Willmore flow of planar networks

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Abstract

Geometric gradient flows for elastic energies of Willmore type play an important role in mathematics and in many applications. The evolution of elastic curves has been studied in detail both for closed as well as for open curves. Although elastic flows for networks also have many interesting features, they have not been studied so far from the point of view of mathematical analysis. So far it was not even clear what are appropriate boundary conditions at junctions. In this paper we give a well-posedness result for Willmore flow of networks in different geometric settings and hence lay a foundation for further mathematical analysis. A main point in the proof is to check whether different proposed boundary conditions lead to a well posed problem. In this context one has to check the Lopatinskii–Shapiro condition in order to apply the Solonnikov theory for linear parabolic systems in Hölder spaces which is needed in a fixed point argument. We also show that the solution we get is unique in a purely geometric sense.

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1 Introduction

A planar network $\mathcal{N}$ is a connected set composed of a finite number of sufficiently smooth curves $\mathcal{N}^i$ that meet at junctions. We consider two types of networks of three curves: Theta–networks and Triods. The three regular curves of a Theta–network intersect each other at their endpoints in two triple junctions. A Triod is composed by three regular curves that intersect each other at a triple junction and have the other three endpoints on the boundary of an open set $\Omega \subset \mathbb{R}^2$.

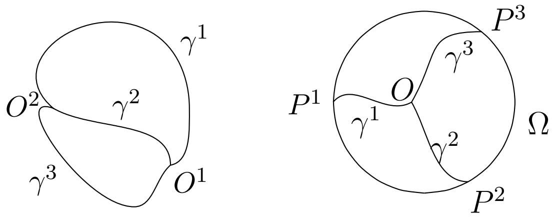

Figure 1: A Theta–network and a Triod

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We consider the elastic energy of the network, penalizing its global length, that is

$$E_\mu(\mathcal{N}) := \sum_{i=1}^{3} \int_{\mathcal{N}^i} \left(\left(k^i\right)^2 + \mu \right) \mathrm{d}s^i \,, \qquad \mu > 0 \,,$$

where s^i denotes the arc length parameter of the curve $\mathcal{N}^i$ and k^i the curvature. We are interested in the L^2–gradient flow of the energy E_μ: we start with an initial network (Triod or Theta) and let it evolve with a normal velocity that induces the steepest descent of the energy with respect to the L^2–inner product. The curves need to stay attached during the evolution but the junctions are allowed to move.

Our result is inspired by several others in the case of curves [2, 5, 6, 7]. The problem for networks was first proposed by Barrett, Garcke and Nürnberg: in [1] it is shown that if $(\mathcal{N}(t))_{t\in[0,T]}$ is a time dependent family of networks evolving according to the L^2–gradient flow of E_μ, then the normal velocity of each of its curves is

$$-2k_{ss} - k^3 + \mu k \,.$$

Depending on which constraints one imposes at the endpoints of the curves one obtains a different flow. We consider two different types of properties at the triple junctions. In one case we only demand that the curves stay attached during the evolution which we call C^0–flow. In the other case we additionally impose an angle condition at the triple junction(s). We require all angles to be equal and name this evolution C^1–flow. We note that there are several variants in the case of a Triod since different boundary conditions are possible at the endpoints on $\partial\Omega$, namely Navier (fixed endpoints and zero curvature) and clamped (position and unit tangent vectors are prescribed). Depending on whether we consider the C^0– or the C^1–flow different natural boundary conditions have to be prescribed at the triple junction(s). These follow from computing the first variation of the energy taking the constraints at the junction into account. The main question we would like to answer is whether the resulting boundary conditions at the triple junction(s) lead to a well-posed evolution problem. The answer will be positive with the exception of one degenerate geometric situation in the C^0–case.
We are looking for classical solutions to the C^0– and C^1–flow: we require that each evolving curve $\mathcal{N}^i$ of the network $\mathcal{N}$ admits a time dependent parametrisation

$$\gamma^i \in C^{\frac{4+\alpha}{4}, 4+\alpha}([0,T] \times [0,1]) \,.$$

In particular the initial network needs to fulfil certain compatibility conditions to be admissible. These will be specified in Definitions 3.2 and 4.1.

The result of this paper is the following:

Theorem. *Let $\mathcal{N}_0$ be a geometrically admissible initial network for the C^0–flow (or the C^1–flow). Then there exists a positive time T such that within the interval $[0,T]$ the C^0–flow (C^1–flow) admits a unique solution (as precisely defined in 3.3 and 4.2).*

We give a detailed proof in the case of the C^0–flow for a Theta–network. In Section 4 we show how to adapt the proof to Triods moving according to the C^1–flow. Combining the techniques used to prove the two results we indeed cover all possible situations.

To prove existence of the C^0–flow we translate the geometric problem to the level of parametrisations. As roughly explained before the notion of L^2–gradient flow gives rise to a boundary value problem for the parametrisations. As the motion in tangential direction is not specified in the formulation of this geometric problem, one needs to fix a tangential velocity in order to obtain a well posed parabolic PDE for the parametrisation. Moreover we have one tangential degree of freedom for each curve at each triple junction which forces us to fix another boundary condition. This has to be chosen in such a way that the geometric problem does not change! It turns out that it is convenient to demand

$$\langle \gamma^i_{xx}, \tau^i \rangle = 0$$

at both triple junctions and at every time which we call tangential second order condition. A key point is to ensure that solving the analytic problem is enough to obtain a unique solution to the original problem formulated for networks as geometric objects. This is shown in Theorem 3.32 following the approach presented in [9]. This justifies to concentrate on the analytic problem which is a matter of solving the system (3.5) of parabolic quasilinear PDEs of fourth order with coupled boundary conditions. To this end we study a linearised problem and deduce existence of the system from the linearised problem by a contraction argument. Our linearised problem (3.7) is derived in 3.3.1 and solved in 3.3.2. As the boundary conditions are of different order, the theory of Solonnikov [18] is the appropriate one to show existence of the linearised problem in Hölder spaces. The system is parabolic in the sense of Solonnikov. To apply [18, Theorem 4.9] we notice that the Lopatinskii–Shapiro condition is satisfied in the case one completes the under determined boundary value problem by the tangential second order condition. We conclude the proof in Section 3.4 giving the contraction argument.

The first short time existence result for the elastic flow of curves was obtained by Polden [17, 16]. The author considered closed curves in the plane and also showed global existence. Afterwards the result was extended to curves in $\mathbb{R}^n$ in [7], where also global existence is proved. For other results in a similar framework we also refer to [10, 11, 20]. More recently the elastic flow of open curves in $\mathbb{R}^n$ has been studied. Especially the long time behaviour has been investigated for various boundary conditions [2, 5, 6, 12, 15]. In [19] short time existence of the elastic flow of open clamped curves is shown.
We emphasize that in the case of networks we are concerned with a system of equations with coupled boundary conditions. Moreover the triple junction is allowed to move which gives rise to tangential degrees of freedom. One carefully has to study the problem in order to choose meaningful extra conditions at the boundary without changing the geometric flow.

We notice that all results are valid not only for $\mu > 0$ but also in the case $\mu \in \mathbb{R}$. This is due to the fact that the linearisation and the existence of the linearised system only depend on the highest order terms. As a consequence our result covers short time existence of the Willmore flow of networks which corresponds to $\mu = 0$.

In all cases presented in the literature on the elastic flow of closed and open curves the evolution exists globally in time [2, 5, 6, 7, 12, 14, 17]. In the case of networks instead recent results on the related minimization problem of the energy E_μ suggest different possible behaviours in the long term of our flow. There exist minimizing sequences of Theta–networks that converge to a point. Thus, in order to have a well–posed problem in the class of Theta–networks, one needs a further constraint. One good option among several others is to prescribe the angles at the triple junctions with at least one positive angle. Recently it has been shown that there exists a minimizer in the class of Theta–networks in the case that all angles are prescribed to be of 120 degrees [3]. To prove this one has to enlarge the class of Theta–networks fulfilling the angle condition to degenerate Theta–networks. A degenerate Theta–network consists of only two curves meeting in one quadruple point forming angles in pairs of 60 and 120 degrees. It is shown in [3] that the global minimizer is not degenerate. But it is not excluded that a degenerate network could be a stationary point. In contrast to the static problem both the C^0- and the C^1–flow are interesting in the evolution. In particular, the ill–posedness of the minimization problem in the C^0–case suggests the possible onset of singularities in the long time behaviour. A dramatic change of topology is possible as one or more curves of the network may collapse to a point.

In [1] weak formulations for the Willmore flow of networks have been proposed to study finite element methods for the flow. Applications to evolving non–linear splines are discussed and several numerical studies demonstrate interesting features of the flow.

In the following section we formulate the geometric evolution problem for elastic networks in a precise way. In Section 3 we show short time existence of a unique smooth solution in the C^0–case using a maximal regularity result for linear parabolic systems due to Solonnikov and a contraction argument. A main ingredient in the proof is to show the Lopatinskii–Shapiro condition facing the difficulty that the boundary conditions mix the unknowns in a non-trivial way. We also show geometric uniqueness in the C^0–case. In the final Section 4 we adapt the result to the C^1–flow.

2 The elastic flow for networks

2.1 Notation and definitions

2.1.1 Networks

Definition 2.1. A network $\mathcal{N}$ is a connected set in the plane consisting of a finite union of sufficiently smooth curves $\mathcal{N}^i$.

Each curve $\mathcal{N}^i$ of the network admits a parametrisation $\gamma^i : [0,1] \to \mathbb{R}^2$.

- Let $k \in \mathbb{N}$ and $\alpha \in (0,1)$. A curve $\mathcal{N}^i$ is of class C^k or $C^{k+\alpha}$ if it admits a parametrisation γ^i such that γ^i is of class C^k or $C^{k+\alpha}$, respectively. It is said to be regular if $\gamma^i_x(x) \neq 0$ for every $x \in [0,1]$.
- For a curve of class C^1 we define its unit tangent vector $\tau^i := \gamma^i_x / |\gamma^i_x|$ and its unit normal vector ν^i as the anticlockwise rotation by $\pi/2$ of its unit tangent vector.

We write $\mathcal{N} = \bigcup_{i=1}^n \mathcal{N}^i$ when we consider a network as a geometric object. Instead when we consider a parametrisation of $\mathcal{N}$, we write $\gamma = (\gamma^1, \ldots, \gamma^n)$.

In this paper we consider only 3–networks, namely networks composed by three curves. In particular we define Triods and Theta–networks as follows:

Definition 2.2. Consider an open set $\Omega \subset \mathbb{R}^2$. A Triod $\mathbb{T}$ is a network in Ω composed by three regular at least C^1–curves with one endpoint belonging to the boundary of Ω and the other three endpoints meeting in a triple junction O.

We call P^1, P^2, P^3 the endpoints at $\partial\Omega$ and without loss of generality we consider a parametrisation $\gamma = (\gamma^1, \gamma^2, \gamma^3)$ of the Triod such that $O = \gamma^1(0) = \gamma^2(0) = \gamma^3(0)$ and $P^i = \gamma^i(1)$.

Definition 2.3. A Theta–network Θ is a network in $\mathbb{R}^2$ composed by three regular at least C^1–curves that intersect each other at their endpoints in two triple junctions.

Also in this case we may choose a parametrisation $\gamma = (\gamma^1, \gamma^2, \gamma^3)$ of a Theta–network in such a way that $O^1 = \gamma^1(0) = \gamma^2(0) = \gamma^3(0)$ and $O^2 = \gamma^1(1) = \gamma^2(1) = \gamma^3(1)$.
We have defined two types of networks whose endpoints meet in junctions and we now give a name to this property.

Definition 2.4 (Concurrency condition)**.** Given either a Triod or a Theta–network we say that the *concurrency condition* is satisfied in a point O of the network if the three curves meet at O in a triple junction.

Definition 2.5 (Angle condition)**.** Given either a Triod or a Theta–network, we say that the curves satisfy the *angle condition* at a junction if the concurrency condition is satisfied and the angles between the unit tangent vectors to the curves are of 120 degrees, namely $\sum_{i=1}^{3} \tau^i = 0$.

To be consistent with [1] we call a C^0–network and a C^1–network a network that satisfies the concurrency and the angle condition, respectively. In particular a Theta–network will also be called C^0–Theta–Network.

2.1.2 Function spaces

In the whole paper we consider parabolic Hölder spaces (see also [18, §11, §13]) which are defined as follows.
For a function $u : [0,T] \times [0,1] \to \mathbb{R}$ we define for $\rho \in (0,1)$ the semi–norms

$$[u]_{\rho,0} := \sup_{(t,x),(\tau,x)} \frac{|u(t,x) - u(\tau,x)|}{|t-\tau|^\rho},$$

and

$$[u]_{0,\rho} := \sup_{(t,x),(t,y)} \frac{|u(t,x) - u(t,y)|}{|x-y|^\rho}.$$

Then we define for $k \in \{0,1,2,3,4\}$, $\alpha \in (0,1)$ the classical parabolic Hölder spaces

$$C^{\frac{k+\alpha}{4}, k+\alpha}([0,T] \times [0,1])$$

of all functions $u : [0,T] \times [0,1] \to \mathbb{R}$ that have continuous derivatives $\partial_t^i \partial_x^j u$ where $i, j \in \mathbb{N}$ are such that $4i + j \leq k$ for which the norm

$$\|u\|_{C^{\frac{k+\alpha}{4}, k+\alpha}} := \sum_{4i+j=0}^{k} \left\|\partial_t^i \partial_x^j u\right\|_\infty + \sum_{4i+j=k} \left[\partial_t^i \partial_x^j u\right]_{0,\alpha} + \sum_{0<k+\alpha-4i-j<4} \left[\partial_t^i \partial_x^j u\right]_{\frac{k+\alpha-4i-j}{4},0}$$

is finite.
We notice that the space $C^{\frac{\alpha}{4},\alpha}([0,T] \times [0,1])$ is equal to the space

$$C^{\frac{\alpha}{4}}\left([0,T]; C^0([0,1])\right) \cap C^0\left([0,T]; C^\alpha([0,1])\right)$$

with equivalent norms.
The boundary data are in spaces of the form $C^{\frac{k+\alpha}{4},k+\alpha}([0,T]\times\{0,1\},\mathbb{R}^m)$ which we identify with $C^{\frac{k+\alpha}{4}}([0,T],\mathbb{R}^{2m})$ via the isomorphism $f\mapsto(f(t,0),f(t,1))^t$.
For $\beta>0$ and $T>0$ we define

$$C_0^{\beta}([0,T];\mathbb{R}^n)=\{f\in C^{\beta}([0,T];\mathbb{R}^n)\text{ such that }f(0)=0\}\,.$$

Lemma 2.6. *For $\beta>\alpha>0$ and $T\in[0,1]$ we have the embedding*

$$C_0^{\beta}([0,T];\mathbb{R}^n)\hookrightarrow C_0^{\alpha}([0,T];\mathbb{R}^n)\,.$$

Moreover for all $f\in C_0^{\beta}([0,T];\mathbb{R}^n)$ it holds that

$$\|f\|_{C^\alpha}\le 2T^{\beta-\alpha}\|f\|_{C^\beta}\,.$$

Proof. This follows directly from the fact that $f(0)=0$ and the definition of the Hölder norm. □

2.2 Definition of the flow

We consider the elastic energy functional E_μ for a 3–network $\mathcal{N}$ defined as

$$E_\mu(\mathcal{N}):=\int_{\mathcal{N}}\left(k^2+\mu\right)ds=\sum_{i=1}^{3}\int_{\mathcal{N}^i}\left((k^i)^2+\mu\right)ds^i\,, \tag{2.1}$$

with $\mu\in(0,\infty)$, where k^i is the curvature of the curve $\mathcal{N}^i$ and s^i its arclength parameter. Here we use a sign convention for the curvature such that $\tau_s=k\nu$.

