# THE SAGA OF A FISH: FROM A SURVIVAL GUIDE TO CLOSING LEMMAS

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Abstract. In a recent paper by D. Burago, S. Ivanov and A. Novikov, "A survival guide for feeble fish", it has been shown that a fish with limited velocity capabilities can reach any point in the (possibly unbounded) ocean provided that the fluid velocity field is incompressible, bounded and has vanishing mean drift. This brilliant result extends some known point-to-point global controllability theorems though being substantially non constructive. We will give a fish a different recipe of how to survive in a turbulent ocean, and show how this is related to structural stability of dynamical systems by providing a constructive way to change slightly a divergence free vector field with vanishing mean drift to produce a non dissipative (i.e. conservative in the sense not having wandering sets of positive measure) dynamics. This immediately leads to closing lemmas for dynamical systems, in particular to C. Pugh's closing lemma, the extension of which to incompressible vector fields over a possibly unbounded domain we provide here. The results are based on an extension of the Poincaré recurrence theorem to some $\sigma$-finite measures an on specially constructed Newtonian potentials.

## 1. Introduction

We consider an autonomous system of ordinary differential equations

$$\dot{x} = V(x), \tag{1.1}$$

where $x(\cdot) \in \mathbb{R}^d$. Together with (1.1) we consider the control problem

$$\dot{x} = V(x) + u(t), \tag{1.2}$$

where $u(\cdot)\colon \mathbb{R}^+ \to \mathbb{R}^d$ is a control function. The classical point-to-point controllability problem is that of finding a control $u(\cdot)$ in some class of admissible controls such that the trajectory $x(\cdot)$ of (1.2) starting at a given $x_0 \in \mathbb{R}^d$ at time $t = 0$ (i.e. satisfying $x(0) = x_0$) arrives at some given $x_1 \in \mathbb{R}^d$ at some finite time $\tau > 0$, i.e. has $x(\tau) = x_1$. The usual choice of the class of admissible controls is that of piecewise continuous functions $u\colon \mathbb{R}^+ \to \mathbb{R}^d$ satisfying $\|u\|_\infty \le \delta$, where $\|\cdot\|_\infty$ is the supremum norm, and $\delta > 0$ is given.

We will assume hereinafter that $V$ is *incompressible*, i.e. $\operatorname{div} V = 0$. In this case the above controllability problem can be naturally interpreted, as in [5] (see also [4] for the extension to the nonautonomous equation), as that of determining whether a fish in a turbulent ocean characterized by the velocity field $V$ and able to move with its own velocity not exceeding in modulus the given value $\delta > 0$, can reach any point starting from an arbitrary one. In case when the phase space of (1.2) instead of $\mathbb{R}^d$ one considers a compact subset of the latter invariant with respect to the flow of (1.1) (or a smooth compact manifold), then the positive answer to this question is provided by the global controllability theorem 4.2.7 in [2] (there it is formulated for analytic vector fields on compact Riemannian manifolds), while in

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the whole $\mathbb{R}^n$ the incompressibility condition of $V$ is clearly not enough for such a theorem to hold as can be seen just taking $V$ to be constant with sufficiently large modulus. This is in fact the only possible obstacle for controllability in the whole $\mathbb{R}^d$, since it has been proven in [5] that if $V$ is incompressible and has *vanishing mean drift* (called small mean drift in [5]) in the sense

$$\lim_{L\to\infty}\sup_{x\in\mathbb{R}^d}\left|\frac{1}{L^d}\int_{[0,L]^d} V(x+y)\,dy\right| = 0,$$

then the above controllability problem in $\mathbb{R}^d$ still is solvable for every couple of points $x_0$ and $x_1$. The respective proof is however by contradiction and hence strongly nonconstructive. In other words, one assures the fish that it can reach any given destination without giving any clue on how to do that.

In search for a more constructive solution one might ask whether one can take the control to be given by a simple feedback $u(\cdot) = W(x(\cdot))$, for some a priori unknown vector field $W\colon \mathbb{R}^d \to \mathbb{R}^d$, that could be explicitly constructed. Once one looks at this problem under such a point of view, one immediately observes that in a particular case of the *return problem* (i.e. with $x_0 = x_1$) the answer is positive when $x_0$ is a nonwandering point of (1.1), as provided by the famous Pugh's closing lemma [10], which says that the perturbation $W$ of the original vector field $V$ can be taken arbitrarily small not even in uniform, but even in $C^1$ (or, to be more precise, Lipschitz) norm. The latter lemma is one of the fundamental results of structural stability theory for smooth dynamical systems, and is the first one of a series of similar results, the most well-known of which are the Mañe ergodic closing lemma [9], the Hayashi connecting lemma [6] and Bonatti and Crovisier connecting lemma [3] (see [1] for the comprehensive overview on the subject). It is important however to emphasize here the simple though quite striking observation that the incompressibility of $V$ which means invariance of the Lebesgue measure under the flow induced by (1.1) does not say anything about existence of nonwandering points for this equation (as can be seen just by the example of a constant vector field $V$), in contrast with the case when (1.1) has a *finite* invariant measure (which holds in particular when it has a compact invariant set): in the latter case all the points of the support of the invariant measure are nonwandering by the classical Poincarè recurrence theorem, and even assuming additionally the small mean drift condition for $V$ does not a priori improve this situation (see Remark 4.9).

Our first principal result (Theorem 4.8) shows that in fact under just incompressibility and vanishing mean drift condition on $V$ one can perturb the latter vector field by a small perturbation $W$ (even with small derivatives) so that *every* point of $\mathbb{R}^d$ becomes nonwandering with respect to the flow of $V + W$. This will be done by an explicit construction using Newtonian potential so as to ensure that the flow of $V + W$ preserve a new invariant measure, the support of which is the whole $\mathbb{R}^d$; this measure will still be not finite so that the classical Poincarè recurrence theorem cannot be applied, but will "grow not too fast at infinity", which will be shown to be enough for the extension of the latter theorem to hold. This result combined with Pugh's closing lemma immediately implies the extension of the latter (Theorem 4.14), namely, that in fact every chosen point of $\mathbb{R}^d$ can be made periodic for a dynamical system provided by an ODE with incompressible vector field with vanishing mean drift at the right hand side up to a small perturbation of the vector field in the Lipschitz norm. Finally, we show that our construction actually implies the Burago-Ivanov-Novikov controllability theorem (Theorem 4.11), with a proof conceptually different from the original one, but very close to that of the classical controllability results for affine control systems with recurrent drift (see theorem 5 from [7, chapter 4] or theorem 4.2.7 in [2]).

## 2. Notation and preliminaries

The finite-dimensional space $\mathbb{R}^d$ is assumed to be equipped with the Euclidean norm $|\cdot|$, and notation $B_r(x) \subset \mathbb{R}^d$ stands for the usual open Euclidean ball of radius $r$ centered at $x$. The volume of the unit ball $B_1(0) \subset \mathbb{R}^d$ will be denoted $\omega_d$. The usual scalar product of $x \in \mathbb{R}^d$ and $y \in \mathbb{R}^d$ is denoted by $x \cdot y$. For any set $D \subset \mathbb{R}^d$, we let $\bar{D}$ be the closure of $D$, $\mathbf{1}_D$ be its characteristic function, $D^c := \mathbb{R}^d \setminus D$.

We denote by $C(\mathbb{R}^d;\mathbb{R}^m)$ (respectively $C^k(\mathbb{R}^d;\mathbb{R}^m)$, $C^{k,\beta}_{loc}(\mathbb{R}^d;\mathbb{R}^m)$, $\mathrm{Lip}(\mathbb{R}^d;\mathbb{R}^m)$, $L^1(\mathbb{R}^d;\mathbb{R}^m)$, $L^\infty(\mathbb{R}^d;\mathbb{R}^m)$) the set of continuous (respectively $k$-times continuously differentiable, $k$-times continuously differentiable with locally $\beta$-Hölder $k$-th derivatives, Lipschitz, Lebesgue integrable, Lebesgue measurable and essentially bounded) functions $f\colon \mathbb{R}^d \to \mathbb{R}^m$), omitting the reference for $\mathbb{R}^m$ when $m = 1$, i.e. for real valued functions. The standard uniform norms in $C(\mathbb{R}^d;\mathbb{R}^m)$ and in $L^\infty(\mathbb{R}^d;\mathbb{R}^m)$ will be both denoted by $\|\cdot\|_\infty$. By $L^1(\partial B_1(0);\mathcal{H}^{d-1})$ we denote the class of functions over $\partial B_1(0)$ integrable with respect to the $(d-1)$-dimensional Hausdorff measure $\mathcal{H}^{d-1}$. For a $V \in \mathrm{Lip}(\mathbb{R}^d;\mathbb{R}^m)$ we denote by $\mathrm{Lip}\, V$ its (least) Lipschitz constant, and also use the notation $\|V\|_{\mathrm{Lip}} := \|V\|_\infty + \mathrm{Lip}\, V$. By $*$ we denote the convolution of functions.

All the measures considered in the sequel are positive Radon measures, not necessarily finite. For a Borel measure $\mu$ over a metric space $X$ we let $\mathrm{supp}\,\mu$ stand for its support, and for a Borel map $T\colon X \to Y$ between metric spaces $X$ and $Y$ we denote by $T_{\#}\mu$ the push-forward of $\mu$, i.e. the measure over $Y$ defined by $(T_{\#}\mu)(B) := \mu(T^{-1}(B))$ for every Borel $B \subset Y$. For a metric space $X$ the set $M \subset X$ will be called invariant for the map $T\colon X \to X$, if $T(M) = M$, and the measure $\mu$ over $X$ will be called invariant for this map, if $T_{\#}\mu = \mu$ (of course, the map is assumed Borel in the latter case).

## 3. Accessibility

In what follows we suppose that (1.1) is uniquely solvable and defined a flow $T_t\colon \mathbb{R}^d \to \mathbb{R}^d$, $t \in \mathbb{R}$ by the formula $T_t(y) := x(t)$, where $x(\cdot)$ is a solution of (1.1) satisfying $x(0) = y$. In the sequel, we will also denote the flow $T_t$ by $\varphi^t_V$ when we need to emphasize that it is produced by the vector field $V$. As the set of admissible controls $U_\delta$ with given $\delta > 0$ we consider, as usual in control theory, the set of piecewise continuous functions $u\colon \mathbb{R} \to \mathbb{R}^d$ with $\|u\|_\infty < \delta$, and set $U_0 := \{0\}$. We recall the following definitions.

**Definition 3.1.** *Given a $\delta > 0$, we say that a point $z \in \mathbb{R}^d$ is $\delta > 0$ accessible from an $x \in \mathbb{R}^d$ in finite time $\tau > 0$, if there is an admissible control $u \in U_\delta$ such that a trajectory of* (1.2) *with initial condition $x(0) = y$ arrives in $z$ before time $\tau$, i.e. $x(s) = z$ for $|s| \le \tau$. The set of such points will be denoted by $\mathcal{A}(y, \tau, U_\delta)$. We will also refer to*

$$\mathcal{A}(y, U_\delta) := \cup_{\tau>0} \mathcal{A}(y, \tau, U_\delta)$$

*as the set of points accessible from $y \in \mathbb{R}^d$ using controls in $U_\delta$, and, for a set $M \subset \mathbb{R}^d$, we denote by*

$$\mathcal{A}(M, U_\delta) := \cup_{y\in M} \mathcal{A}(y, U_\delta),$$

*which is the set of points accessible from $M$ using controls in $U_\delta$.*

We also recall the following classical notions.

