AN EXTENSION OF THE PAIRING THEORY BETWEEN DIVERGENCE-MEASURE FIELDS AND BV FUNCTIONS

GRAZIANO CRASTA AND VIRGINIA DE CICCO

Abstract. In this paper we introduce a nonlinear version of the notion of Anzellotti's pairing between divergence–measure vector fields and functions of bounded variation, motivated by possible applications to evolutionary quasilinear problems. As a consequence of our analysis, we prove a generalized Gauss–Green formula.

1. Introduction

In recent years the pairing theory between divergence-measure fields and BV functions, initially developed by Anzellotti [5, 6], has been extended to more general situations (see e.g. [10–15, 18, 29–31] and the references therein). These extensions are motivated, among others, by applications to hyperbolic conservation laws and transport equations [1, 2, 10, 12–14, 19], problems involving the 1-Laplace operator [4, 27], the prescribed mean curvature problem [28, 29] and to lower semicontinuity problems in BV (see [7, 23, 24]).

Another major related result concerns the Gauss–Green formula and its applications (see e.g. [5, 9, 15, 18, 29]).

Let us describe the problem in more details. Let $\mathcal{DM}^\infty$ denote the class of bounded divergence measure vector fields $\boldsymbol{A}\colon \mathbb{R}^N \to \mathbb{R}^N$, i.e. the vector fields with the properties that $\boldsymbol{A}$ is bounded and $\operatorname{div}\boldsymbol{A}$ is a finite Radon measure. If $\boldsymbol{A} \in \mathcal{DM}^\infty$ and u is a function of bounded variation with precise representative u^*, then Chen and Frid [10] proved that

$$\operatorname{div}(u\boldsymbol{A}) = u^* \operatorname{div}\boldsymbol{A} + \mu,$$

where μ is a Radon measure, absolutely continuous with respect to $|Du|$. This measure μ has been denoted by Anzellotti [5] with the symbol $(\boldsymbol{A}, Du)$ and by Chen and Frid with $\overline{\boldsymbol{A}\cdot Du}$, and it is called the pairing between the divergence–measure field $\boldsymbol{A}$ and the gradient of the BV function u. The characterization of the decomposition of this measure into absolutely continuous, Cantor and jump parts has been studied in [10, 18]. In particular, the analysis of the jump part has been considered in [18] using the notion of weak normal traces of a divergence–measure vector field on oriented countably $\mathcal{H}^{N-1}$-rectifiable sets given by [1].

In view to applications to evolutionary quasilinear problems, our aim is to extend the pairing theory from the product $u\boldsymbol{A}$ to the mixed case $\boldsymbol{B}(x, u)$. Our main assumptions on $\boldsymbol{B}$ are that $\boldsymbol{B}(\cdot, w) \in \mathcal{DM}^\infty$ for every $w \in \mathbb{R}$, $\boldsymbol{B}(x, \cdot)$ is of class C^1 and the least upper bound

$$\sigma := \bigvee_{w\in\mathbb{R}} |\operatorname{div}_x \partial_w \boldsymbol{B}(\cdot, w)|$$

Date: November 29, 2017.
2010 *Mathematics Subject Classification.* 26B30,35L65.
Key words and phrases. Anzellotti's pairing, divergence measure fields, Gauss–Green formula.

is a Radon measure. (See Section 3.1 for the complete list of assumptions.) We remark that, in the case $\boldsymbol{B}(x,w) = w\,\boldsymbol{A}(x)$, with $\boldsymbol{A} \in \mathcal{DM}^\infty$, these assumptions are automatically satisfied with $\sigma = |\operatorname{div}\boldsymbol{A}|$. More precisely, we will prove that, if $u \in BV \cap L^\infty$, then the composite function $\mathbf{v}(x) := \boldsymbol{B}(x, u(x))$ belongs to $\mathcal{DM}^\infty$ and

$$\operatorname{div}[\boldsymbol{B}(x,u(x))] = \frac{1}{2}\left[F(x,u^+(x)) + F(x,u^-(x))\right]\,\sigma + \mu,$$

in the sense of measures. Here $F(\cdot,w)$ denotes the Radon–Nikodým derivative of the measure $\operatorname{div}_x \boldsymbol{B}(\cdot,w)$ with respect to σ, $u^\pm$ are the approximate limits of u and $\mu \equiv (\partial_w \boldsymbol{B}(\cdot,u), Du)$ is again a Radon measure, absolutely continuous with respect to $|Du|$ (see Theorem 4.3). We remark, that, when $\boldsymbol{B}(x,w) = w\,\boldsymbol{A}(x)$, then $\mu = (\boldsymbol{A}, Du)$ is exactly the Anzellotti's pairing between $\boldsymbol{A}$ and Du. Even for general vector fields $\boldsymbol{B}(x,w)$, we can prove the following characterization of the decomposition $\mu = \mu^{ac} + \mu^c + \mu^j$ of this measure into absolutely continuous, Cantor and jump parts (see Theorem 4.6):