Our aim is to study the L^2–gradient flow of E_μ for a Triod or a Theta–network in the case we require at the triple junctions the concurrency or the angle conditions.

Problem 2.7 (C^0–Flow)**.** Either a Triod or a Theta–network evolves by the geometric L^2–gradient flow of the elastic energy functional (2.1) requiring that the concurrency condition is satisfied at the triple junction(s).

Problem 2.8 (C^1–Flow)**.** A Triod or a Theta–network evolves by the geometric L^2–gradient flow of the elastic energy functional (2.1) requiring that the angle condition is satisfied at the triple junction(s).

In both problems the triple junctions are allowed to move. We are looking for classical solutions to both Problem 2.7 and Problem 2.8.

We compute the first variation of the functional E_μ for a Triod or a Theta–network $\mathcal{N}$ admitting a regular $C^{4+\alpha}$–parametrisation $\gamma=(\gamma^1,\gamma^2,\gamma^3)$, compare [1, 4] for details.
We get

$$\begin{aligned}\frac{d}{dt}E_\mu(\widetilde{\mathcal{N}}_t)_{|t=0}=&\sum_{i=1}^{3}\int_{\gamma^i}\left(2k^i_{ss}+\left(k^i\right)^3-\mu k^i\right)\left\langle\psi^i,\nu^i\right\rangle ds\\&+\sum_{i=1}^{3}2\left\langle\psi^i_s,k^i\nu^i\right\rangle\Big|_0^1+\sum_{i=1}^{3}\left\langle\psi^i,-2k^i_s\nu^i-\left(k^i\right)^2\tau^i+\mu\tau^i\right\rangle\Big|_0^1\,,\end{aligned} \tag{2.2}$$

where $\widetilde{\mathcal{N}}_t$ is a variation of the network $\mathcal{N}$ parametrised by $\tilde{\gamma} = (\tilde{\gamma}^1, \tilde{\gamma}^2, \tilde{\gamma}^3)$ with $\tilde{\gamma}^i = \gamma^i + t\psi^i$, $t \in \mathbb{R}$, $\psi^i \in C^\infty([0,1], \mathbb{R}^2)$ and $\psi^1(0) = \psi^2(0) = \psi^3(0)$. In the case of a Theta–network we also require $\psi^1(1) = \psi^2(1) = \psi^3(1)$ whereas in the case of a Triod we prescribe $\psi^i(1) = 0$, $i = 1, 2, 3$, in the Navier case and $\psi^i(1) = 0$, $\psi^i_x(1) = 0$, $i = 1, 2, 3$ in the clamped case. In the C^1–case there is a further condition on ψ at the triple junction(s), see [1, 4] for details.
The previous computation tells us that the steepest descent with respect to the L^2–inner product is given by a normal velocity for each curve γ^i equal to

$$-\left(2k^i_{ss} + \left(k^i\right)^3 - \mu k^i\right)\nu^i =: -A^i\nu^i\,.$$

Computing this quantity in terms of the parametrisation γ^i we get

$$\begin{aligned}
A^i\nu^i &= 2k^i_{ss}\nu^i + \left(k^i\right)^3\nu^i - \mu k^i\nu^i \\
&= 2\frac{\gamma^i_{xxxx}}{|\gamma^i_x|^4} - 12\frac{\gamma^i_{xxx}\left\langle\gamma^i_{xx}, \gamma^i_x\right\rangle}{|\gamma^i_x|^6} - 5\frac{\gamma^i_{xx}\left|\gamma^i_{xx}\right|^2}{|\gamma^i_x|^6} - 8\frac{\gamma^i_{xx}\left\langle\gamma^i_{xxx}, \gamma^i_x\right\rangle}{|\gamma^i_x|^6} + 35\frac{\gamma^i_{xx}\left\langle\gamma^i_{xx}, \gamma^i_x\right\rangle^2}{|\gamma^i_x|^8} \\
&+ \left\langle -2\frac{\gamma^i_{xxxx}}{|\gamma^i_x|^4} + 12\frac{\gamma^i_{xxx}\left\langle\gamma^i_{xx}, \gamma^i_x\right\rangle}{|\gamma^i_x|^6} + 5\frac{\gamma^i_{xx}\left|\gamma^i_{xx}\right|^2}{|\gamma^i_x|^6} + 8\frac{\gamma^i_{xx}\left\langle\gamma^i_{xxx}, \gamma^i_x\right\rangle}{|\gamma^i_x|^6} - 35\frac{\gamma^i_{xx}\left\langle\gamma^i_{xx}, \gamma^i_x\right\rangle^2}{|\gamma^i_x|^8}, \tau^i\right\rangle\tau^i \\
&- \mu\frac{\gamma^i_{xx}}{|\gamma^i_x|^2} + \left\langle\mu\frac{\gamma^i_{xx}}{|\gamma^i_x|^2}, \tau^i\right\rangle\tau^i\,.
\end{aligned} \tag{2.3}$$

3 C^0–Flow

Now we consider the C^0–Flow for a Theta–network.
From now on we consider an initial C^0–Theta network $\Theta_0 = \bigcup_{i=1}^3 \sigma^i$ and we denote with τ^i_0, ν^i_0, k_0 its unit tangent, unit normal vectors and its curvature, respectively. Moreover we call α_3, α_1 and α_2 the angle between τ^1_0 and τ^2_0, τ^2_0 and τ^3_0, and τ^3_0 and τ^1_0, respectively. Then basic trigonometric relations imply that the angles and tangents satisfy

$$\begin{aligned}
\sin(\alpha_1)\tau^1_0 + \sin(\alpha_2)\tau^2_0 + \sin(\alpha_3)\tau^3_0 &= 0\,, \\
\sin(\alpha_1)\nu^1_0 + \sin(\alpha_2)\nu^2_0 + \sin(\alpha_3)\nu^3_0 &= 0\,.
\end{aligned}$$

3.1 Geometric problem

To obtain an L^2–gradient flow of the functional E_μ for a C^0–Theta–network the following two conditions need to be fulfilled at the boundary (which are derived from (2.2), for a detailed proof see [1]) :

$$\sum_{i=1}^{3} 2\left\langle\psi^i_s, k^i\nu^i\right\rangle\Big|_0^1 = 0 \quad\text{and}\quad \left\langle\psi^1, \sum_{i=1}^{3} -2k^i_s\nu^i - \left(k^i\right)^2\tau^i + \mu\tau^i\right\rangle\Big|_0^1 = 0 \tag{3.1}$$

for all $\psi^i \in C^\infty([0,1], \mathbb{R}^2)$ with $\psi^1(y) = \psi^2(y) = \psi^3(y)$ for $y \in \{0, 1\}$.
These conditions lead to natural boundary conditions which we now specify.

Definition 3.1 (Natural boundary conditions)**.** We say that a Theta–network satisfies

- the *curvature condition* if $k^i = 0$ holds at the triple junctions;
- the *third order condition* if $\sum_{i=1}^{3} \left(2k_s^i \nu^i - \mu\tau^i\right) = 0$ holds at the triple junctions.

We notice that (3.1) is satisfied if and only if the curvature and the third order conditions are satisfied. We now specify geometrically admissible initial networks which are needed to solve the evolution problem in the class of smooth parametrisations. In particular we need to specify compatibility conditions between initial and boundary data.

Definition 3.2 (Geometrically admissible initial network)**.** A Theta–network Θ_0 is a geometrically admissible initial network for Problem 2.7 if

- at each triple junction there are at least two curves that form a strictly positive angle;
- at each triple junction each curve has zero curvature (curvature condition);
- at each triple junction it holds $\sum_{i=1}^{3} \left(2k_{0,s}^i \nu_0^i - \mu\tau_0^i\right) = 0$ (third order condition);
- at each triple junction it holds that

$$\sin(\alpha_1)A_0^1 + \sin(\alpha_2)A_0^2 + \sin(\alpha_3)A_0^3 = 0\,,$$

  where $A_0^i = 2k_{0,ss}^i + \left(k_0^i\right)^3 - \mu k_0^i$.
- there exists a parametrisation $\gamma = (\gamma^1, \gamma^2, \gamma^3)$ of Θ_0 such that every curve $\gamma^i : [0,1] \to \mathbb{R}^2$ is of class $C^{4+\alpha}$ and regular.

Definition 3.3 (Solution of the geometric problem)**.** Let Θ_0 be a geometrically admissible initial network. A time dependent family of Theta–networks $\Theta(t)$ is a solution to the C^0–flow with initial data Θ_0 in a time interval $[0,T]$ with $T > 0$ if and only if there exist parametrisations $\gamma(t)$ of $\Theta(t)$, $t \in [0,T]$, $\gamma^i \in C^{\frac{4+\alpha}{4},4+\alpha}([0,T] \times [0,1], \mathbb{R}^2)$, such that for every $t \in [0,T)\,, x \in [0,1]$ and for $i \in \{1,2,3\}$ the system

$$\begin{cases} \left\langle \gamma_t^i(t,x), \nu^i(t,x)\right\rangle \nu^i(t,x) = -A^i(t,x)\nu^i(t,x) & & \text{motion,} \\ \gamma^1(t,y) = \gamma^2(t,y) = \gamma^3(t,y) & \text{for } y \in \{0,1\} & \text{concurrency condition,} \\ k^i(t,y) = 0 & \text{for } y \in \{0,1\} & \text{curvature condition,} \\ \sum_{i=1}^{3} \left(2k_s^i \nu^i - \mu\tau^i\right)(t,y) = 0 & \text{for } y \in \{0,1\} & \text{third order condition,} \\ \Theta(0) = \Theta_0 & & \text{initial data} \end{cases} \tag{3.2}$$

is satisfied.

To specify the geometric evolution of the network it is enough to write the normal velocity of the curves. We remind that in our evolution problem the triple junctions are allowed to move which forces the curves to move also in tangential direction.

Proposition 3.4. *The system (3.2) is a L^2-gradient flow of E_μ and in particular E_μ decreases in time.*

Proof. This follows from [1, Theorem 2.1]. □

3.2 Analytic problem

In this subsection we consider a time dependent family of Theta–networks parametrised by regular curves $\gamma = (\gamma^1, \gamma^2, \gamma^3) \in C^{\frac{4+\alpha}{4}, 4+\alpha}([0,T] \times [0,1])$ with $T > 0$ and $\alpha \in (0,1)$.

Definition 3.5 (Tangential velocity)**.** We define

$$T^i := \left\langle 2\frac{\gamma^i_{xxxx}}{|\gamma^i_x|^4} - 12\frac{\gamma^i_{xxx}\left\langle \gamma^i_{xx}, \gamma^i_x \right\rangle}{|\gamma^i_x|^6} - 5\frac{\gamma^i_{xx}\left|\gamma^i_{xx}\right|^2}{|\gamma^i_x|^6} \right.$$
$$\left. -8\frac{\gamma^i_{xx}\left\langle \gamma^i_{xxx}, \gamma^i_x \right\rangle}{|\gamma^i_x|^6} + 35\frac{\gamma^i_{xx}\left\langle \gamma^i_{xx}, \gamma^i_x \right\rangle^2}{|\gamma^i_x|^8} - \mu\frac{\gamma^i_{xx}}{|\gamma^i_x|^2}, \tau^i \right\rangle . \tag{3.3}$$

Definition 3.6 (Second order condition)**.** We say that the parametrisation γ of a Theta–network satisfies the *second order condition* if each curve satisfies $\gamma^i_{xx} = 0$ at the triple junctions.

Remark 3.7. We want to write the geometric problem as a system of PDEs. This is obtained describing the curves with the help of a parametrisation γ. To have a well posed problem we need to fix a tangential velocity and to impose another condition on the parametrisation of each curve at the boundary. This is due to the fact that we need to specify some tangential degrees of freedom. These conditions have to be chosen in such a way that the geometric problem does not change (see below the discussion about geometric existence and uniqueness in Subsection 3.5). As it turns out the boundary condition

$$\langle \gamma_{xx}, \tau \rangle = 0 \tag{3.4}$$

is convenient in our case (see Lemma 3.14). Together with the curvature condition the condition (3.4) is equivalent to the second order condition.

Hence we consider the following problem:

Problem 3.8. For every $t \in [0,T)$, $x \in [0,1]$ and for $i \in \{1,2,3\}$

$$\begin{cases} \gamma^i_t(t,x) = -A^i(t,x)\nu^i(t,x) - T^i(t,x)\tau^i(t,x) & & \text{motion,} \\ \gamma^1(t,y) = \gamma^2(t,y) = \gamma^3(t,y) & \text{for } y \in \{0,1\} & \text{concurrency condition,} \\ \gamma^i_{xx}(t,y) = 0 & \text{for } y \in \{0,1\} & \text{second order condition,} \\ \sum_{i=1}^3 \left(2k^i_s \nu^i - \mu\tau^i\right)(t,y) = 0 & \text{for } y \in \{0,1\} & \text{third order condition,} \\ \gamma^i(0,x) = \varphi^i(x) & & \text{initial data} \end{cases} \tag{3.5}$$

with φ^i admissible initial parametrisation as defined in Definition 3.10.

Definition 3.9 (Compatibility conditions)**.** A parametrisation φ of an initial network Θ_0 satisfies the compatibility conditions for system (3.5) if it satisfies the concurrency, second and third order conditions and

$$A^i_0(y)\nu^i_0(y) + T^i_0(y)\tau^i_0(y) = A^j_0(y)\nu^j_0(y) + T^j_0(y)\tau^j_0(y)$$

for $i, j \in \{1,2,3\}$ and $y \in \{0,1\}$ where A^i_0 and T^i_0 denote the equations (2.3) and (3.3) applied to φ^i.

