Lectures on curvature flow of networks

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Abstract

We present a collection of results on the evolution by curvature of networks of planar curves. We discuss in particular the existence of a solution and the analysis of singularities.

1 Introduction

These notes have been prepared for a course given by the second author within the IndAM Intensive Period *Contemporary Research in elliptic PDEs and related topics*, organized by Serena Dipierro at the University of Bari from April to June 2017. We warmly thank the organizer for the invitation, the IndAM for the support, and the Department of Mathematics of the University of Bari for the kind hospitality.

The aim of this work is to provide an overview on the motion by curvature of a network of curves in the plane. This evolution problem attracted the attention of several researchers in recent years, see for instance [9–11, 13, 20, 23, 24, 29, 31, 33, 36, 38, 44]. We refer to the extended survey [32] for a motivation and a detailed analysis of this problem.

This geometric flow can be regarded as the L^2-gradient flow of the length functional, which is the sum of the lengths of all the curves of the network (see [10]). From the energetic point of view it is then natural to expect that configurations with multi–points of order greater than three or 3–points with angles different from 120 degrees, being unstable for the length functional, should be present only at a discrete set of times, during the flow. Therefore, we shall restrict our analysis to networks whose junctions are composed by exactly three curves, meeting at 120 degrees. This is the so-called **Herring condition**, and we call **regular** the networks satisfying this condition at each junction.

The existence problem for the curvature flow of a regular network with only one triple junctions was first considered by L. Bronsard and F. Reitich in [11], where they proved the local existence of the flow, and by D. Kinderlehrer and C. Liu in [24], who showed the global existence and convergence of a smooth solution if the initial network is sufficiently close to a minimal configuration (Steiner tree).

We point out that the class of regular networks is not preserved by the flow, since two (or more) triple junctions might collide during the evolution, creating a multiple junction composed by more than three curves. It is then natural to ask what is the subsequent evolution of the network. A possibility is restarting the evolution at the collision time with a different set of curves, describing a non–regular network, with multi–points of order higher than three. A suitable short time existence result has been worked out by T. Ilmanen, A. Neves and F. Schulze in [23], where it is shown that there exists a flow of networks which becomes immediately regular for positive times.

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These notes are organizes as follows: In Section 2 we introduce the notion of regular network and the geometric evolution problem we are interested in. In Section 3 we recall the short time existence and uniqueness result by Bronsard and Reitich, and we sketch its proof. We also show that the embeddedness of the network is preserved by the evolution (till the maximal time of smooth existence). In Section 4 we describe some special solution which evolve self-similarly. More precisely, we discuss translating, rotating and homotetically shrinking solutions. The latter ones are particularly important for our analysis since they describe the blow-up limit of the flow near a singularity point. In Section 5 we derive the evolution equation for the L^2-norm of the curvature and of its derivatives. As a consequence, we show that, at a singular point, either the curvature blows-up or there is a collision of triple junctions. Finally, in Section 6 we recall Huisken's Monotonicity Formula for mean curvature flow, which holds also for the evolution of a network, and we introduce the rescaling procedures used to get blow–up limits at the maximal time of smooth existence, in order to describe the singularities of the flow. In particular, we show that the limits of the rescaled networks are self-similar shrinking solutions of the flow, possibly with multiplicity greater than one, and we identify all the possible limits under the assumption that the length of each curve of the network is uniformly bounded from below.

2 Notation and setting of the problem

2.1 Curves and networks

Given an interval $I \subset \mathbb{R}$, we consider planar curves $\gamma : I \to \mathbb{R}^2$.

The interval I can be both bounded and unbounded depending whether one wants to parametrize a bounded or an unbounded curve. In the first case we restrict to consider $I = [0, 1]$.

By **curve** we mean both image of the curve in $\mathbb{R}^2$ and parametrization of the curve, we will be more specific only when the meaning cannot be got by the context.

- A curve is of class C^k if it admits a parametrization $\gamma : I \to \mathbb{R}^2$ of class C^k.
- A C^1 curve, is **regular** if it admits a regular parametrization, namely $\gamma_x(x) = \frac{d\gamma}{dx}(x) \neq 0$ for every $x \in I$.
- It is then well defined its **unit tangent vector** $\tau = \gamma_x/|\gamma_x|$.
- We define its **unit normal vector** as $\nu = R\tau = R\gamma_x/|\gamma_x|$, where $R : \mathbb{R}^2 \to \mathbb{R}^2$ is the anticlockwise rotation centred in the origin of $\mathbb{R}^2$ of angle $\pi/2$.
- The **arclength parameter** of a curve γ is given by

  $$s := s(x) = \int_0^x |\gamma_x(\xi)|\, d\xi\,.$$

  We use the letter s to indicate the arclength parameter and the letter x for any other parameter. Notice that $\partial_s = |\gamma_x|^{-1}\partial_x$.
- If the curve γ is C^2 and regular, we define the curvature $k := |\tau_s| = |\gamma_{ss}|$ and the **curvature vector** $\boldsymbol{k} := \tau_s = \gamma_{ss}$. We get:

  $$\boldsymbol{k} = \frac{1}{|\gamma_x|}\left(\frac{\gamma_x}{|\gamma_x|}\right)_x = \frac{\gamma_{xx}|\gamma_x|^2 - \gamma_x\langle\gamma_{xx}, \gamma_x\rangle}{|\gamma_x|^4}\,.$$

  As we are in $\mathbb{R}^2$ we remind that $\boldsymbol{k} = \tau_s = k\nu$.

- The **length** L of a curve γ is given by

$$L(\gamma) := \int_I |\gamma_x(x)|\, dx = \int_\gamma 1\, ds\,.$$

A curve is injective if for every $x \neq y \in I$ we have $\gamma(x) \neq \gamma(y)$.
A curve $\gamma : [0,1] \to \mathbb{R}^2$ of class C^k is **closed** if $\gamma(0) = \gamma(1)$ and if γ has a 1–periodic C^k extension to $\mathbb{R}$ (Figure 1).

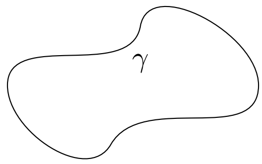

Figure 1: A simple closed curve.

In what follows we will consider time–dependent families of curves $(\gamma(t,x))_{t\in[0,T]}$. We let $\tau = \tau\,(t,x)$ be the unit tangent vector to the curve, $\nu = \nu\,(t,x)$ the unit normal vector and $\boldsymbol{k} = \boldsymbol{k}\,(t,x)$ its curvature vector as previously defined.

We denote with $\partial_x f$, $\partial_s f$ and $\partial_t f$ the derivatives of a function f along a curve γ with respect to the x variable, the arclength parameter s on such curve and the time, respectively. Moreover $\partial_x^n f$, $\partial_s^n f$, $\partial_t^n f$ are the higher order partial derivatives, possibly denoted also by $f_x, f_{xx} \dots$, $f_s, f_{ss}, \dots$ and $f_t, f_{tt}, \dots$.

We adopt the following convention for integrals:

$$\int_{\gamma_t} f(t,\gamma,\tau,\nu,k,k_s,\dots,\lambda,\lambda_s\dots)\, ds = \int_0^1 f(t,\gamma^i,\tau^i,\nu^i,k^i,k^i_s,\dots,\lambda^i,\lambda^i_s\dots)\, |\gamma^i_x|\, dx$$

as the arclength measure is given by $ds = |\gamma^i_x|\, dx$ on every curve γ.

Let now Ω be a smooth, convex, open set in $\mathbb{R}^2$.

Definition 2.1. A **network** $\mathcal{N}$ in $\overline{\Omega}$ is a connected set described by a finite family of regularC^1 curves contained in $\overline{\Omega}$ such that

1. the interior of every curve is injective, a curve can self–intersect only at its end–points;
2. two different curves can intersect each other only at their end–points;
3. a curve is allowed to meet $\partial\Omega$ only at its end–points;
4. if an end–point of a curve coincide with $P \in \partial\Omega$, then no other end–point of any curve can coincide with P.

The curves of a network can meet at **multi–points** in Ω, labeled by $O^1, O^2, \dots, O^m$. We call **end–points** of the network, the vertices (of order one) $P^1, P^2, \dots, P^l \in \partial\Omega$.

Condition 4 keeps things simpler implying that multi–points can be only inside Ω, not on the boundary.

We say that a network is of class C^k with $k \in \{1,2,\dots\}$ if all its curves are of class C^k.

Remark 2.2. With a slightly modification of Definition 2.1 we could also consider networks in the whole $\mathbb{R}^2$ with unbounded curves. In this case we require that every non compact branch of $\mathcal{N}$ is asymptotic to an half line and its curvature is uniformly bounded. We call these unbounded networks **open networks**.

Definition 2.3. We call a network **regular** if all its multi–points are triple and the sum of unit tangent vectors of the concurring curves at each of them is zero.

Example 2.4.

- A network could consists of a single closed embedded curve.
- A network could be composed of a single embedded curve with fixed end–points on $\partial\Omega$.
- There are two possible (topological) structures of networks with only one triple junction: the **triod** $\mathbb{T}$ or the **spoon** $\mathbb{S}$. A triod is a tree composed of three curves that intersects each other at a 3–point and have their other end–points on the boundary of Ω. A spoon is the union of two curves: a closed one attached to the other at a triple junction. The "open" curve of the spoon has an end–point on $\partial\Omega$ (Figure 2).

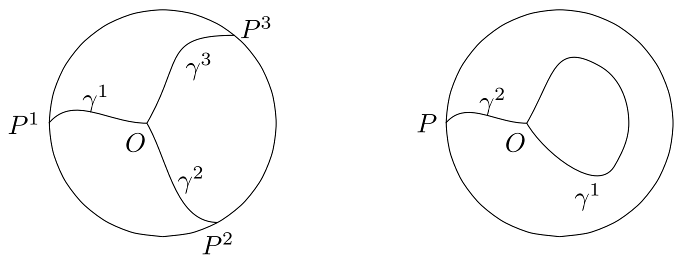

Figure 2: A triod and a spoon.

2.2 The evolution problem

Given a network composed of n curves we define its **global length** as

$$L = L^1 + \cdots + L^n \, .$$

The evolution we have in mind is the L^2–gradient flow of the global length L. Therefore, geometrically speaking, this means that the normal velocity of the curves is the curvature. In the case of the curves (curve shortening flow) this condition fully defines the evolution, at least geometrically. In the case of networks another condition at the junctions comes from the variational formulation of the evolution, as we will see below.

2.2.1 Formal derivation of the gradient flow

We begin by considering one closed embedded C^2 curve, parametrized by $\gamma : [0,1] \to \mathbb{R}^2$. Then $\gamma(0) = \gamma(1)$, $\tau(0) = \tau(1)$ and $k(0) = k(1)$. We want to compute the directional derivative of the length. Given $\varepsilon \in \mathbb{R}$ and $\psi : [0,1] \to \mathbb{R}^2$ a smooth function satisfying $\psi(0) = \psi(1)$, we take $\widetilde{\gamma} = \gamma + \varepsilon\psi$ a variation of γ. From now on we neglect the dependence on the variable x to maintain the notation simpler. We have

$$\frac{\partial}{\partial\varepsilon}L(\widetilde{\gamma})_{|\varepsilon=0} = \frac{\partial}{\partial\varepsilon}\int_0^1 |\gamma_x + \varepsilon\psi_x|\,dx = \int_0^1 \frac{\langle \psi_x, \gamma_x\rangle}{|\gamma_x|}\,dx = \int_\gamma \langle \psi_s, \tau\rangle \; ds$$
$$= -\int_\gamma \langle \psi, \tau_s\rangle \; ds + \langle \psi(1), \tau(1)\rangle - \langle \psi(0), \tau(0)\rangle \; .$$

As γ is a simple closed embedded curve, then the boundary terms are equal zero. We get

$$\frac{\partial}{\partial\varepsilon}L(\widetilde{\gamma})_{|t=0} = \int_0^1 \langle \psi, -\boldsymbol{k}\rangle \; ds\,.$$

Since we have written the directional derivative of L in the direction ψ as the scalar product of ψ and $-\boldsymbol{k}$, we conclude (at least formally) that $-\boldsymbol{k}$ is the gradient of the length. Hence we can understand the curve shortening flow as the gradient flow of the length.

We considering now a triod $\mathbb{T}$ in a convex, open and regular set $\Omega \subset \mathbb{R}^2$, whose curves are parametrized by $\gamma^i : [0,1] \to \mathbb{R}^2$ of class C^2 with $i \in \{1,2,3\}$. Without loss of generality we can suppose that $\gamma^1(0) = \gamma^2(0) = \gamma^3(0)$ and $\gamma^i(1) = P^i \in \partial\Omega$ with $i \in \{1,2,3\}$. We consider again a variation $\tilde{\gamma}^i = \gamma^i + \varepsilon\psi^i$ of each curve with $\psi^i : [0,1] \to \mathbb{R}^2$ three smooth functions. We require that $\psi^1(0) = \psi^2(0) = \psi^3(0)$ and $\psi^i(1) = 0$ because we want that the set $\widetilde{\mathbb{T}}$ parametrized by $\tilde{\gamma} = (\tilde{\gamma}^1, \tilde{\gamma}^2, \tilde{\gamma}^3)$ is a triod with end point on $\partial\Omega$ fixed at P^i. In such a way we are asking two (Dirichlet) boundary conditions. By definition of total length L of a network, we have

$$L(\widetilde{\mathbb{T}}) = \sum_{i=1}^3 L(\gamma^i) = \sum_{i=1}^3 \int_0^1 |\widetilde{\gamma}^i_x|\,dx = \sum_{i=1}^3 \int_0^1 |\gamma^i_x + \varepsilon\psi^i_x|\,dx\,.$$

Repeating the previous computation and using the hypothesis on ψ^i we have

$$\frac{\partial}{\partial\varepsilon}L(\widetilde{\mathbb{T}})_{|\varepsilon=0} = \sum_{i=1}^3 \int_\gamma \left\langle \psi^i, -\boldsymbol{k}^i \;\right\rangle \; ds + \sum_{i=1}^3 \left\langle \psi^i(1), \tau^i(1)\right\rangle - \sum_{i=1}^3 \left\langle \psi^i(0), \tau^i(0)\right\rangle$$
$$= \sum_{i=1}^3 \int_\gamma \left\langle \psi^i, -\boldsymbol{k}^i \;\right\rangle \; ds + \sum_{i=1}^3 - \left\langle \psi^1(0), \tau^i(0)\right\rangle \; .$$

Imposing that the boundary term equals zero we get

$$0 = \sum_{i=1}^3 \left\langle \psi^1(0), \tau^i(0)\right\rangle = \left\langle \psi^1(0), \sum_{i=1}^3 \tau^i(0)\right\rangle \Longrightarrow \sum_{i=1}^3 \tau^i(0) = 0\,.$$

Hence, we have derived a further boundary condition at the junctions.

2.2.2 Geometric problem

We define the **motion by curvature** of regular networks.

Problem 2.5. Given a regular network we let it evolve by the L^2–gradient flow of the (total) length functional L in a maximal time interval $[0,T)$. That is:

- each curve of the network has a normal velocity equal to its curvature at every point and for all times $t \in [0,T)$ – **motion by curvature**;
- the curves that meet at junctions remains attached for all times $t \in [0,T)$ – **concurrency**;

- the sum of the unit tangent vectors of the three curves meeting at a junction is zero for all times $t \in [0,T)$ – **angle condition**.

Moreover we ask that the end–points $P^r \in \partial\Omega$ stay fixed during the evolution – **Dirichlet boundary condition**.

As a possible variant one lets the end–points free to move on the boundary of Ω but asking that the curves intersect orthogonally $\partial\Omega$ – **Neumann boundary condition**.

Although our problem is **geometric** (as we want to describe the flow of a set moving in $\mathbb{R}^2$), to solve we will turn to a **parametric approach**. As a consequence we will work often at the level of parametrization.

Definition 2.6 (Geometric admissible initial data)**.** A network $\mathcal{N}_0$ is a geometrically admissible initial data for the motion by curvature if it is regular, at each junction the sum of the curvature is zero, the curvature at each end–point on $\partial\Omega$ is zero and each of its curve can be parametrized by a regular curve $\gamma_0^i : [0,1] \to \mathbb{R}^2$ of class $C^{2+\alpha}$ with $\alpha \in (0,1)$.

We introduce a way to label the curves: given a network composed by n curves with l end–points $P^1, P^2, \dots, P^l \in \partial\Omega$ (if present) and m triple points $O^1, O^2, \dots O^m \in \Omega$, we denote with γ^{pi} the curves of this network concurring at the multi–point O^p with $p \in \{1,2,\dots,m\}$ and $i \in \{1,2,3\}$.

Definition 2.7. [Solution of the motion by curvature of networks] Consider a geometrically admissible initial network $\mathcal{N}_0$ composed of n curves parametrized by $\gamma_0^i : [0,1] \to \overline{\Omega}$, with m triple points $O^1, O^2, \dots O^m \in \Omega$ and (if present) l end–points $P^1, P^2, \dots, P^l \in \partial\Omega$. A time dependent family of networks $(\mathcal{N}_t)_{t\in[0,T)}$ is a solution of the motion by curvature in the maximal time interval $[0,T)$ with initial data $\mathcal{N}_0$ if it admits a time dependent family of parametrization $\gamma = (\gamma^1, \dots, \gamma^n)$ such that each curve $\gamma^i \in C^{\frac{2+\alpha}{2}, 2+\alpha}([0,T) \times [0,1])$ is regular and the following system of conditions is satisfied for every $x \in [0,1]$, $t \in [0,T)$, $i,j \in \{1,2,\dots,n\}$

$$\begin{cases} (\gamma^i)_t^\perp(t,x) = \boldsymbol{k}^i(t,x) & & \text{motion by curvature,} \\ \gamma^{pi} = \gamma^{pj} & \text{at every 3–point } O^p & \text{concurrency,} \\ \sum_{i=1}^3 \tau^{pi} = 0 & \text{at every 3–point } O^p & \text{angle condition,} \\ \gamma^r(t,1) = P^r & \text{with } 0 \le r \le l & \text{Dirichlet boundary condition,} \end{cases} \tag{2.1}$$

where we assumed conventionally that the end–point P^r of the network is given by $\gamma^r(t,1)$.

