^{\epsilon_n}, v^{\epsilon_n}) \rightharpoonup \int g_{k,2}(u,v)\, d\nu_{t,x} \text{ in } L^2_{loc} \text{ for } k \in \{i,j\}.$$

Applying the Div-Curl Lemma (see Chapter 5 in [Ev 90]) to $(g_{j,1}(u^\epsilon, v^\epsilon), g_{j,2}(u^\epsilon, v^\epsilon))$ and $(-g_{i,2}(u^\epsilon, v^\epsilon), g_{i,1}(u^\epsilon, v^\epsilon))$ we have that

$$\begin{aligned} &g_{j,2}(u^{\epsilon_n}, v^{\epsilon_n})\, g_{i,1}(u^{\epsilon_n}, v^{\epsilon_n}) - g_{i,2}(u^{\epsilon_n}, v^{\epsilon_n})\, g_{j,1}(u^{\epsilon_n}, v^{\epsilon_n}) \\ &\to \left(\int g_{j,2}(u,v)\, d\nu_{t,x}\right)\left(\int g_{i,1}(u,v)\, d\nu_{t,x}\right) - \left(\int g_{i,2}(u,v)\, d\nu_{t,x}\right)\left(\int g_{j,1}(u,v)\, d\nu_{t,x}\right) \end{aligned}$$

in the sense of distributions. Hence by the fundamental theorem of Young measures, for a.e. $(t,x) \in \mathbb{R}^2$, we have that

$$\begin{aligned} &\int \left(g_{j,2}(u,v)\, g_{i,1}(u,v) - g_{i,2}(u,v)\, g_{j,1}(u,v)\right) d\nu_{t,x} \\ &= \int g_{j,2}(u,v)\, d\nu_{t,x} \int g_{i,1}(u,v)\, d\nu_{t,x} - \int g_{i,2}(u,v)\, d\nu_{t,x} \int g_{j,1}(u,v)\, d\nu_{t,x}. \end{aligned}$$

Note that by (21), (164), (165), (13) and (14) we have that

$$P_2(u,v) = \begin{pmatrix} -g_{1,2}(u,v) & g_{1,1}(u,v) \\ -g_{2,2}(u,v) & g_{2,1}(u,v) \\ -g_{3,2}(u,v) & g_{3,1}(u,v) \\ -g_{4,2}(u,v) & g_{4,1}(u,v) \end{pmatrix}.$$

Now as

$$g_{j,2}(u,v)\,g_{i,1}(u,v) - g_{i,2}(u,v)\,g_{j,1}(u,v) = M_{ij}(P_2(u,v)) \text{ for any } 1 \le i < j \le 4$$

and

$$\int g_{j,2}(u,v)\,d\nu_{t,x} \int g_{i,1}(u,v)\,d\nu_{t,x} - \int g_{i,2}(u,v)\,d\nu_{t,x} \int g_{j,1}(u,v)\,d\nu_{t,x}$$
$$= M_{ij}\left(\int P_2(u,v)d\nu_{t,x}\right) \text{ for any } 1 \le i < j \le 4,$$

it follows that for a.e. $(t,x) \in \mathbb{R}^2$, we have $(P_2)_\sharp \nu_{t,x} \in \mathcal{M}^{pc}(\mathcal{K}_2)$. Since $\{(u^\epsilon, v^\epsilon)\}$ has sufficiently small oscillation, we know that $\mathrm{Spt}\nu_{t,x}$ is contained in a sufficiently small neighborhood. It follows from Theorem 7 that $\nu_{t,x}$ is a Dirac measure for a.e. $(t,x) \in \mathbb{R}^2$. Hence $(u^{\epsilon_n}, v^{\epsilon_n})$ converges to (u,v) strongly in L^1_{loc}. □

10. Appendix I: Auxiliary lemma for entropies

Let the functions a and F be as in Subsection 1.1. Define $G : \mathbb{R}^2 \to \mathbb{R}^2$ to be

$$G\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} -a(y_2) \\ -y_1 \end{pmatrix},$$

and consider the system

$$\frac{\partial}{\partial t}\begin{pmatrix} u \\ v \end{pmatrix} + \frac{\partial}{\partial x} G\begin{pmatrix} u \\ v \end{pmatrix} = 0. \tag{166}$$

Define $\Phi, \Psi : \mathbb{R}^2 \to \mathbb{R}^2$ by

$$\Phi\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} \frac{y_1^2}{2} + F(y_2) \\ y_1 y_2 \end{pmatrix} \text{ and } \Psi\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} -y_1 a(y_2) \\ -\left(\frac{y_1^2}{2} + a(y_2)y_2 - F(y_2)\right) \end{pmatrix}.$$

Further define η_1, q_1 and η_2, q_2 in the following way:

$$\eta_1(y_1,y_2) = \frac{y_1^2}{2} + F(y_2), \quad q_1(y_1,y_2) = -y_1 a(y_2) \tag{167}$$

and

$$\eta_2(y_1,y_2) = y_1 y_2, \quad q_2(y_1,y_2) = -\frac{y_1^2}{2} - a(y_2)y_2 + F(y_2). \tag{168}$$