**Definition 3.2.** *We say that a set $\omega_x \subset \mathbb{R}^d$ is the $\omega$-limit set of $x \in \mathbb{R}^d$ for the flow $T_t$, if it is the set of limit points of trajectories of* (1.1) *starting at $x$, i.e. a set of $y \in \mathbb{R}^d$ such that there exist a sequence $t_k \to +\infty$ (depending on $y$) such that $T_{t_k}(x) \to y$ as $k \to \infty$.*

*A point $x \in \mathbb{R}^d$ is called*

(i) *(forward) Poisson stable for the flow* $T_t$*, if* $x \in \omega_x$*, where* $\omega_x \subset \mathbb{R}^d$ *is the* $\omega$*-limit set of* $x$*;*
(ii) *nonwandering for the flow* $T_t$*, if for every open* $U \subset \mathbb{R}^d$*,* $x \in U$ *one has* $T_{t_k}(U) \cap U \neq \emptyset$ *for some sequence* $t_k \to +\infty$ *(depending on* $U$*).*

Clearly the closure of the set of Poisson stable points for the flow $T_t$ is contained in the set of nonwandering points for the latter (called usually *nonwandering set*).

We will further use the following technical and probably folkloric (though not easily found in the literature) result (in a sense, a vaguely similar assertion is theorem 5 from [7, chapter 4]).

**Proposition 3.3.** *Let* $V: \mathbb{R}^d \to \mathbb{R}^d$ *be uniformly continuous and* $M \subset \mathbb{R}^d$ *be a closed set invariant with respect to the flow* $T_t := \varphi_V^t$ *of* (1.1) *such that the set of Poisson stable points of* $T_t$ *is dense in* $M$*. If* $M_0 \subset M$ *is a closed set invariant with respect to* $T_t$ *that cannot be represented as a disjoint union of two non-empty closed subsets invariant with respect to* $T_t$*, then* $M_0 \subset \mathcal{A}(x_0, U_\delta)$ *for every* $\delta > 0$ *and every* $x_0 \in M_0$*.*

*Proof.* Fix an arbitrary $x_0 \in M_0$ and $\delta > 0$. Consider the set

$$M_{x_0,\delta} := M \cap \bigcup_{\sigma<\delta} \mathcal{A}(x_0, U_\sigma)$$

of all points $y \in M$ accessible from $x_0$ using controls in $U_\sigma$ with any $0 \leq \sigma < \delta$. Clearly, $M_{x_0,\delta} \neq \emptyset$ because $\mathcal{A}(x_0, U_0) \subset M$ and $\mathcal{A}(x_0, U_0) \subset M_{x_0,\delta}$. We now prove several consecutive claims.

STEP 1. We show that $M_{x_0,\delta}$ is relatively open in $M$. In fact, if $z \in M \cap \mathcal{A}(x_0, U_\sigma)$ for a $\sigma < \delta$, i.e. $z = x(s)$ for $s > 0$ and some trajectory $x(\cdot)$ of (1.2) with $x(0) = x_0$ and $\|u\|_\infty \leq \sigma$, considering the system of equations equation

$$\dot{y}_i = \tilde{V}_i(y) + u_i(t) + \alpha_i, \qquad i = 1, \dots, d, \tag{3.1}$$

where $\tilde{V} \in C^1(\mathbb{R}^d; \mathbb{R}^d)$, $\|V - \tilde{V}\|_\infty < \varepsilon$ with $\varepsilon < (\delta - \sigma)/2$ to be chosen later (such a vector field exists because $V$ is assumed to be unformly continuous) and $\alpha \in B_\rho(0) \subset \mathbb{R}^d$ is a constant vector, with $\rho \in (0, (\delta - \sigma)/3$ to be chosen later. Letting $R := (\|V\|_\infty + \delta)s$, we have that both $x(s) \in B_R(x_0)$ and the solution $y(\cdot) = y_\alpha(\cdot)$ of (3.1) with the initial condition $y(0) = x_0$ satisfies $y(s) \in B_R(x_0)$. Letting $L > 0$ stand for the Lipschitz constant of $\tilde{V}$ restricted to $B_R(x_0)$, we have

$$\begin{aligned} \frac{d|x(t) - y(t)|}{dt} &\leq |V(x(t)) - V(y(t))| + |\tilde{V}(y(t)) - V(y(t))| + |\alpha| \\ &\leq L|x(t) - y(t)| + \varepsilon + \rho \quad \text{for } t \in (0, s), \end{aligned}$$

and hence, recalling that $x(0) - y(0) = 0$, we get

$$|y_0(s) - z| = |y_0(s) - x(s)| \leq \lambda := (\varepsilon + \rho)\frac{e^{Ls} - 1}{L}.$$

Moreover, $y_\alpha$ is differentiable in $\alpha$ with $\frac{d}{dt} D_\alpha y_\alpha$ the identity matrix. Therefore by implicit function theorem there is an $\varepsilon \in (\delta - \sigma)/3$ and a $\rho \in (0, (\delta - \sigma)/3)$ such that

$$B_{2\lambda}(y_0(s)) \subset \{y_\alpha(s) \colon \alpha \in B_\rho(0)\},$$

and $y_\alpha$ may be regarded as solutions of (1.2) with the new control

$$\tilde{u}(t) := \tilde{V}(y(t)) - V(y(t)) + u(t) + \alpha$$

instead of $u$, and

$$\|\tilde{u}\|_\infty < \tilde{\sigma} := \sigma + 2(\delta - \sigma)/3 < \delta,$$

then $\mathcal{A}(x_0, s, U_{\tilde{\sigma}})$ contains a ball centered in $z$ proving the claim of this step. It is worth observing however that we have proven more: in fact, there is an $r > 0$ such that for every $z \in \mathcal{A}(x_0, s, U_\sigma)$ with $\sigma < \delta$ one has $B_r(z) \bigcap M_0 \subset \mathcal{A}(x_0, s, U_\delta)$, with $r$ possibly depending on $s$ and $\delta$ and $\sigma$ but not on $z$.

STEP 2. We show

$$\overline{T_s(M_{x_0,\delta})} = M_{x_0,\delta} \quad \text{for all } s \in \mathbb{R}^+. \tag{3.2}$$

To show $\overline{T_s(M_{x_0,\delta})} \subset M_{x_0,\delta}$, observe that $T_s(M_{x_0,\delta}) \subset M_{x_0,\delta}$, and hence the desired inclusion for the closure follows from the claim proven on Step 1.

To prove the converse inclusion $M_{x_0,\delta} \subset T_s(M_{x_0,\delta})$, take an arbitrary Poisson stable point $p \in M_{x_0,\delta}$, so that $p = \lim_k x(t_k)$ where $x(\cdot)$ is a solution of (1.1) with $x(0) = p$, for some sequence $t_k \to +\infty$. Clearly $x(t) \in M_{x_0,\delta}$ for every $t \in \mathbb{R}^+$, and hence $q_k := x(t_k - s) \in M_{x_0,\delta}$. By construction $x(t_k) \in T_s(q_k) \subset T_s(M_{x_0,\delta})$, and hence $p$ belongs to the closure of the latter set.

STEP 3. As a result of Step 2 we have that $M_{x_0,\delta}$ is closed. Therefore, $M_0 \subset M_{x_0,\delta}$, since otherwise disjoint sets $M_0 \bigcap M_{x_0,\delta}$ and $M_0 \setminus M_{x_0,\delta}$ are both closed, nonempty and invariant for $T_t$ for all $t \in \mathbb{R}^+$ contrary to the assumption. □

*Remark* 3.4. The above Proposition 3.3 remains true if $V$ is assumed locally Lipschitz (and not necessarily uniformly continuous); it is enough to take in Step 1 of its proof $\tilde{V} := V$ and $\varepsilon := 0$.

## 4. Controllability and closing lemma

4.1. **Poincaré recurrence theorem for not necessarily finite measures.** We need the following generalization of the Poincaré recurrence theorem, which we formulate for injective maps, since this will be more adapted to the use in the sequel.

**Proposition 4.1.** *Let $T\colon \mathbb{R}^d \to \mathbb{R}^d$ be an injective Borel map preserving a $\sigma$-finite Borel measure $\mu$. Suppose also that*

$$\mu\left(\bigcup_{k=0}^{n} T^k(B_\rho(0))\right) = o(n) \quad \text{as } n \to +\infty$$

*for every $\rho > 0$ fixed. Then for every open $U \subset \mathbb{R}^d$ such that $\mu(U) > 0$ one has $T^n(U) \bigcap U \neq \emptyset$ for a subsequence of $n \in \mathbb{N}$ (depending on $U$).*

*Remark* 4.2. The conditions of Proposition 4.1 are in particular satisfied in the following cases which are of practical importance:

(i) $\mu$ is finite (this recovers the classical statement of the Poincarè recurrence theorem;

(ii) for some function $\kappa\colon \mathbb{R}^+ \to \mathbb{R}^+$ one has

$$\begin{aligned} \mu(B_\rho(0)) = o(\kappa(\rho)) &\quad \text{as } \rho \to +\infty, \\ T^n(B_\rho(0)) \subset B_R(0) &\quad \text{with } R = R(\rho, n) > 0,\ R(\rho, \cdot) \text{ nondecreasing} \\ &\quad \text{and } \kappa(R) = O(n) \text{ as } n \to +\infty \text{ for every } \rho > 0 \text{ fixed;} \end{aligned}$$

(iii) $|T^n(x)| \leq A|x| + Bn$ for some $A \geq 0$, $B \geq 0$, and $\mu(B_\rho(0)) = o(\rho)$ as $\rho \to +\infty$, as can be seen from taking $\kappa$ linear in (ii).

*Proof.* Since it is enough to prove the statement for each $U \cap B_\rho(0)$, for an arbitrary $\rho > 0$, we may assume without loss of generality that $U \subset B_\rho(0)$ for some $\rho > 0$. Fix an arbitrary $n \in \mathbb{N}$ and consider the set

$$U_n := \{x \in U : T^k(x) \notin U \quad \text{for all } k \geq n\}.$$

Let $F := T^n$ and since $F^k$ are injectve and preserve $\mu$, we get

$$\mu(F^k(U)) = \mu(F^{-k}(F^k(U))) = \mu(U)$$

for all $k \in \mathbb{N}$. Clearly, $\{F^k(U_n)\}_{k\in\mathbb{N}}$ are pairwise disjoint, since otherwise one would have $D := F^m(U_n) \cap F^i(U_n) \neq \emptyset$ for some $m > i \geq n$, and hence, using again injectivity of $F^k$, we get

$$(F^i)^{-1}(D) = F^{m-i}(U_n) \cap U_n = T^{(m-i)n}(U_n) \cap U_n \neq \emptyset$$

contradicting the definition of $U_n$. Then

$$\begin{aligned} m\mu(U_n) &= \mu\left(\bigcup_{k=0}^{m-1} F^k(U_n)\right) \quad \text{because } F^k(U_n) \text{ are pairwise disjoint} \\ &\leq \mu\left(\bigcup_{k=0}^{(m-1)n} T^k(U_n)\right) = o(m) \end{aligned}$$

as $m \to \infty$, which implies $\mu(U_n) = 0$. □

The following corollary gives a "continuous" version of this statement adapted to our setting.