Remark 2.8. The boundary conditions in system (2.1) are consistent with a second order flow of three curves. Indeed we expect three vectorial conditions at the junctions and one for each curve at the other end points.

Remark 2.9. We have defined solutions in $C^{\frac{2+\alpha}{2}, 2+\alpha}$ but the natural class seems to be $C^{1,2}$. It is indeed possible to define a solution to the motion by curvature of networks asking less regularity on the parametrization. Our choice simplify the proof of the short time existence result. We will see in the sequel that it is based on linearization and on a fixed point argument. The classical theory for system of linear parabolic equations developed by Solonnikov [40] is a Hölder functions setting (see [40, Theorem 4.9]).

Remark 2.10. Suppose that $(\mathcal{N}(t))_{t\in[0,T]}$ is a solution to the motion by curvature as defined in 2.7. We will see later that at $t > 0$ the curvature at the end–points and the sum of the three curvatures at every 3–point are automatically zero. Then a necessary condition for $(\mathcal{N}(t))_{t\in[0,T]}$ to be C^2 in space till $t = 0$ is that these properties are satisfied also by the the initial regular network. These conditions on the curvatures are **geometric**, independent of the parametrizations of the curves, but intrinsic to the **set** and they are not satisfied by a generic regular, C^2 network.

Remark 2.11. Notice that in the geometric problem we specify only the the **normal component** of the velocity of the curves (their curvature). This does not mean that there is not a tangential component of the velocity, rather a tangential motion is needed to allow the junctions move in any direction.

Example 2.12.

- The motion by curvature of a single closed embedded curve was widely studied by many authors [3–5, 16–19, 28]. In particular the curve evolves smoothly, becoming convex and getting rounder and rounder. In finite time it shrinks to a point.

- The case of a curve with either an angle or a cusp can be dealt by the works of Angenent [3–5]. Actually the curve becomes immediately smooth and then for all positive time we come back to the evolution described in the previous example.

- The evolution of a single embedded curve with fixed end–points (Figure 3) is discussed in [22, 41, 42]. The curve converges to the straight segment connecting the two fixed end–points as the time goes to infinity.

- Two curves that concur at a 2–point forming an angle (or a cusp, if they have the same tangent) can be regarded as a single curve with a singular point, which will vanish immediately under the flow (Figure 3).

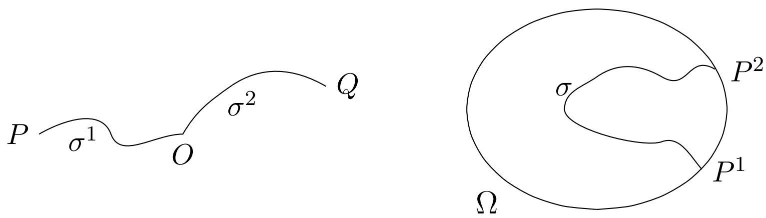

Figure 3: Two special cases: two curves forming an angle at their junction and a single curve with two end–points on the boundary of Ω.

2.2.3 The system of quasilinear PDEs

In this section we actually work by defining the evolution in terms of differential equations for the parametrization of the curves. For sake of presentation we restrict to the case of the triod. This allows us maintaining the notation simpler.

Let us start focusing on the geometric evolution equation $\gamma_t^\perp = \boldsymbol{k}$, that can be equivalently written as

$$\langle \gamma_t(t,x)\,,\,\nu(t,x)\rangle\,\nu(t,x) = \left\langle \frac{\gamma_{xx}(t,x)}{|\gamma_x(t,x)|^2}\,,\,\nu(t,x)\right\rangle \nu(t,x)\,.$$

This equation specify the velocity of each curve only in direction of the normal ν.

Curve shortening flow for closed curve is not affected by tangential velocity. In the evolution by curvature of a smooth closed curve it is well known that any tangential contribution to the velocity actually affects only the "inner motion" of the "single points" (Lagrangian point of view), but it does not affect the motion of the whole curve as a subset of $\mathbb{R}^2$ (Eulerian point of view). Indeed the classical mean curvature flow for hypersurfaces is invariant under tangential perturbations (see for instance [30, Proposition 1.3.4]). In particular in the case of curves it can be shown that a solution of the curve shortening flow satisfying the equation $\gamma_t = k\nu + \lambda\tau$ for some continuous function λ can be globally reparametrized (dynamically in time) in order to satisfy $\gamma_t = k\nu$ and vice versa.

As already anticipated, in the case of networks it is instead **necessary** to consider an extra **tangential term** (as for the case of the single curve that is not closed). It allows the motion of the 3–points. At the junctions the sum of the unit normal vectors is zero. If the velocity would be in normal direction to the three curves concurring at a 3–point, this latter should move in a direction which is normal to all of them, then the only possibility would be that the junction does not move at all.

Saying that a junction cannot move is equivalent to fix it, hence to add a condition in the system (2.1). Thus, from the PDE point of view, the system becomes overdetermined as at the junctions we have already required the concurrency and the angle conditions.

Therefore solving the problem of the motion by curvature of regular networks means that we require the concurrency and the angle condition (regular networks remain regular networks for all the times) and that the main equation for each curve is

$$\gamma^i_t(t,x) = k^i(t,x)\nu^i(t,x) + \lambda^i(t,x)\tau^i(t,x)$$

for some λ continuous function not specified. To the aim of writing a non–degenerate PDE for each curve we consider the tangential velocity

$$\lambda^i = \frac{\langle \gamma^i_{xx} | \tau^i \rangle}{|\gamma^i_x|^2}\,.$$

Then the velocity of the curves is

$$\gamma^i_t(t,x) = \left\langle \frac{\gamma^i_{xx}(t,x)}{|\gamma^i_x(t,x)|^2} \,\middle|\, \nu^i(t,x) \right\rangle \nu^i(t,x) + \left\langle \frac{\gamma^i_{xx}(t,x)}{|\gamma^i_x(t,x)|^2} \,\middle|\, \tau^i(t,x) \right\rangle \tau^i(t,x) = \frac{\gamma^i_{xx}(t,x)}{|\gamma^i_x(t,x|^2}\,. \tag{2.2}$$

A family of networks evolving according to (2.2) will be called a **special flow**.

We are finally able to write explicitly the system of PDE we consider.

Without loss of generality any triod $\mathbb{T}$ can be parametrized by $\gamma = (\gamma^1, \gamma^2, \gamma^3)$ in such a way that the triple junction is $\gamma^1(0) = \gamma^2(0) = \gamma^3(0)$ and that the other end–points P^i on $\partial\Omega$ are given by $\gamma^i(1) = P^i$ with $i \in \{1, 2, 3\}$.

Definition 2.13. Given an admissible initial parametrization $\varphi = (\varphi^1, \varphi^2, \varphi^2)$ of a geometrically admissible initial triod $\mathbb{T}_0$ the family of time–dependent parametrizations $\gamma = (\gamma^1, \gamma^2, \gamma^3)$ is a **solution of the special flow** in the time interval $[0, T]$ if the functions γ^i are of class $C^{\frac{2+\alpha}{2}, 2+\alpha}([0, T] \times [0, 1])$ and the following system is satisfied for every $t \in [0, T]$, $x \in [0, 1]$, $i \in \{1, 2, 3\}$

$$\begin{cases} \gamma^i_t(t,x) = \frac{\gamma^i_{xx}(t,x)}{|\gamma^i_x(t,x)|^2} & \text{motion by curvature,} \\ \gamma^1(t,0) = \gamma^2(t,0) = \gamma^3(t,0) & \text{concurrency,} \\ \sum_{i=1}^3 \tau^i(t,0) = 0 & \text{angle condition,} \\ \gamma^i(t,1) = P^i & \text{Dirichlet boundary condition} \\ \gamma^i(0,x) = \varphi^i(x) & \text{initial data} \end{cases} \tag{2.3}$$

Definition 2.14 (Admissible initial parametrization of a triod)**.** We say that a parametrization $\varphi = (\varphi^1, \varphi^2, \varphi^3)$ is admissible for the system (2.3) if:

1. $\cup_{i=1}^3 \varphi^i([0,1])$ is a triod;
2. each curve φ^i is regular and of class $C^{2+\alpha}([0,1])$;
3. $\varphi^i(0) = \varphi^j(0)$ for every $i, j \in \{1,2,3\}$;
4. $\frac{\varphi^1_x(0)}{|\varphi^1_x(0)|} + \frac{\varphi^2_x(0)}{|\varphi^2_x(0)|} + \frac{\varphi^3_x(0)}{|\varphi^3_x(0)|} = 0$;
5. $\frac{\varphi^i_{xx}(0)}{|\varphi^i_x(0)|^2} = \frac{\varphi^j_{xx}(0)}{|\varphi^j_x(0)|^2}$ for every $i, j \in \{1,2,3\}$;
6. $\varphi^i(1) = P^i$ for every $i \in \{1,2,3\}$;
7. $\varphi^i_{xx}(1) = 0$ for every $i \in \{1,2,3\}$.

Remark 2.15*.* Notice that in the literature one refers to conditions 3. to 7. in Definition 2.14 as **compatibility conditions** for system (2.3). In particular conditions 5. and 7. are called compatibility conditions of order 2.

We want to stress the fact that choosing the tangential velocity and so passing to consider the special flow allows us to turn the geometric problem into a non degenerate PDE's system. The goodness of our choice will be revealed when one verifies the well posedness of the system (2.3).

Once proved existence and uniqueness of solution for the PDE's system, it is then crucial to come back to the geometric problem and show that we have solved it in a "geometrically" unique way. This can be done in two step: first one shows that for any geometrically admissible initial data there exists an admissible initial parametrization for system (2.3) (and consequently a unique solution related to that parametrization). In the second step one supposes that there exist two different solutions of the geometric problem and then proves that it is possible to pass from one to another by time–dependent reparametrization.

However from the previous discussion we have understood that in our situation of motion of networks the invariance under tangential terms of the curve shortening flow is not trivially true. To prove existence and uniqueness of the motion by curvature of networks starting from existence and uniqueness of the PDE's system solution a key role will be played again by our good choice of the tangential velocity.

3 Short time existence and uniqueness

We now deal with the problem of short time existence and uniqueness of the flow.

3.1 Existence and uniqueness for the special flow

We restrict again to a triod in $\Omega \subset \mathbb{R}^2$. We consider first system (2.3). The short time existence result is due to Bronsard and Reitich [11].

We look for classical solutions in the space $C^{\frac{2+\alpha}{2}, 2+\alpha}([0,T] \times [0,1])$ with $\alpha \in (0,1)$. We recall the definition of this function space and of the norm it is endowed with (see also [40, §11, §13]).

For a function $u : [0,T] \times [0,1] \to \mathbb{R}$ we define the semi–norms

$$[u]_{\alpha,0} := \sup_{(t,x),(\tau,x)} \frac{|u(t,x) - u(\tau,x)|}{|t-\tau|^\alpha},$$

and

$$[u]_{0,\alpha} := \sup_{(t,x),(t,y)} \frac{|u(t,x)-u(t,y)|}{|x-y|^\alpha}\,.$$

The classical parabolic Hölder space $C^{\frac{2+\alpha}{2},2+\alpha}([0,T]\times[0,1])$ is the space of all functions $u : [0,T]\times[0,1]\to\mathbb{R}$ that have continuous derivatives $\partial_t^i\partial_x^j u$ (where $i,j\in\mathbb{N}$ are such that $2i+j\leq 2$) for which the norm

$$\|u\|_{C^{\frac{2+\alpha}{2},2+\alpha}} := \sum_{2i+j=0}^{2} \left\|\partial_t^i\partial_x^j u\right\|_\infty + \sum_{2i+j=2} \left[\partial_t^i\partial_x^j u\right]_{0,\alpha} + \sum_{0<2+\alpha-2i-j<2} \left[\partial_t^i\partial_x^j u\right]_{\frac{2+\alpha-2i-j}{2},0}$$

is finite.

The boundary terms are in spaces of the form $C^{\frac{k+\alpha}{2},k+\alpha}([0,T]\times\{0,1\},\mathbb{R}^m)$ with $k\in\{1,2\}$ which we identify with $C^{\frac{k+\alpha}{2}}([0,T],\mathbb{R}^{2m})$ via the isomorphism $f\mapsto(f(t,0),f(t,1))^t$.

Calling B_r the ball of radius r centred at the origin the short time existence result reads as follows:

Theorem 3.1 (Bronsard and Reitich)**.** *For any admissible initial parametrization there exists a positive radius M and a positive time T such that the system (2.3) has a unique solution in $C^{\frac{2+\alpha}{2},2+\alpha}([0,T)\times[0,1])\cap\overline{B}_M$.*

Remark 3.2. Actually in [11] the authors do not consider exactly system (2.3), but the analogous Neumann problem. They require that the end–points of the three curves intersect the boundary of Ω with a prescribed angle (of 90 degrees).

Bronsard and Reitich approach, based on linearising the problem around the initial data, nowadays is considered classical. We explain here their strategy.

Step 1: Linearization

Fix an admissible initial datum $\sigma=(\sigma^1,\sigma^2,\sigma^3)$. We linearise the system (2.3) around σ getting

$$\gamma_t^i - \frac{1}{|\sigma_x^i|^2}\gamma_{xx}^i = \left(\frac{1}{|\gamma_x^i|^2}-\frac{1}{|\sigma_x^i|^2}\right)\gamma_{xx}^i =: f^i(\gamma_{xx}^i,\gamma_x^i)\,. \tag{3.1}$$

The concurrency condition and the Dirichlet boundary condition are already linear. The angle condition instead is not linear, so one has to take into account the linear version of it:

$$-\sum_{i=1}^3 \frac{\gamma_x^i}{|\sigma_x^i|} - \frac{\sigma_x^i\left\langle\gamma_x^i,\sigma_x^i\right\rangle}{|\sigma_x^i|^3} = \sum_{i=1}^3\left(\frac{1}{|\gamma_x^i|}-\frac{1}{|\sigma_x^i|}\right)\gamma_x^i + \frac{\sigma_x^i\left\langle\gamma_x,\sigma_x^i\right\rangle}{|\sigma_x^i|^3} =: b(\gamma_x)\,.$$

The linearized system associated to (2.3) is the following: for $i\in\{1,2,3\}$, $t\in[0,T]$ and $x\in[0,1]$

$$\begin{cases} \gamma_t^i(t,x) - \frac{\gamma_{xx}^i(t,x)}{|\sigma_x^i|^2} & = f^i(t,x) \quad \text{motion,}\\ \gamma^1(t,0)-\gamma^2(t,0) & = 0 \quad \text{concurrency}\\ \gamma^1(t,0)-\gamma^3(t,0) & = 0 \quad \text{concurrency}\\ -\sum_{i=1}^3 \frac{\gamma_x^i(t,0)}{|\sigma_x^i|} - \frac{\sigma_x^i\left\langle\gamma(t,x)_x^i,\sigma_x^i\right\rangle}{|\sigma_x^i|^3} & = b(t,0) \quad \text{angles condition}\\ \gamma^i(t,1) & = P^i \quad \text{Dirichlet boundary condition}\\ \gamma^i(0,x) & = \varphi^i(x) \quad \text{initial data} \end{cases} \tag{3.2}$$

We remind that the initial data for the system has to satisfy some linear compatibility conditions.

Step 2: Existence and uniqueness of solution for the linearized system

We have linearized system (2.3) to obtain system (3.2). We now want to show that this latter admits a unique solution in $C^{\frac{2+\alpha}{2},2+\alpha}([0,T]\times[0,1])$. This is due to general results by Solonnikov [40], provided the so–called **complementary conditions** hold (see [40, p. 11]). The theory of Solonnikov is a generalization to parabolic systems of the elliptic theory by Agmon, Douglis and Nirenberg.

The complementary conditions are algebraic conditions that the matrices that represent the boundary operator and the initial datum have to satisfy (see also [40, p. 97]). Showing this conditions for a particular system can be heavy from the computational point of view. For instance in [15, pages 11–15] it is proved that the complementary condition follows from the **Lopatinskii–Shapiro condition**. We state here the definition of Lopatinskii–Shapiro condition at the triple junction, it is similar at the end–points on $\partial\Omega$.

Definition 3.3. Let $\lambda \in \mathbb{C}$ with $\Re(\lambda) > 0$ be arbitrary. The Lopatinskii–Shapiro condition for system (3.2) is satisfied at the triple junction if every solution $(\gamma^i)_{i=1,2,3} \in C^2([0,\infty),(\mathbb{C}^2)^3)$ to

$$\begin{cases} \lambda\gamma^i(x) - \frac{1}{|\sigma^i_x(0)|^2}\gamma^i_{xx}(x) & = 0 \quad x\in[0,\infty), i\in\{1,2,3\} & \text{motion,}\\ \gamma^1(0)-\gamma^2(0) & = 0 & \text{concurrency,}\\ \gamma^2(0)-\gamma^3(0) & = 0 & \text{concurrency,}\\ \sum_{i=1}^3 \frac{\gamma^i_x(x)}{|\sigma^i_x(0)|} - \frac{\sigma^i_x(0)\left\langle \gamma^i_x(x),\sigma^i_x(0)\right\rangle}{|\sigma^i_x(0)|^3} & = 0 & \text{angle condition,} \end{cases}$$

which satisfies $\lim_{x\to\infty}|\gamma^i(x)| = 0$ is the trivial solution.