$$\begin{aligned}
\mu^{ac} &= \langle \partial_w \boldsymbol{B}(x,u(x))\,,\,\nabla u(x)\rangle\,\mathcal{L}^N,\\
\mu^c &= \left\langle \widetilde{\partial_w \boldsymbol{B}}(x,\widetilde{u}(x))\,,\,D^c u\right\rangle\,,\\
\mu^j &= [\beta^*(x,u^+(x)) - \beta^*(x,u^-(x))]\mathcal{H}^{N-1} \llcorner J_u,
\end{aligned}$$

where, for every $w \in \mathbb{R}$, $\beta^\pm(\cdot,w)$ are the normal traces of $\boldsymbol{B}(\cdot,w)$ on J_u and $\beta^*(\cdot,w) = [\beta^+(\cdot,w) + \beta^-(\cdot,w)]/2$, and μ^c is the Cantor part of the measure. We remark that, to prove the representation formula for μ^c, we need an additional technical assumption (see (32) in Theorem 4.6).

We recall that a similar characterization of $\operatorname{div}[\boldsymbol{B}(x,u(x))]$ has been proved in [19] under stronger assumptions, including the existence of the strong traces of $\boldsymbol{B}(\cdot,w)$.

As a consequence of our analysis we prove that, if $E \subset \mathbb{R}^N$ is a bounded set with finite perimeter, then the following Gauss–Green formula holds:

$$\int_{E^1} \frac{F(x,u^+(x)) + F(x,u^-(x))}{2}\,d\sigma(x) + \mu(E^1) = -\int_{\partial^* E} \beta^+(x,u^+(x))\,d\mathcal{H}^{N-1}\,,$$

where E^1 is the measure theoretic interior of E, and $\partial^* E$ is the reduced boundary of E (see Theorem 6.1). In the particular case $\boldsymbol{B}(x,w) = w\,\boldsymbol{A}(x)$, the analogous formula has been proved in [18, Theorem 5.1]. We remark that the map $x \mapsto \beta^+(x,u^+(x))$ in the last integral coincides with the interior weak normal trace of the vector field $x \mapsto \boldsymbol{B}(x,u(x))$ on $\partial^* E$ (see Proposition 4.5).

The plan of the paper is the following. In Section 2 we recall some known results on functions of bounded variation, divergence-measure vector fields and their normal traces, and the Anzellotti's pairing.

In Section 3 we list the assumptions on $\boldsymbol{B}$ and we prove a number of its basic properties, including some regularity result of the weak normal traces. Finally, we prove that for every $u \in BV \cap L^\infty$, the composite function $x \mapsto \boldsymbol{B}(x,u(x))$ belongs to $\mathcal{DM}^\infty$.

In Section 4 we prove our main results on the pairing measure μ and its properties.

Finally, in Section 5 we describe some gluing construction and we prove an extension theorem that will be used in Section 6 to prove a Gauss–Green formula for weakly regular domains.

2. Preliminaries

In the following Ω will always denote a nonempty open subset of $\mathbb{R}^N$.

Let $u \in L^1_{\rm loc}(\Omega)$. We say that u has an approximate limit at $x_0 \in \Omega$ if exists $z \in \mathbb{R}$ such that

$$\lim_{r\to 0^+} \frac{1}{\mathcal{L}^N\left(B_r(x_0)\right)} \int_{B_r(x_0)} |u(x) - z| \, dx = 0. \tag{1}$$

The set $S_u \subset \Omega$ of points where this property does not hold is called the approximate discontinuity set of u. For every $x_0 \in \Omega \setminus S_u$ the number z, uniquely determined by (1), is called the approximate limit of u at x_0 and denoted by $\widetilde{u}(x_0)$.