**Corollary 4.3.** *Let $V\colon \mathbb{R}^d \to \mathbb{R}^d$ be a bounded locally Lipschitz vector field, and $\mu$ be a $\sigma$-finite Borel measure invariant with respect to the flow $\varphi_V^t$ induced by $V$ and satisfying $\mu(B_R(0)) = o(R)$ as $R \to +\infty$. Then for every open $U \subset \mathbb{R}^d$ and for every $\tau > 0$ one has*

$$\mu\left(\left\{x \in U\colon \varphi_V^t(x) \notin U \text{ for all } t \geq \tau\right\}\right) = 0.$$

*Proof.* Fix an arbitrary $\tau > 0$ and let $T := \varphi_V^\tau$. The statement follows from Proposition 4.1 together with Remark 4.2(iii) (with $A := 1$, $B := \|V\|_\infty$, since $\|\varphi_V^t(x)\| \leq |x| + \|V\|_\infty t$ for $t \geq 0$). □

*Remark* 4.4. The above Corollary 4.3 remains valid for locally Lipschitz but not necessarily bounded vector fileds $V$ satisfying one-sided Osgood estimate

$$\sup_{|x|\leq r} \frac{V(x)\cdot x}{|x|} \leq K(r)$$

for all $r > 0$ and for some measurable function $K\colon \mathbb{R}^+ \to \mathbb{R}^+$ such that

$$\int_0^\infty \frac{ds}{K(s)} = +\infty,$$

under the condition

$$\mu(B_\rho(0)) = o\left(\int_0^\rho \frac{ds}{K(s)}\right),$$

as $\rho \to \infty$. In fact, denoting

$$\kappa(\rho) := \int_0^\rho \frac{ds}{K(s)},$$

we get that $\kappa$ is strictly increasing with $\kappa(t) \to +\infty$ as $t \to +\infty$. Furthermore,

$$|\varphi_V^t(x)| \leq R(|x|, t) := \kappa^{-1}(t + \kappa(|x|)), \tag{4.1}$$

because for $r(t) := |x(t)|$, $x(t) := \varphi_V^t(x)$ one has

$$\dot{r}(t) = \frac{V(x(t))\cdot x(t)}{r(t)} \leq K(r(t)),$$

and integrating the latter inequality one gets $\kappa(r(t)) - \kappa(r(0)) \leq t$, which implies (4.1). Thus, $\varphi_V^t(B_\rho(0)) \subset B_{R(\rho,t)}(0)$ with $\kappa(R(\rho,t)) = O(t)$ as $t \to +\infty$, and $\mu(B_\rho(0)) = o(\kappa(\rho))$ as $\rho \to +\infty$, and hence in the proof of Corollary 4.3 it is enough to refer to Remark 4.2(ii) instead of Remark 4.2(iii).

We will further use also the following statement.

**Corollary 4.5.** *Under conditions of Corollary 4.3 (or, more generally, of the Remark 4.4) one has that Poisson stable points of $V$ are dense in* $\operatorname{supp}\mu$*, and in particular, all the points of* $\operatorname{supp}\mu$ *are nonwandering for the flow generated by $V$.*

*Proof.* We repeat the main idea of the part 1 of proposition 4.1.18 from [8]. Let $\{U_j\}_{j\in\mathbb{N}}$ be a countable base of the topology in $\mathbb{R}^d$. For every $j\in\mathbb{N}$ we consider

$$N_j := \{x\in U_j : \varphi_V^t(x)\notin U_j \text{ for all } t\geq T_0 \text{ and for some } T_0=T_0(x)\in\mathbb{R}\},$$
$$R := \bigcap_j N_j^c.$$

In other words, $N_j$ is the set of all points of $U_j$, the iterations of which leave this set forever, while for every $x\in R\cap U_j$ there is a sequence $t_k\to\infty$ such that $\varphi_V^{t_k}(x)\in U_j$. By Corollary 4.3, $\mu(N_j)=0$ for all $j\in\mathbb{N}$, and thus $\mu(R^c)=0$, which implies density of $R$ in $\operatorname{supp}\mu$.

But if $x\in R$, then for any neighborhood $U$ of $x$ there is a $U_j\subset U$ such that $x\in U_j$, and hence for a sequence $t_k\to+\infty$ one has $\varphi_V^{t_k}(x)\in U_j\subset U$ for sufficiently large $k\in\mathbb{N}$. This means that the $\omega$-limit set $\omega_x$ of the point $x$ intersects with $U$. Since the set $\omega_x$ is closed and the neighborhood $U$ is arbitrary, we have $x\in\omega_x$ concluding the proof. □

4.2. **Correcting the vectorfield.** The basic idea of our construction is as follows. Supposing that the Lebesgue measure over $\mathbb{R}^d$ is invariant under the flow of the vector field $V$, we will "correct" the latter by adding a new vector filed $W$ (referred later as *corrector* such that the sum $V+W$ satisfy Corollary 4.3 (or, more generally, of the Remark 4.4), and hence also Corollary 4.5 for some new $\sigma$-finite measure $\mu$ with $\operatorname{supp}\mu=\mathbb{R}^d$ which will also be explcitly constructed. This will immediately lead to the proof of Burago-Ivanov-Novikov controllability theorem once one shows that $\|W\|_\infty$ may be made arbitrarily small. We will further show that this construction is in fact deeper and provides, for instance, a version of the Pugh closing lemma.

To fulfill this program define a positive function

$$\psi(x) := (|x|^2+\alpha^2)^{-p}, \tag{4.2}$$

where $\alpha$ and $p$ are positive parameters to be defined later. We will define a smooth map $W\colon\mathbb{R}^d\to\mathbb{R}^d$ depending on $\alpha$ and $p$ will so that the measure $\mu:=\psi\,dx$ be invariant under the flow of the perturbed system of ODEs

$$\dot{x} = V(x)+W(x). \tag{4.3}$$

Making $\mu$ invariant with respect to the flow of (4.3) amounts to making

$$\operatorname{div}\psi(V+W)=0$$

or equivalently, recalling that $V$ is incompressible,

$$\operatorname{div}(\psi W) = -\nabla\psi\cdot V \tag{4.4}$$

in the weak sense. Letting $u$ stand for a solution to the Poisson equation

$$-\Delta u = \nabla\psi\cdot V$$

in $\mathbb{R}^d$, we get that (4.4) is satisfied with

$$W := \frac{1}{\psi}\nabla u. \tag{4.5}$$

We may thus take $u$ of the form

$$u := \Phi * (\nabla\psi\cdot V),$$

where $\Phi$ stands for the fundamental solution of the Laplace equation $\mathbb{R}^d$, so that (4.5) reduces then to

$$
\begin{aligned}
W(x) &:= -\frac{c_d}{\psi(x)} \int_{\mathbb{R}^d} \frac{x-y}{|x-y|^d} \nabla\psi(y) \cdot V(y)\, dy \\
&= 2pc_d(|x|^2+\alpha^2)^p \int_{\mathbb{R}^d} \frac{x-y}{|x-y|^d} \frac{y \cdot V(y)}{(|y|^2+\alpha^2)^{p+1}}\, dy,
\end{aligned}
\tag{4.6}
$$

where $c_d := 1/d\omega_d$. Clearly, therefore, the following statement is valid.

**Lemma 4.6.** *If $V\colon \mathbb{R}^d \to \mathbb{R}^d$ is locally Lipschitz, then for every $p$ and $\alpha$ the vector field $W$ defined by (4.6) is $C^{1,\beta}_{loc}$ smooth for every $\beta \in [0,1)$ and the measure $\mu := \psi\, dx$ is invariant under the flow of (4.3).*

*Proof.* The smoothness of $W$ follows immediately from local elliptic Sobolev regularity together with the Sobolev embedding theorem. The invariance of $\mu$ follows from the construction. □

We observe now that with the appropriate choice of the parameters the vector field $W$ can be made arbitrarily small in supremum and, under a bit more requirements on regularity of $V$, even Lipschitz norm; this is the assertion of the following lemma which collects several calculations made in the Appendix B and A.

**Lemma 4.7.** *Let $p \in ((d-1)/2, d/2)$. Suppose that $V \in \mathrm{Lip}_{loc}(\mathbb{R}^d;\mathbb{R}^d)$ is a bounded incompressible vector field with vanishing mean drift. Then, given an $\varepsilon > 0$, there is an $\bar\alpha = \bar\alpha(p,\varepsilon)$ such that for every $\alpha > \bar\alpha$ one has*

(i) $\|W\|_\infty \le \varepsilon$,
(ii) $\|\operatorname{div} W\|_\infty \le \varepsilon$.

*If, moreover, $V \in \mathrm{Lip}(\mathbb{R}^d;\mathbb{R}^d)$ is incompressible vector field with vanishing mean drift, and all $V_{x_j}$, $j = 1,\ldots,d$, are locally Lipschitz still having vanishing mean drift, then one can choose $\bar\alpha$ so as to have additionally*

(iii) $\|W_{x_j}\|_\infty \le \varepsilon$ *for every $\alpha > \bar\alpha$ and for every $j = 1,\ldots,d$.*

*The latter assertion holds in particular when $V \in C^1(\mathbb{R}^d;\mathbb{R}^d)\cap\mathrm{Lip}(\mathbb{R}^d;\mathbb{R}^d)$ and has uniformly continuous first derivatives.*

*Proof.* Assertions (i) and (ii) are Lemma B.1 and Corollary B.2 respectively, (iii) is Lemma B.3. Finally, when $V \in C^1(\mathbb{R}^d;\mathbb{R}^d)\cap\mathrm{Lip}(\mathbb{R}^d;\mathbb{R}^d)$ and has uniformly continuous first derivatives, then the first derivatives of $V$ also are incompressible and have vanishing mean drift by Lemma A.5, and so (iii) still holds. □

4.3. **The end of the game: results.** The following statement is the first principal result of the paper.

**Theorem 4.8.** *Let $V \in \mathrm{Lip}_{loc}(\mathbb{R}^d;\mathbb{R}^d)$ be a bounded incompressible vector field with vanishing mean drift. Then for every $\varepsilon > 0$ there is a $C^{1,\beta}_{loc}$ (for every $\beta > 0$) vector field $W_\varepsilon\colon \mathbb{R}^d \to \mathbb{R}^d$ with $\|W_\varepsilon\|_\infty \le \varepsilon$ such that every $x \in \mathbb{R}^d$ is a nonwandering point of the system of ODEs*

$$
\dot x = V(x) + W_\varepsilon(x), \tag{4.7}
$$

*and $\mathbb{R}^d$ is invariant with respect to the flow defined by the latter. If, moreover, $V \in C^1(\mathbb{R}^d;\mathbb{R}^d)\cap\mathrm{Lip}(\mathbb{R}^d;\mathbb{R}^d)$ and has uniformly continuous first derivatives, then one may find $W_\varepsilon$ as above satisfying even the stronger estimate $\|W_\varepsilon\|_{\mathrm{Lip}} \le \varepsilon$.*

*Proof.* We choose a $p \in ((d-1)/2, d/2)$ and an $\alpha > 0$ so as to have $\|W\|_\infty \le \varepsilon$ (resp. $\|W\|_{\mathrm{Lip}} \le \varepsilon$ under the stronger regularity condition $V \in C^1(\mathbb{R}^d;\mathbb{R}^d)\cap\mathrm{Lip}(\mathbb{R}^d;\mathbb{R}^d)$ with uniformly continuous first derivatives), where $W$ is defined by (4.6): this is possible in view of assertion (i) (resp. (iii)) of Lemma 4.7. Since we have chosen $p > (d-1)/2$, then

$$\mu(B_R(0)) = \int_{B_R(0)} \psi(x)\, dx = d\omega_d \int_0^R \frac{r^{d-1}\, dr}{(r^2+\alpha^2)^p} = o(R),$$

and hence by Corollary 4.5 applied to $V + W$ (in place of $V$) all the points of $\mathbb{R}^d = \operatorname{supp}\mu$ (note that $\operatorname{supp}\mu$ is invariant for the latter flow by Lemma C.1) are nonwandering for the flow generated by the vector field $V + W$. It suffices to take then $W_\varepsilon := W$. □

*Remark* 4.9. It is important to emphasize that in contrast with the case when (1.1) has a *finite* invariant measure (e.g. when it has a compact invariant set), an incompressible smooth vector field $V$ with vanishing mean drift may produce a strongly dissipative dynamics in the sense of having a wandering set of full measure, as, for instance, when $d = 2$ and, say, $V(x_1, x_2) := (0, \sin x_1)$. In view of this observation the Theorem 4.8 is quite striking: it says that one may change the dynamics from strongly dissipative to a conservative one (i.e. with no wandering set of positive measure) over the whole (unbounded) space by an arbitrarily small perturbation (even with small first derivatives) of the vector field.