Definition 3.10 (Admissible initial parametrisation)**.** A parametrisation φ of an initial Theta–network Θ_0 is an admissible initial parametrisation for system (3.5) if

- for $i \in \{1,2,3\}$ and for some $\alpha \in (0,1)$ we have $\varphi^i \in C^{4+\alpha}([0,1])$;
- each curve φ^i is regular;
- for $y \in \{0,1\}$ we have that $\mathrm{span}\{\nu_0^1(y), \nu_0^2(y), \nu_0^3(y)\} = \mathbb{R}^2$;
- the compatibility conditions for system (3.5) are satisfied.

3.3 Existence of the linearised system

From now on we fix an initial Theta–network Θ_0 parametrised by an admissible initial parametrisation φ.

3.3.1 Linearisation

Consider the system defined in (3.5). Omitting the dependence on (t,x) we write our motion equation in the form

$$\gamma_t^i = -2\frac{\gamma_{xxxx}^i}{|\gamma_x^i|^4} + \tilde{f}(\gamma_{xxx}^i, \gamma_{xx}^i, \gamma_x^i)$$

for $i = 1,2,3$. We linearise the highest order terms of the previous equations around the initial parametrisation obtaining

$$\gamma_t^i + \frac{2}{|\varphi_x^i|^4}\gamma_{xxxx}^i = \left(\frac{2}{|\varphi_x^i|^4} - \frac{2}{|\gamma_x^i|^4}\right)\gamma_{xxxx}^i + \tilde{f}(\gamma_{xxx}^i, \gamma_{xx}^i, \gamma_x^i) =: f^i(\gamma_{xxxx}^i, \gamma_{xxx}^i, \gamma_{xx}^i, \gamma_x^i)\,.$$

Omitting the dependence on (t,y) with $y \in \{0,1\}$ we linearise the highest order terms of the boundary conditions. The concurrency condition and the second order condition are already linear:

$$\gamma^1 = \gamma^2 = \gamma^3 \quad \text{and} \quad \gamma_{xx}^i = 0 \quad \text{for } i \in \{1,2,3\}\,.$$

The third order condition is of the form

$$\sum_{i=1}^{3} \frac{1}{|\gamma_x^i|^3}\left\langle \gamma_{xxx}^i, \nu^i \right\rangle \nu^i + h^i(\gamma_x^i) = 0\,. \tag{3.6}$$

The linearised version of the highest order term in the third order condition (3.6) is the following:

$$-\sum_{i=1}^{3} \frac{1}{|\varphi_x^i|^3}\left\langle \gamma_{xxx}^i, \nu_0^i \right\rangle \nu_0^i = -\sum_{i=1}^{3} \frac{1}{|\varphi_x^i|^3}\left\langle \gamma_{xxx}^i, \nu_0^i \right\rangle \nu_0^i + \sum_{i=1}^{3} \frac{1}{|\gamma_x^i|^3}\left\langle \gamma_{xxx}^i, \nu^i \right\rangle \nu^i + h^i(\gamma_x^i) =: b(\gamma)\,.$$

The linearised system associated to 3.5 is given by

$$\begin{cases} \gamma_t^i + \frac{2}{|\varphi_x^i|^4}\gamma_{xxxx}^i & = f^i & \text{motion,} \\ \gamma^1 - \gamma^2 & = 0 & \text{concurrency,} \\ \gamma^1 - \gamma^3 & = 0 & \text{concurrency,} \\ \gamma_{xx}^i & = 0 & \text{second order,} \\ -\sum_{i=1}^3 \frac{1}{|\varphi_x^i|^3}\left\langle \gamma_{xxx}^i, \nu_0^i \right\rangle \nu_0^i & = b & \text{third order,} \\ \gamma^i(0) & = \psi^i & \text{initial condition} \end{cases} \tag{3.7}$$

for $i \in \{1,2,3\}$ where we have omitted the dependence on $(t,x) \in [0,T] \times [0,1]$ in the motion equation, on $(t,y) \in [0,T] \times \{0,1\}$ for the boundary conditions and on $x \in [0,1]$ for the initial condition, respectively. Here (f,b,ψ) is a general right hand side.

Definition 3.11. [Linear compatibility conditions] Let (f,b) be a given right hand side to the linear system (3.7). A function $\psi \in C^{4+\alpha}\left([0,1];(\mathbb{R}^2)^3\right)$ satisfies the linear compatibility conditions for the linear system (3.7) with respect to (f,b) if for $y \in \{0,1\}$ and $i,j \in \{1,2,3\}$ the equalities

$$\begin{aligned} \psi^i(y) &= \psi^j(y)\,, \\ \psi_{xx}^i(y) &= 0\,, \\ -\sum_{i=1}^{3} \frac{1}{|\varphi_x^i|^3}\left\langle \psi_{xxx}^i(y), \nu_0^i \right\rangle \nu_0^i &= b(0,y) \\ \frac{2}{|\varphi_x^i|^4}\psi_{xxxx}^i(y) - f^i(0,y) &= \frac{2}{|\varphi_x^j|^4}\psi_{xxxx}^j(y) - f^j(0,y) \end{aligned}$$

hold.

Remark 3.12. In the end we will be interested in the case $\psi = \varphi$.

We show existence of the linear parabolic boundary value problem (3.7) using the theory of Solonnikov [18].

3.3.2 Parabolic system in the sense of Solonnikov

Following the notation in [18] we let $\gamma^i = (u^i, v^i)$ and $\varphi^i = (\tilde{\varphi}^i, \hat{\varphi}^i)$. As we are working with a fourth order operator we have $b = 2$ and $r = 6$. We write our motion equation in the form

$$\begin{cases} u_t^i + \frac{2}{|\varphi_x^i|^4}u_{xxxx}^i = a^i, \\ v_t^i + \frac{2}{|\varphi_x^i|^4}v_{xxxx}^i = b^i. \end{cases}$$

Notice that $\gamma(t,x)$ is the vector with components $(u^1,v^1,u^2,v^2,u^3,v^3)$ and $f(t,x)$ has components $(a^1,b^1,a^2,b^2,a^3,b^3)$. Introducing the matrix $\mathcal{L}(x,t,\partial_x,\partial_t)$ given by

$$\begin{bmatrix} \partial_t + \frac{2}{|\varphi_x^1|^4}\partial_x^4 & 0 & 0 & 0 & 0 & 0 \\ 0 & \partial_t + \frac{2}{|\varphi_x^1|^4}\partial_x^4 & 0 & 0 & 0 & 0 \\ 0 & 0 & \partial_t + \frac{2}{|\varphi_x^2|^4}\partial_x^4 & 0 & 0 & 0 \\ 0 & 0 & 0 & \partial_t + \frac{2}{|\varphi_x^2|^4}\partial_x^4 & 0 & 0 \\ 0 & 0 & 0 & 0 & \partial_t + \frac{2}{|\varphi_x^3|^4}\partial_x^4 & 0 \\ 0 & 0 & 0 & 0 & 0 & \partial_t + \frac{2}{|\varphi_x^3|^4}\partial_x^4 \end{bmatrix}$$

the previous system of six equations in the unknown functions $u^1, v^1, u^2, v^2, u^3, v^3$ can be written in the form

$$\mathcal{L}\gamma = f\,.$$

With the choice $s_k = 4$ for every $k \in \{1,\dots,6\}$ and $t_j = 0$ for every $j \in \{1,\dots,6\}$ the conditions of [18, page 8] are satisfied. Following the classical theory in [18] we call $\mathcal{L}_0$ the principal part of the matrix $\mathcal{L}$. With our choice of s_k and t_j we have $\mathcal{L}_0 = \mathcal{L}$ and

$$\det\mathcal{L}_0(x,t,i\xi,p) = \prod_{j=1}^{3}\left(\frac{2}{|\varphi^j_x|}\xi^4 + p\right)^2 .$$

This is a polynomial of degree six in p with roots of multiplicity two of the form $p_j = -\frac{2}{|\varphi^j_x|^4}\xi^4$ with $j = 1,2,3$. Choosing $\delta = \min\left\{\frac{2}{|\varphi^j_x|^4} : j = 1,2,3\right\}$ the system is parabolic in the sense of Solonnikov (see [18, page 8]).

3.3.3 Complementary conditions

The complementary condition in [18, page 11] follows from the Lopatinskii–Shapiro condition (see [8, pages 11–15]). We prove that the Lopatinskii–Shapiro condition is satisfied in the boundary point $y = 0$. The case $y = 1$ can be treated analogously.

Definition 3.13. Let $\lambda \in \mathbb{C}$ with $\Re(\lambda) > 0$ be arbitrary. The Lopatinskii–Shapiro condition demands that every solution $(\gamma^i)_{i=1,2,3} \in C^4([0,\infty),(\mathbb{C}^2)^3)$ to the system of ODEs

$$\begin{cases} \lambda\gamma^i(x) + \frac{1}{|\varphi^i_x|^4}\gamma^i_{xxxx}(x) &= 0 \quad x\in[0,\infty), i\in\{1,2,3\} \quad \text{motion,}\\ \gamma^1(0)-\gamma^2(0) &= 0 \quad \text{concurrency,}\\ \gamma^2(0)-\gamma^3(0) &= 0 \quad \text{concurrency,}\\ \gamma^i_{xx}(0) &= 0 \quad i\in\{1,2,3\} \quad \text{second order,}\\ \sum_{i=1}^3 \frac{1}{|\varphi^i_x|^3}\left\langle\gamma^i_{xxx}(0),\nu^i_0\right\rangle\nu^i_0 &= 0 \quad \text{third order} \end{cases} \tag{3.8}$$

which satisfies $\lim_{x\to\infty}|\gamma^i(x)| = 0$ has to be the trivial solution.

In system (3.8) and in the proof of the following lemma we use the notation $\nu^i_0 := \nu^i_0(0)$ and $\varphi^i_x := \varphi^i_x(0)$.

Lemma 3.14. *The Lopatinskii–Shapiro condition is satisfied.*

Proof. Let $(\gamma^i)_{i=1,2,3}$ be a solution to 3.8 such that $\lim_{x\to\infty}|\gamma^i(x)| = 0$. With γ^i also all derivatives decay to zero for x tending to infinity which follows from the specific exponential form of the solutions to (3.8). We test the motion equation by $|\varphi^i_x|\left\langle\overline{\gamma}^i(x),\nu^i_0\right\rangle\nu^i_0$, integrate twice by parts and sum to get

$$\begin{aligned} 0 &= \sum_{i=1}^3 \lambda|\varphi^i_x|\int_0^\infty |\left\langle\gamma^i(x),\nu^i_0\right\rangle|^2\,\mathrm{d}x + \sum_{i=1}^3\frac{1}{|\varphi^i_x|^3}\int_0^\infty|\left\langle\gamma^i_{xx}(x),\nu^i_0\right\rangle|^2\,\mathrm{d}x \\ &+ \sum_{i=1}^3\frac{1}{|\varphi^i_x|^3}\left\langle\overline{\gamma}^i(0),\nu^i_0\right\rangle\left\langle\gamma^i_{xxx}(0),\nu^i_0\right\rangle - \sum_{i=1}^3\frac{1}{|\varphi^i_x|^3}\left\langle\overline{\gamma}^i_x(0),\nu^i_0\right\rangle\left\langle\gamma^i_{xx}(0),\nu^i_0\right\rangle\,, \end{aligned} \tag{3.9}$$

where we have already used the decay at infinity. Using the concurrency and the third order condition in the first boundary term we get

$$\sum_{i=1}^{3}\frac{1}{|\varphi_x^i|^3}\left\langle\overline{\gamma}^i(0),\nu_0^i\right\rangle\left\langle\gamma_{xxx}^i(0),\nu_0^i\right\rangle=\left\langle\overline{\gamma}^1(0),\sum_{i=1}^{3}\frac{1}{|\varphi_x^i|^3}\left\langle\gamma_{xxx}^i(0),\nu_0^i\right\rangle\nu_0^i\right\rangle=0\,.$$

The second boundary term vanishes due to the second order condition. Now we take the real part of (3.9) and we obtain $\left\langle\gamma^i(x),\nu_0^i\right\rangle=0$ for $x\in[0,\infty)$ and $i\in\{1,2,3\}$.
In particular, using the concurrency condition, we find that $\left\langle\gamma^1(x),\sum_{i=1}^3 a^i\nu_0^i\right\rangle=0$ for all $a^i\in\mathbb{C}$. As φ is an admissible initial parametrisation, this enforces $\gamma^i(0)=0$.
In the same way as before, testing the motion equation by $|\varphi_x^i|\left\langle\overline{\gamma}^i(x),\tau_0^i\right\rangle\tau_0^i$ where τ_0^i is the unit tangent vector in $\varphi^i(0)$, we get

$$\begin{aligned}0=&\sum_{i=1}^{3}\lambda|\varphi_x^i|\int_0^\infty|\left\langle\gamma^i(x),\tau_0^i\right\rangle|^2\,\mathrm{d}x+\sum_{i=1}^{3}\frac{1}{|\varphi_x^i|^3}\int_0^\infty|\left\langle\gamma_{xx}^i(x),\tau_0^i\right\rangle|^2\,\mathrm{d}x\\&+\sum_{i=1}^{3}\frac{1}{|\varphi_x^i|^3}\left\langle\overline{\gamma}^i(0),\tau_0^i\right\rangle\left\langle\gamma_{xxx}^i(0),\tau_0^i\right\rangle-\sum_{i=1}^{3}\frac{1}{|\varphi_x^i|^3}\left\langle\overline{\gamma}_x^i(0),\tau_0^i\right\rangle\left\langle\gamma_{xx}^i(0),\tau_0^i\right\rangle\,.\end{aligned}\tag{3.10}$$

Again the second boundary term vanishes due to the second order condition and the first boundary term vanishes due to the fact that we have previously obtained $\gamma^i(0)=0$. Considering again only the real part of (3.10) we find for all $x\in[0,\infty)$ and $i\in\{1,2,3\}$ that $\left\langle\gamma^i(x),\tau_0^i\right\rangle=0$. Hence we conclude that $\gamma^i(x)=0$ for every $x\in[0,\infty)$ and $i\in\{1,2,3\}$. □

Finally to check the complementary conditions stated in [18, page 12] for the initial data we observe, using again the notation of [18], that the 6×6 matrix $[C_{\alpha j}]$ is the identity matrix. With the choice $\gamma_{\alpha j}=0$ for every $\alpha\in\{1,\ldots,6\}$ and $j\in\{1,\ldots,6\}$ we obtain $\rho_\alpha=0$ and $C_0=Id$. Moreover the rows of the matrix $\mathcal{D}(x,p)=\hat{\mathcal{L}}_0(x,0,0,p)=p^5Id$ are linearly independent modulo the polynomial p^6.