We say that $x_0 \in \Omega$ is an approximate jump point of u if there exist $a, b \in \mathbb{R}$ and a unit vector $\nu \in \mathbb{R}^n$ such that $a \neq b$ and

$$\begin{aligned} &\lim_{r\to 0^+} \frac{1}{\mathcal{L}^N(B_r^+(x_0))} \int_{B_r^+(x_0)} |u(y) - a| \, dy = 0, \\ &\lim_{r\to 0^+} \frac{1}{\mathcal{L}^N(B_r^-(x_0))} \int_{B_r^-(x_0)} |u(y) - b| \, dy = 0, \end{aligned} \tag{2}$$

where $B_r^\pm(x_0) := \{y \in B_r(x_0) : \pm(y - x_0) \cdot \nu > 0\}$. The triplet (a, b, ν), uniquely determined by (2) up to a permutation of (a, b) and a change of sign of ν, is denoted by $(u^+(x_0), u^-(x_0), \nu_u(x_0))$. The set of approximate jump points of u will be denoted by J_u.

The notions of approximate discontinuity set, approximate limit and approximate jump point can be obviously extended to the vectorial case (see [3, §3.6]).

In the following we shall always extend the functions $u^\pm$ to $\Omega \setminus (S_u \setminus J_u)$ by setting

$$u^\pm \equiv \widetilde{u} \text{ in } \Omega \setminus S_u.$$

In some occasions it will be useful to choose the orientation of ν in such a way that $u^- < u^+$ in J_u. These particular choices of u^- and u^+ will be called the approximate lower limit and the approximate upper limit of u respectively.

Definition 2.1 (Strong traces)**.** *Let $u \in L^\infty_{\rm loc}(\mathbb{R}^N)$ and let $\mathcal{J} \subset \mathbb{R}^N$ be a countably $\mathcal{H}^{N-1}$-rectifiable set oriented by a normal vector field ν. We say that two Borel functions $u^\pm \colon \mathcal{J} \to \mathbb{R}$ are the strong traces of u on $\mathcal{J}$ if for $\mathcal{H}^{N-1}$-almost every $x \in \mathcal{J}$ it holds*

$$\lim_{r\to 0^+} \int_{B_r^\pm(x)} |u(y) - u^\pm(x)| \, dy = 0,$$

where $B_r^\pm(x) := B_r(x) \cap \{y \in \mathbb{R}^N : \pm\langle y - x, \nu(x)\rangle \geq 0\}$.

Here and in the following we will denote by $\rho \in C_c^\infty(\mathbb{R}^N)$ a symmetric convolution kernel with support in the unit ball, and by $\rho_\varepsilon(x) := \varepsilon^{-N}\rho(x/\varepsilon)$.

In the sequel we will use often the following result.

Proposition 2.2. *Let $u \in L^1_{\rm loc}(\Omega)$ and define*

$$u_\varepsilon\left(x\right) = \rho_\varepsilon * u(x) := \int_\Omega \rho_\varepsilon\left(x - y\right)\, u\left(y\right)\, dy.$$

If $x_0 \in \Omega \setminus S_u$, then $u_\varepsilon(x_0) \to \widetilde{u}(x_0)$ as $\varepsilon \to 0^+$.

Proposition 2.3. *Let E be a Lebesgue measurable subset of $\mathbb{R}^N$ and G a Borel subset of $\mathbb{R}^M$. Let $g \colon E \times G \to \mathbb{R}$ be a Borel function such that for $\mathcal{L}^N$–a.e. $x \in E$ the function $g\left(x, \cdot\right)$ is continuous on G. Then there exists an $\mathcal{L}^N$-null set $\mathcal{M} \subset \mathbb{R}^N$ such that for every $t \in G$ the function $g(\cdot, t)$ is approximately continuous in $E \setminus \mathcal{M}$.*

Proof. (See the proof of Theorem 1.3, p. 539 in [25].) By the Scorza–Dragoni theorem (see [26, Theorem 6.35]), for every $i \in \mathbb{Z}^+$ there exists a closed set $K_i \subset E$, with $\mathcal{L}^N(E \setminus K_i) < 1/i$, such that the restriction of g to $K_i \times G$ is continuous. Let K_i^* be the set of Lebesgue points of K_i, and define

$$\mathcal{E} := \bigcup_{i=1}^{\infty} (K_i \cap K_i^*).$$

Since $\mathcal{L}^N(K_i) = \mathcal{L}^N(K_i \cap K_i^*)$, it holds

$$\mathcal{L}^N(E \setminus \mathcal{E}) \leq \mathcal{L}^N(E \setminus K_i) \leq \frac{1}{i} \qquad \forall i \in \mathbb{Z}^+,$$

hence $\mathcal{L}^N(E \setminus \mathcal{E}) = 0$.