The angle condition in the previous system can be equivalently written as

$$\sum_{i=1}^{3} \frac{1}{|\sigma_x(0)^i|^3}\left\langle \gamma^i_x(x),\nu^i_0(0)\right\rangle \nu^i_0(0) = 0\,.$$

It can be proved that Lopatinskii–Shapiro condition for system (3.2) is satisfied testing the motion equation by $|\sigma(0)^i_x|\overline{\langle\gamma^i(x),\nu^i(0)\rangle}\nu^i(0)$ and then by $|\sigma(0)^i_x|\overline{\langle\gamma^i(x),\tau^i(0)\rangle}\tau^i(0)$ and using the concurrency and the angle conditions.

Once it is shown that the complementary conditions are fulfilled, then [40, Theorem 4.9] guarantees existence and uniqueness of a solution of system (3.2).

For $T>0$ we define the map $L_T : X_T \to Y_T$ as

$$L(\gamma) = \begin{pmatrix} \left(\gamma^i_t - \frac{1}{|\sigma^i_x|^2}\gamma^i_{xx}\right)_{i\in\{1,2,3\}} \\ -\sum_{i=1}^3 \frac{\gamma^i_x}{|\sigma^i_x|} - \frac{\sigma^i_x\left\langle\gamma^i_x,\sigma^i_x\right\rangle}{|\sigma^i_x|^3}\Big|_{x=0} \\ \gamma^i_{|x=1} \\ \gamma_{|t=0} \end{pmatrix}$$

where the linear spaces X_T and Y_T are

$$\begin{aligned} X_T &:= \{\gamma\in C^{\frac{2+\alpha}{2},2+\alpha}([0,T]\times[0,1];(\mathbb{R}^2)^3) \text{ such that for } t\in[0,T]\,, i\in\{1,2,3\} \\ &\qquad \text{it holds}\, \gamma^1(t,1)=\gamma^2(t,2)=\gamma^3(t,3)\}\,,\\ Y_T &:= \{(f,b,\psi)\in C^{\frac{\alpha}{4},\alpha}([0,T]\times[0,1];(\mathbb{R}^2)^3)\times C^{\frac{1+\alpha}{2}}([0,T];\mathbb{R}^4)\times C^{2+\alpha}\left([0,1];\left(\mathbb{R}^2\right)^3\right) \\ &\qquad \text{such that the linear compatibility conditions hold}\}\,, \end{aligned}$$

endowed with the induced norms. Then as a consequence of the existence and uniqueness of a solution of system (3.2) we get that L_T is a continuous isomorphism.

Remark 3.4. The linearized version of $\frac{\gamma_{xx}}{|\gamma_x|^2}$ (linearising around σ) is

$$\frac{1}{|\sigma_x|^2}\gamma_{xx} - 2\frac{\sigma_{xx}\langle\gamma_x, \sigma_x\rangle}{|\sigma_x|^4}\,. \tag{3.3}$$

As the well posedness of system (3.2) depends only on the highest order term we can restrict to consider (3.1) instead of (3.3).

Step 3: Fixed point argument

In the last step of the proof we deduce existence of a solution for system (2.3) from the linear problem by a **contraction argument**.

Let us define the operator N that "contains the information" about the non–linearity of our problem. The two components of this map are the following:

$$N_1 : \begin{cases} X_T^{\varphi,P} & \to C^{\frac{\alpha}{2},\alpha}([0,T]\times[0,1];(\mathbb{R}^2)^3),\\ \gamma & \mapsto f(\gamma), \end{cases}$$
$$N_2 : \begin{cases} X_T^{\varphi,P} & \to C^{\frac{1+\alpha}{2}}([0,T];\mathbb{R}^4),\\ \gamma & \mapsto b(\gamma) \end{cases}$$

where $X_T^{\varphi,P} = \{\gamma \in X_T \,\text{such that}\; \gamma_{|t=0} = \varphi \;\text{and}\, \gamma^i(t,1) = P^i \;\text{for}\; i \in \{1,2,3\}\}$.

Then γ is a solution for system (2.3) if and only if $\gamma \in X_T^{\varphi}$ and

$$L_T(\gamma) = N_T(\gamma) \qquad \Longleftrightarrow \qquad \gamma = L_T^{-1}N_T(\gamma) := K_T(\gamma)\,.$$

Hence there exists a unique solution to system (2.3) if and only if $K_T : X_T^{\varphi,P} \to X_T^{\varphi,P}$ has a unique fixed point. By the contraction mapping principle it is enough to show that K is a contraction. This result conclude the proof of Theorem 3.1. □

The method of Bronsard and Reitich extends to the case of a networks with several 3–points and end–points. Indeed such method relies on the uniform parabolicity of the system (which is the same) and on the fact that the complementary and compatibility conditions are satisfied.

We have only to define what is an admissible initial parametrization of a network.

Definition 3.5 (Admissible initial parametrization of a network)**.** We say that a parametrization $\varphi = (\varphi^1, \dots, \varphi^n)$ of a geometric admissible network $\mathcal{N}_0$ composed by n curves (hence such that $\cup_{i=1}^n \varphi^i([0,1]) = \mathcal{N}_0$) is an admissible initial one if each curve φ^i is regular and of class $C^{2+\alpha}([0,1])$, at the end—points $\varphi^i(1) = P^i$ it holds $\varphi^i_{xx}(1) = 0$ and at any 3–point O^p we have

$$\varphi^{p1}(O^p) = \varphi^{p2}(O^p) = \varphi^{p3}(O^p)\,,$$
$$\frac{\varphi^{p1}_x(O^p)}{|\varphi^{p1}_x(O^p)|} + \frac{\varphi^{p2}_x(O^p)}{|\varphi^{p2}_x(O^p)|} + \frac{\varphi^{p3}_x(O^p)}{|\varphi^{p3}_x(O^p)|} = 0\,,$$
$$\frac{\varphi^{p1}_{xx}(O^p)}{|\varphi^{p1}_x(O^p)|^2} = \frac{\varphi^{p2}_{xx}(O^p)}{|\varphi^{p2}_x(O^p)|^2} = \frac{\varphi^{p3}_{xx}(O^p)}{|\varphi^{p3}_x(O^p)|^2}$$

where we abused a little the notation as in Definition 2.13.

Theorem 3.6. *Given an admissible initial parametrization $\varphi = (\varphi^1, \dots, \varphi^n)$ of a geometric admissible network $\mathcal{N}_0$, there exists a unique solution $\gamma = (\gamma^1, \dots, \gamma^n)$ in $C^{\frac{2+\alpha}{2}, 2+\alpha}([0,T] \times [0,1])$ of the following system*

$$\begin{cases} \gamma^i_t(t,x) = \frac{\gamma^i_{xx}(t,x)}{|\gamma^i_x(t,x)|^2} & & \textit{motion by curvature} \\ \gamma^{pj}(t, O^p) = \gamma^{pk}(t, O^p) & \textit{at every } 3\textit{–point } O^p & \textit{concurrency} \\ \sum_{j=1}^3 \frac{\gamma^{pj}_x(t,O^p)}{|\gamma^{pj}_x(t,O^p)|} = 0 & \textit{at every } 3\textit{–point } O^p & \textit{angles condition} \\ \gamma^r(t,1) = P^r & \textit{with } 0 \le r \le l & \textit{Dirichlet boundary condition} \\ \gamma^i(x,0) = \varphi^i(x) & & \textit{initial data} \end{cases} \tag{3.4}$$

(where we used the notation of Definition 2.13) for every $x \in [0,1]$, $t \in [0,T]$ and $i \in \{1,2,\dots,n\}$, $j \neq k \in \{1,2,3\}$ in a positive time interval $[0,T]$.

3.2 Existence and uniqueness

In the previous section we have explained how to obtain a unique solution for short time to system (2.3) and more in general to system (3.4), but till now we have not solved our original problem yet. Indeed in Definition 2.7 of solution of the motion by curvature appears a slightly different system. Moreover Theorem 3.1 (and Theorem 3.6) provides a solution given an **admissible initial parametrization** but in Definition 2.7 we speak of **geometrically admissible initial network**. It is then clear that we have to establish a relation between this two notions.

To this aim the following lemma will be useful.

Lemma 3.7. *Consider a triple junction O where the curves γ^1, γ^2 and γ^3 concur forming angles of 120 degrees (that is $\sum_{i=1}^3 \tau^i = \sum_{i=1}^3 \nu^i = 0$). Then*

$$k^1\nu^1 + \lambda^1\tau^1 = k^2\nu^2 + \lambda^2\tau^2 = k^3\nu^3 + \lambda^3\tau^3\,,$$

is satisfied if and only if

$$k^1 + k^2 + k^3 = 0 \qquad \textit{and} \qquad \lambda^1 + \lambda^2 + \lambda^3 = 0\,.$$

Proof. Suppose that for $i \neq j \in \{1,2,3\}$ we have

$$k^i\nu^i + \lambda^i\tau^i = k^j\nu^j + \lambda^j\tau^j\,.$$

Multiplying these vector equalities by τ^l and ν^l and varying i, j, l, thanks to the conditions $\sum_{i=1}^3 \tau^{pi} = \sum_{i=1}^3 \nu^{pi} = 0$, we get the relations

$$\begin{aligned} \lambda^i &= -\lambda^{i+1}/2 - \sqrt{3}k^{i+1}/2 \\ \lambda^i &= -\lambda^{i-1}/2 + \sqrt{3}k^{i-1}/2 \\ k^i &= -k^{i+1}/2 + \sqrt{3}\lambda^{i+1}/2 \\ k^i &= -k^{i-1}/2 - \sqrt{3}\lambda^{i-1}/2 \end{aligned}$$

with the convention that the second superscripts are to be considered "modulus 3". Solving this system we get

$$\begin{aligned} \lambda^i &= \frac{k^{i-1} - k^{i+1}}{\sqrt{3}} \\ k^i &= \frac{\lambda^{i+1} - \lambda^{i-1}}{\sqrt{3}} \end{aligned}$$

which implies

$$\sum_{i=1}^{3} k^i = \sum_{i=1}^{3} \lambda^i = 0\,. \tag{3.5}$$

□

It is also possible to prove that at each triple junction the following properties hold

$$\begin{aligned}
&\sum_{i=1}^{3}(k^i)^2 = \sum_{i=1}^{3}(\lambda^i)^2 \qquad \text{and} \qquad \sum_{i=1}^{3} k^i\lambda^{pi} = 0\,,\\
&\partial_t^l \sum_{i=1}^{3} k^{pi} = \sum_{i=1}^{3} \partial_t^l k^{pi} = \partial_t^l \sum_{i=1}^{3} \lambda^{pi} = \sum_{i=1}^{3} \partial_t^l \lambda^{pi} = \partial_t \sum_{i=1}^{3} k^{pi}\lambda^{pi} = 0\,,,\\
&\sum_{i=1}^{3}(\partial_t^l k^{pi})^2 = \sum_{i=1}^{3}(\partial_t^l \lambda^{pi})^2 \quad \text{for every } l \in \mathbb{N},\\
&\partial_t^m(k_s^{pi} + \lambda^{pi}k^{pi}) = \partial_t^m(k_s^{pj} + \lambda^{pj}k^{pj}) \quad \text{for every pair } i,j \text{ and } m \in \mathbb{N}.,\\
&\sum_{i=1}^{3} \partial_t^l k^{pi}\, \partial_t^m(k_s^{pi} + \lambda^{pi}k^{pi}) = \sum_{i=1}^{3} \partial_t^l \lambda^{pi}\, \partial_t^m(k_s^{pi} + \lambda^{pi}k^{pi}) = 0 \quad \text{for every } l,m \in \mathbb{N}.
\end{aligned} \tag{3.6}$$

We are ready now to establish the relation between geometrically admissible initial networks and admissible parametrizations.

Lemma 3.8. *Suppose that $\mathbb{T}_0$ is a geometrically admissible initial triod parametrized by $\gamma = (\gamma^1, \gamma^2, \gamma^3)$. Then there exist three smooth functions $\theta^i : [0,1] \to [0,1]$ such that the reparametrization $\varphi := \left(\gamma^1 \circ \theta^1, \gamma^2 \circ \theta^2, \gamma^3 \circ \theta^3\right)$ is an admissible initial parametrization.*

Proof. Consider $\gamma = (\gamma^1, \gamma^2, \gamma^3)$ the parametrization of class $C^{2+\alpha}$ of $\mathbb{T}_0$ (that exists as $\mathbb{T}_0$ is a geometrically admissible initial triod). It is not restrictive to suppose that $\gamma^1(0) = \gamma^2(0) = \gamma^3(0)$ is the triple junction and that $\gamma^i(1) = P^i \in \partial\Omega$ with $i \in \{1,2,3\}$.

We look for smooth maps $\theta^i : [0,1] \to [0,1]$ such that $\theta^i_x(x) \neq 0$ for every $x \in [0,1]$, $\theta^i(0) = 0$ and $\theta^i(1) = 1$. Then conditions 1. 2. 3. and 6. of Definition 2.14 are satisfied.

Condition 4. at the triple junction is true for any choice of the θ^i as it involves the unit tangent vectors that are invariant under reparametrization.

We pass now to Condition 5. namely we want that

$$\frac{\varphi^1_{xx}}{|\varphi^1_x|^2} = \frac{\varphi^2_{xx}}{|\varphi^2_x|^2} = \frac{\varphi^3_{xx}}{|\varphi^3_x|^2}\,. \tag{3.7}$$

We indicate with the subscript γ or φ the geometric quantities computed for the parametrization γ or φ, respectively. We define $\lambda^i := \frac{\langle \varphi^i_{xx} | \varphi^i_x \rangle}{|\varphi^i_x|^3}$. Then (3.7) can be equivalently written as

$$k^1_\varphi \nu^1_\varphi + \lambda^1 \tau^1_\varphi = k^2_\varphi \nu^2_\varphi + \lambda^2 \tau^2_\varphi = k^3_\varphi \nu^3_\varphi + \lambda^3 \tau^3_\varphi\,, \tag{3.8}$$

and, as all the geometric quantities involved are invariant under reparametrization, the equality (3.8) is nothing else than

$$k^1_\gamma \nu^1_\gamma + \lambda^1 \tau^1_\gamma = k^2_\gamma \nu^2_\gamma + \lambda^2 \tau^2_\gamma = k^3_\gamma \nu^3_\gamma + \lambda^3 \tau^3_\gamma\,,$$

that by Lemma 3.7 is satisfied if and only if

$$k_\gamma^1 + k_\gamma^2 + k_\gamma^3 = 0 \qquad \text{and} \qquad \lambda^1 + \lambda^2 + \lambda^3 = 0\,. \tag{3.9}$$

To satisfy Condition 7. we need a similar request. Indeed $\varphi_{xx}^i = 0$ at every end–point of the network is equivalent to the condition $k_\gamma^i \nu_\gamma^i + \lambda^i \tau_\gamma^i = 0$, that is satisfied if and only if

$$k_\gamma^i = 0 \qquad \text{and} \qquad \lambda^i = 0 \tag{3.10}$$

at every end–point of the network.

Hence, we only need to find C^∞ reparametrizations θ^i such that at the borders of $[0,1]$ the values of λ^i are given by the relations in (3.9) and (3.10). This can be easily done since at the borders of the interval $[0,1]$ we have $\theta^i(0) = 0$ and $\theta^i(1) = 1$, hence

$$\lambda^i = \frac{\langle \varphi_{xx}^i \,|\, \varphi_x^i \rangle}{|\varphi_x^i|^3} = -\partial_x \frac{1}{|\varphi_x^i|} = -\partial_x \frac{1}{|\gamma_x^i \circ \theta^i| \theta_x^i} = \frac{\langle \gamma_{xx}^i \,|\, \gamma_x^i \rangle}{|\gamma_x^i|^3} + \frac{\theta_{xx}^i}{|\sigma_x^i||\theta_x^i|^2} = \lambda_\gamma^i + \frac{\theta_{xx}^i}{|\sigma_x^i||\theta_x^i|^2}$$

where $\lambda_\gamma^i = \frac{\langle \gamma_{xx}^i \,|\, \gamma_x^i \rangle}{|\gamma_x^i|^3}$.

Choosing any C^∞ functions θ^i with $\theta_x^i(0) = \theta_x^i(1) = 1$, $\theta(1)_{xx}^i = -\lambda_\gamma^i |\gamma_x^i||\theta_x^i|^2$ and

$$\theta(0)_{xx}^i = \left(\frac{k_\gamma^{i-1} - k_\gamma^{i+1}}{\sqrt{3}} - \lambda_\gamma^i \right) |\gamma_x^i||\theta_x^i|^2$$

(for instance, one can use a polynomial function) the reparametrization $\varphi = (\varphi^1, \varphi^2, \varphi^3)$ satisfies Conditions 1. to 7. of Definition 2.14 and the proof is completed. □

Remark 3.9. Vice versa if φ is an admissible initial parametrization, then the triod $\cup_{i=1}^3 \varphi^i([0,1])$ is clearly a geometrically admissible initial network. Indeed one uses Lemma 3.7 to get that the sum of the curvature at the junction is zero. The other properties are trivially verified.

We are ready now to discuss existence and uniqueness of solution of the geometric problem. We need to introduce the notion of geometric uniqueness because even if the solution γ of system (2.3) is unique, there are anyway several solutions of Problem 2.5 obtained by reparametrizing γ.

Definition 3.10. We say that Problem 2.5 admits a **geometrically unique** solution if there exists a unique family of time–dependent networks (sets) $(\mathcal{N}_t)_{t\in[0,T]}$ satisfying the definition of solution 2.7.

In particular this means that all the solutions (functions) satisfying system (2.1) can be obtained one from each other by means of time–depending reparametrization.

Theorem 3.11 (Geometric uniqueness)**.** *Let $\mathbb{T}_0$ be a geometrically admissible initial triod. Then there exists a geometrically unique solution of Problem* (2.3) *in a positive time interval* $[0, \widetilde{T}]$.