Lastly, for all $\alpha \in \mathbb{R}^2$, define

$$Q_\alpha \eta_i(y_1,y_2) := \eta_i(y_1,y_2) - \eta_i(\alpha_1,\alpha_2) - \nabla\eta_i(\alpha_1,\alpha_2)\cdot(y_1-\alpha_1, y_2-\alpha_2) \tag{169}$$

and

$$Q^*_\alpha q_i(y_1,y_2) := q_i(y_1,y_2) - q_i(\alpha_1,\alpha_2) - \nabla\eta_i(\alpha_1,\alpha_2)\cdot(-a(y_2)+a(\alpha_2), -y_1+\alpha_1). \tag{170}$$

The main purpose of the following lemma is to show that the entropy/entropy flux pairs $(\eta_i, q_i), i = 1,2$, defined in (167)-(168) are the correct ones associated to the system (166). More precisely, we prove

Lemma 28. *The pair (Φ, Ψ) is an entropy/entropy flux pair for the system given by (166), i.e., (Φ, Ψ) satisfies*

$$D\Phi(y_1,y_2)DG(y_1,y_2) = D\Psi(y_1,y_2). \tag{171}$$

Further, we have that

$$Q_\alpha\eta_1(y_1,y_2) = \frac{(y_1-\alpha_1)^2}{2} + F(y_2) - F(\alpha_2) - a(\alpha_2)(y_2-\alpha_2), \tag{172}$$

$$Q^*_\alpha q_1(y_1,y_2) = -(y_1-\alpha_1)\left(a(y_2)-a(\alpha_2)\right), \tag{173}$$

$$Q_\alpha\eta_2(y_1,y_2) = (y_1-\alpha_1)(y_2-\alpha_2), \tag{174}$$

and

$$Q^*_\alpha q_2(y_1, y_2) = -\left(\frac{(y_1-\alpha_1)^2}{2} + F(\alpha_2) - F(y_2) - a(y_2)(\alpha_2 - y_2)\right). \tag{175}$$

Hence (η_1, q_1) and (η_2, q_2) are the entropy/entropy flux pairs that give Equations (9.8) of DiPerna [DP 85].

Proof. Note that there is a discrepancy between the entropy/entropy flux pairs (η_1, q_1) and (η_2, q_2) defined in (167)-(168) and those given by DiPerna [DP 85]. Part of the discrepancy may be down to what DiPerna means by Legendre transform.

We can find an explicit formula for the Legendre transform in the following way. Given a convex function f, for each p we want to find the x value that maximizes $q(x) := xp - f(x)$. So we need $q'(x) = p - f'(x) = 0$, and thus $p = f'(x) =: g(x)$. Since f is convex, f' is monotonic increasing and therefore $x = (f')^{-1}(p) = g^{-1}(p)$. So the Legendre transform of f, denoted by Lf, is given by

$$Lf(p) = pg^{-1}(p) - f\left(g^{-1}(p)\right).$$

In the context of entropies in [DP 85], DiPerna defines $\tilde{\tau} := LF$ and thus $\tilde{\tau}(p) = pa^{-1}(p) - F\left(a^{-1}(p)\right)$. Further he defines

$$\tilde{q}_2(y_1, y_2) := \frac{y_1^2}{2} + \tilde{\tau}(y_2).$$

However, with $\tilde{q}_2$ defined as above along with η_2 given in (168), the pair $(\eta_2, \tilde{q}_2)$ does not form an entropy/entropy flux pair or satisfy (174) and (175), which are Equations (9.8) of DiPerna [DP 85]. However, if we define the "Legendre transform" to be

$$\tau(p) := \tau\left(a(p)\right) = a(p)p - F(p), \tag{176}$$

then up to a sign we obtain the definition of entropies given by DiPerna (see Equation (1.11) in [DP 85]), because with the definition of τ, we have

$$q_2(y_1, y_2) \overset{(168),(176)}{=} -\frac{y_1^2}{2} - \tau(y_2).$$

Note that

$$\frac{d}{dp}\left(\tau(p)\right) \overset{(176)}{=} \frac{d}{dp}\left(a(p)p - F(p)\right) = a'(p)p + a(p) - a(p) = a'(p)p. \tag{177}$$

Step 1. We establish (171) to show that $(\eta_i, q_i), i = 1, 2$, are entropy/entropy flux pairs associated to the system (166).

Proof of Step 1. We calculate

$$D\Phi(y_1, y_2) = \begin{pmatrix} y_1 & a(y_2) \\ y_2 & y_1 \end{pmatrix},$$

$$D\Psi(y_1, y_2) \overset{(177)}{=} \begin{pmatrix} -a(y_2) & -y_1 a'(y_2) \\ -y_1 & -y_2 a'(y_2) \end{pmatrix}$$

and

$$DG(y_1, y_2) = \begin{pmatrix} 0 & -a'(y_2) \\ -1 & 0 \end{pmatrix}.$$

Thus

$$D\Phi(y_1,y_2)DG(y_1,y_2) = \begin{pmatrix} y_1 & a(y_2) \\ y_2 & y_1 \end{pmatrix}\begin{pmatrix} 0 & -a'(y_2) \\ -1 & 0 \end{pmatrix} = \begin{pmatrix} -a(y_2) & -a'(y_2)y_1 \\ -y_1 & -a'(y_2)y_2 \end{pmatrix} = D\Psi(y_1,y_2).$$

This establishes (171).

