Regularity results for vectorial minimizers of a class of degenerate convex integrals

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Abstract

We establish the higher differentiability and the higher integrability for the gradient of vectorial minimizers of integral functionals with (p, q)-growth conditions. We assume that the non-homogeneous densities are uniformly convex and have a radial structure, with respect to the gradient variable, only at infinity. The results are obtained under a possibly discontinuous dependence on the spatial variable of the integrand.

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1 Introduction

In this paper we study the regularity of vectorial local minimizers of functionals of the form

$$\mathcal{F}(u;\Omega) := \int_\Omega f(x, Du)\,dx, \tag{1.1}$$

where $\Omega \subset \mathbb{R}^n$, $n > 2$, is a bounded open set, $u : \Omega \to \mathbb{R}^N$, $N > 1$, is a Sobolev map and $f : \Omega \times \mathbb{R}^{nN} \to [0, +\infty)$ is a Carathéodory function, convex and of class C^2 with respect to the second variable. The main features of the energy densities $f(x, \xi)$ considered here are the following:

- they satisfy the so-called (p, q)-*growth conditions*
- they are uniformly convex only for large values of $|\xi|$
- the partial map $x \to f(x, \xi)$ is possibly discontinuous.

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The interest in the study of non autonomous degenerate convex functionals relies on the fact that they are strictly connected with the study of optimal transport problems with congestion effects (for more details we refer to [2], [3], [4]). Actually, the Euler- Lagrange equation of the non-autonomous model functional with (p, q)-growth

$$\int_\Omega (|Du| - \delta)_+^p + a(x)(|Du| - \delta)_+^q \, dx$$

with a nonnegative and bounded function $a(x)$, $\delta > 0$, can be interpreted as the flow in a origin-destination network, in traffic problems that take into account the behavior of the flow in every position x of the path and with different constrained capacities.

Let us now list our assumptions.

The integrand f will satisfy the so-called (p, q)-*growth conditions*, i.e.

(A1) there exist exponents $2 \leq p \leq q < \infty$ and constants $c_1, c_2, L > 0$ such that

$$c_1|\xi|^p - c_2 \leq f(x, \xi) \leq L\left(1 + |\xi|^q\right),$$

for a.e. $x \in \Omega$ and for every $\xi \in \mathbb{R}^{nN}$.

The further properties of f will be assumed to hold only for large values of the modulus of the gradient variable ξ; i.e., for all $\xi \in \mathbb{R}^{nN}$, such that $|\xi| \geq R$ for some positive R. For the sake of simplicity and without loss of generality, we will assume, from now on, $R = 1$.

As it is well known since the famous example by De Giorgi [16] (see also [40], [41], [45]), in order to avoid the irregularity phenomena peculiar of the vectorial minimizers, the dependence of the energy density on the modulus of the gradient variable is assumed. Precisely:

(A2) there exists a function $\tilde{f} : \Omega \times [1, +\infty) \to [0, +\infty)$, nondecreasing in the last variable, such that

$$f(x, \xi) = \tilde{f}(x, |\xi|),$$

for a.e. $x \in \Omega$ and for every $\xi \in \mathbb{R}^{nN} \setminus B_1(0)$.

The p-*uniform convexity* will be assumed only at infinity. More precisely:

(A3) there exists $\nu > 0$, such that

$$\langle D_{\xi\xi} f(x, \xi)\lambda, \lambda\rangle \geq \nu\left(1 + |\xi|^2\right)^{\frac{p-2}{2}}|\lambda|^2,$$

for a.e. $x \in \Omega$, and for every $\xi \in \mathbb{R}^{nN} \setminus B_1(0)$ and for every $\lambda \in \mathbb{R}^{nN}$.

Moreover, we will assume that

(A4) there exists $L_1 > 0$ such that

$$|D_{\xi\xi} f(x, \xi)| \leq L_1(1 + |\xi|^2)^{\frac{q-2}{2}},$$

for a.e. $x \in \Omega$ and for every $\xi \in \mathbb{R}^{nN} \setminus B_1(0)$.

As we already mentioned, we will not ask a regular dependence of f on the x-variable. Indeed, the function $x \to D_\xi f(x, \xi)$ will be required to be weakly differentiable for every $\xi \in \mathbb{R}^{nN} \setminus B_1(0)$ and it will be assumed that

(A5) there exists a nonnegative function $k \in L^{\sigma}_{\text{loc}}(\Omega)$, with $\sigma \geq p+2$, such that

$$|D_{\xi x} f(x,\xi)| \leq k(x)(1+|\xi|^2)^{\frac{q-1}{2}},$$

for a.e. $x \in \Omega$ and for every $\xi \in \mathbb{R}^{nN} \setminus B_1(0)$.

Our last assumption is the following

(A6) there exists a constant $\mu > 1$, such that

$$f(x,\lambda\xi) \leq \lambda^{\mu} f(x,\xi) \qquad \text{for every} \quad \lambda > 1,$$

for a.e. $x \in \Omega$ and for every $\xi \in \mathbb{R}^{nN} \setminus B_1(0)$.

Note that the previous assumption will be used only to obtain the local boundedness of the minimizers ([12]).

In order to explain the difficulties of our problem, we now recall some class of functionals included in our setting and some known regularity properties of the related minimizers.

The study of the regularity of local minimizers of not uniformly convex functionals started in [7] (see also [30]), where integrals of the form

$$\mathcal{G}(v,\Omega) = \int_{\Omega} G(x,Dv)\,dx,$$

with integrand G enjoying a p-Laplacian type structure at infinity were considered. This means that the partial map

$$x \to G(x,\xi) \;\; \text{is Lipschitz continuous} \tag{1.2}$$

and the following ellipticity

$$\langle D_{\xi\xi} G(x,\xi)\eta,\eta\rangle \geq \nu|\xi|^{p-2}|\eta|^2 \qquad\qquad |\xi| > M \tag{1.3}$$

and growth condition

$$|D_{\xi} G(x,\xi)| \leq C|\xi|^{p-1} \qquad\qquad |\xi| > M \tag{1.4}$$

hold true for some $M > 0$.

Under these assumptions it has been proven that the minimizers of the functional $\mathcal{G}$ are locally Lipschitz continuous (see [19], [24], [25], [26], [44]). The Lipschitz continuity of local minimizers of the functional $\mathcal{G}(v,\Omega)$ with integrand G satisfying assumptions (1.2), (1.3), but under the growth assumption

$$|D_{\xi} G(x,\xi)| \leq C|\xi|^{q-1} \qquad\qquad |\xi| > M \tag{1.5}$$

with $q > p$ has been widely investigated (see [6], [11], [20], [22], [23]). Obviously, the ellipticity assumption (1.3), together with the growth assumption (1.5), entails a (p,q)-growth condition for the integrand G, i.e. there exist positive constants c, C such that

$$c|\xi|^p \leq G(x,\xi) \leq C(1+|\xi|^q).$$

The study of regularity of minimizers of functionals satisfying (p,q)-growth conditions began with the pioneering papers by Marcellini ([36], [37], [38]) and later on widely investigated.

From the very beginning it has been clear that if p and q are too far apart no regularity can be expected for the local minimizers (see [29], [35]), while if p and q are sufficiently close many regularity results are available. Actually, the research in this field is so intense that it is almost impossible to give an exhaustive and comprehensive list of references; we confine ourselves to refer the interested reader to the survey [39] and the more recent papers [1], [8], [9] and the references therein.
It is well known that the presence of the x-variable in the integrand of functionals with non standard growth conditions may yield the so-called Lavrentiev phenomenon, even under a continuity assumption on the coefficients. Such phenomenon does not occur in case of standard growth condition, i.e. if $p = q$. Actually, it has been proven that minimizers of the functional $\mathcal{G}$ under assumptions (1.3) and (1.4) are regular even weakening the Lipschitz regularity of the integrand G expressed by (1.2) in a Sobolev type regularity on the partial map $x \mapsto G(x, \xi)$. Indeed, if there exists a nonnegative function $k(x) \in L^n(\Omega)$ such that

$$|D_x D_\xi G(x, \xi)| \leq k(x)|\xi|^{p-1},$$

then the minimizer possesses second derivatives and for its gradient a higher integrability result holds true (see e.g. [33], [42], [43] and, for other regularity results under a Sobolev type assumption with respect to the x-variable, see [18], [27], [28], [31]).

A Sobolev type assumption, combined with a degenerate ellipticity and nonstandard growth conditions, has been used in [17] , where it is proven the local Lipschitz continuity of minimizers of $\mathcal{G}$, under assumptions (1.3), (1.4) and

$$|D_x D_\xi G(x, \xi)| \leq k(x)|\xi|^{q-1},$$

with $k \in L^r(\Omega)$, $r > n$, with $q \geq p$ satisfying

$$\frac{q}{p} < 1 + \frac{1}{n} - \frac{1}{r}.$$

Note that such a Sobolev assumption yields the Hölder continuity of the partial map $x \mapsto D_\xi G(x, \xi)$ through the Sobolev imbedding Theorem. If $r \to n$, the inequality above implies that $q \to p$. This case, i.e. $k \in L^n(\Omega)$, has been faced in [10], where we proved that, the weakening the Sobolev regularity of the coefficients, leads not only to deal with standard growth conditions, but also to the loss of the Lipschitz regularity of the minimizers. Actually, we were able to prove that the gradient of the minimizer belongs to L^t for every $t \geq p$, thus that the minimizers are α-Hölder continuous, for every $\alpha \in (0, 1)$.

When treating the regularity of minimizers, the proof of their local boundedness can be seen as a first step in the analysis of further regularity properties (see [12], [13] for recent contributions to the (p, q) growth case). It is now clear that the Sobolev conjugate exponent of p denoted by p^* is a threshold level for q to expect local bounded minimizers: actually if q is below p^* than the local boundedness is attained, otherwise, if $q > p^*$, irregularity occurs. In the present paper we use the bound $q < p^*$ in order to deal with locally bounded minimizers and we take advantage from this information to prove higher regularity properties of minimizers of functionals with non standard growth conditions, not uniformly convex and with a Sobolev x-dependence that can be even weaker than $W^{1,n}$.

More precisely, our main result is the following.

Theorem 1.1. *Let $f : \Omega \times \mathbb{R}^{nN} \to [0, +\infty)$ be Carathéodory function satisfying the assumptions* **(A1)–(A6)**. *Let $u \in W^{1,p}_{\mathrm{loc}}(\Omega; \mathbb{R}^N)$ be a local minimizer of* (1.1). *If*

$$2 \leq p \leq q \leq p + 1 - \frac{p+2}{\sigma} \qquad \textit{and} \qquad q < p^*,$$

then

$$u \in L^{\infty}_{\mathrm{loc}}(\Omega;\mathbb{R}^N) \quad \textit{and} \quad Du \in L^{p+2}_{\mathrm{loc}}(\Omega;\mathbb{R}^{nN}).$$

Moreover, setting

$$\mathcal{G}(t) := \int_0^t s(2+s)^{\frac{p-4}{2}}\, ds,$$

we have

$$\mathcal{G}((|Du|-2)_+) \in W^{1,2}_{\mathrm{loc}}(\Omega) \quad \textit{and} \quad (|Du|-2)_+ \in W^{1,2}_{\mathrm{loc}}(\Omega) \tag{1.6}$$

and the following Caccioppoli type inequality holds

$$\begin{aligned} &\int_{B_\rho(x_0)} ((|Du|-2)_+)^2(2+(|Du|-2)_+)^{-2}|Du|^{p-2}|D^2u|^2\, dx \\ \leq\ & C\left(1+\|k\|_{L^\sigma(B_r)}\right)^{\frac{p+2}{p-q+1}} \left(\int_{B_r(x_0)} (1+f(x,Du))\, dx\right)^{\gamma}, \end{aligned} \tag{1.7}$$

for every ball $B_r(x_0) \Subset \Omega$*, every* $0 < \rho < r$*, and for some* $C = C(n,N,p,q,\sigma,c_1,c_2,L,L_1,\nu,\rho,r,\operatorname{diam}\Omega)$ *and* $\gamma = \gamma(p,q,n) > 1$.

The first step in the proof is an a priori estimate for local minimizers u of the functional $\mathcal{F}$ which are Lipschitz continuous and such that $(|Du|-1)_+ \in W^{1,2}_{\mathrm{loc}}(\Omega)$.

Such a priori estimate is achieved by the use of test functions that vanish on the set where the Euler Lagrange system associated to $\mathcal{F}$ fails to be uniformly elliptic. It is worth pointing out that the key tools in the proof are the local boundedness of the minimizers, which is known to hold true by virtue of the bound $q < p^*$ (see Theorems 2.2 and 2.3) and a suitable interpolation inequality (see Proposition 2.5). In particular, the interpolation inequality together with the a priori assumption $(|Du|-1)_+ \in W^{1,2}(\Omega)$, allows us to prove an a priori estimate. Indeed the gradient of the minimizers is locally in L^{p+2} and this higher integrability reveals to be useful in the estimates of terms with critical summability (see the proof of Theorem 3.2, Step 2).

Once the a priori estimate is established we have to use an approximation procedure. Usually the approximating sequence of functionals are constructed regularizing the integrand of $\mathcal{F}$ in order that their minimizers are sufficiently regular to satisfy the a priori estimate.

Since the regularity of minimizers of the approximating functionals can be proven only in the region where the energy densities are uniformly convex, which may vary terms by terms, one of the difficulties that we have to face is to transfer this regularity information to the limit. This difficulty is overcome by using Proposition 2.6, a measure theory result, which may be of interest by itself.

Another difficulty comes from the lack of a global uniform convexity of the integrand that yields a lack of uniqueness of the minimizers. Therefore, in principle, we could only say that the sequence of the minimizers of the approximating functionals converge to a regular minimizer, but not that all the minimizers of $\mathcal{F}$ have the desired regularity. In order to overcome this problem, following [6], we introduce in the approximating problem a penalization term that does not effect the a priori estimate and that forces the approximating sequence to converge to an arbitrarily fixed local minimizer.

Finally, we note that, even in the case $\sigma = p+2$, that implies $q = p$, our result improves a very recent one obtained in [32] concerning the regularity of local minimizers of functionals uniformly convex in the whole $\mathbb{R}^{nN}$. There the local minimizers are assumed to be a priori locally bounded, so no structure assumptions, such as **(A2)** and **(A6)** is needed.

The plan of the paper is the following: in Section 2 we fix the notations and collect preliminary results; in Section 3 we establish the a priori estimate and in Section 4 we give the proof of the main result.

2 Preliminaries

In this section we introduce some notations and collect several results that we shall use to establish our main result.

We will follow the usual convention and denote by c or C a general constant that may vary on different occasions, even within the same line of estimates. Relevant dependencies on parameters and special constants will be suitably emphasized using parentheses or subscripts. All the norms we use on $\mathbb{R}^n$, $\mathbb{R}^N$ and $\mathbb{R}^{nN}$ will be the standard Euclidean ones and denoted by $|\cdot|$ in all cases. In particular, for matrices ξ, $\eta \in \mathbb{R}^{nN}$ we write $\langle \xi, \eta \rangle := \mathrm{trace}(\xi^T \eta)$ for the usual inner product of ξ and η, and $|\xi| := \langle \xi, \xi \rangle^{\frac{1}{2}}$ for the corresponding euclidean norm.

In what follows, $B_r(x)$ will denote the ball in $\mathbb{R}^n$ centered at x of radius r. The integral mean of a function u over a ball $B_r(x)$ will be denoted by $u_{x,r}$; i.e.,

$$u_{x,r} := \frac{1}{|B_r(x)|} \int_{B_r(x)} u(y)\, dy,$$

where $|B_r(x)|$ is the Lebesgue measure of the ball in $\mathbb{R}^n$. If no confusion may arise we shall omit the dependence on the center.

The following lemma is an important application in the so called hole-filling method. Its proof can be found for example in [34, Lemma 6.1] .

Lemma 2.1. *Let $h : [r, R_0] \to \mathbb{R}$ be a nonnegative bounded function and $0 < \vartheta < 1$, $A, B \geq 0$ and $\beta > 0$. Assume that*

$$h(s) \leq \vartheta h(t) + \frac{A}{(t-s)^{\beta}} + B,$$

for all $r \leq s < t \leq R_0$. Then

$$h(r) \leq \frac{cA}{(R_0 - r)^{\beta}} + cB,$$

where $c = c(\vartheta, \beta) > 0$.