Let us fix $t \in G$ and let us prove that the function $u(x) := g(x,t)$ is approximately continuous on $\mathcal{E}$. Let $\varepsilon > 0$ and $x_0 \in \mathcal{E}$. By definition of $\mathcal{E}$, there exists an index $i \in \mathbb{Z}^+$ such that $x_0 \in K_i \cap K_i^*$. Since the restriction of u to K_i is continuous, there exists $\delta > 0$ such that

$$|u(x) - u(x_0)| < \varepsilon \qquad \forall x \in B_\delta(x_0) \cap K_i.$$

As a consequence, for every $r \in (0, \delta)$, it holds

$$\mathcal{L}^N(B_r(x_0) \cap \{x \in E : \ |u(x) - u(x_0)| \geq \varepsilon\}) \leq \mathcal{L}^N(B_r(x_0) \setminus K_i),$$

hence the conclusion follows since x_0 is a Lebesgue point of K_i. □

Corollary 2.4. *Let E be a Lebesgue measurable subset of $\mathbb{R}^N$ and G a Borel subset of $\mathbb{R}^M$. Let $g \colon E \times G \to \mathbb{R}$ be a Borel function such that for $\mathcal{L}^N$–a.e. $x \in E$ the function $g(x, \cdot)$ is continuous on G. Let us define*

$$g_\varepsilon(x,t) := \int_{\mathbb{R}^N} \rho_\varepsilon(x-y)\, g(y,t)\, dy.$$

Then there exists an $\mathcal{L}^N$-null set $Z \subset \mathbb{R}^N$ such that for every $t \in G$ and for every $x \in E \setminus Z$ we have $g_\varepsilon(x_0,t) \to g(x_0,t)$, as $\varepsilon \to 0^+$.

2.1. **Functions of bounded variation and sets of finite perimeter.** We say that $u \in L^1(\Omega)$ is a function of bounded variation in Ω if the distributional derivative Du of u is a finite Radon measure in Ω. The vector space of all functions of bounded variation in Ω will be denoted by $BV(\Omega)$. Moreover, we will denote by $BV_{\rm loc}(\Omega)$ the set of functions $u \in L^1_{\rm loc}(\Omega)$ that belongs to $BV(A)$ for every open set $A \Subset \Omega$ (i.e., the closure $\overline{A}$ of A is a compact subset of Ω).

If $u \in BV(\Omega)$, then Du can be decomposed as the sum of the absolutely continuous and the singular part with respect to the Lebesgue measure, i.e.

$$Du = D^a u + D^s u, \qquad D^a u = \nabla u \, \mathcal{L}^N,$$

where ∇u is the approximate gradient of u, defined $\mathcal{L}^N$-a.e. in Ω. On the other hand, the singular part $D^s u$ can be further decomposed as the sum of its Cantor and jump part, i.e.

$$D^s u = D^c u + D^j u, \qquad D^c u := D^s u \llcorner (\Omega \setminus S_u), \quad D^j u := D^s u \llcorner J_u,$$

where the symbol $\mu \llcorner B$ denotes the restriction of the measure μ to the set B. We will denote by $D^d u := D^a u + D^c u$ the diffuse part of the measure Du.

In the following, we will denote by $\theta_u \colon \Omega \to S^{N-1}$ the Radon–Nikodým derivative of Du with respect to $|Du|$, i.e. the unique function $\theta_u \in L^1(\Omega, |Du|)^N$ such that the polar decomposition $Du = \theta_u \, |Du|$ holds. Since all parts of the derivative of u are mutually singular, we have

$$D^a u = \theta_u \, |D^a u|, \quad D^j u = \theta_u \, |D^j u|, \quad D^c u = \theta_u \, |D^c u|$$

as well. In particular $\theta_u(x) = \nabla u(x)/|\nabla u(x)|$ for $\mathcal{L}^N$-a.e. $x \in \Omega$ such that $\nabla u(x) \neq 0$ and $\theta_u(x) = \operatorname{sign}(u^+(x) - u^-(x)) \, \nu_u(x)$ for $\mathcal{H}^{N-1}$-a.e. $x \in J_u$.

Let E be an $\mathcal{L}^N$-measurable subset of $\mathbb{R}^N$. For every open set $\Omega \subset \mathbb{R}^N$ the perimeter $P(E, \Omega)$ is defined by

$$P(E, \Omega) := \sup \left\{ \int_E \operatorname{div} \varphi \, dx : \ \varphi \in C^1_c(\Omega, \mathbb{R}^N), \ \|\varphi\|_\infty \leq 1 \right\}.$$

We say that E is of finite perimeter in Ω if $P(E, \Omega) < +\infty$.