Proof. Let $\mathbb{T}_0$ be a geometrically admissible initial triod parametrized by $\gamma_0 = (\gamma_0^1, \gamma_0^2, \gamma_0^3)$ admissible initial parametrization (that always exists thanks to Lemma 3.8). Then by Theorem 3.1 there exists a unique solution $\gamma = (\gamma^1, \gamma^2, \gamma^3)$ to system (2.3) with initial data $\gamma_0 = (\gamma_0^1, \gamma_0^2, \gamma_0^3)$ in a positive time interval $[0,T]$. In particular $(\mathbb{T}_t)_{t\in[0,T]} = (\cup_{i=1}^3 \gamma^i([0,1])_t)_{t\in[0,T]}$ is a solution of the motion by curvature.

Suppose by contradiction that there exists another solution $(\widetilde{\mathbb{T}}_t)_{t\in[0,T']}$ to Problem 2.5 with the same initial $\mathbb{T}_0$. Let this solution be parametrized by $\tilde{\gamma} = (\tilde{\gamma}^1, \tilde{\gamma}^2, \tilde{\gamma}^3)$ with

$$\widetilde{\gamma}^i \in C^{\frac{2+\alpha}{2}, 2+\alpha}([0, T'] \times [0,1])\,.$$

We want to show that the sets $\mathbb{T}$ and $\widetilde{\mathbb{T}}$ coincide, namely that $\tilde{\gamma}$ coincides to γ up to a reparametrization of the curves $\widetilde{\gamma}(\cdot, t)$ for every $t \in [0, \min\{T, T'\})$.

Let $\varphi^i : [0, \min\{T, T'\}] \times [0,1] \to [0,1]$ be in $C^{\frac{2+\alpha}{2}, 2+\alpha}([0, \min\{T, T'\}] \times [0,1])$ and consider the reparametrizations $\overline{\gamma}^i(t,x) = \widetilde{\gamma}^i(t, \varphi^i(t,x))$. We have $\overline{\gamma}^i \in C^{\frac{2+\alpha}{2}, 2+\alpha}([0, \min\{T, T'\}) \times [0,1])$ and

$$\begin{aligned}
\overline{\gamma}^i_t(t,x) =& \,\partial_t[\widetilde{\gamma}^i(t, \varphi^i(t,x))]\\
=& \,\widetilde{\gamma}^i_t(t, \varphi^i(t,x)) + \widetilde{\gamma}^i_x(t, \varphi^i(t,x))\varphi^i_t(t,x)\\
=& \,\widetilde{k}^i(t, \varphi^i(t,x))\widetilde{\nu}^i(t, \varphi^i(t,x)) + \widetilde{\lambda}^i(t, \varphi^i(t,x))\widetilde{\tau}^i(t, \varphi^i(t,x))\\
&+ \widetilde{\gamma}^i_x(t, \varphi^i(t,x))\varphi^i_t(t,x)\\
=& \left\langle \frac{\widetilde{\gamma}^i_{xx}\left(t, \varphi^i(t,x)\right)}{\left|\widetilde{\gamma}^i_x\left(t, \varphi^i(t,x)\right)\right|^2} \,\middle|\, \widetilde{\nu}^i(t, \varphi^i(t,x)) \right\rangle \widetilde{\nu}^i(t, \varphi^i(t,x))\\
&+ \widetilde{\lambda}^i(t, \varphi^i(t,x)) \frac{\widetilde{\gamma}^i_x(t, \varphi^i(t,x))}{\left|\widetilde{\gamma}^i_x\left(t, \varphi^i(t,x)\right)\right|} + \widetilde{\gamma}^i_x(t, \varphi^i(t,x))\varphi^i_t(t,x)\,.
\end{aligned}$$

We ask now the maps φ^i to be solutions for some positive interval of time $[0, T'']$ of the following quasilinear PDE's

$$\begin{aligned}
\varphi^i_t(t,x) =& \frac{1}{\left|\widetilde{\gamma}^i_x\left(t, \varphi^i(t,x)\right)\right|} \left\langle \frac{\widetilde{\gamma}^i_{xx}\left(t, \varphi^i(t,x)\right)}{\left|\widetilde{\gamma}^i_x\left(t, \varphi^i(t,x)\right)\right|^2} \,\middle|\, \frac{\widetilde{\gamma}^i_x(t, \varphi^i(t,x))}{\left|\widetilde{\gamma}^i_x\left(t, \varphi^i(t,x)\right)\right|} \right\rangle\\
&- \frac{\widetilde{\lambda}^i(t, \varphi^i(t,x))}{\left|\widetilde{\gamma}^i_x\left(t, \varphi^i(t,x)\right)\right|} + \frac{\varphi^i_{xx}(t,x)}{\left|\widetilde{\gamma}^i_x\left(t, \varphi^i(t,x)\right)\right|^2 \left|\varphi^i_x(t,x)\right|^2}\,,
\end{aligned}$$

with $\varphi^i(t,0) = 0$, $\varphi^i(t,1) = 1$, $\varphi^i(0,x) = x$ (hence, $\overline{\gamma}^i(0,x) = \gamma^i(0,x) = \sigma^i(x)$) and $\varphi_x(t,x) \neq 0$. The existence of such solutions follows by standard theory of second order quasilinear parabolic equations (see [25, 27]). Then we have

$$\begin{aligned}
\overline{\gamma}^i_t(t,x) =& \left\langle \frac{\widetilde{\gamma}^i_{xx}\left(t, \varphi^i(t,x)\right)}{\left|\widetilde{\gamma}^i_x\left(t, \varphi^i(t,x)\right)\right|^2} \,\middle|\, \widetilde{\nu}^i(t, \varphi^i(t,x)) \right\rangle \widetilde{\nu}^i(t, \varphi^i(t,x))\\
&+ \left\langle \frac{\widetilde{\gamma}^i_{xx}\left(t, \varphi^i(t,x)\right)}{\left|\widetilde{\gamma}^i_x\left(t, \varphi^i(t,x)\right)\right|^2} \,\middle|\, \frac{\widetilde{\gamma}^i_x(t, \varphi^i(t,x))}{\left|\widetilde{\gamma}^i_x\left(t, \varphi^i(t,x)\right)\right|} \right\rangle \frac{\widetilde{\gamma}^i_x\left(t, \varphi^i(t,x)\right)}{\left|\widetilde{\gamma}^i_x\left(t, \varphi^i(t,x)\right)\right|}\\
&+ \frac{\varphi^i_{xx}(t,x)\widetilde{\gamma}^i_x\left(t, \varphi^i(t,x)\right)}{\left|\widetilde{\gamma}^i_x\left(t, \varphi^i(t,x)\right)\right|^2 \left|\varphi^i_x(t,x)\right|^2}\\
=& \frac{\widetilde{\gamma}^i_{xx}\left(t, \varphi^i(t,x)\right)}{\left|\widetilde{\gamma}^i_x\left(t, \varphi^i(t,x)\right)\right|^2} + \frac{\varphi^i_{xx}(t,x)\widetilde{\gamma}^i_x\left(t, \varphi^i(t,x)\right)}{\left|\widetilde{\gamma}^i_x\left(t, \varphi^i(t,x)\right)\right|^2 \left|\varphi^i_x(t,x)\right|^2}\\
=& \frac{\overline{\gamma}^i_{xx}(t,x)}{|\overline{\gamma}^i_x(t,x)|^2}\,.
\end{aligned}$$

By the uniqueness result of Theorem 3.6 we can then conclude that $\overline{\gamma}^i = \gamma^i$ for every $i \in \{1, 2, \ldots, n\}$, hence $\gamma^i(t,x) = \widetilde{\gamma}^i(t, \varphi^i(t,x))$ in the time interval $[0, \widetilde{T}]$ where $\widetilde{T} := \min\{T, T', T''\}$. □

3.3 Geometric properties of the flow

In Definition 2.1 of network we require that the curves are injective and regular. The second assumption is needed to define the flow because $|\gamma_x|$ appears at the denominator. For the short time existence of the flow we did not require that the curves are embedded. We now show that if the initial network is embedded then the evolving networks stay embedded and intersect the boundary of Ω only at the fixed end–points (transversally).

Proposition 3.12. *Let $\mathcal{N}_t$ be the curvature flow of a regular network in a smooth, convex, bounded, open set Ω, with fixed end–points on the boundary of Ω, for $t \in [0,T)$. Then, for every time $t \in [0,T)$, the network $\mathcal{N}_t$ intersects the boundary of Ω only at the end–points and such intersections are transversal for every positive time. Moreover, $\mathcal{N}_t$ remains embedded.*

Proof. By continuity, the 3–points cannot hit the boundary of Ω at least for some time $T' > 0$. The convexity of Ω and the strong maximum principle (see [37]) imply that the network cannot intersect the boundary for the first time at an inner regular point. As a consequence, if $t_0 > 0$ is the "first time" when the $\mathcal{N}_t$ intersects the boundary at an inner point, this latter has to be a 3–point. The minimality of t_0 is then easily contradicted by the convexity of Ω, the 120 degrees condition and the nonzero length of the curves of $\mathcal{N}_{t_0}$.
Even if some of the curves of the initial network are tangent to $\partial\Omega$ at the end–points, by the strong maximum principle, as Ω is convex, the intersections become immediately transversal and stay so for every subsequent time.
Finally, if the evolution $\mathcal{N}_t$ loses embeddedness for the first time, this cannot happen neither at a boundary point, by the argument above, nor at a 3–point, by the 120 degrees condition. Hence it must happen at interior regular points, but this contradicts the strong maximum principle. □

Proposition 3.13. *In the same hypotheses of the previous proposition, if the smooth, bounded, open set Ω is strictly convex, for every fixed end–point P^r on the boundary of Ω, for $r \in \{1, 2, \dots, l\}$, there is a time $t_r \in (0,T)$ and an angle α_r smaller than $\pi/2$ such that the curve of the network arriving at P^r form an angle less that α_r with the inner normal to the boundary of Ω, for every time $t \in (t_r, T)$.*

Proof. We observe that the evolving network $\mathcal{N}_t$ is contained in the convex set $\Omega_t \subset \Omega$, obtained by letting $\partial\Omega$ (which is a finite set of smooth curves with end–points P^r) move by curvature keeping fixed the end–points P^r (see [22, 41, 42]). By the strict convexity of Ω and strong maximum principle, for every positive $t > 0$, the two curves of the boundary of Ω concurring at P^r form an angle smaller that π which is not increasing in time. Hence, the statement of the proposition follows. □

4 Self-similar solutions

Once established the existence of solution for a short time, we want to analyse the behavior of the flow in the long time. A good way to understand more about the flow is looking for examples of solutions.

A straight line is perhaps the easiest example. It is also easy to see that an infinite flat triod with the triple junction at the origin (called **standard triod**) is a solution (Figure 4).

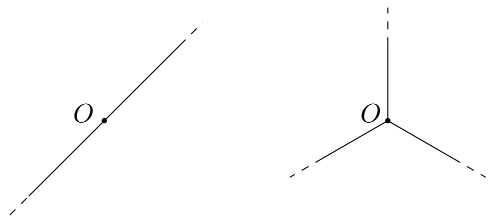

Figure 4: A straight line and a standard triod are solutions of the motion by curvature

In both these examples the existence is global in time and the set does not change shape during the evolution. From this last observation one could guess that there is an entire class of solutions that preserve their shape in time. We try to classify now self–similar solution in a systematic way.

Let us start looking for self–similar **translating** solutions.

Suppose that we have a translating curve γ solving the motion by curvature with initial data σ. We can write $\gamma(t,x) = \eta(x) + w(t)$. The motion by curvature equation $k(t,x) = \langle \gamma_t(t,x), \nu(t,x)\rangle$ in this case reads as $k(x) = \langle w'(t), \nu(x)\rangle$. As a consequence $w(t)$ is constant, hence we are allowed to write $\gamma(t,x) = \eta(x) + t\boldsymbol{v}$ with $\boldsymbol{v} \in \mathbb{R}^2$, and we obtain

$$k(x) = \langle \boldsymbol{v}, \nu(x)\rangle \ .$$

The reverse is also true: if a curve γ satisfies $k(x) = \langle \boldsymbol{v}, \nu(x)\rangle$, then γ is a translating solution of the curvature flow. By integrating this ODE (with $\boldsymbol{v} = \boldsymbol{e}_1$) one can see that the only translating curve is given by the graph of the function $x = -\log\cos y$ in the interval $(-\frac{\pi}{2}, \frac{\pi}{2})$. Grayson in [19] named this curve the **grim reaper** (Figure 5).

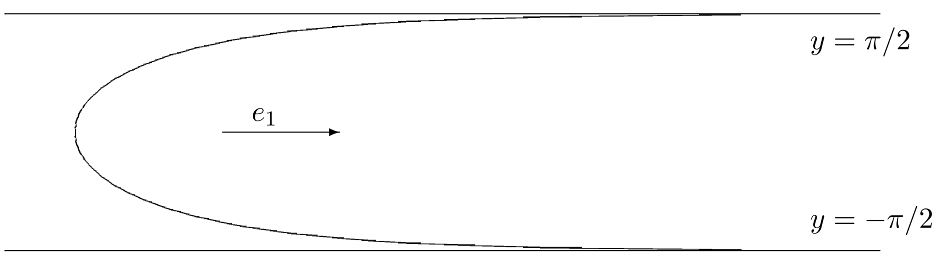

Figure 5: The *grim reaper* relative to e_1.

Passing from a single curve to a regular network, the situation becomes more delicate. Every curve of the translating network has to satisfies $k^i(x) = \left\langle \boldsymbol{v}, \nu^i(x)\right\rangle$. A result for translating triods can be found in [33, Lemma 5.8]: a closed, unbounded and embedded regular triod $\mathbb{R}^2$ self–translating with velocity $\boldsymbol{v} \neq 0$ is composed by halflines parallel to $\boldsymbol{v}$ or translated copies of pieces of the grim reaper relative to $\boldsymbol{v}$, meeting at the 3–point with angles of 120 degrees. Notice that at most one curve is a halfline (Figure 6).

Among curves there are also **rotating** solutions. Suppose indeed that γ is of the form $\gamma(t,x) = R(t)\eta(x)$ with $R(t)$ a rotation. The motion equation becomes

$$k(x) = \left\langle R'(t)\eta(x), R(t)\nu(x)\right\rangle = \left\langle R^t(t)R'(t)\eta(x), \nu(x)\right\rangle \ .$$

We get that $R^t(t)R'(t)$ is constant. By straightforward computations one also get that $R(t)$ is a anticlockwise rotation by ωt, where ω is a given constant. Then

$$k(x) = \omega \left\langle \eta(x), \tau(x)\right\rangle \ .$$

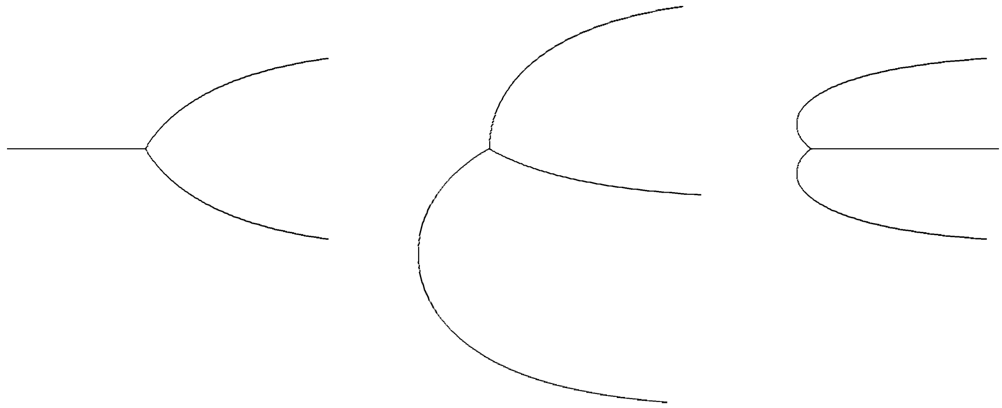

Figure 6: Some examples of translating triods.

A fascinating example can be found in [2]: the Yin–Yang curve.

We have left at last the more significant case: the **self–similarly shrinking networks**.

Suppose that a solution of the motion by curvature evolves homothetically shrinking in time with center of homothety the origin, namely $\gamma(t,x) = \alpha(t)\eta(x)$ with $\alpha(t) > 0$ and $\alpha'(t) < 0$. Being a solution of the flow the curve γ satisfies $k(t,x) = \langle \gamma_t(t,x), \nu(t,x)\rangle$. Then

$$k(x) = \alpha(t)\alpha'(t)\,\langle \eta(x), \nu(x)\rangle\ .$$

We have that $\alpha(t)\alpha'(t)$ is equal to some constant. Up to rescaling we can suppose $\alpha(t)\alpha'(t) = -1$. Then for every $t \in (-\infty, 0]$ we have $\alpha(t) = 2\sqrt{t-T}$ and $k(x) = -\langle \eta(x), \nu(x)\rangle$, or equivalently $\boldsymbol{k}(x) + \eta^{\perp}(x) = 0$.

Definition 4.1. A regular C^2 open network $\mathcal{S}$ union of n curves parametrized by η^i is called a **regular shrinker** if for every curve there holds

$$\boldsymbol{k}^i + (\eta^i)^{\perp} = 0\,.$$

Remark 4.2. Every curve of a regular shrinker satisfies the equation $\boldsymbol{k} + \eta^{\perp} = 0$. As a consequence it must be a piece of a line though the origin or of the so called **Abresch–Langer curves**. Their classification results in [1] imply that any of these non straight pieces is compact. Hence any unbounded curve of a shrinker must be a line or an halfline pointing towards the origin. Moreover, it also follows that if a curve contains the origin, then it is a straight line through the origin or a halfline from the origin.

By the work of Abresch and Langer [1] it follows that the only regular shrinkers without triple junctions (curves) are the lines for the origin and the unit circle. There are two shrinkers with one triple junction [20]: the standard triod and the **Brakke spoon**. The Brakke spoon is a regular shrinker composed by a halfline which intersects a closed curve, forming angles of 120 degrees. It was first mentioned in [10] as an example of evolving network with a loop shrinking down to a point, leaving a halfline that then, in the framework of Brakke flows, vanishes instantaneously. Up to rotation, this particular spoon–shaped network is unique [13] (Figure 7).