2.1 Some useful regularity results

The following local boundedness result is a straightforward consequence of Theorem 2.1 in [12], see also [14].

Theorem 2.2. *Assume* **(A1)–(A4)** *and* **(A6)**, *with $1 < p \leq q < p^*$. Then a local minimizer u of (1.1) is locally bounded. Moreover, for every $B_r(x_0) \Subset \Omega$ there exists $c > 0$, depending on the data, such that*

$$\|u - u_{x_0,r}\|_{L^\infty\left(B_{r/(2\sqrt{n})}(x_0)\right)} \leq c \left\{ 1 + \int_{B_r(x_0)} f(x, Du)\, dx \right\}^{\vartheta},$$

with a positive $\vartheta = \vartheta(q, p, n)$.

A similar result holds for a suitable perturbation of the functional.

Theorem 2.3. *Assume* **(A1)–(A4)** *and* **(A6)***, with* $1 < p \le q < p^*$. *Fix* $B_r(x_0) \Subset \Omega$ *and consider* $u \in W^{1,p}(B_r(x_0);\mathbb{R}^N)$ *and* $\bar{u} \in W^{1,p}(B_r(x_0);\mathbb{R}^N) \cap L^\infty(B_r(x_0);\mathbb{R}^N)$. *Define*

$$\tilde{\mathcal{F}}(v;B_r(x_0)) := \mathcal{F}(v;B_r(x_0)) + \int_{B_r(x_0)} \arctan(|v(x)-\bar{u}(x)|^2)\,dx, \tag{2.1}$$

where $\mathcal{F}$ *is defined in* (1.1). *Then any minimizer* $v \in u + W_0^{1,p}(B_r(x_0);\mathbb{R}^N)$ *of* $\tilde{\mathcal{F}}(v;B_r(x_0))$ *is locally bounded and for every* $B_\rho(\bar{x}) \Subset B_r(x_0)$

$$\|v - v_{\bar{x},\rho}\|_{L^\infty(B_{\rho/(2\sqrt{n})}(\bar{x}))} \le c\left\{1 + \int_{B_\rho(\bar{x})} f(x,Dv)\,dx\right\}^{\vartheta},$$

where $\vartheta = \vartheta(q,p,n)$.

Proof. Let us define $g(x,v) := \arctan(|v-\bar{u}(x)|^2)$. Since v is a local minimizer of the functional $\tilde{\mathcal{F}}$, then v satisfies the Euler's system

$$\int_{B_r(x_0)} \sum_{i=1}^n f_{\xi_i^\alpha}(x,Dv)\varphi^\alpha_{x_i}(x)\,dx = \int_{B_r(x_0)} g(x,v)\varphi^\alpha(x)\,dx \quad \forall \varphi \in C_0^\infty(B_r(x_0);\mathbb{R}^N),\ \alpha = 1,\dots,N.$$

By our choice of g, we can conclude following the proof of Theorem 2.1 in [12], with the obvious modifications. □

We now recall a regularity result for minimizers of functionals with standard growth conditions and smooth dependence on the x-variable. This is a well known result, if the perturbation term $\int_\Omega \arctan(|w(x)-\bar{u}(x)|^2)\,dx$ in the integral in (2.2) is absent. In this case, we refer to [30] for the higher differentiability result and to Theorem 1.1 in [11] as far as the Lipschitz continuity of the local minimizers is concerned. In presence of the perturbation term $\int_\Omega \arctan(|w-\bar{u}(x)|^2)\,dx$, the proofs can be easily adapted because of the boundedness of its integrand and of its derivative with respect to the variable w. More precisely, we have the following.

Theorem 2.4. *Let* $g : \Omega \times \mathbb{R}^{nN} \to [0,+\infty)$, $g \in C^2(\Omega \times \mathbb{R}^{nN})$, *and define the functional*

$$\int_\Omega \Big(g(x,Dw) + \arctan(|w-\bar{u}(x)|^2)\Big)\,dx \tag{2.2}$$

with $\bar{u} \in C^2(\Omega;\mathbb{R}^N)$. *Assume that there exists* $p \ge 2$ *such that for every* $x \in \Omega$ *and every* $\xi, \lambda \in \mathbb{R}^{nN}$,

$$c_1|\xi|^p - c_2 \le g(x,\xi) \le L(1+|\xi|)^p,$$
$$\nu\,(1+|\xi|)^{p-2}|\lambda|^2 \le \langle D_{\xi\xi}g(x,\xi)\lambda,\lambda\rangle,$$
$$|D_{\xi\xi}g(x,\xi)| \le L_1\,(1+|\xi|)^{p-2},$$
$$|D_{\xi x}g(x,\xi)| \le K(1+|\xi|)^{p-1},$$

with positive constants c_1, c_2, L, L_1, ν, K. *Then any local minimizer* v *of* (2.2) *is in* $W^{2,2}_{\mathrm{loc}}(\Omega;\mathbb{R}^N)$ *and*

$$(1+|Dv|^2)^{\frac{p-2}{2}}|D^2v|^2 \in L^1_{\mathrm{loc}}(\Omega).$$

Moreover, if g *satisfies* **(A2)***, then* $v \in W^{1,\infty}_{\mathrm{loc}}(\Omega;\mathbb{R}^N)$.

2.2 An interpolation inequality

We take advantage from the local boundedness of minimizers of the functional $\mathcal{F}$ in (1.1) through the following Gagliardo-Nirenberg type interpolation inequality, see also [5].

Proposition 2.5. *Let $p \geq 2$ and define*

$$\mathcal{G}(t) := \int_0^t s(1+s)^{\frac{p-4}{2}}\, ds. \tag{2.3}$$

If $u \in L^\infty_{\rm loc}(\Omega;\mathbb{R}^N) \cap W^{1,p+2}_{\rm loc}(\Omega;\mathbb{R}^N)$ is such that $(|Du|-1)_+ \in W^{1,2}_{\rm loc}(\Omega;\mathbb{R}^N)$ and $\mathcal{G}((|Du|-1)_+) \in W^{1,2}_{\rm loc}(\Omega)$, then, for every $\psi \in C^1_c(\Omega)$ and every $\kappa \in \mathbb{R}^N$, we have

$$\begin{aligned} &\int_\Omega \psi^2 (1+(|Du|-1)_+)^p ((|Du|-1)_+)^2\, dx \\ \leq\ & c\left(1+\|u-\kappa\|^2_{L^\infty(\mathrm{supp}\psi)}\right)\int_\Omega (\psi^2+|D\psi|^2)(1+(|Du|-1)_+)^p\, dx \\ +\ & c\|u-\kappa\|^2_{L^\infty(\mathrm{supp}\psi)}\int_\Omega \psi^2\left|D(\mathcal{G}((|Du|-1)_+))\right|^2\, dx, \end{aligned} \tag{2.4}$$

for a positive constant $c = c(p)$.

Proof. Consider $\kappa \in \mathbb{R}^N$ and the nonnegative increasing function

$$G(t) = t(1+t)^{\frac{p-2}{2}}, \qquad t \geq 0.$$

An integration by parts yields

$$\begin{aligned} &\int_\Omega \psi^2(1+(|Du|-1)_+)^{p-2}((|Du|-1)_+)^2|Du|^2\, dx \\ =\ & \int_\Omega \psi^2\big(G((|Du|-1)_+)\big)^2|Du|^2\, dx = \int_\Omega \langle \psi^2\big(G((|Du|-1)_+)\big)^2 Du, Du\rangle\, dx \\ =\ & -\sum_{i,\alpha}\int_\Omega (u^\alpha-\kappa^\alpha)\big[\psi^2\big(G((|Du|-1)_+)\big)^2 Du^\alpha\big]_{x_i}\, dx \\ \leq\ & 2\|u-\kappa\|_{L^\infty(\mathrm{supp}\psi)}\int_\Omega |\psi||D\psi|(G((|Du|-1)_+))^2|Du|\, dx \\ +\ & 2\|u-\kappa\|_{L^\infty(\mathrm{supp}\psi)}\int_\Omega \psi^2 G((|Du|-1)_+)G'((|Du|-1)_+)|D^2u||Du|\, dx \\ +\ & \|u-\kappa\|_{L^\infty(\mathrm{supp}\psi)}\int_\Omega \psi^2(G((|Du|-1)_+))^2|D^2u|\, dx =: I_1+I_2+I_3. \end{aligned} \tag{2.5}$$

Recalling the definition of G and using Young's inequality, we obtain

$$\begin{aligned} I_1 \ \leq\ & c_\varepsilon\|u-\kappa\|^2_{L^\infty(\mathrm{supp}\psi)}\int_\Omega |D\psi|^2(1+(|Du|-1)_+))^p\, dx \\ +\ & \varepsilon\int_\Omega \psi^2(1+(|Du|-1)_+)^{p-2}((|Du|-1)_+)^2|Du|^2\, dx, \end{aligned}$$

for every $\varepsilon \in (0,1)$.

Taking into account that $G(t)G'(t) \le \frac{p}{2} t(1+t)^{p-2}$, by Young's inequality we have

$$\begin{aligned} I_2 &\le c(p)\|u-\kappa\|_{L^\infty(\mathrm{supp}\psi)} \int_\Omega \psi^2((|Du|-1)_+)(1+(|Du|-1)_+)^{p-2}|D^2u||Du|\,dx \\ &\le c_\varepsilon\|u-\kappa\|^2_{L^\infty(\mathrm{supp}\psi)} \int_\Omega \psi^2(1+(|Du|-1)_+)^{p-4}((|Du|-1)_+)^2|D^2u|^2\,dx \\ &+ \varepsilon\int_\Omega \psi^2(1+(|Du|-1)_+))^p|Du|^2\,dx \\ &\le c_\varepsilon\|u-\kappa\|^2_{L^\infty(\mathrm{supp}\psi)} \int_\Omega \psi^2(1+(|Du|-1)_+)^{p-4}((|Du|-1)_+)^2|D^2u|^2\,dx \\ &+ 2\varepsilon\int_\Omega \psi^2(1+(|Du|-1)_+)^{p-2}((|Du|-1)_+)^2|Du|^2\,dx \\ &+ 2\varepsilon\int_\Omega \psi^2(1+(|Du|-1)_+)^{p-2}|Du|^2\,dx, \end{aligned}$$

where in the last estimate we used the elementary inequality

$$(1+(|Du|-1)_+))^2 \le 2+2((|Du|-1)_+)^2.$$

Moreover, by the definition of G and by Young's inequality again

$$\begin{aligned} I_3 &\le \|u-\kappa\|_{L^\infty(\mathrm{supp}\psi)} \int_\Omega \psi^2(1+(|Du|-1)_+)^{p-2}((|Du|-1)_+)^2|D^2u|\,dx \\ &\le c_\varepsilon\|u-\kappa\|^2_{L^\infty(\mathrm{supp}\psi)} \int_\Omega \psi^2(1+(|Du|-1)_+)^{p-4}((|Du|-1)_+)^2|D^2u|^2\,dx \\ &+ \varepsilon\int_\Omega \psi^2(1+(|Du|-1)_+)^p((|Du|-1)_+)^2\,dx \\ &\le c_\varepsilon\|u-\kappa\|^2_{L^\infty(\mathrm{supp}\psi)} \int_\Omega \psi^2(1+(|Du|-1)_+)^{p-4}((|Du|-1)_+)^2|D^2u|^2\,dx \\ &+ \varepsilon\int_\Omega \psi^2(1+(|Du|-1)_+)^{p-2}((|Du|-1)_+)^2|Du|^2\,dx, \end{aligned}$$

where we used

$$1+(|Du|-1)_+ = |Du| \qquad \text{a.e in the set } \{|Du|>1\}.$$

Inserting the estimates of I_1, I_2, I_3, in (2.5), we obtain that

$$\begin{aligned} &\int_\Omega \psi^2(1+(|Du|-1)_+)^{p-2}((|Du|-1)_+)^2|Du|^2\,dx \\ \le\ & c_\varepsilon\|u-\kappa\|^2_{L^\infty(\mathrm{supp}\psi)} \int_\Omega |D\psi|^2(1+(|Du|-1)_+)^p\,dx \\ +\ & 4\varepsilon\int_\Omega \psi^2(1+(|Du|-1)_+)^{p-2}((|Du|-1)_+)^2|Du|^2\,dx \\ +\ & c_\varepsilon\|u-\kappa\|^2_{L^\infty(\mathrm{supp}\psi)} \int_\Omega \psi^2(1+(|Du|-1)_+)^{p-4}((|Du|-1)_+)^2|D^2u|^2\,dx \end{aligned}$$

$$+ \quad 2\varepsilon \int_\Omega \psi^2 (1+(|Du|-1)_+)^{p-2}|Du|^2\,dx.$$

The integrability assumptions on u imply that the right hand side is finite, in particular

$$|D(\mathcal{G}((|Du|-1)_+))|^2 = (1+(|Du|-1)_+)^{p-4}((|Du|-1)_+)^2|D^2u|^2 \qquad \text{a.e. in } \Omega \tag{2.6}$$

is in $L^1_{\mathrm{loc}}(\Omega)$. Choosing $\varepsilon = \frac{1}{8}$, we can reabsorb the second integral in the right hand side by the left hand side, thus getting

$$\begin{aligned} &\int_\Omega \psi^2 (1+(|Du|-1)_+)^{p-2}((|Du|-1)_+)^2|Du|^2\,dx \\ \leq\ & c\left(1+\|u-\kappa\|^2_{L^\infty(\mathrm{supp}\psi)}\right)\int_\Omega (\psi^2+|D\psi|^2)(1+(|Du|-1)_+)^p\,dx \\ +\ & c\|u-\kappa\|^2_{L^\infty(\mathrm{supp}\psi)}\int_\Omega \psi^2\left|D(\mathcal{G}((|Du|-1)_+))\right|^2\,dx, \end{aligned}$$

where we used (2.6) in the last integral. The inequality (2.4) follows. □

2.3 A measure theory result

This subsection is devoted to the proof of a result, which is of interest by itself, that will be used in the approximating procedure (see Step 3 in the proof of Theorem 1.1).

Proposition 2.6. *Let $f_k \in W^{1,p}(\Omega;\mathbb{R}^N)$ be a sequence and $f \in W^{1,p}(\Omega;\mathbb{R}^N)$, $p \geq 2$, $N \geq 1$, satisfying the following assumptions:*

(a) f_k is weakly convergent to f in $W^{1,p}(\Omega;\mathbb{R}^N)$,

(b) $(|Df_k|-1)_+ \in W^{1,2}(\Omega)$,

(c) there exists $\varepsilon > 0$ such that

$$\sup_k \|(|Df_k|-1)_+\|_{L^{p+\varepsilon}(\Omega)} < \infty. \tag{2.7}$$

Let $\mathcal{G}$ be the function defined in (2.3). Assume that

(d) $\mathcal{G}((|Df_k|-1)_+) \in W^{1,2}(\Omega)$ and

$$\int_\Omega |D(\mathcal{G}((|Df_k|-1)_+))|^2\,dx \leq c \tag{2.8}$$

for some c independent of k.

Then

$$(|Df|-1)_+ \in W^{1,2}(\Omega)$$

and there exists a not relabeled subsequence f_k such that

$$\lim_{k\to\infty} |Df_k| = |Df| \quad \textit{strongly in } L^p(\Omega \cap \{|Df| > \lambda\}) \textit{ for a.e. } \lambda > 1,$$

and

$$\lim_{k\to\infty} |Df_k| = |Df| \quad \textit{a.e. in } \Omega \cap \{|Df| > 1\}.$$

Proof. By (2.8) we have that

$$\begin{aligned}&\int_\Omega (1+(|Df_k|-1)_+)^{-2}((|Df_k|-1)_+)^p|D^2f_k|^2\,dx\\&=\int_\Omega |D(\mathcal{G}((|Df_k|-1)_+))|^2\,dx\le c,\end{aligned}\tag{2.9}$$

with c independent of k. This implies that, for every $\tau>1$,

$$\begin{aligned}&\int_{\Omega\cap\{|Df_k|>\tau\}}|Df_k|^{-2}(|Df_k|-1)^p|D^2f_k|^2\,dx\\&=\int_{\Omega\cap\{|Df_k|>\tau\}}(1+(|Df_k|-1)_+)^{-2}((|Df_k|-1)_+)^2((|Df_k|-1)_+)^{p-2}|D^2f_k|^2\,dx\\&\le\int_\Omega|D(\mathcal{G}((|Df_k|-1)_+))|^2\,dx\le c,\end{aligned}\tag{2.10}$$

with c independent of k.

