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The Lavrentiev phenomenon in the Calculus of Variations

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Part I

Introduction

New Results in the Calculus of Variations

The words “Calculus of Variations” were first used by L. Euler in 1760 ([25]) after he had been astonished by a letter from J. Lagrange, where a new method (based on what we would call now variations) for the study of isoperimetric problems was proposed ([30]). But the birth of the Calculus of Variations is attributed to Johann Bernoulli when many years earlier, in 1696, he challenged mathematicians like Jakob Bernoulli, I. Newton, E. Tschirnhaus, G. De l’Hôpital and G. Liebniz with the Brachistochrone problem ([45], [43], [47]):

> *“... If in a vertical plane two points A and B are given, then it is required to specify the orbit AMB of the movable point M, along which it, starting from A, and under the influence of its own weight, arrives at B in the shortest possible time...”*

The basic problem (P) of the *Calculus of Variations* consist in minimizing the *action* functional

$$\mathcal{I}(x) = \int_a^b L(t, x(t), x'(t))dt,$$

where $[a, b] \subset \mathbb{R}$ is an interval of the real line, on a suitable set $\mathbf{X}$ of *trajectories* $x : [a, b] \to \mathbb{R}^N$ satisfying the boundary conditions $x(a) = A$, $x(b) = B$. The function $L : [a, b] \times \mathbb{R}^N \times \mathbb{R}^N \to \mathbb{R}$ it is called the *Lagrangian* function.

The problem (P) arises three natural questions: does it *exist* a minimizer? Does a minimizer gain *regularity*? Does a minimizer satisfy *necessary conditions*? Certainly, the answers are negative in general, without further assumptions.

With respect to the first question, the standard hypotheses on the Lagrangian under which it is possible to prove an existence result, in the set $\mathbf{X} = \mathbf{AC}[a, b]$ of the absolutely continuous trajectories, are: some lower semi-continuity of L in its domain, the *convexity* in its last variable x' and a *super-linear growth* condition, i.e. the requirement that there exists a function θ such that $\theta(s)/s$ converges to $+\infty$, as s tends to $+\infty$, and $L(t, x, x') \geq \theta(|x'|)$.

This method of proof is called the *Direct Method* of the Calculus of Variations and it was mainly developed by L. Tonelli (1915, [48]). Roughly speaking, what one does in this method is the following: starting from a minimizing sequence for the action $\mathcal{I}$, i.e. a sequence of trajectories $\{x_n\}_n$ such that $\mathcal{I}(x_n)$ converges to the infimum of $\mathcal{I}$, and, by using the fact that the super-linear growth condition implies the weak precompactness of $\{x_n\}_n$, one selects a subsequence that converges weakly to a trajectory $\hat{x}$ in $\mathbf{AC}[a, b]$. Since the convexity, together with some continuity, ensures the weak lower semi-continuity of $\mathcal{I}$, one concludes that $\hat{x}$ is a minimizer.

In section 4.1 we present an original existence theorem ([11]) in the case of an autonomous Lagrangian, i.e. a Lagrangian which does not depend explicitly on t, where a variant of the Direct Method can be applied under more general growth assumptions that include the classical super-linear growth but also some cases of Lagrangians with linear growth. For instance, for the Lagrangian defined by$L(x, x') = |x'| - \sqrt{|x'|}$, which has not super-linear growth, by this result it can be proved that there exists a minimizer. We obtain also that this minimizer is Lipschitz. Results on the existence of Lipschitz minimizer to autonomous problems, under conditions of super-linear growth, were established in [5], [20] and, under weaker regularity conditions on L, in [10].

Assuming the existence of a minimizer $\hat{x}$, another task is to find appropriate necessary conditions. A basic principle of analysis is that, giving a minimum point belonging to the interior of the domain of a differential function, one obtains a necessary condition for this point exploring its neighbourhood; one finds out that it is a stationary point, i.e. the gradient of the function calculated on the point must be zero. Is it possible to apply this principle in the Calculus of Variations? The answer is positive, and it leads to the fundamental *Euler-Lagrange equation*.

Fix an admissible *variation*, i.e. a smooth function $\eta : [a, b] \to \mathbb{R}^N$, equal to zero at the boundary, consider the action $\mathcal{I}$ evaluated on the trajectory $\hat{x} + \epsilon\eta$, where ϵ is a real number, and consider the function obtained $\mathcal{I}(\hat{x} + \epsilon\eta)$ as a real valued function of ϵ. For this function the point 0 must be a stationary point. Whenever it is possible to pass to the limit under the integral sign, one obtains the sought Euler-Lagrange equation (E-L): for any variation η,

$$\int_a^b [\langle \nabla_{x'} L(t, \hat{x}(t), \hat{x}'(t)), \eta'(t)\rangle + \langle \nabla_x L(t, \hat{x}(t), \hat{x}'(t)), \eta(t)\rangle] dt = 0,$$

or, considering d/dt as a weak derivative,

$$\frac{d}{dt} \nabla_{x'} L(t, \hat{x}(t), \hat{x}'(t)) = \nabla_x L(t, \hat{x}(t), \hat{x}'(t)).$$

L. Tonelli was able to prove the validity of the Euler-Lagrange equation with a Lagrangian L belonging to $\mathbf{C}^3$, or belonging to $\mathbf{C}^2$ provided that $N = 1$ and $L_{x'x'}$ is strictly positive. Many efforts have been performed in order to weaken the regularity assumptions on L ([5], [16], [19], [20], [33], [38], [40], [50]).

In [5], J. M. Ball and V. J. Mizel presented an example of Lagrangian such that the integrability of $\nabla_x L(\cdot, \hat{x}(\cdot), \hat{x}'(\cdot))$ does not hold and, as a consequence, (E-L) is not true along the minimizer. Hence, some condition on the term $\nabla_x L(\cdot, \hat{x}(\cdot), \hat{x}'(\cdot))$ has to be imposed in order to ensure the validity of (E-L). In [16], [19], [38], [50], under assumptions like the existence of an integrable function $S(t)$ such that, for any y in a neighbourhood of the minimizer, $|\nabla_x L(t, y, \hat{x}'(t))|$ is bounded by $S(t)$, the Euler-Lagrange equation is proved to be valid. This condition implies that, locally along the minimizer, $L(t, \cdot, \hat{x}'(t))$ is Lipschitzian.

In section 6.2 we provide a result on the validity of (E-L) ([27]) that is satisfied by Lagrangians that are Lipschitzian in x, but that applies as well to some non-Lipschitzian cases. For instance, for the problem (P) with the (non-Lipschitzian)

Lagrangian $L(t,x,x') = (x'\sqrt{|x|} - 2/3)^2$, on the interval $[0,1]$ and with boundary conditions $x(0) = 0$, $x(1) = 1$, by this result it can be proved the validity of (E-L).

It would be easier to show the validity of the Euler-Lagrange equation if one considers the problem (P) on Lipschitzian trajectories, i.e. $\mathbf{X} = \mathbf{Lip}[a,b]$, instead of absolutely continuous. In fact, in this case, it can be proved directly, from the Lebesgue dominated convergence theorem, the validity of the $(E-L)$ for a Lipschitzian minimizer, just requiring the Lagrangian to be $\mathbf{C}^1$.

The set $\mathbf{X}$ of the trajectories plays an important role in the problem (P), that it is not only technical. A measure of the importance of the choice of $\mathbf{X}$ is given by the *Lavrentiev phenomenon* (1926, [34]): a Lagrangian L exhibits the Lavrentiev phenomenon if the infimum taken over the set of absolutely continuous trajectories $\mathbf{AC}[a,b]$ is strictly lower than the infimum taken over the set of Lipschitzian trajectories $\mathbf{Lip}[a,b]$, with fixed boundary conditions. The occurrence of this phenomenon prevents the possibility of computing the minimum, and the minimizer, by a standard finite-element scheme (anyway, in [4], [35], alternative numerical methods for computing minimizer in this situation was presented) and, in case that the action represents the energy of some physical system, it points out that the set of trajectories is a fundamental part of the physical model. Furthermore, it says something about the regularity of the minimizer (if this exists).

One of the surprising feature of this phenomenon is that it occurs for simple Lagrangians: $L(t,x,x') = (x^3 - t)^2 x'^6$, on the interval $[0,1]$, with boundary conditions $x(0) = 0$, $x(1) = 1$, exhibits the Lavrentiev phenomenon (B. Manià, 1934, [37]). Moreover, in [29] it was given a physical action of a nonlinear elastic material where the occurrence of the Lavrentiev phenomenon is the occurrence of a meaningful physical event.

In the autonomous case, sufficient conditions to prevent the occurrence of this phenomenon were given by several authors, by imposing enough growth conditions [2], [10], [20], or assuming some regularity conditions on the Lagrangian [1], [3].

In section 2.1 we prove a general approximation theorem for the action ([12]) that implies the non non-occurrence of the Lavrentiev Phenomenon for autonomous and a class of non-autonomous Lagrangians, without assuming any growth condition.

Besides the problem (P), one can consider the problem with a higher-order action ([16], [21]): minimize

$$\int_a^b L(t, x(t), x'(t), \cdots, x^{(\nu+1)}(t))dt,$$

on a suitable set $\mathbf{X}$ of trajectory satisfying the boundary conditions

$$\begin{array}{ll} x(a) = A, & x(b) = B, \\ x'(a) = A^{(1)}, & x'(b) = B^{(1)}, \\ \vdots & \\ x^{(\nu)}(a) = A^{(\nu)}, & x^{(\nu)}(b) = B^{(\nu)}. \end{array}$$

The Lavrentiev phenomenon occurs as well in this case. In [17], it was given an example of second-order action that presents a restricted Lavrentiev phenomenon in which the gap occurs for a dense subset of the absolutely continuous non-negative trajectories, and it was proved that even autonomous Lagrangian can exhibit it. Some years later, A. V. Sarychev ([43]) provided a class of higher-order Lagrangians, including autonomous cases, that exhibits the Lavrentiev phenomenon in the classical sense.

In section 2.2 we prove the non-occurrence of the Lavrentiev phenomenon for a class of Lagrangians of the Calculus of Variations with higher-order derivatives ([26]).

Moreover, in section 2.3 we infer a *necessary condition for the occurrence of the Lavrentiev phenomenon* for our class of actions: if the action $\mathcal{I}$ assumes only finite values in a neighbourhood of a minimizer, then $\mathcal{I}$ does not exhibit the Lavrentiev phenomenon ([26]). This result applies to the actions of Manià and Sarychev, for instance, and to the examples of actions exhibiting the Lavrentiev phenomenon proposed in [5], [6], [34], [36], [37], [39], [43]. This necessary condition is also related to the *repulsion property* of the action of Manià ([5], [34]), i.e. the action $\mathcal{I}$ evaluated on the trajectories of an absolutely continuous minimizing sequence is divergent to $+\infty$.

A New Result in Minimum Time Control Theory

Around 1950, L. Pontriagin worked on an innovative minimum problem ([41]), that of minimize

$$\int_a^b L(t, x(t), u(t))dt,$$

on the *state* functions $x : [a, b] \to \mathbb{R}^N$ and the *control* functions $u : [a, b] \to U \subset \mathbb{R}^N$ satisfying the differential equation

$$x'(t) = f(t, x(t), u(t)),$$

with boundary conditions $x(a) = A$, $x(b) = B$. This is the basic problem (Q) of the *Optimal Control* theory.

In the special case when $f(t, x, u) = u$, the differential equation becomes $x'(t) = u(t)$ and we recognize the problem of the Calculus of Variations with a variations: it appears the new conditions that $x'(t)$ belongs to the set U. Hence, two are the novelties of the problem: the *dynamics* f and the *set of controls* U.

The Optimal Control theory contains problems of the Calculus of Variations and, at the opposite extreme, *Minimum Time Control* problems, i.e. problems where the Lagrangian L is identically 1.

In 1959 A. F. Filippov proved the first general theorem on the existence of solutions to Minimum Time Control problems of the form $x'(t) = f(x(t), u(t))$, with $u(t)$ belonging to $U(x(t))$, requiring that the set valued map U be upper semi-continuous (with respect to the inclusion) and that the values $F(x) = f(x, U(x))$ be compact and convex ([28]).

In section 4.2, we prove the existence of solutions to Minimum Time problems for differential inclusions, under assumptions that do not require the convexity of the images $F(x) = f(x, U(x))$. Furthermore, we present a model for the Brachistochrone problem and we show that our model (a non-convex control problem) satisfies the assumptions required for the existence of solutions to Minimum Time problems. The case of Brachistochrone as a Minimum Time Control problem has already been amply treated in [46], [47].

Part II

The Lavrentiev phenomenon - A necessary condition

Chapter 1

Preliminaries

In section 1.1 we present some results in convex analysis involving the polar function and the sub-gradient of a convex function that we will use in the proof of our new results ([12]).

In section 1.2 we discuss the example of Manià ([37]) of a Lagrangian that exhibits the Lavrentiev phenomenon. We show also the persistence of the Lavrentiev phenomenon by perturbation and the repulsion property for the Lagrangian of Manià, i.e. the action evaluated on the trajectories of an absolutely continuous minimizing sequence is divergent to $+\infty$.

1.1 Some results in Convex Analysis

The proofs of the new results in this Part II are based on some properties of the *polar* function $f^*(p)$ of a *convex* function $f : \mathbb{R}^N \to \mathbb{R}$, i.e.

$$f^*(p) = \sup_{\xi \in \mathbb{R}^N} [\langle p, \xi \rangle - f(\xi)].$$

We state these properties in the propositions below ([12]).

In what follows, the *sub-gradient* evaluated in ξ, of a convex function f, i.e. the set $\{y \in \mathbb{R}^N : f(\cdot) - f(\xi) \geq \langle \cdot - \xi, y \rangle\}$, is denoted by $\partial f(\xi)$

Propositon 1.1. *Let f be a convex function and $p \in \partial f(\xi)$. Then f^* is finite at p and $f^*(p) = \langle \xi, p \rangle - f(\xi)$.*

Proof. See [42], page 218. □

Propositon 1.2. *i) Let $f, f_n : \mathbb{R}^N \to \mathbb{R}$ be convex and let f_n converge pointwise to f; let $p_n \in \partial f_n(\xi)$. Then the sequence $\{p_n\}$ admits a subsequence converging to some $p \in \partial f(\xi)$.*

ii) Let $L : C \times \mathbb{R}^N \to \mathbb{R}$, where C is a closed subset of $\mathbb{R}^N$, be continuous and such that $L(x, \cdot)$ is convex; let $x_n \to x \in C$ and set $f(\xi) = L(x, \xi)$, $f_n(\xi) = L(x_n, \xi)$. Then the same conclusion as in i) holds for $p_n \in \partial f_n(\xi_n) = \partial_\xi L(x_n, \xi_n)$.

Proof. We prove i) and ii) at once setting, in case i), $\xi_n = \xi$ and noticing that, in both cases, we have that, for every z, $f_n(\xi_n + z) \to f(\xi + z)$.

The sequence $\{p_n\}$ cannot be unbounded; if it were, along a subsequence we would have $|p_n| \to \infty$; choose a further subsequence so that $p_n/|p_n| \to p_0$, where $|p_0| = 1$. We have

$$f(\xi + p_0) = \lim_{n\to\infty} f_n(\xi_n + p_0) \geq \limsup_{n\to\infty} [f_n(\xi_n) + \langle p_n, p_0\rangle] = +\infty$$

a contradiction, since f is finite at $\xi + p_0$. Hence the sequence $\{p_n\}$ is bounded and we can select a subsequence converging to p_*. If it were $p_* \notin \partial f(\xi)$ there would exist ξ' such that $f(\xi') < f(\xi) + \langle p_*, \xi' - \xi\rangle$. Since

$$\begin{aligned} f(\xi') &= f(\xi + (\xi' - \xi)) = \lim_{n\to\infty} f_n(\xi_n + (\xi' - \xi)) \\ &\geq \limsup_{n\to\infty} [f_n(\xi_n) + \langle p_n, \xi' - \xi\rangle] = f(\xi) + \langle p_*, \xi' - \xi\rangle \end{aligned}$$

we would have a contradiction. □

Propositon 1.3. *Let $f : \mathbb{R}^N \to \mathbb{R}$ be convex. The map $t \to \{\langle \xi t, p\rangle - f(\xi t) : p \in \partial f(\xi t)\}$, from $[0, +\infty)$ to the closed convex subsets of $\mathbb{R}$, is monotonically increasing.*

Proof. a) Assume, in addition, that f is smooth; then we have $\nabla(\langle \xi, \nabla f(\xi)\rangle - f(\xi)) = \xi^T H$, where H is the Hessian matrix of f. Hence

$$\frac{d}{dt}(\langle \xi t, \nabla f(\xi t)\rangle - f(\xi t)) = t\xi^T H\xi \geq 0,$$

so that $t_2 \geq t_1$ implies

$$\langle \xi t_2, \nabla f(\xi t_2)\rangle - f(\xi t_2) \geq \langle \xi t_1, \nabla f(\xi t_1)\rangle - f(\xi t_1).$$

b) In general, the map $\phi(t) = f(\xi t)$, being convex, is differentiable for a.e. t. Let $t_1^+ > t_1$ and $t_2^- < t_2$ be points where ϕ is differentiable. Approximate f by a sequence $\{f_n\}$ of convex smooth maps, converging pointwise to f. Set $\phi_n(t) = f_n(\xi t)$: in particular, applying the previous Proposition 1.2, we have that $\phi_n'(t)$ converges to $\phi'(t)$ both at t_1^+ and at t_2^-. Applying point a) to f_n we obtain that

$$t_2^- \phi_n'(t_2^-) - \phi_n(t_2^-) \geq t_1^+ \phi_n'(t_1^+) - \phi_n(t_1^+)$$

so that, passing to the limit as $n \to \infty$,

$$t_2^- \phi'(t_2^-) - \phi(t_2^-) \geq t_1^+ \phi'(t_1^+) - \phi(t_1^+).$$

By the monotonicity of the sub-differential of ϕ, for every $a_1 \in \partial\phi(t_1)$ and $a_2 \in \partial\phi(t_2)$, we have

$$t_2 a_2 - \phi(t_2^-) \geq t_2^- \phi'(t_2^-) - \phi(t_2^-) \geq t_1^+ \phi'(t_1^+) - \phi(t_1^+) \geq t_1 a_1 - \phi(t_1^+)$$

and passing to the limit as $t_2^+ \to t_2$ and $t_1^- \to t_1$, by the continuity of ϕ, one has

$$t_2 a_2 - \phi(t_2) \geq t_1 a_1 - \phi(t_1).$$

Since ([32], page 257), $\partial\phi(t) = \{\langle \xi, p\rangle : p \in \partial f(\xi t)\}$, the claim is proved. □

Propositon 1.4. *Let $f : \mathbb{R}^N \to \mathbb{R}$ be convex. Then the function $f(\xi/(1+\cdot))(1+\cdot)$ is convex in $(-1,\infty)$. Moreover, given δ there are θ, $0 \leq \theta \leq 1$ and $p_\theta \in \partial f(\xi/(1+\theta\delta))$, such that*

$$f\left(\frac{\xi}{1+\delta}\right)(1+\delta) - f(\xi) = -\delta f^*(p_\theta).$$

Proof. a) Assume, in addition, that f is $\mathbf{C}^2$. Then, computing the derivatives, one obtains

$$\frac{d}{d\alpha}\left(f\left(\frac{\xi}{1+\alpha}\right)(1+\alpha)\right) = \langle \nabla f\left(\frac{\xi}{1+\alpha}\right), \frac{-\xi}{1+\alpha}\rangle + f\left(\frac{\xi}{1+\alpha}\right) = -f^*\left(\nabla f\left(\frac{\xi}{1+\alpha}\right)\right)$$

and

$$\frac{d^2}{d\alpha^2}\left(f\left(\frac{\xi}{1+\alpha}\right)(1+\alpha)\right) = \frac{1}{(1+\alpha)^3}\xi^T H \xi$$

where H is the Hessian matrix of f computed at $\xi/(1+\alpha)$, so that the second derivative is non-negative, and the map $f(\xi/(1+\cdot))(1+\cdot)$ is convex.

b) In the general case, approximate the convex map f by a sequence of convex differentiable maps f_n converging pointwise to f to obtain the required convexity and to have:

$$\begin{aligned} f\left(\frac{\xi}{1+\delta}\right)(1+\delta) - f(\xi) &= \lim_{n\to\infty} f_n\left(\frac{\xi}{1+\delta}\right)(1+\delta) - f_n(\xi) \\ &= \lim_{n\to\infty} \delta\left[-\langle \frac{\xi}{1+\theta_n\delta}, \nabla f_n\left(\frac{\xi}{1+\theta_n\delta}\right)\rangle + f_n\left(\frac{\xi}{1+\theta_n\delta}\right)\right]. \end{aligned}$$

c) Applying Proposition 2, let $p_\theta \in \partial f(\xi/(1+\theta\delta))$ be the limit of a converging subsequence of $\{\nabla f_n(\xi/(1+\theta_n\delta))\}$. We have

$$f\left(\frac{\xi}{1+\delta}\right)(1+\delta) - f(\xi) = \delta\left[-\langle \frac{\xi}{1+\theta\delta}, p\rangle + f\left(\frac{\xi}{1+\theta\delta}\right)\right] = -\delta f^*(p).$$

□

1.2 The Manià example

We present in this section the example due to B. Manià (1934, [37]) of a Lagrangian that exhibits the Lavrentiev phenomenon (1926, [34]). In the next chapter we discuss further this example in relation with the new results obtained ([12], [26]).

We recall that a Lagrangian L exhibits the Lavrentiev phenomenon if the infimum taken over the set of absolutely continuous trajectories $\mathbf{AC}[a,b]$ is strictly lower than the infimum taken over the set of Lipschitzian trajectories $\mathbf{Lip}[a,b]$, with fixed boundary conditions.

Consider the problem of minimize the action

$$\mathcal{I}(x) = \int_0^1 [x^3(t) - t]^2 x'^6(t)dt,$$

on the trajectories x satisfying the boundary conditions $x(0) = 0$, $x(1) = 1$.

Theorem 1.5 (Manià, 1934)**.** *The Lagrangian $L(t, x, \xi) = (x^3 - t)^2 \xi^6$ exhibits the Lavrentiev phenomenon, i.e.*

$$\inf_{x \in \mathbf{AC}_*[0,1]} \mathcal{I}(x) < \inf_{x \in \mathbf{Lip}_*[0,1]} \mathcal{I}(x),$$

where $\mathbf{AC}_*[0,1] = \{x \in \mathbf{AC}[0,1] : x(0) = 0, x(1) = 1\}$ and $\mathbf{Lip}_*[0,1] = \{x \in \mathbf{Lip}[0,1] : x(0) = 0, x(1) = 1\}$.

Proof. By definition, the Lagrangian L and the action $\mathcal{I}$ have non-negative values. By the fact that $\mathcal{I}$ evaluated in $\hat{x}(t) = \sqrt[3]{t}$ is zero, we have that $\hat{x}$ is a minimizer of $\mathcal{I}$ on $\mathbf{AC}_*[0,1]$.

Let x be any trajectory in $\mathbf{Lip}_*[0,1]$ and consider the function $f(t) = \sqrt[3]{t}/2$. By the regularity of x, there exists a real number a in $(0,1)$ such that $x(t) \leq f(t)$, for any t in $[0,a]$, and $x(a) = f(a)$. Hence,

$$[x^3(t) - t]^2 \xi^6 \geq [f^3(t) - t]^2 \xi^6 = \frac{7^2}{8^2} t^2 \xi^6,$$

for any t in $[0,a]$, any ξ in $\mathbb{R}$. By the Hölder inequality, we have

$$\frac{\sqrt[3]{a}}{2} = \int_0^a \frac{\sqrt[3]{t}}{\sqrt[3]{t}} x'(t)dt \leq \left(\int_0^a t^{-2/5}dt\right)^{5/6} \left(\int_0^a t^2 x'^6(t)dt\right)^{1/6} = \frac{5^{5/6}}{3^{5/6}} a^{1/2} \left(\int_0^a t^2 x'^6(t)dt\right)^{1/6}.$$

We conclude that, for any x in $\mathbf{Lip}_*[0,1]$,

$$\mathcal{I}(x) \geq \int_0^a [x^3(t) - t]^2 x'^6(t)dt \geq \frac{7^2 3^5}{8^2 5^5 2^6 a} \geq \frac{7^2 3^5}{8^2 5^5 2^6} > 0.$$

□

Perturbing the Lagrangian of Manià it is possible to construct Lagrangians that exhibit the Lavrantiev phenomenon. This is true for a generic Lagrangian ([5], [16], [34]):

Propositon 1.6. *Let L be a Lagrangian that exhibits the Lavrentiev phenomenon. Suppose that $\mathcal{I} \geq 0$ and that there exists a minimizer $\hat{x}$ with $\mathcal{I}(\hat{x}) = 0$.*

Then, for any action $\mathcal{P}(x) = \int_a^b P(t, x, x')$ such that $\mathcal{P} \geq 0$ and $\mathcal{P}(\hat{x})$ is finite, there exists $\hat{\epsilon} > 0$ such that, for any ϵ in $[0, \hat{\epsilon}]$, the Lagrangian $L + \epsilon P$ exhibits the Lavrentiev phenomenon.