*Remark* 4.10. It is worth observing that the perturbation $W_\varepsilon$, and hence the perturbed vector field $V + W_\varepsilon$ constructed in the proof of the above Theorem 4.8 are in general not incompressible. However in view of Lemma 4.7(ii) one can ensure that the divergence of $W_\varepsilon$ (hence also that of $V + W_\varepsilon$) be arbitrarily small in the uniform norm.

The first corollary of the above theorem is the following global point-to-point controllability result, which is a reformulation of the Burago-Ivanov-Novikov controllability theorem 1.1 from [5] (and a partial extension of theorem 4.2.7 in [2] formulated for compact manifolds, although for possibly more general control affine systems), however proven now by a completely different and direct method.

**Theorem 4.11.** *Let $V \in \mathrm{Lip}_{loc}(\mathbb{R}^d;\mathbb{R}^d)$ be a bounded incompressible vector field with vanishing mean drift. Then for every couple of points $\{x_0, y_0\} \subset \mathbb{R}^d$ and every $\varepsilon > 0$ there is a piecewise continuous function $u\colon \mathbb{R}^+ \to \mathbb{R}^d$ ("control") with $\|u\|_\infty \le \varepsilon$ such that the trajectory of the system of ODEs*

$$\dot{x} = V(x) + u(t), \tag{4.8}$$

*satisfying $x(0) = x_0$ passes through $y_0$, i.e. $x(T) = y_0$ for some $T > 0$.*

*Proof.* Fixed an $\varepsilon > 0$, by Theorem 4.8 we find a smooth vector field $W_\varepsilon\colon \mathbb{R}^d \to \mathbb{R}^d$ with $\|W_\varepsilon\|_\infty \le \varepsilon/2$ such that every $x \in \mathbb{R}^d$ is nonwandering with respect to the flow defined by $V+W_\varepsilon$ and $\mathbb{R}^d$ is invariant with respect to the latter flow. Proposition 3.3 and Remark 3.4 applied with $M_0 = M := \mathbb{R}^d$ and $V+W_\varepsilon$ instead of $V$ (in particular, $T_t$ standing for the flow generated by $V + W_\varepsilon$) implies now the existence of a piecewise continuous control $\tilde{u}\colon \mathbb{R}^+ \to \mathbb{R}$ with $\|\tilde{u}\|_\infty \le \varepsilon/2$ such that the trajectory $x(\cdot)$ of the system

$$\dot{x} = V(x) + W_\varepsilon(x) + \tilde{u}(t)$$

starting at $x_0 \in \mathbb{R}^d$ arrives at $x_1 \in \mathbb{R}^d$ in finite time (in alternative to Proposition 3.3 and Remark 3.4 one could have used here theorem 5 from [7, chapter 4]). It suffices to take now $u(t) := W_\varepsilon(x(t)) + \tilde{u}(t)$. □

*Remark* 4.12. If one extends a bit Proposition 3.3 showing that one can achieve any given point in a compact set from a another point in the same set in finite time depending on the compact set, then under global Lipschitz continuity of $V$ one would have also the estimate on arrival time to the destination as in theorem 1.2

of [5] (and with exactly the same proof), though this is beyond the scope of this paper.

The following easy corollary slightly extends Theorem 4.11 to velocity fields which are just uniformly continuous (hence possibly even not provide unique solvablity of (1.1)).

**Corollary 4.13.** *If $V\colon \mathbb{R}^d \to \mathbb{R}^d$ is a bounded uniformly continuous (not necessarily locally Lipschitz) incompressible (in the weak sense) vector field with vanishing mean drift, then for every couple of points $\{x_0, y_0\} \subset \mathbb{R}^d$ and every $\varepsilon > 0$ there is a piecewise continuous control $u\colon \mathbb{R}^+ \to \mathbb{R}^d$ with $\|u\|_\infty \leq \varepsilon$ such that there is a trajectory of the system of ODEs* (4.8) *satisfying $x(0) = x_0$ and $x(T) = y_0$ for some $T > 0$.*

*Proof.* Fixed an $\varepsilon > 0$, by means of a convolution with an appropriate smooth approximate identity with compact support we may find a smooth $V_\varepsilon\colon \mathbb{R}^d \to \mathbb{R}^d$ satisfying $\|V - V_\varepsilon\|_\infty \leq \varepsilon/2$ (this is possible because $V$ is assumed to be uniformly continuous). Clearly, $V_\varepsilon$ is still incompressible and has vanishing mean drift. Since now $V_\varepsilon$ is smooth, we may apply Theorem 4.11 to find a piecewise continuous control $\tilde{u}\colon \mathbb{R}^+ \to \mathbb{R}$ with $\|\tilde{u}\|_\infty \leq \varepsilon/2$ such that the trajectory $x(\cdot)$ of the system

$$\dot{x} = V_\varepsilon(x) + \tilde{u}(t)$$

starting at $x_0 \in \mathbb{R}^d$ arrives at $x_1 \in \mathbb{R}^d$ in finite time. It suffices to take then $u(t) := V_\varepsilon(x(t)) - V(x(t)) + \tilde{u}(t)$. □

At last, we are able to prove the following version of the Pugh closing lemma which is the second principal result of the paper.

**Theorem 4.14.** *Let $V \in C^1(\mathbb{R}^d;\mathbb{R}^d) \cap \mathrm{Lip}(\mathbb{R}^d;\mathbb{R}^d)$ be a bounded incompressible vector field with uniformly continuous first derivatives and satisfying vanishing mean drift condition. Then for every $x_0 \in \mathbb{R}^d$ and every $\varepsilon > 0$ there is a $C^1$ vector field $Y_\varepsilon\colon \mathbb{R}^d \to \mathbb{R}^d$ with $\|Y_\varepsilon\|_{\mathrm{Lip}} \leq \varepsilon$ such that $x_0$ is a periodic (in particular, stationary) point of the system of ODEs*

$$\dot{x} = V(x) + Y_\varepsilon(x). \tag{4.9}$$

*Proof.* Fixed an $\varepsilon > 0$, by Theorem 4.8 we find a smooth vector field $W\colon \mathbb{R}^d \to \mathbb{R}^d$ with $\|W\|_\infty \leq \varepsilon/2$ such that every $x \in \mathbb{R}^d$ is nonwandering with respect to the flow defined by $V + W$. Observe that perturbation $W$ of the vector field $V$ is global and does not depend on the point $x_0$ which we want to make periodic. It suffices now to use Pugh's closing Lemma [10] to construct a local small perturbation $\tilde{W}$ of $V + W$ with $\|\tilde{W}\|_{\mathrm{Lip}} \leq \varepsilon/2$, so that $x_0$ becomes periodic with respect to the flow of $V + Y$, where $Y := W + \tilde{W}$, and observe that $\|Y\|_{\mathrm{Lip}} \leq \varepsilon$ as claimed. □

## Appendix A. Vanishing mean drift condition

We start with the following statement.

**Lemma A.1.** *Suppose $V\colon \mathbb{R}^d \to \mathbb{R}^d$ is a bounded locally Lipschitz vector field, and for each $\varepsilon > 0$ there is an $L_0 > 0$ such that for every $(d-1)$-dimensional box $Q$ of sidelength $L \geq L_0$ one has that the mean flux*

$$\frac{1}{L^{d-1}} \left| \int_Q V(x) \cdot n \, d\mathcal{H}^{d-1}(x) \right| \leq \varepsilon, \tag{A.1}$$

*where $n$ stands for a normal vector to the hyperplane containing $Q$ (we will say that $V$ has vanishing mean flux). Then $V$ has vanishing mean drift.*

*Vice versa, if $V$ has vanishing mean drift and is incompressible, then it satisfies the above property (i.e. has vanishing mean flux). In particular, for incompressible vector fields the having vanishing mean drift is equivalent to having vanishing mean flux.*

*Remark* A.2. It is worth observing that incompressibility condition is essential for a vector field $V$ having vanishing mean drift to have vanishing mean flux. In fact, the vector field $V(x,y) := (f(x),0)$ in $\mathbb{R}^2$, where $f$ is a smooth function with compact support in $\mathbb{R}$, clearly has vanishing mean drift, but not vanishing mean flux as can be seen by computing the flux of $V$ through one-dimensional segments $\{(x,0\} \times [-L/2, L/2]$ as $L \to \infty$; in fact, $V$ is in general not incompressible.

*Proof.* For a fixed $\varepsilon > 0$ let $L_0$ be such that the vector field (in fact, even not necessarily incompressible) $V$ satisfies (A.1). Then for every $i \in \{1,\dots,d\}$ and every $L > L_0$ one has

$$\begin{aligned}\left|\int_{x+[-L/2,L/2]^d} V_i(y)\,dy\right| &= \left|\int_{-L/2}^{L/2} dt \int_{(x+[-L/2,L/2]^d)\cap\{x_i=t\}} V_i(s)\,d\mathcal{H}^{d-1}(s)\right| \\ &\le \int_{-L/2}^{L/2} dt \left|\int_{(x+[-L/2,L/2]^d)\cap\{x_i=t\}} V_i(s)\,d\mathcal{H}^{d-1}(s)\right| \\ &\le L\int_{-L/2}^{L/2} \varepsilon L^{d-1}\,dt = \varepsilon L^d,\end{aligned}$$

i.e. $V$ has vanishing mean drift as claimed. The reverse statement for incompressible $V$ is lemma 3.1 from [5]. $\square$

It seems quite intuitive that the $(d-1)$-dimensional boxes in the vanishing mean flux condition of Lemma A.1 may be replaced by more general increasing sequences of sets. We give here only two particular examples to be used later.

*Example* A.3. If $V\colon \mathbb{R}^d \to \mathbb{R}^d$ is a bounded locally Lipschitz vector field with vanishing mean flux, then for each $\varepsilon > 0$ there is an $R_0 > 0$ such that for every $(d-1)$-dimensional ball $B$ of radius $R \ge R_0$ one has that the mean flux

$$\frac{1}{\mathcal{H}^{d-1}(B)}\left|\int_B V(x)\cdot n\,d\mathcal{H}^{d-1}(x)\right| \le \varepsilon, \tag{A.2}$$

where $n$ stands for a normal vector to the hyperplane containing $Q$. In fact, given an $\varepsilon > 0$, covering a unit $(d-1)$-dimensional ball $B$ with $N = N(\varepsilon) \in \mathbb{N}$ disjoint $(d-1)$-dimensional open boxes $Q_i$ so that $\mathcal{H}^{d-1}(B\setminus \cup_i Q_i) \le \mathcal{H}^{d-1}(\cup_i Q_i)/\|V\|_\infty$, and taking an $R_0 > 0$ such that

$$\left|\int_{RQ_i} V(x)\cdot n\,d\mathcal{H}^{d-1}(x)\right| \le \varepsilon\mathcal{H}^{d-1}(RQ_i) = \varepsilon R^{d-1}\mathcal{H}^{d-1}(Q_i)$$

for all $i = 1,\dots,N$ and all $R > R_0$, we get

$$\begin{aligned}\left|\int_{RB} V(x)\cdot n\,d\mathcal{H}^{d-1}(x)\right| &\le \left|\int_{\cup_i RQ_i} V(x)\cdot n\,d\mathcal{H}^{d-1}(x)\right| + \|V\|_\infty \mathcal{H}^{d-1}(RB\setminus \cup_i RQ_i) \\ &\le \varepsilon R^{d-1}\mathcal{H}^{d-1}(\cup_i Q_i) + R^{d-1}\mathcal{H}^{d-1}(B\setminus \cup_i Q_i) \\ &= R^{d-1}\mathcal{H}^{d-1}(B)\end{aligned}$$

for $R > R_0$ proving the claim.