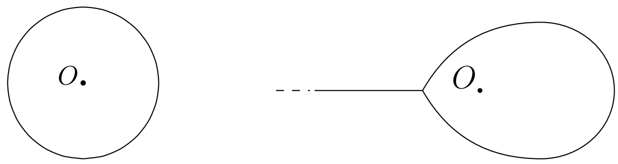

Figure 7: A circle and a Brakke spoon. Together with a straight line and a standard triod, they are all possible regular shrinker with at most one triple junction.

Also the classification of shrinkers with two triple junctions is complete. It is not difficult to show [6,7] that there are only two possible topological shapes for a complete embedded, regular shrinker: one is the “lens/fish” shape and the other is the shape of the Greek “Theta” letter (or “double cell”). It is well known that there exist unique (up to a rotation) lens–shaped or fish–shaped, embedded, regular shrinkers which are symmetric with respect to a line through the origin of $\mathbb{R}^2$ [13,38] (Figure 8). Instead, there are no regular Θ–shaped shrinkers [8].

Figure 8: The **standard lens** is a shrinker with two triple junctions symmetric with respect to two perpendicular axes, composed by two halflines pointing the origin, posed on a symmetry axis and opposite with respect to the other. Each halfline intersects two equal curves forming an angle of 120 degrees. The **fish** is a shrinker with the same topology of the standard lens, but symmetric with respect to only one axis. The two halfines, pointing the origin, intersect two different curves, forming angles of 120 degrees.

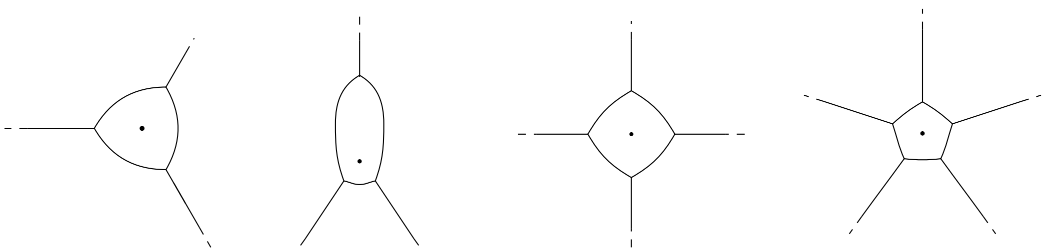

Figure 9: The regular shrinkers with a single bounded region.

The classification of (embedded) regular shrinkers is completed for the shrinkers with a single bounded region [8,12,13,38], see Figure 9.

Several questions (also of independent interest) arise in trying to classify the regular shrinkers. We just mention an open question: does there exist a regular shrinker with more than five unbounded halflines?

Numerical computations, partial results and conjectures can be found in [20].

5 Integral estimates

A good way to understand what happens during the evolution of a network by curvature is to describe the changing in time of the geometric quantities related to the network. For instance we can write the evolution law of the length of the curves or of area enclosed by the curves. In several situations estimating the evolution of the curvature has revealed a winning strategy to pass from short time to long time existence results.

Differently from the case of the curve shortening flow (and of the mean curvature flow) here to obtain our a priori estimates we cannot use the maximum principle and a comparison principle is not valid because of the presence of junctions. Therefore integral estimates are computed in [33, Section 3] in [32, Section 5] in the case of a triod and a regular network, respectively. An outline for the estimates appeared in [23, Section 7], where the authors consider directly the evolution $\gamma_t = k\nu + \lambda\tau$. We summarise here these calculations focusing on the easier cases.

Form now on we suppose that all the derivatives of the functions that appear exist.

We start showing that if a curve moves by curvature, then its time derivative ∂_t and the arclength derivative ∂_s do not commute.

We have already mentioned that the motion by curvature $\gamma_t^{\perp} = \boldsymbol{k}$ can be written as

$$\gamma_t = k\nu + \lambda\tau\,,$$

for some continuous function λ.

Lemma 5.1. *If γ is a curve moving by $\gamma_t = k\nu + \lambda\tau$, then we have the following commutation rule:*

$$\partial_t\partial_s = \partial_s\partial_t + (k^2 - \lambda_s)\partial_s\,. \tag{5.1}$$

Proof. Let $f : [0,1]\times[0,T) \to \mathbb{R}$ be a smooth function, then

$$\begin{aligned}\partial_t\partial_s f - \partial_s\partial_t f &= \frac{f_{tx}}{|\gamma_x|} - \frac{\langle \gamma_x \,|\, \gamma_{xt}\rangle f_x}{|\gamma_x|^3} - \frac{f_{tx}}{|\gamma_x|} = -\langle \tau \,|\, \partial_s\gamma_t\rangle\partial_s f \\ &= -\langle \tau \,|\, \partial_s(\lambda\tau + k\nu)\rangle\partial_s f = (k^2 - \lambda_s)\partial_s f\end{aligned}$$

and the formula is proved. ☐

In all this section we will consider a C^∞ solution of the special flow. Hence each curve is moving by

$$\gamma_t^i(t,x) = \frac{\gamma_{xx}^i\,(t,x)}{|\gamma_x^i\,(t,x)|^2}\,,$$

and $\lambda = \frac{\langle \gamma_{xx} \,|\, \gamma_x\rangle}{|\gamma_x|^3}$.

Using the rule in the previous lemma we can compute

$$
\begin{aligned}
\partial_t \tau &= \partial_t \partial_s \gamma = \partial_s \partial_t \gamma + (k^2 - \lambda_s)\partial_s \gamma = \partial_s(\lambda\tau + k\nu) + (k^2 - \lambda_s)\tau = (k_s + k\lambda)\nu\,,\\
\partial_t \nu &= \partial_t(\mathrm{R}\tau) = \mathrm{R}\,\partial_t \tau = -(k_s + k\lambda)\tau\,,\\
\partial_t k &= \partial_t \langle \partial_s \tau \,|\, \nu\rangle = \langle \partial_t \partial_s \tau \,|\, \nu\rangle = \langle \partial_s \partial_t \tau \,|\, \nu\rangle + (k^2 - \lambda_s)\langle \partial_s \tau \,|\, \nu\rangle\\
&= \partial_s \langle \partial_t \tau \,|\, \nu\rangle + k^3 - k\lambda_s = \partial_s(k_s + k\lambda) + k^3 - k\lambda_s\\
&= k_{ss} + k_s\lambda + k^3\,,\\
\partial_t \lambda &= -\,\partial_t \partial_x \frac{1}{|\gamma_x|} = \partial_x \frac{\langle \gamma_x \,|\, \gamma_{tx}\rangle}{|\gamma_x|^3} = \partial_x \frac{\langle \tau \,|\, \partial_s(\lambda\tau + k\nu)\rangle}{|\gamma_x|} = \partial_x \frac{(\lambda_s - k^2)}{|\gamma_x|}\\
&= \partial_s(\lambda_s - k^2) - \lambda(\lambda_s - k^2) = \lambda_{ss} - \lambda\lambda_s - 2kk_s + \lambda k^2\,.
\end{aligned}
$$

5.1 Evolution of length and volume

We now compute the evolution in time of the total length.

By the commutation formula (5.1) the time derivative of the measure ds on any curve γ^i of the network is given by the measure $(\lambda^i_s - (k^i)^2)\,ds$. Then the evolution law for the length of one curve is

$$
\frac{dL^i(t)}{dt} = \frac{d}{dt}\int_{\gamma^i(\cdot,t)} 1\,ds = \int_{\gamma^i(\cdot,t)} (\lambda^i_s - (k^i)^2)\,ds = \lambda^i(1,t) - \lambda^i(0,t) - \int_{\gamma^i(\cdot,t)} (k^i)^2\,ds\,.
$$

We remind that by relation (3.5) the contributions of λ^{pi} at every 3–point O^p vanish. Suppose that the network has l end–points on the boundary of Ω. With a little abuse of notation we call $\lambda(t, P^r)$ the tangential velocity at the end–point P^r for any $r \in \{1, 2, \dots, l\}$. Since the total length is the sum of the lengths of all the curves, we get

$$
\frac{dL(t)}{dt} = \sum_{r=1}^{l} \lambda(t, P^r) - \int_{\mathcal{N}_t} k^2\,ds\,,
$$

In particular if the end–points P^r of the network are fixed during the evolution all the terms $\lambda(t, P^r)$ are zero and we have

$$
\frac{dL(t)}{dt} = -\int_{\mathcal{N}_t} k^2\,ds\,.
$$

The total length $L(t)$ is decreasing in time and uniformly bounded above by the length of the initial network.

We now discuss the behavior of the area of the regions enclosed by some curves of the evolving regular network. Let us suppose that a region $\mathcal{A}(t)$ is bounded by m curves $\gamma^1, \gamma^2, \dots, \gamma^m$ and let $A(t)$ be its area. We call **loop** ℓ the union of these m curves. The loop ℓ can be regarded as a single piecewise C^2 closed curve parametrized anticlockwise (possibly after reparametrization of the curves that composed it). Hence the curvature of ℓ is positive at the convexity points of the boundary of $\mathcal{A}(t)$. Then we have

$$
A'(t) = -\sum_{i=1}^{m} \int_{\gamma^i} \langle x_t \,|\, \nu\rangle\,ds = -\sum_{i=1}^{m} \int_{\gamma^i} \langle k\nu \,|\, \nu\rangle\,ds = -\sum_{i=1}^{m} \int_{\gamma^i} k\,ds = -\sum_{i=1}^{m} \Delta\theta_i\,, \tag{5.2}
$$

where $\Delta\theta_i$ is the difference in the angle between the unit tangent vector τ and the unit coordinate vector $e_1 \in \mathbb{R}^2$ at the final and initial point of the curve γ^i. Indeed supposing the unit tangent vector of the curve γ^i "lives" in the second quadrant of $\mathbb{R}^2$ (the other cases are analogous) there holds

$$
\partial_s \theta_i = \partial_s \arccos\langle \tau \,|\, e_1\rangle = -\frac{\langle \tau_s \,|\, e_1\rangle}{\sqrt{1 - \langle \tau \,|\, e_1\rangle^2}} = k\,,
$$

so

$$A'(t) = -\sum_{i=1}^{m} \int_{\gamma^i} \partial_s \theta_i \, ds = -\sum_{i=1}^{m} \Delta\theta_i \, .$$

Considering that the curves γ^i form angles of 120 degrees, we have

$$m\pi/3 + \sum_{i=1}^{m} \Delta\theta_i = 2\pi \, .$$

We then obtain the equality (see [34])

$$A'(t) = -(2 - m/3)\pi \, . \tag{5.3}$$

An immediate consequence of (5.3) is that the area of every region bounded by the curves of the network evolves linearly. More precisely it increases if the region has more than six edges, it is constant with six edges and it decreases if its edges are less than six. This implies that if less than six curves of the initial network enclose a region of area A_0, then the maximal time T of existence of a smooth flow is finite and

$$T \leq \frac{A_0}{(2 - m/3)\pi} \leq \frac{3A_0}{\pi} \, .$$

5.2 Evolution of the curvature and its derivatives

We want to estimate the L^2 norm of the curvature and its derivatives, that will result crucial in the analysis of the motion. The main consequence of these computation indeed is that the flow of a regular smooth network with "controlled" end–points exists smooth as long as the curvature stays bounded and none of the lengths of the curves goes to zero (Theorem 5.7).

We consider a regular C^∞ network $\mathcal{N}_t$ in Ω, composed by n curves γ^i with m triple–points $O^1, O^2, \dots, O^m$ and l end–points $P^1, P^2, \dots, P^l$. We suppose that it is a C^∞ solution of the system (3.4). We assume that either the end–points are fixed (the Dirichlet boundary condition in (3.4) is satisfied) or that there exist uniform (in time) constants C_j, for every $j \in \mathbb{N}$, such that

$$|\partial_s^j k(P^r, t)| + |\partial_s^j \lambda(t, P^r)| \leq C_j \, , \tag{5.4}$$

for every $t \in [0, T)$ and $r \in 1, 2, \dots, l$. This second possibility will allow us to localise the estimates if needed.

We are now ready to compute $\frac{d}{dt} \int_{\mathcal{N}_t} |k|^2 \, ds$. We get

$$\frac{d}{dt} \int_{\mathcal{N}_t} |k|^2 \, ds = 2 \int_{\mathcal{N}_t} k \, \partial_t k \, ds + \int_{\mathcal{N}_t} |k|^2 (\lambda_s - k^2) \, ds \, .$$

Using that $\partial_t k = k_{ss} + k_s \lambda + k^3$ we get

$$\frac{d}{dt} \int_{\mathcal{N}_t} |k|^2 \, ds = \int_{\mathcal{N}_t} 2k \, k_{ss} + 2\lambda k \, k_s + k^2 \lambda_s + k^4 \, ds = \int_{\mathcal{N}_t} 2k \, k_{ss} + \partial_s(\lambda \, k^2) + k^4 \, ds \, .$$

Integrating by parts and estimating the contributions given by the end–points P^r by means of assumption (5.4) we can write

$$
\begin{aligned}
\frac{d}{dt}\int_{\mathcal{N}_t}|k|^2\,ds = &-2\int_{\mathcal{N}_t}|k_s|^2\,ds+\int_{\mathcal{N}_t}\partial_s(\lambda\,k^2)\,ds+\int_{\mathcal{N}_t}k^4\,ds\\
&-2\sum_{p=1}^{m}\sum_{i=1}^{3}k^{pi}\,k_s^{pi}\,\Big|_{\text{at the 3–point } O^p}+2\sum_{r=1}^{l}k^r\,k_s^r\,\Big|_{\text{at the end–point } P^r}\\
\leq &-2\int_{\mathcal{N}_t}|k_s|^2\,ds+\int_{\mathcal{N}_t}k^4\,ds+lC_0C_1\\
&-\sum_{p=1}^{m}\sum_{i=1}^{3}2k^{pi}\,k_s^{pi}+\lambda^{pi}|k^{pi}|^2\,\Big|_{\text{at the 3–point } O^p}\,.
\end{aligned}
$$

Then recalling relation (3.6) at the 3–points we have

$$
\sum_{i=1}^{3}k^ik_s^i+\lambda^i|k^i|^2\;=0\,.
$$

Substituting it above we lower the maximum order of the space derivatives of the curvature in the 3–point terms

$$
\frac{d}{dt}\int_{\mathcal{N}_t}k^2\,ds\leq-2\int_{\mathcal{N}_t}|k_s|^2\,ds+\int_{\mathcal{N}_t}k^4\,ds+\sum_{p=1}^{m}\sum_{i=1}^{3}\lambda^{pi}|k^{pi}|^2\,\Big|_{\text{at the 3–point } O^p}+lC_0C_1\,.
$$

We notice that we can estimate the boundary terms at each 3–point of the form $\sum_{i=1}^3\lambda^i|k^i|^2$ by $\sum_{i=1}^3\lambda^i|k^i|^2\leq\|k^3\|_{L^\infty}$ (see [33, Remark 3.9]). Hence

$$
\frac{d}{dt}\int_{\mathcal{N}_t}k^2\,ds\leq-2\int_{\mathcal{N}_t}|k_s|^2\,ds+\int_{\mathcal{N}_t}k^4\,ds+\|k\|^3_{L^\infty}+lC_0C_1\,. \tag{5.5}
$$

From now on we do not use any geometric property of our problem. We suppose that the lengths of curves of the networks are equibounded from below by some positive value. We reduce to estimate the L^4 and L^∞ norm of the curvature of any curve γ^i, seen as a Sobolev function defined on the interval $[0,L(\gamma^i)]$.

Lemma 5.2. *Let $0<\mathrm{L}<+\infty$ and $u\in C^\infty([0,\mathrm{L}],\mathbb{R})$. Then there exists a uniform constant C, depending on* L*, such that*

$$
\|u\|^4_{L^4}+\|u\|^3_{L^\infty}-2\|u'\|^2_{L^2}\leq C\left(\|u\|^2_{L^2}+1\right)^3\,.
$$

Proof. The key estimates of the proof are Gagliardo–Nirenberg interpolation inequalities [35, Section 3, pp. 257–263] written in the form (see also [33, Proposition 3.11])

$$
\begin{aligned}
\|u\|_{L^p}&\leq C_p\|u'\|^{\frac12-\frac1p}_{L^2}\|u\|^{\frac12+\frac1p}_{L^2}+\frac{B_p}{\mathrm{L}^{\frac12-\frac1p}}\|u\|_{L^2}\,,\\
\|u\|_{L^\infty}&\leq C\|u'\|^{\frac12}_{L^2}\|u\|^{\frac12}_{L^2}+\frac{B}{\mathrm{L}^{\frac12}}\|u\|_{L^2}\,.
\end{aligned}
$$

We first focus on the term $\|u\|^4_{L^4}$. We have

$$
\|u\|_{L^4}\leq C\left(\|u'\|^{1/4}_{L^2}\|u\|^{3/4}_{L^2}+\frac{\|u\|_{L^2}}{\mathrm{L}^{\frac12}}\right)\,,
$$

and so

$$\|u\|_{L^4}^4 \leq \widetilde{C}\left(\|u'\|_{L^2}\|u\|_{L^2}^3 + \|u\|_{L^2}^4\right) .$$

Using Young inequality

$$\|u\|_{L^4}^4 \leq \widetilde{C}\left(\varepsilon\|u'\|_{L^2}^2 + c_\varepsilon\|u\|_{L^2}^6 + \|u\|_{L^2}^4\right) . \tag{5.6}$$

Similarly we estimate the term $\|u\|_{L^\infty}^3$ by

$$\|u\|_{L^\infty}^3 \leq C\left(\|u'\|_{L^2}^{\frac{3}{2}}\|u\|_{L^2}^{\frac{3}{2}} + \|u\|_{L^2}^3\right) \leq C\left(\varepsilon\|u'\|_{L^2}^2 + c_\varepsilon\|u\|_{L^2}^6 + \|u\|_{L^2}^3\right) . \tag{5.7}$$

Putting (5.6) and (5.7) together and choosing appropriately ε we obtain

$$\|u\|_{L^4}^4 + \|u\|_{L^\infty}^3 - 2\|u'\|_{L^2}^2 \leq \tilde{c}_\varepsilon\|u\|_{L^2}^6 + \widetilde{C}\|u\|_{L^2}^4 + C\|u\|_{L^2}^3 \leq C\left(\|u\|_{L^2}^2 + 1\right)^3 .$$

□

Applying Lemma 5.2 to the curvature k^i of each curve γ^i of the network the estimate (5.5) becomes

$$\left|\frac{d}{dt}\int_{\mathcal{N}_t} k^2\,ds\right| \leq C\left(\int_{\mathcal{N}_t} k^2\,ds\right)^3 + C + lC_0C_1 . \tag{5.8}$$

The aim now is to repeat the previous computation for $\partial_s^j k$ with $j \in \mathbb{N}$.