Denoting by χ_E the characteristic function of E, if E is a set of finite perimeter in Ω, then $D\chi_E$ is a finite Radon measure in Ω and $P(E, \Omega) = |D\chi_E|(\Omega)$.

If $\Omega \subset \mathbb{R}^N$ is the largest open set such that E is locally of finite perimeter in Ω, we call reduced boundary $\partial^* E$ of E the set of all points $x \in \Omega$ in the support of $|D\chi_E|$ such that the limit

$$\widetilde{\nu}_E(x) := \lim_{\rho \to 0^+} \frac{D\chi_E(B_\rho(x))}{|D\chi_E|(B_\rho(x))}$$

exists in $\mathbb{R}^N$ and satisfies $|\widetilde{\nu}_E(x)| = 1$. The function $\widetilde{\nu} \colon \partial^* E \to S^{N-1}$ is called the measure theoretic unit interior normal to E.

A fundamental result of De Giorgi (see [3, Theorem 3.59]) states that $\partial^* E$ is countably $(N-1)$-rectifiable and $|D\chi_E| = \mathcal{H}^{N-1} \llcorner \partial^* E$.

Let E be an $\mathcal{L}^N$-measurable subset of $\mathbb{R}^N$. For every $t \in [0, 1]$ we denote by E^t the set

$$E^t := \left\{ x \in \mathbb{R}^N : \ \lim_{\rho \to 0^+} \frac{\mathcal{L}^N(E \cap B_\rho(x))}{\mathcal{L}^N(B_\rho(x))} = t \right\}$$

of all points where E has density t. The sets E^0, E^1, $\partial^e E := \mathbb{R}^N \setminus (E^0 \cup E^1)$ are called respectively the measure theoretic exterior, the measure theoretic interior and the essential boundary of E. If E has finite perimeter in Ω, Federer's structure theorem states that $\partial^* E \cap \Omega \subset E^{1/2} \subset \partial^e E$ and $\mathcal{H}^{N-1}(\Omega \setminus (E^0 \cup \partial^e E \cup E^1)) = 0$ (see [3, Theorem 3.61]).

2.2. **Divergence–measure fields.** We will denote by $\mathcal{DM}^\infty(\Omega)$ the space of all vector fields $\boldsymbol{A} \in L^\infty(\Omega, \mathbb{R}^N)$ whose divergence in the sense of distribution is a bounded Radon measure in Ω. Similarly, $\mathcal{DM}^\infty_{\text{loc}}(\Omega)$ will denote the space of all vector fields $\boldsymbol{A} \in L^\infty_{\text{loc}}(\Omega, \mathbb{R}^N)$ whose divergence in the sense of distribution is a Radon measure in Ω. We set $\mathcal{DM}^\infty = \mathcal{DM}^\infty(\mathbb{R}^N)$.

We recall that, if $\boldsymbol{A} \in \mathcal{DM}^\infty_{\text{loc}}(\Omega)$, then $|\operatorname{div} \boldsymbol{A}| \ll \mathcal{H}^{N-1}$ (see [10, Proposition 3.1]). As a consequence, the set

$$\Theta_{\boldsymbol{A}} := \left\{ x \in \Omega : \ \limsup_{r \to 0+} \frac{|\operatorname{div} \boldsymbol{A}|(B_r(x))}{r^{N-1}} > 0 \right\}, \tag{3}$$

is a Borel set, σ-finite with respect to $\mathcal{H}^{N-1}$, and the measure $\operatorname{div} \boldsymbol{A}$ can be decomposed as

$$\operatorname{div} \boldsymbol{A} = \operatorname{div}^a \boldsymbol{A} + \operatorname{div}^c \boldsymbol{A} + \operatorname{div}^j \boldsymbol{A},$$

where $\operatorname{div}^a \boldsymbol{A}$ is absolutely continuous with respect to $\mathcal{L}^N$, $\operatorname{div}^c \boldsymbol{A}(B) = 0$ for every set B with $\mathcal{H}^{N-1}(B) < +\infty$, and

$$\operatorname{div}^j \boldsymbol{A} = h \, \mathcal{H}^{N-1} \llcorner \Theta_{\boldsymbol{A}}$$

for some Borel function h (see [2, Proposition 2.5]).

2.3. **Normal traces.** The traces of the normal component of the vector field $\boldsymbol{A} \in \mathcal{DM}^\infty_{\mathrm{loc}}(\Omega)$ can be defined as distributions $\mathrm{Tr}^\pm(\boldsymbol{A}, \Sigma)$ on every oriented countably $\mathcal{H}^{N-1}$–rectifiable set $\Sigma \subset \Omega$ in the sense of Anzellotti (see [1, 5, 10]).