*Example* A.4. For an $x \in \partial B_1(0)$ and $r \le 2$ denote $D_r(x) := \partial B_1(0) \cap B_r(x)$ (i.e. a ball in the natural inner metric of $\partial B_1(0)$). If $V\colon \mathbb{R}^d \to \mathbb{R}^d$ is a bounded incompressible locally Lipschitz vector field with vanishing mean drift (hence with vanishing mean flux by Lemma A.1), then for each $\varepsilon > 0$ there is an $R_0 > 0$ such that for every $R \ge R_0$ and every $x \in \partial B_1(0)$ one has that the mean flux

$$\frac{1}{\mathcal{H}^{d-1}(RD_r(x))}\left|\int_{RD_r(x)} V(y)\cdot n(y)\,d\mathcal{H}^{d-1}(y)\right| \le \varepsilon, \tag{A.3}$$

where $n(y)$ stands for the external unit normal to $\partial B_1(0)$ at $y$. In fact, given an $\varepsilon > 0$, consider the segment $\Omega$ of the unit ball $B_1(0)$ bounded by $D_r(x)$ and the $(d-1)$-dimensional ball $C$ in the section of $B_1(0)$ by the plane containing the relative boundary of $D_r(x)$ in $\partial B_1(0)$ (i.e. the set $\partial B_1(0) \cap \partial B_r(x)$). Letting $n$ stand for the external normal to the boundary of $\Omega$, we get

$$\left| \int_{RD_r(x)} V(y) \cdot n(y)\, d\mathcal{H}^{d-1}(y) \right| = \left| -\int_{RC} V(y) \cdot n(y)\, d\mathcal{H}^{d-1}(y) + \int_{R\Omega} \operatorname{div} V(y)\, dy \right|$$
$$= \left| \int_{RC} V(y) \cdot n(y)\, d\mathcal{H}^{d-1}(y) \right| \leq \varepsilon \mathcal{H}^{d-1}(D_r(x)) R^{d-1}$$

for all $R > R_0$, where $R_0$ is chosen (depending on $\varepsilon$) so that the latter inequality be satisfied (which is possible by Example A.3), and therefore

$$\frac{1}{\mathcal{H}^{d-1}}(RD_r(x)) \left| \int_{RD_r(x)} V(y) \cdot n(y)\, d\mathcal{H}^{d-1}(y) \right| \leq \varepsilon$$

as claimed.

**Lemma A.5.** *Suppose $V \in C^1(\mathbb{R}^d; \mathbb{R}^d)$ is a vector field with vanishing mean drift having uniformly continuous first derivatives. Then the latter also have vanishing mean drift.*

*Proof.* For an arbitrary $i = 1, \dots, d$ we have that

$$\left| \frac{V_j(x + te_j) - V_i(t)}{t} - V_{i,x_j}(x) \right| = \left| V_{i,x_j}(x + \theta e_j) - V_{i,x_j}(x) \right|$$

for some $\theta \in [0, t]$ (depending possibly on $x \in \mathbb{R}^d$), and hence given an $\varepsilon > 0$ we may choose a $t$ such that the above estimate does not exceed $\varepsilon/2$. Since $V_t^j := (V_j(x + te_j) - V_i(t))/t$ clearly has vanishing mean drift, then there is an $L_0 > 0$ (depending on $t$ which is fixed) such that for every $d$-dimensional box $Q \subset \mathbb{R}^d$ of side length $L > L_0$ one has

$$\left| \int_Q V_t^j(x)\, dx \right| \leq \varepsilon L^d / 2,$$

and therefore one has

$$\left| \int_Q V_{x_j}(x)\, dx \right| \leq \left| \int_Q V_t^j(x)\, dx \right| + \int_Q \left| V_t^j(x) - V_{i,x_j}(x) \right| dx \leq \varepsilon L^d/2 + \varepsilon L^d = \varepsilon L^d,$$

proving the claim. □

**Proposition A.6.** *Let $F \subset L^1(\partial B_1(0); \mathcal{H}^{d-1})$ be a compact family of functions and $V \colon \mathbb{R}^d \to \mathbb{R}^d$ be a bounded incompressible locally Lipschitz vector field with vanishing mean drift. Then*

$$\int_{\partial B_1(0)} f(x) x \cdot V(\alpha x)\, d\mathcal{H}^{d-1}(x) \to 0$$

*as $\alpha \to +\infty$, uniformly over $f \in F$, i.e. for every $\varepsilon > 0$ there exists an $\alpha_0 > 0$ such that*

$$\left| \int_{\partial B_1(0)} f(x) x \cdot V(\alpha x)\, d\mathcal{H}^{d-1}(x) \right| < \varepsilon$$

*for all $f \in F$, $\alpha \geq \alpha_0$.*

*Proof.* If not, there is a sequence $\{\alpha_n\} \subset \mathbb{R}$, $\lim_n \alpha_n = +\infty$, and $f_n \subset F$, such that

$$\left| \int_{\partial B_1(0)} f_n(x) x \cdot V(\alpha_n x)\, d\mathcal{H}^{d-1}(x) \right| \geq \varepsilon_0$$

for some $\varepsilon_0 > 0$ and all $n \in \mathbb{N}$. By eventually passing to a subsequence of $n$ (not relabeled) we may assume that $f_n \to f$ in $L^1(\partial B_1(0))$ as $n \to \infty$. But then

$$\begin{aligned}\left|\int_{\partial B_1(0)} f_n(x)x \cdot V(\alpha_n x)\, d\mathcal{H}^{d-1}(x)\right| &\leq \int_{\partial B_1(0)} |f_n(x) - f(x)| x \cdot V(\alpha_n x)\, d\mathcal{H}^{d-1}(x)\\ &\quad + \left|\int_{\partial B_1(0)} f(x)x \cdot V(\alpha_n x)\, d\mathcal{H}^{d-1}(x)\right|\\ &\leq \|V\|_\infty \|f_n - f\|_1\\ &\quad + \left|\int_{\partial B_1(0)} f(x)x \cdot V(\alpha_n x)\, d\mathcal{H}^{d-1}(x)\right| \to 0\end{aligned}$$

since the latter integral is vanishing by Lemma A.7, a contradiction. □

The following lemma has been used in the above proof.

**Lemma A.7.** *Let $f \in L^1(\partial B_1(0); \mathcal{H}^{d-1})$ and $V\colon \mathbb{R}^d \to \mathbb{R}^d$ be a bounded incompressible locally Lipschitz vector field with vanishing mean drift. Then*

$$\int_{\partial B_1(0)} f(x)x \cdot V(\alpha x)\, d\mathcal{H}^{d-1}(x) \to 0$$

*as $\alpha \to +\infty$.*

*Proof.* It is enough to proof the statement for $f$ from a family of functions having dense linear span in $L^1(\partial B_1(0); \mathcal{H}^{d-1})$, in particular, for $f$ just a characteristic function of the form $f = \mathbf{1}_{D_r(x)}$ for some $r \in (0, 2]$, $x \in \partial B_r(0)$. The claim for this case follows then from the change of variables

$$\int_{D_r(x)} y \cdot V(\alpha y)\, d\mathcal{H}^{d-1}(y) = \frac{1}{\alpha^{d-1}} \int_{\alpha D_r(x)} V(y) \cdot n(y)\, d\mathcal{H}^{d-1}(y),$$

where $n$ stands for the external normal to $\partial B_1(0)$, and from Example A.4. □

A.1. **Auxiliary computations.** We will also need the following technical assertions.

**Lemma A.8.** *Let $p \in ((d-1)/2, d/2)$, and $K\colon \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}$ be a function Lipschitz outside every $R$-neighborhood of the diagonal (for every $R > 0$), with*

$$K(x, y) = \frac{A(x, y)}{|x - y|^k}, \quad k < d,$$

*for some $A \in L^\infty(\mathbb{R}^d \times \mathbb{R}^d)$ and for all $(x, y) \in \mathbb{R}^d \times \mathbb{R}^d$. Suppose that $V : \mathbb{R}^d \to \mathbb{R}^d$ is a bounded incompressible vector field with vanishing mean drift. Then, with the notation $n(x) := x/|x|$ one has*

$$\Psi(x) := \int_{\mathbb{R}^d} K(n(x), y) \frac{y \cdot V(\alpha|x|y)}{(y^2 + |x|^{-2})^{p+1}}\, dy \to 0 \tag{A.4}$$

*as $\alpha \to +\infty$ uniformly in $x \in \bar{B}_1^c(0) \subset \mathbb{R}^d$.*

*Proof.* Without loss of generality (up to a small increase of $k$) we may assume $A$ to be continuous. Let $x \in \bar{B}_1^c(0) \subset \mathbb{R}^d$, choose an arbitrary $\varepsilon > 0$ and denote for brevity

$$C(p) := \sup_{y \in \mathbb{R}^d} \frac{|y|}{(|y|^2 + 1)^{p+1}}$$

(note that $C(p) < +\infty$ because $p > -1/2$). Consider the function $F\colon B_1(0) \to \mathbb{R}^d$ defined by

$$F(y) := K(n(x), y) - K(n(x), 0).$$

Clearly, $F(0) = 0$ and $F$ is Lipschitz in a neighborhood of zero, so that for some $\rho > 0$ one has in particular $|F(y)| \le C|y|$ for some $C > 0$ whenever $|y| \le \rho$. Therefore,

$$\begin{aligned}\left|\int_{B_\rho(0)} (K(n(x),y) - K(n(x),0))\,\frac{y\cdot V(\alpha|x|y)}{(y^2+|x|^{-2})^{p+1}}\,dy\right| \\ \le C(p)C\|V\|_\infty \int_{B_\rho(0)} \frac{|y|^2}{(|y|^2+|x|^{-2})^{p+1}}\,dy \\ \le C(p)C\|V\|_\infty \int_{B_\rho(0)} \frac{|y|^2}{|y|^{2p+1}}\,dy = C(p)Cd\omega_d\|V\|_\infty \int_0^\rho \frac{dr}{r^{2p-d+1}},\end{aligned}$$

and recalling that $p < d/2$ we conclude that one can choose a $\rho > 0$ so that

$$\left|\int_{B_\rho(0)} (K(n(x),y) - K(n(x),0))\,\frac{y\cdot V(\alpha|x|y)}{(|y|^2+|x|^{-2})^{p+1}}\,dy\right| \le \varepsilon/3.$$

But by Lemma A.9 one has

$$\begin{aligned}\int_{B_\rho(0)} (K(n(x),y) - K(n(x),0))\frac{y\cdot V(\alpha|x|y)}{(|y|^2+|x|^{-2})^{p+1}}\,dy \\ = \int_{B_\rho(0)} K(n(x),y)\frac{y\cdot V(\alpha|x|y)}{(|y|^2+|x|^{-2})^{p+1}}\,dy,\end{aligned}$$

and hence

$$\left|\int_{B_\rho(0)} K(n(x),y)\frac{y\cdot V(\alpha|x|y)}{(y^2+|x|^{-2})^{p+1}}\,dy\right| \le \varepsilon/3. \tag{A.5}$$