Since the function $t\mapsto\frac{t-1}{t}$ is positive, increasing and bounded in $[\tau,\infty)$, there exists $c(\tau)>0$ such that

$$\left(\frac{t-1}{t}\right)^2\ge c(\tau)\quad\forall t\in[\tau,\infty),\tag{2.11}$$

which implies

$$\frac{(t-1)^p}{t^2}=(t-1)^{p-2}\left(\frac{t-1}{t}\right)^2\ge(\tau-1)^{p-2}c(\tau)\quad\forall t\in[\tau,\infty).$$

The above inequality with $t=|Df_k(x)|$, gives

$$|Df_k|^{-2}(|Df_k|-1)^p\ge(\tau-1)^{p-2}c(\tau)\qquad\text{a.e. in }\Omega\cap\{|Df_k|>\tau\}.$$

Thus, (2.10) implies

$$\int_{\Omega\cap\{|Df_k|>\tau\}}|D^2f_k|^2\,dx\le\frac{c}{(\tau-1)^{p-2}c(\tau)},$$

or, equivalently, there exists $C(\tau,p)$ such that

$$\int_\Omega\Big|D\big((|Df_k|-\tau)_+\big)\Big|^2\,dx\le C(\tau,p).$$

This implies that the sequence $(|Df_k|-\tau)_+$ converges weakly in $W^{1,2}(\Omega)$ and strongly in $L^2(\Omega)$ to a function $G(\tau)\in W^{1,2}(\Omega)$. Therefore, up to a not relabeled subsequence, we also have that

$$(|Df_k|-\tau)_+\to G(\tau)\qquad\text{a.e. in }\Omega.$$

Analogously, by (2.11),

$$|Df_k|^{-2}(|Df_k|-1)^2\ge c(\tau)\qquad\text{a.e. in }\Omega\cap\{|Df_k|>\tau\}.$$

Therefore, by (2.9)

$$
\begin{aligned}
& c(\tau) \int_{\Omega\cap\{|Df_k|>\tau\}} (|Df_k|-\tau)^{p-2}|D^2 f_k|^2\,dx \\
& \le \int_{\Omega\cap\{|Df_k|>\tau\}} |Df_k|^{-2}(|Df_k|-1)^2(|Df_k|-\tau)^{p-2}|D^2 f_k|^2\,dx \\
& \le \int_{\Omega\cap\{|Df_k|>\tau\}} |Df_k|^{-2}(|Df_k|-1)^2(|Df_k|-1)^{p-2}|D^2 f_k|^2\,dx \\
& = \int_{\Omega\cap\{|Df_k|>\tau\}} (1+(|Df_k|-1)_+)^{-2}((|Df_k|-1)_+)^{p}|D^2 f_k|^2\,dx \\
& \le \int_{\Omega} |D(\mathcal{G}((|Df_k|-1)_+))|^2\,dx \le c.
\end{aligned}
$$

This imply

$$\int_\Omega ((|Df_k|-\tau)_+)^{p-2}|D^2 f_k|^2\,dx \le C(\tau)$$

that yields

$$\int_\Omega \left|D\Big(((|Df_k|-\tau)_+)^{p/2}\Big)\right|^2 dx \le C(\tau).$$

Thus, up to subsequences, the sequence

$$((|Df_k|-\tau)_+)^{p/2}$$

converges strongly in $L^2(\Omega)$ as $k\to\infty$. This is equivalent to say that there exists a function $F(\tau)\in L^p(\Omega)$, such that

$$(|Df_k|-\tau)_+ \to F(\tau) \qquad \text{strongly in } L^p(\Omega),$$

that is

$$\lim_k \int_\Omega |(|Df_k|-\tau)_+ - F(\tau)|^p = 0. \tag{2.12}$$

We also have that

$$(|Df_k|-\tau)_+ \to F(\tau) \qquad\qquad \text{a.e. in } \Omega \tag{2.13}$$

and

$$(|Df_k|-\tau)_+ \to F(\tau) \qquad\qquad \text{in measure.} \tag{2.14}$$

Note that the uniqueness of the a.e. limit implies that $G(\tau)=F(\tau)$ a.e. in Ω and so $F(\tau)\in W^{1,2}(\Omega)$. For every $t>0$ and $k\in\mathbb{N}$ define

$$
\begin{aligned}
A_{k,\tau}(t) &:= \{x\in\Omega:\ (|Df_k|-\tau)_+ > t\} = \{x\in\Omega:\ |Df_k|>\tau+t\},\\
A_\tau(t) &:= \{x\in\Omega\ :\ F(\tau)>t\}.
\end{aligned}
$$

The function $t\mapsto |A_\tau(t)|$ is continuous a.e., where $|A_\tau(t)|$ denotes the Lebesgue measure of the set $A_\tau(t)$. It can be easily checked that if t is a continuity point of $|A_\tau(\cdot)|$, then (2.14) implies

$$\lim_{k\to\infty} |A_{k,\tau}(t)| = |A_\tau(t)|$$

or, equivalently,

$$\|\chi_{A_{k,\tau}(t)}\|_{L^1(\Omega)} \to \|\chi_{A_\tau(t)}\|_{L^1(\Omega)}.$$

Moreover, by (2.13),

$$\chi_{A_{k,\tau}(t)} \to \chi_{A_\tau(t)} \qquad \text{a.e. in } \Omega.$$

Thus, for a.e. $t > 0$

$$\chi_{A_{k,\tau}(t)} \to \chi_{A_\tau(t)} \qquad \text{strongly in } L^1(\Omega). \tag{2.15}$$

By (2.12) and the definition of $A_{k,\tau}(t)$, we have

$$\lim_{k\to\infty} \int_{A_{k,\tau}(t)} ||Df_k| - \tau - F(\tau)|^p = 0. \tag{2.16}$$

Let us prove that

$$\lim_{k\to\infty} \int_{A_\tau(t)} ||Df_k| - \tau - F(\tau)|^p \, dx = 0. \tag{2.17}$$

To this aim, notice that (2.16) implies

$$\begin{aligned}
& \lim_{k\to\infty} \int_{A_\tau(t)} ||Df_k| - \tau - F(\tau)|^p \, dx \\
= & \lim_{k\to\infty} \int_{A_\tau(t)} ||Df_k| - \tau - F(\tau)|^p \, dx - \lim_k \int_{A_{k,\tau}(t)} ||Df_k| - \tau - F(\tau)|^p \, dx \\
= & \lim_{k\to\infty} \int_\Omega ||Df_k| - \tau - F(\tau)|^p (\chi_{A_\tau(t)}(x) - \chi_{A_{k,\tau}(t)}(x)) \, dx. \qquad (2.18)
\end{aligned}$$

By (2.7)

$$\int_\Omega |(|Df_k| - \tau)_+|^{p+\varepsilon} \, dx \le C \tag{2.19}$$

with a constant C independent of k and τ. This yields that, up to subsequences, $(|Df_k| - \tau)_+$ weakly converges in $L^{p+\varepsilon}(\Omega)$ to a function $H(\tau) \in L^{p+\varepsilon}(\Omega)$.

Due to (2.12), $H(\tau) = F(\tau)$ a.e. in Ω and so $F(\tau) \in L^{p+\varepsilon}(\Omega)$. Therefore by Hölder's inequality and thanks to (2.19) and (2.15), it follows that

$$\begin{aligned}
& \lim_k \left| \int_\Omega ||Df_k| - \tau - F(\tau)|^p (\chi_{A_\tau(t)}(x) - \chi_{A_{k,\tau}(t)}(x)) \, dx \right| \\
\le & \lim_k \int_\Omega ||Df_k| - \tau - F(\tau)|^p |\chi_{A_\tau(t)}(x) - \chi_{A_{k,\tau}(t)}(x)| \, dx \\
\le & \lim_k \left(\int_\Omega ||Df_k| - \tau - F(\tau)|^{p+\varepsilon} \, dx \right)^{\frac{p}{p+\varepsilon}} \left(\int_\Omega |\chi_{A_\tau(t)} - \chi_{A_{k,\tau}(t)}|^{\frac{p+\varepsilon}{\varepsilon}} \, dx \right)^{\frac{\varepsilon}{p+\varepsilon}} \\
\le & C(\tau) \lim_k \left(\int_\Omega |\chi_{A_\tau(t)} - \chi_{A_{k,\tau}(t)}| \, dx \right)^{\frac{\varepsilon}{p+\varepsilon}} = 0. \qquad (2.20)
\end{aligned}$$

By (2.20) and (2.18), we get (2.17), i.e.,

$$|Df_k| \to F(\tau) + \tau \qquad \text{strongly in } L^p(A_\tau(t)) \tag{2.21}$$

as $k \to \infty$ and, of course, also weakly and, up to subsequences, almost everywhere.

By assumption, f_k is weakly convergent in $W^{1,p}(\Omega;\mathbb{R}^N)$ to f, so, by the essential uniqueness of the weak limit we conclude that

$$|Df| = F(\tau) + \tau \qquad \text{a.e. in } A_\tau(t). \tag{2.22}$$

Therefore

$$A_\tau(t) = \{x \in \Omega : |Df| > \tau + t\}.$$

By (2.21) and (2.22) and the equality above, there exists a subsequence (not relabeled) of f_k such that

$$|Df_k| \to |Df| \qquad \text{strongly in } L^p(\{x \in \Omega : |Df| > \tau + t\}), \tag{2.23}$$

as $k \to \infty$. Recalling that $F(\tau) \in W^{1,2}(\Omega)$, from the equality in (2.22) we deduce also that

$$(|Df| - \tau)_+ \in W^{1,2}(\Omega).$$

The conclusion follows by the arbitrariness of $\tau > 1$, and the arbitrariness of t, that can be chosen, up to a negligible set, arbitrarily in $(0,\infty)$. □

3 The a priori estimate

This section is devoted to the proof of an a priori estimate which is the main tool in the proof of Theorem 1.1. More precisely, we establish the following result.

Theorem 3.1. *Assume* **(A1)–(A5)** *with p, q such that*

$$2 \le p \le q \le p + 1 - \frac{p+2}{\sigma}.$$

Fix $B_r(x_0) \Subset \Omega$ and consider $u, \bar{u} \in W^{1,p}(B_r(x_0);\mathbb{R}^N)$. Define $\tilde{\mathcal{F}}(v; B_r(x_0))$ as in (2.1) *i.e.,*

$$\tilde{\mathcal{F}}(v; B_r(x_0)) := \mathcal{F}(v; B_r(x_0)) + \int_{B_r(x_0)} \arctan(|v - \bar{u}(x)|^2)\, dx.$$

Let $v \in u + W_0^{1,p}(B_r(x_0);\mathbb{R}^N)$ be a minimizer of $\tilde{\mathcal{F}}$, satisfying

$$v \in W^{2,2}_{\mathrm{loc}}(B_r(x_0);\mathbb{R}^N) \cap W^{1,\infty}_{\mathrm{loc}}(B_r(x_0);\mathbb{R}^N) \quad \text{and} \quad (1 + |Dv|^2)^{\frac{p-2}{2}} |D^2 v|^2 \in L^1_{\mathrm{loc}}(B_r(x_0)).$$

Then, for every $B_{\bar{r}}(\bar{x}) \Subset B_r(x_0)$, every $0 < \rho < r' \le \bar{r}$ and every $\kappa \in \mathbb{R}^N$,

$$\begin{aligned}
&\int_{B_\rho(\bar{x})} (1 + (|Dv| - 1)_+)^{p-4} ((|Dv| - 1)_+)^2 |D^2 v|^2\, dx \\
&\le \frac{c}{(r' - \rho)^4} \int_{B_{r'}(\bar{x})} (1 + (|Dv| - 1)_+)^p\, dx \\
&+ C \left(1 + \|v - \kappa\|^\gamma_{L^\infty(B_{r'}(\bar{x}))}\right) \left(\int_{B_{r'}(\bar{x})} (1 + k(x))^\sigma\, dx\right)^{\frac{p+2}{\sigma(p-q+1)}}
\end{aligned} \tag{3.1}$$

for some constant $C = C(n, N, p, q, L_1, \nu, \operatorname{diam}\Omega)$ and for some exponent $\gamma = \gamma(p, q)$ both positive.

As far as the proof is concerned, the integral $\int_{B_r(x_0)} \arctan(|v-\bar{u}|^2)\,dx$ is a perturbation of $\mathcal{F}(v;B_r(x_0))$ that provides no difficulties. Indeed, denoted $g(x,v) := \arctan(|v-\bar{u}(x)|^2)$, we have that g and its derivatives g_{v^α}, $\alpha = 1,\dots,m$, are bounded. Thus, for the sake of clarity, we prefer to drop this perturbation term, and to state, and prove an a priori estimate for local minimizers of $\mathcal{F}(\cdot;\Omega)$ only, see Theorem 3.2 below. Its proof, that can be easily adapted to obtain Theorem 3.1, follows the one of [17, Proposition 3.1], but it is drastically different in the second step, where we have to take into account the weaker assumption on the function $k(x)$ appearing in **(A5)**.

Theorem 3.2. *Let $u \in W^{1,p}_{\mathrm{loc}}(\Omega;\mathbb{R}^N)$ be a local minimizer of the functional $\mathcal{F}$ at* (1.1) *with the integrand f satisfying assumptions* **(A1)-(A5)** *with p, q such that*

$$2 \le p \le q \le p+1-\frac{p+2}{\sigma}. \tag{3.2}$$

If

$$u \in W^{2,2}_{\mathrm{loc}}(\Omega;\mathbb{R}^N) \cap W^{1,\infty}_{\mathrm{loc}}(\Omega;\mathbb{R}^N) \quad \text{and} \quad (1+|Du|^2)^{\frac{p-2}{2}}|D^2u|^2 \in L^1_{\mathrm{loc}}(\Omega),$$

then, for every $B_{r'}(x_0) \Subset \Omega$, for every ρ, such that $0<\rho<r'$, and for every $\kappa \in \mathbb{R}^N$,

$$\begin{aligned}
&\int_{B_\rho(x_0)} (1+(|Du|-1)_+)^{p-4}((|Du|-1)_+)^2|D^2u|^2\,dx \\
&\le \frac{c}{(r'-\rho)^4}\int_{B_{r'}(x_0)} (1+(|Du|-1)_+)^p\,dx \\
&+C\left(1+\|u-\kappa\|^\gamma_{L^\infty(B_{r'}(x_0))}\right)\left(\int_{B_{r'}(x_0)} (1+k(x))^\sigma\,dx\right)^{\frac{p+2}{\sigma(p-q+1)}}.
\end{aligned} \tag{3.3}$$

for some constant $C = C(n,N,p,q,L_1,\nu,\operatorname{diam}\Omega)$ and for some exponent $\gamma = \gamma(p,q)$, both positive.

Proof. We shall divide the proof into two steps.