3.3.4 Existence theorem for the linearised system

Adapting [18, Theorem 4.9] to our situation we obtain the following existence result for the system (3.7):

Theorem 3.15. *Let $\alpha\in(0,1)$. For every $T>0$ the Problem (3.7) has a unique solution $\gamma\in C^{\frac{4+\alpha}{4},4+\alpha}([0,T]\times[0,1];(\mathbb{R}^2)^3)$ provided that*

- $f^i\in C^{\frac{\alpha}{4},\alpha}([0,T]\times[0,1];\mathbb{R}^2)$ *for* $i\in\{1,2,3\}$*;*
- $b\in C^{\frac{1+\alpha}{4}}([0,T];\mathbb{R}^4)$*;*
- $\psi\in C^{4+\alpha}\left([0,1];\left(\mathbb{R}^2\right)^3\right)$ *and*
- ψ *satisfies the linear compatibility conditions 3.11 with respect to* (f,b).

Moreover, for all $T > 0$ there exists a $C(T) > 0$ such that for all solutions the inequality

$$\sum_{i=1}^{3} \|\gamma^i\|_{\frac{(4+\alpha)}{4},4+\alpha} \leq C(T) \left(\sum_{i=1}^{3} \|f^i\|_{\frac{\alpha}{4},\alpha} + \|b\|_{\frac{1+\alpha}{4}} + \|\psi\|_{4+\alpha} \right)$$

holds.

3.4 Short time existence of the analytic problem

In the following we fix a coefficient $\alpha \in (0,1)$ and an admissible initial parametrisation φ with $\|\varphi\|_{C^{4+\alpha}([0,1])} = R$.

Definition 3.16. For $T > 0$ we consider the linear spaces

$$\begin{aligned}\mathbb{E}_T &:= \{\gamma \in C^{\frac{4+\alpha}{4},4+\alpha}([0,T]\times[0,1];(\mathbb{R}^2)^3) \text{ such that for } i \in \{1,2,3\}, t \in [0,T]\,,\\ &\qquad y \in \{0,1\} \text{ it holds } \gamma^1(t,y) = \gamma^2(t,y) = \gamma^3(t,y), \gamma^i_{xx}(t,y) = 0\}\,,\\ \mathbb{F}_T &:= \{(f,b,\psi) \in C^{\frac{\alpha}{4},\alpha}([0,T]\times[0,1];(\mathbb{R}^2)^3) \times C^{\frac{1+\alpha}{4}}([0,T];\mathbb{R}^4) \times C^{4+\alpha}\left([0,1];\left(\mathbb{R}^2\right)^3\right)\\ &\qquad \text{such that the linear compatibility conditions hold}\}\,,\end{aligned}$$

endowed with the induced norms.

Lemma 3.17. *The map $L_T : \mathbb{E}_T \to \mathbb{F}_T$ defined by*

$$L_T(\gamma) = \begin{pmatrix} \left(\gamma^i_t + \frac{2}{|\varphi^i_x|^4}\gamma^i_{xxxx}\right)_{i\in\{1,2,3\}} \\ -\mathrm{tr}_{\partial[0,1]} \sum_{i=1}^{3} \frac{1}{|\varphi^i_x|^3} \left\langle \gamma^i_{xxx}, \nu^i_0 \right\rangle \nu^i_0 \\ \gamma_{|t=0} \end{pmatrix}$$

is a continuous isomorphism.

Proof. The map L_T is clearly well defined, bijective and continuous due to Theorem 3.15. □

Remark 3.18. Notice that as φ is regular, there exists a constant $C > 0$ such that

$$\inf_{x\in[0,1]} |\varphi_x(x)| \geq C\,.$$

As a consequence we have

$$\sup_{x\in[0,1]} \frac{1}{|\varphi_x(x)|} \leq \frac{1}{C}\,,$$

and for every $j \in \mathbb{N}$ there hold

$$\left\| \frac{1}{|\varphi^i_x|^j} \right\|_{C^\alpha([0,1])} \leq \left(\frac{\|\varphi_x\|_{C^\alpha([0,1])}}{C^2} \right)^j \leq \left(\frac{R}{C^2} \right)^j\,,$$

and

$$\left\| \frac{1}{|\varphi^i_x|^j} \right\|_{C^{1+\alpha}([0,1])} \leq \widetilde{C}(R,C)\,.$$

Definition 3.19. We define the affine spaces

$$\mathbb{E}_T^\varphi = \{\gamma \in \mathbb{E}_T \text{ such that } \gamma_{|t=0} = \varphi\}\,,$$
$$\mathbb{F}_T^\varphi = \{(f,b) \text{ such that } (f,b,\varphi) \in \mathbb{F}_T\} \times \{\varphi\}\,.$$

In the following we denote by B_M the open ball of radius M and center 0 in $\mathbb{E}_T$.

Lemma 3.20. *There exists $\tilde{t}(M,C) \in (0,1]$ such that for all $\gamma \in \mathbb{E}_T^\varphi \cap B_M$ and for all $t \in [0,\tilde{t}(M,C)]$ we have*

$$\inf_{x\in[0,1]} |\gamma_x(t,x)| \geq \frac{C}{2}\,.$$

Proof. We have

$$|\gamma_x(t,x)| \geq |\varphi_x(x)| - |\gamma_x(t,x) - \varphi_x(x)|$$

with $|\varphi_x(x)| \geq C$ and $|\gamma_x(t,x) - \varphi_x(x)| \leq [\gamma_x]_{\beta,0}\, t^\beta \leq Mt^\beta$ with $\beta = \frac{3}{4} + \frac{\alpha}{4}$. Passing to the infimum one gets the claim. □

As a consequence every $\gamma \in \mathbb{E}_T^\varphi \cap B_M$ is a regular curve for $T \in [0,\tilde{t}(M,C)]$. Moreover we find for every $\gamma \in \mathbb{E}_T^\varphi \cap B_M$ and $t \in [0,\tilde{t}(M,C)]$ that

$$\sup_{x\in[0,1]} \frac{1}{|\gamma_x(t,x)|} \leq \frac{2}{C}\,. \tag{3.11}$$

Lemma 3.21. *Let M be positive and $\tilde{t}(M,C)$ be as above. Then for every $\gamma \in \mathbb{E}_T^\varphi \cap B_M$ with $T \in [0,\tilde{t}(M,C)]$*

$$\left\| \frac{1}{|\gamma_x^i|^j} \right\|_{C^{\frac{\alpha}{4},\alpha}([0,T]\times[0,1])} \leq \left(\frac{4M}{C^2}\right)^j.$$

Proof. For $j = 1$ the result follows directly combining the estimate (3.11) with the definition of the norm $\|\cdot\|_{C^{\frac{\alpha}{4},\alpha}([0,T]\times[0,1])}$. The case $j \geq 2$ follows from multiplicativity of the norm. □

Remark 3.22. Let again M be positive and $\tilde{t}(M,C)$ be as above.
Then for every $T \in (0,\tilde{t}(M,C)]$, $\gamma \in \mathbb{E}_T^\varphi \cap B_M$, $y \in \{0,1\}$ and $j \in \mathbb{N}$ it holds

$$\left\| \frac{1}{|\gamma_x^i(y)|^j} \right\|_{C^{\frac{1+\alpha}{4}}([0,T])} \leq \tilde{C}(R,C)\,. \tag{3.12}$$

This can be shown as in Lemma 3.21 using the definition of the norm of $C^{\frac{1+\alpha}{4}}([0,T])$, estimate (3.11) and the inequality $[f]_{\frac{1+\alpha}{4},0} \leq [f]_{\frac{3+\alpha}{4},0}$.

Lemma 3.23. *Let $T \in (0,\tilde{t}(M,C)]$. Then the maps*

$$N_{T,1} : \begin{cases} \mathbb{E}_T^\varphi & \to C^{\frac{\alpha}{4},\alpha}([0,T]\times[0,1];(\mathbb{R}^2)^3), \\ \gamma & \mapsto f(\gamma) := (f^i(\gamma^i))_{i=1,2,3}, \end{cases}$$
$$N_{T,2} : \begin{cases} \mathbb{E}_T^\varphi & \to C^{\frac{1+\alpha}{4}}([0,T];\mathbb{R}^4), \\ \gamma & \mapsto b(\gamma) \end{cases}$$

are well defined where the functions $f^i(\gamma^i) := f^i(\gamma^i_{xxxx},\gamma^i_{xxx},\gamma^i_{xx},\gamma^i_x)$ and $b(\gamma) := b(\gamma_{xxx},\gamma_x)$ are defined in the linearisation 3.3.1.

Proof. As f^i and b are given by

$$f^i(\gamma^i_{xxxx},\gamma^i_{xxx},\gamma^i_{xx},\gamma^i_x)=\left(\frac{2}{|\varphi^i_x|^4}-\frac{2}{|\gamma^i_x|^4}\right)\gamma^i_{xxxx}+12\frac{\gamma_{xxx}\left\langle\gamma_{xx},\gamma_x\right\rangle}{|\gamma_x|^6}+5\frac{\gamma_{xx}\left|\gamma_{xx}\right|^2}{|\gamma_x|^6}$$
$$+8\frac{\gamma^i_{xx}\left\langle\gamma^i_{xxx},\gamma^i_x\right\rangle}{|\gamma^i_x|^6}-35\frac{\gamma^i_{xx}\left\langle\gamma^i_{xx},\gamma^i_x\right\rangle^2}{|\gamma^i_x|^8}+\mu\frac{\gamma^i_{xx}}{|\gamma^i_x|^2}\,,$$

and

$$b(\gamma^i_{xxx},\gamma^i_x)=-\sum_{i=1}^{3}\frac{1}{|\varphi^i_x|^3}\left\langle\gamma^i_{xxx},\nu^i_0\right\rangle\nu^i_0+\sum_{i=1}^{3}\frac{1}{|\gamma^i_x|^3}\left\langle\gamma^i_{xxx},\nu^i\right\rangle\nu^i-\frac{\mu}{2}\sum_{i=1}^{3}\frac{\gamma^i_x}{|\gamma^i_x|}\,,$$

the Banach–algebra property of the Hölder spaces immediately implies that $N_{T,1}$ and $N_{T,2}$ are well–defined. ☐

Lemma 3.24. *The map N_T given by $\gamma\mapsto(N_{T,1},N_{T,2},\cdot|_{t=0})(\gamma)$ is a mapping from $\mathbb{E}^\varphi_T$ to $\mathbb{F}^\varphi_T$.*

Proof. We have only to show that φ satisfies the linear compatibility conditions with respect to $(N_{T,1},N_{T,2})(\gamma)$. As φ satisfies the third order condition, we have for $y\in\{0,1\}$

$$\begin{aligned}b(\gamma^i_{xxx},\gamma^i_x)(0,y)&=-\sum_{i=1}^{3}\frac{1}{|\varphi^i_x|^3}\left\langle\varphi^i_{xxx},\nu^i_0\right\rangle\nu^i_0+\sum_{i=1}^{3}\frac{1}{|\varphi^i_x|^3}\left\langle\varphi^i_{xxx},\nu^i_0\right\rangle\nu^i_0-\frac{\mu}{2}\sum_{i=1}^{3}\frac{\varphi^i_x}{|\varphi^i_x|}\\&=-\sum_{i=1}^{3}\frac{1}{|\varphi^i_x|^3}\left\langle\varphi^i_{xxx},\nu^i_0\right\rangle\nu^i_0\,.\end{aligned}$$

Moreover, the admissible initial parametrisation φ satisfies for $i,j\in\{1,2,3\}$ and $y\in\{0,1\}$

$$\begin{aligned}\frac{2}{|\varphi^i_x|^4}\varphi^i_{xxxx}(y)-f^i(\gamma^i)(0,y)&=A^i_0(y)\nu^i_0(y)+T^i_0(y)\tau^i_0(y)=A^j_0(y)\nu^j_0(y)+T^j_0(y)\tau^j_0(y)\\&=\frac{2}{|\varphi^j_x|^4}\varphi^j_{xxxx}(y)-f^j(\gamma^j)(0,y)\,.\end{aligned}$$

☐

3.4.1 Contraction estimates

Definition 3.25. We define the mapping $K_T:\mathbb{E}^\varphi_T\to\mathbb{E}^\varphi_T$ as $K_T:=L_T^{-1}N_T$.

Clearly the map K_T is well defined for all $T\in(0,\tilde{t}(M,C)]$ since $L_T^{-1}(\mathbb{F}^\varphi_T)\subset\mathbb{E}^\varphi_T$.

Lemma 3.26. *Let $k\in\{1,2,3\}$, $T\in[0,1]$ and $\gamma,\tilde\gamma\in\mathbb{E}^\varphi_T$. We denote by $\gamma^{(4-k)}$, $\tilde\gamma^{(4-k)}$ the $(4-k)$th space derivative of γ and $\tilde\gamma$, respectively. Then there exist $\varepsilon_k>0$ and a constant $\tilde C$ independent of T such that*

$$\left\|\gamma^{(4-k)}-\tilde\gamma^{(4-k)}\right\|_{C^{\frac{\alpha}{4},\alpha}}\le\tilde C T^{\varepsilon_k}\left\|\gamma^{(4-k)}-\tilde\gamma^{(4-k)}\right\|_{C^{\frac{k+\alpha}{4},k+\alpha}}\le\tilde C T^{\varepsilon_k}\left\|\gamma-\tilde\gamma\right\|_{\mathbb{E}_T}\,.$$

Proof. We let $k \in \{1, 2, 3\}$ be fixed and define

$$X_T^k := \{g \in C^{\frac{k+\alpha}{4}, k+\alpha}([0, T] \times [0, 1]) : g(0, x) = 0 \text{ for all } x \in [0, 1]\}.$$

Note that for $\gamma, \tilde{\gamma} \in \mathbb{E}_T^\varphi$ we have that $\left(\gamma^{(4-k)} - \tilde{\gamma}^{(4-k)}\right) \in X_T^k$. We fix some $\theta_k \in \left[\frac{\alpha}{k+\alpha}, \frac{k}{k+\alpha}\right)$ such that $\theta_k(k + \alpha)$ is not an integer and let $\beta_k := (1 - \theta_k)\frac{k+\alpha}{4} > \frac{\alpha}{4}$. By [13, Proposition 1.1.3] the space $C^{\theta_k(k+\alpha)}([0, 1])$ belongs to the class J_{θ_k} between $C([0, 1])$ and $C^{k+\alpha}([0, 1])$. Using [13, Proposition 1.1.4] this implies

$$B\left([0, T]; C^{k+\alpha}([0, 1])\right) \cap C^{\frac{k+\alpha}{4}}([0, T]; C([0, 1])) \hookrightarrow C^{\beta_k}\left([0, T]; C^{\theta_k(k+\alpha)}([0, 1])\right)$$

and hence also

$$X_T^k \hookrightarrow C_0^{\beta_k}([0, T]; C^{\theta_k(k+\alpha)}([0, 1])) \hookrightarrow C_0^{\beta_k}([0, T]; C^\alpha([0, 1])) \hookrightarrow C_0^{\frac{\alpha}{4}}([0, T]; C^\alpha([0, 1])),$$

where

$$C_0^\delta([0, T]; C^\mu([0, 1])) := \{g \in C_0^\delta([0, T]; C^\mu([0, 1])) : g(0, x) = 0 \text{ for all } x \in [0, 1]\}.$$