More precisely, let us briefly recall the construction given in [1] (see Propositions 3.2, 3.4 and Definition 3.3). First of all, given a domain $\Omega' \Subset \Omega$ of class C^1, we define the trace of the normal component of $\boldsymbol{A}$ on $\partial\Omega'$ as a distribution as follows:

$$\left\langle \mathrm{Tr}(\boldsymbol{A}, \partial\Omega'), \varphi \right\rangle := \int_{\Omega'} \boldsymbol{A} \cdot \nabla\varphi \, dx + \int_{\Omega'} \varphi \, d \operatorname{div} \boldsymbol{A}, \qquad \forall \varphi \in C^\infty_c(\Omega). \tag{4}$$

It turns out that this distribution is induced by an L^∞ function on $\partial\Omega'$, still denoted by $\mathrm{Tr}(\boldsymbol{A}, \partial\Omega')$, and

$$\| \mathrm{Tr}(\boldsymbol{A}, \partial\Omega')\|_{L^\infty(\partial\Omega')} \leq \|\boldsymbol{A}\|_{L^\infty(\Omega')}.$$

Since Σ is oriented and countably $\mathcal{H}^{N-1}$–rectifiable, we can find countably many *oriented* C^1 hypersurfaces Σ_i, with classical normal ν_{Σ_i}, and pairwise disjoint Borel sets $N_i \subseteq \Sigma_i$ such that $\mathcal{H}^{N-1}(\Sigma \setminus \bigcup_i N_i) = 0$.

Moreover, it is not restrictive to assume that, for every i, there exist two open bounded sets Ω_i, Ω_i' with C^1 boundary and exterior normal vectors ν_{Ω_i} and $\nu_{\Omega_i'}$ respectively, such that $N_i \subseteq \partial\Omega_i \cap \partial\Omega_i'$ and

$$\nu_{\Sigma_i}(x) = \nu_{\Omega_i}(x) = -\nu_{\Omega_i'}(x) \qquad \forall x \in N_i.$$

At this point we choose, on Σ, the orientation given by $\nu_\Sigma(x) := \nu_{\Sigma_i}(x)$ $\mathcal{H}^{N-1}$-a.e. on N_i.

Using the localization property proved in [1, Proposition 3.2], we can define the normal traces of $\boldsymbol{A}$ on Σ by

$$\mathrm{Tr}^-(\boldsymbol{A}, \Sigma) := \mathrm{Tr}(\boldsymbol{A}, \partial\Omega_i), \quad \mathrm{Tr}^+(\boldsymbol{A}, \Sigma) := -\mathrm{Tr}(\boldsymbol{A}, \partial\Omega_i'), \qquad \mathcal{H}^{N-1} - \text{a.e. on } N_i.$$

These two normal traces belong to $L^\infty(\Sigma, \mathcal{H}^{N-1} \llcorner \Sigma)$ (see [1, Proposition 3.2]) and

$$\operatorname{div} \boldsymbol{A} \llcorner \Sigma = \left[\mathrm{Tr}^+(\boldsymbol{A}, \Sigma) - \mathrm{Tr}^-(\boldsymbol{A}, \Sigma)\right] \mathcal{H}^{N-1} \llcorner \Sigma \,.$$

2.4. **Anzellotti's pairing.** As in Anzellotti [5] (see also [10]), for every $\boldsymbol{A} \in \mathcal{DM}^\infty_{\mathrm{loc}}(\Omega)$ and $u \in BV_{\mathrm{loc}}(\Omega) \cap L^\infty_{\mathrm{loc}}(\Omega)$ we define the linear functional $(\boldsymbol{A}, Du) \colon C^\infty_0(\Omega) \to \mathbb{R}$ by

$$\langle (\boldsymbol{A}, Du), \varphi \rangle := -\int_\Omega u^* \varphi \, d \operatorname{div} \boldsymbol{A} - \int_\Omega u \, \boldsymbol{A} \cdot \nabla\varphi \, dx. \tag{5}$$

The distribution $(\boldsymbol{A}, Du)$ is a Radon measure in Ω, absolutely continuous with respect to $|Du|$ (see [5, Theorem 1.5] and [10, Theorem 3.2]), hence the equation

$$\operatorname{div}(u\boldsymbol{A}) = u^* \operatorname{div} \boldsymbol{A} + (\boldsymbol{A}, Du) \tag{6}$$

holds in the sense of measures in Ω. (We remark that, in [10], the measure $(\boldsymbol{A}, Du)$ is denoted by $\overline{\boldsymbol{A} \cdot Du}$.) Furthermore, Chen and Frid in [10] proved that the absolutely continuous part of this measure with respect to the Lebesgue measure is given by $(\boldsymbol{A}, Du)^a = \boldsymbol{A} \cdot \nabla u \, \mathcal{L}^N$.