Step 1. Our first aim is to prove that for every cut off function $\eta \in C^\infty_0(\Omega)$, the following integral estimate holds

$$\begin{aligned}
&\int_\Omega \eta^4(1+(|Du|-1)_+)^{p-4}((|Du|-1)_+)^2|D^2u|^2\,dx \\
\le &C\int_\Omega \eta^4(1+k^2(x))(1+(|Du|-1)_+)^{2q-p}\,dx + C\int_\Omega |D\eta|^4(1+(|Du|-1)_+)^p\,dx\,,
\end{aligned} \tag{3.4}$$

with $C = C(n,N,q,L_1,\nu)$.
Since u is a local minimizer of the functional $\mathcal{F}(u,\Omega)$, then u satisfies the Euler's system

$$\int_\Omega \sum_{i,\alpha} f_{\xi_i^\alpha}(x,Du)\varphi^\alpha_{x_i}(x)\,dx = 0 \qquad \forall \varphi \in C^\infty_0(\Omega;\mathbb{R}^N),$$

and, using the second variation, for every $s = 1,\dots,n$ it holds

$$\int_\Omega \left\{\sum_{i,j,\alpha,\beta} f_{\xi_i^\alpha\xi_j^\beta}(x,Du)\varphi^\alpha_{x_i}u^\beta_{x_sx_j} + \sum_{i,\alpha} f_{\xi_i^\alpha x_s}(x,Du)\varphi^\alpha_{x_i}\right\}dx = 0 \qquad \forall \varphi \in C^\infty_0(\Omega;\mathbb{R}^N). \tag{3.5}$$

Fix $s = 1, \dots, n$, a cut off function $\eta \in C_0^\infty(\Omega)$ and consider the function

$$\varphi^\alpha := \eta^4 u^\alpha_{x_s}(1 + (|Du| - 1)_+)^{-2}((|Du| - 1)_+)^2 \quad \alpha = 1, \dots, N.$$

Therefore, denoting

$$\Phi(t) := (1 + t)^{-2} t^2,$$

we have

$$\begin{aligned}\varphi^\alpha_{x_i} =& 4\eta^3 \eta_{x_i} u^\alpha_{x_s} \Phi((|Du| - 1)_+) \\ &+ \eta^4 u^\alpha_{x_s x_i} \Phi((|Du| - 1)_+) + \eta^4 u^\alpha_{x_s} \Phi'((|Du| - 1)_+)[(|Du| - 1)_+]_{x_i}.\end{aligned}$$

Thanks to our assumptions on the minimizer u, through a standard density argument, we can use φ as test function in the equation (3.5), thus getting

$$\begin{aligned}0 =& \int_\Omega 4\eta^3 \Phi((|Du| - 1)_+) \sum_{i,j,s,\alpha,\beta} f_{\xi^\alpha_i \xi^\beta_j}(x, Du) \eta_{x_i} u^\alpha_{x_s} u^\beta_{x_s x_j}\, dx \\ &+ \int_\Omega \eta^4 \Phi((|Du| - 1)_+) \sum_{i,j,s,\alpha,\beta} f_{\xi^\alpha_i \xi^\beta_j}(x, Du) u^\alpha_{x_s x_i} u^\beta_{x_s x_j}\, dx \\ &+ \int_\Omega \eta^4 \Phi'((|Du| - 1)_+) \sum_{i,j,s,\alpha,\beta} f_{\xi^\alpha_i \xi^\beta_j}(x, Du) u^\alpha_{x_s} u^\beta_{x_s x_j} [(|Du| - 1)_+]_{x_i}\, dx \\ &+ \int_\Omega 4\eta^3 \Phi((|Du| - 1)_+) \sum_{i,s,\alpha} f_{\xi^\alpha_i x_s}(x, Du) \eta_{x_i} u^\alpha_{x_s}\, dx \\ &+ \int_\Omega \eta^4 \Phi((|Du| - 1)_+) \sum_{i,s,\alpha} f_{\xi^\alpha_i x_s}(x, Du) u^\alpha_{x_s x_i}\, dx \\ &+ \int_\Omega \eta^4 \Phi'((|Du| - 1)_+) \sum_{i,s,\alpha} f_{\xi^\alpha_i x_s}(x, Du) u^\alpha_{x_s} [(|Du| - 1)_+]_{x_i}\, dx \\ =&: I_1 + I_2 + I_3 + I_4 + I_5 + I_6. \end{aligned} \tag{3.6}$$

Let us now estimate the integrals I_j, $j = 1, \dots, 6$. In the estimates we will use some properties of Φ. In particular, notice that since $\Phi(0) = 0$ and $\Phi(t) \le 1$ for every t, then

$$\Phi((|Du| - 1)_+)|Du|^\tau \le (1 + (|Du| - 1)_+)^\tau \quad \forall \tau \ge 0. \tag{3.7}$$

ESTIMATE OF I_1

By assumptions **(A3)** and **(A4)**, we can estimate the integral I_1 by the use of Cauchy-Schwartz and Young's inequalities as follows

$$\begin{aligned}|I_1| \le& 4 \int_\Omega \Phi((|Du| - 1)_+) \left\{ \eta^2 \sum_{i,j,s,\alpha,\beta} f_{\xi^\alpha_i \xi^\beta_j}(x, Du) \eta_{x_i} u^\alpha_{x_s} \eta_{x_j} u^\beta_{x_s} \right\}^{\frac{1}{2}} \times \\ & \times \left\{ \eta^4 \sum_{i,j,s,\alpha,\beta} f_{\xi^\alpha_i \xi^\beta_j}(x, Du) u^\alpha_{x_s x_i} u^\beta_{x_s x_j} \right\}^{\frac{1}{2}} dx\end{aligned}$$

$$
\begin{aligned}
&\leq \quad C(\varepsilon, q, L_1)\int_\Omega \eta^2|D\eta|^2\Phi((|Du|-1)_+)|Du|^q\,dx \\
&+ \quad \varepsilon\int_\Omega \eta^4\Phi((|Du|-1)_+)\sum_{i,j,s,\alpha,\beta} f_{\xi_i^\alpha\xi_j^\beta}(x,Du)u^\alpha_{x_sx_i}u^\beta_{x_sx_j}\,dx,
\end{aligned}
$$

where $\varepsilon > 0$ will be chosen later. Using (3.7) we get

$$
\begin{aligned}
|I_1| \quad &\leq \quad C(\varepsilon, q, L_1)\int_\Omega \eta^2|D\eta|^2(1+(|Du|-1)_+)^q\,dx \\
&+ \quad \varepsilon\int_\Omega \eta^4\Phi((|Du|-1)_+)\sum_{i,j,s,\alpha,\beta} f_{\xi_i^\alpha\xi_j^\beta}(x,Du)u^\alpha_{x_sx_i}u^\beta_{x_sx_j}\,dx. \qquad (3.8)
\end{aligned}
$$

ESTIMATE OF I_3
Since for every $\xi \in \mathbb{R}^{nN}\setminus\{0\}$

$$
f_{\xi_i^\alpha\xi_j^\beta}(x,\xi) = \left(\frac{\tilde f_{tt}(x,|\xi|)}{|\xi|^2} - \frac{\tilde f_t(x,|\xi|)}{|\xi|^3}\right)\xi_i^\alpha\xi_j^\beta + \frac{\tilde f_t(x,|\xi|)}{|\xi|}\delta_{\xi_i^\alpha\xi_j^\beta}
$$

and

$$
[(|Du|-1)_+]_{x_i} = (|Du|)_{x_i} = \frac{1}{|Du|}\sum_{\alpha,s} u^\alpha_{x_ix_s}u^\alpha_{x_s} \qquad \text{a.e. in } \{|Du|\geq 1\} \qquad (3.9)
$$

then, for a.e. $x \in \{|Du| \geq 1\}$,

$$
\begin{aligned}
&\sum_{i,j,s,\alpha,\beta} f_{\xi_i^\alpha\xi_j^\beta}(x,Du)u^\alpha_{x_s}u^\beta_{x_sx_j}[(|Du|-1)_+]_{x_i} \\
&= \left(\frac{\tilde f_{tt}(x,|Du|)}{|Du|^2} - \frac{\tilde f_t(x,|Du|)}{|Du|^3}\right)\sum_{i,j,s,\alpha,\beta} u^\alpha_{x_s}u^\beta_{x_sx_j}u^\alpha_{x_i}u^\beta_{x_j}[(|Du|-1)_+]_{x_i} \\
&+ \frac{\tilde f_t(x,|Du|)}{|Du|}\sum_{i,s,\alpha} u^\alpha_{x_s}u^\alpha_{x_sx_i}[(|Du|-1)_+]_{x_i} \\
&= \left(\frac{\tilde f_{tt}(x,|Du|)}{|Du|} - \frac{\tilde f_t(x,|Du|)}{|Du|^2}\right)\sum_{i,s,\alpha} u^\alpha_{x_s}(|Du|)_{x_s}u^\alpha_{x_i}(|Du|)_{x_i} \\
&+ \frac{\tilde f_t(x,|Du|)}{|Du|}\sum_{i,s,\alpha} u^\alpha_{x_s}u^\alpha_{x_sx_i}(|Du|)_{x_i} \\
&= \left(\frac{\tilde f_{tt}(x,|Du|)}{|Du|} - \frac{\tilde f_t(x,|Du|)}{|Du|^2}\right)\sum_\alpha\left(\sum_i u^\alpha_{x_i}(|Du|)_{x_i}\right)^2 \\
&+ \tilde f_t(x,|Du|)|D(|Du|)|^2.
\end{aligned}
$$

Thus,

$$
I_3 = \int_\Omega \eta^4\Phi'((|Du|-1)_+)\frac{\tilde f_{tt}(x,|Du|)}{|Du|}\sum_\alpha\left(\sum_i u^\alpha_{x_i}(|Du|)_{x_i}\right)^2\,dx
$$

$$+\int_\Omega \eta^4 \Phi'((|Du|-1)_+)\tilde f_t(x,|Du|)\left(|D(|Du|)|^2 - \frac{\sum_\alpha \left(\sum_i u^\alpha_{x_i}(|Du|)_{x_i}\right)^2}{|Du|^2}\right)\,dx.$$

Now, if we use the Cauchy-Schwartz inequality, we have

$$\sum_\alpha \left(\sum_i u^\alpha_{x_i}(|Du|)_{x_i}\right)^2 \le |Du|^2|D(|Du|)|^2.$$

Since

$$\Phi'(t) = \frac{2t}{(1+t)^3}, \tag{3.10}$$

is nonnegative for every $t \ge 0$ and, by **(A2)**, $\tilde f_t(x,|Du|) \ge 0$, then we conclude that

$$I_3 \ge \int_\Omega \eta^4 \Phi'((|Du|-1)_+)\frac{\tilde f_{tt}(x,|Du|)}{|Du|}\sum_\alpha \left(\sum_i u^\alpha_{x_i}(|Du|)_{x_i}\right)^2 dx \ge 0. \tag{3.11}$$

ESTIMATE OF I_4

By using assumption **(A5)** we obtain

$$|I_4| \le c(q)\int_\Omega \eta^3|D\eta|k(x)\Phi((|Du|-1)_+)|Du|^q\,dx.$$

Hence, by (3.7) we get

$$|I_4| \le c(q)\int_\Omega \eta^3|D\eta|k(x)(1+(|Du|-1)_+)^q\,dx. \tag{3.12}$$

ESTIMATE OF I_5

Using assumption **(A5)** and Young's inequality we have that

$$\begin{aligned}
|I_5| &\le c(q)\int_\Omega \eta^4\Phi((|Du|-1)_+)k(x)|Du|^{q-1}|D^2u|\,dx \\
&\le \varepsilon\int_\Omega \eta^4\Phi((|Du|-1)_+)|Du|^{p-2}|D^2u|^2\,dx \\
&+ C(\varepsilon,q)\int_\Omega \eta^4\Phi((|Du|-1)_+)k^2(x)|Du|^{2q-p}\,dx,
\end{aligned}$$

that implies, by (3.7),

$$\begin{aligned}
|I_5| &\le \varepsilon\int_\Omega \eta^4\Phi((|Du|-1)_+)|Du|^{p-2}|D^2u|^2\,dx \\
&+ C(\varepsilon,q)\int_\Omega \eta^4 k^2(x)(1+(|Du|-1)_+)^{2q-p}\,dx.
\end{aligned} \tag{3.13}$$

ESTIMATE OF I_6

Using assumption **(A5)** and equality (3.9), we get

$$|I_6| \le c(q)\int_\Omega \eta^4\Phi'((|Du|-1)_+)k(x)|Du|^{q-1}|Du|\,|D((|Du|-1)_+)|\,dx$$

$$\leq \quad c(n,N,q)\int_\Omega \eta^4\Phi'((|Du|-1)_+)k(x)|Du|^q|D^2u|\,dx.$$

Multiplying and dividing the integrand in the right hand side by $2^{-1/2}(\delta+(|Du|-1)_+)^{1/2}$, with $0<\delta<1$ to be chosen later, we have

$$\begin{aligned}
|I_6| \quad &\leq \quad \int_\Omega \eta^4\left\{\frac{1}{2}\Phi'((|Du|-1)_+)(\delta+(|Du|-1)_+)|Du|^{p-2}|D^2u|^2\right\}^{\frac12}\\
&\times \quad \left\{2c^2\Phi'((|Du|-1)_+)k^2(x)(\delta+(|Du|-1)_+)^{-1}|Du|^{2q-p+2}\right\}^{\frac12}\,dx\\
&\leq \quad \frac{\varepsilon}{2}\int_\Omega \eta^4\Phi'((|Du|-1)_+)\,(\delta+(|Du|-1)_+)\,|Du|^{p-2}|D^2u|^2\,dx\\
&+ \quad C_\varepsilon\int_\Omega \eta^4\Phi'((|Du|-1)_+)k^2(x)\,(\delta+(|Du|-1)_+)^{-1}\,|Du|^{2q-p+2}\,dx\,, \qquad (3.14)
\end{aligned}$$

where we used Young's inequality. In order to estimate the first integral in the right hand side of (3.14) we note that, by virtue of (3.10),

$$t\Phi'(t)=\frac{2t^2}{(1+t)^3}=\frac{2}{1+t}\Phi(t)\leq 2\Phi(t) \qquad \text{for all } t\geq 0.$$

Therefore we have

$$\Phi'((|Du|-1)_+)(|Du|-1)_+\leq 2\Phi((|Du|-1)_+).$$

This also implies, recalling that $\delta<1$,

$$\delta\Phi'((|Du|-1)_+)\chi_{\{|Du|\geq 2\}}\leq \Phi'((|Du|-1)_+)(|Du|-1)_+\chi_{\{|Du|\geq 2\}}\leq 2\Phi((|Du|-1)_+)\chi_{\{|Du|\geq 2\}}.$$

Moreover, since $\Phi'(t)\leq 1$ in $[0,1]$ (see (3.10)), we have that

$$\Phi'((|Du|-1)_+)\chi_{\{1\leq|Du|\leq 2\}}\leq \chi_{\{1\leq|Du|\leq 2\}}.$$

Hence, the following estimate holds:

$$\begin{aligned}
&\frac{\varepsilon}{2}\int_\Omega \eta^4\Phi'((|Du|-1)_+)\,(\delta+(|Du|-1)_+)\,|Du|^{p-2}|D^2u|^2\,dx\\
= \quad &\frac{\varepsilon}{2}\int_\Omega \eta^4\Phi'((|Du|-1)_+)(|Du|-1)_+|Du|^{p-2}|D^2u|^2\,dx\\
+ \quad &\frac{\varepsilon\,\delta}{2}\int_{\{|Du|\geq 2\}}\eta^4\Phi'((|Du|-1)_+)|Du|^{p-2}|D^2u|^2\,dx\\
+ \quad &\frac{\varepsilon\,\delta}{2}\int_{\{1\leq|Du|\leq 2\}}\eta^4\Phi'((|Du|-1)_+)|Du|^{p-2}|D^2u|^2\,dx\\
\leq \quad &\varepsilon\int_\Omega \eta^4\Phi((|Du|-1)_+)|Du|^{p-2}|D^2u|^2\,dx\\
+ \quad &\varepsilon\int_{\{|Du|\geq 2\}}\eta^4\Phi((|Du|-1)_+)|Du|^{p-2}|D^2u|^2\,dx
\end{aligned}$$

$$
\begin{aligned}
&+ \quad \frac{\varepsilon\,\delta}{2}\int_{\{1\le|Du|\le 2\}}\eta^4|Du|^{p-2}|D^2u|^2\,dx \\
&\le \quad 2\varepsilon\int_\Omega \eta^4\Phi((|Du|-1)_+)|Du|^{p-2}|D^2u|^2\,dx+\varepsilon\,\delta\int_\Omega\eta^4|Du|^{p-2}|D^2u|^2\,dx. \qquad (3.15)
\end{aligned}
$$

Let us now estimate the second integral in the right hand side of (3.14). By (3.10) we get

$$
\begin{aligned}
&\frac{\Phi'((|Du|-1)_+)}{\delta+(|Du|-1)_+}=\frac{2}{(1+(|Du|-1)_+)^3}\frac{(|Du|-1)_+}{\delta+(|Du|-1)_+}\\
&\le\frac{2}{(1+(|Du|-1)_+)^3}\le\frac{2}{(1+(|Du|-1)_+)^2}.
\end{aligned}
$$