Here all embedding constants are independent of $T \in [0, 1]$. In particular we find for all $\gamma, \tilde{\gamma} \in \mathbb{E}_T^\varphi$ with $\varepsilon_k := \beta_k - \frac{\alpha}{4}$ that

$$\begin{aligned}
\|\gamma^{(4-k)} - \tilde{\gamma}^{(4-k)}\|_{C^{\frac{\alpha}{4},\alpha}} &\leq \|\gamma^{(4-k)} - \tilde{\gamma}^{(4-k)}\|_{C^{\frac{\alpha}{4}}([0,T];C^\alpha[0,1])} \\
&\leq T^{\varepsilon_k}\|\gamma^{(4-k)} - \tilde{\gamma}^{(4-k)}\|_{C^{\beta_k}([0,T];C^\alpha[0,1])} \\
&\leq T^{\varepsilon_k}\|\gamma^{(4-k)} - \tilde{\gamma}^{(4-k)}\|_{C^{\beta_k}([0,T];C^{\theta_k(k+\alpha)}([0,1])} \\
&\leq \tilde{C}T^{\varepsilon_k}\|\gamma^{(4-k)} - \tilde{\gamma}^{(4-k)}\|_{X_T^k}.
\end{aligned}$$

□

Lemma 3.27. *For all $T_0 > 0$ there exists a constant $c(T_0)$ such that*

$$\sup_{T \in (0, T_0]} \|L_T^{-1}\|_{\mathcal{L}(\mathbb{F}_T, \mathbb{E}_T)} \leq c(T_0).$$

Proof. For $(f, b, \psi) \in \mathbb{F}_T$ we define the extension $E(f, b, \psi)$ as $E(f, b, \psi) := (E_1 f, E_2 b, \psi)$ with

$$E_1 f(t) := \begin{cases} f(t) & \text{for } t \in [0, T], \\ f\left(T\frac{T_0 - t}{T_0 - T}\right) & \text{for } t \in (T, T_0], \end{cases} \qquad E_2 b(t) := \begin{cases} b(t) & \text{for } t \in [0, T], \\ b\left(T\frac{T_0 - t}{T_0 - T}\right) & \text{for } t \in (T, T_0]. \end{cases}$$

It is clear that $E(f, b, \psi) \in \mathbb{F}_{T_0}$ and that $\|E\|_{\mathcal{L}(\mathbb{F}_T, \mathbb{F}_{T_0})} \leq 1$.
As $L_{T_0}^{-1}(E(f, b, \psi))_{|[0,T]} = L_T^{-1}(f, b, \psi)$ we find that

$$\begin{aligned}
\|L_T^{-1}(f, b, \psi)\|_{\mathbb{E}_T} &\leq \|L_{T_0}^{-1}(E(f, b, \psi))\|_{\mathbb{E}_{T_0}} \\
&\leq \|L_{T_0}^{-1}\|_{\mathcal{L}(\mathbb{F}_{T_0}, \mathbb{E}_{T_0})}\|E(f, b, \psi)\|_{\mathbb{F}_{T_0}} \leq c(T_0)\|(f, b, \psi)\|_{\mathbb{F}_T}.
\end{aligned}$$

□

From now on we fix $T_0 \in (0, 1]$.

Proposition 3.28. *For any positive radius M there exists a time $T(M) \in (0, \min\{\tilde{t}(M,C), T_0\})$ such that for all $T \in (0, T(M)]$ the map $K_T : \mathbb{E}_T^{\varphi} \cap \overline{B_M} \to \mathbb{E}_T^{\varphi}$ is a contraction.*

Proof. We fix a positive radius M and let T be smaller than $\min\{\tilde{t}(M,C), T_0\}$. For $\gamma, \tilde{\gamma} \in \mathbb{E}_T^{\varphi} \cap \overline{B_M}$ we now estimate the different components of the expression $\|N_T(\gamma) - N_T(\tilde{\gamma})\|_{\mathbb{F}_T}$, namely

$$\|N_{T,1}(\gamma) - N_{T,1}(\tilde{\gamma})\|_{C^{\frac{\alpha}{4},\alpha}} = \|f(\gamma) - f(\tilde{\gamma})\|_{C^{\frac{\alpha}{4},\alpha}},$$
$$\|N_{T,2}(\gamma) - N_{T,2}(\tilde{\gamma})\|_{C^{\frac{1+\alpha}{4}}} = \|b(\gamma) - b(\tilde{\gamma})\|_{C^{\frac{1+\alpha}{4}}}.$$

Step 1: Estimate on $\|f(\gamma) - f(\tilde{\gamma})\|_{C^{\frac{\alpha}{4},\alpha}}$
The highest order term is given by

$$\left(\frac{2}{|\varphi_x^i|^4} - \frac{2}{|\gamma_x^i|^4}\right)(\gamma_{xxxx}^i - \tilde{\gamma}_{xxxx}^i) + \left(\frac{2}{|\tilde{\gamma}_x^i|^4} - \frac{2}{|\gamma_x^i|^4}\right)\tilde{\gamma}_{xxxx}^i.$$

Combining the identity

$$\frac{1}{|a|^4} - \frac{1}{|b|^4} = (|b| - |a|)\left(\frac{1}{|a|^2|b|} + \frac{1}{|a||b|^2}\right)\left(\frac{1}{|a|^2} + \frac{1}{|b|^2}\right)$$

with Lemma 3.21 and 3.26 we get

$$\left\|\left(\frac{2}{|\varphi_x^i|^4} - \frac{2}{|\gamma_x^i|^4}\right)(\gamma_{xxxx}^i - \tilde{\gamma}_{xxxx}^i)\right\|_{C^{\alpha,\frac{\alpha}{4}}} \leq \tilde{C}(M,C,R)T^{\varepsilon_3}\|\gamma - \tilde{\gamma}\|_{\mathbb{E}_T}$$

and

$$\left\|\left(\frac{2}{|\tilde{\gamma}_x^i|^4} - \frac{2}{|\gamma_x^i|^4}\right)\tilde{\gamma}_{xxxx}^i\right\|_{C^{\alpha,\frac{\alpha}{4}}} \leq \tilde{C}(M,C)T^{\varepsilon_3}\|\gamma - \tilde{\gamma}\|_{\mathbb{E}_T}.$$

All the other terms of $f(\gamma) - f(\tilde{\gamma})$ are of the form

$$\frac{a\langle b, c\rangle}{|d|^j} - \frac{\tilde{a}\langle \tilde{b}, \tilde{c}\rangle}{|\tilde{d}|^j} \tag{3.13}$$

with $j \in \{2, 6, 8\}$ and with $a, b, c, d, \tilde{a}, \tilde{b}, \tilde{c}, \tilde{d}$ spacial derivatives up to order three of γ^i and $\tilde{\gamma}^i$, respectively.
Adding and subtracting the expression

$$\frac{\tilde{a}\langle b, c\rangle}{|d|^j} + \frac{\tilde{a}\left\langle \tilde{b}, c\right\rangle}{|d|^j} + \frac{\tilde{a}\left\langle \tilde{b}, \tilde{c}\right\rangle}{|d|^j}$$

to (3.13), we get

$$\frac{(a - \tilde{a})\langle b, c\rangle}{|d|^j} + \frac{\tilde{a}\left\langle (b - \tilde{b}), c\right\rangle}{|d|^j} + \frac{\tilde{a}\left\langle \tilde{b}, (c - \tilde{c})\right\rangle}{|d|^j} + \left(\frac{1}{|d|^j} - \frac{1}{|\tilde{d}|^j}\right)\tilde{a}\left\langle \tilde{b}, \tilde{c}\right\rangle. \tag{3.14}$$

With the help of Lemma 3.26 we can estimate the first term of (3.14) in the following way:

$$\left\|\frac{(a - \tilde{a})\langle b, c\rangle}{|d|^j}\right\|_{C^{\frac{\alpha}{4},\alpha}} \leq \tilde{C}(M,C)\|a - \tilde{a}\|_{C^{\frac{\alpha}{4},\alpha}} \leq \tilde{C}(M,C)T^{\varepsilon}\|\gamma - \tilde{\gamma}\|_{\mathbb{E}_T}.$$

The second and the third term of (3.14) can be estimated similarly using Hölder inequality. Considering the last term of (3.14) we get with the help of the identities

$$\frac{1}{|d|^2} - \frac{1}{|\tilde{d}|^2} = \left(|\tilde{d}| - |d|\right)\left(\frac{1}{|d|^2|\tilde{d}|} + \frac{1}{|d||\tilde{d}|^2}\right),$$
$$\frac{1}{|d|^j} - \frac{1}{|\tilde{d}|^j} = \left(|\tilde{d}| - |d|\right)\left(\frac{1}{|d|^2|\tilde{d}|} + \frac{1}{|d||\tilde{d}|^2}\right)\left(\frac{1}{|d|^2} + \frac{1}{|\tilde{d}|^2}\right)\left(\frac{1}{|d|^{j-4}} + \frac{1}{|\tilde{d}|^{j-4}}\right),$$

for $j \in \{6, 8\}$ and Lemmata 3.21 and 3.26 that

$$\left\|\left(\frac{1}{|d|^j} - \frac{1}{|\tilde{d}|^j}\right)\tilde{a}\left\langle\tilde{b}, \tilde{c}\right\rangle\right\|_{C^{\frac{\alpha}{4},\alpha}} \leq \tilde{C}(M,C)T^{\varepsilon}\|d - \tilde{d}\|_{C^{\frac{\alpha}{4},\alpha}} \leq \tilde{C}(M,C)T^{\varepsilon}\|\gamma - \tilde{\gamma}\|_{\mathbb{E}_T}.$$

Then we conclude that

$$\|f(\gamma) - f(\tilde{\gamma})\|_{C^{\frac{\alpha}{4},\alpha}} \leq \tilde{C}(M,C)T^{\varepsilon}\|\gamma - \tilde{\gamma}\|_{\mathbb{E}_T}.$$

Step 2: Estimate on $\|b(\gamma) - b(\tilde{\gamma})\|_{C^{\frac{1+\alpha}{4}}}$

We now consider the boundary terms. We remind that we use the identification

$$C^{\frac{1+\alpha}{4},1+\alpha}([0,T]\times\{0,1\};\mathbb{R}^2) \simeq C^{\frac{1+\alpha}{4}}([0,T];\mathbb{R}^4)$$

introduced in 2.1.2.

The expression $b(\gamma) - b(\tilde{\gamma})$ is given by

$$\sum_{i=1}^{3}\frac{1}{|\varphi_x^i|^3}\left\langle(\tilde{\gamma}_{xxx}^i - \gamma_{xxx}^i), \nu_0^i\right\rangle\nu_0^i + \sum_{i=1}^{3}\frac{1}{|\gamma_x^i|^3}\left\langle\gamma_{xxx}^i, \nu^i\right\rangle\nu^i$$
$$-\sum_{i=1}^{3}\frac{1}{|\tilde{\gamma}_x^i|^3}\left\langle\tilde{\gamma}_{xxx}^i, \tilde{\nu}^i\right\rangle\tilde{\nu}^i - \frac{\mu}{2}\sum_{i=1}^{3}\left(\frac{\gamma_x^i}{|\gamma_x^i|} - \frac{\tilde{\gamma}_x^i}{|\tilde{\gamma}_x^i|}\right). \tag{3.15}$$

Writing the fourth term of (3.15) as

$$-\frac{\mu}{2}\sum_{i=1}^{3}\frac{\gamma_x^i - \tilde{\gamma}_x^i}{|\gamma_x^i|} + \left(\frac{1}{|\gamma_x^i|} - \frac{1}{|\tilde{\gamma}_x^i|}\right)\tilde{\gamma}_x^i, \tag{3.16}$$

we find, combining Lemma 2.6 with (3.12), that

$$\left\|\frac{\gamma_x^i - \tilde{\gamma}_x^i}{|\gamma_x^i|}\right\|_{C^{\frac{1+\alpha}{4}}} \leq \tilde{C}(M,C)T^{\varepsilon}\|\gamma - \tilde{\gamma}\|_{\mathbb{E}_T},$$
$$\left\|\left(\frac{1}{|\gamma_x^i|} - \frac{1}{|\tilde{\gamma}_x^i|}\right)\tilde{\gamma}_x^i\right\|_{C^{\frac{1+\alpha}{4}}} \leq C\left\|\frac{|\gamma_x^i - \tilde{\gamma}_x^i|}{|\gamma_x^i||\tilde{\gamma}_x^i|}\right\|_{C^{\frac{1+\alpha}{4}}}\|\tilde{\gamma}_x^i\|_{C^{\frac{1+\alpha}{4}}} \leq \tilde{C}(M,C)T^{\varepsilon}\|\gamma - \tilde{\gamma}\|_{\mathbb{E}_T}.$$

Adding and subtracting

$$\sum_{i=1}^{3}\frac{1}{|\varphi_x^i|^3}\langle\tilde{\gamma}_{xxx}^i - \gamma_{xxx}^i, \nu^i\rangle\nu_0^i + \sum_{i=1}^{3}\frac{1}{|\varphi_x^i|^3}\langle\tilde{\gamma}_{xxx}^i - \gamma_{xxx}^i, \nu^i\rangle\nu^i + \sum_{i=1}^{3}\frac{1}{|\gamma_x^i|^3}\left\langle\tilde{\gamma}_{xxx}^i, \nu^i\right\rangle\nu^i$$
$$+\sum_{i=1}^{3}\frac{1}{|\gamma_x^i|^3}\left\langle\tilde{\gamma}_{xxx}^i, \tilde{\nu}^i\right\rangle\nu^i + \sum_{i=1}^{3}\frac{1}{|\gamma_x^i|^3}\left\langle\tilde{\gamma}_{xxx}^i, \tilde{\nu}^i\right\rangle\tilde{\nu}^i$$

to the remaining terms of (3.15) we obtain

$$\sum_{i=1}^{3} \frac{1}{|\gamma_x^i|^3} \left\langle \tilde{\gamma}_{xxx}^i, \nu^i - \tilde{\nu}^i \right\rangle \nu^i + \sum_{i=1}^{3} \frac{1}{|\gamma_x^i|^3} \left\langle \tilde{\gamma}_{xxx}^i, \tilde{\nu}^i \right\rangle \left(\nu^i - \tilde{\nu}^i\right) \tag{3.17}$$

$$+ \sum_{i=1}^{3} \frac{1}{|\varphi_x^i|^3} \left\langle (\tilde{\gamma}_{xxx}^i - \gamma_{xxx}^i), \nu_0^i - \nu^i \right\rangle \nu_0^i + \sum_{i=1}^{3} \frac{1}{|\varphi_x^i|^3} \left\langle (\tilde{\gamma}_{xxx}^i - \gamma_{xxx}^i), \nu^i \right\rangle \left(\nu_0^i - \nu^i\right) \tag{3.18}$$

$$+ \sum_{i=1}^{3} \left(\frac{1}{|\gamma_x^i|^3} - \frac{1}{|\tilde{\gamma}_x^i|^3} \right) \left\langle \tilde{\gamma}_{xxx}^i, \tilde{\nu}^i \right\rangle \tilde{\nu}^i + \sum_{i=1}^{3} \left(\frac{1}{|\varphi_x^i|^3} - \frac{1}{|\gamma_x^i|^3} \right) \left\langle \tilde{\gamma}_{xxx}^i - \gamma_{xxx}^i, \nu^i \right\rangle \nu^i \,. \tag{3.19}$$