Proof. Let c be a positive constant such that $\mathcal{I}(x) \geq c$, for any Lipschitz trajectory x. Setting $\hat{\epsilon} = c/[2\mathcal{P}(\hat{x})]$, we have, for any ϵ in $[0, \hat{\epsilon}]$, $\mathcal{I}(\hat{x}) + \epsilon\mathcal{P}(\hat{x}) \leq c/2$ and $\mathcal{I}(x) + \epsilon\mathcal{P}(x) \geq c$, for any Lipschitz trajectory x. Hence, $L + \epsilon P$ exhibits the Lavrentiev phenomenon. □

Consider the Lagrangian $P(t, x, \xi) = |\xi|^{5/4}$. From the previous proposition it follows that the problem of the Calculus of Variations with action

$$\int_0^1 \{[x^3(t) - t]^2 x'^6(t) + \epsilon |x'(t)|^{5/4}\} dt,$$

and boundary conditions $x(0) = 0$, $x(1) = 1$, presents the Lavrentiev phenomenon. This is significant because the Lagrangian in this problems is strictly convex in its last variable and with super-linear growth.

The Lagrangian of Manià presents also another interesting phenomenon: the repulsion property ([5], [34]).

Theorem 1.7. *For any sequence of Lipschitz trajectories $\{x_n\}_n$ such that x_n tends to $\hat{x}$, as n tends to ∞, almost everywhere on $[0, 1]$, we have that $\mathcal{I}(x_n)$ diverges to ∞.*

Proof. For any integer n, let a_n in $(0, 1)$ be such that $x_n(t) \leq \sqrt[3]{t}/2$, for any t in $[0, a_n]$, and $x(a_n) = \sqrt[3]{a_n}/2$. By the convergence of $x_n(t)$ to $\hat{x}(t)$, for almost every t in $[0, 1]$, we have that a_n tends to 0.

Using the inequality obtained in the proof of Theorem 1.5, we have

$$\mathcal{I}(x_n) \geq \int_0^{a_n} [x^3(t) - t]^2 x'^6(t) dt \geq \frac{7^2 3^5}{8^2 5^5 2^6 a_n}.$$

Hence, $\mathcal{I}(x_n)$ tends to ∞, as n tends to ∞. □

Chapter 2

New Results

In section 2.1 we prove a general approximation theorem for the action ([12]) that implies the non non-occurrence of the Lavrentiev Phenomenon for autonomous and a class of non-autonomous Lagrangians, without assuming any growth condition.

In section 2.2 we prove the non-occurrence of the Lavrentiev phenomenon for a class of Lagrangians of the Calculus of Variations with higher-order derivatives ([26]).

In section 2.3 we infer a necessary condition for the occurrence of the Lavrentiev phenomenon for our class of actions: if the action $\mathcal{I}$ assumes only finite values in a neighbourhood of a minimizer, then $\mathcal{I}$ does not exhibit the Lavrentiev phenomenon ([26]). This result applies to the actions of Manià and Sarychev, for instance, and to the examples of actions exhibiting the Lavrentiev phenomenon proposed in [5], [6], [34], [36], [37], [39], [43]. This necessary condition is also related to the repulsion property of the action of Manià ([5], [34]), i.e. the action evaluated on the trajectories of an absolutely continuous minimizing sequence is divergent to $+\infty$.

2.1 Non-occurrence of the Lavrentiev phenomenon

The purpose of this section is to prove a general theorem on reparameterizations of an interval onto itself which states that, given an absolutely continuous function x on an interval $[a, b]$ and $\epsilon > 0$, under appropriate conditions on L and ψ, there exists a reparameterization $s = s_\epsilon(t)$ of $[a, b]$ such that the composition $x_\epsilon = x \circ s_\epsilon$ is at once Lipschitzian and is such that

$$\int_a^b \sum_{i=1}^m L_i(x_\epsilon(t), x'_\epsilon(t))\psi_i(t, x_\epsilon(t))dt \leq \int_a^b \sum_{i=1}^m L_i(x(t), x'(t))\psi_i(t, x(t))dt + \epsilon.$$

An application of our Theorem 2.1 is the non-occurence of the Lavrentiev Phenomenon for a class of functionals of the Calculus of Variations.

We recall that in 1926 M. Lavrentiev [34] published an example of a functional of the kind

$$\int_a^b L(t, x(t), x'(t))dt, \quad x(a) = A, x(b) = B$$

whose infimum taken over the space of absolutely continuous functions was strictly lower than the infimum taken over the space of Lipschitzian functions. The occurrence of this phenomenon prevents the possibility of computing the minimum, and the minimizer, by a standard finite-element scheme (anyway, in [4], [35], alternative numerical methods for computing minimizer in this situation was presented) and, in case that the action represents the energy of some physical system, it points out that the set of trajectories is a fundamental part of the physical model. Furthermore, it says something about the regularity of the minimizer (if this exists).

One of the surprising feature of this phenomenon is that it occurs for simple Lagrangians: $L(t, x, x') = (x^3 - t)^2 x'^6$, on the interval $[0, 1]$, with boundary conditions $x(0) = 0$, $x(1) = 1$, exhibits the Lavrentiev phenomenon (B. Maniá, 1934, [37]). Moreover, in [29] it was given a physical action of a nonlinear elastic material where the occurrence of the Lavrentiev phenomenon is the occurrence of a meaningful physical event.

In the autonomous case, sufficient conditions to prevent the occurrence of this phenomenon were given by several authors, by imposing enough growth conditions [2], [10], [20], or assuming some regularity conditions on the Lagrangian [1], [3].

Our result applies to non-autonomous problems; it applies to multidimensional rotationally invariant problems, where the measure is $r^D dr$, and, even in the simple autonomous case, it applies to problems with obstacles or with other constraints.

The following is our main theorem, a reparameterization theorem.

Theorem 2.1. *Let $x : [a, b] \to \mathbb{R}^N$ be absolutely continuous and set $C = \{x(t) : t \in [a, b]\}$. Let $L_1, \cdots, L_m : C \times \mathbb{R}^N \to \mathbb{R}$ be continuous and such that $L_i(x, \cdot)$ are convex, and let $\psi_1, \cdots, \psi_m : [a, b] \times C \to [c, +\infty)$ be continuous, with $c > 0$. Then:*

i) $\mathcal{I}(x) = \displaystyle\int_a^b \sum_{i=1}^m L_i(x(t), x'(t))\psi_i(t, x(t))dt > -\infty$;

ii) given any $\epsilon > 0$, there exists a Lipschitzian function x_ϵ, a reparameterization of x, such that $x(a) = x_\epsilon(a)$, $x(b) = x_\epsilon(b)$ and $\mathcal{I}(x_\epsilon) \leq \mathcal{I}(x) + \epsilon$.

Remark 2.2. The only technical assumption of theorem 2.1 is the hypothesis that ψ_i are bounded below by a positive constant. However, in the proof of Theorem 2.1, this assumption is used only to infer that $\int_a^b L_i(x(t), x'(t))dt$ are finite, provided that $\mathcal{I}$ is finite (point *ii)* in the proof). The theorem holds under the following more general assumption: $\psi_i(t, x) \geq 0$ and $\int_a^b L_i(x(t), x'(t))dt < +\infty$, for every i.

To verify how sharp our assumptions are, consider the following example of

Manià ([37], [16] and [22]). Consider the problem of minimizing the functional

$$\int_0^1 [t - x(t)^3]^2 [x'(t)]^6 dt, \quad x(0) = 0, x(1) = 1.$$

Then the infimum taken over the space of absolutely continuous functions (assumed in $x(t) = \sqrt[3]{t}$) is strictly lower than the infimum taken over the space of Lipschitzian functions.

As a consequence, the result of Theorem 2.1 cannot hold for the functional of Manià evaluated along $x(t) = \sqrt[3]{t}$.

Setting $\psi(t, x) = [t - x^3]^2$ and $L(x, \xi) = \xi^6$, we see that $\psi \geq 0$ (but not $\psi \geq c > 0$) and that

$$\int_0^1 [x'(t)]^6 dt = \int_0^1 1/(3^6 t^4) dt = +\infty.$$

Hence the assumption $\psi(t, x) \geq 0$ and $\int_a^b L(x(s), x'(s)) ds < +\infty$ cannot possibly be dropped.

Proof. i) For every $t \in [a, b]$, $L_i(x(t), x'(t)) \geq L_i(x(t), 0) + \langle p_0(t), x'(t) \rangle$, where $p_0(t)$ is any selection from $\partial_\xi L_i(x(t), 0)$. Let $E_i = \{t \in [a, b] : [L_i(x(t), x'(t))]^- \neq 0\}$. Hence,

$$\int_a^b [L_i(x(t), x'(t)) \psi_i(t, x(t))]^- dt \leq - \int_{E_i} [L_i(x(t), 0) + \langle p_0(t), x'(t) \rangle] \psi_i(t, x(t)) dt,$$

for any i. Since ψ_i is bounded and, by Proposition 1.2 in section 1.1, $p_0(t)$ is bounded, the claim follows by Hölder's inequality.

ii) In case

$$\int_a^b L_i(x(t), x'(t)) \psi_i(t, x(t)) dt = +\infty,$$

for some i, any parameterization $t : [a, b] \to [a, b]$ that would make $x \circ t$ Lipschitzian, is acceptable as x_ϵ. Hence from now on we shall assume, for every i,

$$\int_a^b L_i(x(t), x'(t)) \psi_i(t, x(t)) dt < +\infty.$$

We have also

$$+\infty > \int_a^b |L_i(x(t), x'(t))| \psi_i(t, x(t)) dt \geq c \int_a^b |L_i(x(t), x'(t))| dt.$$

a) $C = \{x(t) : t \in [a, b]\}$ is a compact subset of $\mathbb{R}^N$: consider the set

$$V_i = \{(x, p) : x \in C, p \in \partial_\xi L_i(x, \xi), |\xi| \leq 1\}.$$

By Proposition 1.2, arguing by contradiction, we obtain that V_i is compact. Then, $\min_{V_i} L_i^*(x,p)$ is attained and is finite. Applying Proposition 1.3, we obtain that $L_i^*(x,p) \geq \min_{V_i} L_i^*(x,p)$, any $x \in C$ and any $p \in \partial_\xi L_i(x,\xi)$, for any $\xi \in \mathbb{R}^N$. Set $\eta = \min\{\min_{V_1} L_1^*, \cdots, \min_{V_m} L_m^*\}$.

Consider $\tilde{L}_i(x,\xi) = L_i(x,\xi) + \eta$. Since $\partial_\xi L_i(x,\xi) = \partial_\xi \tilde{L}_i(x,\xi)$, we have that $\tilde{L}_i^*(x,p) \geq 0$, for any i.

b) Set $\ell_i = \int_a^b |\tilde{L}_i(x(s), x'(s))| ds$, $\ell = \max\{\ell_1, \cdots, \ell_m\}$, and let Ψ be such that $|\psi_i(s,x)| \leq \Psi$, $\forall (s,x) \in [a,b] \times C$, for any i.

From the uniform continuity of $\psi_i(\cdot, x(\cdot))$ on $[a,b]\times[a,b]$, we infer that we can fix $k \in \mathbb{N}$ such that $\forall (s_1,t_1),(s_2,t_2) \in [a,b]\times[a,b]$, with $|s_2 - s_1| \leq (b-a)/2^k$ and $|t_2 - t_1| \leq (b-a)/2^k$ we have

$$|\psi_i(s_2, x(t_2)) - \psi_i(s_1, x(t_1))| \leq \min\left\{\frac{\epsilon}{4m\ell}, \frac{\epsilon}{2m(|\eta|+1)(b-a)}\right\},$$

for any i.

For $j = 0, \cdots, 2^k - 1$ set $I_j = [(b-a)j/2^k, (b-a)(j+1)/2^k]$, $H_j = \int_{I_j} |x'(s)| ds$, $\mu = \max\{2^{k+1} H_j/(b-a) : j = 0, \cdots, 2^k - 1\}$ and

$$T_{H_j} = \left\{ s \in I_j : |x'(s)| \leq \frac{2^{k+1} H_j}{b-a} \right\};$$

it follows that $|T_{H_j}| \geq (b-a)/2^{k+1}$. Set also $T = \bigcup_{j=0}^{2^k-1} T_{H_j}$.

Since $\{(x(s), x'(s)) : s \in T\}$ belongs to a compact set and $L_1, \cdots, L_m$ are continuous, there exists a constant M, such that

$$\left| \tilde{L}_i(x(s), 2x'(s))\frac{1}{2} - \tilde{L}_i(x(s), x'(s)) \right| \leq M,$$

for all $s \in T$, for any i.

c) For every $n \in \mathbb{N}$ set $S_n^j = \{s \in I_j : |x'(s)| > n\}$, $\epsilon_n^j = \displaystyle\int_{S_n^j} \left(\frac{|x'(s)|}{n} - 1\right) ds$ and $\epsilon_n = \displaystyle\sum_{j=0}^{2^k-1} \epsilon_n^j$. From the integrability of $|x'|$, we have that $\lim_{n\to\infty} \epsilon_n = 0$.

d) Having defined ϵ_n^j, for all n such that $\epsilon_n^j \leq (b-a)/2^{k+2}$, choose $\Sigma_n^j \subset T_{H_j}$ such that $|\Sigma_n^j| = 2\epsilon_n^j$. This is possible from point c).

e) Define the absolutely continuous functions t_n by $t_n(s) = a + \int_a^s t_n'(\tau) d\tau$, where

$$t_n'(s) = \begin{cases} 1 + \left(\frac{|x'(s)|}{n} - 1\right) & , s \in S_n = \bigcup_{i=0}^{2^k-1} S_n^j \\ 1 - \frac{1}{2} & , s \in \Sigma_n = \bigcup_{j=0}^{2^k-1} \Sigma_n^j \\ 1 & , \text{otherwise} \end{cases}$$

One verifies that, $\forall j = 0, \cdots, 2^k - 1$, the restriction of t_n to I_j is an invertible map from I_j onto itself (in particular, each t_n is an invertible map from $[a, b]$ onto itself). It follows that $|t_n(s) - s| \leq (b-a)/2^k$.

e) We have

$$\int_a^b \tilde{L}_i\left(x(s), \frac{x'(s)}{t_n'(s)}\right) t_n'(s)\psi_i(t_n(s), x(s))ds - \int_a^b \tilde{L}_i(x(s), x'(s))\psi_i(s, x(s))ds =$$
$$\int_a^b \left[\tilde{L}_i\left(x(s), \frac{x'(s)}{t_n'(s)}\right) t_n'(s) - \tilde{L}_i(x(s), x'(s))\right] \psi_i(t_n(s), x(s))ds +$$
$$\int_a^b \tilde{L}_i(x(s), x'(s))[\psi_i(t_n(s), x(s)) - \psi_i(s, x(s))]ds,$$

and, from the definition of t_n',

$$\int_a^b \left[\tilde{L}_i\left(x(s), \frac{x'(s)}{t_n'(s)}\right) t_n'(s) - \tilde{L}_i(x(s), x'(s))\right] \psi_i(t_n(s), x(s))ds =$$
$$\int_{S_n} \left[\tilde{L}_i\left(x(s), n\frac{x'(s)}{|x'(s)|}\right) \frac{|x'(s)|}{n} - \tilde{L}_i(x(s), x'(s))\right] \psi_i(t_n(s), x(s))ds + \ .$$
$$\int_{\Sigma_n} \left[\tilde{L}_i\left(x(s), 2x'(s)\right) \frac{1}{2} - \tilde{L}_i(x(s), x'(s))\right] \psi_i(t_n(s), x(s))ds.$$

We wish to estimate the above integrals. Since $\Sigma_n \subset T$, we obtain

$$\int_{\Sigma_n} \left[\tilde{L}_i\left(x(s), 2x'(s)\right) \frac{1}{2} - \tilde{L}_i(x(s), x'(s))\right] \psi_i(t_n(s), x(s))ds \leq 2M\Psi\epsilon_n.$$

Moreover, for every $s \in S_n$

$$\tilde{L}_i\left(x(s), n\frac{x'(s)}{|x'(s)|}\right) \frac{|x'(s)|}{n} - \tilde{L}_i(x(s), x'(s)) \leq$$
$$-\left(\frac{|x'(s)|}{n} - 1\right) \tilde{L}_i^*\left(x(s), p\left(x(s), n\frac{x'(s)}{|x'(s)|}\right)\right) \leq 0;$$

hence

$$\int_{S_n} \left[\tilde{L}_i\left(x(s), n\frac{x'(s)}{|x'(s)|}\right) \frac{|x'(s)|}{n} - \tilde{L}_i(x(s), x'(s))\right] \psi_i(t_n(s), x(s))ds \leq 0.$$

f) The choice of k implies that

$$\int_a^b \tilde{L}_i(x(s), x'(s))[\psi_i(t_n(s), x(s)) - \psi_i(s, x(s))]ds \leq \frac{\epsilon}{4m}.$$

We have obtained

$$\int_a^b \tilde{L}_i\left(x(s), \frac{x'(s)}{t_n'(s)}\right) t_n'(s)\psi_i(t_n(s), x(s))ds - \int_a^b \tilde{L}_i(x(s), x'(s))\psi_i(s, x(s))ds \leq 2M\Psi\epsilon_n + \frac{\epsilon}{4m}.$$

g) Fix n such that $2M\Psi\epsilon_n \leq \epsilon/(4m)$.

Then, the conclusion of f) proves the Theorem; in fact, defining $x_\epsilon = x \circ s_n$, where s_n is the inverse of the function t_n, we obtain, by the change of variable formula [44], that

$$\begin{aligned}
\int_a^b \tilde{L}_i(x_\epsilon(t), x'_\epsilon(t))\psi_i(t, x_\epsilon(t))dt &= \int_a^b \tilde{L}_i\left(x_\epsilon(t_n(s)), \frac{dx_\epsilon}{dt}(t_n(s))\right) t'_n(s)\psi_i(t_n(s), x_\epsilon(t_n(s)))ds \\
&= \int_a^b \tilde{L}_i\left(x(s), \frac{x'(s)}{t'_n(s)}\right) t'_n(s)\psi_i(t_n(s), x(s))ds \\
&\leq \int_a^b \tilde{L}_i(x(s), x'(s))\psi_i(s, x(s))ds + \frac{\epsilon}{2m}.
\end{aligned}$$

so that

$$\begin{aligned}
&\int_a^b L_i(x_\epsilon(t), x'_\epsilon(t))\psi_i(t, x_\epsilon(t))dt - \int_a^b L_i(x(s), x'(s))\psi_i(s, x(s))ds \leq \\
&\int_a^b [L_i(x_\epsilon(t), x'_\epsilon(t)) + \eta]\psi_i(t, x_\epsilon(t))dt - \int_a^b [L_i(x(s), x'(s)) + \eta]\psi_i(s, x(s))ds + \frac{\epsilon}{2m} = \\
&\int_a^b \tilde{L}_i(x_\epsilon(t), x'_\epsilon(t))\psi_i(t, x_\epsilon(t))dt - \int_a^b \tilde{L}_i(x(s), x'(s))\psi_i(s, x(s))ds + \frac{\epsilon}{2m} \leq \frac{\epsilon}{m}.
\end{aligned}$$

Hence, $\mathcal{I}(x_\epsilon) - \mathcal{I}(x) \leq \epsilon$.

Moreover, x_ϵ is Lipschitzian. In fact, consider the equality $x'_\epsilon(t_n(s)) = x'(s)/t'_n(s)$ and fix s where $t'_n(s)$ exists; we obtain

$$\left|\frac{dx_\epsilon}{dt}(t_n(s))\right| \begin{cases} = n & , s \in S_n \\ \leq \mu & , s \in \Sigma_n \\ \leq n & , otherwise \end{cases} \quad ;$$

hence, at almost every s, the norm of the derivative of x_ϵ is bounded by n. This completes the proof. $\square$

The theorems below present some applications of Theorem 2.1 to prevent the occurrence of the Lavrentiev phenomenon to different classes of Minimum Problems.

Denote by $\mathbf{Lip}[a,b]$ and by $\mathbf{AC}[a,b]$, respectively, the space of all Lipschitzian and absolutely continuous functions from $[a,b]$ to $\mathbb{R}^N$. Let $E \subset \mathbb{R}^N$ and consider the functional

$$\mathcal{I}(x) = \int_a^b L(x(s), x'(s))\psi(s, x(s))ds.$$

Call $\inf(P)_\infty$ the infimum of $\{\mathcal{I}(x) : x \in \mathbf{Lip}[a,b], x(t) \in E, x(a) = A, x(b) = B\}$ and $\inf(P)_1$ the infimum of $\{\mathcal{I}(x) : x \in \mathbf{AC}[a,b], x(t) \in E, x(a) = A, x(b) = B\}$.

Theorem 2.3. *Let $L : E \times \mathbb{R}^N \to \mathbb{R}$ be continuous and such that $L(x, \cdot)$ is convex and let $\psi : [a,b] \times E \to [c, +\infty)$ be continuous, with $c > 0$; then* $\inf(P)_\infty = \inf(P)_1$.

In the previous Theorem E can be any subset of $\mathbb{R}^N$ such that the set of absolutely continuous functions with values in E and satisfying the boundary conditions is non-empty. In particular, $x \in E$ can describe a problem with an obstacle.

As an application to a problem with a constraint different from an obstacle, let $E = \mathbb{R}^2 \setminus \{0\}$ and call $\inf(P^i)_\infty$ the infimum of $\{\mathcal{I}(x) : x \in \mathbf{Lip}[a,b], x(t) \in E, x(a) = x(b)\}$ and having prescribed *rotation number* $i(x) = k$. Call $\inf(P^i)_1$ the infimum of the same problem but for $x \in \mathbf{AC}[a,b]$.

Theorem 2.4. *Let $L : E \times \mathbb{R}^2 \to \mathbb{R}$ be continuous and such that $L(x,\cdot)$ is convex and let $\psi : [a,b] \times E \to [c,+\infty)$ be continuous, with $c > 0$; then* $\inf(P^i)_\infty = \inf(P^i)_1$.

Proof. As it is well known the rotation number i is independent of the parameterization of x. □

Theorem 2.4 applies in particular to the case $L(x,\xi) = |\xi|^2/2 + 1/|x|$, the case of the Newtonian potential generated by a body fixed at the origin. Gordon in [31] proved that Keplerian orbits are minima to this problem with $k = 1$.

As a further application, we consider a vectorial case. Let $L : E \times \mathbb{R}^N \to \mathbb{R}$ be a continuous function such that $L(u,\cdot)$ is convex (we shall assume that the Lagrangian is independent of the integration variable). Suppose that $L(u,\cdot)$ has the symmetry of being rotationally invariant, i.e. assuming that there exists a function $h : E \times [0,\infty) \to \mathbb{R}$ such that $L(u,\xi) = h(u,|\xi|)$.

Consider the functional

$$\mathcal{I}(u) = \int_{S[a,b]} L(u(x), \nabla u(x))dx$$

where $S[a,b] = \{x \in \mathbb{R}^{D+1} : a \le |x| \le b\}$. Denote by $\inf(P)_\infty$ the infimum of $\{\mathcal{I}(u) : u \in \mathbf{Lip}(S[a,b]), u(x) \in E, u \text{ radial}, u|_{\partial B(0,a)} = A, u|_{\partial B(0,b)} = B\}$ and $\inf(P)_1$ the infimum of $\{\mathcal{I}(u) : u \in \mathbf{W}^{1,1}(S[a,b]), u(x) \in E, u \text{ radial}, u|_{\partial B(0,a)} = A, u|_{\partial B(0,b)} = B\}$. It is our purpose to prove that $\inf(P)_\infty = \inf(P)_1$.