Although the calculations are much harder, it is possible to conclude that for every even $j \in \mathbb{N}$ there holds

$$\int_{\mathcal{N}_t} |\partial_s^j k|^2\,ds \leq C\int_0^t \left(\int_{\mathcal{N}_\xi} k^2\,ds\right)^{2j+3} d\xi + C\left(\int_{\mathcal{N}_t} k^2\,ds\right)^{2j+1} + Ct + lC_jC_{j+1}t + C .$$

Passing from integral to L^∞ estimates we have the following proposition.

Proposition 5.3. *If assumption (5.4) holds, the lengths of all the curves are uniformly positively bounded from below and the L^2 norm of k is uniformly bounded on $[0,T)$, then the curvature of $\mathcal{N}_t$ and all its space derivatives are uniformly bounded in the same time interval by some constants depending only on the L^2 integrals of the space derivatives of k on the initial network $\mathcal{N}_0$.*

We now derive a second set of estimates where everything is controlled – still under the assumption (5.4) – only by the L^2 norm of the curvature and the inverses of the lengths of the curves at time zero.

As before we consider the C^∞ special curvature flow $\mathcal{N}_t$ of a smooth network $\mathcal{N}_0$ in the time interval $[0,T)$, composed by n curves $\gamma^i(\cdot,t) : [0,1] \to \overline{\Omega}$ with m triple junctions $O^1, O^2, \ldots, O^m$ and l end–points $P^1, P^2, \ldots, P^l$, satisfying assumption (5.4).

As shown above, the evolution equations for the lengths of the n curves are given by

$$\frac{dL^i(t)}{dt} = \lambda^i(1,t) - \lambda^i(0,t) - \int_{\gamma^i(\cdot,t)} k^2\,ds .$$

Then, proceeding as in the computations above, we get

$$
\begin{aligned}
\frac{d}{dt}\left(\int_{\mathcal{N}_t} k^2\,ds + \sum_{i=1}^n \frac{1}{L^i}\right) \leq & -2\int_{\mathcal{N}_t} k_s^2\,ds + \int_{\mathcal{N}_t} k^4\,ds + 6m\|k\|_{L^\infty}^3 + lC_0C_1 - \sum_{i=1}^n \frac{1}{(L^i)^2}\frac{dL^i}{dt} \\
= & -2\int_{\mathcal{N}_t} k_s^2\,ds + \int_{\mathcal{N}_t} k^4\,ds + 6m\|k\|_{L^\infty}^3 + lC_0C_1 \\
& -\sum_{i=1}^n \frac{\lambda^i(1,0)-\lambda^i(0,t)+\int_{\gamma^i(\cdot,t)} k^2\,ds}{(L^i)^2} \\
\leq & -2\int_{\mathcal{N}_t} k_s^2\,ds + \int_{\mathcal{N}_t} k^4\,ds + 6m\|k\|_{L^\infty}^3 + lC_0C_1 \\
& + 2\sum_{i=1}^n \frac{\|k\|_{L^\infty}+C_0}{(L^i)^2} + \sum_{i=1}^n \frac{\int_{\mathcal{N}_t} k^2\,ds}{(L^i)^2} \\
\leq & -2\int_{\mathcal{N}_t} k_s^2\,ds + \int_{\mathcal{N}_t} k^4\,ds + (6m+2n/3)\|k\|_{L^\infty}^3 + lC_0C_1 + 2nC_0^3/3 \\
& + \frac{n}{3}\left(\int_{\mathcal{N}_t} k^2\,ds\right)^3 + \frac{2}{3}\sum_{i=1}^n \frac{1}{(L^i)^3}\,,
\end{aligned}
$$

where we used Young inequality in the last passage. Proceeding as before, but keeping track of the terms where the inverse of the length appear, it is possible to obtain

$$
\begin{aligned}
\frac{d}{dt}\left(\int_{\mathcal{N}_t} k^2\,ds + \sum_{i=1}^n \frac{1}{L^i}\right) \leq & -\int_{\mathcal{N}_t} k_s^2\,ds + C\left(\int_{\mathcal{N}_t} k^2\,ds\right)^3 + C\sum_{i=1}^n \frac{\left(\int_{\mathcal{N}_t} k^2\,ds\right)^2}{L^i} \\
& + C\sum_{i=1}^n \frac{\left(\int_{\mathcal{N}_t} k^2\,ds\right)^{3/2}}{(L^i)^{3/2}} + C\sum_{i=1}^n \frac{1}{(L^i)^3} + C \\
\leq & \, C\left(\int_{\mathcal{N}_t} k^2\,ds\right)^3 + C\sum_{i=1}^n \frac{1}{(L^i)^3} + C \\
\leq & \, C\left(\int_{\mathcal{N}_t} k^2\,ds + \sum_{i=1}^n \frac{1}{L^i} + 1\right)^3,
\end{aligned}
\tag{5.9}
$$

with a constant C depending only on the structure of the network and on the constants C_0 and C_1 in assumption (5.4).

5.3 Consequences of the estimates

Thanks to the just computed estimates on the curvature and on the inverse of the length one can obtain the following result:

Proposition 5.4. *For every $M > 0$ there exists a time $T_M \in (0, T)$, depending only on the* structure *of the network and on the constants C_0 and C_1 in assumption* (5.4)*, such that if the square of the L^2 norm of the curvature and the inverses of the lengths of the curves of $\mathcal{N}_0$ are bounded by M, then the square of the L^2 norm of k and the inverses of the lengths of the curves of $\mathcal{N}_t$ are smaller than $2(n+1)M + 1$, for every time $t \in [0, T_M]$.*

Proof. Consider the positive function $f(t) = \int_{\mathcal{N}_t} k^2\,ds + \sum_{i=1}^n \frac{1}{L^i(t)} + 1$. Then by inequality (5.9) f satisfies the differential inequality $f' \leq Cf^3$. After integration it reads as

$$f^2(t) \leq \frac{f^2(0)}{1 - 2Ctf^2(0)} \leq \frac{f^2(0)}{1 - 2Ct[(n+1)M + 1]}\,,$$

then if $t \leq T_M = \frac{3}{8C[(n+1)M+1]}$ we get $f(t) \leq 2f(0)$. Hence

$$\int_{\mathcal{N}_t} k^2\,ds + \sum_{i=1}^n \frac{1}{L^i(t)} \leq 2\int_{\mathcal{N}_0} k^2\,ds + 2\sum_{i=1}^n \frac{1}{L^i(0)} + 1 \leq 2[(n+1)M] + 1\,.$$

□

The combination of these estimates implies estimates on all the derivatives of the maps γ^i, stated in the next proposition.

Proposition 5.5. *If $\mathcal{N}_t$ is a C^∞ special evolution of the initial network $\mathcal{N}_0 = \bigcup_{i=1}^n \sigma^i$, satisfying assumption* (5.4)*, such that the lengths of the n curves are uniformly bounded away from zero and the L^2 norm of the curvature is uniformly bounded by some constants in the time interval $[0,T)$, then*

- *all the derivatives in space and time of k and λ are uniformly bounded in $[0,1]\times[0,T)$,*
- *all the derivatives in space and time of the curves $\gamma^i(t,x)$ are uniformly bounded in $[0,1]\times[0,T)$,*
- *the quantities $|\gamma^i_x(t,x)|$ are uniformly bounded from above and away from zero in $[0,1]\times[0,T)$.*

All the bounds depend only on the uniform controls on the L^2 norm of k, on the lengths of the curves of the network from below, on the constants C_j in assumption (5.4)*, on the L^∞ norms of the derivatives of the curves σ^i and on the bound from above and below on $|\sigma^i_x(t,x)|$, for the curves describing the initial network $\mathcal{N}_0$.*

By means of Proposition 5.4 we can strengthen the conclusion of Proposition 5.5.

Corollary 5.6. *In the hypothesis of the previous proposition, in the time interval $[0,T_M]$ all the bounds in Proposition 5.5 depend only on the L^2 norm of k on $\mathcal{N}_0$, on the constants C_j in assumption* (5.4)*, on the L^∞ norms of the derivatives of the curves σ^i, on the bound from above and below on $|\sigma^i_x(t,x)|$ and on the lengths of the curves of the initial network $\mathcal{N}_0$.*

By means of the a priori estimates we can work out some results about the smooth flow of an initial regular geometrically smooth network $\mathcal{N}_0$.

Theorem 5.7. *If $[0,T)$, with $T < +\infty$, is the maximal time interval of existence of a C^∞ curvature flow of an initial geometrically smooth network $\mathcal{N}_0$, then*

1. *either the inferior limit of the length of at least one curve of $\mathcal{N}_t$ is zero, as $t \to T$,*
2. *or $\overline{\lim}_{t\to T} \int_{\mathcal{N}_t} k^2\,ds = +\infty$.*

Proof. We can C^∞ reparametrize the flow $\mathcal{N}_t$ in order that it becomes a special smooth flow $\widetilde{\mathcal{N}}_t$ in $[0,T)$. If the lengths of the curves of $\mathcal{N}_t$ are uniformly bounded away from zero and the L^2 norm of k is bounded, the same holds for the networks $\widetilde{\mathcal{N}}_t$. Then, by Proposition 5.5 and Ascoli–Arzelà Theorem, the network $\widetilde{\mathcal{N}}_t$ converges in C^∞ to a smooth network $\widetilde{\mathcal{N}}_T$ as $t \to T$. We could hence restart the flow obtaining a C^∞ special curvature flow in a longer time interval. Reparametrizing back this last flow, we get a C^∞ "extension" in time of the flow $\mathcal{N}_t$, hence contradicting the maximality of the interval $[0,T)$. □

Proposition 5.8. *If $[0,T)$, with $T<+\infty$, is the maximal time interval of existence of a C^∞ curvature flow of an initial geometrically smooth network $\mathcal{N}_0$. If the lengths of the n curves are uniformly positively bounded from below, then this superior limit is actually a limit and there exists a positive constant C such that*

$$\int_{\mathcal{N}_t} k^2\,ds \geq \frac{C}{\sqrt{T-t}}\,,$$

for every $t\in[0,T)$.

Proof. Considering the flow $\widetilde{\mathcal{N}}_t$ introduced in the previous theorem. By means of differential inequality (5.8), we have

$$\frac{d}{dt}\int_{\widetilde{\mathcal{N}}_t}\widetilde{k}^2\,ds \leq C\left(\int_{\widetilde{\mathcal{N}}_t}\widetilde{k}^2\,ds\right)^3 + C \leq C\left(1+\int_{\widetilde{\mathcal{N}}_t}\widetilde{k}^2\,ds\right)^3,$$

which, after integration between $t,r\in[0,T)$ with $t<r$, gives

$$\frac{1}{\left(1+\int_{\widetilde{\mathcal{N}}_t}\widetilde{k}^2\,ds\right)^2} - \frac{1}{\left(1+\int_{\widetilde{\mathcal{N}}_r}\widetilde{k}^2\,ds\right)^2} \leq C(r-t)\,.$$

Then, if case (1) does not hold, we can choose a sequence of times $r_j\to T$ such that $\int_{\widetilde{\mathcal{N}}_{r_j}}\widetilde{k}^2\,ds\to+\infty$. Putting $r=r_j$ in the inequality above and passing to the limit, as $j\to\infty$, we get

$$\frac{1}{\left(1+\int_{\widetilde{\mathcal{N}}_t}\widetilde{k}^2\,ds\right)^2} \leq C(T-t)\,,$$

hence, for every $t\in[0,T)$,

$$\int_{\widetilde{\mathcal{N}}_t}\widetilde{k}^2\,ds \geq \frac{C}{\sqrt{T-t}} - 1 \geq \frac{C}{\sqrt{T-t}}\,,$$

for some positive constant C and $\lim_{t\to T}\int_{\widetilde{\mathcal{N}}_t} k^2\,ds=+\infty$.
By the invariance of the curvature by reparametrization, this last estimate implies the same estimate for the flow $\mathcal{N}_t$. □

This theorem obviously implies the following corollary.

Corollary 5.9. *If $[0,T)$, with $T<+\infty$, is the maximal time interval of existence of a C^∞ curvature flow of an initial geometrically smooth network $\mathcal{N}_0$ and the lengths of the curves are uniformly bounded away from zero, then*

$$\max_{\mathcal{N}_t} k^2 \geq \frac{C}{\sqrt{T-t}} \to +\infty\,,$$

as $t\to T$.

In the case of the evolution γ_t of a single closed curve in the plane there exists a constant $C>0$ such that if at time $T>0$ a singularity develops, then

$$\max_{\gamma_t} k^2 \geq \frac{C}{T-t}$$

for every $t\in[0,T)$ (see [21]). It is unknown if this lower bound on the rate of blow-up of the curvature holds also in the case of the evolution of a network.

Remark 5.10. Using more refine estimates it is possible to weaken the assumption of Theorem 5.7: one can suppose to have a $C^{\frac{2+\alpha}{2},2+\alpha}$ curvature flow (see [32, p. 33]).

We conclude this section with the following estimate from below on the maximal time of smooth existence.

Proposition 5.11. *For every $M > 0$ there exists a positive time T_M such that if the L^2 norm of the curvature and the inverses of the lengths of the geometrically smooth network $\mathcal{N}_0$ are bounded by M, then the maximal time of existence $T > 0$ of a C^∞ curvature flow of $\mathcal{N}_0$ is larger than T_M.*

Proof. As before, considering again the reparametrized special curvature flow $\widetilde{\mathcal{N}}_t$, by Proposition 5.4 in the interval $[0, \min\{T_M, T\})$ the L^2 norm of $\widetilde{k}$ and the inverses of the lengths of the curves of $\widetilde{\mathcal{N}}_t$ are bounded by $2M^2 + 6M$.
Then, by Theorem 5.7, the value $\min\{T_M, T\}$ cannot coincide with the maximal time of existence of $\widetilde{\mathcal{N}}_t$ (hence of $\mathcal{N}_t$), so it must be $T > T_M$. □

6 Analysis of singularities

6.1 Huisken's Monotonicity Formula

We shall use the following notation for the evolution of a network in $\Omega \subset \mathbb{R}^2$: let $\mathcal{N} \subset \mathbb{R}^2$ be a network homeomorphic to the all $\mathcal{N}_t$, we consider a map

$$F : (0, T) \times \mathcal{N} \to \mathbb{R}^2$$

given by the union of the maps $\gamma^i : (0, T) \times I_i \to \overline{\Omega}$ (with I_i the intervals $[0, 1], (0, 1], [1, 0)$ or $(0, 1))$ describing the curvature flow of the network in the time interval $(0, T)$, that is $\mathcal{N}_t = F(t, \mathcal{N})$.

Let us start from the easiest case in which the network is composed by a unique closed simple smooth curve. Let $t_0 \in (0, +\infty), x_0 \in \mathbb{R}^2$ and $\rho_{t_0,x_0} : [0, t_0) \times \mathbb{R}^2$ be the one–dimensional **backward heat kernel** in $\mathbb{R}^2$ relative to (t_0, x_0), that is

$$\rho_{t_0,x_0}(t, x) = \frac{e^{-\frac{|x-x_0|^2}{4(t_0-t)}}}{\sqrt{4\pi(t_0 - t)}}.$$

Theorem 6.1 (Monotonicity Formula)**.** *Assume $t_0 > 0$. For every $t \in [0, \min\{t_0, T\})$ and $x_0 \in \mathbb{R}^2$ we have*

$$\frac{d}{dt} \int_{\mathcal{N}_t} \rho_{t_0,x_0}(t, x)\, ds = - \int_{\mathcal{N}_t} \left| \boldsymbol{k} + \frac{(x - x_0)^\perp}{2(t_0 - t)} \right|^2 \rho_{t_0,x_0}(t, x)\, ds$$

Proof. See [21]. □

Then one can wonder if a modified version of this formula holds for networks. Clearly one needs a way to deal with the boundary points (the triple junctions). In [33] the authors gave a positive answer to this question in the case of a triod. With a slight modification of the computation in [33, Lemma 6.3] one can extend the result to any regular network. As before, with a little abuse of notation, we will write $\tau(t, P^r)$ and $\lambda(t, P^r)$ respectively for the unit tangent vector and the tangential velocity at the end–point P^r of the curve of the network getting at such point, for any $r \in \{1, 2, \ldots , l\}$.