3. Assumptions on the vector field and preliminary results

As we have explained in the Introduction, we are willing to compute the divergence of the composite function $\mathbf{v}(x) := \boldsymbol{B}(x, u(x))$ with $u \in BV$, where $\boldsymbol{B}(\cdot, t) \in \mathcal{DM}^\infty$ and $\boldsymbol{B}(x, \cdot) \in C^1$. Nevertheless, it will be convenient to state our assumptions on the vector field $\boldsymbol{b}(x, t) := \partial_t \boldsymbol{B}(x, t)$.

In Section 3.1 we list the assumptions on $\boldsymbol{b}$ and we prove a number of basic properties of $\boldsymbol{b}$ and $\boldsymbol{B}$.

Then, in Section 3.2 we prove some regularity result of the weak normal traces of $\boldsymbol{B}$. Finally, in Section 3.3, we prove that $\mathbf{v} \in \mathcal{DM}^\infty$.

3.1. **Assumptions on the vector field $\boldsymbol{B}$.** In this section we list and comment all the assumptions on the vector field $\boldsymbol{B}(x,t)$.

Let $\Omega \subset \mathbb{R}^N$ be a non-empty open set. Let $\boldsymbol{b}\colon \Omega \times \mathbb{R} \to \mathbb{R}^N$ be a function satisfying the following assumptions:

(i) $\boldsymbol{b}$ is a locally bounded Borel function;
(ii) for $\mathcal{L}^N$–a.e. $x \in \Omega$, the function $\boldsymbol{b}(x,\cdot)$ is continuous in $\mathbb{R}$;
(iii) for every $t \in \mathbb{R}$, $\boldsymbol{b}(\cdot,t) \in \mathcal{DM}^\infty_{\mathrm{loc}}(\Omega)$;
(iv) the least upper bound

$$\sigma := \bigvee_{t\in\mathbb{R}} |\operatorname{div}_x \boldsymbol{b}(\cdot,t)|$$

is a Radon measure.

We remark that, since $\operatorname{div}_x \boldsymbol{b}(\cdot,t) \ll \mathcal{H}^{N-1}$ for every $t \in \mathbb{R}$, then also $\sigma \ll \mathcal{H}^{N-1}$.

From (i) and Proposition 2.3 it follows that there exists a set $Z_1 \subset \mathbb{R}^N$ such that $\mathcal{L}^N(Z_1) = 0$ and, for every $t \in \mathbb{R}$, the function $x \mapsto \boldsymbol{b}(x,t)$ is approximately continuous on $\mathbb{R}^N \setminus Z_1$.

By definition of least upper bound of measures, we have that $\operatorname{div}_x \boldsymbol{b}(\cdot,t) \ll \sigma$ for every $t \in \mathbb{R}$. If we denote by $f(\cdot,t)$ the Radon–Nikodým derivative of $\operatorname{div}_x \boldsymbol{b}(\cdot,t)$ with respect to σ, we thus have

$$\operatorname{div}_x \boldsymbol{b}(\cdot,t) = f(\cdot,t)\,\sigma, \quad f(\cdot,t) \in L^1(\sigma), \qquad \forall t \in \mathbb{R}.$$

Moreover, since $|\operatorname{div}_x \boldsymbol{b}(\cdot,t)| \leq \sigma$, we have that

$$\forall t \in \mathbb{R}: \quad |f(x,t)| \leq 1 \quad \text{for } \sigma\text{-a.e. } x \in \Omega. \tag{7}$$

Let us extend $\boldsymbol{b}$ to 0 in $(\mathbb{R}^N \setminus \Omega) \times \mathbb{R}$, so that the vector field

$$\boldsymbol{B}(x,t) := \int_0^t \boldsymbol{b}(x,s)\,ds, \qquad x \in \mathbb{R}^N,\ t \in \mathbb{R}, \tag{8}$$

is defined for all $(x,t) \in \mathbb{R}^N \times \mathbb{R}$. Moreover $\boldsymbol{B}(x,0) = 0$ for every $x \in \mathbb{R}^N$ and, from (ii), for every $x \in \mathbb{R}^n$ one has $\boldsymbol{b}(x,t) = \partial_t \boldsymbol{B}(x,t)$ for every $t \in \mathbb{R}$.