Observe that if $w : [a,b] \to E$ is such that $u(x) = w(|x|)$ then

$$\mathcal{I}(u) = C_D \int_a^b L(w(r), w'(r)) r^D dr, \quad w(a) = A, w(b) = B,$$

where $C_D = \dfrac{\pi^{(D+1)/2}}{\Gamma((D+3)/2)}(b^{D+1} - a^{D+1})$.

Theorem 2.5. *Let $L : E \times \mathbb{R}^N \to \mathbb{R}$ be continuous and such that $L(u,\cdot)$ is convex; then* $\inf(P)_\infty = \inf(P)_1$.

2.2 The case of higher-order action functionals

Besides the first-order case, the Lavrentiev phenomenon occurs as well in the case with $(\nu+1)$-order derivatives, $\mathcal{I}(x) = \int_a^b L(t, x, x', \cdots, x^{(\nu+1)})$. For $\nu = 1$, in 1994 C. W. Cheng and V. J. Mizel [17] described a restricted Lavrentiev phenomenon in which the gap occurs for a dense subset of the absolutely continuous non-negative functions, and they proved that even autonomous Lagrangian $L(x, x', x'')$ can exhibit it. Some years later A. V. Sarychev [43] proved that a class of Lagrangians of the form

$$L_1(x'')\psi_1(x, x') + L_2(x'')$$

exhibits the Lavrentiev phenomenon provided that $\psi_1(x, x') = \phi(kx - k|x'-1|^{k-1} - (k-1)|x'-1|^k)$ for appropriate constants k, that L_1, L_2, ϕ satisfy certain growth conditions, and that $\phi(0) = 0$. For example, $L_1(x'') = |x''|^7$, $L_2(x'') = \alpha|x''|^{3/2}$, $\phi_1(\cdot) = (\cdot)^2$, $k = 3$ and $\alpha > 0$ sufficiently small yield a Lagrangian whose integral exhibits the Lavrentiev phenomenon when the boundary values are $x(0) = 0$,$x(1) = \ 5/3$, $x'(0) = 1$,$x'(1) = 2$.

The Lagrangians proposed by Manià and Sarychev have the property that L_1 evaluated along the minimizer x is not integrable (this is possible because there exists at least one point t in $[a, b]$ such that ψ_1 evaluated along x in t is 0). A condition avoiding the occurrence of this fact will turn out, in this section, to be essential for the non-occurrence of the Lavrentiev phenomenon.

We prove the following general approximation theorem: let $x : [a, b] \to \mathbb{R}^N$ be a function in $\mathbf{W}^{\nu+1,1}$ (independently on whether is a minimizer or not), then the integrability of L_i evaluated along x (or the assumption that $\psi_i > 0$), for every i, implies that, given $\epsilon > 0$, there exists a function x_ϵ in $\mathbf{W}^{\nu+1,\infty}$ with the same boundary values of x in a and in b, i.e. $x_\epsilon(a) = x(a)$, $x_\epsilon(b) = x(b)$, $x'_\epsilon(a) = x'(a)$, $x'_\epsilon(b) = x'(b)$,$\cdots$, $x_\epsilon^{(\nu)}(a) = x^{(\nu)}(a)$, $x_\epsilon^{(\nu)}(b) = x^{(\nu)}(b)$, such that

$$\int_a^b \sum_{i=1}^m L_i(x_\epsilon^{(\nu)}, x_\epsilon^{(\nu+1)})\psi_i(t, x_\epsilon, x'_\epsilon, \cdots, x_\epsilon^{(\nu)}) < \int_a^b \sum_{i=1}^m L_i(x^{(\nu)}, x^{(\nu+1)})\psi_i(t, x, x', \cdots, x^{(\nu)}) + \epsilon.$$

We underline that an application of this result is the non-occurrence of the Lavrentiev phenomenon for a class of functionals of the Calculus of Variations with $(\nu+1)$-order derivatives, $\nu \geq 1$. The case $\nu = 0$, $m = 1$ is covered in the previous section. The case $\nu = 0$, $m > 1$ can be obtained modifying slightly the proof of the main result of the previous section.

For $\delta > 0$, $B[c, \delta]$ denotes the closed ball in $\mathbb{R}^N$ centred in c with radius δ. For a function x in $\mathbf{C}^\nu[a, b]$, with values in $\mathbb{R}^N$, the closed δ-tube along $(x, \cdots, x^{(\nu)})$

$$\mathsf{T}_\delta^\nu[x] = \{(t, z_0, \cdots, z_\nu) \in [a, b] \times \mathbb{R}^{(\nu+1)N} : (z_0, \cdots, z_\nu) \in B[x(t), \delta] \times \cdots \times B[x^{(\nu)}(t), \delta], t \in [a, b]\}$$

and the closed δ-neighbourhood of the image $\mathrm{Im}(x^{(\nu)})$ of $x^{(\nu)}$

$$\mathsf{I}_\delta[x^{(\nu)}] = \{z \in \mathbb{R}^N : \mathrm{dist}(z, \mathrm{Im}(x^{(\nu)})) \le \delta\}$$

are compact sets.

We recall that the space $\mathbf{W}^{\nu+1,p}(a,b)$ can be seen as the space of functions x in $\mathbf{C}^\nu[a,b]$ such that $x^{(\nu)}$ is absolutely continuous with derivative in $\mathbf{L}^p(a,b)$, $p \ge 1$.

The following approximation theorem is our main result:

Theorem 2.6. *Let x be a function in $\mathbf{W}^{\nu+1,1}(a,b)$, $\nu \ge 1$, and let the real-valued functions $L_1, \cdots, L_m$ and $\psi_1, \cdots, \psi_m$ be continuous on $\mathsf{I}_\delta[x^{(\nu)}] \times \mathbb{R}^N$ and on $\mathsf{T}^\nu_\delta[x]$ respectively, for some $\delta > 0$.*

Assume that, for every i in $\{1, \cdots, m\}$,

- *$L_i(\xi, \cdot)$ is convex, for every ξ in $\mathsf{I}_\delta[x^{(\nu)}]$,*
- *ψ_i is non-negative, and $\psi_i(t, x(t), x'(t), \cdots, x^{(\nu)}(t)) > 0$, for every t in $[a,b]$.*

Then:

(i) $\mathcal{I}(x) = \displaystyle\int_a^b \sum_{i=1}^m L_i(x^{(\nu)}(t), x^{(\nu+1)}(t))\psi_i(t, x(t), x'(t), \cdots, x^{(\nu)}(t))dt > -\infty$;

(ii) given any $\epsilon > 0$, there exists a function x_ϵ in $\mathbf{W}^{\nu+1,\infty}(a,b)$ such that

$$\mathcal{I}(x_\epsilon) < \mathcal{I}(x) + \epsilon,$$

and

$$\begin{array}{ll} x_\epsilon(a) = x(a), & x_\epsilon(b) = x(b), \\ x'_\epsilon(a) = x'(a), & x'_\epsilon(b) = x'(b), \\ \vdots & \\ x^{(\nu)}_\epsilon(a) = x^{(\nu)}(a), & x^{(\nu)}_\epsilon(b) = x^{(\nu)}(b). \end{array}$$

As a corollary we obtain the non-occurrence of the Lavrentiev phenomenon:

Theorem 2.7. *Let $\Omega_0, \cdots, \Omega_\nu$ be open sets in $\mathbb{R}^N$, $\nu \ge 1$, such that the set $E = \{x \in \mathbf{W}^{\nu+1,1}(a,b) : x(t) \in \Omega_0, \cdots, x^{(\nu)}(t) \in \Omega_\nu, \forall t \in [a,b]\}$ is non-empty.*

Let $L_1, \cdots, L_m : \Omega_\nu \times \mathbb{R}^N \to \mathbb{R}$ and $\psi_1, \cdots, \psi_m : [a,b] \times \Omega_0 \times \cdots \times \Omega_\nu \to (0, +\infty)$ be continuous and such that $L_i(\xi, \cdot)$ is convex, for any ξ in Ω_ν, any i in $\{1, \cdots, m\}$.

Then, for every boundary values $A, B \in \Omega_0$, $A^{(1)}, B^{(1)} \in \Omega_1$,$\cdots$, $A^{(\nu)}, B^{(\nu)} \in \Omega_\nu$, the infimum of

$$\mathcal{I}(x) = \int_a^b \sum_{i=1}^m L_i(x^{(\nu)}(t), x^{(\nu+1)}(t))\psi_i(t, x(t), x'(t), \cdots, x^{(\nu)}(t))dt$$

over the space $E_{a,b} = \{x \in E : x(a) = A, x(b) = B,\ x'(a) = A^{(1)}, x'(b) = B^{(1)}$, $\cdots$,$x^{(\nu)}(a) = A^{(\nu)}, x^{(\nu)}(b) = B^{(\nu)}\}$ is equal to the infimum of the same functional $\mathcal{I}$ over the space $E_{a,b} \cap \mathbf{W}^{\nu+1,\infty}(a,b)$.

Proof. Let $\{x_n\}_n \subset E_{a,b}$ be a minimizing sequence for $\mathcal{I}$: by the fact that $\psi_i > 0$, for every i, the theorem follows from Theorem 2.6 applied to any x_n, with $\epsilon = 1/n$. $\square$

Setting $m = 1$, $\psi_1 = 1$ and $L_1 = L$, we obtain that a Lagrangian depending only on $x^{(\nu)}$ and $x^{(\nu+1)}$ satisfies the assumptions of Theorem 2.7. Hence, the integral functional

$$\int_a^b L(x^{(\nu)}(t), x^{(\nu+1)}(t))dt$$

does not exhibit the Lavrentiev phenomenon, for any boundary values

$$\begin{array}{ll} x(a) = A, & x(b) = B, \\ x'(a) = A^{(1)}, & x'(b) = B^{(1)}, \\ \vdots & \\ x^{(\nu)}(a) = A^{(\nu)}, & x^{(\nu)}(b) = B^{(\nu)}. \end{array}$$

This extends some previous results ([1], [6]), where functionals without boundary conditions, or with boundary conditions only in a, have been considered.

We point out that the assumption $\psi_i(t, x(t), x'(t), \cdots, x^{(\nu)}(t)) \neq 0$, $\forall t \in [a, b]$, in Theorem 2.6 will be used only to infer that $\int_a^b L_i(x^{(\nu)}, x^{(\nu+1)})$ is finite, provided that $\mathcal{I}(x)$ is finite (point *(a)* in the proof). The theorem holds under the weaker assumption $\int_a^b |L_i(x^{(\nu)}, x^{(\nu+1)})| < +\infty$, for every i.

To verify how sharp our assumptions are, consider the following example of A. V. Sarychev [43]: for $\nu = 1$, $m = 1$, minimize the functional

$$\int_0^1 |x''(t)|^7 [3x(t) - 3|x'(t) - 1|^2 - 2|x'(t) - 1|^3]^2 dt,$$

with boundary conditions $x(0) = 0$, $x(1) = 5/3$, $x'(0) = 1$, $x'(1) = 2$. He proved that the infimum taken over the space $\mathbf{W}^{2,1}(0,1)$, assumed in $\bar{x}(t) = (2/3)\sqrt[2]{t^3} + t$, is strictly lower than the infimum taken over the space $\mathbf{W}^{2,\infty}(0,1)$.

The assumption $\int_a^b |L_1(x', x'')| < +\infty$ along $\bar{x}$ is not verified. Indeed, setting $\psi_1(t, x, \xi) = [3x - 3|\xi - 1|^2 - 2|\xi - 1|^3]^2$ and $L_1(\xi, w) = |w|^7$, we see that $\psi_1 \geq 0$ (but, for example, $\psi_1(0, x(0), x'(0)) = 0$) and that

$$\int_0^1 |\bar{x}''(t)|^7 dt = \int_0^1 \frac{1}{(2\sqrt{t})^7} dt = +\infty.$$

Proof. of Theorem 2.6. In what follows, $\mathbf{x}$ denotes the matrix $(x, \cdots, x^{(\nu-1)})$ and $\mathsf{x} = x^{(\nu-1)}$, so that $\mathsf{x}' = x^{(\nu)}$, $\mathsf{x}'' = x^{(\nu+1)}$ (similarly, $\mathbf{z} = (z, \cdots, z^{(\nu-1)})$ and $\mathsf{z} = z^{(\nu-1)}$). The Lagrangian we consider takes the form

$$\sum_{i=1}^m L_i(\mathsf{x}'(t), \mathsf{x}''(t)) \psi_i(t, \mathbf{x}(t), \mathsf{x}'(t)).$$

(In case $\nu = 1$, $\mathbf{x}$, x', x'' coincide with x, x', x'' respectively.)

(i) For every $t \in [a,b]$, $L_i(\mathsf{x}'(t), \mathsf{x}''(t)) \geq L_i(\mathsf{x}'(t), 0) + \langle p_0(t), \mathsf{x}''(t)\rangle$, where $p_0(t)$ is any selection from the sub-differential $\partial_w L_i(\mathsf{x}'(t), 0)$ of L_i with respect to its second variable. Set $E_i = \{t \in [a,b] : [L_i(\mathsf{x}'(t), \mathsf{x}''(t))]^- \neq 0\}$, so that

$$\int_a^b [L_i(\mathsf{x}'(t), \mathsf{x}''(t))\psi_i(t, \mathbf{x}(t), \mathsf{x}'(t))]^- dt \leq -\int_{E_i} [L_i(\mathsf{x}'(t), 0) + \langle p_0(t), \mathsf{x}''(t)\rangle]\psi_i(t, \mathbf{x}(t), \mathsf{x}'(t))dt,$$

for any i. Since ψ_i is bounded and, by Proposition 1.2 in section 1.1, $p_0(t)$ is bounded, the claim follows by Hölder's inequality.

(ii) Fix $\epsilon > 0$; set $\bar{\epsilon} = \epsilon/m$. Without loss of generality, we shall assume $\epsilon < 1$, and also $\delta < 1$.

In case $\int_a^b L_i(\mathsf{x}'(t), \mathsf{x}''(t))\psi_i(t, \mathbf{x}(t), \mathsf{x}'(t))dt = +\infty$, for some i, any Lipschitz function x_ϵ satisfying the boundary conditions is acceptable. Hence we can assume, for every i,

$$\int_a^b L_i(\mathsf{x}'(t), \mathsf{x}''(t))\psi_i(t, \mathbf{x}(t), \mathsf{x}'(t))dt < +\infty.$$

The proof is in three steps. In *Step (1)* of the proof we introduce new functions $\tilde{L}_i$ such that $\tilde{L}_i = L_i + \text{const}$ and such that their polar functions $\tilde{L}_i^*$ (with respect to the second variable) are non-negative. In *Step (3)* we define a variation z_n in $\mathbf{W}^{\infty,1}(a,b)$, with the same boundary values of x in a and in b, such that $\mathcal{I}(z_n) < \mathcal{I}(x) + \epsilon$. In order to define z_n, in *Step (2)* we define a sequence of reparameterizations s_n of $[a,b]$.

Step (1) We claim that there exists functions $\tilde{L}_i$ and a constant η such that $\tilde{L}_i = L_i + \eta$ and $\tilde{L}_i^* \geq 0$, for any i.

In fact, consider the set

$$V_i = \{(\xi, p) : \xi \in \mathsf{I}_\delta[x^{(\nu)}], p \in \partial_w L_i(\xi, w), |w| \leq 1\}.$$

By Proposition 1.2, arguing by contradiction, we obtain that V_i is compact. Let $L_i^*(\xi, p) = \sup_{w \in \mathbb{R}^N} \langle p, w\rangle - L_i(\xi, w)$ be the polar function of L_i with respect to its second variable. Then, $\min_{V_i} L_i^*$ is attained and is finite. Applying Proposition 1.3, we obtain that $L_i^*(\xi, p) \geq \min_{V_i} L_i^*$, for every $\xi \in \mathsf{I}_\delta[x^{(\nu)}]$, for every $p \in \partial_w L_i(\xi, w)$ and for every $w \in \mathbb{R}^N$. Set $\eta = \min\{\min_{V_1} L_1^*, \cdots, \min_{V_m} L_m^*\}$.

Consider $\tilde{L}_i(\xi, w) = L_i(\xi, w) + \eta$. Since $\partial_w L_i(\xi, w) = \partial_w \tilde{L}_i(\xi, w)$, we have that $\tilde{L}_i^*(\xi, p) \geq 0$, for any i. (We denote $\tilde{\mathcal{I}}_i$ the functional $\int_a^b \tilde{L}_i \psi_i$.)

(a) We set some constants, depending on $\bar{\epsilon}$ fixed, that we shall use in the following steps.

By the condition on ψ_i, there exists a $c > 0$ such that $\psi_i(t, \mathbf{x}(t), \mathsf{x}'(t)) \geq c$, for every t in $[a,b]$, and we obtain

$$\begin{aligned} +\infty \ &> \int_a^b |L_i(\mathsf{x}'(t),\mathsf{x}''(t))\psi_i(t,\mathbf{x}(t),\mathsf{x}'(t))|dt + \eta \int_a^b \psi_i(t,\mathbf{x}(t),\mathsf{x}'(t))dt \\ &\geq \int_a^b |\tilde{L}_i(\mathsf{x}'(t),\mathsf{x}''(t))\psi_i(t,\mathbf{x}(t),\mathsf{x}'(t))|dt \geq c\int_a^b |\tilde{L}_i(\mathsf{x}'(t),\mathsf{x}''(t))|dt. \end{aligned}$$

Set $\ell_i = \int_a^b |\tilde{L}_i(\mathsf{x}'(s),\mathsf{x}''(s))|ds$, $\ell = \max\{\ell_1,\cdots,\ell_m\}$, and Ψ and $\tilde{\mathbf{L}}$ the maximum value of $|\psi_1|,\cdots,|\psi_m|$ over $\mathsf{T}^\nu_\delta[x]$ and of $|\tilde{L}_1|,\cdots,|\tilde{L}_m|$ over $\mathsf{I}_\delta[x^{(\nu)}]\times B[0,|\mathsf{x}''(\tau)|+\delta]$ respectively. Denote $\alpha = \max\{1,(b-a)^\nu\}$.

From the uniform continuity of $\psi_1,\cdots,\psi_m$ on $\mathsf{T}^\nu_\delta[x]$, we infer that we can fix $h\in\mathbb{N}$, $1/2^h<\delta$, such that whenever $(t_1,\mathbf{x}_1,\xi_1),(t_2,\mathbf{x}_2,\xi_2)\in\mathsf{T}^\nu_\delta[x]$ and

$$|t_1-t_2| \leq \frac{b-a}{2^h}, \quad |\mathbf{x}_{1,j}-\mathbf{x}_{2,j}| \leq \frac{1}{2^h},\ \forall j\in\{0,\cdots,\nu-1\}, \quad |\xi_1-\xi_2| \leq \frac{1}{2^h},$$

we have

$$|\psi_i(t_1,\mathbf{x}_1,\xi_1)-\psi_i(t_2,\mathbf{x}_2,\xi_2)| < \min\left\{\frac{\bar{\epsilon}}{8(\ell+\tilde{\mathbf{L}}+1)}, \frac{\bar{\epsilon}}{2(|\eta|+1)(b-a)}\right\},$$

for any i.

Let $\theta:\mathbb{R}\to[0,1]$ be a $\mathbf{C}^\infty$ increasing function with value 0 on $(-\infty,0]$ and 1 on $[1,+\infty)$. Observe that $1\leq||\theta^{(j)}||_\infty \leq ||\theta^{(j+1)}||_\infty$, for any $j\geq 0$. Set $\Theta = ||\theta^{(\nu+1)}||_\infty$.

There exists a point τ in (a,b) which is a Lebesgue point for the functions $\tilde{L}_1(\mathsf{x}'(\cdot),\mathsf{x}''(\cdot))\psi_1(\cdot,\mathbf{x}(\cdot),\mathsf{x}'(\cdot)),\cdots,\tilde{L}_m(\mathsf{x}'(\cdot),\mathsf{x}''(\cdot))\psi_m(\cdot,\mathbf{x}(\cdot),\mathsf{x}'(\cdot))$ and x'', $\mathsf{x}''(\tau)$ in $\mathbb{R}^N$. By definition of Lebesgue point, there exists a positive number ρ less than

$$\min\left\{\frac{1}{2^{h+4}(\nu+2)(\nu+1)\nu\Theta\alpha^2}, \frac{\bar{\epsilon}}{32\tilde{\mathbf{L}}\Psi}\right\}$$

such that, for any λ^-,λ^+ in $(0,\rho)$,

$$\int_{\tau-\lambda^-}^{\tau+\lambda^+} |\tilde{L}_i(\mathsf{x}'(t),\mathsf{x}''(t))\psi_i(t,\mathbf{x}(t),\mathsf{x}'(t)) - \tilde{L}_i(\mathsf{x}'(\tau),\mathsf{x}''(\tau))\psi_i(\tau,\mathbf{x}(\tau),\mathsf{x}'(\tau))|dt \leq (\lambda^++\lambda^-)\bar{\epsilon}$$

for any i, and

$$\int_{\tau-\lambda^-}^{\tau+\lambda^+} |\mathsf{x}''(t)-\mathsf{x}''(\tau)|dt \leq (\lambda^++\lambda^-)\frac{1}{2^{h+4}(\nu+1)\nu\Theta\alpha}.$$

Fix $t_0^- = (b-a)v^-/2^\gamma$, $t_0^+ = (b-a)v^+/2^\gamma$, where $\gamma\in\mathbb{N}$, $v^-,v^+\in\{0,1,\cdots,2^\gamma\}$, $v^-<v^+$, are such that $\tau\in(\tau^-,\tau^+)\subset(\tau-\rho,\tau+\rho)$.

We define the absolutely continuous function $\mathsf{z}' : [a,b] \to \mathbb{R}^N$ by $\mathsf{z}'(t) = x^{(\nu)}(a) + \int_a^t \mathsf{z}''$, where

$$\mathsf{z}''(t) = \begin{cases} x^{(\nu+1)}(\tau) + \dfrac{1}{\tau^+ - \tau^-} \displaystyle\int_{\tau^-}^{\tau^+} [\mathsf{x}'' - \mathsf{x}''(\tau)] & , t \in [\tau^-, \tau^+] \\ x^{(\nu+1)}(t) & , \text{otherwise} \end{cases} .$$

By definition, $\mathsf{z}''(t) = \mathsf{x}''(t)$, $\mathsf{z}'(t) = \mathsf{x}'(t)$, for any t in $[a, \tau^-] \cup [\tau^+, b]$. For any t in $[\tau^-, \tau^+]$, we have that $\mathsf{z}''(t) \in B[0, |\mathsf{x}''(\tau)| + \delta/2]$ and

$$|\mathsf{z}'(t) - \mathsf{x}'(t)| \le 2 \int_{\tau^-}^{\tau^+} |\mathsf{x}''(\tau) - \mathsf{x}''| < (\tau^+ - \tau^-) \frac{1}{2^{h+3}(\nu+1)\nu\Theta\alpha}.$$

Step (2) Our purpose is to show that there exists a sequence of reparameterizations s_n of $[a,b]$ into itself such that $\mathsf{z}' \circ s_n$ is Lipschitz continuous on $[a,b]$.

From the uniform continuity of $x, \cdots, x^{(\nu)}$ on $[a, \tau^-] \cup [\tau^+, b]$, we infer that we can fix $k \in \mathbb{N}$, such that whenever $|s_1 - s_2| \le (b-a)/2^k$, we have $|x^{(j)}(s_1) - x^{(j)}(s_2)| < (\tau^+ - \tau^-)^{\nu+2}$, for any j in $\{1, \cdots, \nu\}$.

For $v = 0, \cdots, 2^k - 1$ set $I_v = [(b-a)v/2^k, (b-a)(v+1)/2^k]$, $H_v = \int_{I_v} |\mathsf{z}''(s)| ds$, $\mu = \max\{2^{k+1} H_v/(b-a) : v = 0, \cdots, 2^k - 1\}$ and

$$T_{H_v} = \left\{ s \in I_v : |\mathsf{z}''(s)| \le \frac{2^{k+1} H_v}{b-a} \right\};$$

we have that $|T_{H_v}| \ge (b-a)/2^{k+1}$.