Proposition 6.2 (Monotonicity Formula)**.** *Assume $t_0 > 0$. For every $t \in [0, \min\{t_0, T\})$ and $x_0 \in \mathbb{R}^2$ the following identity holds*

$$\frac{d}{dt}\int_{\mathcal{N}_t} \rho_{t_0,x_0}(t,x)\,ds = -\int_{\mathcal{N}_t}\left|\boldsymbol{k} + \frac{(x-x_0)^\perp}{2(t_0-t)}\right|^2 \rho_{t_0,x_0}(t,x)\,ds \\ + \sum_{r=1}^{l}\left[\left\langle \frac{P^r - x_0}{2(t_0-t)}\,\middle|\,\tau(t,P^r)\right\rangle - \lambda(t,P^r)\right]\rho_{t_0,x_0}(t,P^r)\,.$$

Integrating between t_1 and t_2 with $0 \le t_1 \le t_2 < \min\{t_0, T\}$ we get

$$\int_{t_1}^{t_2}\int_{\mathcal{N}_t}\left|\boldsymbol{k} + \frac{(x-x_0)^\perp}{2(t_0-t)}\right|^2 \rho_{t_0,x_0}(t,x)\,ds\,dt = \int_{\mathcal{N}_{t_1}} \rho_{t_0,x_0}(x,t_1)\,ds - \int_{\mathcal{N}_{t_2}} \rho_{t_0,x_0}(x,t_2)\,ds \\ + \sum_{r=1}^{l}\int_{t_1}^{t_2}\left[\left\langle \frac{P^r - x_0}{2(t_0-t)}\,\middle|\,\tau(t,P^r)\right\rangle - \lambda(t,P^r)\right]\rho_{t_0,x_0}(t,P^r)\,dt\,.$$

We need the following lemma in order to estimate the end–points contribution (see [33, Lemma 6.5]).

Lemma 6.3. *For every $r \in \{1, 2, \dots, l\}$ and $x_0 \in \mathbb{R}^2$, the following estimate holds*

$$\left|\int_{t}^{t_0}\left[\left\langle \frac{P^r - x_0}{2(t_0-\xi)}\,\middle|\,\tau(\xi,P^r)\right\rangle - \lambda(\xi,P^r)\right]\rho_{t_0,x_0}(\xi,P^r)\,d\xi\right| \le C\,,$$

where C is a constant depending only on the constants C_l in assumption (5.4).
Then for every point $x_0 \in \mathbb{R}^2$, we have

$$\lim_{t\to t_0}\sum_{r=1}^{l}\int_{t}^{t_0}\left[\left\langle \frac{P^r - x_0}{2(t_0-\xi)}\,\middle|\,\tau(\xi,P^r)\right\rangle - \lambda(\xi,P^r)\right]\rho_{t_0,x_0}(\xi,P^r)\,d\xi = 0\,.$$

As a consequence, the following definition is well posed.

Definition 6.4 (Gaussian densities)**.** For every $t_0 \in (0, +\infty)$, $x_0 \in \mathbb{R}^2$ we define the **Gaussian density** function $\Theta_{t_0,x_0} : [0, \min\{t_0, T\}) \to \mathbb{R}$ as

$$\Theta_{t_0,x_0}(t) = \int_{\mathcal{N}_t} \rho_{t_0,x_0}(t,\cdot)\,ds$$

and provided $t_0 \le T$ the **limit Gaussian density** function $\widehat{\Theta} : (0, +\infty) \times \mathbb{R}^2 \to \mathbb{R}$ as

$$\widehat{\Theta}(t_0, x_0) = \lim_{t\to t_0} \Theta_{t_0,x_0}(t)\,.$$

For every $(t_0, x_0) \in (0, T] \times \mathbb{R}^2$, the limit $\widehat{\Theta}(t_0, x_0)$ exists (by the monotonicity of Θ_{t_0,x_0}) it is finite and non negative. Moreover the map $\widehat{\Theta} : \mathbb{R}^2 \to \mathbb{R}$ is upper semicontinuous [29, Proposition 2.12].

6.2 Dynamical rescaling

We introduce the rescaling procedure of Huisken in [21] at the maximal time T.
Fixed $x_0 \in \mathbb{R}^2$, let $\widetilde{F}_{x_0} : [-1/2 \log T, +\infty) \times \mathcal{N} \to \mathbb{R}^2$ be the map

$$\widetilde{F}_{x_0}(\mathfrak{t}, p) = \frac{F(t,p) - x_0}{\sqrt{2(T-t)}} \qquad \mathfrak{t}(t) = -\frac{1}{2}\log(T-t)$$

then, the rescaled networks are given by

$$\widetilde{\mathcal{N}}_{\mathfrak{t},x_0} = \frac{\mathcal{N}_t - x_0}{\sqrt{2(T-t)}} \tag{6.1}$$

and they evolve according to the equation

$$\frac{\partial}{\partial \mathfrak{t}} \widetilde{F}_{x_0}(\mathfrak{t},p) = \widetilde{\boldsymbol{v}}(\mathfrak{t},p) + \widetilde{F}_{x_0}(\mathfrak{t},p)$$

where

$$\widetilde{\boldsymbol{v}}(\mathfrak{t},p) = \sqrt{2(T-t(\mathfrak{t}))} \cdot \boldsymbol{v}(t(\mathfrak{t}),p) = \widetilde{k}\nu + \widetilde{\lambda}\tau \qquad \text{and} \qquad t(\mathfrak{t}) = T - e^{-2\mathfrak{t}}\,.$$

Notice that we did not put the sign "$\sim$" over the unit tangent and normal, since they remain the same after the rescaling.

When there is no ambiguity on the point x_0, we will write $\widetilde{P}^r(\mathfrak{t}) = \widetilde{F}_{x_0}(\mathfrak{t}, P^r)$ for the end–points of the rescaled network $\widetilde{\mathcal{N}}_{\mathfrak{t},x_0}$.

The rescaled curvature evolves according to the following equation,

$$\partial_{\mathfrak{t}} \widetilde{k} = \widetilde{k}_{\mathfrak{s}\mathfrak{s}} + \widetilde{k}_{\mathfrak{s}}\widetilde{\lambda} + \widetilde{k}^3 - \widetilde{k}$$

which can be obtained by means of the commutation law

$$\partial_{\mathfrak{t}}\partial_{\mathfrak{s}} = \partial_{\mathfrak{s}}\partial_{\mathfrak{t}} + (\widetilde{k}^2 - \widetilde{\lambda}_{\mathfrak{s}} - 1)\partial_{\mathfrak{s}}\,,$$

where we denoted with $\mathfrak{s}$ the arclength parameter for $\widetilde{\mathcal{N}}_{\mathfrak{t},x_0}$.

By straightforward computations (see [21]) we have the following rescaled version of the Monotonicity Formula.

Proposition 6.5 (Rescaled Monotonicity Formula)**.** *Let $x_0 \in \mathbb{R}^2$ and set*

$$\widetilde{\rho}(x) = e^{-\frac{|x|^2}{2}}$$

For every $\mathfrak{t} \in [-1/2 \log T, +\infty)$ the following identity holds

$$\frac{d}{d\mathfrak{t}} \int_{\widetilde{\mathcal{N}}_{\mathfrak{t},x_0}} \widetilde{\rho}(x)\, d\mathfrak{s} = -\int_{\widetilde{\mathcal{N}}_{\mathfrak{t},x_0}} |\,\widetilde{\boldsymbol{k}} + x^{\perp}|^2 \widetilde{\rho}(x)\, d\mathfrak{s} + \sum_{r=1}^{l} \Big[\Big\langle\, \widetilde{P}^r(\mathfrak{t}) \,\Big|\, \tau(t(\mathfrak{t}), P^r) \Big\rangle - \widetilde{\lambda}(\mathfrak{t}, P^r)\Big]\, \widetilde{\rho}(\widetilde{P}^r(\mathfrak{t}))$$

where $\widetilde{P}^r(\mathfrak{t}) = \frac{P^r - x_0}{\sqrt{2(T-t(\mathfrak{t}))}}$.

Integrating between $\mathfrak{t}_1$ and $\mathfrak{t}_2$ with $-1/2 \log T \leq \mathfrak{t}_1 \leq \mathfrak{t}_2 < +\infty$ we get

$$\begin{aligned} \int_{\mathfrak{t}_1}^{\mathfrak{t}_2} \int_{\widetilde{\mathcal{N}}_{\mathfrak{t},x_0}} |\,\widetilde{\boldsymbol{k}} + x^{\perp}|^2 \widetilde{\rho}(x)\, d\mathfrak{s}\, d\mathfrak{t} = & \int_{\widetilde{\mathcal{N}}_{\mathfrak{t}_1,x_0}} \widetilde{\rho}(x)\, d\mathfrak{s} - \int_{\widetilde{\mathcal{N}}_{\mathfrak{t}_2,x_0}} \widetilde{\rho}(x)\, d\mathfrak{s} \\ & + \sum_{r=1}^{l} \int_{\mathfrak{t}_1}^{\mathfrak{t}_2} \Big[\Big\langle\, \widetilde{P}^r(\mathfrak{t}) \,\Big|\, \tau(t(\mathfrak{t}), P^r) \Big\rangle - \widetilde{\lambda}(\mathfrak{t}, P^r)\Big]\, \widetilde{\rho}(\widetilde{P}^r(\mathfrak{t}))\, d\mathfrak{t}\,. \end{aligned} \tag{6.2}$$

We have also the analog of Lemma 6.3 (see [33, Lemma 6.7]).

Lemma 6.6. *For every $r \in \{1, 2, \dots, l\}$ and $x_0 \in \mathbb{R}^2$, the following estimate holds for all $\mathfrak{t} \in \big[-\frac{1}{2}\log T, +\infty\big)$,*

$$\left| \int_{\mathfrak{t}}^{+\infty} \Big[\Big\langle\, \widetilde{P}^r(\xi) \,\Big|\, \tau(t(\xi), P^r) \Big\rangle - \widetilde{\lambda}(\xi, P^r)\Big]\, d\xi \right| \leq C\,,$$

where C is a constant depending only on the constants C_l in assumption (5.4).

As a consequence, for every point $x_0 \in \mathbb{R}^2$, we have

$$\lim_{\mathfrak{t} \to +\infty} \sum_{r=1}^{l} \int_{\mathfrak{t}}^{+\infty} \Big[\Big\langle\, \widetilde{P}^r(\xi) \,\Big|\, \tau(t(\xi), P^r) \Big\rangle - \widetilde{\lambda}(\xi, P^r)\Big]\, d\xi = 0\,.$$

6.3 Blow–up limits

We now discuss the possible blow–up limits of an evolving network at the maximal time of existence. This analysis can be seen as a tool to exclude the possible arising of singularity in the evolution and to obtain (if possible) global existence of the flow.

Thanks to Theorem 5.7 we know what happens when the evolution approaches the singular time T: either the length of at least one curve of the network goes to zero, or the L^2-norm of the curvature blows-up. When the curvature does not remain bounded, we look at the possible limit networks after (Huisken's dynamical) rescaling procedure. The rescaled Monotonicity Formula 6.5 will play a crucial role. We first suppose that the length of all the curves of the network remains strictly positive during the evolution. In this case the classification of the limits is complete (Proposition 6.8). Without a bound from below on the length of the curves the situation is more involved, and we will see that in general the limit sets are no longer regular networks. For this purpose, we shall introduce the notion of degenerate regular network.

We now describe the blow–up limit of networks under the assumption that the length of each curve is bounded below by a positive constant independent of time. We start with a lemma due to A. Stone [43].

Lemma 6.7. *Let $\widetilde{\mathcal{N}}_{\mathfrak{t},x_0}$ be the family of rescaled networks obtained via Huisken's dynamical procedure around some $x_0 \in \mathbb{R}^2$ as defined in formula (6.1).*

1. *There exists a constant $C = C(\mathcal{N}_0)$ such that, for every $\overline{x}, \in \mathbb{R}^2$, $\mathfrak{t} \in \big[-\frac{1}{2}\log T, +\infty\big)$ and $R > 0$ there holds*
$$\mathcal{H}^1(\widetilde{\mathcal{N}}_{\mathfrak{t},x_0} \cap B_R(\overline{x})) \leq CR\,.$$

2. *For any $\varepsilon > 0$ there is a uniform radius $R = R(\varepsilon)$ such that*
$$\int_{\widetilde{\mathcal{N}}_{\mathfrak{t},x_0} \setminus B_R(\overline{x})} e^{-|x|^2/2}\, ds \leq \varepsilon\,,$$
that is, the family of measures $e^{-|x|^2/2}\,\mathcal{H}^1 \llcorner \widetilde{\mathcal{N}}_{\mathfrak{t},x_0}$ is tight *(see [14]).*

Proposition 6.8. *Let $\mathcal{N}_t = \bigcup_{i=1}^n \gamma^i(t,[0,1])$ be a $C^{1,2}$ curvature flow of regular networks with fixed end–points in a smooth, strictly convex, bounded open set $\Omega \subset \mathbb{R}^2$ in the time interval $[0,T)$. Assume that the lengths $L^i(t)$ of the curves of the networks are uniformly in time bounded away from zero for every $i \in \{1,2,\dots,n\}$. Then for every $x_0 \in \mathbb{R}^2$ and for every subset $\mathcal{I}$ of $[-1/2\log T, +\infty)$ with infinite Lebesgue measure, there exists a sequence of rescaled times $\mathfrak{t}_j \to +\infty$, with $\mathfrak{t}_j \in \mathcal{I}$, such that the sequence of rescaled networks $\widetilde{\mathcal{N}}_{\mathfrak{t}_j,x_0}$ (obtained via Huisken's dynamical procedure) converges in $C^{1,\alpha}_{\mathrm{loc}} \cap W^{2,2}_{\mathrm{loc}}$, for any $\alpha \in (0,1/2)$, to a (possibly empty) limit, which is (if non-empty)*

- *a straight line through the origin with multiplicity $m \in \mathbb{N}$ (in this case $\widehat{\Theta}(x_0) = m$);*
- *a standard triod centered at the origin with multiplicity 1 (in this case $\widehat{\Theta}(x_0) = 3/2$).*
- *a halfline from the origin with multiplicity 1 (in this case $\widehat{\Theta}(x_0) = 1/2$).*

Moreover the L^2–norm of the curvature of $\widetilde{\mathcal{N}}_{\mathfrak{t}_j,x_0}$ goes to zero in every ball $B_R \subset \mathbb{R}^2$, as $j \to \infty$.

Proof. We divide the proof into three steps. We take for simplicity $x_0 = 0$.

Step 1: Convergence to $\widetilde{\mathcal{N}}_\infty$.
Consider the rescaled Monotonicity Formula (6.2) and let $\mathfrak{t}_1 = -1/2 \log T$ and $\mathfrak{t}_2 \to +\infty$. Then thanks to Lemma 6.6 we get

$$\int_{-1/2\log T}^{+\infty} \int_{\widetilde{\mathcal{N}}_{\mathfrak{t},x_0}} |\,\widetilde{\boldsymbol{k}} + x^\perp|^2 \widetilde{\rho}\, d\sigma\, d\mathfrak{t} < +\infty\,,$$

which implies

$$\int_{\mathcal{I}} \int_{\widetilde{\mathcal{N}}_{\mathfrak{t},x_0}} |\,\widetilde{\boldsymbol{k}} + x^\perp|^2 \widetilde{\rho}\, d\sigma\, d\mathfrak{t} < +\infty\,.$$

Being the last integral finite and being the integrand a non negative function on a set of infinite Lebesgue measure, we can extract within $\mathcal{I}$ a sequence of times $\mathfrak{t}_j \to +\infty$, such that

$$\lim_{j\to+\infty} \int_{\widetilde{\mathcal{N}}_{\mathfrak{t}_j,x_0}} |\,\widetilde{\boldsymbol{k}} + x^\perp|^2 \widetilde{\rho}\, d\sigma = 0\,. \tag{6.3}$$

It follows that for every ball B_R of radius R in $\mathbb{R}^2$ the networks $\widetilde{\mathcal{N}}_{\mathfrak{t}_j,x_0}$ have curvature uniformly bounded in $L^2(B_R)$. Moreover, by Lemma 6.7, for every ball B_R centered at the origin of $\mathbb{R}^2$ we have the uniform bound $\mathcal{H}^1(\widetilde{\mathcal{N}}_{\mathfrak{t}_j,x_0} \cap B_R) \leq CR$, for some constant C independent of $j \in \mathbb{N}$. Then reparametrizing the rescaled networks by arclength, we obtain curves with uniformly bounded first derivatives and with second derivatives uniformly bounded in L^2_{loc}.
By a standard compactness argument (see [21, 26]), the sequence $\widetilde{\mathcal{N}}_{\mathfrak{t}_j,x_0}$ of reparametrized networks admits a subsequence $\widetilde{\mathcal{N}}_{\mathfrak{t}_{j_l},x_0}$ which converges, weakly in $W^{2,2}_{\mathrm{loc}}$ and strongly in $C^{1,\alpha}_{\mathrm{loc}}$, to a (possibly empty) limit $\widetilde{\mathcal{N}}_\infty$ (possibly with multiplicity). The strong convergence in $W^{2,2}_{\mathrm{loc}}$ is implied by the weak convergence in $W^{2,2}_{\mathrm{loc}}$ and equation (6.3).

Step 2: The limit $\widetilde{\mathcal{N}}_\infty$ is a regular shrinker.
We first notice that the bound from below on the lengths prevents any "collapsing" along the rescaled sequence. Since the integral functional

$$\widetilde{\mathcal{N}} \mapsto \int_{\widetilde{\mathcal{N}}} |\,\widetilde{\boldsymbol{k}} + x^\perp|^2 \widetilde{\rho}\, d\sigma$$

is lower semicontinuous with respect to this convergence (see [39], for instance), the limit $\widetilde{\mathcal{N}}_\infty$ satisfies $\widetilde{\boldsymbol{k}}_\infty + x^\perp = 0$ in the sense of distributions.
A priori, the limit network is composed by curves in $W^{2,2}_{\mathrm{loc}}$, but from the relation $\widetilde{\boldsymbol{k}}_\infty + x^\perp = 0$, it follows that the curvature $\widetilde{\boldsymbol{k}}_\infty$ is continuous. By a bootstrap argument, it is then easy to see that $\widetilde{\mathcal{N}}_\infty$ is actually composed by C^∞ curves.