To estimate (3.17) we use

$$\begin{aligned}\left\|\nu^i - \tilde{\nu}^i\right\|_{C^{\frac{1+\alpha}{4}}} &= \left\| R\left(\frac{\gamma_x^i}{|\gamma_x^i|}\right) - R\left(\frac{\tilde{\gamma}_x^i}{|\tilde{\gamma}_x^i|}\right) \right\|_{C^{\frac{1+\alpha}{4}}} \leq c \left\| \frac{\gamma_x^i}{|\gamma_x^i|} - \frac{\tilde{\gamma}_x^i}{|\tilde{\gamma}_x^i|} \right\|_{C^{\frac{1+\alpha}{4}}} \\ &\leq \tilde{C}(M,C) T^{\varepsilon} \|\gamma - \tilde{\gamma}\|_{\mathbb{E}_T} \,.\end{aligned}$$

To gain the last inequality we have repeated the argument done for the term (3.16). Combining $\|\tilde{\gamma}_{xxx} - \gamma_{xxx}\|_{C^{\frac{1+\alpha}{4}}} \leq \|\tilde{\gamma} - \gamma\|_{\mathbb{E}_T}$ with

$$\left\|\nu_0^i - \nu^i\right\|_{C^{\frac{1+\alpha}{4}}} \leq \overline{C}(M,C) T^{\varepsilon} \|\varphi - \gamma\|_{\mathbb{E}_T} \leq \tilde{C}(M,C) T^{\varepsilon}$$

the expression of (3.18) is bounded by $\tilde{C}(M,C)T^{\varepsilon}\|\gamma - \tilde{\gamma}\|_{\mathbb{E}_T}$.
For the terms in (3.19) we use the equality

$$\left(\frac{1}{|a|^3} - \frac{1}{|b|^3}\right) = \left(\frac{1}{|a|} - \frac{1}{|b|}\right)\left(\frac{1}{|a|^2} + \frac{1}{|a||b|} + \frac{1}{|b|^2}\right)$$

with $a, b \in \mathbb{R}^2$. In particular we have

$$\left\| \left(\frac{1}{|\gamma_x^i|^3} - \frac{1}{|\tilde{\gamma}_x^i|^3} \right) \right\|_{C^{\frac{1+\alpha}{4}}} \leq \overline{C}(M,C) \left\| \frac{|\gamma_x^i - \tilde{\gamma}_x^i|}{|\gamma_x^i||\tilde{\gamma}_x^i|} \right\|_{C^{\frac{1+\alpha}{4}}} \leq \tilde{C}(M,C) T^{\varepsilon} \|\gamma - \tilde{\gamma}\|_{\mathbb{E}_T} \,.$$

Also the last term in (3.19) is bounded from above by $\tilde{C}(M,C)T^{\varepsilon}\|\gamma - \tilde{\gamma}\|_{\mathbb{E}_T}$ using

$$\left\| \left(\frac{1}{|\varphi^i|^3} - \frac{1}{|\gamma_x^i|^3} \right) \right\|_{C^{\frac{1+\alpha}{4}}} \leq \overline{C}(M,C) T^{\varepsilon} \|\varphi - \gamma\|_{\mathbb{E}_T} \leq \tilde{C}(M,C) T^{\varepsilon}$$

and $\left\|\tilde{\gamma}_{xxx}^i - \gamma_{xxx}^i\right\|_{C^{\frac{1+\alpha}{4}}} \leq \|\tilde{\gamma} - \gamma\|_{\mathbb{E}_T}$.
Hence we have shown that

$$\|b(\gamma) - b(\tilde{\gamma})\|_{C^{\frac{1+\alpha}{4}}} \leq \tilde{C}(M,C) T^{\varepsilon} \|\gamma - \tilde{\gamma}\|_{\mathbb{E}_T} \,.$$

Step 3: Contraction estimate
The estimates presented in ***Step 1*** and ***Step 2*** imply for all $T \in (0, \min\{\tilde{t}(M,C), T_0\}]$

$$\begin{aligned}\|K_T(\gamma) - K_T(\tilde{\gamma})\|_{\mathbb{E}_T} &= \|L_T^{-1}(N_T(\gamma)) - L^{-1}(N_T(\tilde{\gamma}))\|_{\mathbb{E}_T} \\ &= \|L_T^{-1}(N_T(\gamma) - N_T(\tilde{\gamma}))\|_{\mathbb{E}_T} \\ &\leq \sup_{T \in [0,T_0]} \|L_T^{-1}\|_{\mathcal{L}(\mathbb{F}_T, \mathbb{E}_T)} \|N_T(\gamma) - N_T(\tilde{\gamma})\|_{\mathbb{F}_T} \\ &\leq \tilde{C}(M,C,R,T_0) T^{\varepsilon} \|\gamma - \tilde{\gamma}\|_{\mathbb{E}_T}\end{aligned}$$

with $0 < \varepsilon < 1$. Choosing $T(M) \in (0, \min\{\tilde{t}(M,C), T_0\}]$ small enough we can conclude that K_T is a contraction for every $T \in (0, T(M)]$. □

3.4.2 Short time existence Theorem

Proposition 3.29. *There exists a positive radius $M = M(R, C, T_0)$ and a positive time $\tilde{T}(M)$ such that for all $T \in (0, \tilde{T}(M)]$ the map $K_T : \mathbb{E}_T^{\varphi} \cap \overline{B_M} \to \mathbb{E}_T^{\varphi} \cap \overline{B_M}$ is well defined.*

Proof. We define
$$M := 2c(T_0) \max\{R, \|N_{T_0}(\varphi)\|_{\mathbb{F}_{T_0}}\}$$
and note that the quantity $\|N_{T_0}(\varphi)\|_{\mathbb{F}_{T_0}}$ does not depend on T_0 but only on $R = \|\varphi\|_{C^{4+\alpha}([0,1])}$ and C. In particular we have for any $T \in (0, T_0]$

$$\|K_T(\varphi)\|_{\mathbb{E}_T} \leq \sup_{T\in(0,T_0]} \|L_T^{-1}\|_{\mathcal{L}(\mathbb{F}_T,\mathbb{E}_T)} \|N_T(\varphi)\|_{\mathbb{F}_T} \leq c(T_0)\|N_{T_0}(\varphi)\|_{\mathbb{F}_{T_0}} \leq \frac{M}{2}\,. \tag{3.20}$$

Let $T(M)$ be the corresponding time as in Proposition 3.28 such that for all $T \in (0, T(M)]$ the map K_T is a contraction on $\overline{B_M}$ and let $\gamma \in \mathbb{E}_T^{\varphi} \cap \overline{B_M}$. As φ lies as well in $\overline{B_M}$ we have for all $T \in (0, T(M)]$ that

$$\|K_T(\gamma) - K_T(\varphi)\|_{\mathbb{E}_T} \leq \tilde{C}(M, C, R, T_0)T^{\varepsilon}\|\gamma - \varphi\|_{\mathbb{E}_T} \leq \tilde{C}(M, C, R, T_0)2MT^{\varepsilon}$$

for some $0 < \varepsilon < 1$. We choose $\tilde{T}(M) \in (0, T(M)]$ so small that $\tilde{C}(M, C, R, T_0)2MT^{\varepsilon} \leq \frac{M}{2}$ for all $T \in (0, \tilde{T}(M)]$. Combining this with (3.20) implies for all $T \in (0, \tilde{T}(M)]$ and $\gamma \in \mathbb{E}_T^{\varphi} \cap \overline{B_M}$ that
$$\|K_T(\gamma)\|_{\mathbb{E}_T} \leq \|K_T(\varphi)\|_{\mathbb{E}_T} + \|K_T(\gamma) - K_T(\varphi)\|_{\mathbb{E}_T} \leq M.$$

□

Theorem 3.30. *Let φ be an admissible initial parametrisation. There exists a positive radius M and a positive time T such that the system (3.5) has a unique solution in $C^{\frac{4+\alpha}{4},4+\alpha}([0,T] \times [0,1]) \cap \overline{B_M}$.*

Proof. Let M and $\tilde{T}(M)$ be the radius and time as in Proposition 3.29 and let $T \in (0, \tilde{T}(M)]$. Then
$$K_T : \mathbb{E}_T^{\varphi} \cap \overline{B_M} \to \mathbb{E}_T^{\varphi} \cap \overline{B_M}$$
is a contraction of the complete metric space $\mathbb{E}_T^{\varphi} \cap \overline{B_M}$ and has a unique fixed point in $\mathbb{E}_T^{\varphi} \cap \overline{B_M}$ by the Contraction Mapping Principle. As the solutions of (3.5) in $C^{\frac{4+\alpha}{4},4+\alpha}([0,T] \times [0,1]) \cap \overline{B_M}$ are precisely the fixed points of K_T in $\mathbb{E}_T^{\varphi} \cap \overline{B_M}$, the claim follows. □

3.5 Geometric evolution and geometric uniqueness

In the previous subsection we have shown that there exists a unique solution to the Analytic Problem provided that the initial data is admissible. It remains to establish a relation between geometrically admissible initial networks and admissible initial parametrisations. This is necessary to solve the question of geometric existence and uniqueness of the flow.

Lemma 3.31. *Suppose that Θ_0 is a geometrically admissible initial network in the sense of Definition 3.2 parametrised by σ^i. Then there exist three smooth functions $\theta^i : [0,1] \to [0,1]$ such that the reparametrisation $(\sigma^i \circ \theta^i)$ of Θ_0 by (θ^i) is an admissible initial parametrisation for the Analytic Problem (3.5).*

Proof. Without loss of generality we may assume that $\sigma^1(0) = \sigma^2(0) = \sigma^3(0)$ and $\sigma^1(1) = \sigma^2(1) = \sigma^3(1)$.
We look for some smooth maps $\theta^i : [0,1] \to [0,1]$ such that $\theta^i_x(x) \neq 0$ for every $x \in [0,1]$, $\theta^i(0) = 0$ and $\theta^i(1) = 1$. Then $\varphi^i := \sigma^i \circ \theta^i : [0,1] \to \mathbb{R}^2$ is regular and of class $C^{4+\alpha}([0,1])$ and the concurrency condition is satisfied. Since the unit normal vectors are invariant under reparametrisation, we have $\mathrm{span}\{\nu_0^1, \nu_0^2, \nu_0^3\} = \mathbb{R}^2$ at both triple junctions. Also the curvature and third order condition remain true being invariant under reparametrisation. In order to fulfil the second order condition $\varphi^i_{xx}(0) = \varphi^i_{xx}(1) = 0$ we demand that $\theta^i_x(y) = 1$ and $\theta^i_{xx}(y) = \frac{-\sigma^i_{xx}(y)}{\sigma^i_x(y)}$ for $y \in \{0,1\}$. Indicating with subscript φ the quantities computed for the reparametrised object the last property we have to prove is that at $y \in \{0,1\}$ it holds

$$A^1_\varphi \nu^1_\varphi + T^1_\varphi \tau^1_\varphi = A^2_\varphi \nu^2_\varphi + T^2_\varphi \tau^2_\varphi = A^3_\varphi \nu^3_\varphi + T^3_\varphi \tau^3_\varphi \,.$$

Since the geometric quantities are invariant under reparametrisation, this is equivalent to

$$A^1_0 \nu^1_0 + T^1_\varphi \tau^1_0 = A^2_0 \nu^2_0 + T^2_\varphi \tau^2_0 = A^3_0 \nu^3_0 + T^3_\varphi \tau^3_0 \,.$$

This is satisfied if and only if it holds

$$\begin{aligned}
&\sin(\alpha^1) A^1_0 + \sin(\alpha^2) A^2_0 + \sin(\alpha^3) A^3_0 = 0\,,\\
&\sin(\alpha^1) T^1_\varphi = \cos(\alpha^2) A^2_0 - \cos(\alpha^3) A^3_0\,,\\
&\sin(\alpha^2) T^2_\varphi = \cos(\alpha^3) A^3_0 - \cos(\alpha^1) A^1_0\,,\\
&\sin(\alpha^3) T^3_\varphi = \cos(\alpha^1) A^1_0 - \cos(\gamma^2) A^2_0\,.
\end{aligned}$$

The first equation is fulfilled as Θ_0 is a geometrically admissible initial network. Asking that $\theta^i_{xxx}(0) = 1 = \theta^i_{xxx}(1)$ the remaining three equations are of the form

$$a^i(\sigma^i_x)(y)\, \theta^i_{xxxx}(y) + b^i(\sigma^i_{xxxx}, \sigma^i_{xxx}, \sigma^i_{xx}, \sigma^i_x)(y) = 0\,,$$

where a^i and b^i are given non–linear functions. Hence $\theta^i_{xxxx}(0)$ and $\theta^i_{xxxx}(1)$ are uniquely determined for every $i \in \{1,2,3\}$. Thus we may choose θ^i to be the fourth Taylor polynomial near each triple junction determined by the values of the derivatives at the triple junctions that we have just derived. □

Theorem 3.32. *[Geometric existence and uniqueness] Let Θ_0 be a geometrically admissible initial network. Then there exists a positive time T such that within the time interval $[0,T]$ the Problem 2.7 admits a unique solution $\Theta(t)$ in the sense of Definition 3.3.*

Proof. Let Θ_0 be a geometrically admissible initial network. Thanks to Lemma 3.31 there exists an admissible initial parametrisation φ for Θ_0. Then by Theorem 3.30 there exists a solution $\tilde{\gamma}$ to system (3.5) on some time interval $[0,\tilde{T}]$. In particular the family of networks $(\tilde{\Theta}(t))_{t\in[0,\tilde{T}]}$ with $\tilde{\Theta}(t) = \cup_{i=1}^3 \tilde{\gamma}^i(t)$ is a solution to Problem 2.7 in the sense of Definition 3.3.

Suppose that there exists another solution $(\Theta(t))_{t\in[0,T]}$ to the C^0–flow in the sense of Definition 3.3 in a certain time interval $[0,T]$ with $T > 0$ and with the same initial network Θ_0. We assume that $(\Theta(t))$ is parametrised by γ.