Since $\{(\mathsf{z}'(s), \mathsf{z}''(s)) : s \in \bigcup_{v=0}^{2^k-1} T_{H_v}\}$ belongs to a compact set and $L_1, \cdots, L_m$ are continuous, there exists a constant M, such that

$$\left| \tilde{L}_i(\mathsf{z}'(s) + \xi, 2\mathsf{z}''(s) + w) \frac{1}{2} - \tilde{L}_i \left(\mathsf{z}'(s) + \xi, \mathsf{z}''(s) + \frac{w}{2} \right) \right| \le M,$$

for any $s \in \bigcup_{v=0}^{2^k-1} T_{H_v}$, any $|\xi| \le \delta$, any $|w| \le \delta$, and for any i.

For every $n \in \mathbb{N}$, set $S_n^v = \{s \in I_v : |\mathsf{z}''(s)| > n\}$. From the integrability of z'' it follows that $\int_{S_n^v} (|\mathsf{z}''(s)|/n - 1) ds$ converges to 0, as n goes to ∞. Hence, we can fix a subset Σ_n^v of T_{H_v} such that $|\Sigma_n^v| = 2 \int_{S_n^v} (|\mathsf{z}''(s)|/n - 1) ds$.

We define the absolutely continuous functions t_n by $t_n(s) = a + \int_a^s t_n'$, where

$$t_n'(s) = \begin{cases} 1 + (|\mathsf{z}''(s)|/n - 1) & , s \in S_n = \bigcup_{v=0}^{2^k-1} S_n^v \\ 1 - 1/2 & , s \in \Sigma_n = \bigcup_{v=0}^{2^k-1} \Sigma_n^v \\ 1 & , \text{otherwise} \end{cases} .$$

One verifies that t_n admits inverse function s_n on the interval $[a,b]$. Furthermore, for any v in $\{0, \cdots, 2^k - 1\}$, the restriction of t_n to I_v maps I_v onto itself. Hence,

$|t_n(s) - s| \le (b-a)/2^k$, for any s in $[a,b]$. If n is greater than $|\mathsf{x}''(\tau)| + \delta/2$, the restriction of t_n to $[\tau^-, \tau^+]$ is the identity.

The function $\mathsf{z}' \circ s_n$ is Lipschitz continuous on $[a,b]$. In fact, fix t where $s_n'(t)$ exists: we obtain

$$\left|\frac{d(\mathsf{z}' \circ s_n)}{dt}(t)\right| = |\mathsf{z}''(s_n(t))s_n'(t)| \begin{cases} = n & , t \in S_n \\ \le \mu & , t \in \Sigma_n \\ \le n & , \text{otherwise} \end{cases} .$$

Step (3) We construct a function $z_n : [a,b] \to \mathbb{R}^N$, with the same boundary values of x in a and in b, such that z_n belongs to $\mathbf{W}^{\nu+1,\infty}(a,b)$ and $\tilde{\mathcal{I}}_i(z_n) < \tilde{\mathcal{I}}_i(x) + \bar{\epsilon}/2$.

Set $f'(t) = \theta((t-\tau^-)/(\tau^+ - \tau^-))$, for any t in $[a,b]$ (the function θ as defined in point *(a)*): then f' is identically 0 on $[a,\tau^-]$, it is identically 1 on $[\tau^+, b]$, and $||f^{(j+1)}||_\infty = ||\theta^{(j)}||_\infty/(\tau^+ - \tau^-)^j$, for any $j \ge 0$.

We define ν absolutely continuous functions $z_{n,\nu-1}, \cdots, z_{n,0} : [a,b] \to \mathbb{R}^N$ by

$$\begin{aligned} z_{n,\nu-1}(t) &= x^{(\nu-1)}(a) + \int_a^t \mathsf{z}' \circ s_n + f'(t)D_{\nu-1}, \\ z_{n,\nu-2}(t) &= x^{(\nu-2)}(a) + \int_a^t z_{n,\nu-1} + f'(t)D_{\nu-2}, \\ &\vdots \\ z_{n,0}(t) &= x(a) + \int_a^t z_{n,1} + f'(t)D_0, \end{aligned}$$

where, for any j in $\{0, \cdots, \nu-2\}$,

$$D_j = x^{(j)}(b) - x^{(j)}(a) - \int_a^b z_{n,j+1}, \qquad D_{\nu-1} = x^{(\nu-1)}(b) - x^{(\nu-1)}(a) - \int_a^b \mathsf{z}' \circ s_n.$$

Set $z_n = z_{n,0}$. The derivatives of z_n up to the order $\nu+1$ are

$$\begin{aligned} & z_n'(t) = z_{n,1}(t) + f''(t)D_0, \\ & z_n''(t) = z_{n,2}(t) + f'''(t)D_0 + f''(t)D_1, \\ & \vdots \\ & z_n^{(\nu-1)}(t) = z_{n,\nu-1}(t) + \textstyle\sum_{j=0}^{\nu-2} f^{(\nu-j)}(t)D_j, \\ & z_n^{(\nu)}(t) = \mathsf{z}'(s_n(t)) + \textstyle\sum_{j=0}^{\nu-1} f^{(\nu-j+1)}(t)D_j, \\ & z_n^{(\nu+1)}(t) = \mathsf{z}''(s_n(t))s_n'(t) + \textstyle\sum_{j=0}^{\nu-1} f^{(\nu-j+2)}(t)D_j. \end{aligned}$$

We denote H' the function $\sum_{j=0}^{\nu-1} f^{(\nu-j+1)}D_j$. By the properties of $f^{(j)}$ and s_n, we have that z_n belongs to $\mathbf{W}^{\nu+1,\infty}(a,b)$, with $||z_n^{(\nu+1)}||_\infty \le n + ||\mathsf{H}''||_\infty$ (where $||\cdot||_\infty$ is the essential supremum on (a,b)), and it has the same boundary values of x in a and in b.

(b) We claim that $||z_n^{(j)} - x^{(j)}||_\infty \leq 1/2^h$ and $||z_n^{(j)} \circ t_n - x^{(j)}||_\infty \leq 1/2^h$, for any j in $\{0, \cdots, \nu\}$, eventually in n.

In fact, for any n greater than $|\mathsf{x}''(\tau)| + \delta/2$, we have

$$
\begin{aligned}
|D_{\nu-1}| &\leq \int_a^{\tau^-} |\mathsf{x}' - \mathsf{x}' \circ s_n| + \int_{\tau^-}^{\tau^+} |\mathsf{x}' - \mathsf{z}'| + \int_{\tau^+}^b |\mathsf{x}' - \mathsf{x}' \circ s_n| \\
&\leq (\tau^+ - \tau^-)^2 \left[3\alpha(\tau^+ - \tau^-)^\nu + \frac{1}{2^{h+3}(\nu+1)\nu\Theta\alpha}\right] \leq (\tau^+ - \tau^-)^2 \frac{1}{2^{h+2}(\nu+1)\nu\Theta\alpha}, \\
|D_{\nu-2}| &\leq \int_a^{\tau^+} \left|\mathsf{x}'(t) - \mathsf{x}'(a) - \int_a^t \mathsf{z}' \circ s_n - f'(t) D_{n,\nu-1}\right| dt \\
&\quad + \int_{\tau^+}^b \left|\mathsf{x}'(t) - \mathsf{x}'(b) + \int_t^b \mathsf{z}' \circ s_n - [1 - f'(t)] D_{n,\nu-1}\right| dt \\
&\leq \int_a^{\tau^+} \int_a^t |\mathsf{x}' - \mathsf{z}' \circ s_n| dt + (\tau^+ - \tau^-)|D_{n,\nu-1}| + \int_{\tau^+}^b \int_t^b |\mathsf{x}' - \mathsf{z}' \circ s_n| dt \\
&\leq (\tau^+ - \tau^-)^3 \left[4\alpha(\tau^+ - \tau^-)^{\nu-1} + \frac{1}{2^{h+3}(\nu+1)\nu\Theta\alpha}\right] + (\tau^+ - \tau^-)|D_{\nu-1}| \\
&\leq (\tau^+ - \tau^-)^3 \frac{2}{2^{h+2}(\nu+1)\nu\Theta\alpha}, \\
&\vdots \\
|D_j| &\leq (\tau^+ - \tau^-)^{\nu-j+1} \frac{\nu - j}{2^{h+2}(\nu+1)\nu\Theta\alpha} \leq (\tau^+ - \tau^-)^{\nu-j+1} \frac{1}{2^{h+2}(\nu+1)\Theta}, \quad \forall j \in \{0, \cdots, \nu-1\},
\end{aligned}
$$

so that $||\mathsf{H}'||_\infty \leq \sum_{j=0}^{\nu-1} ||f^{(\nu-j+1)}||_\infty (\tau^+ - \tau^-)^{\nu-j+1}/[2^{h+2}(\nu+1)\Theta] \leq (\tau^+ - \tau^-)/2^{h+2}$, $||\mathsf{H}''||_\infty \leq 1/2^{h+2}$, and

$$
\begin{aligned}
|\mathsf{z}'(s_n(t)) - \mathsf{x}'(t)| &\leq (\tau^+ - \tau^-) \left[3\alpha(\tau^+ - \tau^-)^{\nu+1} + \frac{1}{2^{h+3}(\nu+1)\nu\Theta\alpha}\right] \leq \frac{1}{2^{h+2}(\nu+1)\nu\Theta\alpha}, \\
|z_{n,\nu-1}(t) - x^{(\nu-1)}(t)| &\leq \int_a^b |\mathsf{z}' \circ s_n - \mathsf{x}'| + (b-a)|D_{\nu-1}| \leq (1 + b - a)|D_{\nu-1}| \\
&\leq \frac{2\alpha}{2^{h+2}(\nu+1)\nu\Theta\alpha}, \\
&\vdots \\
|z_{n,j}(t) - x^{(j)}(t)| &\leq \frac{(\nu - j + 1)\alpha}{2^{h+2}(\nu+1)\nu\Theta\alpha} \leq \frac{1}{2^{h+2}}, \qquad \forall j \in \{0, \cdots, \nu-1\}.
\end{aligned}
$$

Hence, we can fix n such that $M\Psi|\Sigma_n| < \bar{\epsilon}/8$, $||z_n^{(j)} - x^{(j)}||_\infty \leq 1/2^{h+1}$ and

$$
||z_n^{(j)} \circ t_n - x^{(j)}||_\infty \leq ||z_n^{(j)} \circ t_n - x^{(j)} \circ t_n||_\infty + ||x^{(j)} \circ t_n - x^{(j)}||_\infty \leq 1/2^h,
$$

for any j in $\{0, \cdots, \nu\}$. The graph of the function $(\mathbf{z}_n, \mathsf{z}_n')$ is included in $\mathsf{T}_\delta^\nu[x]$, and $\mathsf{z}_n''(t) \in B[0, |\mathsf{x}''(\tau)| + \delta]$, for any t in $[\tau^-, \tau^+]$. (From what follows, it turns out that z_n is the sought variation x_ϵ.)

(c) We show that $\tilde{\mathcal{I}}_i(z_n) < \tilde{\mathcal{I}}_i(x) + \bar{\epsilon}/2$, for any i.

Using the change of variable formula [44], we compute $\tilde{\mathcal{I}}_i(z_n) - \tilde{\mathcal{I}}_i(x)$ as the sum of three appropriate terms:

$$\int_a^b \tilde{L}_i\left(\mathsf{z}_n'(t_n(s)), \mathsf{z}_n''(t_n(s)\right)\psi_i(t_n(s), \mathbf{z}_n(t_n(s)), \mathsf{z}_n'(t_n(s)))t_n'(s)ds - \int_a^b \tilde{L}_i(\mathsf{x}'(s), \mathsf{x}''(s))\psi_i(s, \mathbf{x}(s), \mathsf{x}'(s))ds$$
$$= \int_a^b \left[\tilde{L}_i\left(\mathsf{z}_n'(t_n(s)), \mathsf{z}_n''(t_n(s))\right)t_n'(s) - \tilde{L}_i(\mathsf{z}_n'(t_n(s)), \mathsf{z}''(s) + t_n'(s)\mathsf{H}''(t_n(s)))\right]\psi_i(t_n(s), \mathbf{z}_n(t_n(s)), \mathsf{z}_n'(t_n(s)))ds$$
$$+ \int_a^b \tilde{L}_i(\mathsf{z}_n'(t_n(s)), \mathsf{z}''(s) + t_n'(s)\mathsf{H}''(t_n(s)))[\psi_i(t_n(s), \mathbf{z}_n(t_n(s)), \mathsf{z}_n'(t_n(s))) - \psi_i(s, \mathbf{x}(s), \mathsf{x}'(s))]ds$$
$$+ \int_a^b \left[\tilde{L}_i\left(\mathsf{z}_n'(t_n(s)), \mathsf{z}''(s) + t_n'(s)\mathsf{H}''(t_n(s))\right) - \tilde{L}_i(\mathsf{x}'(s), \mathsf{x}''(s))\right]\psi_i(s, \mathbf{x}(s), \mathsf{x}'(s))ds = I_i^1 + I_i^2 + I_i^3.$$

To estimate I_i^1, it is enough to estimate its integrand over the sets S_n and Σ_n (because it is identically 0 elsewhere). Since $\Sigma_n \subset T$ and $||\mathsf{H}''||_\infty \leq \delta$, we obtain that

$$\tilde{L}_i\left(\mathsf{z}'(s) + \mathsf{H}'(t_n(s)), 2\mathsf{z}''(s) + \mathsf{H}''(t_n(s))\right)\frac{1}{2} - \tilde{L}_i\left(\mathsf{z}'(s) + \mathsf{H}'(t_n(s)), \mathsf{z}''(s) + \frac{\mathsf{H}''(t_n(s))}{2}\right) \leq M,$$

for every s in Σ_n. By Proposition 3 and 4 in [12], for every s in S_n,

$$\tilde{L}_i\left(\mathsf{z}_n'(t_n(s)), n\frac{\mathsf{z}''(s) + t_n'(s)\mathsf{H}''(t_n(s))}{|\mathsf{z}''(s)|}\right)\frac{|\mathsf{z}''(s)|}{n} - \tilde{L}_i(\mathsf{z}_n'(t_n(s)), \mathsf{z}''(s) + t_n'(s)\mathsf{H}''(t_n(s)))$$
$$\leq -\left(\frac{|\mathsf{z}''(s)|}{n} - 1\right)\tilde{L}_i^*(\mathsf{z}_n'(t_n(s)), p) \leq 0,$$

where $p \in \partial_w L_i(\mathsf{z}_n'(t_n(s)), n(\mathsf{z}''(s) + t_n'(s)\mathsf{H}''(t_n(s)))/|\mathsf{z}''(s)|)$. Using the fact that ψ_i is positive and bounded by Ψ, we have $I_i^1 \leq M\Psi|\Sigma_n| < \bar{\epsilon}/8$.

To estimate I_i^2, we observe that

$$\tilde{L}_i(\mathsf{z}_n'(t_n(s)), \mathsf{z}''(s) + t_n'(s)\mathsf{H}''(t_n(s))) = \begin{cases} \tilde{L}_i(\mathsf{z}_n'(s), \mathsf{z}''(s) + \mathsf{H}''(s)) & , s \in [\tau^-, \tau^+] \\ \tilde{L}_i(\mathsf{x}'(s), \mathsf{x}''(s)) & , \text{otherwise} \end{cases}.$$

By the fact that $|\psi_i(t_n(s), \mathbf{z}_n(t_n(s)), \mathsf{z}_n'(t_n(s))) - \psi_i(s, \mathbf{x}(s), \mathsf{x}'(s))| \leq \bar{\epsilon}/[8(\ell + \tilde{\mathbf{L}} + 1)]$, for any s in $[a, b]$, and that $\mathsf{z}'' + \mathsf{H}'' \in B[0, |\mathsf{x}''(\tau)| + \delta]$ on $[\tau^-, \tau^+]$, we have $I_i^2 \leq \bar{\epsilon}/8$.

To estimate I_i^3, it is enough to estimate the integrals over $[\tau^-, \tau^+]$ (because it is identically 0 elsewhere). Recalling that τ is a Lebesgue point for $\tilde{L}_i(\mathsf{x}'(\cdot), \mathsf{x}''(\cdot))\psi_i(\cdot, \mathbf{x}(\cdot), \mathsf{x}'(\cdot))$, we have

$$\begin{aligned} I_i^3 &\leq \int_{\tau^-}^{\tau^+} \tilde{L}_i(\mathsf{z}_n'(s), \mathsf{z}''(s) + \mathsf{H}''(s))\psi_i(s, \mathbf{x}(s), \mathsf{x}'(s)) - \tilde{L}_i(\mathsf{x}'(\tau), \mathsf{x}''(\tau))\psi_i(\tau, \mathbf{x}(\tau), \mathsf{x}'(\tau))]ds + \frac{\bar{\epsilon}}{8} \\ &\leq 4\rho\tilde{\mathbf{L}}\Psi + \frac{\bar{\epsilon}}{8} < \frac{\bar{\epsilon}}{4}. \end{aligned}$$

Hence, $I_i^1 + I_i^2 + I_i^3 < \bar{\epsilon}/2$, for any i.

Conclusion. We have obtained

$$\int_a^b L_i(\mathsf{z}_n'(t), \mathsf{y}_n''(t))\psi_i(t, \mathbf{z}_n(t), \mathsf{z}_n'(t))dt - \int_a^b L_i(\mathsf{x}'(t), \mathsf{x}''(t))\psi_i(t, \mathbf{x}(t), \mathsf{x}'(t))dt$$
$$< \int_a^b [L_i(\mathsf{z}_n'(t), \mathsf{z}_n''(t)) + \eta]\psi_i(t, \mathbf{z}_n(t), \mathsf{z}_n'(t))dt - \int_a^b [L_i(\mathsf{x}'(t), \mathsf{x}''(t)) + \eta]\psi_i(t, \mathbf{x}(t), \mathsf{x}'(t))dt + \frac{\bar{\epsilon}}{2}$$
$$= \int_a^b \tilde{L}_i(\mathsf{z}_n'(t), \mathsf{z}_n''(t))\psi_i(t, \mathbf{z}_n(t), \mathsf{z}_n'(t))dt - \int_a^b \tilde{L}_i(\mathsf{x}'(s), \mathsf{x}''(s))\psi_i(s, \mathbf{x}(s), \mathsf{x}'(s))ds + \frac{\bar{\epsilon}}{2} < \bar{\epsilon}.$$

Hence, $\mathcal{I}(z_n) - \mathcal{I}(x) < \sum_{i=1}^m \bar{\epsilon} = \epsilon$.

So, setting $x_\epsilon = z_n$, we have proved the theorem. □

2.3 A necessary condition for the Lavrentiev phenomenon

The content of this section is to show the following necessary condition: a functional

$$\mathcal{I}(x) = \int_a^b \sum_{i=1}^m L_i(x^{(\nu)}, x^{(\nu+1)})\psi_i(t, x, x', \cdots, x^{(\nu)}),$$

with $\nu \geq 0$, exhibiting the Lavrentiev phenomenon takes the value $+\infty$ in any neighbourhood of a minimizer $\bar{x}$; or equivalently if $\mathcal{I}$ assumes only finite values in a neighbourhood of $\bar{x}$, then $\mathcal{I}$ does not exhibit the Lavrentiev phenomenon.

The corollary above applies to the functionals of Manià and Sarychev, for instance, and to the examples of functionals exhibiting the Lavrentiev phenomenon proposed in [5], [6], [34], [36], [37], [39], [43]. This necessary condition is also related to the *repulsion property* of the action of Maniá ([5], [34]), i.e. the action $\mathcal{I}$ evaluated on the trajectories of an absolutely continuous minimizing sequence is divergent to ∞.

This is proved in the following corollary to Theorem 2.6 and of the main Theorem of the previous section:

Corollary 2.8. *Let $\Omega_0, \cdots, \Omega_\nu$ be open sets in $\mathbb{R}^N$, $\nu \geq 0$, such that the set $E = \{x \in \mathbf{W}^{\nu+1,1}(a,b) : x(t) \in \Omega_0, \cdots, x^{(\nu)}(t) \in \Omega_\nu, \forall t \in [a,b]\}$ is non-empty. Let $A, B \in \Omega_0$, $A^{(1)}, B^{(1)} \in \Omega_1, \cdots$, $A^{(\nu)}, B^{(\nu)} \in \Omega_\nu$ be given boundary values.*

Let $L_1, \cdots, L_m : \Omega_\nu \times \mathbb{R}^N \to \mathbb{R}$ and $\psi_1, \cdots, \psi_m : [a,b] \times \Omega_0 \times \cdots \times \Omega_\nu \to [0, +\infty)$ be continuous and such that $L_i(\xi, \cdot)$ is convex, for any ξ in Ω_ν, any i in $\{1, \cdots, m\}$.

Let

$$\mathcal{I}(x) = \int_a^b \sum_{i=1}^m L_i(x^{(\nu)}(t), x^{(\nu+1)}(t))\psi_i(t, x(t), x'(t), \cdots, x^{(\nu)}(t))dt$$

be a functional exhibiting the Lavrentiev phenomenon, and let $\bar{x}$ be a minimum of $\mathcal{I}$ over $E_{a,b} = \{x \in E : x(a) = A, x(b) = B,\ x'(a) = A^{(1)}, x'(b) = B^{(1)},\ \cdots, x^{(\nu)}(a) = A^{(\nu)}, x^{(\nu)}(b) = B^{(\nu)}\}$

Assume that, for any $\delta > 0$, there exists $\sigma_\delta > 0$ such that $\sigma_\delta \to 0$, for $\delta \to 0$, and that ψ_i restricted to $\mathsf{T}^{\nu}_{\delta}[\bar{x}]$ may vanish only on the graph of $(\bar{x}, \bar{x}', \cdots, \bar{x}^{(\nu)})$ or on a σ_δ-neighbourhood of $(a, A, \cdots, A^{(\nu)})$ or on a σ_δ-neighbourhood of $(b, B, \cdots, B^{(\nu)})$, for any i in $\{1, \cdots, m\}$.

Then, for any $\epsilon > 0$, there exists x_ϵ in $E_{a,b}$ such that the graph of $(x_\epsilon, x'_\epsilon, \cdots, x_\epsilon^{(\nu)})$ is included in $\mathsf{T}^{\nu}_{\epsilon}[\bar{x}]$ and $\mathcal{I}(x_\epsilon) = +\infty$.

Proof. Fix $\epsilon > 0$. From the theorem 2.6 and 2.1, it follows that $\int_a^b |L_i(\bar{x}^{(\nu)}, \bar{x}^{(\nu+1)})| = +\infty$, for at least one i in $\{1, \cdots, m\}$.

Without loss of generality, we suppose that $\int_a^{(a+b)/2} |L_i(\bar{x}^{(\nu)}, \bar{x}^{(\nu+1)})| = +\infty$.

Let $g : (-\infty, +\infty) \to [0,1]$ be a $\mathbf{C}^\infty$ increasing function with value 1 on $[b, +\infty)$ and 0 on $(-\infty, (a+b)3/4]$. We define the integrable function $x_{\delta,\nu+1} : [a,b] \to \mathbb{R}^N$ by

$$x_{\delta,\nu+1}(t) = \begin{cases} 0 & , t \in [a, a+\sigma_\delta) \\ \bar{x}^{(\nu+1)}(t - \sigma_\delta) & , \text{otherwise} \end{cases},$$

and ν absolutely continuous functions $x_{\delta,j}(t) = A^{(j)} + \int_a^t x_{\delta,j+1} + g(t) D_{\delta,j}$, for any t in $[a,b]$, where $D_{\delta,j} = B^{(j)} - A^{(j)} - \int_a^b x_{\delta,j+1}$, for any j in $\{0, \cdots, \nu\}$.