Choose now an $R \in (0, 1-\rho)$ such that

$$\int_{B_R(n(x))} \frac{dy}{|n(x)-y|^k} < \delta,$$

for some $\delta > 0$ to be chosen later (it is only here that we use the assumption on $k$). Denoting

$$K_R(n(x),y) := \begin{cases} K(n(x),y), & y \in B_R^c(n(x)),\\ A(n(x),y)/R^k, & y \in B_R(x),\end{cases}$$

we get that $(x,y) \mapsto K_R(n(x),y)$ is bounded and uniformly continuous in $\bar{B}_1^c(0)\times\mathbb{R}^d$. Now, recalling that $\rho < |y| < 1$ when $y \in B_R(n(x))$, we get

$$\begin{aligned}\left|\int_{B_\rho^c(0)} K(n(x),y)\frac{y\cdot V(\alpha|x|y)}{(|y|^2+|x|^{-2})^{p+1}}\,dy\right| \\ \le \left|\int_{B_\rho^c(0)\cap B_R^c(n(x))} K(n(x),y)\frac{y\cdot V(\alpha|x|y)}{(|y|^2+|x|^{-2})^{p+1}}\,dy\right| + \frac{\delta\|V\|_\infty\|A\|_\infty}{\rho^{2(p+1)}} \\ \le \left|\int_{B_\rho^c(0)} K_R(n(x),y)\frac{y\cdot V(\alpha|x|y)}{(y^2+|x|^{-2})^{p+1}}\,dy\right| \\ + \left|\int_{B_R(0)} \frac{A(n(x),y)}{R^k}\,\frac{y\cdot V(\alpha|x|y)}{(y^2+|x|^{-2})^{p+1}}\,dy\right| + \frac{\delta\|V\|_\infty\|A\|_\infty}{\rho^{2(p+1)}} \\ \le \left|\int_{B_\rho^c(0)} K_R(n(x),y)\frac{y\cdot V(\alpha|x|y)}{(y^2+|x|^{-2})^{p+1}}\,dy\right| \\ + \omega_d R^{d-k}\|V\|_\infty\|A\|_\infty\frac{1}{\rho^{2(p+1)}} + \frac{\delta\|V\|_\infty\|A\|_\infty}{\rho^{2(p+1)}},\end{aligned}$$

so that choosing $\delta$ (depending on $\rho$ and $\varepsilon$) and $R$ (depending on $\delta$, $\rho$ and $\varepsilon$) sufficiently small we will have

$$\begin{aligned}\left|\int_{B^c_\rho(0)} K(n(x),y)\ \frac{y\cdot V(\alpha|x|y)}{(y^2+|x|^{-2})^{p+1}}\,dy\right| \\ \leq \left|\int_{B^c_\rho(0)} K_R(n(x),y)\frac{y\cdot V(\alpha|x|y)}{(y^2+|x|^{-2})^{p+1}}\,dy\right| + \varepsilon/3.\end{aligned} \tag{A.6}$$

Finally,

$$\begin{aligned}&\int_{B^c_\rho(0)} K_R(n(x),y)\frac{y\cdot V(\alpha|x|y)}{(y^2+|x|^{-2})^{p+1}}\,dy \\ &\qquad = \int_\rho^{+\infty}\frac{r^d\,dr}{(r^2+|x|^{-2})^{p+1}}\int_{\partial B_1(0)} K_R(n(x),rs)s\cdot V(\alpha|x|rs)\,d\mathcal{H}^{d-1}(s).\end{aligned} \tag{A.7}$$

Recalling that the family of functions $\{s\in\partial B_1(0)\mapsto K_R(n(x),rs)\colon r\geq\rho, x\in B^c_\rho(0)\}$ on $\partial B_1(0)$ is bounded and equicontinuous, we get from Proposition A.6 that

$$\int_{\partial B_1(0)} K_R(n(x),rs)s\cdot V(\beta s)\,d\mathcal{H}^{d-1}(s)\to 0$$

uniformly in $x\in\bar{B}^c_1(0)$ and $r\geq\rho$ as $\beta\to+\infty$, so that in particular

$$\int_{\partial B_1(0)} K_R(n(x),rs)s\cdot V(\alpha|x|rs)\,d\mathcal{H}^{d-1}(s)\to 0$$

uniformly in $x\in\bar{B}^c_1(0)$ and $r\geq\rho$ as $\alpha\to+\infty$. Thus, observing that

$$\frac{r^d}{(r^2+|x|^{-2})^{p+1}}\leq\frac{r^d}{(r^2+1)^{p+1}}$$

for $x\in\bar{B}^c_1(0)$ and $r\geq 0$, the function on the right-hand side of the above inequality being integrable over $(0,+\infty)$ (because $p>(d-1)/2$), we get from (A.7) the estimate

$$\left|\int_{B^c_\rho(0)} K_R(n(x),y)\ \frac{y\cdot V(\alpha|x|y)}{(y^2+|x|^{-2})^{p+1}}\,dy\right|\ \leq\varepsilon/3, \tag{A.8}$$

once $\alpha$ is sufficiently large independent on $x$. Combining the estimates (A.5), (A.6) and (A.8) we arrive from (A.4) to $|\Psi(x)|\leq\varepsilon$ for sufficiently large $\alpha$ independent on $x\in B_1(0)^c$ as claimed. □

The following lemma has been used in the above proof.

**Lemma A.9.** *For an incompressible vector field $V$ one has*

$$\int_{B_\rho(0)}\frac{y\cdot V(\alpha|x|y)}{(y^2+|x|^{-2})^p}=0.$$

*for every $\rho>0$.*

*Proof.* We write

$$\int_{B_\rho(0)}\frac{y\cdot V(\alpha|x|y)}{(y^2+|x|^{-2})^p}=\int_0^\rho\frac{r^k\,dr}{(r^2+|x|^{-2})^p}\int_{\partial B_1(0)} s\cdot V(\alpha|x|rs)\,d\mathcal{H}^{d-1}(s),$$

and recalling that $V$ is divergence-free, and hence so is the vector field $y\mapsto V(\alpha|x|ry)$ (with $\alpha$, $r$ and $x$ fixed), one has

$$\int_{\partial B_1(0)} s\cdot V(\alpha|x|rs)\,d\mathcal{H}^{d-1}(s)=0$$

implying the thesis. □

At last we need the following computation.

**Lemma A.10.** *Let $p > (d-1)/2$, and $K\colon \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}$ be a function uniformly continuous outside every $R$-neighborhood of the diagonal (for every $R > 0$), with*

$$K(x,y) = \frac{A(x,y)}{|x-y|^k}, \quad k < d,$$

*for some $A \in L^\infty(\mathbb{R}^d \times \mathbb{R}^d)$ and for all $(x,y) \in \mathbb{R}^d \times \mathbb{R}^d$. Suppose that $V\colon \mathbb{R}^d \to \mathbb{R}^d$ is a bounded incompressible vector field with vanishing mean drift. Then*

$$\Phi(x) := \int_{\mathbb{R}^d} K(x,y) \frac{y \cdot V(\alpha y)}{(|y|^2+1)^{p+1}}\, dy \to 0 \tag{A.9}$$

*as $\alpha \to +\infty$ uniformly in $x \in \bar{B}_1(0) \subset \mathbb{R}^d$.*

*Proof.* Consider an arbitrary $x \in \bar{B}_1(0) \subset \mathbb{R}^d$ and an arbitrary $\varepsilon > 0$. We retain the notation on the constant $C(p)$ used in the proof of Lemma A.8. Finding a $\rho \in [0,1]$ such that

$$\int_{B_\rho(x)} \frac{dy}{|x-y|^k} < \delta, \tag{A.10}$$

for some $\delta > 0$ to be chosen later, we get

$$\begin{aligned} |\Phi(x)| &\leq \left( \|A\|_\infty \omega_d \rho^d \delta C(p) \|V\|_\infty + |I(\alpha,\rho)| \right), \quad \text{where} \\ I(\alpha,\rho) &:= \int_{B_\rho^c(x)} K(x,y) \frac{y \cdot V(\alpha y)}{(y^2+1)^{p+1}}\, dy. \end{aligned} \tag{A.11}$$

Denoting

$$K_\rho(x,y) := \left\{ \begin{array}{cc} K(x,y), & y \in B_\rho^c(x), \\ A(x,y)/\rho^k, & y \in B_\rho(x), \end{array} \right. \tag{A.12}$$

we get that $K_\rho$ is uniformly continuous and bounded in $\mathbb{R}^d \times \mathbb{R}^d$. We have now

$$\begin{aligned} |I(\alpha,\rho)| &\leq \left| \int_{\mathbb{R}^d} K_\rho(x,y) \frac{y \cdot V(\alpha y)}{(y^2+1)^{p+1}}\, dy \right| + \left| \int_{B_\rho(x)} \frac{A(x,y)}{\rho^k} \frac{y \cdot V(\alpha y)}{(y^2+1)^{p+1}}\, dy \right| \\ &\leq \left| \int_{\mathbb{R}^d} K_\rho(x,y) \frac{y \cdot V(\alpha y)}{(y^2+1)^{p+1}}\, dy \right| + \|A\|_\infty C(p) \|V\|_\infty \omega_d \rho^{d-k}. \end{aligned} \tag{A.13}$$

Recall that

$$\begin{aligned} &\int_{\mathbb{R}^d} K_\rho(x,y) \frac{y \cdot V(\alpha y)}{(y^2+1)^{p+1}}\, dy \\ &\qquad = \int_0^{+\infty} \frac{r^d}{(r^2+1)^{p+1}}\, dr \int_{\partial B_1(0)} K_\rho(x,rs) s \cdot V(\alpha r s)\, d\mathcal{H}^{d-1}(s). \end{aligned}$$

We choose $\delta > 0$ and $\rho > 0$ to be sufficiently small (depending on $\varepsilon$) so as to have

$$\|A\|_\infty \omega_d \rho^d \delta C(p) \|V\|_\infty < \varepsilon/5, \quad \|A\|_\infty C(p) \|V\|_\infty \omega_d \rho^{d-k} < \varepsilon/5,$$

so that in view of (A.13) the estimate (A.11) becomes

$$|\Phi(x)| \leq 2\varepsilon/5 + \left| \int_{\mathbb{R}^d} K_\rho(x,y) \frac{y \cdot V(\alpha y)}{(y^2+1)^{p+1}}\, dy \right|. \tag{A.14}$$

We choose now $r_0 > 0$ and $r_1 > r_0$ (depending on $\varepsilon$ and $\rho$) such that

$$\int_0^{r_0} \frac{r^d}{(r^2+1)^{p+1}}\,dr \int_{\partial B_1(0)} |K_\rho(x, rs)s \cdot V(\alpha rs)|\, d\mathcal{H}^{d-1}(s) \le \|K_\rho\|_\infty d\omega_d \|V\|_\infty \int_0^{r_0} \frac{r^d}{(r^2+1)^{p+1}}\,dr \le \varepsilon/5,$$

$$\int_{r_1}^{+\infty} \frac{r^d}{(r^2+1)^{p+1}}\,dr \int_{\partial B_1(0)} |\varphi_\rho(x, rs)s \cdot V(\alpha rs)|\, d\mathcal{H}^{d-1}(s) \le \|K_\rho\|_\infty d\omega_d \|V\|_\infty \int_{r_1}^{+\infty} \frac{r^d}{(r^2+1)^{p+1}}\,dr \le \varepsilon/5,$$

(recall that we assumed $p > (d-1)/2$). Since by Proposition A.6 one has

$$\int_{\partial B_1(0)} K_\rho(x, rs)s \cdot V(\alpha rs)\, d\mathcal{H}^{d-1}(s) \to 0$$

uniformly in $x \in B_1(0)$ as $\alpha \to +\infty$ for all $r \in [r_0, r_1]$, we get

$$\left| \int_{\mathbb{R}^d} K_\rho(x,y) \frac{y \cdot V(\alpha y)}{(y^2+1)^{p+1}}\,dy \right| \le 3\varepsilon/5,$$

for all sufficiently large $\alpha > 0$ and hence, by (A.14), $|\Phi(x)| \le \varepsilon$ for such $\alpha$ as claimed. □

## Appendix B. Estimates on the corrector

B.1. **Uniform estimates.** We show that with the appropriate choice of the parameters the vector field $W$ defined by the formula (4.6) can be made arbitrarily small in supremum norm, namely, that the following lemma is valid.