It remains to prove that there exists a $\overline{T} \in (0, \min\{\tilde{T}, T\}]$ such that the networks $(\tilde{\Theta}(t))$ and $(\Theta(t))$ coincide for every $t \in [0,\overline{T}]$. To show this it is enough to find a time dependent

family of reparametrisations $\psi : [0,\overline{T}] \times [0,1] \to [0,1]$ such that $\tilde{\gamma}(t,x) = \gamma(t,\psi(t,x))$ for all $(t,x) \in [0,\overline{T}] \times [0,1]$.
Following the approach presented in [9, Theorem 5.3] to prove the existence of ψ it is sufficient to solve the following initial boundary value problem on $[0,\overline{T}] \times [0,1]$:

$$\begin{cases} \psi_t(t,x) + \frac{2\psi_{xxxx}(t,x)}{|\gamma_x(t,\psi(t,x))|^4|\psi_x(t,x)|^4} + g(\psi_{xxx},\psi_{xx},\psi_x,\psi,\gamma_{xxxx},\gamma_{xxx},\gamma_{xx},\gamma_x,\gamma_t) = 0, \\ \psi(t,y) = 0, \\ \psi_{xx}(t,y) = -\frac{\gamma_{xx}(t,y)\psi_x^2(t,y)}{\gamma_x(t,y)}, \\ \gamma(0,\psi(0,x)) = \varphi(x) \end{cases}$$

with $y \in \{0,1\}$ and $(t,x) \in [0,\overline{T}] \times [0,1]$.

Indeed assume that ψ exists. Then the family of parametrisations $Z(t,x) := \gamma(t,\psi(t,x))$ is a solution to the Analytic Problem (3.5). The motion equation follows from the PDE satisfied by ψ and the geometric evolution of Θ in normal direction. The geometric boundary conditions (concurrency, curvature and third order) are satisfied as Θ is a solution to the Geometric Problem. The boundary condition on ψ_{xx} guarantees that Z satisfies the second order condition. By uniqueness of the Analytic Problem for short time Z and $\tilde{\gamma}$ need to coincide on a small time interval.

We note that the parabolic boundary value problem for ψ^i has a similar structure as the system of PDEs we studied before. The associated linearised system satisfies the Lopatinskii–Shapiro condition and a contraction argument allows us to conclude that the desired function ψ^i exists and is unique. □

4 C^1–Flow

In this section we briefly discuss an analogous result to Theorem 3.32 in the case of a C^1 Triod evolving with respect to the gradient flow of the elastic energy E_μ.

In contrast to the C^0–case we now demand that at the triple junction the curves form an angle of 120 degrees. Moreover we ask that the other three endpoints P^i remain fixed on $\partial\Omega$ during the evolution. This corresponds to the Navier case.

Definition 4.1. A Triod $\mathbb{T}_0$ is called geometrically admissible for Problem 2.8 if

- the curves meet in one triple junction forming angles of 120 degrees and satisfying $\sum_{i=1}^3 k_0^i = 0$ and $\sum_{i=1}^3 \left(2k_{0,s}^i \nu_0^i - (k_0^i)^2 \tau_0^i\right) = 0$;
- each curve has zero curvature at the endpoint P^i on $\partial\Omega$;
- at the triple junction it holds that $A_0^1 + A_0^2 + A_0^3 = 0$ where $A_0^i = 2k_{0,ss}^i + \left(k_0^i\right)^3 - \mu k_0^i$;
- there exists a parametrisation $\gamma = (\gamma^1,\gamma^2,\gamma^3)$ of $\mathbb{T}_0$ such that every curve is regular and an element of $C^{4+\alpha}([0,1];\mathbb{R}^2)$.

Definition 4.2. Let $\mathbb{T}_0$ be a geometrically admissible initial Triod. A time dependent family of Triods $\mathbb{T}(t)$ is a solution to the C^1–Flow with initial data $\mathbb{T}_0$ in a time interval $[0,T]$ with $T > 0$ if and only if there exist parametrisations $\gamma(t) = (\gamma^1(t),\gamma^2(t),\gamma^3(t))$ of $\mathbb{T}(t)$, $t \in [0,T]$, with

$\gamma^i \in C^{\frac{4+\alpha}{4}, 4+\alpha}([0,T]\times[0,1],\mathbb{R}^2)$ such that for every $t\in[0,T]$, $x\in[0,1]$ and for $i\in\{1,2,3\}$ the following system

$$\begin{cases} \left\langle \gamma^i_t(t,x), \nu^i(t,x)\right\rangle \nu^i(t,x) = -A^i(t,x)\nu^i(t,x) & \text{motion,}\\ \gamma^1(t,0)=\gamma^2(t,0)=\gamma^3(t,0) & \text{concurrency condition,}\\ \tau^1(t,0)+\tau^2(t,0)+\tau^3(t,0)=0 & \text{angle condition,}\\ k^1(t,0)+k^2(t,0)+k^3(t,0)=0 & \text{curvature condition,}\\ \sum_{i=1}^3 \left(2k^i_s\nu^i-(k^i)^2\tau^i\right)(t,0)=0 & \text{third order condition,}\\ \gamma^i(t,1)=P^i & \text{fixed endpoints,}\\ k^i(t,1)=0 & \text{curvature condition,}\\ \mathbb{T}(0)=\mathbb{T}_0 & \text{initial data} \end{cases}$$

is satisfied.

Our aim is to prove geometric existence and uniqueness of the C^1–flow.

Theorem 4.3. *Let $\mathbb{T}_0$ be a geometrically admissible initial Triod. Then there exists a positive time T such that within the time interval $[0,T]$ the Problem 2.8 admits a unique solution $(\mathbb{T}(t))$ in the sense of Definition 4.2.*

4.1 Proof of Theorem 4.3

Similar arguments as used in the previous section on geometric existence and uniqueness justify that to find a unique geometric solution to the C^1–flow of a Triod we may focus on the Analytic Problem formulated for an appropriate time dependent parametrisation of the flow. To this end we now sketch how to prove a short time existence result for the associated system of PDEs.
We again choose the same tangential velocity and we impose the tangential second order condition at both boundary points which leads to the following definition of admissible initial parametrisation and to the subsequent problem.

Definition 4.4. A parametrisation φ of an initial Triod $\mathbb{T}_0$ is admissible if

- each curve φ^i is regular and in $C^{4+\alpha}([0,1])$ for some $\alpha\in(0,1)$;
- at the triple junction $O=\gamma^i(0)$ the curves satisfy the concurrency, angle and tangential second order condition as well as $\sum_{i=1}^3 k^i_0=0$, $\sum_{i=1}^3\left(2k^i_{0,s}\nu^i_0-(k^i_0)^2\tau^i_0\right)=0$ and

  $$A^i_0\nu^i_0+T^i_0\tau^i_0=A^j_0\nu^j_0+T^j_0\tau^j_0$$

  for $i,j\in\{1,2,3\}$ where A^i_0 and T^i_0 denote the equations (2.3) and (3.3) applied to φ^i;
- at the endpoints $P^i=\gamma^i(1)$ it holds that $\gamma^i_{xx}=0$.

Problem 4.5. For every $t \in [0,T)$, $x \in [0,1]$ and for $i \in \{1,2,3\}$

$$\begin{cases} \gamma_t^i(t,x) = -A^i(t,x)\nu^i(t,x) - T^i(t,x)\tau^i(t,x) & \text{motion,} \\ \gamma^1(t,0) = \gamma^2(t,0) = \gamma^3(t,0) & \text{concurrency condition,} \\ \tau^1(t,0) + \tau^2(t,0) + \tau^3(t,0) = 0 & \text{angle condition,} \\ k^1(t,0) + k^2(t,0) + k^3(t,0) = 0 & \text{curvature condition,} \\ \left\langle \gamma_{xx}^i(t,0), \tau^i(t,0) \right\rangle = 0 & \text{tangential second order condition,} \\ \sum_{i=1}^3 \left(2k_s^i \nu^i - (k^i)^2 \tau^i \right)(t,0) = 0 & \text{third order condition,} \\ \gamma^i(t,1) = \varphi^i(1) & \text{fixed endpoints,} \\ \gamma_{xx}^i(t,1) = 0 & \text{second order condition,} \\ \gamma^i(0,x) = \varphi^i(x) & \text{initial data} \end{cases} \tag{4.1}$$

with φ^i admissible initial parametrisation as defined in 4.4.

Theorem 4.6. *Let φ be an admissible initial parametrisation. There exists a positive radius M and a positive time T such that the system (4.1) has a unique solution in $C^{\frac{4+\alpha}{4}, 4+\alpha}([0,T] \times [0,1]) \cap \overline{B_M}$.*

To prove existence of a unique solution $\gamma \in C^{\frac{4+\alpha}{4}, 4+\alpha}([0,T] \times [0,1])$ to (4.1) one linearises the system around the initial data, proves existence of the linear system and provides contraction estimates which guarantee the existence of a solution to the original system. As many arguments are exactly analogous to the case of C^0–Theta networks, we focus on the relevant changes.

Hence we prove that the Lopatinskii–Shapiro condition is fulfilled and we describe the contraction estimates for the boundary terms that are remarkably different.

4.1.1 Linearisation

We state here the linearisation of the remaining boundary conditions that appear in the system 4.1 (and do not in system 3.5).
Notice that all the conditions at $x = 1$ are already linear. We linearise the conditions at $x = 0$. The angle condition becomes

$$-\sum_{i=1}^{3} \left(\frac{\gamma_x^i}{|\varphi_x^i|} - \frac{\varphi_x^i \left\langle \gamma_x^i, \varphi_x^i \right\rangle}{|\varphi_x^i|^3} \right) = \sum_{i=1}^{3} \left(\left(\frac{1}{|\gamma_x^i|} - \frac{1}{|\varphi_x^i|} \right) \gamma_x^i + \frac{\varphi_x^i \left\langle \gamma_x^i, \varphi_x^i \right\rangle}{|\varphi_x^i|^3} \right) =: b^1(\gamma)\,.$$

Considering the unit tangent vector τ_0^i and normal vector ν_0^i at time zero,

$$\tau_0^i = \frac{1}{|\varphi_x^i|} \begin{pmatrix} \tilde{\varphi}_x^i \\ \hat{\varphi}_x^i \end{pmatrix} \qquad \nu_0^i = \frac{1}{|\varphi_x^i|} \begin{pmatrix} \hat{\varphi}_x^i \\ -\tilde{\varphi}_x^i\,, \end{pmatrix}$$

the linearised version of the curvature condition is of the form:

$$\sum_{i=1}^{3} \frac{1}{|\varphi_x^i|^2} \left\langle \gamma_{xx}^i, \nu_0^i \right\rangle = \sum_{i=1}^{3} \frac{1}{|\varphi_x^i|^2} \left\langle \gamma_{xx}^i, \nu_0^i \right\rangle - \sum_{i=1}^{3} \frac{1}{|\gamma_x^i|^2} \left\langle \gamma_{xx}^i, \nu^i \right\rangle =: b^2(\gamma,)$$

and the linearised tangential second order condition becomes:

$$\left\langle \gamma_{xx}^i, \tau_0^i \right\rangle = \left\langle \gamma_{xx}^i, \tau_0^i - \tau^i \right\rangle =: b^{3,i}(\gamma^i)\,.$$

As we are linearising only the highest order part of each condition, the linearised third order condition that will appear in the linearised system for 4.1 is the same as in the case of C^0 Theta–networks:

$$\sum_{i=1}^{3} \frac{1}{|\varphi_x^i|^3} \left\langle \gamma_{xxx}^i, \nu_0^i \right\rangle \nu_0^i = \sum_{i=1}^{3} \frac{1}{|\varphi_x^i|^3} \left\langle \gamma_{xxx}^i, \nu_0^i \right\rangle \nu_0^i - \sum_{i=1}^{3} \frac{1}{|\gamma_x^i|^3} \left\langle \gamma_{xxx}^i, \nu^i \right\rangle \nu^i - h^i(\gamma_{xx}^i, \gamma_x^i) =: -b^4 \,.$$

Thus the linearised system around the initial data associated to 4.1 is

$$\begin{cases} \gamma_t^i + \frac{2}{|\varphi_x^i|^4}\gamma_{xxxx}^i & = f^i & \text{motion,} \\ \gamma^1(t,0) - \gamma^2(t,0) & = 0 & \text{concurrency,} \\ \gamma^1(t,0) - \gamma^3(t,0) & = 0 & \text{concurrency,} \\ -\sum_{i=1}^3 \left(\frac{\gamma_x^i}{|\varphi_x^i|} - \frac{\varphi_x^i \langle \gamma_x^i, \varphi_x^i \rangle}{|\varphi_x^i|^3} \right) & = b^1(t,0) & \text{angle condition,} \\ \sum_{i=1}^3 \frac{1}{|\varphi_x^i(0)|^2} \left\langle \gamma_{xx}^i(t,0), \nu_0^i(0) \right\rangle & = b^2(t,0) & \text{curvature condition,} \\ \left\langle \gamma_{xx}^i(t,0), \tau_0^i(0) \right\rangle & = b^{3,i}(t,0) & \text{tangential second order,} \\ -\sum_{i=1}^3 \frac{1}{|\varphi_x^i(0)|^3} \left\langle \gamma_{xxx}^i(t,0), \nu_0^i(0) \right\rangle \nu_0^i(0) & = b^4(t,0) & \text{third order,} \\ \gamma^i\,(t,1) & = P^i & \text{fixed endpoints,} \\ \gamma_{xx}^i(t,1) & = 0 & \text{second order,} \\ \gamma^i(0) & = \psi^i & \text{initial condition} \end{cases}$$

for $i \in \{1,2,3\}$ where have omitted the dependence on $(t,x) \in [0,T] \times [0,1]$ in the motion equation and on $x \in [0,1]$ in the initial condition.