Set $x_\delta = x_{\delta,0}$. The derivatives of x_δ up to the order $\nu+1$ are

$$\begin{aligned} &x'_\delta(t) = x_{\delta,1}(t) + g'(t) D_{\delta,0}, \\ &x''_\delta(t) = x_{\delta,2}(t) + g''(t) D_{\delta,0} + g'(t) D_{\delta,1}, \\ &\vdots \\ &x_\delta^{(\nu+1)}(t) = x_{\delta,\nu+1}(t) + \textstyle\sum_{j=0}^{\nu} g^{(\nu-j+1)}(t) D_{\delta,j}. \end{aligned}$$

By definition, x_δ belongs to $\mathbf{W}^{\nu+1,1}(a,b)$, it has the same boundary values of $\bar{x}$ in a and in b, and, for j in $\{\nu, \nu+1\}$, for any t in $[a+\sigma_\delta, (a+b)3/4]$, we have $x_\delta^{(j)}(t) = \bar{x}^{(j)}(t - \sigma_\delta)$. Furthermore, there exists constants c_j, d_j, independent on δ, such that $|D_{\delta,j}| \le c_j \int_{b-\sigma_\delta}^b |\bar{x}^{(\nu+1)}|$ and $||x_{\delta,j} - x_\delta^{(j)}||_\infty \le d_j \int_{b-\sigma_\delta}^b |\bar{x}^{(\nu+1)}|$. Hence, for any j in $\{0, \cdots, \nu\}$,

$$||\bar{x}^{(j)} - x_\delta^{(j)}||_\infty \le \left(c_j + ||g^{(\nu+1)}||_\infty \sum_{j=0}^{\nu} d_j \right) \int_{b-\sigma_\delta}^b |\bar{x}^{(\nu+1)}|.$$

By hypothesis, we can choose $\bar{\delta} > 0$ such that $(c_j + ||g^{(\nu+1)}||_\infty \sum_{j=0}^{\nu} d_j) \int_{b-\sigma_{\bar{\delta}}}^b |\bar{x}^{(\nu+1)}| < \epsilon$ and $\sigma_{\bar{\delta}} < (b-a)/4$.

Set $\Psi_i = \min\{\psi_i(t, x_{\bar\delta}(t), \cdots, x_{\bar\delta}^{(\nu)}(t)) : t \in [a + \sigma_{\bar\delta}, (a+b)3/4]\}$: by hypothesis, Ψ_i is positive. We have obtained that the graph of $(x_{\bar\delta}, x'_{\bar\delta}, \cdots, x_{\bar\delta}^{(\nu)})$ belongs to $\mathsf{T}_\epsilon^\nu[\bar{x}]$ and

$$\begin{aligned}
&\int_a^b |L_i(x_{\bar\delta}^{(\nu)}(t), x_{\bar\delta}^{(\nu+1)}(t))\psi_i(t, x_{\bar\delta}(t), \cdots, x_{\bar\delta}^{(\nu)}(t))|dt \\
&\geq \int_{a+\sigma_{\bar\delta}}^{(a+b)3/4} |L_i(\bar{x}^{(\nu)}(t-\sigma_{\bar\delta}), \bar{x}^{(\nu+1)}(t-\sigma_{\bar\delta}))|\psi_i(t, x_{\bar\delta}(t), \cdots, x_{\bar\delta}^{(\nu)}(t))dt \\
&\geq \Psi_i \int_a^{(a+b)/2} |L_i(\bar{x}^{(\nu)}(t), \bar{x}^{(\nu+1)}(t))|dt = +\infty.
\end{aligned}$$

From *(i)* in the proof of Theorem 2.6 and Theorem 1 in [12], we infer that $\mathcal{I}(x_{\bar\delta}) = +\infty$.

So, setting $x_\epsilon = x_{\bar\delta}$, we have proved the corollary. □

Part III

Existence and regularity of minimizers

Chapter 3

Preliminaries

3.1 The Direct Method of the Calculus of Variations

The standard hypotheses on the Lagrangian under which it is possible to prove an existence result, in the set $\mathbf{X} = \mathbf{AC}[a, b]$ of the absolutely continuous trajectories, are: some lower semi-continuity of L in its domain, the convexity in its last variable x' and a super-linear growth condition, i.e. the requirement that there exists a function θ such that $\theta(s)/s$ converges to $+\infty$, as s tends to $+\infty$, and $L(t, x, x') \geq \theta(|x'|)$.

This method of proof is called the Direct Method of the Calculus of Variations and it was mainly developed by L. Tonelli (1915, [48]). Roughly speaking, what one does in this method is the following: starting from a minimizing sequence for the action $\mathcal{I}$, i.e. a sequence of trajectories $\{x_n\}_n$ such that $\mathcal{I}(x_n)$ converges to the infimum of $\mathcal{I}$, and, by using the fact that the super-linear growth condition implies the weak precompactness of $\{x_n\}_n$, one selects a subsequence that converges weakly to a trajectory $\hat{x}$ in $\mathbf{AC}[a, b]$. Since the convexity, together with some continuity, ensures the weak lower semi-continuity of $\mathcal{I}$, one concludes that $\hat{x}$ is a minimizer.

This method is based on the two theorems stated below due to Mazur, De la Vallee-Poussin, Dunford, Pettis, and Tonelli ([16], [22]):

Theorem 3.1. *Let $\{\xi_n\}_n$ be a sequence of $\mathbb{R}^N$-valued measurable functions belonging to $\mathbf{L}^1(a, b)$.*

Then, $\{\xi_n\}_n$ is sequentially weakly precompact in $\mathbf{L}^1(a, b)$ if and only if there exists a constant M and a function $\theta : [0, \infty) \to [0, \infty)$ such that $\theta(s)/s \to \infty$, as $s \to \infty$, and $\int_a^b \theta(|\xi_n(t)|)dt \leq M$, for any n.

Theorem 3.2 (Mazur lemma)**.** *Let $\{\xi_n\}_n$ be a sequence of $\mathbb{R}^N$-valued measurable functions weakly convergent to ξ in $\mathbf{L}^1(a, b)$.*

Then, for any integer n there exists a convex combination $a_{1,n}, \cdots, a_{n,n}$ of $\xi_1, \cdots, \xi_n$ such that

$$\left\| \sum_{i=1}^{n} a_{i,n}\xi_i - \xi \right\|_1 \to 0,$$

as n tends to ∞.

In section 4.1 we present an original existence theorem ([11]) in the case of an autonomous Lagrangian, i.e. a Lagrangian which does not depend explicitly on t, where a variant of the Direct Method can be applied under more general growth assumptions that include the classical super-linear growth but also some cases of Lagrangians with linear growth. We drop the super-linear growth condition and hence the fact that all the minimizing sequences of absolutely continuous trajectories admit a subsequence weakly convergent in $\mathbf{L}^1$. Instead, from any minimizing sequence of absolutely continuous trajectories, we construct a minimizing sequence of equi-Lipschitzian trajectories; therefore we apply theorems 3.2 and the following (partially due to Ascoli-Arzelà) to the obtained sequence ([16], [22]).

Theorem 3.3. *Let $\{x_n\}_n$ be a sequence of $\mathbb{R}^N$-valued measurable functions belonging to $\mathbf{Lip}[a,b]$. Suppose that there exists a positive constant M such that $||x_n'||_\infty \leq M$.*

Then, $\{x_n\}_n$ is sequentially precompact in $\mathbf{C}[a,b]$, and $\{x_n'\}_n$ is sequentially weakly precompact in $\mathbf{L}^1(a,b)$.

If follows also that this minimizer is Lipschitz. Results on the existence of Lipschitz minimizer to autonomous problems, under conditions of super-linear growth, were established in [5], [20] and, under weaker regularity conditions on the Lagrangian, in [10].

Chapter 4

New Results

In section 4.1 we present an original existence theorem ([11]) in the case of an autonomous Lagrangian, i.e. a Lagrangian which does not depend explicitly on t, where a variant of the Direct Method can be applied under more general growth assumptions that include the classical super-linear growth but also some cases of Lagrangians with linear growth.

In section 4.2, we prove the existence of solutions to Minimum Time problems for differential inclusions, under assumptions that do not require the convexity of the images $F(x) = f(x, U(x))$. Furthermore, we present a model for the Brachistochrone problem and we show that our model (a non-convex control problem) satisfies the assumptions required for the existence of solutions to Minimum Time problems. The case of Brachistochrone as a Minimum Time Control problem has already been amply treated in [46], [47].

4.1 Existence of Lipschitzian minimizers

The purpose of this section is to show that in the case of autonomous problems, where the Lagrangian does not depend explicitly on the integration variable t, a variant of the Direct Method can be applied under more general growth assumptions. More precisely, we consider Problem (P), the problem of minimizing the integral

$$\int_a^b L(x(s), x'(s))ds$$

for $x : [a, b] \to \mathbb{R}^N$ absolutely continuous and satisfying $x(a) = A, x(b) = B$. Under more general growth conditions, that include the classical super-linear growth but also some cases of Lagrangians with linear growth, we show that, from any sequence $\{x_n\}_{n\in\mathbb{N}}$, minimizing for the functional, one can derive another sequence $\{\bar{x}_n\}_{n\in\mathbb{N}}$, each function $\bar{x}_n$ obtained from x_n by reparameterizing the interval $[a, b]$, that is again minimizing, and consists of equi-Lipschitzian functions. As a consequence, in the case the Lagrangian $L(x,\xi)$ is convex in ξ, one can prove the existence of a solution to Problem (P), that, in particular, is a Lipschitzian function. A result on the regularity (Lipschitzianity) of solutions to autonomous minimum problems, under conditions of super-linear growth, was established in [20] and, under weaker growth conditions, in [10].

The growth assumption we consider is expressed in terms of the polar of the Lagrangian L with respect to ξ (for the properties of the polar see, e.g., [24]). The same condition was already introduced in [15] to prove existence of solutions for a rather special class of Lagrangians. The results we present apply to different classes of Lagrangians, that can possibly be extended valued and either convex or differentiable in ξ. Two simple examples of a convex everywhere defined Lagrangian satisfying the assumptions of our main Theorem, in particular the growth condition, are the maps, having linear growth, $L(\xi) = |\xi| - \sqrt{|\xi|}$ and

$$L(\xi) = \begin{cases} |\xi| - \ln(|\xi|) & , |\xi| \geq 1 \\ 1 & , otherwise \end{cases} .$$

In what follows $L(x,\xi) : \mathbb{R}^N \times \mathbb{R}^N \to \bar{\mathbb{R}} = \mathbb{R} \cup \{+\infty\}$ is an extended valued function, continuous and bounded below, not identically $+\infty$. $L^*(x,p)$ is the *polar* of L with respect to its second variable [42], i.e.

$$L^*(x,p) = \sup_{\xi \in \mathbb{R}^N} \langle p, \xi \rangle - L(x,\xi).$$

We denote by $\mathsf{dom} = \{(x,\xi) \in \mathbb{R}^N \times \mathbb{R}^N : L(x,\xi) \in \mathbb{R}\}$ its effective domain. Since the assumptions on L for the case where $\mathsf{dom} = \mathbb{R}^N \times \mathbb{R}^N$ are somewhat simpler than the assumptions needed in the general case, we shall state separately the results for the two cases. For each case, L, as a function of ξ, may be either convex or not; in this second case, we shall need the extra assumption of differentiability of L with respect to ξ. This assumption is not needed in the convex case, since, in this case, the existence of a sub-differential is enough for the proof. Hence, we will provide four different statements of the what is basically the same result; the proof will be one proof for the four different Theorems. We first present the results for the simpler case where $\mathsf{dom} = \mathbb{R}^N \times \mathbb{R}^N$.

Theorem 4.1 (Convex case)**.** *Assume that:*

(1) $\mathsf{dom} = \mathbb{R}^N \times \mathbb{R}^N$ *and* $L(x,\cdot)$ *is convex,* $\forall x \in \mathbb{R}^N$*;*

(2) for every selection $p(x,\cdot) \in \partial_\xi L(x,\cdot)$ *we have*

$$L^*(x, p(x,\xi)) \to +\infty$$

as $|\xi|$ *tends to* $+\infty$*, uniformly in* x*.*

Then: given any minimizing sequence $\{x_n\}_{n\in\mathbb{N}}$ for the functional in (P), there exists a constant Λ and a sequence of reparameterizations s_n of the interval $[a,b]$ onto itself, such that $\{\bar{x}_n\}_{n\geq n_1} = \{x_n \circ s_n\}_{n\geq n_1}$ is again a minimizing sequence and each $\bar{x}_n$ is Lipschitzian with Lipschitz constant Λ.

The convex Lagrangian $L(\xi)$ described in the Introduction is such that $L^*(p(\xi)) = \ln(|\xi|) - 1 \to +\infty$.

Theorem 4.2 (Differentiable case)**.** *Assume that:*

(1) $\mathsf{dom} = \mathbb{R}^N \times \mathbb{R}^N$ and $\forall x \in \mathbb{R}^N$, $L(x,\cdot)$ is differentiable;

(2) $L^*(x, \nabla_\xi L(x,\xi)) \to +\infty$ as $|\xi|$ tends to $+\infty$, uniformly in x.

Then the conclusion of Theorem 4.1 holds.

The following are the analogous result in the more complex case where $\mathsf{dom} \neq \mathbb{R}^N \times \mathbb{R}^N$. In this case it is not necessarily true that the functional in (P) is not identically $+\infty$.

Theorem 4.3 (Convex case)**.** *Assume that:*

(1) $L(x,\cdot)$is a convex extended valued map and $(x,0) \in \mathsf{dom}$ whenever there exists ξ such that $(x,\xi) \in \mathsf{dom}$;

(2) for every selection $p(x,\cdot) \in \partial_\xi L(x,\cdot)$ we have

$$L^*(x, p(x,\xi)) \to +\infty$$

as $|\xi|$ tends to $+\infty$, with $(x,\xi) \in \mathsf{dom}$, uniformly in x;

(3) for every $M > 0$, $\exists \delta > 0$ such that $L(x,\xi) > M$, for every $(x,\xi) \in \mathsf{dom}$ with $d((x,\xi), \partial\mathsf{dom}) < \delta$;

(4) the functional in (P) is not identically $+\infty$.

Then the conclusion of Theorem 4.1 holds.

Theorem 4.4 (Differentiable case)**.** *Assume that:*

(1) $L(x,\cdot)$is differentiable and $\mathsf{dom} \cap (\{x\} \times \mathbb{R}^N)$ is star shaped with respect to $(x,0)$ whenever there exists ξ such that $(x,\xi) \in \mathsf{dom}$;

(2) $L^*(x, \nabla_\xi L(x,\xi)) \to +\infty$ as $|\xi|$ tends to $+\infty$, with $(x,\xi) \in \mathsf{dom}$, uniformly in x;

(3) for every $M > 0$, $\exists \delta > 0$ such that $L(x,\xi) > M$, for every $(x,\xi) \in \mathsf{dom}$ with $d((x,\xi), \partial\mathsf{dom}) < \delta$;

(4) the functional in (P) *is not identically* $+\infty$.

Then the conclusion of Theorem 4.1 holds.

Remark 4.5. Consider the case $N = 1$ and L independent on x. The growth condition *(2)* in the statement of the Theorems above means that the Lagrangian does not admit an oblique asymptote at infinite.

We shall need the following Proposition on the existence of a lower bound for the Lagrangian L under the conditions stated in any of the theorems above.

Propositon 4.6. *Let* L *satisfy assumptions* (1) *and* (2) *of any of the Theorems 4.1, 4.2, 4.3 or 4.4. Then there exist* $\alpha > 0$ *and* $\beta \in \mathbb{R}$ *such that* $L(x,\xi) \geq \alpha|\xi| + \beta$, $\forall (x,\xi) \in \mathit{dom}$.

Proof. Set $\ell = \inf\{L(x,\xi)\}$; assumption (2) implies that there exists $r > 0$ such that $-L^*(x, p(x,\xi)) \leq \ell - 1$, for every $(x,\xi) \in \mathit{dom}$ with $|\xi| \geq r$, where $p(x,\cdot)$ is either $\nabla_\xi L(x,\cdot)$ or any selection from the sub-differential of $L(x,\cdot)$. We claim that we can choose $\alpha = 1/(2r)$ and $\beta = \ell - 1$

Fix $(x,\xi) \in \mathit{dom}$. When $|\xi| \leq r$, we have $L(x,\xi) \geq \ell > |\xi|/(2r) + \ell - 1$, and the claim is true in this case.

Consider the case $|\xi| > r$. Set $\psi(s) = s/(2r) + \ell - 1$; assumption (1) implies that the convex function $\mathcal{L}(s) = L(x, s\xi/|\xi|)$ is well defined for $s \in [r, |\xi|]$, hence the selection $p_{\mathcal{L}}(s) = \langle \xi/|\xi|, p(x, s\xi/|\xi|)\rangle \in \partial\mathcal{L}(s)$ is increasing and we have $p_{\mathcal{L}}(s) \geq p_{\mathcal{L}}(r)$; moreover, from the inequality

$$L\left(x, r\frac{\xi}{|\xi|}\right) - \langle r\frac{\xi}{|\xi|}, p\left(x, r\frac{\xi}{|\xi|}\right)\rangle \leq \ell - 1,$$

we obtain $p_{\mathcal{L}}(r) \geq 1/r > 1/(2r) = \psi'(s)$. From $\mathcal{L}(r) > \psi(r)$, we obtain $\mathcal{L}(s) > \psi(s)$, for every $s \in [r, |\xi|]$; setting $s = |\xi|$, the claim is proved.

Now, assume the validity of (1), (2) of the Differentiable cases. Again, let $r > 0$ be such that for every $(x,\xi) \in \mathit{dom}$ with $|\xi| \geq r$ we have $-L^*(x, \nabla_\xi L(x,\xi)) \leq \ell - 1$. As before, it follows that the Claim is true for $(x,\xi) \in \mathit{dom}$, $|\xi| \leq r$. Fix $\xi, |\xi| > r$. By assumption (1), $\mathcal{L}(s)$ is defined for $s \in [r, |\xi|]$, and we infer

$$L\left(x, s\frac{\xi}{|\xi|}\right) - \langle s\frac{\xi}{|\xi|}, \nabla_\xi L\left(x, s\frac{\xi}{|\xi|}\right)\rangle \leq \ell - 1,$$

so that

$$\begin{aligned} \mathcal{L}(s) - \psi(s) &= L\left(x, s\frac{\xi}{|\xi|}\right) - \left[\frac{s}{2r} + \ell - 1\right] \\ &\leq \langle s\frac{\xi}{|\xi|}, \nabla_\xi L\left(x, s\frac{\xi}{|\xi|}\right)\rangle + \frac{s}{2r} = s[\mathcal{L}'(s) - \psi'(s)]. \end{aligned}$$

Assume that the set $\{s \in (r, |\xi|] : \mathcal{L}(s) - \psi(s) < 0\}$ is non-empty, and let s_0 be its infimum. By continuity, $\mathcal{L}(s_0) - \psi(s_0) = 0$, so that $s_0 > r$. From the Mean Value Theorem we infer the existence of $s_1 \in (r, s_0)$ such that $\mathcal{L}'(s_1) - \psi'(s_1) < 0$, that in turn implies $\mathcal{L}(s_1) - \psi(s_1) < 0$, a contradiction to the definition of s_0. Hence $\mathcal{L}(s) \geq \psi(s)$, $\forall s \in (r, |\xi|]$, in particular for $s = |\xi|$. □

Lemma 4.7. *Let $f : \mathbb{R}^N \to \bar{\mathbb{R}}$ be convex, and such that* dom *contains the origin. Then, for every ξ in the domain of f, the function $f(\xi/(1+\cdot))(1+\cdot)$ from $[0, +\infty)$ to $\mathbb{R}$ is convex. Moreover, there exists a selection $p(\cdot) \in \partial f(\cdot)$ such that*

$$f\left(\frac{\xi}{1+s}\right)(1+s) - f(\xi) \leq -sf^*\left(p\left(\frac{\xi}{1+s}\right)\right), \qquad \forall s \in [0, +\infty).$$

Proof. For the proof see Proposition 1.4. □

Proof. of Theorems 4.1–4.4. Set m be the infimum of the values of

$$\int_a^b L(x(s), x'(s))ds$$

for x as in Problem (P). Proposition 4.6 and the assumptions of Theorems 4.1–4.4 imply that m is finite. Let $\{x_n\}_{n\in\mathbb{N}}$ be a minimizing sequence for Problem (P). By Proposition 4.6 we obtain that there exists $H > 0$ such that $\|x'_n\|_1 \leq H$ so that, for every n, for every s in $[a, b]$, we have $x_n(s) \in B[0, A + H]$.

Next point $a)$ reaches a conclusion with an argument that differs in the cases where L is or is not extended valued, so the argument is presented separately in the two cases.

a) (Case $\mathsf{dom} = \mathbb{R}^N \times \mathbb{R}^N$) For every n, consider the subset of $[a, b]$ defined by

$$T_n^H = \{s \in [a, b] : |x'_n(s)| \leq 4H/3(b-a)\};$$

one verifies that the Lebesgue measure of any such set is larger or equal to $(b-a)/4$.

Fix $\delta > 0$. Since $(x_n(s), (1+\delta)x'_n(s)) \in B[0, A+H] \times B[0, 4(1+\delta)H/3(b-a)]$, and L is continuous, we infer that: there exists $\mu \in \mathbb{R}$ such that, for every $n \in \mathbb{N}$ and $s \in T_n^H$,

$$L(x_n(s), (1+\delta)x'_n(s))\frac{1}{1+\delta} - L(x_n(s), x'_n(s)) \leq \mu.$$

a) (Case $\mathsf{dom} \neq \mathbb{R}^N \times \mathbb{R}^N$) Consider a real positive M. Assumption (3) implies that there exists $\delta(M) > 0$ such that $L(x, \xi) > M$, $\forall (x, \xi) \in \mathsf{dom}$ with $d((x, \xi), \partial\mathsf{dom}) < 2\delta(M)$.

Consider the subsets of $[a, b]$

$$J_n^{\delta(M)} = \{s \in [a, b] : d((x_n(s), x'_n(s)), \partial\mathsf{dom}) \geq 2\delta(M)\};$$

we have the inequality

$$\begin{aligned}\int_a^b L(x_n(s),x_n'(s))ds &= \int_{J_n^{\delta(M)}} L(x_n(s),x_n'(s))ds + \int_{[a,b]\setminus J_n^{\delta(M)}} L(x_n(s),x_n'(s))ds \\ &\geq |J_n^{\delta(M)}|\ell + |[a,b]\setminus J_n^{\delta(M)}|M = (b-a)M + |J_n^{\delta(M)}|(\ell - M),\end{aligned}$$

so that

$$\liminf_{n\to+\infty} |J_n^{\delta(M)}| \geq \frac{(b-a)M-m}{M-\ell}.$$

Since $\lim_{M\to+\infty}[(b-a)M-m]/(M-\ell) = b-a$, we can choose $\bar{M}>0$ such that

$$[(b-a)\bar{M}-m]/(\bar{M}-\ell) > (b-a)3/4.$$

Set $\delta = \delta(\bar{M})$. We have obtained that there exists $n_1 \in \mathbb{N}$ such that $n \geq n_1$ implies

$$|J_n^\delta| \geq (b-a)3/4.$$

We have also obtained that the sets $\{(x_n(s),(1+\delta)x_n'(s)) : s \in J_n^\delta\}$ are contained in dom. Finally, consider the sets

$$I_n^H = \{s \in [a,b] : |x_n'(s)| \leq 4H/(b-a)\};$$

the measure of each I_n^H is at least $(b-a)3/4$ so that, defining $T_n^H = I_n^H \cap J_n^\delta$, we obtain $|T_n^H| = |I_n^H \cap J_n^\delta| \geq (b-a)/4$, $\forall n \geq n_1$.

Since $(x_n(s),(1+\delta)x_n'(s))$ belongs to $B[0,A+H] \times B[0,4(1+\delta)H/(b-a)] \cap \{(x,\xi) \in \mathsf{dom} : d((x,\xi),\partial\mathsf{dom}) \geq \delta\}$, a compact subset of dom and L is continuous on dom, we infer that: there exists $\mu \in \mathbb{R}$ such that, for every $n \in \mathbb{N}$ and $s \in T_n^H$,

$$L(x_n(s),(1+\delta)x_n'(s))\frac{1}{1+\delta} - L(x_n(s),x_n'(s)) \leq \mu.$$

b) Consider a real positive ν and set $S_n^\nu = \{s \in [a,b] : |x_n'(s)| > \nu\}$. From $\|x_n'\|_1 \leq H$, we easily obtain that both the measure of S_n^ν and

$$\epsilon_n^\nu = \int_{S_n^\nu} \left[\frac{|x_n'(s)|}{\nu} - 1\right] ds$$

converge to 0 as $\nu \to +\infty$, uniformly with respect to $n \in \mathbb{N}$.