Hence, we obtain

$$
\begin{aligned}
&\int_\Omega\eta^4\Phi'((|Du|-1)_+)k^2(x)\,(\delta+(|Du|-1)_+)^{-1}\,|Du|^{2q-p+2}\,dx.\\
\le\quad & 2\int_\Omega\eta^4k^2(x)(1+(|Du|-1)_+)^{2q-p}\,dx. \qquad (3.16)
\end{aligned}
$$

Inserting the estimates (3.15) and (3.16) in (3.14), we obtain that

$$
\begin{aligned}
|I_6| \quad\le\quad & C_\varepsilon\int_\Omega\eta^4k^2(x)(1+(|Du|-1)_+)^{2q-p}\,dx\\
+\quad & 2\varepsilon\int_\Omega\eta^4\Phi((|Du|-1)_+)|Du|^{p-2}|D^2u|^2\,dx\\
+\quad & \varepsilon\,\delta\int_\Omega\eta^4|Du|^{p-2}|D^2u|^2\,dx. \qquad (3.17)
\end{aligned}
$$

Observe that equality (3.6) can be written as follows

$$
I_2+I_3=-I_1-I_4-I_5-I_6\,,
$$

and that, by virtue of (3.11), we get

$$
I_2\le|I_1|+|I_4|+|I_5|+|I_6|\,.
$$

Therefore, recalling the estimates (3.8), (3.12), (3.13) and (3.17), we get

$$
\begin{aligned}
&(1-\varepsilon)\int_\Omega\eta^4\Phi((|Du|-1)_+)\sum_{i,j,s,\alpha,\beta}f_{\xi_i^\alpha\xi_j^\beta}(x,Du)u^\alpha_{x_sx_i}u^\beta_{x_sx_j}\,dx\\
&\le C(\varepsilon,q,L_1)\int_\Omega\eta^2|D\eta|^2(1+(|Du|-1)_+)^q\,dx\\
&+C(q)\int_\Omega\eta^3|D\eta|k(x)(1+(|Du|-1)_+)^q\,dx\\
&+C(\varepsilon,q)\int_\Omega\eta^4k^2(x)(1+(|Du|-1)_+)^{2q-p}\,dx\\
&+3\varepsilon\int_\Omega\eta^4\Phi((|Du|-1)_+)|Du|^{p-2}|D^2u|^2\,dx
\end{aligned}
$$

$$+\varepsilon\,\delta\int_\Omega \eta^4|Du|^{p-2}|D^2u|^2\,dx. \tag{3.18}$$

By assumption **(A3)**, the integral in the left hand side of (3.18) can be estimated as

$$\int_\Omega \eta^4\Phi((|Du|-1)_+)\sum_{i,j,s,\alpha,\beta} f_{\xi_i^\alpha\xi_j^\beta}(x,Du)u^\alpha_{x_sx_i}u^\beta_{x_sx_j}\,dx \geq \nu\int_\Omega \eta^4\Phi((|Du|-1)_+)|Du|^{p-2}|D^2u|^2\,dx.$$

Hence

$$\begin{aligned}
&(1-\varepsilon)\nu\int_\Omega \eta^4\Phi((|Du|-1)_+)|Du|^{p-2}|D^2u|^2\,dx\\
\leq\ & 3\varepsilon\int_\Omega \eta^4\Phi((|Du|-1)_+)|Du|^{p-2}|D^2u|^2\,dx\\
+\ & C\int_\Omega \eta^2|D\eta|^2(1+(|Du|-1)_+)^q\,dx + C\int_\Omega \eta^3|D\eta|k(x)(1+(|Du|-1)_+)^q\,dx\\
+\ & C\int_\Omega \eta^4k^2(x)(1+(|Du|-1)_+)^{2q-p}\,dx + \varepsilon\,\delta\int_\Omega \eta^4|Du|^{p-2}|D^2u|^2\,dx,
\end{aligned}$$

with $C=C(n,N,q,\nu,L_1,\varepsilon)$. Choosing $\varepsilon=\frac{\nu}{6+\nu}$, we can reabsorb the first integral in the right hand side of the previous inequality by the left hand side, thus getting

$$\begin{aligned}
&\int_\Omega \eta^4\Phi((|Du|-1)_+)|Du|^{p-2}|D^2u|^2\,dx\\
\leq\ & C\int_\Omega \eta^2|D\eta|^2(1+(|Du|-1)_+)^q\,dx + C\int_\Omega \eta^3|D\eta|k(x)(1+(|Du|-1)_+)^q\,dx\\
+\ & C\int_\Omega \eta^4k^2(x)(1+(|Du|-1)_+)^{2q-p}\,dx + C\delta\int_\Omega \eta^4|Du|^{p-2}|D^2u|^2\,dx.
\end{aligned} \tag{3.19}$$

Since $|Du|^{p-2}|D^2u|^2\in L^1_{\text{loc}}(\Omega)$ by assumption, we can let δ go to 0 in (3.19), and so the last term goes to 0. Moreover

$$\begin{aligned}
&\int_\Omega \eta^3|D\eta|k(x)(1+(|Du|-1)_+)^q\,dx\\
&\leq \frac{1}{2}\int_\Omega \eta^4k^2(x)(1+(|Du|-1)_+)^{2q-p}\,dx + \frac{1}{2}\int_\Omega \eta^2|D\eta|^2(1+(|Du|-1)_+)^p\,dx,
\end{aligned}$$

Thus estimate (3.19) implies

$$\begin{aligned}
&\int_\Omega \eta^4\Phi((|Du|-1)_+)|Du|^{p-2}|D^2u|^2\,dx\\
&\leq C\int_\Omega \eta^2|D\eta|^2(1+(|Du|-1)_+)^q\,dx + C\int_\Omega \eta^4k^2(x)(1+(|Du|-1)_+)^{2q-p}\,dx,
\end{aligned}$$

with $C=C(n,N,q,L_1,\nu)$. Using Young's inequality again in the first integral of the right hand side of previous inequality we get

$$\begin{aligned}
&\int_\Omega \eta^4\Phi((|Du|-1)_+)|Du|^{p-2}|D^2u|^2\,dx\\
&\leq C\int_\Omega |D\eta|^4(1+(|Du|-1)_+)^p\,dx + C\int_\Omega \eta^4(1+k^2(x))(1+(|Du|-1)_+)^{2q-p}\,dx
\end{aligned}$$

Estimate (3.4) follows recalling the definition of Φ.

Step 2. In this step we conlcude the proof.

Fix radii $0 < \rho < s < t \leq r'$ and a cut off function $\eta \in C_0^\infty(B_t)$ such that $\eta \equiv 1$ on B_s and $|D\eta| \leq \frac{2}{t-s}$. By using the assumption $k \in L^\sigma_{\mathrm{loc}}(\Omega)$ and Hölder's inequality with exponents $\frac{\sigma}{2}$, $\frac{\sigma}{\sigma-2}$ in the first integral of the right hand side of (3.4), we get

$$\begin{aligned}
&\int_\Omega \eta^4(1+(|Du|-1)_+)^{p-4}((|Du|-1)_+)^2|D^2u|^2\,dx \\
\leq\ & C\left(\int_\Omega \eta^4(1+k(x))^\sigma\,dx\right)^{\frac{2}{\sigma}}\left(\int_\Omega \eta^4(1+(|Du|-1)_+)^{\frac{(2q-p)\sigma}{\sigma-2}}\,dx\right)^{\frac{\sigma-2}{\sigma}} \\
+\ & C\int_\Omega |D\eta|^4(1+(|Du|-1)_+)^p\,dx.
\end{aligned} \tag{3.20}$$

Note that, by the assumptions on q, we have that $\frac{(2q-p)\sigma}{\sigma-2} \leq p+2$. In the case $\frac{(2q-p)\sigma}{\sigma-2} < p+2$, we can apply Hölder's inequality with exponents $\frac{(p+2)(\sigma-2)}{(2q-p)\sigma}$ and $\frac{(p+2)(\sigma-2)}{(p+2)(\sigma-2)-\sigma(2q-p)}$ thus obtaining

$$\begin{aligned}
&\int_\Omega \eta^4(1+(|Du|-1)_+)^{p-4}((|Du|-1)_+)^2|D^2u|^2\,dx \\
&\leq C_1\left(\int_\Omega \eta^4(1+k(x))^\sigma\,dx\right)^{\frac{2}{\sigma}}\left(\int_\Omega \eta^4(1+(|Du|-1)_+)^{p+2}\,dx\right)^{\frac{2q-p}{p+2}} \\
&+C_2\int_\Omega |D\eta|^4(1+(|Du|-1)_+)^p\,dx,
\end{aligned} \tag{3.21}$$

where $C_1 = C_1(n,p,q,\sigma,\operatorname{diam}\Omega)$ and $C_2 = C_2(n,N,q,L_1,\nu)$. In the case $\frac{(2q-p)\sigma}{\sigma-2} = p+2$, the estimate (3.20) obviously gives (3.21), without applying Hölder's inequality.

Since by virtue of assumption (3.2) we have that $\frac{2q-p}{p+2} < 1$, we can use Young's inequality in the first term in the right hand side of previous estimate and the inequality $(1+|a|)^{p+2} \leq 2(1+|a|)^p|a|^2 + 2(1+|a|)^p$, thus getting

$$\begin{aligned}
&\int_\Omega \eta^4(1+(|Du|-1)_+)^{p-4}((|Du|-1)_+)^2|D^2u|^2\,dx \\
&\leq \frac{\varepsilon}{2}\int_\Omega \eta^4(1+(|Du|-1)_+)^{p+2}\,dx + C_2\int_\Omega |D\eta|^4(1+(|Du|-1)_+)^p\,dx \\
&+\frac{C_3}{\varepsilon^{\gamma/2}}\left(\int_\Omega \eta^4(1+k(x))^\sigma\,dx\right)^{\frac{p+2}{\sigma(p-q+1)}} \\
&\leq \varepsilon\int_\Omega \eta^4(1+(|Du|-1)_+)^p((|Du|-1)_+)^2\,dx \\
&+(\varepsilon+C_2)\int_\Omega (\eta^4+|D\eta|^4)(1+(|Du|-1)_+)^p\,dx \\
&+\frac{C_3}{\varepsilon^{\gamma/2}}\left(\int_\Omega \eta^4(1+k(x))^\sigma\,dx\right)^{\frac{p+2}{\sigma(p-q+1)}},
\end{aligned}$$

where $\gamma = \gamma(p,q)$ and $C_3 = C_3(n,N,p,q,L_1,\nu,\sigma,\operatorname{diam}\Omega)$ are positive. By the interpolation inequality of Proposition 2.5, with $\psi = \eta^2$, and using equality (2.6), we deduce that for every $\kappa \in \mathbb{R}^N$

$$\int_\Omega \eta^4(1+(|Du|-1)_+)^{p-4}((|Du|-1)_+)^2|D^2u|^2\,dx$$

$$
\begin{aligned}
&\leq c\varepsilon\|u-\kappa\|^2_{L^\infty(B_{r'})}\int_\Omega \eta^4(1+(|Du|-1)_+)^{p-4}((|Du|-1)_+)^2|D^2u|^2\,dx,\\
&+c\varepsilon\left(1+\|u-\kappa\|^2_{L^\infty(B_{r'})}\right)\int_\Omega(\eta^4+|D\eta|^4)(1+(|Du|-1)_+)^p\,dx\\
&+(\varepsilon+C_2)\int_\Omega(\eta^4+|D\eta|^4)(1+(|Du|-1)_+)^p\,dx\\
&+\frac{C_3}{\varepsilon^{\gamma/2}}\left(\int_\Omega\eta^4(1+k(x))^\sigma\,dx\right)^{\frac{p+2}{\sigma(p-q+1)}}.
\end{aligned}
$$

By the properties of the cut off function η, previous estimate gives

$$
\begin{aligned}
&\int_{B_s}(1+(|Du|-1)_+)^{p-4}((|Du|-1)_+)^2|D^2u|^2\,dx\\
&\leq c\varepsilon\|u-\kappa\|^2_{L^\infty(B_{r'})}\int_{B_t}(1+(|Du|-1)_+)^{p-4}((|Du|-1)_+)^2|D^2u|^2\,dx\\
&+c\left(\varepsilon\left(1+\|u-\kappa\|^2_{L^\infty(B_{r'})}\right)\left(1+\frac{1}{(t-s)^4}\right)+1\right)\int_{B_{r'}}(1+(|Du|-1)_+)^p\,dx\\
&+\frac{C_3}{\varepsilon^{\gamma/2}}\left(\int_\Omega\eta^4(1+k(x))^\sigma\,dx\right)^{\frac{p+2}{\sigma(p-q+1)}}.
\end{aligned}
$$

If $\|u-\kappa\|_{L^\infty(B_{r'})}=0$ the claim follows, otherwise, since c is a constant independent of ε, we can choose

$$
\varepsilon=\frac{1}{2c\left(1+\|u-\kappa\|^2_{L^\infty(B_{r'})}\right)}
$$

thus getting

$$
\begin{aligned}
&\int_{B_s}(1+(|Du|-1)_+)^{p-4}((|Du|-1)_+)^2|D^2u|^2\,dx\\
&\leq\frac{1}{2}\int_{B_t}(1+(|Du|-1)_+)^{p-4}((|Du|-1)_+)^2|D^2u|^2\,dx\\
&+c\left(1+\frac{1}{(t-s)^4}\right)\int_{B_{r'}}(1+(|Du|-1)_+)^p\,dx\\
&+C(1+\|u-\kappa\|^\gamma_{L^\infty(B_{r'})})\left(\int_{B_{r'}}(1+k(x))^\sigma\,dx\right)^{\frac{p+2}{\sigma(p-q+1)}}.
\end{aligned}
$$

Taking into account that

$$
1+\frac{1}{(t-s)^4}\leq\frac{c(\operatorname{diam}\Omega)}{(t-s)^4}
$$

and using the iteration Lemma 2.1 we get (3.3). □

4 Proof of Theorem 1.1

Before going on with the proof of the main Theorem, we state two approximation results for the energy density f. More precisely, the first one approximates f through an increasing sequence (f_j) of integrands satisfying p-growth conditions, the second one approximates each f_j through a sequence of uniformly elliptic integrands having smooth dependence on the x-variable.

Lemma 4.1. *Let $f : \Omega\times\mathbb{R}^{nN} \to [0, +\infty)$ be a Carathéodory function which satisfies assumptions* **(A1)**–**(A6)**. *Then there exists a sequence of Carathéodory functions $f_j : \Omega \times \mathbb{R}^{nN} \to [0, +\infty)$, monotonically convergent to f, such that*

(I) *for a.e. $x \in \Omega$ and every $\xi \in \mathbb{R}^{nN} \setminus B_1(0)$, $f_j(x, \xi) = \tilde{f}_j(x, |\xi|)$, $t \mapsto \tilde{f}_j(x, t)$ nondecreasing,*

(II) *for a.e. $x \in \Omega$, for every $\xi \in \mathbb{R}^{nN}$, and for every j, $f_j(x, \xi) \leq f_{j+1}(x, \xi) \leq f(x, \xi)$,*

(III) *for a.e. $x \in \Omega$, every $\xi \in \mathbb{R}^{nN} \setminus B_1(0)$ and for all $\lambda \in \mathbb{R}^{nN}$ we have*

$$\langle D_{\xi\xi} f_j(x, \xi)\lambda, \lambda\rangle \geq \bar{\nu}(1 + |\xi|^2)^{\frac{p-2}{2}}|\lambda|^2$$

with $\overline{\nu}$ depending only on p and ν, where ν is the constant in **(A3)**.