**Lemma B.1.** *Let $p \in ((d-1)/2, d/2)$. Suppose that $V \in \mathrm{Lip}_{loc}(\mathbb{R}^d; \mathbb{R}^d)$ is a bounded incompressible vector field with vanishing mean drift. Then, given an $\varepsilon > 0$, there is an $\bar\alpha = \bar\alpha(p, \varepsilon)$ such that*

$$\|W\|_\infty \le \varepsilon \quad \text{for every } \alpha > \bar\alpha.$$

*Proof.* We calculate

$$\begin{aligned} W_\alpha(x) := W(\alpha x) &= 2pc_d \alpha^{2p}(x^2+1)^p \int_{\mathbb{R}^d} \frac{\alpha x - y}{|\alpha x - y|^d} \frac{y \cdot V(y)}{(y^2+\alpha^2)^{p+1}}\,dy \\ &= 2pc_d (x^2+1)^p \int_{\mathbb{R}^d} \frac{x-y}{|x-y|^d} \frac{y \cdot V(\alpha y)}{(y^2+1)^{p+1}}\,dy, \end{aligned} \tag{B.1}$$

and note that $\|W\|_\infty = \|W_\alpha\|_\infty$, so that we may estimate the latter.

Case 1: $|x| \le 1$. The desired estimate follows immediately from Lemma A.10 applied with

$$K(x,y) := \frac{x_i - y_i}{|x-y|^d}$$

(i.e. $A(x,y) := (x_i - y_i)/|x-y)|$, $k = d-1$) for every $i = 1, \ldots, d$.

Case 2: $|x| > 1$. In this case, denoting $n(x) := x/|x|$, we further calculate

$$\begin{aligned} W_\alpha(x) &= 2pc_d(1+|x|^{-2})^p |x|^{2p} \int_{\mathbb{R}^d} \frac{n(x)|x| - y}{|n(x)|x| - y|^d} \frac{y \cdot V(\alpha y)}{(y^2+1)^{p+1}}\,dy \\ &= 2pc_d(1+|x|^{-2})^p \int_{\mathbb{R}^d} \frac{n(x) - y}{|n(x) - y|^d} \frac{y \cdot V(\alpha |x| y)}{(y^2+|x|^{-2})^{p+1}}\,dy, \end{aligned} \tag{B.2}$$

so that the necessary estimate follows immediately from Lemma A.8 applied with the same data as in Case 1. □

**Corollary B.2.** *Under conditions of Lemma B.1 one has that for every $\varepsilon > 0$, there is an $\bar\alpha = \bar\alpha(p,\varepsilon)$ such that*

$$\|\operatorname{div} W\|_\infty \le \varepsilon \quad \text{for every } \alpha > \bar\alpha.$$

*Proof.* From (4.4) one has

$$\psi \operatorname{div} W = \operatorname{div}(\psi W) - \nabla\psi\cdot W = -\nabla\psi\cdot V - \nabla\psi\cdot W,$$

so that

$$\operatorname{div} W = -\nabla\log\psi\cdot(V+W) = 2p\frac{|x|}{|x|^2+\alpha^2}\frac{x}{|x|}\cdot(V+W).$$

This gives

$$|\operatorname{div} W| \le \frac{2p}{\alpha}\frac{|x/\alpha|}{|x/\alpha|^2+1}(\|V\|_\infty + \|W\|_\infty) \le \frac{p}{\alpha}(\|V\|_\infty + \|W\|_\infty),$$

since $|t|/(t^2+1)\le 1/2$, concluding the proof. □

B.2. **Estimates on the derivatives.** We show now that if $V$ is sufficiently smooth, then the derivatives of $W$ can also be made as small as desired with the appropriate choice of parameters.

**Lemma B.3.** *Let $p \in ((d-1)/2, d/2)$. Assume that $V \in \operatorname{Lip}(\mathbb{R}^d;\mathbb{R}^d)$ is incompressible vector field with vanishing mean drift, and all $V_{x_j}$, $j = 1,\dots,d$, are locally Lipschitz still having vanishing mean drift. Given an $\varepsilon > 0$, there is an $\bar\alpha = \bar\alpha(p,\varepsilon)$ such that*

$$\|W_{x_j}\|_\infty \le \varepsilon \quad \text{for every } \alpha > \bar\alpha$$

*for every $j = 1,\dots,d$.*

*Proof.* Fix an arbitrary $j \in \{1,\dots,d\}$. Rewriting (4.6) in the form

$$W(x) = 2p(-1)^d c_d(|x|^2+\alpha^2)^p \int_{\mathbb{R}^d} \frac{z}{|z|^d}\frac{(x-z)\cdot V(x-z)}{(|x-z|^2+\alpha^2)^{p+1}}\,dz,$$

we get the following relationships

$$W_{x_j}(x) = Z_1(x) + Z_2(x) + Z_3(x), \quad \text{where} \tag{B.3}$$

$$\begin{aligned} Z_1(x) &= 4p^2(-1)^d c_d x_j(x^2+\alpha^2)^{p-1}\int_{\mathbb{R}^d}\frac{z}{|z|^d}\frac{(x-z)\cdot V(x-z)}{((x-z)^2+\alpha^2)^p}\,dz \\ &= 4p^2 c_d x_j (x^2+\alpha^2)^{p-1}\textstyle\int_{\mathbb{R}^d}\frac{x-y}{|x-y|^d}\frac{y\cdot V(y)}{(y^2+\alpha^2)^p}\,dy \\ &= \frac{2px_j}{x^2+\alpha^2}W(x), \end{aligned} \tag{B.4}$$

$$\begin{aligned} Z_2(x) &= 2p(-1)^d c_d(x^2+\alpha^2)^p\int_{\mathbb{R}^d}\frac{z}{|z|^d}\frac{(x-z)\cdot V_{x_j}(x-z)}{((x-z)^2+\alpha^2)^{p+1}}\,dz \\ &= 2pc_d(x^2+\alpha^2)^p\int_{\mathbb{R}^d}\frac{x-y}{|x-y|^d}\frac{y\cdot V_{y_j}(y)}{(y^2+\alpha^2)^{p+1}}\,dy, \end{aligned} \tag{B.5}$$

$$\begin{aligned} Z_3(x) &= 2p(-1)^d c_d(x^2+\alpha^2)^p\int_{\mathbb{R}^d}\frac{z}{|z|^d}V(x-z)\cdot\frac{\partial}{\partial x_i}\frac{x-z}{((x-z)^2+\alpha^2)^{p+1}}\,dz = \\ &= 2pc_d(x^2+\alpha^2)^p\int_{\mathbb{R}^d}\frac{x-y}{|x-y|^d}V(y)\cdot\frac{\partial}{\partial y_i}\frac{y}{(y^2+\alpha^2)^{p+1}}\,dy. \end{aligned} \tag{B.6}$$

We now estimate separately the three terms $Z_1$, $Z_2$ and $Z_3$.

Estimate of $Z_1$. Since the function $x \mapsto x_j/(x^2+\alpha^2)$ is uniformly bounded over $\mathbb{R}^d$, then from (B.4) by Lemma B.1 we get that $\|Z_1\|_\infty \le \varepsilon/3$ once $\alpha$ is sufficiently large (depending on $p$ and $\varepsilon$).

Estimate of $Z_2$. From (B.5) we see that the expression for $Z_2$ is exactly the same as the definition (4.6) of $W$, but with $V_{x_j}$ instead of $V$. Since $V_{x_j}$ is still bounded, incompressible and has vanishing mean drift, then again by Lemma B.1 we get that $\|Z_2\|_\infty \le \varepsilon/3$ once $\alpha$ is sufficiently large (depending on $p$ and $\varepsilon$).

Estimate of $Z_3$. We let

$$Z_{3,\alpha}(x) := Z_3(\alpha x) = \frac{2pc_d}{\alpha}(x^2+1)^p \int_{\mathbb{R}^d} \frac{x-y}{|x-y|^d} V(\alpha y) \cdot \frac{\partial}{\partial y_j} \frac{y}{(y^2+1)^{p+1}}\, dy, \tag{B.7}$$

and note that $\|Z_3\|_\infty = \|Z_{3,\alpha}\|_\infty$, so that we may estimate the latter. As in the proof of Lemma B.1 we separate this estimate in two cases.

*Case* $|x| \le 1$. In this case we repeat line by line the proof of Lemma A.10, but with the vector field $\frac{\partial}{\partial y_j}\frac{y}{(y^2+1)^{p+1}}$ in place of just $y/(y^2+1)^{p+1}$, observing that this vector field is still uniformly bounded by a constant depending only on $p$ that we will call $C(p)$. Then choosing an arbitrary $\delta > 0$ and a $\rho > 0$ so as to satisfy (A.10), instead of (A.11) and (A.13) we will get, with the notation of $\varphi_\rho$ defined in (A.12), the estimate

$$\begin{aligned} |Z_{3,\alpha}(x)| \le{}& \frac{2pc_d 2^p}{\alpha}\delta C(p)\|V\|_\infty + \frac{2pc_d 2^p}{\alpha} C(p)\|V\|_\infty \omega_d \rho \\ &+ \frac{2pc_d 2^p}{\alpha}\left| \int_{\mathbb{R}^d} \varphi_\rho(x,y) V(\alpha y) \cdot \frac{\partial}{\partial y_j}\frac{y}{(y^2+1)^{p+1}}\, dy \right|. \end{aligned} \tag{B.8}$$

Writing

$$\begin{aligned} &\int_{\mathbb{R}^d} \varphi_\rho(x,y) V(\alpha y) \cdot \frac{\partial}{\partial y_j}\frac{y}{(y^2+1)^{p+1}}\, dy \\ &\quad= \int_0^{+\infty} \frac{r^{d-1}}{(r^2+1)^{p+1}}\, dr \int_{\partial B_1(0)} \varphi_\rho(x, rs) V_j(\alpha r s)\, d\mathcal{H}^{d-1}(s) \\ &\quad\quad - 2(p+1)\int_0^{+\infty} \frac{r^{d+1}}{(r^2+1)^{p+2}}\, dr \int_{\partial B_1(0)} \varphi_\rho(x, rs) s_j s \cdot V(\alpha r s)\, d\mathcal{H}^{d-1}(s), \end{aligned} \tag{B.9}$$

and estimating

$$\begin{aligned} &\left| \int_0^{+\infty} \frac{r^{d-1}}{(r^2+1)^{p+1}}\, dr \int_{\partial B_1(0)} \varphi_\rho(x, rs) V_j(\alpha r s)\, d\mathcal{H}^{d-1}(s) \right| \\ &\qquad\qquad \le d\omega_d \frac{1}{\rho^{d-1}}\|V\|_\infty \int_0^{+\infty} \frac{r^{d-1}}{(r^2+1)^{p+1}}\, dr, \\ &\left| \int_0^{+\infty} \frac{r^{d+1}}{(r^2+1)^{p+2}}\, dr \int_{\partial B_1(0)} \varphi_\rho(x, rs) s_j s \cdot V(\alpha r s)\, d\mathcal{H}^{d-1}(s) \right| \\ &\qquad\qquad \le d\omega_d \frac{1}{\rho^{d-1}}\|V\|_\infty \int_0^{+\infty} \frac{r^{d+1}}{(r^2+1)^{p+2}}\, dr, \end{aligned}$$

the integrals on the right-hand sides of the above inequalities being convergent (because $p > d/2 - 1$), we get that the integral on the left-hand side of (B.9) is estimated by a constant depending only on $d$, $p$, $\|V\|_\infty$ and $\rho$. Recalling that $\alpha$ is in the denominator in the right-hand side of (B.8), we get from (B.8) together with (B.9) that $|Z_{3,\alpha}(x)| \le \varepsilon/3$ for $\alpha$ sufficiently large (depending only on $p$ and $\varepsilon$).