Consider first the Convex case; let $p(x,\cdot) \in \partial_\xi L(x,\cdot)$ be the selection provided by Lemma 4.7. By Assumption (2) of this case, there exists a map $M : \mathbb{N} \to \mathbb{R}$, $\lim_{\nu\to+\infty} M(\nu) = +\infty$, such that

$$L^*(x,p(x,\xi)) \geq M(\nu)$$

for every $(x,\xi) \in \mathsf{dom} \cap [\mathbb{R}^N \times (B[0,\nu])^c]$; in particular

$$L^*(x_n(x), p(x_n(s), x_n'(s))) \geq M(\nu)$$

for every $n \in \mathbb{N}$ and $s \in S_n^\nu$. Analogously, under assumption (2) of the Differentiable case, there exists a map $M : \mathbb{N} \to \mathbb{R}$, $\lim_{\nu \to +\infty} M(\nu) = +\infty$, such that

$$L^*(x, \nabla L(x,\xi)) \geq M(\nu)$$

for every $(x,\xi) \in \mathsf{dom} \cap [\mathbb{R}^N \times (B[0,\nu])^c]$; in particular

$$L^*(x_n(x), \nabla L(x_n(s), x_n'(s))) \geq M(\nu)$$

for every $n \in \mathbb{N}$ and $s \in S_n^\nu$.

Hence, both in the Convex and in the Differentiable case, we have obtained that there exists an integer $\bar{\nu}$ such that at once we have: $\bar{\nu} \geq 4H/(b-a)$, $M(\bar{\nu}) \geq (1+\delta)\mu$ and $\epsilon_n^{\bar{\nu}} \leq (b-a)/[4(1+\delta)]$, $\forall n \in \mathbb{N}$.

c) For every $n \geq n_1$, there exists Σ_n^H , a subset of T_n^H, having measure $(1+\delta)\epsilon_n^{\bar{\nu}}$. Define the absolutely continuous functions $t_n(s) = a + \int_a^s t_n'(\tau)d\tau$ by setting

$$t_n'(s) = \begin{cases} 1 + \left[\frac{|x_n'(s)|}{\bar{\nu}} - 1\right] & , s \in S_n^{\bar{\nu}} \\ 1 - \frac{1}{1+\delta} & , s \in \Sigma_n^H \\ 1 & , otherwise \end{cases} ;$$

each t_n is an invertible map from $[a,b]$ onto itself.

d) From the definition of t_n' we have that

$$\begin{aligned} &\int_a^b L\left(x_n(s), \frac{x_n'(s)}{t_n'(s)}\right) t_n'(s)ds - \int_a^b L(x_n(s), x_n'(s))ds = \\ &\int_{S_n^{\bar{\nu}}} L\left(x_n(s), \bar{\nu}\frac{x_n'(s)}{|x_n'(s)|}\right) \frac{|x_n'(s)|}{\bar{\nu}} ds - \int_{S_n^{\bar{\nu}}} L(x_n(s), x_n'(s))ds + \\ &\int_{\Sigma_n^H} L(x_n(s), (1+\delta)x_n'(s))\frac{1}{1+\delta}ds - \int_{\Sigma_n^H} L(x_n(s), x_n'(s))ds. \end{aligned}$$

We wish to estimate the above integrals. Since $\Sigma_n^H \subset T_n^H$, we easily obtain

$$\int_{\Sigma_n^H} \left[L(x_n(s), (1+\delta)x_n'(s))\frac{1}{1+\delta} - L(x_n(s), x_n'(s))\right] ds \leq (1+\delta)\epsilon_n^{\bar{\nu}}\mu.$$

To conclude the estimate we have to consider separately the Convex and the Differentiable case.

e) (Convex case) The choice of p implies that

$$L\left(x_n(s), \bar{\nu}\frac{x_n'(s)}{|x_n'(s)|}\right)\frac{|x_n'(s)|}{\bar{\nu}} - L(x_n(s), x_n'(s)) \leq$$
$$-\left[\frac{|x_n'(s)|}{\bar{\nu}} - 1\right] L^*\left(x_n(s), p\left(x_n(s), \bar{\nu}\frac{x_n'(s)}{|x_n'(s)|}\right)\right)$$

for every $s \in S_n^{\bar{\nu}}$.

e) (Differentiable case) The mean value Theorem implies that there exists $\alpha_n(s) \in [0, |x_n'(s)|/\bar{\nu} - 1]$ such that

$$L\left(x_n(s), \bar{\nu}\frac{x_n'(s)}{|x_n'(s)|}\right)\frac{|x_n'(s)|}{\bar{\nu}} - L(x_n(s), x_n'(s)) =$$
$$-\left[\frac{|x_n'(s)|}{\bar{\nu}} - 1\right]\left[\langle\nabla_\xi L\left(x_n(s), \frac{x_n'(s)}{1+\alpha_n(s)}\right), \frac{x_n'(s)}{1+\alpha_n(s)}\rangle - L\left(x_n(s), \frac{x_n'(s)}{1+\alpha_n(s)}\right)\right] =$$
$$-\left[\frac{|x_n'(s)|}{\bar{\nu}} - 1\right] L^*\left(x_n(s), \nabla_\xi L\left(x_n(s), \frac{x_n'(s)}{1+\alpha_n(s)}\right)\right)$$

for every $s \in S_n^{\bar{\nu}}$.

f) Since both $|\bar{\nu}x_n'(s)/|x_n'(s)|| \geq \bar{\nu}$ and $|x_n'(s)|/(1+\alpha_n(s)) \geq \bar{\nu}$, by the definition of $M(\bar{\nu})$ we obtain

$$\int_{S_n^{\bar{\nu}}} L\left(x_n(s), \bar{\nu}\frac{x_n'(s)}{|x_n'(s)|}\right)\frac{|x_n'(s)|}{\bar{\nu}}ds - \int_{S_n^{\bar{\nu}}} L(x_n(s), x_n'(s))ds \leq -\epsilon_n^{\bar{\nu}} M(\bar{\nu}),$$

hence our estimate becomes: $n \geq n_1$ implies that

$$\int_a^b L\left(x_n(s), \frac{x_n'(s)}{t_n'(s)}\right) t_n'(s)ds - \int_a^b L(x_n(s), x_n'(s))ds \leq \epsilon_n^{\bar{\nu}}[-M(\bar{\nu}) + (1+\delta)\mu] \leq 0.$$

g) The conclusion of f) proves the Theorem; in fact, defining $\bar{x}_n = x_n \circ s_n$, where s_n is the inverse of the function t_n, we obtain, by the change of variable formula [44], that $\{\bar{x}_n\}_{n\geq n_1} = \{x_n \circ s_n\}_{n \geq n_1}$ is a minimizing sequence, since

$$\begin{aligned}\int_a^b L(\bar{x}_n(t), \bar{x}_n'(t))dt &= \int_a^b L\left(\bar{x}_n(t_n(s)), \frac{d\bar{x}_n}{dt}(t_n(s))\right) t_n'(s)ds \\ &= \int_a^b L\left(x_n(s), \frac{x_n'(s)}{t_n'(s)}\right) t_n'(s)ds \leq \int_a^b L(x_n(s), x_n'(s))ds.\end{aligned}$$

Moreover, we claim that $\bar{x}_n$ are Lipschitzian functions, with the same Lipschitz constant $\Lambda = (1+1/\delta)\bar{\nu}$. In fact, consider the equality $\bar{x}_n'(t_n(s)) = x_n'(s)/t_n'(s)$ and fix s where $t_n'(s)$ exists; we obtain

$$\left|\frac{d\bar{x}_n}{dt}(t_n(s))\right| \begin{cases} = \bar{\nu} & , s \in S_n^{\bar{\nu}} \\ \leq (1+1/\delta)\bar{\nu} & , s \in \Sigma_n^H \\ \leq \bar{\nu} & , otherwise \end{cases} ;$$

hence, at almost every point $t_n(s)$, the norm of the derivative of $\bar{x}_n$ is bounded by Λ. This completes the proof. $\square$

4.2 An existence result for a class of Minimum Time problems

Around 1696 Jakob Bernoulli raised the following question: find the path from an initial point x^0 to a target point x^f such that a body, subject to gravity only, starting from x^0 with initial velocity zero, would reach x^f in minimum time.

In 1959 [26] A. F. Filippov proved the first general theorem on the existence of solutions to minimum time control problems of the form

$$x'(t) = f(x(t), u(t)); \quad u(t) \in U(x(t))$$

requiring that the set valued map $x \to U(x)$ be upper semi-continuous (with respect to the inclusion) and that the values $F(x) = f(x, U(x))$ be compact and convex.

In Theorem 4.9 of this section, we prove the existence of solutions to minimum time problems for differential inclusions, under assumptions that do not require the convexity of the images $F(x) = f(x, U(x))$. Furthermore, we present a model for the problem raised by Bernoulli and we show that our model (a non-convex control problem) satisfies the assumptions required by our Theorem 4.9 for the existence of solutions to minimum time problems. The case of Brachistochrone as a minimum time control problem has already been amply treated in [43, 47].

For a compact subset $A \subset \mathbb{R}^N$, set coA be its convex hull. For basic results relating to solutions to differential inclusions, measurable selections and properties of set-valued maps we refer to any standard text on the subject.

Lemma 4.8. *Let $x : [0, t^*] \to \mathbb{R}^N$ be absolutely continuous; let $X = \{x(t) : t \in [0, t^*]\}$ and let F be defined on X. Assume that, for almost every $t \in [0, t^*]$,*

$$x'(t) \in F(x(t))$$

and that there exists $E \subset [0, t^*]$ of positive measure such that $x'(t) = 0$ on E. Then there exist $\tau^* < t^*$ and an absolutely continuous function $\tilde{x} : [0, \tau^*] \to X$, such that $\tilde{x}(0) = x(0) = x^0$, $\tilde{x}(\tau^*) = x(t^*)$ and

$$\tilde{x}'(t) \in F(\tilde{x}(t))$$

for almost every $\tau \in [0, \tau^*]$.

Proof. a) It follows from the assumptions that there exists a closed subset K of E of positive measure. The complement of K consists of at most countably many open non-overlapping intervals $(a_i, b_i), i \in I$. Since the intervals (a_i, b_i) are disjoint, we must have that $\tau^* = \sum_{i \in I}(b_i - a_i) < t^*$. For each $i \in I$, set

$$\tau(b_i) = \sum_{j \in I : b_j < b_i} (b_j - a_j).$$

If $b_l > b_m$, we have that $\tau(b_l) \geq \tau(b_m) + (b_m - a_m)$.

Consider the intervals $(\tau(b_i), \tau(b_i) + b_i - a_i)$; they are disjoint, since in case we had $(\tau(b_l), \tau(b_l) + b_l - a_l) \cap (\tau(b_m), \tau(b_m) + b_m - a_m) \neq \emptyset$ with $b_l > b_m$, we would obtain $\tau(b_m) + (b_m - a_m) > \tau(b_l)$, a contradiction. Set $T = \cup_{i \in I}(\tau(b_i), \tau(b_i) + b_i - a_i)$; since $\tau \leq \sum_{i \in I}(b_j - a_j) = \tau^*$, $\forall \tau \in T$, we have that $T \subset [0, \tau^*]$. Moreover, the measure of T equals τ^*, in fact

$$m(T) = \sum_{i \in I}(b_i - a_i) = \tau^*.$$

b) Define the absolutely continuous function $\tilde{x} : [0, \tau^*] \to \mathbb{R}^N$ by $\tilde{x}(\tau) = x^0 + \int_0^\tau \tilde{x}'(s)ds$, where

$$\tilde{x}'(s) = \begin{cases} x'(s + a_i - \tau(b_i)) & , s \in (\tau(b_i), \tau(b_i) + b_i - a_i), i \in I \\ 0 & , s \in [0, \tau^*] \setminus T \end{cases} ;$$

we have $\tilde{x}(0) = x^0$.

Fix $\tau \in T$, there exists $i \in I$ such that $\tau \in (\tau(b_i), \tau(b_i) + b_i - a_i)$. Notice that $\tau \in (\tau(b_i), \tau(b_i) + b_i - a_i)$ if and only if $\tau + a_i - \tau(b_i) \in (a_i, b_i)$. We have

$$\begin{aligned} \tilde{x}(\tau) - x^0 &= \int_0^{\tau(b_i)} \tilde{x}'(s)ds + \int_{\tau(b_i)}^{\tau} \tilde{x}'(s)ds \\ &= \sum_{j:\tau(b_j)+b_j-a_j \leq \tau(b_i)} \int_{\tau(b_j)}^{\tau(b_j)+b_j-a_j} \tilde{x}'(s)ds + \int_{\tau(b_i)}^{\tau} \tilde{x}'(s)ds \\ &= \sum_{j:\tau(b_j)+b_j-a_j \leq \tau(b_i)} \int_{\tau(b_j)}^{\tau(b_j)+b_j-a_j} x'(s + a_j - \tau(b_j))ds + \int_{\tau(b_i)}^{\tau} x'(s + a_i - \tau(b_i))ds \\ &= \sum_{j:\tau(b_j)+b_j-a_j \leq \tau(b_i)} \int_{a_j}^{b_j} x'(s)ds + \int_{a_i}^{\tau+a_i-\tau(b_i)} x'(s)ds. \end{aligned}$$

Notice that

$$\sum_{j:\tau(b_j)+b_j-a_j \leq \tau(b_i)} \int_{a_j}^{b_j} x'(s)ds = \sum_{j:b_j<b_i} \int_{a_j}^{b_j} x'(s)ds;$$

in fact, by the definition, $\tau(b_j) + b_j - a_j \leq \tau(b_i)$ if and only if $\tau(b_j) < \tau(b_i)$. Hence

$$\begin{aligned} \tilde{x}(\tau) - x^0 &= \sum_{j:b_j<b_i} \int_{a_j}^{b_j} x'(s)ds + \int_{a_i}^{\tau+a_i-\tau(b_i)} x'(s)ds \\ &= \int_0^{a_i} x'(s)\chi_{\{[0,t^*]\setminus K\}}(s)ds + \int_{a_i}^{\tau+a_i-\tau(b_i)} x'(s)ds \\ &= x(\tau + a_i - \tau(b_i)) - x_0. \end{aligned}$$

The previous equality implies that $\tilde{x}$ is a solution to the differential inclusion. In fact we have that, almost everywhere in $[0, \tau^*]$,

$$\tilde{x}'(\tau) = x'(\tau + a_i - \tau(b_i)) \in F(x(\tau + a_i - \tau(b_i)) = F(\tilde{x}(\tau)).$$

c) Set $B = \sup\{b_i\}$. Then either the supremum is attained or it is not. In the first case, for some $j_{\tilde{k}}$, $B = b_{j_{\tilde{k}}}$ and $\tau(b_{j_{\tilde{k}}}) + (b_{j_{\tilde{k}}} - a_{j_{\tilde{k}}}) = \tau^*$. From b) we have that $x(t) = \tilde{x}(t - a_{j_{\tilde{k}}} + \tau(b_{j_{\tilde{k}}}))$, $\forall t \in [a_{j_{\tilde{k}}}, b_{j_{\tilde{k}}}]$. Since $x'(t) = 0$ on $[B, t^*]$, by continuity we have that $x(t^*) = x(B) = x(b_{j_{\tilde{k}}}) = \tilde{x}(b_{j_{\tilde{k}}} - a_{j_{\tilde{k}}} + \tau(b_{j_{\tilde{k}}})) = \tilde{x}(\tau^*)$.

In the second case, let $\{b_{j_k}\}$ be an increasing sequence, converging to B. From

$$|x(t^*) - x(a_{j_k})| = \left| \int_{a_{j_k}}^{B} x'(s)ds \right| \leq \left| \sum_{\{j \in I : b_j > b_{j_{k-1}}\}} \int_{(a_j, b_j)} x'(s)ds \right|$$

it follows that $x(a_{j_k}) \to x(t^*)$, while from

$$\sum_{\{j \in I : b_j > b_{j_k}\}} b_j - a_j < B - b_{j_k} \quad \text{and} \quad \tau^* = \sum_{j \in I} b_j - a_j = \tau(b_{j_k}) + \sum_{j \in I : b_j \geq b_{j_k}} b_j - a_j$$

we obtain that $\tau(b_{j_k}) \to \tau^*$. By the previous point b) we have that $\tilde{x}(\tau(b_{j_k})) = x(a_{j_k})$ and by continuity we infer that $x(t^*) = \tilde{x}(\tau^*)$. $\square$

In what follows we shall consider the following *minimum time problem* for solutions to a differential inclusion: X and S are closed subset of $\mathbb{R}^N$, $S \subset X$, $x^0 \in X$ and F a set valued map. Consider the problem of reaching the target set S from x^0, satisfying the constraint $x(t) \in X$, and $x(.)$ is a solution to the differential inclusion

$$x'(t) \in F(x(t)).$$

Theorem 4.9. *Let $X \subset \mathbb{R}^N$ be closed and let F be an upper semi-continuous set-valued map defined on X with compact non empty images, with the additional property that, for every $x \in X$, for every $\xi \in coF(x)$, with $\xi \neq 0$, there exists $\lambda \geq 1$ such that $\lambda\xi \in F(x)$. Assume that there exists $\tilde{t} > 0$ and a solution x to*

$$x'(t) \in coF(x(t)) \quad x(0) = x^0$$

such that $x(t) \in X$ for every $t \in [0, \tilde{t}]$ and that $x(\tilde{t}) \in S$. Then the minimum time problem for

$$x'(t) \in F(x(t))$$

has a solution.

Proof. a) Let $t^* = \inf\{t : A_0^{co}(t) \cap S \neq \emptyset\}$. Let (t_n) be decreasing to t^* and let x_n be solutions to the differential inclusion

$$x'(t) \in coF(x(t))$$

such that $x_n(0) = 0$ and $x(t_n) \in S$, $x(t) \in X$ for $t \in [0, t_n]$. A subsequence of this sequence converges uniformly to x_*. Clearly, $x_*(0) = 0$, $x_*(t^*) \in S$ and $x_*(t) \in X$ for $t \in [0, t^*]$. It is known that, under the assumptions of the Theorem, x_* is again a solution to

$$x'(t) \in coF(x(t)).$$

Hence, x_* is a solution to the convexified problem that reaches S in minimum time, and t^* is the value of the minimum time for the convexified problem.

b) Applying Lemma 4.8, we infer that $x'_*(t) \neq 0$ for a.e. $t \in [0, t^*]$. In fact, otherwise, we could define a different solution to the convexified differential inclusion, defined on an interval $[0, \tau^*]$ with $\tau^* < t^*$, having the same initial and final point: hence t^* would not be the value of the minimum time for the convexified problem.

c) By the previous point and the assumption on $F(x)$, for almost every t there exists a non empty set $\Lambda(t)$ such that $\lambda \in \Lambda(t)$ implies $\lambda x'_*(t) \in F(x_*(t))$ and $\lambda \geq 1$. Reasoning as in [14], we obtain that $\Lambda(.)$ is measurable on $[0, t^*]$, hence, by standard arguments, that there exists a measurable selection $\lambda(.)$ from $\Lambda(.)$. Define the absolutely continuous map s by $s(0) = 0$ and $s'(t) = 1/\lambda(t)$: s is an increasing map and maps $[0, t^*]$ onto $[0, s^*]$, where $s^* \leq t^*$. Let $t = t(s)$ be its inverse and consider the map $\tilde{x}(s) = x_*(t(s))$. We obtain in particular that $\tilde{x}(0) = x_*(0)$ and that $\tilde{x}(s^*) = x_*(t(s^*)) = x_*(t^*)$. We also have

$$\begin{aligned} \frac{d}{ds}\tilde{x}(s) &= x'_*(t(s))t'(s) = x'_*(t(s))\frac{1}{s'(t(s))} \\ &= x'_*(t(s))\lambda_1(t(s)) \in F(x_*(t(s))) = F(\tilde{x}(s)). \end{aligned}$$

Hence we have obtained that $\tilde{x}$ is a solution to the original differential inclusion but also a solution to the minimum time problem for I_{co}. Since every solution to the original problem is also a solution to the convexified problem, the infimum of the times needed to reach S along the solutions to the original problem cannot be lesser than the minimum time for the convexified problem. Hence $\tilde{x}$ is a solution to the minimum time problem for the original differential inclusion. □

The following result is on the existence of solutions to initial value problems for differential inclusions with non-convex right hand side.

Theorem 4.10. *Let Ω be open, $x^0 \in \Omega$ and let F be as in Theorem 4.9. Assume that there exists $t^* > 0$ such that, on $[0, t^*]$, the Cauchy Problem*

$$x'(t) \in coF(x(t)) \quad x(0) = x^0$$

admits a solution $x \not\equiv x^0$. *Then the Cauchy Problem*

$$x'(t) \in F(x(t)) \quad x(0) = x^0$$

admits a solution on some interval $[0, \tau^*]$

Proof. Since $x \not\equiv x^0$, there exists $t^1 \in [0, t^*]$ such that $x(t^1) \neq x^0$. Consider the minimum time problem for the convexified inclusion, with target set $S = \{x(t^1)\}$. This problem has a solution with minimum time τ^*, where $0 < \tau^* \leq t^1$. By Theorem 4.9, the original non-convexified problem has a solution on $[0, \tau^*]$. □

The scheme commonly used to attack the minimum time problem raised by Bernoulli is to transform it into the problem of minimizing

$$\int \frac{\sqrt{1 + (\xi_2')^2}}{\sqrt{-\xi_2}} dt$$

then to apply necessary conditions to find candidates for the solution; finally to show that the family of such candidates covers the plane, hence that there is one that passes through the final point. This approach, besides being very indirect, is somewhat unsatisfactory, mainly because the integrand is *not* defined at the initial condition $(0, 0)$ and the necessary conditions are applied without a previous proof of existence of solutions. Instead, Bernoulli's problem can be stated as follows: in the plane, an initial condition (ξ_1^0, ξ_2^0) is given; consider all the possible oriented rectifiable curves passing through it. Each such curve is defined by assigning $(u_1(.), u_2(.))$, a unit vector describing the direction of its (oriented) tangent. In this way, the parameter t is the arc-length parameterization of the curve.

The system of equations

$$\begin{cases} \xi_1' = u_1 \xi_3 \\ \xi_2' = u_2 \xi_3 \\ \xi_3' = -g u_2 \end{cases} \tag{B}$$

where the maps $u_1(.)$ and $u_2(.)$ are measurable and $u_1^2(t) + u_2^2(t) = 1$ a.e., describes the motion of a body in the plane, defined by the coordinates $(\xi_1(t), \xi_2(t))$, with (scalar) velocity $\xi_3(t) = \sqrt{(\xi_1'(t))^2 + (\xi_2'(t))^2}$, along a curve identified assigning the direction of its tangent vector $(u_1(.), u_2(.))$, subject to the gravity g.

Hence, the problem raised by Bernoulli can be stated as the following minimum time control problem:

The Brachistochrone Minimum Time Problem: find a solution to the control system

$$\begin{cases} \xi_1' = u_1 \xi_3 \\ \xi_2' = u_2 \xi_3 \\ \xi_3' = -g u_2 \end{cases}$$

$(\xi_1(0), \xi_2(0), \xi_3(0)) = (\xi_1^0, \xi_2^0, 0)$, subject to the constraint $\xi_3 \geq 0$, with control set

$$U = \{(u_1, u_2) : u_1^2 + u_2^2 = 1)\},$$

that would reach the target set $S = (\xi_1^f, \xi_2^f, \mathbb{R}^+)$, where $\xi_2^f \leq \xi_2^0$, in minimum time.

Theorem 4.11. *The Brachistochrone Minimum Time Problem admits a solution.*

Proof. Let $x = (\xi_1, \xi_2, \xi_3)$, $X = \{\xi_3 \geq 0\}$, $S = (\xi_1^f, \xi_2^f, \mathbb{R}^+)$; let $f(x, u)$ be the right hand side of (B) and set $F(x) = f(x, U)$. Then, as one can check, $F(x)$ satisfies the assumptions of Theorem 4.9. Hence it is enough to show that there are solutions $x(t)$ issuing from $x^0 = (0, 0, 0)$ with $\xi_3(t) \geq 0$ and such that, at some finite time t^*, $\xi_1(t^*) = \xi_1^f, \xi_2(t^*) = \xi_2^f$.