(IV) *for a.e. $x \in \Omega$ and for every $\xi \in \mathbb{R}^{nN}$, there exist K_0, independent of j, and K_1, depending on j, such that*

$$K_0(|\xi|^p - 1) \leq f_j(x, \xi) \leq L(1 + |\xi|)^q,$$

and

$$f_j(x, \xi) \leq K_1(j)(1 + |\xi|)^p, \tag{4.1}$$

where L is the constant in **(A1)**,

(V) *using the same notation as in* **(A4)**, *for every $\xi \mapsto f_j(x, \xi)$ the vector field $x \mapsto D^+_\xi f_j(x, \xi)$ is weakly differentiable and for a.e. $x \in \Omega$ and for every $\xi \in \mathbb{R}^{nN} \setminus B_1(0)$*

$$|D_x D^+_\xi f_j(x, \xi)| \leq k(x)(1 + |\xi|^2)^{\frac{q-1}{2}},$$
$$|D_x D^+_\xi f_j(x, \xi)| \leq C(j)k(x)(1 + |\xi|^2)^{\frac{p-1}{2}},$$

where k is the function in **(A5)**,

(VI) *for a.e. $x \in \Omega$, for every $\xi \in \mathbb{R}^{nN} \setminus B_1(0)$ and for every $\lambda > 1$,*

$$f_j(x, \lambda\xi) \leq \lambda^\mu f_j(x, \xi),$$

where μ is the constant in **(A6)**.

For the proof we refer to Lemma 4.1 in [11]. In particular, the presence of the function $k(x)$ in **(V)** is due to assumption **(A5)**. The proof of **(VI)**, absent in [11] trivially follows by **(A6)** and the definition of f_j.

Lemma 4.2. *Let $f : \Omega \times \mathbb{R}^{nN} \to [0, +\infty)$, be a Carathéodory function which satisfies assumptions* **(A1)–(A6)**. *Let f_j be the sequence of functions in Lemma 4.1. Fixed an open set $\Omega' \Subset \Omega$, for every j there exists a sequence of C^2-functions $f_{j,h} : \Omega' \times \mathbb{R}^{nN} \to [0, +\infty)$, $f_{j,h}$ convex in the last variable, such that $f_{j,h}$ converges to f_j pointwise a.e. in Ω' and everywhere in $\mathbb{R}^{nN}$. Moreover, $f_{j,h}$ satisfies the following properties:*

($\mathcal{A}1$) there exist $\tilde{c}_1, \tilde{c}_2 > 0$, independent of j and of h, and $\tilde{K}_1(j) > 0$, independent of h, such that for all $(x, \xi) \in \Omega' \times \mathbb{R}^{nN}$

$$f_{j,h}(x, \xi) \leq (L + 1)(1 + |\xi|)^q,$$

and

$$\tilde{c}_1 |\xi|^p - \tilde{c}_2 \leq f_{j,h}(x, \xi) \leq \tilde{K}_1(j)(1 + |\xi|)^p, \tag{4.2}$$

($\mathcal{A}2$) for every $x \in \Omega'$ and $\xi \in \mathbb{R}^{nN} \setminus B_2(0)$, $f_{j,h}(x, \xi) = \tilde{f}_{j,h}(x, |\xi|)$, with $t \mapsto \tilde{f}_{j,h}(x, t)$ nondecreasing,

($\mathcal{A}3$) there exists $\tilde{\nu}$, depending on ν and p, but not on j nor h, such that for every $x \in \Omega'$, $\xi \in \mathbb{R}^{nN} \setminus B_2(0)$ and $\lambda \in \mathbb{R}^{nN}$

$$\tilde{\nu}(1 + |\xi|)^{p-2} |\lambda|^2 \leq \langle D_{\xi\xi} f_{j,h}(x, \xi)\lambda, \lambda\rangle,$$

($\mathcal{A}4$) there exists $\overline{L_1}(j) > 0$, independent of h, such that for all $(x, \xi) \in \Omega' \times \mathbb{R}^{nN} \setminus B_2(0)$

$$|D_{\xi\xi} f_{j,h}(x, \xi)| \leq \overline{L_1}(j)(1 + |\xi|)^{p-2},$$

($\mathcal{A}5$) there exists $\overline{C}(j) > 0$, independent of h, such that for every $x \in \Omega'$ and $\xi \in \mathbb{R}^{nN} \setminus B_2(0)$,

$$|D_{\xi x} f_{j,h}(x, \xi)| \leq \overline{C}(j) k_h(x)(1 + |\xi|)^{p-1},$$

where $k_h \in C^\infty(\Omega')$ is a nonnegative function, such that $k_h \to k$ in $L^\sigma(\Omega')$, where $\sigma \geq p + 2$ is the integrability exponent in **(A5)**,

($\mathcal{A}6$) for every $\lambda > 1$, for every $x \in \Omega'$ and for every $\xi \in \mathbb{R}^{nN} \setminus B_2(0)$

$$f_{j,h}(x, \lambda\xi) \leq \lambda^\mu f_{j,h}(x, \xi),$$

($\mathcal{H}1$) there exists $\Lambda(j, h) > 0$ such that for every $x \in \Omega'$ and $\xi \in \overline{B_2(0)}$

$$|D_{\xi x} f_{j,h}(x, \xi)| \leq \Lambda(j, h)(1 + |\xi|)^{p-1}.$$

($\mathcal{H}2$) there exists $\mu(h) > 0$, independent of j, such that for all $(x, \xi) \in \Omega' \times \mathbb{R}^{nN}$ and for all $\lambda \in \mathbb{R}^{nN}$

$$\mu(h)(1 + |\xi|)^{p-2} |\lambda|^2 \leq \langle D_{\xi\xi} f_{j,h}(x, \xi)\lambda, \lambda\rangle.$$

($\mathcal{H}3$) there exists $\sigma(j, h) > 0$ such that for all $(x, \xi) \in \Omega' \times \mathbb{R}^{nN}$

$$|D_{\xi\xi} f_{j,h}(x, \xi)| \leq \sigma(j, h)(1 + |\xi|)^{p-2}.$$

For the proof we refer to Proposition 4.1 in [10].

Now we are in position to give the proof of our main result.

Proof of Theorem 1.1. Let u be a local minimizer of $\mathcal{F}$ in (1.1). By Theorem 2.2, $u \in L^{\infty}_{\rm loc}(\Omega;\mathbb{R}^N)$ and so we have to prove that $Du \in L^{p+2}_{\rm loc}(\Omega;\mathbb{R}^{nN})$, the higher differentiability result (1.6) and the estimate (1.7).

Step 1. The approximation

Fixed an open set $\Omega' \Subset \Omega$, let f_j be defined as in Lemma 4.1 and $f_{j,h}$ be the sequence defined in Lemma 4.2. Fix a ball $B_r(x_0) \Subset \Omega'$. Consider the smooth functions $u_{\frac{1}{i}} \in C^{\infty}(\Omega')$,

$$u_{\frac{1}{i}}(x) = \int_{B_1} \sigma(y) u\left(x + \frac{1}{i}y\right)\, dy$$

where $\sigma \in C^{\infty}_c(B_1)$ is the usual nonnegative, radially symmetric mollifier and $i \in \mathbb{N}$, $i \geq i_0$, with i_0 sufficiently large so that $B_{r+\frac{1}{i_0}}(x_0) \Subset \Omega'$. Without loss of generality we can assume $i_0 = 1$.

Inspired by [6], we introduce the functional

$$\tilde{\mathcal{F}}_{i,j,h}(w; B_r(x_0)) := \int_{B_r(x_0)} f_{j,h}(x, Dw)\, dx + \int_{B_r(x_0)} \arctan(|w - u_{\frac{1}{i}}(x)|^2)\, dx,$$

and the corresponding minimization problem

$$\min\left\{\tilde{\mathcal{F}}_{i,j,h}(w; B_r(x_0)) \,:\, w \in u + W^{1,p}_0(B_r(x_0);\mathbb{R}^N)\right\}.$$

By virtue of (4.2) and ($\mathcal{H}2$), we have the existence of a minimizer $v_{i,j,h} \in u + W^{1,p}_0(B_r(x_0);\mathbb{R}^N)$ of the problem above.
From the left inequality in (4.2) and the minimality of $v_{i,j,h}$, there exists $c > 0$, independent of i, j and h, such that

$$\begin{aligned}\int_{B_r(x_0)} |Dv_{i,j,h}|^p\, dx &\leq c\int_{B_r(x_0)} (1 + f_{j,h}(x, Dv_{i,j,h}) + \arctan(|v_{i,j,h}(x) - u_{\frac{1}{i}}(x)|^2))\, dx \\ &\leq c\int_{B_r(x_0)} (1 + f_{j,h}(x, Du) + \arctan(|u(x) - u_{\frac{1}{i}}(x)|^2))\, dx. \qquad (4.3)\end{aligned}$$

By the right inequality in (4.2) and the pointwise convergence of $f_{j,h}$ to f_j, we can use the dominated convergence theorem to deduce that

$$\limsup_{h\to+\infty} \int_{B_r(x_0)} f_{j,h}(x, Du)\, dx = \int_{B_r(x_0)} f_j(x, Du)\, dx.$$

The properties of $\limsup$ imply that there exists $h_0 = h_0(j) \in \mathbb{N}$, such that

$$\begin{aligned}\int_{B_r(x_0)} f_{j,h}(x, Du)\, dx &\leq \limsup_{h\to+\infty} \int_{B_r(x_0)} (1 + f_{j,h}(x, Du))\, dx \\ &= \int_{B_r(x_0)} (1 + f_j(x, Du))\, dx \qquad \forall j,\ \forall h \geq h_0(j). \qquad (4.4)\end{aligned}$$

Since $f_j \leq f$, we have

$$\int_{B_r(x_0)} f_{j,h}(x, Du)\, dx \leq \int_{B_r(x_0)} (1 + f(x, Du))\, dx \qquad \forall j,\ \forall h \geq h_0(j). \qquad (4.5)$$

Inserting (4.5) in (4.3), we get that there exists c, independent of i, j, h, such that

$$\begin{aligned}\int_{B_r(x_0)} |Dv_{i,j,h}|^p \, dx &\leq \frac{c}{3} \int_{B_r(x_0)} (1 + f(x, Du) + \arctan(|u(x) - u_{\frac{1}{i}}(x)|^2) \, dx \\ &\leq c \int_{B_r(x_0)} (1 + f(x, Du)) \, dx \qquad \forall i, \ \forall j, \ \forall h \geq h_0(j). \end{aligned} \tag{4.6}$$

So there exists $v_{j,h} \in u + W_0^{1,p}(B_r(x_0); \mathbb{R}^N)$ such that, up to a subsequence, as $i \to \infty$ we have

$$v_{i,j,h} \rightharpoonup v_{j,h} \text{ weakly in } W^{1,p}(B_r(x_0); \mathbb{R}^N) \quad \forall j, \ \forall h \geq h_0(j) \tag{4.7}$$

$$v_{i,j,h} \to v_{j,h} \text{ a.e. in } B_r(x_0) \quad \forall j, \ \forall h \geq h_0(j). \tag{4.8}$$

By the first inequality in (4.2) and by Proposition 8.7, Corollary 8.8, Proposition 8.10 in [15], see also Proposition 2.20 in [21], $\{\tilde{\mathcal{F}}_{i,j,h}(\cdot; B_r(x_0))\}$ Γ-converges, as $i \to \infty$, to

$$\tilde{\mathcal{F}}_{j,h}(w; B_r(x_0)) := \int_{B_r(x_0)} f_{j,h}(x, Dw) \, dx + \int_{B_r(x_0)} \arctan(|w - u(x)|^2) \, dx \quad \forall j, \ \forall h \geq h_0(j)$$

with respect to the weak $W^{1,p}$-topology induced on $u + W_0^{1,p}(B_r(x_0); \mathbb{R}^N)$. This implies that for every j and every $h \geq h_0(j)$,

$$v_{j,h} \text{ is a minimizer of } \tilde{\mathcal{F}}_{j,h}(\cdot; B_r(x_0)) \text{ in } u + W_0^{1,p}(B_r(x_0); \mathbb{R}^N),$$

see Corollary 7.20 in [15].
From the first inequality in (4.2) and the minimality of $v_{j,h}$, there exists $c > 0$, independent of j and h, such that

$$\begin{aligned}\int_{B_r(x_0)} |Dv_{j,h}|^p \, dx &\leq c \int_{B_r(x_0)} \Big(1 + f_{j,h}(x, Dv_{j,h}) + \arctan(|v_{j,h}(x) - u(x)|^2)\Big) \, dx \\ &\leq c \int_{B_r(x_0)} (1 + f_{j,h}(x, Du)) \, dx.\end{aligned}$$

The inequalities above, the right inequality in (4.2), the condition (4.1), the dominated convergence theorem and the assertion **(II)** in Lemma 4.1 imply

$$\begin{aligned}&\limsup_{h \to +\infty} \int_{B_r(x_0)} |Dv_{j,h}|^p \, dx \leq c \limsup_{h \to +\infty} \int_{B_r(x_0)} (1 + f_{j,h}(x, Du)) \, dx \\ \leq \ & c \int_{B_r(x_0)} (1 + f_j(x, Du)) \, dx \leq c \int_{B_r(x_0)} (1 + f(x, Du)) \, dx.\end{aligned}$$

Therefore, there exists a function $v_j \in u + W_0^{1,p}(B_r(x_0))$, such that, up to a subsequence, as $h \to +\infty$,

$$v_{j,h} \rightharpoonup v_j \text{ weakly in } W^{1,p}(B_r(x_0); \mathbb{R}^N) \quad \forall j, \tag{4.9}$$

$$v_{j,h} \to v_j \text{ a.e. in } B_r(x_0) \quad \forall j. \tag{4.10}$$

By the p-growth conditions (4.1) and (4.2), by the convexity of the functions $f_{j,h}(x, \cdot)$ and f_j, the pointwise convergence a.e. in Ω' of $f_{j,h}(\cdot, \xi)$ to $f_j(\cdot, \xi)$, we have that the sequence $\{\tilde{\mathcal{F}}_{j,h}(\cdot; B_r(x_0))\}_{h \geq h_0(j)}$ Γ-converges to

$$\tilde{\mathcal{F}}_j(w; B_r(x_0)) := \int_{B_r(x_0)} f_j(x, Dw) \, dx + \int_{B_r(x_0)} \arctan(|w - u(x)|^2) \, dx,$$

with respect to the weak $W^{1,p}$-topology induced on $u + W_0^{1,p}(B_r(x_0);\mathbb{R}^N)$, see Proposition 6.21, Theorem 5.14 in [15] and Theorem 2.8, Proposition 2.20 in [21]. Moreover, by the minimality of $v_{j,h}$ and (4.9), we have that for every j

$$v_j \text{ is a minimizer of } \tilde{\mathcal{F}}_j(\cdot; B_r(x_0)) \text{ in } u + W_0^{1,p}(B_r(x_0);\mathbb{R}^N) \tag{4.11}$$

and

$$\lim_{h\to\infty} \tilde{\mathcal{F}}_{j,h}(v_{j,h}; B_r(x_0)) = \tilde{\mathcal{F}}_j(v_j; B_r(x_0)), \tag{4.12}$$

see Corollary 7.20 in [15].

By the weak lower semicontinuity of the L^p-norm, (4.9) implies

$$\int_{B_r(x_0)} |Dv_j|^p \, dx \le c \int_{B_r(x_0)} (1 + f(x, Du)) \, dx.$$

As a consequence we have, up to a subsequence, that there exists a function $v \in u + W_0^{1,p}(B_r(x_0);\mathbb{R}^N)$, such that, up to a subsequence, as $j \to +\infty$,

$$v_j \rightharpoonup v \;\text{ weakly in } W^{1,p}(B_r(x_0);\mathbb{R}^N), \tag{4.13}$$

$$v_j \to v \;\text{ a.e. in }\; B_r(x_0). \tag{4.14}$$

We claim that v is a minimizer of $\mathcal{F}(\cdot\,; B_r(x_0))$.