*Case* $|x| > 1$. We rewrite

$$Z_{3,\alpha}(x) = \frac{2pc_d}{\alpha|x|}(1+|x|^{-2})^p \int_{\mathbb{R}^d} \frac{n(x)-y}{|n(x)-y|^d} V(\alpha|x|y) \cdot \frac{\partial}{\partial y_j}\frac{y}{(y^2+1)^{p+1}}\, dy, \tag{B.10}$$

where $n(x) := x/|x|$. Let now

$$\Lambda_1(x) := \frac{2pc_d}{\alpha|x|}(1+|x|^{-2})^p \int_{\mathbb{R}^d} \left( \frac{n(x)-y}{|n(x)-y|^d} - \frac{n(x)-y}{|n(x)-y|^{d-1}} \right) V(\alpha|x|y) \cdot \frac{\partial}{\partial y_j} \frac{y}{(y^2+|x|^{-2})^{p+1}}\, dy$$

$$\Lambda_2(x) := \frac{2pc_d}{\alpha|x|}(1+|x|^{-2})^p \int_{\mathbb{R}^d} \frac{n(x)-y}{|n(x)-y|^{d-1}} V(\alpha|x|y) \cdot \frac{\partial}{\partial y_j} \frac{y}{(y^2+|x|^{-2})^{p+1}}\, dy,$$

so that $Z_{3,\alpha}(x) = \Lambda_1(x) + \Lambda_2(x)$. We estimate the terms $\Lambda_1$ and $\Lambda_2$ separately.

To estimate $\Lambda_1$, consider the function $F\colon B_1(0) \to \mathbb{R}^d$ defined by

$$F(y) := \frac{n(x)-y}{|n(x)-y|^d} - \frac{n(x)-y}{|n(x)-y|^{d-1}}.$$

Clearly, $F(0) = 0$ and $F$ is smooth, so that $|F(y)| \le C|y|$ for some $C > 0$ whenever $|y| \le \rho$, for some $\rho > 0$. Therefore, one gets

$$\begin{aligned} |\Lambda_1(x)| &\le \frac{2pc_d}{\alpha} C\|V\|_\infty \int_{B_\rho(0)} |y| \cdot \left| \frac{\partial}{\partial y_j} \frac{y}{(y^2+|x|^{-2})^{p+1}} \right| dy + \frac{2pc_d}{\alpha} |J_\alpha|, \\ &\le \frac{2pc_d}{\alpha} C\|V\|_\infty C(p)\omega_d \rho^{d+1} + \frac{2pc_d}{\alpha} |J_\alpha|, \end{aligned} \tag{B.11}$$

where

$$J_\alpha := \int_{B^c_\rho(0)} f(n(x),y) V(\alpha|x|y) \cdot \frac{\partial}{\partial y_j} \frac{y}{(y^2+|x|^{-2})^{p+1}}\, dy,$$

$$f(x,y) := \frac{x-y}{|x-y|^d} - \frac{x-y}{|x-y|^{d-1}}.$$

To estimate $J_\alpha$, we write

$$\begin{aligned} J_\alpha &= \int_{B^c_\rho(0)} f(n(x),y) V(\alpha|x|y) \cdot \frac{\partial}{\partial y_j} \frac{y}{(y^2+|x|^{-2})^{p+1}}\, dy \\ &= \int_{B^c_\rho(0) \cap B^c_R(n(x))} \eta_R(n(x),y) V(\alpha|x|y) \cdot \frac{\partial}{\partial y_j} \frac{y}{(y^2+|x|^{-2})^{p+1}}\, dy \\ &\quad + \int_{B_R(n(x))} f(n(x),y) V(\alpha|x|y) \cdot \frac{\partial}{\partial y_j} \frac{y}{(y^2+|x|^{-2})^{p+1}}\, dy \\ &= \int_\rho^{+\infty} \frac{r^{d-1}}{(r^2+|x|^{-2})^{p+1}}\, dr \int_{\partial B_1(0)} \eta_R(n(x),rs) V_j(\alpha rs)\, d\mathcal{H}^{d-1}(s) \\ &\quad - 2(p+1) \int_\rho^{+\infty} \frac{r^{d+1}}{(r^2+|x|^{-2})^{p+2}}\, dr \\ &\qquad \int_{\partial B_1(0)} \eta_R(n(x),rs) s_j s \cdot V(\alpha rs)\, d\mathcal{H}^{d-1}(s) \\ &\quad + \int_{B_R(n(x))} f(n(x),y) V(\alpha|x|y) \cdot \frac{\partial}{\partial y_j} \frac{y}{(y^2+|x|^{-2})^{p+1}}\, dy, \end{aligned} \tag{B.12}$$

where $R \in (0, 1-\rho)$ will be chosen later, and

$$\eta_R(x,y) := \begin{cases} \frac{x-y}{|x-y|^d} - \frac{x-y}{|x-y|^{d-1}}, & y \in B^c_R(x), \\ \frac{x-y}{R^d} - \frac{x-y}{R^{d-1}}, & y \in B_R(x). \end{cases}$$

Since for $y \in B_R(n(x))$ one has

$$\begin{aligned} \left| \frac{\partial}{\partial y_j} \frac{y_i}{(y^2+|x|^{-2})^{p+1}} \right| &= \frac{|\delta_{ij} - 2(p+1)y_i y_j|}{y^2+|x|^{-2})^{p+2}} \\ &\le \frac{1+2(p+1)R^2}{|y|^{2(p+2)}} \le C_R := \frac{1+2(p+1)R^2}{|1-R|^{2(p+2)}}, \end{aligned}$$

choosing $R$ so that

$$\int_{B_R(n(x)} |f(n(x),y)|\,dy < \delta$$

for some $\delta > 0$, we get

$$\left|\int_{B_R(n(x))} f(n(x),y)V(\alpha|x|y)\cdot \frac{\partial}{\partial y_j}\frac{y}{(y^2+|x|^{-2})^{p+1}}\,dy\right| \le \|V\|C_R\delta. \tag{B.13}$$

But

$$\left|\int_\rho^{+\infty} \frac{r^{d-1}}{(r^2+|x|^{-2})^{p+1}}\,dr \int_{\partial B_1(0)} \eta_R(n(x),rs)V_j(\alpha rs)\,d\mathcal{H}^{d-1}(s)\right|$$
$$\le \|V\|_\infty\|\eta_R\|_\infty d\omega_d \int_\rho^{+\infty} \frac{r^{d-1}}{r^{2(p+1)}}\,dr,$$
$$\left|\int_\rho^{+\infty} \frac{r^{d+1}}{(r^2+|x|^{-2})^{p+2}}\,dr \int_{\partial B_1(0)} \eta_R(n(x),rs)s_j s\cdot V(\alpha rs)\,d\mathcal{H}^{d-1}(s)\right|$$
$$\le \|V\|_\infty\|\eta_R\|_\infty d\omega_d \int_\rho^{+\infty} \frac{r^{d+1}}{r^{2(p+2)}}\,dr,$$

the integrals on the right-hand side of the above inequalities being convergent (because $p > d/2-1$), so that combining this with (B.13) we get from (B.12) that $|J_\alpha|$ is bounded by a constant independent on $\alpha$ (hence depending only on $p$, $d$, $\|V\|_\infty$). Therefore, from (B.11) we have that $|\Lambda_1(x)| \le \varepsilon/6$ for $\alpha$ sufficiently large (depending on $p$ and $\varepsilon$).

Finally, it remains to estimate $\Lambda_2$. To this aim integrating by parts the expression for $\Lambda_2$, we get

$$\begin{aligned}
\Lambda_2(x) &= -\frac{2pc_d}{\alpha|x|}(1+|x|^{-2})^p \int_{\mathbb{R}^d} \frac{\partial}{\partial y_j}\left(\frac{n(x)-y}{|n(x)-y|^{d-1}}V(\alpha|x|y)\right)\cdot\frac{y\,dy}{(y^2+|x|^{-2})^{p+1}}\\
&= -\frac{2pc_d}{\alpha|x|}(1+|x|^{-2})^p \int_{\mathbb{R}^d} \frac{\partial}{\partial y_j}\left(\frac{n(x)-y}{|n(x)-y|^{d-1}}\right)V(\alpha|x|y)\cdot\frac{y\,dy}{(y^2+|x|^{-2})^{p+1}}\\
&\quad - 2pc_d(1+|x|^{-2})^p \int_{\mathbb{R}^d} \frac{n(x)-y}{|n(x)-y|^{d-1}}V_{y_j}(\alpha|x|y)\cdot\frac{y\,dy}{(y^2+|x|^{-2})^{p+1}}\\
&= \frac{2pc_d}{\alpha|x|}(1+|x|^{-2})^p \int_{\mathbb{R}^d} \frac{e_j}{|n(x)-y|^{d-1}}V(\alpha|x|y)\cdot\frac{y\,dy}{(y^2+|x|^{-2})^{p+1}}\\
&\quad - \frac{2pc_d}{\alpha|x|}(1+|x|^{-2})^p\\
&\qquad \int_{\mathbb{R}^d}\left(\frac{n(x)-y}{|n(x)-y|^{d}}(n_j(x)-y_j)\right)V(\alpha|x|y)\cdot\frac{y\,dy}{(y^2+|x|^{-2})^{p+1}}\\
&\quad - 2pc_d(1+|x|^{-2})^p \int_{\mathbb{R}^d} \frac{n(x)-y}{|n(x)-y|^{d-1}}V_{y_j}(\alpha|x|y)\cdot\frac{y\,dy}{(y^2+|x|^{-2})^{p+1}},
\end{aligned}$$

and the desired estimate for $\Lambda_2$ follows from Lemma A.8 applied to each of the three integrals in the right-hand side of the above equality. □

## Appendix C. Auxiliary lemmata

**Lemma C.1.** *If $T\colon X \to X$ is continuous, has an invariant measure $\mu$, and $T(\operatorname{supp}\mu)$ is closed (in particular, this is true if $\mu$ has compact support). Then $\operatorname{supp}\mu$ is invariant for $T$.*

*Proof.* Denoting for brevity $M := \operatorname{supp}\mu$, we have that $T^{-1}(M)$ is closed and $\mu((T^{-1}(M))^c) = \mu(T^{-1}(M^c)) = 0$, in other words, $\mu$ is concentrated on $T^{-1}(M)$, which implies $M \subset T^{-1}(M)$, thus $T(M) \subset M$. On the other hand,

$$\begin{aligned}\mu((T(M))^c) &= (T_\# \mu)((T(M))^c) \\ &= \mu(T^{-1}((T(M))^c) = \mu((T^{-1}(T(M))^c) \leq \mu(M^c) = 0,\end{aligned}$$

the inequaity in the above chain being due to $M \subset T^{-1}(T(M))$. Thus $\mu$ is also concentrated on $T(M)$ and since $T(M)$ is also closed, we have $M \subset T(M)$ concluding the proof. □

Note that continuity of $T$ in the above Lemma is essential.

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