This is so in the case where $\xi_2^f < 0$. In fact, in this case we have that, by choosing the constant control

$$\begin{cases} u_1 = \frac{\xi_1^f}{\sqrt{(\xi_1^f)^2 + (\xi_2^f)^2}} \\ u_2 = \frac{\xi_2^f}{\sqrt{(\xi_1^f)^2 + (\xi_2^f)^2}} \end{cases}$$

we obtain that

$$\begin{cases} \xi_1(t) = u_1(-gu_2\frac{t^2}{2}) \\ \xi_2(t) = u_2(-gu_2\frac{t^2}{2}) \end{cases}$$

so that $\xi_1(t^f) = \xi_1^f$ and $\xi_2(t^f) = \xi_2^f$ for $t^f = \sqrt{\frac{2}{g}}\sqrt{\frac{(\xi_1^f)^2 + (\xi_2^f)^2}{-\xi_2^f}}$. We also obtain that $\xi_3(t) > 0$ on $(0, t^f)$ and that $\xi_3(t^f) = \sqrt{-2g\xi_2^f}$.

Consider the case $\xi_2^f = 0$. We can assume $\xi_1^f \neq 0$, otherwise $t^* = 0$ is the solution to the minimum time problem. In the case $\xi_1^f > 0$, consider the solution with the constant control $u_1 = \frac{1}{\sqrt{2}}, u_2 = -\frac{1}{\sqrt{2}}$ on $[0, \sqrt{2\xi_1^f/g}]$. At time $t = \sqrt{2\xi_1^f/g}$, we have that $\xi_1(\sqrt{2\xi_1^f/g}) = \frac{\xi_1^f}{2}$, $\xi_2(\sqrt{2\xi_1^f/g}) = -\frac{\xi_1^f}{2}$ and $\xi_3(\sqrt{2\xi_1^f}) = \sqrt{g\xi_1^f}$. The solution with constant control $u_1 = \frac{1}{\sqrt{2}}$, $u_2 = \frac{1}{\sqrt{2}}$ on the interval $(\sqrt{2\xi_1^f/g}, 2\sqrt{2\xi_1^f/g})$, with initial conditions $\xi_1(\sqrt{2\xi_1^f/g}) = \frac{\xi_1^f}{2}$, $\xi_2(\sqrt{2\xi_1^f/g}) = -\frac{\xi_1^f}{2}$ and $\xi_3(\sqrt{2\xi_1^f/g}) = \sqrt{g\xi_1^f}$ is such that at $t = 2\sqrt{2\xi_1^f/g}$, $\xi_1(2\sqrt{2\xi_1^f/g}) = \xi_1^f$ and $\xi_2(2\sqrt{2\xi_1^f/g}) = 0$. Hence $t^f = 2\sqrt{2\xi_1^f/g}$ and $\xi_3(t) > 0$ on $(0, t^f)$. □

Part IV

The Euler-Lagrange equation

Chapter 5

Preliminaries

5.1 From the stationary points to the Euler-Lagrange equation

A basic principle of analysis is that, giving a minimum point belonging to the interior of the domain of a differential function, one obtains a necessary condition for this point exploring its neighbourhood; one finds out that it is a stationary point, i.e. the gradient of the function calculated on the point must be zero. It is possible to apply this principle in the Calculus of Variations.

Let $\hat{x}$ be a minimizer to the basic problem of the Calculus of Variations consisting in minimizing the action

$$\mathcal{I}(x) = \int_I L(t, x(t), x'(t))dt,$$

where $I = [a, b] \subset \mathbb{R}$, on the set of those absolutely continuous functions $x : I \to \mathbb{R}^N$ satisfying the boundary conditions $x(a) = A, x(b) = B$.

Fix an admissible variation, i.e. a smooth function $\eta : [a, b] \to \mathbb{R}^N$, equal to zero at the boundary, consider the action $\mathcal{I}$ evaluated on the trajectory $\hat{x} + \epsilon\eta$, where ϵ is a real number, and consider the function obtained $\mathcal{I}(\hat{x}+\epsilon\eta)$ as a real valued function of ϵ. For this function the point 0 must be a stationary point. Whenever it is possible to pass to the limit under the integral sign, one obtains the Euler-Lagrange equation (E-L): for any variation η,

$$\int_a^b [\langle \nabla_{x'} L(t, \hat{x}(t), \hat{x}'(t)), \eta'(t)\rangle + \langle \nabla_x L(t, \hat{x}(t), \hat{x}'(t)), \eta(t)\rangle]dt = 0,$$

or, considering d/dt as a weak derivative,

$$\frac{d}{dt}\nabla_{x'} L(t, \hat{x}(t), \hat{x}'(t)) = \nabla_x L(t, \hat{x}(t), \hat{x}'(t)).$$

L. Tonelli was able to prove the validity of the Euler-Lagrange equation with a Lagrangian L belonging to $\mathbf{C}^3$, or belonging to $\mathbf{C}^2$ provided that $N = 1$ and $L_{x'x'}$ is strictly positive. Many efforts have been performed in order to weaken the regularity assumptions on L ([5], [16], [19], [20], [33], [38], [40], [50]).

In [5], J. M. Ball and V. J. Mizel presented an example of Lagrangian such that the integrability of $\nabla_x L(\cdot, \hat{x}(\cdot), \hat{x}'(\cdot))$ does not hold and, as a consequence, (E-L) is not true along the minimizer. Hence, some condition on the term $\nabla_x L(\cdot, \hat{x}(\cdot), \hat{x}'(\cdot))$ has to be imposed in order to ensure the validity of (E-L).

In [16], [19], [38], [50], under assumptions like the existence of an integrable function $S(t)$ such that, for any y in a neighbourhood of the minimizer, $|\nabla_x L(t, y, \hat{x}'(t))|$ is bounded by $S(t)$, the Euler-Lagrange equation is proved to be valid. This condition implies that, locally along the minimizer, $L(t, \cdot, \hat{x}'(t))$ is Lipschitzian. However, there are simple and meaningful examples of variational problems where this Lipschitzianity condition is not verified. In section 6.1, we present an example of this kind.

In section 6.2 we provide a result on the validity of (E-L) ([27]) that is satisfied by Lagrangians that are Lipschitzian in x, but that applies as well to some non-Lipschitzian cases as the example in section 6.1.

Chapter 6

New Results

In section 6.2 we provide a result on the validity of (E-L) ([27]) that is satisfied by Lagrangians that are Lipschitzian in x, but that applies as well to some non-Lipschitzian cases. For instance, for the problem (P) with the (non-Lipschitzian) Lagrangian $L(t, x, x') = (x'\sqrt{|x|} - 2/3)^2$, on the interval $[0, 1]$ and with boundary conditions $x(0) = 0$, $x(1) = 1$, by this result it can be proved the validity of (E-L). This example is presented in section 6.1.

6.1 An example involving a non-Lipschitz Lagrangian

A result of F. H. Clarke ([19]) implies that the assumption of the existence of an integrable function $S(t)$ such that, for y in a neighbourhood of the minimizer, $|\nabla_x L(t, y, \hat{x}'(t))| \leq S(t)$, is sufficient to establish the validity of the Euler-Lagrange equation. This condition implies that, locally along the minimizer, $L(t, \cdot, \hat{x}'(t))$ is Lipschitzian of Lipschitz constant $S(t)$. However, there are simple and meaningful examples of variational problems where this Lipschitzianity condition is not verified.

Consider the Lagrangian defined by $L(x, \xi) = (\xi\sqrt{|x|} - 2/3)^2$, and the problem (P) of minimizing

$$\int_0^1 \left[x'(t)\sqrt{|x(t)|} - \frac{2}{3}\right]^2 dt$$

over the absolutely continuous functions x with $x(0) = 0$, $x(1) = 1$.

One can easily verify that $\hat{x}(t) = t^{2/3}$ is a minimizer for (P). Indeed, $L(\hat{x}(t), \hat{x}'(t)) = 0$ on $[0, 1]$, and L is non negative everywhere. The Lagrangian L is non-Lipschitzian locally along $\hat{x}$. In this case, although L is not differentiable everywhere, $L_x(\hat{x}(t), \hat{x}'(t))$ exists a.e. (it is a.e. zero) and it is integrable.

6.2 The Euler-Lagrange equation for non-Lipschitz Lagrangians

In this section we provide (under Caratheodory's conditions) a result on the validity of the Euler-Lagrange equation that is satisfied by Lagrangians that are Lipschitzian in x, but that applies as well to the non-Lipschitzian cases as the example in the previous section.

In the proof we first show that the fact that $\hat{x}$ is a minimizer implies the integrability of $\nabla_\xi L(\cdot, \hat{x}(\cdot), \hat{x}'(\cdot))$. Then, using this result, we establish the validity of (E-L) under Caratheodory's condition.

Note that we do not assume any convexity hypothesis on the Lagrangian. Moreover, no growth condition whatsoever is assumed so that, as far as we know, relaxation theorems cannot be applied.

Consider the problem of minimizing the action

$$\mathcal{I}(x) = \int_I L(t, x(t), x'(t))dt,$$

where I is the interval $[a, b]$, on the set of those absolutely continuous functions $x : I \to \mathbb{R}^N$ satisfying the boundary conditions $x(a) = A$, $x(b) = B$. Let $\hat{x}$ be a (weak local) minimizer yielding a finite value for the action $\mathcal{I}$, and set $\mu = \sup_{t\in[a,b]} |\hat{x}(t)|$.

Our results will depend on the following assumption:

ASSUMPTION A. *i) L is differentiable in x along $\hat{x}$, for a.e. t, and the map $\nabla_x L(\cdot, \hat{x}(\cdot), \hat{x}'(\cdot))$ is integrable on I;*

ii) there exists a function $S(t)$ integrable on I such that, for any $y \in B(0, \mu+1)$,

$$L(t, y, \hat{x}'(t)) \le L(t, \hat{x}(t), \hat{x}'(t)) + S(t)|y - \hat{x}(t)|.$$

Consider problem (P) as presented in the previous section. The Lagrangian L and $\hat{x}$ satisfy the assumption A: $L_x(\hat{x}(t), \hat{x}'(t))$ exists a.e. (identically zero, hence integrable), $S(t) = t^{-2/3}$ verifies the inequality

$$L(y, \hat{x}'(t)) \le S(t)|y - \hat{x}(t)|,$$

since

$$\left(\frac{2}{3}t^{-1/3}\sqrt{|y|} - \frac{2}{3}\right)^2 = \frac{4(\sqrt{|y|} - t^{1/3})}{9t^{2/3}(\sqrt{|y|} + t^{1/3})}(|y| - t^{2/3}) \le \frac{4}{9t^{2/3}}|y - t^{2/3}|.$$

In what follows, $\overline{\mathbb{R}}$ denotes $\mathbb{R} \cup \{+\infty\}$.

This is our first result on the term $\nabla_\xi L(\cdot, \hat{x}(\cdot), \hat{x}'(\cdot))$:

Theorem 6.1. *Suppose that $L : I \times \mathbb{R}^N \times \mathbb{R}^N \to \overline{\mathbb{R}}$ is an extended valued function, finite on its effective domain of the form $domL = I \times \mathbb{R}^N \times G$, where $G \subset \mathbb{R}^N$ is an open set, and that it satisfies Carathéodory's conditions, i.e., $L(\cdot, x, \xi)$ is measurable for fixed (x, ξ) and $L(t, \cdot, \cdot)$ is continuous for almost every t. Moreover assume that L is differentiable in ξ on $domL$ and that $\nabla_\xi L$ satisfies Carathéodory's conditions on $domL$.*

Suppose that assumption A holds. Then,

$$\int_I |\nabla_\xi L(t, \hat{x}(t), \hat{x}'(t))| dt < +\infty.$$

Proof. 1) By assumption, $L(\cdot, \hat{x}(\cdot), \hat{x}'(\cdot)) \in \mathbf{L}^1(I)$, hence setting $S_0 = \{t \in I : \hat{x}'(t) \notin G\}$, we have $m(S_0) = 0$. Given $\epsilon > 0$, we can cover S_0 by an open set O_1 of measure $m(O_1) < \epsilon/2$. We have also that $\nabla_\xi L$ is a Carathéodory's function and that $\hat{x}'$ is measurable in I. Hence, by the theorems of Scorza Dragoni and of Lusin, for the given $\epsilon > 0$ there exists an open set O_2 such that $m(O_2) < \epsilon/2$ and at once $\hat{x}'$ is continuous in $I \setminus O_2$, and $\nabla_\xi L$ is continuous in $(I \setminus O_2) \times \mathbb{R}^N \times G$. By taking $K_\epsilon = I \setminus (O_1 \cup O_2)$, we have that K_ϵ is a closed set such that on it $\hat{x}'$ is continuous with values in G, $\nabla_\xi L$ is continuous on $K_\epsilon \times \mathbb{R}^N \times G$ and $m(I \setminus K_\epsilon) < \epsilon$. For $n \geq 1$ set $\epsilon_n = (b-a)/2^{n+1}$ and $K_n = K_{\epsilon_n}$; set also $C_n = \cup_{j=1}^n K_j$. Then C_n are closed sets, $C_n \subset C_{n+1}$, $\hat{x}'$ is continuous on C_n with values in G, $\nabla_\xi L$ is continuous in $C_n \times \mathbb{R}^N \times G$ and $\lim_{n \to +\infty} m(I \setminus C_n) = 0$.

From these properties it follows that there exists $k_n > 0$ such that, for all $t \in C_n$,

$$|\nabla_\xi L(t, \hat{x}(t), \hat{x}'(t))| < k_n.$$

There is no loss of generality in assuming $k_n \geq k_{n-1}$. Moreover, we have that $m(C_1) \geq (b-a)/2$ and $\sum_{n=2}^\infty m(C_n \setminus C_{n-1}) \leq (b-a)/2$.

For all $n > 1$, we set $A_n = C_n \setminus C_{n-1}$. Hence we obtain that $C_m = C_1 \cup_{n=2}^m A_n$ and that $I = E \cup C_1 \cup_{n>1} A_n$, where $m(E) = 0$.

2) Consider the function

$$\theta(t) = \begin{cases} 0 & \text{if } \nabla_\xi L(t, \hat{x}(t), \hat{x}'(t)) = 0, \\ \frac{\nabla_\xi L(t, \hat{x}(t), \hat{x}'(t))}{|\nabla_\xi L(t, \hat{x}(t), \hat{x}'(t))|} & \text{otherwise,} \end{cases}$$

and

$$v_n = \int_{A_n} \theta(t) dt,$$

so that $|v_n| \leq m(A_n)$. There exists a closed set $B_n \subseteq C_1$ such that $m(B_n) = |v_n|$. Set

$$\theta_n{}'(t) = -\theta(t) \chi_{A_n}(t) + \frac{v_n}{|v_n|} \chi_{B_n}(t).$$

We have that
$$\int_I {\theta_n}'(t)dt = -\int_{A_n} \theta(t)dt + v_n = 0.$$
Hence, setting $\theta_n(t) = \int_a^t {\theta_n}'(\tau)d\tau$, we see that the functions $\theta_n(t)$ are admissible variations. Moreover we obtain
$$||\theta_n||_\infty \leq \sup_{t\in I}\int_a^t |{\theta_n}'(\tau)|d\tau \leq \int_I |{\theta_n}'(\tau)|d\tau \leq 2m(A_n).$$

3) For t in A_n, we have $|\nabla_\xi L(t, \hat{x}(t), \hat{x}'(t))| < k_n$; for t in B_n, $|\nabla_\xi L(t, \hat{x}(t), \hat{x}'(t))| < k_1 \leq k_n$. Recalling that $\overline{A_n} \subset C_n$, we infer that, for all $t \in \overline{A_n} \cup B_n$,
$$|\nabla_\xi L(t, \hat{x}(t), \hat{x}'(t))| \leq k_n.$$
We wish to obtain an uniform bound for $|\nabla_\xi L|$ computed in a suitable neighbourhood of the minimizer $(\hat{x}(\cdot), \hat{x}'(\cdot))$. Consider the set $(\overline{A_n} \cup B_n) \times \mathbb{R}^N \times G$ as a metric space M_n with distance $d((t,x,\xi),(t',x',\xi')) = \sup(|t-t'|, |x-x'|, |\xi-\xi'|)$. On M_n, $\nabla_\xi L$ is continuous. Moreover, its subset
$$G_n = \{(t, \hat{x}(t), \hat{x}'(t)) : t \in \overline{A_n} \cup B_n\}$$
is compact and, on G_n, $|\nabla_\xi L|$ is bounded by k_n. Hence there exists $\delta_n > 0$ such that, for $(t,x,\xi) \in M_n$ with $d\left((t,x,\xi),(t,\hat{x}(t),\hat{x}'(t))\right) < \delta_n$, we have $|\nabla_\xi L(t,x,\xi)| < k_n + 1$.

4) For $|\lambda| < \min\{1/2m(A_n), \delta_n/2m(A_n), \delta_n\}$, consider the integrals
$$\int_I \frac{1}{\lambda}[L(t, \hat{x}(t) + \lambda\theta_n(t), \hat{x}'(t) + \lambda\theta_n'(t)) - L(t, \hat{x}(t), \hat{x}'(t))]dt =$$
$$\int_{A_n\cup B_n} \frac{1}{\lambda}[L(t, \hat{x}(t) + \lambda\theta_n(t), \hat{x}'(t) + \lambda\theta_n'(t)) - L(t, \hat{x}(t) + \lambda\theta_n(t), \hat{x}'(t))]dt +$$
$$\int_I \frac{1}{\lambda}[L(t, \hat{x}(t) + \lambda\theta_n(t), \hat{x}'(t)) - L(t, \hat{x}(t), \hat{x}'(t))]dt.$$
For every $t \in A_n \cup B_n$ there exists $\zeta_\lambda(t) \in (0, \lambda)$ such that
$$\frac{1}{\lambda}[L(t, \hat{x}(t) + \lambda\theta_n(t), \hat{x}'(t) + \lambda\theta_n'(t)) - L(t, \hat{x}(t) + \lambda\theta_n(t), \hat{x}'(t))] =$$
$$\langle \nabla_\xi L(t, \hat{x}(t) + \lambda\theta_n(t), \hat{x}'(t) + \zeta_\lambda(t)\theta_n'(t)), \theta_n'(t)\rangle \leq$$
$$|\nabla_\xi L(t, \hat{x}(t) + \lambda\theta_n(t), \hat{x}'(t) + \zeta_\lambda(t)\theta_n'(t))|$$
and from the choice of λ, $|\nabla_\xi L(t, \hat{x}(t) + \lambda\theta_n(t), \hat{x}'(t) + \zeta_\lambda(t)\theta_n'(t))| < k_n + 1$. Hence, we can apply the Dominated Convergence Theorem to obtain that
$$\lim_{\lambda\to 0}\int_{A_n\cup B_n} \frac{1}{\lambda}[L(t, \hat{x}(t) + \lambda\theta_n(t), \hat{x}'(t) + \lambda\theta_n'(t)) - L(t, \hat{x}(t) + \lambda\theta_n(t), \hat{x}'(t))]dt =$$

$$\int_{A_n \cup B_n} \langle \nabla_\xi L(t, \hat{x}(t), \hat{x}'(t)), \theta_n'(t) \rangle dt.$$

5) Set $f^+(s) = \max\{0, f(s)\}$, $f^-(s) = \max\{0, -f(s)\}$. Since

$$0 \le \frac{1}{\lambda}[L(t, \hat{x}(t) + \lambda\theta_n(t), \hat{x}'(t)) - L(t, \hat{x}(t), \hat{x}'(t))]^+ \le S(t)|\theta_n(t)|,$$

by the Dominated Convergence Theorem,

$$\lim_{\lambda\to 0} \int_I \frac{1}{\lambda}[L(t, \hat{x}(t) + \lambda\theta_n(t), \hat{x}'(t)) - L(t, \hat{x}(t), \hat{x}'(t))]^+ dt =$$

$$\int_I \lim_{\lambda\to 0} \frac{1}{\lambda}[L(t, \hat{x}(t) + \lambda\theta_n(t), \hat{x}'(t)) - L(t, \hat{x}(t), \hat{x}'(t))]^+ dt.$$

By the Fatou's Lemma,

$$\liminf_{\lambda\to 0} \int_I \frac{1}{\lambda}[L(t, \hat{x}(t) + \lambda\theta_n(t), \hat{x}'(t)) - L(t, \hat{x}(t), \hat{x}'(t))]^- dt \ge$$

$$\int_I \liminf_{\lambda\to 0} \frac{1}{\lambda}[L(t, \hat{x}(t) + \lambda\theta_n(t), \hat{x}'(t)) - L(t, \hat{x}(t), \hat{x}'(t))]^- dt.$$

We have obtained that

$$\limsup_{\lambda\to 0} \int_I \frac{1}{\lambda}[L(t, \hat{x}(t) + \lambda\theta_n(t), \hat{x}'(t)) - L(t, \hat{x}(t), \hat{x}'(t))] dt \le$$

$$\lim_{\lambda\to 0} \int_I (\frac{1}{\lambda}[L(t, \hat{x}(t) + \lambda\theta_n(t), \hat{x}'(t)) - L(t, \hat{x}(t), \hat{x}'(t))]^+ -$$

$$\liminf_{\lambda\to 0} \int_I \frac{1}{\lambda}[L(t, \hat{x}(t) + \lambda\theta_n(t), \hat{x}'(t)) - L(t, \hat{x}(t), \hat{x}'(t))]^- dt \le$$

$$\int_I \limsup_{\lambda\to 0} \frac{1}{\lambda}[L(t, \hat{x}(t) + \lambda\theta_n(t), \hat{x}'(t)) - L(t, \hat{x}(t), \hat{x}'(t))] dt =$$

$$\int_I \langle \nabla_x L(t, \hat{x}(t), \hat{x}'(t)), \theta_n(t) \rangle] dt.$$

6) Since $\hat{x}$ is a minimizer, we have

$$\begin{aligned} 0 \;\; \le \;\; & \int_{A_n \cup B_n} \langle \nabla_\xi L(t, \hat{x}(t), \hat{x}'(t)), \theta_n'(t) \rangle dt \\ & + \limsup_{\lambda\to 0} \int_I \frac{1}{\lambda}[L(t, \hat{x}(t) + \lambda\theta_n(t), \hat{x}'(t)) - L(t, \hat{x}(t), \hat{x}'(t))] dt \\ \le \;\; & \int_{A_n \cup B_n} \langle \nabla_\xi L(t, \hat{x}(t), \hat{x}'(t)), \theta_n'(t) \rangle dt + \int_I \langle \nabla_x L(t, \hat{x}(t), \hat{x}'(t)), \theta_n(t) \rangle dt. \end{aligned} \tag{6.2.1}$$

Since $\theta_n'(t) = -\nabla_\xi L(t, \hat{x}(t), \hat{x}'(t))/|\nabla_\xi L(t, \hat{x}(t), \hat{x}'(t))|$, for any t in A_n, it follows that $-\langle \nabla_\xi L(t, \hat{x}(t), \hat{x}'(t)), \theta_n'(t)\rangle \chi_{A_n}(t) = |\nabla_\xi L(t, \hat{x}(t), \hat{x}'(t))|\chi_{A_n}(t)$. Hence, we obtain that (6.2.1) can be written as

$$\int_{A_n} |\nabla_\xi L(t, \hat{x}(t), \hat{x}'(t))| dt \leq$$

$$\int_{B_n} \langle \nabla_\xi L(t, \hat{x}(t), \hat{x}'(t)), \theta_n'(t)\rangle dt + \int_I \langle \nabla_x L(t, \hat{x}(t), \hat{x}'(t)), \theta_n(t)\rangle dt.$$

On B_n, $|\nabla_\xi L(t, \hat{x}(t), \hat{x}'(t))|$ is bounded by k_1; from Hölder's inequality and the estimate on $||\theta_n||_\infty$ obtained in 2) we have that there exists a constant C (independent of n) such that

$$\int_{A_n} |\nabla_\xi L(t, \hat{x}(t), \hat{x}'(t))| dt \leq C m(A_n).$$

7) As $m \to +\infty$, the sequence of functions $\left(|\nabla_\xi L(t, \hat{x}(t), \hat{x}'(t))|\chi_{\{\cup_{n=2}^m A_n\}}(t)\right)_m$ converges monotonically to the function $|\nabla_\xi L(t, \hat{x}(t), \hat{x}'(t))|\chi_{\{\cup_{n>1} A_n\}}(t)$. From the estimate above and monotone convergence, we obtain

$$\int_{I \setminus C_1} |\nabla_\xi L(t, \hat{x}(t), \hat{x}'(t))| dt = \int_I |\nabla_\xi L(t, \hat{x}(t), \hat{x}'(t))|\chi_{\{\cup_{n>1} A_n\}} dt \leq C m(\cup_{n>1} A_n).$$

On C_1, $|\nabla_\xi L(t, \hat{x}(t), \hat{x}'(t))| < k_1$. Hence

$$\int_I |\nabla_\xi L(t, \hat{x}(t), \hat{x}'(t))| dt < +\infty.$$

$\square$

Corollary 6.2. *Under the same assumptions as in Theorem 6.1, for every variation η, $\eta(a) = 0$, $\eta(b) = 0$ and $\eta' \in L^\infty(I)$, we have*

$$\int_I [\langle \nabla_\xi L(t, \hat{x}(t), \hat{x}'(t)), \eta'(t)\rangle + \langle \nabla_x L(t, \hat{x}(t), \hat{x}'(t)), \eta(t)\rangle] dt = 0.$$

Proof. We shall prove that, for every η in $\mathbf{AC}(I)$ with bounded derivative, such that $\eta(a) = \eta(b) = 0$, we have

$$\int_I [\langle \nabla_\xi L(t, \hat{x}(t), \hat{x}'(t)), \eta'(t)\rangle + \langle \nabla_x L(t, \hat{x}(t), \hat{x}'(t)), \eta(t)\rangle] dt \geq 0.$$

Fix η, let $|\eta'(t)| \leq K$ for almost every t in I.