Using (4.12) and the minimality of $v_{j,h}$, we get

$$\begin{aligned} \int_{B_r(x_0)} f_j(x, Dv_j)\, dx &\le \int_{B_r(x_0)} \big(f_j(x, Dv_j) + \arctan(|v_j - u|^2)\big)\, dx \\ &= \lim_{h\to\infty} \int_{B_r(x_0)} \big(f_{j,h}(x, Dv_{j,h}) + \arctan(|v_{j,h} - u|^2)\big)\, dx \\ &\le \lim_{h\to\infty} \int_{B_r(x_0)} f_{j,h}(x, Du)\, dx \end{aligned}$$

By the dominated convergence theorem and using that $f_j \le f$, see (**I**) in Lemma 4.1, we have

$$\lim_{h\to\infty} \int_{B_r(x_0)} f_{j,h}(x, Du)\, dx = \int_{B_r(x_0)} f_j(x, Du)\, dx \le \int_{B_r(x_0)} f(x, Du)\, dx.$$

Collecting the above inequalities we get

$$\int_{B_r(x_0)} f_j(x, Dv_j)\, dx \le \int_{B_r(x_0)} f(x, Du)\, dx. \tag{4.15}$$

Now, fix $j_0 \in \mathbb{N}$. By (**I**) in Lemma 4.1 and (4.15)

$$\int_{B_r(x_0)} f_{j_0}(x, Dv_j)\, dx \le \int_{B_r(x_0)} f_j(x, Dv_j)\, dx \le \int_{B_r(x_0)} f(x, Du)\, dx \quad \forall j \ge j_0. \tag{4.16}$$

Therefore, (4.13) and the lower semicontinuity of the functional $v \mapsto \int_{B_r(x_0)} f_{j_0}(x, Dv)\, dx$ with respect to the weak topology in $W^{1,p}$ imply

$$\int_{B_r(x_0)} f_{j_0}(x, Dv)\, dx \le \liminf_{j\to\infty} \int_{B_r(x_0)} f_{j_0}(x, Dv_j)\, dx \le \int_{B_r(x_0)} f(x, Du)\, dx.$$

By (**I**) in Lemma 4.1 we can use the monotone convergence theorem, so to obtain, as $j_0 \to \infty$,

$$\int_{B_r(x_0)} f(x, Dv)\, dx \le \int_{B_r(x_0)} f(x, Du)\, dx.$$

Since u is a minimizer of $\mathcal{F}(\cdot; B_r(x_0))$ and $v \in u + W_0^{1,p}(B_r(x_0))$ we conclude that v is a minimizer of $\mathcal{F}(\cdot; B_r(x_0))$.

Step 2. In this step we prove that $v = u$.

Using that u and v are minimizers of $\mathcal{F}(\cdot\,; B_r(x_0))$ in $u + W_0^{1,p}(B_r(x_0); \mathbb{R}^N)$, the assertion **(II)** in Lemma 4.1, the convergence (4.13), the semicontinuity of the functionals $\tilde{\mathcal{F}}_j(\cdot; B_r(x_0))$ with respect to the weak topology in $u+W_0^{1,p}(B_r(x_0); \mathbb{R}^N)$ and using also that v_j minimizes $\tilde{\mathcal{F}}_j(\cdot\,; B_r(x_0))$, see (4.11), we deduce that

$$\begin{aligned}
&\int_{B_r(x_0)} f(x, Dv)\, dx \le \int_{B_r(x_0)} \left(f(x, Dv) + \arctan(|v-u|^2)\right)\, dx \\
= &\lim_{j_0\to\infty} \int_{B_r(x_0)} \left(f_{j_0}(x, Dv) + \arctan(|v-u|^2)\right)\, dx \\
\le &\lim_{j_0\to\infty} \liminf_{j\to\infty} \int_{B_r(x_0)} \left(f_{j_0}(x, Dv_j) + \arctan(|v_j-u|^2)\right)\, dx \\
\le &\liminf_{j\to\infty} \int_{B_r(x_0)} \left(f_{j}(x, Dv_j) + \arctan(|v_j-u|^2)\right)\, dx \\
\le &\liminf_{j\to\infty} \int_{B_r(x_0)} f_j(x, Du)\, dx \le \int_{B_r(x_0)} f(x, Du)\, dx = \int_{B_r(x_0)} f(x, Dv)\, dx.
\end{aligned}$$

Since f is nonnegative, this obviously implies that

$$\int_{B_r(x_0)} \arctan(|v-u|^2)\, dx = 0$$

and so $u \equiv v$ a.e. in $B_r(x_0)$.

Step 3. In this step we prove the regularity of u and the estimate (1.7).

By virtue of Lemma 4.2, the integrands of the functionals $\tilde{\mathcal{F}}_{i,j,h}$ satisfy the assumptions of Theorems 2.4 and 3.1. By Theorem 2.4,

$$v_{i,j,h} \in W^{2,2}_{\mathrm{loc}}(B_r(x_0); \mathbb{R}^N) \cap W^{1,\infty}_{\mathrm{loc}}(B_r(x_0); \mathbb{R}^N) \tag{4.17}$$

and

$$(1 + |Dv_{i,j,h}|^2)^{\frac{p-2}{4}} |D^2 v_{i,j,h}| \in L^2_{\mathrm{loc}}(B_r(x_0)).$$

Let $\mathcal{G}$ be the function defined by

$$\mathcal{G}(t) := \int_0^t s(2+s)^{\frac{p-4}{2}}\, ds.$$

Of course, $\mathcal{G}((|Dv_{i,j,h}| - 2)_+) \in L^2_{\mathrm{loc}}(B_r(x_0))$. We claim that

$$\mathcal{G}((|Dv_{i,j,h}| - 2)_+) \in W^{1,2}(B_{\frac{r}{8\sqrt{n}}}(x_0)) \tag{4.18}$$

and the following estimate holds:

$$\int_{B_{\frac{r}{8\sqrt{n}}}(x_0)} |D(\mathcal{G}((|Dv_{i,j,h}|-2)_+))|^2\,dx \le C\left(\int_{B_r(x_0)} (1+f(x,Du))\,dx\right)$$
$$+\quad C\left(1+\int_{B_r(x_0)} f(x,Du)\,dx\right)^{\tilde\vartheta}\left(\int_{B_r(x_0)}(1+k_h^\sigma)\,dx\right)^{\frac{p+2}{\sigma(p-q+1)}}\qquad \forall i,\forall j,\forall h\ge h_0(j)\tag{4.19}$$

where $\tilde\vartheta=\tilde\vartheta(q,p,n)$ and $C=C(n,N,p,q,L_1,\nu,r,\operatorname{diam}\Omega)$ are positive constants independent of i, j and h and $h_0(j)$ is as in (4.4).

To prove the claim we start by applying Theorem 3.1 to the functional $\tilde{\mathcal{F}}=\tilde{\mathcal{F}}_{i,j,h}$, with

$$\bar x=x_0,\quad \rho=\frac{r}{8\sqrt{n}},\quad r'=\frac{r}{4\sqrt{n}},\quad \bar r=\frac{r}{2},\quad \kappa=(v_{i,j,h})_{x_0,\frac{r}{2}}.$$

Notice that $\tilde{\mathcal{F}}_{i,j,h}$ has an integrand $f_{j,k}$ that satisfies, with respect to the gradient variable, the radial condition and the p-ellipticity property outside $B_2(0)$.

Since

$$\frac{(|Dv_{i,j,h}|-2)_+}{2+(|Dv_{i,j,h}|-2)_+}|Dv_{i,j,h}|^{\frac{p-2}{2}}=(|Dv_{i,j,h}|-2)_+(2+(|Dv_{i,j,h}|-2)_+)^{\frac{p-4}{2}}$$

a.e. in $B_r(x_0)$, then Theorem 3.1, implies (4.18) and, by (3.1), for every i,j,h,

$$\int_{B_{\frac{r}{8\sqrt{n}}}(x_0)} |D(\mathcal{G}((|Dv_{i,j,h}|-2)_+))|^2\,dx$$
$$\le\quad C\int_{B_{\frac{r}{4\sqrt{n}}}(x_0)}(2+(|Dv_{i,j,h}|-2)_+)^p\,dx$$
$$+\quad C\left(1+\|v_{i,j,h}-(v_{i,j,h})_{x_0,\frac{r}{2}}\|^\gamma_{L^\infty(B_{\frac{r}{4\sqrt{n}}}(x_0))}\right)\left(\int_{B_{\frac{r}{4\sqrt{n}}}(x_0)}(1+k_h^\sigma)\,dx\right)^{\frac{p+2}{\sigma(p-q+1)}},\tag{4.20}$$

where $\gamma=\gamma(p,q)$ and $C=C(n,N,p,q,L_1,\nu,r,\operatorname{diam}\Omega)$ are positive constants, independent of i, j and h, and k_h is the function in $(\mathcal{A}5)$ (see Lemma 4.2).
Let us now estimate the right hand side of (4.20).
Since $u_{\frac{1}{i}}\in W^{1,p}(B_r(x_0);\mathbb{R}^N)\cap L^\infty(B_r(x_0);\mathbb{R}^N)$ and $f_{j,h}$ satisfies the properties listed in Lemma 4.2, we can apply Theorem 2.3 to the functions $v_{i,j,h}$, so obtaining

$$\|v_{i,j,h}-(v_{i,j,h})_{x_0,\frac{r}{2}}\|_{L^\infty(B_{\frac{r}{4\sqrt{n}}}(x_0))}\le c\left\{1+\int_{B_{\frac{r}{2}}(x_0)} f_{j,h}(x,Dv_{i,j,h})\,dx\right\}^\vartheta,\tag{4.21}$$

where $\vartheta=\vartheta(q,p,n)$.

By the minimality of $v_{i,j,h}$ we have

$$\int_{B_{\frac{r}{2}}(x_0)} f_{j,h}(x,Dv_{i,j,h})\,dx\le\int_{B_r(x_0)}\left(f_{j,h}(x,Dv_{i,j,h})+\arctan(|v_{i,j,h}-u_{\frac{1}{i}}|^2)\right)\,dx$$

$$\leq \int_{B_r(x_0)} \left(f_{j,h}(x,Du) + \arctan(|u-u_{\frac{1}{i}}|^2)\right)\,dx \leq c\int_{B_r(x_0)} (f_{j,h}(x,Du)+1)\,dx.$$

Using (4.5) to estimate the last integral, we get that for every j there exists $h_0(j)$ such that

$$\int_{B_{\frac{r}{2}}(x_0)} f_{j,h}(x,Dv_{i,j,h})\,dx \leq c\int_{B_r(x_0)} (1+f(x,Du))\,dx \qquad \forall i,\forall j,\forall h\geq h_0(j). \tag{4.22}$$

Therefore by (4.21) and (4.22) we have

$$\|v_{i,j,h}-(v_{i,j,h})_{x_0,\frac{r}{2}}\|_{L^\infty(B_{\frac{r}{4\sqrt{n}}}(x_0))} \leq c\left\{1+\int_{B_{\frac{r}{2}}(x_0)} f(x,Du)\,dx\right\}^{\vartheta}, \quad \forall i,\forall j,\forall h\geq h_0(j), \tag{4.23}$$

with c possibly depending on r. To estimate the first integral at the right hand side of (4.20) we use (4.6), which implies that there exists $c>0$ such that

$$\begin{aligned}
&\int_{B_{\frac{r}{4\sqrt{n}}}(x_0)} (2+(|Dv_{i,j,h}|-2)_+)^p\,dx \leq 2^p\int_{B_{\frac{r}{4\sqrt{n}}}(x_0)} (1+|Dv_{i,j,h}|)^p\,dx\\
&\leq c\int_{B_r(x_0)} (1+f(x,Du))\,dx \qquad \forall i,\forall j,\forall h\geq h_0(j).
\end{aligned} \tag{4.24}$$

Therefore, by (4.20), (4.23) and (4.24) we get (4.18) and (4.19). The claim is proved.

Setting $w_{i,j,h} := \mathcal{G}((|Dv_{i,j,h}|-2)_+)$, we deduce that there exists $w_{j,h}\in W^{1,2}(B_{\frac{r}{8\sqrt{n}}}(x_0))$, such that, for a subsequence, as $i\to\infty$

$$w_{i,j,h} \rightharpoonup w_{j,h} \qquad \text{in } W^{1,2}\left(B_{\frac{r}{8\sqrt{n}}}(x_0)\right) \qquad \forall j,\ \forall h\geq h_0(j), \tag{4.25}$$

and

$$w_{i,j,h}\to w_{j,h} \qquad \text{strongly in } L^2\left(B_{\frac{r}{8\sqrt{n}}}(x_0)\right) \text{ and a.e. in } B_{\frac{r}{8\sqrt{n}}}(x_0) \qquad \forall j,\ \forall h\geq h_0(j). \tag{4.26}$$

We note that $v_{i,j,h}$ satisfies the assumptions of Proposition 2.5, see (4.17) and (4.18).

Thus, applying Proposition 2.5 to the function $v_{i,j,h}$ with $\Omega = B_{\frac{r}{8\sqrt{n}}}(x_0)$, $\kappa = (v_{i,j,h})_{x_0,\frac{r}{2}}$ and with $\psi\in C_c^\infty(B_{\frac{r}{8\sqrt{n}}}(x_0))$,

$$\psi\equiv 1 \quad \text{in } B_{\frac{r}{16\sqrt{n}}}(x_0), \qquad |D\psi|\leq \frac{2}{\frac{r}{8\sqrt{n}}-\frac{r}{16\sqrt{n}}},$$

we get

$$\begin{aligned}
&\int_{B_{\frac{r}{16\sqrt{n}}}(x_0)} (2+(|Dv_{i,j,h}|-2)_+)^p((|Dv_{i,j,h}|-2)_+)^2\,dx\\
\leq\ & c\left(1+\|v_{i,j,h}-(v_{i,j,h})_{x_0,\frac{r}{2}}\|^2_{L^\infty(\mathrm{supp}\psi)}\right)\int_{B_{\frac{r}{8\sqrt{n}}}(x_0)} (2+(|Dv_{i,j,h}|-2)_+)^p\,dx\\
+\ & c\|v_{i,j,h}-(v_{i,j,h})_{x_0,\frac{r}{2}}\|^2_{L^\infty(\mathrm{supp}\psi)}\int_{B_{\frac{r}{8\sqrt{n}}}(x_0)} |D(\mathcal{G}((|Dv_{i,j,h}|-2)_+))|^2\,dx.
\end{aligned}$$

By (4.23), (4.24) and (4.19) the right hand side is finite and uniformly bounded w.r.t. i. More precisely, we get that, for every i, j and every $h \geq h_0(j)$,

$$\begin{aligned}&\int_{B_{\frac{r}{16\sqrt{n}}}(x_0)} (2 + (|Dv_{i,j,h}| - 2)_+)^p ((|Dv_{i,j,h}| - 2)_+)^2 \, dx \\ \leq\ & c \left(1 + \int_{B_r(x_0)} (1 + f(x, Du)) \, dx\right)^{\bar{\vartheta}} \left(1 + \int_{B_r(x_0)} k_h^\sigma \, dx\right)^{\frac{p+2}{\sigma(p-q+1)}} \end{aligned} \tag{4.27}$$

with c and $\bar{\vartheta}$ positive constants independent of i, j and $h \geq h_0(j)$.