Step 2. We establish (172)-(175) to show that $(\eta_i, q_i), i = 1,2$, give Equations (9.8) of [DP 85]. *Proof of Step 2.* First note that $\nabla\eta_1(\alpha_1,\alpha_2) = (\alpha_1, a(\alpha_2))$. We calculate

$$\begin{aligned} Q_\alpha\eta_1(y_1,y_2) &\overset{(169),(167)}{=} \frac{y_1^2}{2} + F(y_2) - \frac{\alpha_1^2}{2} - F(\alpha_2) - (\alpha_1, a(\alpha_2))\cdot(y_1-\alpha_1, y_2-\alpha_2) \\ &= \frac{y_1^2}{2} + F(y_2) - \frac{\alpha_1^2}{2} - F(\alpha_2) - \alpha_1(y_1-\alpha_1) - a(\alpha_2)(y_2-\alpha_2) \\ &= \frac{(y_1-\alpha_1)^2}{2} + F(y_2) - F(\alpha_2) - a(\alpha_2)(y_2-\alpha_2), \end{aligned}$$

and hence we have (172). Further we have

$$\begin{aligned} Q_\alpha^* q_1(y_1,y_2) &\overset{(170),(167)}{=} -y_1 a(y_2) + \alpha_1 a(\alpha_2) - (\alpha_1, a(\alpha_2))\cdot(-a(y_2)+a(\alpha_2), -y_1+\alpha_1) \\ &= -y_1 a(y_2) + \alpha_1 a(\alpha_2) + \alpha_1 a(y_2) - \alpha_1 a(\alpha_2) + a(\alpha_2)y_1 - a(\alpha_2)\alpha_1 \\ &= -(y_1-\alpha_1)\left(a(y_2)-a(\alpha_2)\right), \end{aligned}$$

and thus we have (173). Next note that $\nabla\eta_2(\alpha_1,\alpha_2) = (\alpha_2,\alpha_1)$, and we have

$$\begin{aligned} Q_\alpha\eta_2(y_1,y_2) &\overset{(168),(169)}{=} y_1y_2 - \alpha_1\alpha_2 - (\alpha_2,\alpha_1)\cdot(y_1-\alpha_1, y_2-\alpha_2) \\ &= y_1y_2 - \alpha_1\alpha_2 - \alpha_2 y_1 + \alpha_2\alpha_1 - \alpha_1 y_2 + \alpha_1\alpha_2 \\ &= (y_1-\alpha_1)(y_2-\alpha_2), \end{aligned}$$

which gives (174). Finally we have

$$\begin{aligned} Q_\alpha^* q_2(y_1,y_2) &\overset{(168),(170)}{=} -\frac{y_1^2}{2} - y_2 a(y_2) + F(y_2) + \frac{\alpha_1^2}{2} + \alpha_2 a(\alpha_2) \\ &\qquad - F(\alpha_2) - (\alpha_2,\alpha_1)\cdot(-a(y_2)+a(\alpha_2), -y_1+\alpha_1) \\ &= -\frac{y_1^2}{2} - y_2 a(y_2) + F(y_2) + \frac{\alpha_1^2}{2} + \alpha_2 a(\alpha_2) - F(\alpha_2) + \alpha_2 a(y_2) \\ &\qquad - \alpha_2 a(\alpha_2) + \alpha_1 y_1 - \alpha_1^2 \\ &= -\left(\frac{(y_1-\alpha_1)^2}{2} + F(\alpha_2) - F(y_2) - a(y_2)(\alpha_2-y_2)\right), \end{aligned}$$

and thus we have (175). □

11. Appendix II: Auxiliary lemma on linear algebra

Lemma 29. *Suppose $A \in M^{m\times n}$ and $m \le n$. Then* $\mathrm{Rank}(A) = m$ *if and only if* $\det\left(AA^T\right) \neq 0$.

Proof. By singular value decomposition $A = PBQ$ where $P \in O(m)$, $Q \in O(n)$ and B is a diagonal matrix in $M^{m\times n}$. Let $\mathrm{Rank}(B) = p$.

First assume $\mathrm{Rank}(A) = m$. As Q is invertible, we have $\mathrm{Rank}(AQ^{-1}) = m$ and it follows that $p = \mathrm{Rank}(PB) = m$. Now

$$\det(AA^T) = \det\left(PBQQ^TB^TP^T\right) = \det\left(PBB^TP^T\right) = \det\left(BB^T\right) \neq 0. \tag{178}$$

Conversely if $\det(AA^T) \neq 0$, by calculations in (178) we have that $\det(BB^T) \neq 0$ and thus $p = m$. Therefore we have

$$m = \mathrm{Rank}(PB) = \mathrm{Rank}(AQ^{-1}) = \mathrm{Rank}(A).$$

This completes the proof of the lemma. □

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