1) Define C_n and k_n as in point 1) of the proof of Theorem 6.1. Set

$$v_n = \int_{I \setminus C_n} \eta'(t)dt.$$

We have that $\lim_{n \to +\infty} |v_n| = 0$. In particular, for $n \geq \nu$, there exists $B_n \subseteq C_1$ such that $m(B_n) = |v_n|$. Set

$$(\eta_n)'(t) = \begin{cases} 0 & \text{for } t \in I \setminus C_n, \\ \eta'(t) & \text{for } t \in C_n \setminus B_n, \\ \frac{v_n}{|v_n|} + \eta'(t) & \text{for } t \in B_n \end{cases}$$

We obtain

$$\int_I \eta_n'(t)dt = \int_{C_n \setminus B_n} \eta'(t)dt + \int_{B_n} \left[\frac{v_n}{|v_n|} + \eta'(t) \right] dt = \int_{C_n} \eta'(t)dt + v_n = \int_I \eta'(t)dt = 0.$$

Hence, setting $\eta_n(t) = \int_a^t \eta_n'(\tau)d\tau$, we have that the functions $\eta_n(t)$ are variations and that, for almost every t in I, $|\eta_n'(t)| \leq (1+K)$, so that $||\eta_n||_\infty \leq (1+K)(b-a)$.

2) As in point 3) of the proof of Theorem 6.1, there exists $\delta_n > 0$ such that for $(t, x, \xi) \in C_n \times \mathbb{R}^N \times G$, with $d\left((t, x, \xi), (t, \hat{x}(t), \hat{x}'(t))\right) < \delta_n$, we have $|\nabla_\xi L(t, x, \xi)| < k_n + 1$.

3) For $|\lambda| < \min\{1/(1+K)(b-a), \delta_n/(1+K)(b-a), \delta_n/(1+K)\}$, consider the integrals

$$\int_I \frac{1}{\lambda}[L(t, \hat{x}(t) + \lambda\eta_n(t), \hat{x}'(t) + \lambda\eta_n'(t)) - L(t, \hat{x}(t), \hat{x}'(t))]dt =$$

$$\int_{C_n} \frac{1}{\lambda}[L(t, \hat{x}(t) + \lambda\eta_n(t), \hat{x}'(t) + \lambda\eta_n'(t)) - L(t, \hat{x}(t) + \lambda\eta_n(t), \hat{x}'(t))]dt +$$

$$\int_I \frac{1}{\lambda}[L(t, \hat{x}(t) + \lambda\eta_n(t), \hat{x}'(t)) - L(t, \hat{x}(t), \hat{x}'(t))]dt.$$

For almost every $t \in C_n$, there exists $\zeta_\lambda(t) \in (0, \lambda)$ such that

$$\frac{1}{\lambda}[L(t, \hat{x}(t) + \lambda\eta_n(t), \hat{x}'(t) + \lambda\eta_n'(t)) - L(t, \hat{x}(t) + \lambda\eta_n(t), \hat{x}'(t))] =$$

$$\langle \nabla_\xi L(t, \hat{x}(t) + \lambda\eta_n(t), \hat{x}'(t) + \zeta_\lambda(t)\eta_n'(t)), \eta_n'(t) \rangle \leq$$

$$|\nabla_\xi L(t, \hat{x}(t) + \lambda\eta_n(t), \hat{x}'(t) + \zeta_\lambda(t)\eta_n'(t))|(1+K) < (k_n+1)(1+K).$$

Hence, we can apply the Dominated Convergence Theorem to obtain that

$$\lim_{\lambda\to 0}\int_I \frac{1}{\lambda}[L(t,\hat{x}(t)+\lambda\eta_n(t),\hat{x}'(t)+\lambda\eta_n'(t))-L(t,\hat{x}(t)+\lambda\eta_n(t),\hat{x}'(t))]dt =$$

$$\int_{C_n}\langle\nabla_\xi L(t,\hat{x}(t),\hat{x}'(t)),\eta_n'(t)\rangle dt = \int_I\langle\nabla_\xi L(t,\hat{x}(t),\hat{x}'(t)),\eta_n'(t)\rangle dt.$$

4) Following the point 5) of Theorem 6.1, we obtain that

$$\limsup_{\lambda\to 0}\int_I \frac{1}{\lambda}[L(t,\hat{x}(t)+\lambda\eta_n(t),\hat{x}'(t))-L(t,\hat{x}(t),\hat{x}'(t))]dt \le$$

$$\int_I\langle\nabla_x L(t,\hat{x}(t),\hat{x}'(t)),\eta_n(t)\rangle]dt.$$

5) Since $\hat{x}$ is a minimizer, we have

$$0\le \int_I\langle\nabla_\xi L(t,\hat{x}(t),\hat{x}'(t)),\eta_n'(t)\rangle dt+\limsup_{\lambda\to 0}\int_I \frac{1}{\lambda}[L(t,\hat{x}(t)+\lambda\eta_n(t),\hat{x}'(t))-L(t,\hat{x}(t),\hat{x}'(t))]dt \le$$

$$\int_I[\langle\nabla_\xi L(t,\hat{x}(t),\hat{x}'(t)),\eta_n'(t)\rangle+\langle\nabla_x L(t,\hat{x}(t),\hat{x}'(t)),\eta_n(t)\rangle]dt.$$

6) Since

$$\lim_{n\to+\infty}\eta_n'(t)=\eta'(t) \text{ and } |\nabla_\xi L(t,\hat{x}(t),\hat{x}'(t))||\eta_n'(t)|\le|\nabla_\xi L(t,\hat{x}(t),\hat{x}'(t))|(1+K),$$

$$\lim_{n\to+\infty}\eta_n(t)=\eta(t) \text{ and } |\nabla_x L(t,\hat{x}(t),\hat{x}'(t))||\eta_n(t)|\le|\nabla_x L(t,\hat{x}(t),\hat{x}'(t))|(1+K)(b-a),$$

by the dominated convergence we obtain

$$\lim_{n\to+\infty}\int_I\langle\nabla_\xi L(t,\hat{x}(t),\hat{x}'(t)),\eta_n'(t)\rangle dt = \int_I\langle\nabla_\xi L(t,\hat{x}(t),\hat{x}'(t)),\eta'(t)\rangle dt.$$

and

$$\lim_{n\to+\infty}\int_I\langle\nabla_x L(t,\hat{x}(t),\hat{x}'(t)),\eta_n(t)\rangle dt = \int_I\langle\nabla_x L(t,\hat{x}(t),\hat{x}'(t)),\eta(t)\rangle dt.$$

It follows that

$$\int_I[\langle\nabla_\xi L(t,\hat{x}(t),\hat{x}'(t)),\eta'(t)\rangle+\langle\nabla_x L(t,\hat{x}(t),\hat{x}'(t)),\eta(t)\rangle]dt\ge 0.$$

□

Even if the validity of the Euler-Lagrange equations already follows by the previous Corollary 6.2 and the DuBois-Reymond's lemma ([8]), we give an alternative proof in Corollary 6.4.

In the following theorem we prove an additional regularity result for the Lagrangian evaluated along the minimizer.

Theorem 6.3. *Under the same assumptions as in Theorem 6.1, the map* $\nabla_\xi L(\cdot, \hat{x}(\cdot), \hat{x}'(\cdot))$ *is in* $\mathbf{L}^\infty(I)$.

Proof. Using an iteration process, we shall prove that for every p in $\mathbb{N}$, $\nabla_\xi L(\cdot, \hat{x}(\cdot), \hat{x}'(\cdot))$ is in $\mathbf{L}^p(I)$. Since the $\mathbf{L}^p$ are nested, this proves that $\nabla_\xi L(\cdot, \hat{x}(\cdot), \hat{x}'(\cdot)) \in \cap_{p\geq 1}\mathbf{L}^p(I)$. At the same time, we shall prove that there exists a constant $K > 0$ such that, for every $1 \leq p < +\infty$, $||\nabla_\xi L||_p \leq K$, thus proving that $\nabla_\xi L(\cdot, \hat{x}(\cdot), \hat{x}'(\cdot))$ is in $\mathbf{L}^\infty(I)$.

Suppose that

1) From Theorem 6.1, we know that $\nabla_\xi L(\cdot, \hat{x}(\cdot), \hat{x}'(\cdot))$ is in $\mathbf{L}^1(I)$. Starting the iteration process, fix $p \in \mathbb{N}$ and suppose that $\nabla_\xi L(\cdot, \hat{x}(\cdot), \hat{x}'(\cdot)) \in \mathbf{L}^p(I)$, to prove that $\nabla_\xi L(\cdot, \hat{x}(\cdot), \hat{x}'(\cdot)) \in \mathbf{L}^{p+1}(I)$.

2) We can assume $||\nabla_\xi L||_p \neq 0$. Define C_n, A_n, k_n as in point 1) of the proof of Theorem 6.1. For all $n > 1$, set

$$v_n^p = \frac{(b-a)}{2||\nabla_\xi L||_p^p} \int_{A_n} \nabla_\xi L(t, \hat{x}(t), \hat{x}'(t)) |\nabla_\xi L(t, \hat{x}(t), \hat{x}'(t))|^{p-1} dt.$$

Since

$$m(C_1) \geq (b-a)/2 \text{ and } |v_n^p| \leq \frac{(b-a)}{2||\nabla_\xi L||_p^p} \int_{A_n} |\nabla_\xi L(t, \hat{x}(t), \hat{x}'(t))|^p \leq (b-a)/2,$$

there exists a set $B_n^p \subset C_1$ such that $m(B_n^p) = |v_n^p|$, so that

$$\frac{2||\nabla_\xi L||_p^p}{(b-a)} \int_{B_n^p} \frac{v_n^p}{|v_n^p|} dt = \int_{A_n} \nabla_\xi L(t, \hat{x}(t), \hat{x}'(t)) |\nabla_\xi L(t, \hat{x}(t), \hat{x}'(t))|^{p-1}.$$

Set

$$(\theta_n^p)'(t) = -\nabla_\xi L(t, \hat{x}(t), \hat{x}'(t)) |\nabla_\xi L(t, \hat{x}(t), \hat{x}'(t))|^{p-1} \chi_{A_n}(t) + \frac{2||\nabla_\xi L||_p^p}{(b-a)} \frac{v_n^p}{|v_n^p|} \chi_{B_n^p}(t)$$

and $\theta_n^p(t) = \int_a^t (\theta_n^p)'(\tau) d\tau$. We obtain that $||\theta_n^p||_\infty \leq 2\int_{A_n} |\nabla_\xi L(t, \hat{x}(t), \hat{x}'(t))|^p dt$. The variations θ_n^p have bounded derivatives so we can apply Corollary 6.2 to obtain that

$$\int_I [\langle \nabla_\xi L(t, \hat{x}(t), \hat{x}'(t)), (\theta_n^p)'(t) \rangle + \langle \nabla_x L(t, \hat{x}(t), \hat{x}'(t)), \theta_n^p(t) \rangle] dt = 0.$$

It follows that

$$\int_{A_n} |\nabla_\xi L(t, \hat{x}(t), \hat{x}'(t))|^{p+1} dt =$$

$$\frac{2||\nabla_\xi L||_p^p}{(b-a)} \int_{B_n^p} \langle \nabla_\xi L(t, \hat{x}(t), \hat{x}'(t)), v_n^p/|v_n^p| \rangle dt + \int_I \langle \nabla_x L(t, \hat{x}(t), \hat{x}'(t)), \theta_n^p(t) \rangle dt \leq$$

$$k_1 \int_{A_n} |\nabla_\xi L(t, \hat{x}(t), \hat{x}'(t))|^p dt + 2 \int_{A_n} |\nabla_\xi L(t, \hat{x}(t), \hat{x}'(t))|^p dt \int_I |\nabla_x L(t, \hat{x}(t), \hat{x}'(t))| dt \leq$$

$$\tilde{C} \int_{A_n} |\nabla_\xi L(t, \hat{x}(t), \hat{x}'(t))|^p dt$$

where $\tilde{C}$ is independent of n and p (suppose $\tilde{C} \geq 1$). The sequence of maps

$$\left(|\nabla_\xi L(t, \hat{x}(t), \hat{x}'(t))|^{p+1} \chi_{\cup_{n=2}^m A_n}(t) \right)_m$$

converges monotonically to $|\nabla_\xi L(t, \hat{x}(t), \hat{x}'(t))|^{p+1} \chi_{\{\cup_{n>1} A_n\}}(t)$, and each integral $\int_I |\nabla_\xi L(t, \hat{x}(t), \hat{x}'(t))|^{p+1} \chi_{\{\cup_{n=2}^m A_n\}}(t) dt$ is bounded by the same constant

$$\tilde{C} \sum_{n=2}^{\infty} \int_{A_n} |\nabla_\xi L(t, \hat{x}(t), \hat{x}'(t))|^p dt = \tilde{C} \int_{I \setminus C_1} |\nabla_\xi L(t, \hat{x}(t), \hat{x}'(t))|^p dt.$$

Hence, by the Monotone Convergence Theorem,

$$\int_{I \setminus C_1} |\nabla_\xi L(t, \hat{x}(t), \hat{x}'(t))|^{p+1} dt \leq \tilde{C} \int_{I \setminus C_1} |\nabla_\xi L(t, \hat{x}(t), \hat{x}'(t))|^p dt < +\infty.$$

On C_1, $|\nabla_\xi L(t, \hat{x}(t), \hat{x}'(t))| < k_1$, proving that $\nabla_\xi L(\cdot, \hat{x}(\cdot), \hat{x}'(\cdot)) \in L^{p+1}(I)$. Moreover, we have also obtained that

$$\int_{I \setminus C_1} |\nabla_\xi L(t, \hat{x}(t), \hat{x}'(t))|^{p+1} dt \leq \tilde{C}^p \int_{I \setminus C_1} |\nabla_\xi L(t, \hat{x}(t), \hat{x}'(t))| dt,$$

so that

$$\left(\int_{I \setminus C_1} |\nabla_\xi L(t, \hat{x}(t), \hat{x}'(t))|^{p+1} dt \right)^{1/(p+1)} \leq \tilde{C} S,$$

where $S = \max\{1, \int_{I \setminus C_1} |\nabla_\xi L(t, \hat{x}(t), \hat{x}'(t))| dt\}$. Setting $T = \max\{1, m(C_1)\}$ we have that, for all $p \in \mathbb{N}$,

$$||\nabla_\xi L||_{(p+1)} \leq \left(k_1^{(p+1)} m(C_1) + \int_{I \setminus C_1} |\nabla_\xi L(t, \hat{x}(t), \hat{x}'(t))|^{p+1} dt \right)^{1/(p+1)} \leq$$

$$k_1 m(C_1)^{1/(p+1)} + \left(\int_{I \setminus C_1} |\nabla_\xi L(t, \hat{x}(t), \hat{x}'(t))|^{p+1} dt \right)^{1/(p+1)} \leq k_1 T + \tilde{C} S = K.$$

□

Corollary 6.4. *Under the same conditions as in Theorem 6.1, for every variation η, $\eta(a) = 0$, $\eta(b) = 0$ and $\eta' \in \mathbf{L}^1(I)$, we have*

$$\int_I [\langle \nabla_\xi L(t, \hat{x}(t), \hat{x}'(t)), \eta'(t)\rangle + \langle \nabla_x L(t, \hat{x}(t), \hat{x}'(t)), \eta(t)\rangle] dt = 0.$$

As a consequence, $t \to \nabla_\xi L(t, \hat{x}(t), \hat{x}'(t))$ is absolutely continuous.

Proof. We shall prove that, for every η in $\mathbf{AC}(I)$, such that $\eta(a) = \eta(b) = 0$, we have

$$\int_I [\langle \nabla_\xi L(t, \hat{x}(t), \hat{x}'(t)), \eta'(t)\rangle + \langle \nabla_x L(t, \hat{x}(t), \hat{x}'(t)), \eta(t)\rangle] dt \geq 0.$$

1) Fix η. Through the same steps as in point 1) of the proof of Theorem 6.1, for every $n \in \mathbb{N}$ we can define a closed set C_n such that on it η' is continuous, $\hat{x}'$ is continuous with values in G, $\nabla_\xi L$ is continuous in $C_n \times \mathbb{R}^N \times G$ and $\lim_{n\to+\infty} m(I \setminus C_n) = 0$. In particular, it follows that there are constants k_n and $c_n > 0$ such that, for all $t \in C_n$,

$$|\nabla_\xi L(t, \hat{x}(t), \hat{x}'(t))| < k_n \text{ and } |\eta'(t)| < c_n.$$

Define v_n, B_n, η'_n and η_n as in the proof of Corollary 6.2. Since, for all $t \in I$, $|\eta'_n(t)| \leq 1 + |\eta'(t)|$, it follows that $||\eta'_n||_1 \leq (b-a) + ||\eta'||_1$. Moreover, $||\eta_n||_\infty \leq ||\eta'_n||_1 \leq (b-a) + ||\eta'||_1$.

2) As in point 3) of the proof of Theorem 6.1, there exists $\delta_n > 0$ such that for $(t, x, \xi) \in C_n \times \mathbb{R}^N \times G$, with $d\left((t, x, \xi), (t, \hat{x}(t), \hat{x}'(t))\right) < \delta_n$, we have $|\nabla_\xi L(t, x, \xi)| < k_n + 1$.

3) For $|\lambda| < \min\{1/((b-a) + ||\eta'||_1), \delta_n/((b-a) + ||\eta'||_1), \delta_n/(c_n + 1)\}$, consider the integrals

$$\int_I \frac{1}{\lambda}[L(t, \hat{x}(t) + \lambda\eta_n(t), \hat{x}'(t) + \lambda\eta'_n(t)) - L(t, \hat{x}(t), \hat{x}'(t))] dt =$$

$$\int_I \frac{1}{\lambda}[L(t, \hat{x}(t) + \lambda\eta_n(t), \hat{x}'(t) + \lambda\eta'_n(t)) - L(t, \hat{x}(t) + \lambda\eta_n(t), \hat{x}'(t))] dt +$$

$$\int_I \frac{1}{\lambda}[L(t, \hat{x}(t) + \lambda\eta_n(t), \hat{x}'(t)) - L(t, \hat{x}(t), \hat{x}'(t))] dt =$$

$$\int_{C_n} \frac{1}{\lambda}[L(t, \hat{x}(t) + \lambda\eta_n(t), \hat{x}'(t) + \lambda\eta'_n(t)) - L(t, \hat{x}(t) + \lambda\eta_n(t), \hat{x}'(t))] dt +$$

$$\int_I \frac{1}{\lambda}[L(t, \hat{x}(t) + \lambda\eta_n(t), \hat{x}'(t)) - L(t, \hat{x}(t), \hat{x}'(t))] dt.$$

For every $t \in C_n$, there exists $\zeta_\lambda(t) \in (0, \lambda)$ such that

$$\frac{1}{\lambda}[L(t, \hat{x}(t) + \lambda\eta_n(t), \hat{x}'(t) + \lambda\eta_n'(t)) - L(t, \hat{x}(t) + \lambda\eta_n(t), \hat{x}'(t))] =$$
$$\langle \nabla_\xi L(t, \hat{x}(t) + \lambda\eta_n(t), \hat{x}'(t) + \zeta_\lambda(t)\eta_n'(t)), \eta_n'(t)\rangle \leq$$
$$|\nabla_\xi L(t, \hat{x}(t) + \lambda\eta_n(t), \hat{x}'(t) + \zeta_\lambda(t)\eta_n'(t))|c_n < (k_n + 1)c_n.$$

Hence, we can apply the Dominated Convergence Theorem to obtain that

$$\lim_{\lambda\to 0}\int_I \frac{1}{\lambda}[L(t, \hat{x}(t) + \lambda\eta_n(t), \hat{x}'(t) + \lambda\eta_n'(t)) - L(t, \hat{x}(t) + \lambda\eta_n(t), \hat{x}'(t))]dt =$$
$$\int_{C_n}\langle \nabla_\xi L(t, \hat{x}(t), \hat{x}'(t)), \eta_n'(t)\rangle dt = \int_I \langle \nabla_\xi L(t, \hat{x}(t), \hat{x}'(t)), \eta_n'(t)\rangle dt.$$

4) Following the point 5) of Theorem 6.1, we obtain that

$$\limsup_{\lambda\to 0}\int_I \frac{1}{\lambda}[L(t, \hat{x}(t) + \lambda\eta_n(t), \hat{x}'(t)) - L(t, \hat{x}(t), \hat{x}'(t))]dt \leq$$
$$\int_I \langle \nabla_x L(t, \hat{x}(t), \hat{x}'(t)), \eta_n(t)\rangle dt$$

and, since $\hat{x}$ is a minimizer, we have

$$0 \leq \int_I \langle \nabla_\xi L(t, \hat{x}(t), \hat{x}'(t)), \eta_n'(t)\rangle + \limsup_{\lambda\to 0}\int_I \frac{1}{\lambda}[L(t, \hat{x}(t) + \lambda\eta_n(t), \hat{x}'(t)) - L(t, \hat{x}(t), \hat{x}'(t))]dt \leq$$
$$\int_I [\langle \nabla_\xi L(t, \hat{x}(t), \hat{x}'(t)), \eta_n'(t)\rangle + \langle \nabla_x L(t, \hat{x}(t), \hat{x}'(t)), \eta_n(t)\rangle]dt.$$

5) Finally we have:

$\lim_{n\to+\infty} \eta_n'(t) = \eta'(t)$ and $|\nabla_\xi L(t, \hat{x}(t), \hat{x}'(t))||\eta_n'(t)| \leq ||\nabla_\xi L(\cdot, \hat{x}(\cdot), \hat{x}'(\cdot))||_\infty (1 + |\eta'(t)|)$,

and

$\lim_{n\to+\infty} \eta_n(t) = \eta(t)$ and $|\nabla_x L(t, \hat{x}(t), \hat{x}'(t))||\eta_n(t)| \leq |\nabla_x L(t, \hat{x}(t), \hat{x}'(t))|((b-a) + ||\eta'||_1)$,

so that, by dominated convergence, we obtain

$$\lim_{n\to+\infty}\int_I \langle \nabla_\xi L(t, \hat{x}(t), \hat{x}'(t)), \eta_n'(t)\rangle dt = \int_I \langle \nabla_\xi L(t, \hat{x}(t), \hat{x}'(t)), \eta'(t)\rangle dt.$$

and

$$\lim_{n\to+\infty}\int_I \langle \nabla_x L(t, \hat{x}(t), \hat{x}'(t)), \eta_n(t)\rangle dt = \int_I \langle \nabla_x L(t, \hat{x}(t), \hat{x}'(t)), \eta(t)\rangle dt.$$

Hence, it follows that

$$\int_I [\langle \nabla_\xi L(t, \hat{x}(t), \hat{x}'(t)), \eta'(t)\rangle + \langle \nabla_x L(t, \hat{x}(t), \hat{x}'(t)), \eta(t)\rangle]dt \geq 0.$$

The absolute continuity of $t \to \nabla_\xi L(t, \hat{x}(t), \hat{x}'(t))$ is classical (e.g. [7]). $\square$

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