Since $k_h \to k$ in $L^\sigma(B_r(x_0))$, the inequality above implies that

$$\sup_i \|(|Dv_{i,j,h}| - 2)_+\|_{L^{p+2}(B_{\frac{r}{16\sqrt{n}}}(x_0))} \leq c, \tag{4.28}$$

with c independent of j and h. By (4.7), (4.17), (4.18) and (4.28) the assumptions of Proposition 2.6 are satisfied with $\Omega = B_{\frac{r}{16\sqrt{n}}}(x_0)$, $f = v_{j,h}$ and $(f_k)_k$ to be read as $(v_{i,j,h})_i$. Thus

$$(|Dv_{j,h}| - 2)_+ \in W^{1,2}(B_{\frac{r}{16\sqrt{n}}}(x_0)) \tag{4.29}$$

and, up to subsequences,

$$|Dv_{i,j,h}| \to_{i\to\infty} |Dv_{j,h}| \qquad \text{strongly in } L^p(B_{\frac{r}{16\sqrt{n}}}(x_0) \cap \{|Dv_{j,h}| > \lambda\}), \qquad \text{for a.e. } \lambda > 2,$$

$$\begin{aligned} |Dv_{i,j,h}| \to_{i\to\infty} |Dv_{j,h}| \qquad & \text{a.e. in } B_{\frac{r}{16\sqrt{n}}}(x_0) \cap \{|Dv_{j,h}| > 2\}, \\ \mathcal{G}((|Dv_{i,j,h}| - 2)_+) \to_{i\to\infty} \mathcal{G}((|Dv_{j,h}| - 2)_+) \qquad & \text{a.e. in } B_{\frac{r}{16\sqrt{n}}}(x_0) \cap \{|Dv_{j,h}| > 2\}, \end{aligned} \tag{4.30}$$

and, by (4.28),

$$(|Dv_{i,j,h}| - 2)_+ \rightharpoonup_{i\to\infty} (|Dv_{j,h}| - 2)_+ \qquad \text{weakly in } L^{p+2}(B_{\frac{r}{16\sqrt{n}}}(x_0)). \tag{4.31}$$

This last convergence, the sequential weak lower semicontinuity of the norm and (4.27) imply

$$\begin{aligned} &\int_{B_{\frac{r}{16\sqrt{n}}}(x_0)} |Dv_{j,h}|^{p+2} \, dx \leq \int_{B_{\frac{r}{16\sqrt{n}}}(x_0)} (2 + (|Dv_{j,h}| - 2)_+)^{p+2} \, dx \\ \leq\ & c \left(1 + \int_{B_{\frac{r}{16\sqrt{n}}}(x_0)} (2 + (|Dv_{j,h}| - 2)_+)^p ((|Dv_{j,h}| - 2)_+)^2 \, dx\right) \\ \leq\ & c \left(1 + \int_{B_r(x_0)} (1 + f(x, Du)) \, dx\right)^{\bar{\vartheta}} \left(1 + \int_{B_r(x_0)} k_h^\sigma \, dx\right)^{\frac{p+2}{\sigma(p-q+1)}} \qquad \forall j, \forall h \geq h_0(j). \end{aligned} \tag{4.32}$$

Let us recall that

$$w_{i,j,h} := \mathcal{G}((|Dv_{i,j,h}| - 2)_+),$$

and, by (4.26),

$$w_{i,j,h} \to w_{j,h} \text{ a.e. in } B_{\frac{r}{8\sqrt{n}}}(x_0).$$

Therefore, (4.30) implies that

$$w_{j,h} = \mathcal{G}((|Dv_{j,h}| - 2)_+) \qquad \text{a.e. in } B_{\frac{r}{16\sqrt{n}}}(x_0) \cap \{|Dv_{j,h}| > 2\}.$$

By using (4.25), we have

$$D\mathcal{G}((|Dv_{i,j,h}| - 2)_+) \rightharpoonup_{i\to\infty} D\mathcal{G}((|Dv_{j,h}| - 2)_+) \qquad \text{in } L^2(B_{\frac{r}{16\sqrt{n}}}(x_0) \cap \{|Dv_{j,h}| > 2\}).$$

The sequential weak lower semicontinuity of the norm and (4.19) give

$$\begin{aligned}
&\int_{B_{\frac{r}{16\sqrt{n}}}(x_0)\cap\{|Dv_{j,h}|>2\}} |D(\mathcal{G}((|Dv_{j,h}| - 2)_+))|^2 \, dx \\
\leq\ & \liminf_{i\to\infty} \int_{B_{\frac{r}{16\sqrt{n}}}(x_0)\cap\{|Dv_{j,h}|>2\}} |D(\mathcal{G}((|Dv_{i,j,h}| - 2)_+))|^2 \, dx \\
\leq\ & C\left(\int_{B_r(x_0)} (1 + f(x, Du))\, dx\right) \\
+\ & C\left(1 + \int_{B_r(x_0)} f(x, Du)\, dx\right)^{\tilde{\vartheta}} \left(1 + \int_{B_r(x_0)} k_h^\sigma \, dx\right)^{\frac{p+2}{\sigma(p-q+1)}}. \qquad (4.33)
\end{aligned}$$

This clearly implies

$$\begin{aligned}
&\int_{B_{\frac{r}{16\sqrt{n}}}(x_0)} |D(\mathcal{G}((|Dv_{j,h}| - 2)_+))|^2 \, dx \leq C\left(\int_{B_r(x_0)} (1 + f(x, Du))\, dx\right) \\
&+C\left(1 + \int_{B_r(x_0)} f(x, Du)\, dx\right)^{\tilde{\vartheta}} \left(1 + \int_{B_r(x_0)} k_h^\sigma \, dx\right)^{\frac{p+2}{\sigma(p-q+1)}} \qquad \forall j,\ \forall h \geq h_0(j). (4.34)
\end{aligned}$$

Since $k_h \to k$ in $L^\sigma(B_r(x_0))$ and (4.9), (4.28), (4.29), (4.32), (4.34) and Proposition 2.5 hold, we obtain

$$\sup_{h\geq h_0(j)} \|(|Dv_{j,h}| - 2)_+\|_{L^{p+2}(B_{\frac{r}{32\sqrt{n}}}(x_0))} \leq c \qquad \forall j$$

and we are in position to apply Proposition 2.6 to the sequence of functions $(v_{j,h})_h$. Thus, we get

$$(|Dv_j| - 2)_+ \in W^{1,2}(B_{\frac{r}{32\sqrt{n}}}(x_0))$$

and, up to subsequences, as $h \to \infty$,

$$|Dv_{j,h}| \to |Dv_j| \qquad \text{strongly in } L^p(B_{\frac{r}{32\sqrt{n}}}(x_0) \cap \{|Dv_j| > \lambda\}), \qquad \text{for a.e. } \lambda > 2,$$

$$|Dv_{j,h}| \to |Dv_j| \qquad \text{a.e. in } B_{\frac{r}{32\sqrt{n}}}(x_0) \cap \{|Dv_j| > 2\},$$

$$\mathcal{G}((|Dv_{j,h}| - 2)_+) \to \mathcal{G}((|Dv_j| - 2)_+) \qquad \text{a.e. in } B_{\frac{r}{32\sqrt{n}}}(x_0) \cap \{|Dv_j| > 2\},$$

and

$$(|Dv_{j,h}| - 2)_+ \rightharpoonup (|Dv_j| - 2)_+ \qquad \text{weakly in } L^{p+2}(B_{\frac{r}{32\sqrt{n}}}(x_0)).$$

Reasoning as in (4.33), we obtain

$$\int_{B_{\frac{r}{32\sqrt{n}}}(x_0)\cap\{|Dv_j|>2\}} |D(\mathcal{G}((|Dv_j|-2)_+))|^2\,dx$$

$$\leq \quad C\left(\int_{B_r(x_0)}(1+f(x,Du))\,dx\right)+C\left(1+\int_{B_r(x_0)}f(x,Du)\,dx\right)^{\tilde{\vartheta}}\left(1+\int_{B_r(x_0)}k^\sigma\,dx\right)^{\frac{p+2}{\sigma(p-q+1)}},$$

where we used the strong convergence of k_h to k in L^σ. Arguing similarly it can be shown that the assumptions of Proposition 2.6 are satisfied by the sequence of functions (v_j), thus obtaining

$$\int_{B_{\frac{r}{64\sqrt{n}}}(x_0)\cap\{|Du|>2\}} |D(\mathcal{G}((|Du|-2)_+))|^2\,dx$$

$$\leq \quad C\left(\int_{B_r(x_0)}(1+f(x,Du))\,dx\right)+C\left(1+\int_{B_r(x_0)}f(x,Du)\,dx\right)^{\tilde{\vartheta}}\left(1+\int_{B_r(x_0)}k^\sigma\,dx\right)^{\frac{p+2}{\sigma(p-q+1)}} \quad (4.35)$$

and $(|Du|-2)_+\in L^{p+2}(B_{\frac{r}{64\sqrt{n}}}(x_0))$, that trivially implies $|Du|\in L^{p+2}(B_{\frac{r}{64\sqrt{n}}}(x_0))$. These facts, together with a covering argument, allow to conclude. □

References

[1] P. Baroni, M. Colombo, G. Mingione: Harnack inequalities for double phase functionals, *Nonlinear Analysis* 121 (2015) 206-222.

[2] P. Bousquet, L. Brasco: Global Lipschitz continuity for minima of degenerate problems, *Math. Ann.* 366 (2016) 1403-1450.

[3] L. Brasco: Global L^∞ gradient estimates for solutions to a certain degenerate elliptic equation, *Nonlinear Anal.* 74 (2011) 516-531.

[4] L. Brasco, G. Carlier: On certain anisotropic elliptic equations arising in congested optimal transport: local gradient bounds, *Adv. Calc. Var.* 7 (2014) 379-407.

[5] M. Carozza, J. Kristensen, A. Passarelli di Napoli: Higher differentiability of minimizers of convex variational integrals, *Ann. Inst. H. Poincaré Anal. Non Linéaire* 28 (2011) 395-411.

[6] P. Celada, G. Cupini, M. Guidorzi: Existence and regularity of minimizers of nonconvex integrals with p-q growth, *ESAIM Control Optim. Calc. Var.* 13 (2007) 343-358.

[7] M. Chipot, L.C. Evans: Linearisation at infinity and Lipschitz estimates for certain problems in the calculus of variations. *Proc. Roy. Soc. Edinburgh Sect. A* 102 (1986) no. 3-4, 291-303.

[8] M. Colombo, G. Mingione: Regularity for double phase variational problems, *Arch. Rat. Mech. Anal.* 215 (2015) 443-496.

[9] M. Colombo, G. Mingione: Bounded minimisers of double phase variational integrals, *Arch. Rat. Mech. Anal.* 218 (2015) 219-273.

[10] G. Cupini, F. Giannetti, R. Giova, A Passarelli di Napoli: Higher integrability for minimizers of asymptotically convex integrals with discontinuous coefficients, *Nonlinear Anal.* 154 (2017) 7-24.

[11] G. Cupini, M. Guidorzi, E. Mascolo: Regularity of minimizers of vectorial integrals with $p - q$ growth *Nonlinear Anal.* 54 (2003) 591-616.

[12] G. Cupini, P. Marcellini, E. Mascolo: Local boundedness of solutions to some anisotropic elliptic systems, *Recent trends in nonlinear partial differential equations. II. Stationary problems* 169-186, Contemp. Math. 595, Amer. Math. Soc., Providence, RI, 2013.

[13] G. Cupini, P. Marcellini, E. Mascolo: Regularity of minimizers under limit growth conditions. Nonlinear Anal. 153 (2017) 294-310.

[14] A. Dall'Aglio, E. Mascolo: L^∞ estimates for a class of nonlinear elliptic systems with nonstandard growth, *Atti Sem. Mat. Fis. Univ. Modena* 50 (2002) 65-83.

[15] G. Dal Maso: *An introduction to* Γ*-convergence*, Progr. Nonlinear Differential Equations Appl., no. 8, Birkhäuser, Boston, 1993.

[16] E. De Giorgi: Un esempio di estremali discontinue per un problema variazionale di tipo ellittico, *Boll. Unione Mat. Ital.* IV. Ser. 1 (1968) 135-137.

[17] M. Eleuteri, P. Marcellini, E. Mascolo: Lipschitz estimates for systems with ellipticity conditions at infinity, *Ann. Mat. Pura e Appl. (4)* 195 (2016) 1575-1603.

[18] M. Eleuteri, P. Marcellini, E. Mascolo: Lipschitz continuity for energy integrals with variable exponents. *Atti Accad. Naz. Lincei Rend. Lincei Mat. Appl.* 27 (2016) 61-87.

[19] L. Esposito, G. Mingione, C. Trombetti: On the Lipschitz regularity for certain elliptic problems *Forum Math.* 18 (2006) 263-292.

[20] K. Fey, M. Foss: Morrey regularity results for asymptotically convex variational problems with (p, q) growth. *J. Differential Equations* 246 (2009) 4519-4551.

[21] M. Focardi: Γ-convergence: a tool to investigate physical phenomena across scales. *Math. Methods Appl. Sci.* 35 (2012) 1613-1658.

[22] I. Fonseca, N. Fusco, P. Marcellini: An existence result for a nonconvex variational problem via regularity, *ESAIM: Control, Optim. Calc. Var.* 7 (2002) 69-96.

[23] M. Foss, C.S. Goodrich: Partial Hölder continuity of minimizers of functionals satisfying a general asymptotic relatedness condition *J*. Convex Anal. 22 (2015) 219246.

[24] M. Foss, A. Passarelli di Napoli, A. Verde: Morrey regularity and continuity results for almost minimizers of asymptotically convex integrals. *Appl. Math.* (Warsaw) 35 (2008) 335-353.

[25] M. Foss, A. Passarelli di Napoli, A. Verde: Global Morrey regularity results for asymptotically convex variational problems. *Forum Math.* 20 (2008) 921-953.

[26] M. Foss, A. Passarelli di Napoli, A. Verde: Global Lipschitz regularity for almost minimizers of asymptotically convex variational problems. *Ann. Mat. Pura Appl.* (4) 189 (2010) 127-162.

[27] F. Giannetti: A $C^{1,\alpha}$ partial regularity result for integral functionals with $p(x)$- growth condition. *Adv. Calc. Var.*, 9 (2016) 395-407.

[28] F. Giannetti, A. Passarelli di Napoli: Higher differentiability of minimizers of variational integrals with variable exponents, *Math. Z.* 280 (2015) 873-892.

[29] M. Giaquinta: Growth conditions and regularity: a counterexample, *Manuscripta Math.* **59** (1987) 245-248.

[30] M. Giaquinta, G. Modica: Remarks on the regularity of the minimizers of certain degenerate functionals, *Manuscripta Math.* 57 (1986) 55-99.

[31] R. Giova: Regularity results for non-autonomous functionals with $L \log L$ -growth and Orlicz Sobolev coefficients *NoDEA Nonlinear Differential Equations Appl.* Doi: 10.1007/s00030-016-0419-5

[32] R. Giova, A. Passarelli di Napoli: Regularity results for a priori bounded minimizers of non autonomous functionals with discontinuous coefficients, *Adv. Calc. Var.*, to appear.

[33] R. Giova: Higher differentiability for n-harmonic systems with Sobolev coefficients *J. Diff. Equations* 259 (2015) 5667-5687.

[34] E. Giusti: *Direct methods in the calculus of variations,* World Scientific Publishing Co., Inc., River Edge, NJ, 2003.

[35] P. Marcellini: Un example de solution discontinue d'un problème variationnel dans le cas scalaire, *Preprint 11, Istituto Matematico "U.Dini",* Università di Firenze, 1987.

[36] P. Marcellini: Regularity of minimizers of integrals of the calculus of variations with non standard growth conditions, *Arch. Rat. Mech. Anal.* 105 (1989) 267-284.

[37] P. Marcellini: Regularity and existence of solutions of elliptic equations with p, q-growth conditions, *J. Differential Equations* 90 (1991) 1-30.

[38] P. Marcellini: Everywhere regularity for a class of elliptic systems without growth conditions, *Ann. Scuola Norm. Sup. Pisa Cl. Sci.* **23** (1996) 1-25.

[39] G. Mingione: *Regularity of minima: an invitation to the dark side of calculus of variations*, Appl. Math. **51**, (2006).

[40] C. Mooney, O. Savin: Some singular minimizers in low dimensions in the calculus of variations, *Arch. Ration. Mech. Anal.* 221 (2016) 1-22.

[41] J. Nečas: Example of an irregular solution to a nonlinear elliptic system with analytic coefficients and conditions for regularity, *Theory Nonlin. Oper., (Proc. Fourth Internat. Summer School, Acad. Sci., Berlin, 1975)*, 197-206. Abh. Akad. Wiss. DDR Abt. Math.-Natur.-Tech., Jahrgang 1977, 1, Akademie-Verlag, Berlin, 1977.

[42] A. Passarelli di Napoli: Higher differentiability of minimizers of variational integrals with Sobolev coefficients. *Adv. Cal. Var.* 7 (1) (2014) 59-89.

[43] A. Passarelli di Napoli: Higher differentiability of solutions of elliptic systems with Sobolev coefficients: the case $p = n = 2$. *Pot. Anal.* 41 (3) (2014) 715–735

[44] A. Passarelli di Napoli, A. Verde: A regularity result for asymptotically convex problems with lower order terms. *J. Convex Anal.* 15 (2008) 131-148.

[45] V. Sveràk, X. Yan: Non-Lipschitz minimizers of smooth uniformly convex functionals *Proc. Natl. Acad. Sci. USA* 99 (24) (2002) 15269-15276.

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