Step 3: Classification of the possible limits.
If the point $x_0 \in \mathbb{R}^2$ is distinct from all the end–points P^r, then $\widetilde{\mathcal{N}}_\infty$ has no end–points, since they go to infinity along the rescaled sequence. If $x_0 = P^r$ for some r, the set $\widetilde{\mathcal{N}}_\infty$ has a single end–point at the origin of $\mathbb{R}^2$.
Moreover, from the lower bound on the length of the original curves it follows that all the curves of $\widetilde{\mathcal{N}}_\infty$ have infinite length, hence, by Remark 4.2, they must be pieces of straight lines from

the origin.
This implies that every connected component of the graph underlying $\widetilde{\mathcal{N}}_\infty$ can contain at most one 3–point and in such case such component must be a standard triod (the 120 degrees condition must be satisfied) with multiplicity one since the converging networks are all embedded (to get in the C^1_{loc}–limit a triod with multiplicity higher than one it is necessary that the approximating networks have self–intersections). Moreover, since the converging networks are embedded, if both a standard triod and a straight line or another triod are present, they would intersect transversally. Hence if a standard triod is present, a straight line cannot be present and conversely if a straight line is present, a triod cannot be present.

If no end–point is present, that is, we are rescaling around a point in Ω (not on its boundary), and no 3–point is present, the only possibility is a straight line (possibly with multiplicity) through the origin.

If an end–point is present, we are rescaling around an end–point of the evolving network, hence, by the convexity of Ω (which contains all the networks) the limit $\widetilde{\mathcal{N}}_\infty$ must be contained in a halfplane with boundary a straight line H for the origin. This exclude the presence of a standard triod since it cannot be contained in any halfplane. Another halfline is obviously excluded, since they "come" only from end–points and they are all distinct. In order to exclude the presence of a straight line, we observe that the argument of Proposition 3.13 implies that, if $\Omega_t \subset \Omega$ is the evolution by curvature of $\partial\Omega$ keeping fixed the end–points P^r, the blow–up of Ω_t at an end–point must be a cone spanning angle strictly less then π (here we use the fact that three end–points are not aligned) and $\widetilde{\mathcal{N}}_\infty$ is contained in such a cone. It follows that $\widetilde{\mathcal{N}}_\infty$ cannot contain a straight line.

In every case the curvature of $\widetilde{\mathcal{N}}_\infty$ is zero everywhere and the last statement follows by the $W^{2,2}_{\mathrm{loc}}$–convergence.

□

Remark 6.9. In the previous proposition the hypothesis on the length of the curve can be replace by the weaker assumption that the lengths $L^i(t)$ of the curves satisfy

$$\lim_{t\to T} \frac{L^i(t)}{\sqrt{T-t}} = +\infty\,,$$

for every $i \in \{1, 2, \dots, n\}$.

Lemma 6.10. *Under the assumptions of Proposition 6.8, there holds*

$$\lim_{j\to\infty} \frac{1}{\sqrt{2\pi}} \int_{\widetilde{\mathcal{N}}_{\mathfrak{t}_j,x_0}} \widetilde{\rho}\, d\sigma = \frac{1}{\sqrt{2\pi}} \int_{\widetilde{\mathcal{N}}_\infty} \widetilde{\rho}\, d\overline{\sigma} = \Theta_{\widetilde{\mathcal{N}}_\infty} = \widehat{\Theta}(T, x_0)\,. \tag{6.4}$$

where $d\overline{\sigma}$ denotes the integration with respect to the canonical measure on $\widetilde{\mathcal{N}}_\infty$, counting multiplicities

Proof. By means of the second point of Lemma 6.7, we can pass to the limit in the Gaussian integral and we get

$$\lim_{j\to\infty} \frac{1}{\sqrt{2\pi}} \int_{\widetilde{\mathcal{N}}_{\mathfrak{t}_j,x_0}} \widetilde{\rho}\, d\sigma = \frac{1}{\sqrt{2\pi}} \int_{\widetilde{\mathcal{N}}_\infty} \widetilde{\rho}\, d\overline{\sigma} = \Theta_{\widetilde{\mathcal{N}}_\infty}\,.$$

Recalling that

$$\frac{1}{\sqrt{2\pi}} \int_{\widetilde{\mathcal{N}}_{\mathfrak{t}_j,x_0}} \widetilde{\rho}\, d\sigma = \int_{\mathcal{N}_{t(\mathfrak{t}_j)}} \rho_{T,x_0}(\tau(\mathfrak{t}_j), \cdot)\, ds = \Theta_{x_0}(t(\mathfrak{t}_j)) \to \widehat{\Theta}(T, x_0)$$

as $j \to \infty$, equality (6.4) follows. □

Remark 6.11. If the three end–points P^{r-1}, P^r, P^{r+1} are aligned the argument of Proposition 3.13 does not work and we cannot conclude that the only blow–up at P^r is a halfline with multiplicity 1. It could also be possible that a straight line (possibly with higher multiplicity) is present.

We describe now how Proposition 6.8 allows us to obtain a (conditional) global existence result when the lengths of all the curves of the networks are strictly positive.

Suppose that $T < +\infty$. As we have assumed that the lengths of all the curves of the network are uniformly positively bounded from below, the curvature blows-up as $t \to T$ (Theorem 5.7 and Proposition 5.8). Performing a Huisken's rescaling at an interior point x_0 of Ω, we obtain as blow–up limit (if not empty) a standard triod or a straight line with multiplicity $m \in \mathbb{N}$. One can argue as in [29] to show that when such limit is a regular triod, the curvature is locally bounded around such point x_0. For the case of a straight line, if we suppose that the multiplicity m is equal to 1, by White's local regularity theorem [45] we conclude that the curvature is bounded uniformly in time, in a neighborhood of the point x_0. If we instead rescale at an end–point P^r we get a halfline. This case can be treated as above by means of a reflection argument. Indeed for the flow obtain by the union of the original network and the reflection of this latter, the point P^r is no more an end–point. A blow–up at P^r give a straight line, implying that the curvature is locally bounded also around P^r as before by White's theorem.

Supposing that the lengths of the curves of the network are strictly positive and supposing also that any blow–up limit has multiplicity one, it follows that the original network $\mathcal{N}_t$ has bounded curvature as $t \to T$. Hence T cannot be a singular time, and we have therefore global existence of the flow.

In the previous reasoning a key point is the hypothesis that the blow–ups have multiplicity one. Unfortunately, for a general regular network, this is still conjectural and possibly the major open problem in the subject.

Multiplicity–One Conjecture (M1). *Every possible C^1_{loc}–limit of rescalings of networks of the flow is an embedded network with multiplicity one.*

However, in some special situations one can actually prove **M1**.

Proposition 6.12. *If Ω is strictly convex and the evolving network $\mathcal{N}_t$ has at most two triple junctions, every C^1_{loc}–limit of rescalings of networks of the flow is embedded and has multiplicity one.*

Proof. See [31, Section 4,Corollary 4.7]. □

Proposition 6.13. *If during the curvature flow of a tree $\mathcal{N}_t$ the triple junctions stay uniformly far from each other and from the end–points, then every C^1_{loc}–limit of rescalings of networks of the flow is embedded and has multiplicity one.*

Proof. See [32, Proposition 14.14]. □

We now remove the hypothesis on the lengths of the curves of the network. In this case, nothing prevents a length to go to zero in the limit.

In order to describe the possible limits, we introduce the notion of degenerate regular networks. First of all we define the **underlying graph**, which is an oriented graph G with n edges E^i, that can be bounded and unbounded. Every vertex of G can either have order one (and in this case it is called end–points of G) or order three.

For every edge E^i we introduce an orientation preserving homeomorphisms $\varphi^i : E^i \to I^i$ where I^i is the interval $(0,1)$, $[0,1)$, $(0,1]$ or $[0,1]$. If E^i is a segment, then $I^i = [0,1]$. If it is an halfline, we choose $I^i = [0,1)$ or $I^i = (0,1]$. Notice that the interval $(0,1)$ can only appear if it is associated to an unbounded edge E^i without vertices, which is clearly a single connected component of G.

We then consider a family of C^1 parametrizations $\sigma^i : I^i \to \mathbb{R}^2$. In the case I^i is $(0,1)$, $[0,1)$ or $(0,1]$, the map σ^i is a regular C^1 curve with unit tangent vector τ^i. If instead $I^i = [0,1]$ the map σ^i can be either a regular C^1 curve with unit tangent vector τ^i, or a constant map (**degenerate curves**). In this last case we assign a constant unit vector $\tau^i : I^i \to \mathbb{R}^2$ to the curve σ^i. At the points 0 and 1 of I^i the **assigned exterior unit tangents** are $-\tau^i$ and τ^i, respectively. The exterior unit tangent vectors (real or assigned) at the relative borders of the intervals I^i, I^j, I^k of the concurring curves σ^i, σ^j σ^k have zero sum (**degenerate** 120 **degrees condition**). We require that the map $\Gamma : G \to \mathbb{R}^2$ given by the union $\Gamma = \bigcup_{i=1}^n (\sigma^i \circ \varphi^i)$ is well defined and continuous.

We define a **degenerate regular network** $\mathcal{N}$ as the the union of the sets $\sigma^i(I^i)$. If one or several edges E^i of G are mapped under the map $\Gamma : G \to \mathbb{R}^2$ to a single point $p \in \mathbb{R}^2$, we call this sub–network given by the union G' of such edges E^i the **core** of $\mathcal{N}$ at p.

We call multi–points of the degenerate regular network $\mathcal{N}$ the images of the vertices of multiplicity three of the graph G by the map Γ and end–points of $\mathcal{N}$ the images of the vertices of multiplicity one of the graph G, by the map Γ.

A degenerate regular network $\mathcal{N}$ with underlying graph G, seen as a subset in $\mathbb{R}^2$, is a C^1 network, not necessarily regular, that can have end–points and/or unbounded curves. Moreover, self–intersections and curves with integer multiplicities can be present. Anyway, at every image of a multi–point of G the sum (possibly with multiplicities) of the exterior unit tangents is zero.

Definition 6.14. We say that a sequence of regular networks $\mathcal{N}_k = \bigcup_{i=1}^n \sigma_k^i(I_k^i)$ converges in C^1_{loc} to a degenerate regular network $\mathcal{N} = \bigcup_{j=1}^l \sigma_\infty^j(I_\infty^j)$ with underlying graph $G = \bigcup_{j=1}^l E^j$ if:

- letting $O^1, O^2, \dots, O^m$ the multi–points of $\mathcal{N}$, for every open set $\Omega \subset \mathbb{R}^2$ with compact closure in $\mathbb{R}^2 \setminus \{O^1, O^2, \dots, O^m\}$, the networks $\mathcal{N}_k$ restricted to Ω, for k large enough, are described by families of regular curves which, after possibly reparametrizing them, converge to the family of regular curves given by the restriction of $\mathcal{N}$ to Ω;

- for every multi–point O^p of $\mathcal{N}$, image of one or more vertices of the graph G (if a core is present), there is a sufficiently small $R > 0$ and a graph $\widetilde{G} = \bigcup_{r=1}^s F^r$, with edges F^r associated to intervals J^r, such that:

  - the restriction of $\mathcal{N}$ to $B_R(O^p)$ is a regular degenerate network described by a family of curves $\widetilde{\sigma}_\infty^r : J^r \to \mathbb{R}^2$ with (possibly "assigned", if the curve is degenerate) unit tangent $\widetilde{\tau}_\infty^r$,
  - for k sufficiently large, the restriction of $\mathcal{N}_k$ to $B_R(O^p)$ is a regular network with underlying graph $\widetilde{G}$, described by the family of regular curves $\widetilde{\sigma}_k^r : J^r \to \mathbb{R}^2$,
  - for every j, possibly after reparametrization of the curves, the sequence of maps $J^r \ni x \mapsto \big(\widetilde{\sigma}_k^r(x), \widetilde{\tau}_k^r(x)\big)$ converge in C^0_{loc} to the maps $J^r \ni x \mapsto \big(\widetilde{\sigma}_\infty^r(x), \widetilde{\tau}_\infty^r(x)\big)$, for every $r \in \{1, 2, \dots, s\}$.

We will say that $\mathcal{N}_k$ converges to $\mathcal{N}$ in $C^1_{\mathrm{loc}} \cap E$, where E is some function space, if the above curves also converge in the topology of E.

Removing the hypothesis on the lengths of the curves, we get that the limit networks are degenerate regular networks which are homothetically shrinking under the flow.

Proposition 6.15. *Let $\mathcal{N}_t = \bigcup_{i=1}^n \gamma^i(t,[0,1])$ be a $C^{1,2}$ curvature flow of regular networks in the time interval $[0,T]$, then, for every $x_0 \in \mathbb{R}^2$ and for every subset $\mathcal{I}$ of $[-1/2\log T, +\infty)$ with infinite Lebesgue measure, there exists a sequence of rescaled times $\mathfrak{t}_j \to +\infty$, with $\mathfrak{t}_j \in \mathcal{I}$, such that the sequence of rescaled networks $\widetilde{\mathcal{N}}_{\mathfrak{t}_j,x_0}$ (obtained via Huisken's dynamical procedure) converges in $C^{1,\alpha}_{\mathrm{loc}} \cap W^{2,2}_{\mathrm{loc}}$, for any $\alpha \in (0,1/2)$, to a (possibly empty) limit network, which is a degenerate regular shrinker $\widetilde{\mathcal{N}}_\infty$ (possibly with multiplicity greater than one).*

Moreover, we have

$$\lim_{j\to\infty} \frac{1}{\sqrt{2\pi}} \int_{\widetilde{\mathcal{N}}_{\mathfrak{t}_j,x_0}} \widetilde{\rho}\, d\sigma = \frac{1}{\sqrt{2\pi}} \int_{\widetilde{\mathcal{N}}_\infty} \widetilde{\rho}\, d\overline{\sigma} = \Theta_{\widetilde{\mathcal{N}}_\infty} = \widehat{\Theta}(T,x_0)\,.$$

where $d\overline{\sigma}$ denotes the integration with respect to the canonical measure on $\widetilde{\mathcal{N}}_\infty$, counting multiplicities.

Remark 6.16. Notice that the blow–up limit degenerate shrinker obtained by this proposition a priori depends on the chosen sequence of rescaled times $\mathfrak{t}_j \to +\infty$.

Remark 6.17. Thanks to Proposition 6.12, if the network $\mathcal{N}$ has at most two triple junctions, the degenerate regular shrinker $\widetilde{\mathcal{N}}_\infty$ has multiplicity one.

Assuming that the length of at least one curve of $\mathcal{N}_t$ goes to zero, as $t \to T$, there are two possible situations:

- The curvature stays bounded.
- The curvature is unbounded as $t \to T$.

Suppose that the curvature remains bounded in the maximal time interval $[0,T)$. As $t \to T$ the networks $\mathcal{N}_t$ converge in C^1 (up to reparametrization) to a unique limit degenerate regular network $\widehat{\mathcal{N}}_T$ in Ω. This network can be **non–regular** seen as a subset of $\mathbb{R}^2$: multi–points can appear, but anyway the sum of the exterior unit tangent vectors of the concurring curves at every multi–point must be zero. Every triple junction satisfies the angle condition. The non–degenerate curves of $\widehat{\mathcal{N}}_T$ belong to $C^1 \cap W^{2,\infty}$ and they are smooth outside the multi–points (for the proof see [32, Proposition 10.11]).

We have seen in Section 5.1 that if a region is bounded by less than six curves then its area decreases linearly in time going to zero at $\overline{T}$. Not only the area goes to zero in a finite time, but also the lengths of all the curves that bound the region. Moreover when the lengths of all the curves of the loop go to zero, then the curvature blows up. Let us call the loop ℓ. Combing (5.2) with (5.3) there is a positive constant c such that $\int_\ell |k|\, ds \geq c$. By Hölder inequality

$$c \leq \int_\ell |k|\, ds \leq \left(\int_\ell k^2\, ds\right)^{1/2} L(\ell)^{1/2}\,,$$

where $L(\ell)$ is the total length of the loop. Hence

$$\|k\|_{L^2} \geq \frac{c^2}{L(\ell)} \to \infty \quad \text{as} \quad L(\ell) \to 0\,.$$

Then at time $\overline{T}$ we have a singularity where both the length goes to zero and the curvature explodes.

Developing careful a priori estimates of the curvature one can show that if two triple junctions collapse into a 4–point, then the curvature remains bounded (see [32]). The interest of this result relies on the fact that it describes the formation of a “type zero” singularity: a singularity due to the change of topology, not to the blow up of the curvature. This is a new phenomenon with respect to the classical curve shortening flow and the mean curvature flow more in general. Thanks to this result it is possible to show that given an initial network without loops (a tree), if Multiplicity–One Conjecture **M1** is valid, then the curvature is uniformly bounded during the flow. The only possible “singularities” are given by the collapse of a curve with two triple junctions going to collide. Moreover in the case of a tree we are able to show the uniqueness of the blow up limit (see Remark 6.16).

Although one can find example of global existence of the flow (consider for instance an initial triod contained in the triangle with vertices its three end–points and with all angles less than 120 degrees) our analysis underlines the generic presence of singularities. Then a natural question is if it is possible to go beyond the singularity.

There are results on the short time existence of the flow for non-regular networks, that is, networks with multi–points (not only 3-points), or networks that do not satisfy the 120 degrees condition at the 3-points. Till now the most general result of this kind is the one by Ilmanen, Neves and Schulze [23], which provides short time existence of the flow starting from a non-regular network with **bounded curvature**. Notice that the network arising after the collapse of (exactly) two triple junctions has bounded curvature, and therefore fits with the hypotheses this result.

An ambitious project should be constructing a bridge between the analysis of the long time behavior of networks moving by curvature and short time existence results for non-regular initial data: one can interpret the short time existence results for non-regular data as a “restarting” theorem for the flow after the onset of the first singularity.

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