4.1.2 Lopatinskii–Shapiro condition

Definition 4.7. Let $\lambda \in \mathbb{C}$ with $\Re(\lambda) > 0$ be arbitrary. The Lopatinskii–Shapiro condition in $x = 0$ demands that every solution $(\gamma^i)_{i=1,2,3} \in C^4([0,\infty), (\mathbb{C}^2)^3)$ to the system of ODEs

$$\begin{cases} \lambda\gamma^i(x) + \frac{1}{|\varphi_x^i|^4}\gamma_{xxxx}^i(x) & = 0 & x \in [0,\infty), i \in \{1,2,3\} & \text{motion,} \\ \gamma^1(0) - \gamma^2(0) & = 0 & & \text{concurrency,} \\ \gamma^2(0) - \gamma^3(0) & = 0 & & \text{concurrency,} \\ \sum_{i=1}^3 \frac{1}{|\varphi_x^i|} \langle \gamma_x^i(0), \nu_0^i \rangle \nu_0^i & = 0 & & \text{angle condition,} \\ \sum_{i=1}^3 \frac{1}{|\varphi_x^i|^2} \left\langle \gamma_{xx}^i(0), \nu_0^i \right\rangle & = 0 & & \text{curvature condition,} \\ \left\langle \gamma_{xx}^i(0), \tau_0^i \right\rangle & = 0 & i \in \{1,2,3\} & \text{tangential second order,} \\ \sum_{i=1}^3 \frac{1}{|\varphi_x^i|^3} \left\langle \gamma_{xxx}^i(0), \nu_0^i \right\rangle \nu_0^i & = 0 & & \text{third order} \end{cases}$$

which satisfies $\lim_{x\to\infty}|\gamma^i(x)| = 0$ has to be the trivial solution. In addition, the Lopatinskii–Shapiro condition in $x = 1$ demands that every solution $(\gamma^i)_{i=1,2,3} \in C^4([0,\infty), (\mathbb{C}^2)^3)$ to the system of ODEs

$$\begin{cases} \lambda\gamma^i(x) + \frac{1}{|\varphi_x^i|^4}\gamma_{xxxx}^i(x) & = 0 & x \in [0,\infty), i \in \{1,2,3\} & \text{motion,} \\ \gamma^i(0) & = 0 & i \in \{1,2,3\} & \text{fixed endpoints,} \\ \gamma_{xx}^i(0) & = 0 & i \in \{1,2,3\} & \text{second order} \end{cases}$$

which satisfies $\lim_{x\to\infty}|\gamma^i(x)| = 0$ has to be the trivial solution. Notice that we have omitted the dependence of $\varphi^i_x, \tau^i_0, \nu^i_0$ on $x = 0$ in the first system and $x = 1$ in the second system, respectively.

Lemma 4.8. *The Lopatinskii–Shapiro conditions are satisfied.*

Proof. We show that the condition is satisfied at $x = 0$. Let $\lambda \in \mathbb{C}$ with $\Re(\lambda) > 0$ and let γ be a solution to the system which satisfies $\lim_{x\to\infty}|\gamma^i(x)| = 0$. Since φ is an admissible initial parametrisation, the angle condition $\sum_{i=1}^3 \tau^i_0 = 0$ implies that it holds $\sum_{i=1}^3 \nu^i_0 = 0$. Combining this with the third order condition we have

$$\frac{1}{|\varphi^1_x|^3}\left\langle \gamma^1_{xxx}(0), \nu^1_0\right\rangle = \frac{1}{|\varphi^2_x|^3}\left\langle \gamma^2_{xxx}(0), \nu^2_0\right\rangle = \frac{1}{|\varphi^3_x|^3}\left\langle \gamma^3_{xxx}(0), \nu^3_0\right\rangle . \tag{4.2}$$

Multiplying the motion equation by $\frac{1}{|\varphi^i_x|}\overline{\langle\gamma^i_{xx}(x), \nu^i_0\rangle}\nu^i_0$, summing and integrating we obtain

$$\begin{aligned}
0 &= \lambda\sum_{i=1}^3\frac{1}{|\varphi^i_x|}\int_0^\infty \langle\gamma^i, \nu^i_0\rangle\overline{\langle\gamma^i_{xx}, \nu^i_0\rangle}\mathrm{d}x + \sum_{i=1}^3\frac{1}{|\varphi^i_x|^5}\int_0^\infty \langle\gamma^i_{xxxx}, \nu^i_0\rangle\overline{\langle\gamma^i_{xx}, \nu^i_0\rangle}\mathrm{d}x \\
&= -\lambda\sum_{i=1}^3\frac{1}{|\varphi^i_x|}\int_0^\infty \left|\langle\gamma^i_x, \nu^i_0\rangle\right|^2\mathrm{d}x - \sum_{i=1}^3\frac{1}{|\varphi^i_x|^5}\int_0^\infty \left|\langle\gamma^i_{xxx}, \nu^i_0\rangle\right|^2\mathrm{d}x \\
&\quad + \lambda\sum_{i=1}^3\frac{1}{|\varphi^i_x|}\langle\gamma^i(0), \nu^i_0\rangle\overline{\langle\gamma^i_x(0), \nu^i_0\rangle} + \sum_{i=1}^3\frac{1}{|\varphi^i_x|^5}\langle\gamma^i_{xxx}(0), \nu^i_0\rangle\overline{\langle\gamma^i_{xx}(0), \nu^i_0\rangle} .
\end{aligned} \tag{4.3}$$

The concurrency and the angle condition imply that the first boundary term vanishes:

$$\left\langle \gamma^1(0), \sum_{i=1}^3\frac{1}{|\varphi^i_x|}\overline{\langle\gamma^i_x(0), \nu^i_0\rangle}\nu^i_0\right\rangle = 0 .$$

Using (4.2) and the curvature condition we see that the other boundary term is given by

$$\frac{1}{|\varphi^1_x|^3}\left\langle \gamma^1_{xxx}(0), \nu^1_0\right\rangle\sum_{i=1}^3\frac{1}{|\varphi^i_x|^2}\overline{\langle\gamma^i_{xx}(0), \nu^i_0\rangle} = 0 .$$

Considering the real part of (4.3) we conclude that $\langle\gamma^i_x(x), \nu^i_0\rangle = 0$ for all $x \geq 0$ which implies $\langle\gamma^i_{xxxx}(x), \nu^i_0\rangle = 0$ for all $x \geq 0$. Using the motion equation and the fact that $\Re(\lambda) > 0$ we obtain $\langle\gamma^i(x), \nu^i_0\rangle = 0$ for all $x \geq 0$ and in particular $\langle\gamma^1(0), \sum_{i=1}^3 a^i\nu^i_0\rangle = 0$ for all $a \in \mathbb{R}^3$. As the vectors ν^i_0 span all of $\mathbb{R}^2$ due to the angle condition at initial time, we conclude that $\gamma^i(0) = 0$ for all $i \in \{1, 2, 3\}$.
We now proceed similarly as in the C^0-case 3.14.
Testing the motion equation by $\frac{1}{|\varphi^i_x|}\overline{\langle\gamma^i_{xx}, \tau^i_0\rangle}\tau^i_0$, summing and integrating by part gives

$$\begin{aligned}
0 &= -\lambda\sum_{i=1}^3\frac{1}{|\varphi^i_x|}\int_0^\infty \left|\langle\gamma^i_x, \tau^i_0\rangle\right|^2\mathrm{d}x - \sum_{i=1}^3\frac{1}{|\varphi^i_x|^5}\int_0^\infty \left|\langle\gamma^i_{xxx}, \tau^i_0\rangle\right|^2\mathrm{d}x \\
&\quad + \lambda\sum_{i=1}^3\frac{1}{|\varphi^i_x|}\langle\gamma^i(0), \tau^i_0\rangle\overline{\langle\gamma^i_x(0), \tau^i_0\rangle} + \sum_{i=1}^3\frac{1}{|\varphi^i_x|^5}\langle\gamma^i_{xxx}(0), \tau^i_0\rangle\overline{\langle\gamma^i_{xx}(0), \tau^i_0\rangle} .
\end{aligned}$$

The two boundary terms vanish due to $\gamma^i(0) = 0$ and the tangential second order condition. Considering the real part we find $\langle \gamma_x^i(x), \tau_0^i \rangle = 0$ for all $x \geq 0$ and hence $\gamma_x^i(x) = 0$ for all $x \geq 0$. Differentiating and using the motion equation and $\Re(\lambda) > 0$ implies that γ is the trivial solution.
In the case $x = 1$ the solution to the ODE can be calculated explicitly and the result is trivial.

□

4.1.3 Contraction estimates

We now give some remarks on the contraction estimates that are needed to deduce well–posedness of (4.1) from existence of the linearised system. As the motion equation is unchanged and all boundary terms at $x = 1$ are linear, we only provide arguments for the non–linear boundary conditions at $x = 0$.
In all the estimates we use the identification of the function spaces presented in 2.1.2.

As the third order conditions in (4.1) and (3.5) only differ in their lower order terms, it remains to estimate the expression

$$\frac{\gamma_x^i |\gamma_{xx}^i|^2}{|\gamma_x^i|^5} - \frac{\tilde{\gamma}_x^i |\tilde{\gamma}_{xx}^i|^2}{|\tilde{\gamma}_x^i|^5} \tag{4.4}$$

in the norm of $C^{\frac{1+\alpha}{4}}\left([0,T];(\mathbb{R}^3)^2\right)$. By adding and subtracting and using

$$\frac{1}{a^5} - \frac{1}{b^5} = \left(\frac{1}{a} - \frac{1}{b}\right)\left(\frac{1}{a^4} + \frac{1}{a^3 b} + \frac{1}{a^2 b^2} + \frac{1}{a b^3} + \frac{1}{b^4}\right).$$

the expression (4.4) can be written in the following form:

$$\left(\gamma_x^i - \tilde{\gamma}_x^i\right) h(\gamma_x, \gamma_{xx}, \tilde{\gamma}_x, \tilde{\gamma}_{xx}) + \left(\gamma_{xx}^i - \gamma_{xx}^{\tilde{i}}\right) g(\gamma_x, \gamma_{xx}, \tilde{\gamma}_x, \tilde{\gamma}_{xx}).$$

Using Lemma 2.6 the whole term can be estimated by $\tilde{C}(M,C)T^{\varepsilon}\|\gamma - \tilde{\gamma}\|_{\mathbb{E}_T}$.
Moreover we need to estimate

$$\langle \gamma_{xx}^i, \tau_0^i - \tau^i \rangle - \langle \tilde{\gamma}_{xx}^i, \tau_0^i - \tilde{\tau}^i \rangle = \langle \gamma_{xx}^i - \tilde{\gamma}_{xx}^i, \tau_0^i - \tau^i \rangle + \langle \tilde{\gamma}_{xx}^i, \tilde{\tau}^i - \tau^i \rangle$$

and

$$\begin{aligned}
&\sum_{i=1}^{3} \frac{1}{|\varphi_x^i|^2} \left\langle \gamma_{xx}^i, \nu_0^i \right\rangle - \sum_{i=1}^{3} \frac{1}{|\gamma_x^i|^2} \left\langle \gamma_{xx}^i, \nu^i \right\rangle - \sum_{i=1}^{3} \frac{1}{|\varphi_x^i|^2} \left\langle \tilde{\gamma}_{xx}^i, \nu_0^i \right\rangle + \sum_{i=1}^{3} \frac{1}{|\tilde{\gamma}_x^i|^2} \left\langle \tilde{\gamma}_{xx}^i, \tilde{\nu}^i \right\rangle \\
&= \sum_{i=1}^{3} \left(\frac{1}{|\varphi_x^i|^2} - \frac{1}{|\gamma_x^i|^2}\right) \langle \gamma_{xx}^i - \tilde{\gamma}_{xx}^i, \nu_0^i \rangle + \sum_{i=1}^{3} \frac{1}{|\gamma_x^i|^2} \langle \gamma_{xx}^i - \tilde{\gamma}_{xx}^i, \nu_0^i - \nu^i \rangle \\
&+ \sum_{i=1}^{3} \left(\frac{1}{|\tilde{\gamma}_x^i|^2} - \frac{1}{|\gamma_x^i|^2}\right) \langle \tilde{\gamma}_{xx}^i, \nu^i \rangle + \sum_{i=1}^{3} \frac{1}{|\tilde{\gamma}_x^i|^2} \langle \tilde{\gamma}_{xx}^i, \tilde{\nu}^i - \nu^i \rangle
\end{aligned}$$

in the norm of $C^{\frac{2+\alpha}{4}}\left([0,T];(\mathbb{R}^3)^2\right)$ which can be done as before using 2.6.

It remains to study the angle condition. We let $f : \mathbb{R}^2 \to \mathbb{R}^2$ be a smooth function such that $f(p) = \frac{p}{|p|}$ on $\mathbb{R}^2 \setminus B_{C/4}(0)$. As $\gamma_x^i(t), \tilde{\gamma}_x^i(t)$ and φ_x^i all lie in $\mathbb{R}^2 \setminus B_{C/2}(0)$ for $t \in (0, \tilde{t}(M, C)]$, we can write the expression $b^1(\gamma) - b^1(\tilde{\gamma})$ as

$$\begin{aligned}
&\sum_{i=1}^{3} f(\gamma_x^i) - f(\tilde{\gamma}_x^i) - Df(\varphi_x^i)\gamma_x^i + Df(\varphi_x^i)\tilde{\gamma}_x^i \\
=&\sum_{i=1}^{3} \left(\int_0^1 Df(\xi\gamma_x^i + (1-\xi)\tilde{\gamma}_x^i)\,\mathrm{d}\xi - Df(\varphi_x^i) \right)(\gamma_x^i - \tilde{\gamma}_x^i)\,.
\end{aligned}$$

The function

$$t \mapsto g^i(t) := \int_0^1 Df(\xi\gamma_x^i(t) + (1-\xi)\tilde{\gamma}_x^i(t))\,\mathrm{d}\xi$$

is in $C^{\frac{3+\alpha}{4}}\left([0,T]; \left(\mathbb{R}^{2\times 2}\right)^2\right)$ and $g(0) = Df(\varphi_x)$ which implies

$$\sup_{t\in[0,T]} \|g^i(t) - g^i(0)\| \le T^{\frac{3+\alpha}{4}} \|g^i\|_{C^{\frac{3+\alpha}{4}}([0,T])}$$

for any matrix norm $\|\cdot\|$. As both Df and $D^2 f$ are bounded on $\overline{B_M(0)}$, we have for all $T \in (0, \tilde{t}(M, C)]$ that $\|g^i\|_{C^{\frac{3+\alpha}{4}}([0,T])} \le C(M, C)$. Thus we conclude that

$$\begin{aligned}
&\|\left(g^i - g^i(0)\right)\left(\gamma_x^i - \tilde{\gamma}_x^i\right)\|_{C^{\frac{3+\alpha}{4}}([0,T])} \\
&\le \sup_{t\in[0,T]} \|g^i(t) - g^i(0)\| \|\gamma_x^i - \tilde{\gamma}_x^i\|_{C^{\frac{3+\alpha}{4}}([0,T])} + \|g^i - g^i(0)\|_{C^{\frac{3+\alpha}{4}}([0,T])} \sup_{t\in[0,T]} |\gamma_x^i(t) - \tilde{\gamma}_x^i(t)| \\
&\le C(M, C) T^{\varepsilon} \|\gamma - \tilde{\gamma}\|_{\mathbb{E}_T}
\end{aligned}$$

where we used

$$\sup_{t\in[0,T]} |\gamma_x^i(t) - \tilde{\gamma}_x^i(t)| \le \|\gamma_x^i - \tilde{\gamma}_x^i\|_{C^{\frac{\alpha}{4},\alpha}([0,T]\times[0,1])} \le T^{\varepsilon} \|\gamma - \tilde{\gamma}\|_{\mathbb{E}_T}$$

which follows from Lemma 3.26.
As all other arguments are similar to the C^0–case we hence established Theorem 4.3.

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