Lemma 3.1. *For every $t \in \mathbb{R}$ it holds $\boldsymbol{B}(\cdot,t) \in \mathcal{DM}^\infty_{loc}(\Omega)$ and $\operatorname{div}_x \boldsymbol{B}(\cdot,t) \ll \sigma$. If we denote by $F(\cdot,t)$ the Radon–Nikodým derivative of $\operatorname{div}_x \boldsymbol{B}(\cdot,t)$ with respect to σ, we have that*

$$F(x,t) = \int_0^t f(x,s)\,ds, \qquad \text{for } \sigma\text{–a.e. } x \in \Omega, \tag{9}$$

and, for every $t, s \in \mathbb{R}$,

$$|F(x,t) - F(x,s)| \leq |t-s| \qquad \text{for } \sigma\text{–a.e. } x \in \Omega. \tag{10}$$

Proof. Let $\varphi \in C_c^1(\Omega)$. We have that

$$\begin{aligned}\int_\Omega \nabla\varphi(x)\cdot \boldsymbol{B}(x,t)\,dx &= \int_\Omega \nabla\varphi(x)\cdot \int_0^t \boldsymbol{b}(x,s)\,ds\,dx \\ &= -\int_0^t \left(\int_\Omega \varphi(x)\,f(x,s)\,d\sigma(x)\right)\,ds \\ &= -\int_\Omega \varphi(x)\left(\int_0^t f(x,s)\,ds\right)\,d\sigma(x)\,.\end{aligned}$$

Hence $\operatorname{div}_x \boldsymbol{B}(\cdot,t)$ is a Radon measure, it is absolutely continuous with respect to σ and its Radon–Nikodým derivative with respect to σ is $F(\cdot,t)$.

For every nonnegative $\varphi \in C_c^1(\Omega)$ and for every $t,s\in\mathbb{R}$, taking into account that for every $s\in\mathbb{R}$, $|f(x,s)|\le 1$ for σ–a.e. $x\in\Omega$, an analogous integration gives

$$\begin{aligned}\left|\int_\Omega \nabla\varphi(x)\cdot[\boldsymbol{B}(x,t)-\boldsymbol{B}(x,s)]\,dx\right| &= \left|\int_s^t \left(\int_\Omega \varphi(x)\,f(x,w)\,d\sigma(x)\right)\,dw\right| \\ &\le |t-s|\int_\Omega \varphi(x)d\sigma(x)\,,\end{aligned}$$

so that

$$|\operatorname{div}_x \boldsymbol{B}(\cdot,t)(A) - \operatorname{div}_x \boldsymbol{B}(\cdot,s)(A)| \le |t-s|\,\sigma(A)$$

for every Borel set $A \Subset \Omega$, hence (10) follows. □

Lemma 3.2. *There exists a set $Z_2\subset\Omega$, with $\sigma(Z_2)=0$ and $\mathcal{H}^{N-1}(Z_2)=0$, such that every $x_0\in\Omega\setminus Z_2$ is a Lebesgue point of $F(\cdot,t)$ with respect to the measure σ for every $t\in\mathbb{R}$.*

Proof. Let $S\subset\mathbb{R}$ be a countable dense subset of $\mathbb{R}$ and, for every $s\in S$, let Ω_s be the set of the points $x_0\in\Omega$ such that the Radon-Nikodým derivative $F(\cdot,s)$ of $\operatorname{div}_x \boldsymbol{B}(\cdot,s)$ exists in x_0 and x_0 is a Lebesgue point of the function $F(\cdot,s)$ with respect to the measure σ. Define

$$Z_2 := \mathbb{R}^N \setminus \bigcap_{s\in S} \Omega_s.$$

Clearly $\sigma(Z_2)=0$ and $\mathcal{H}^{N-1}(Z_2)=0$. We claim that every $x_0\in\Omega\setminus Z_2$ is a Lebesgue point of the function $F(\cdot,w)$ for all $w\in\mathbb{R}$.

Fix $x_0\in\Omega\setminus Z_2$ and $w\in\mathbb{R}$. Since the set S is dense in $\mathbb{R}$ we may find a sequence $\{w_n\}$ of points in S such that $w_n\to w$. Then, by (10),

$$\begin{aligned}|F(x_0,w)-F(x,w)| &\le |F(x_0,w)-F(x_0,w_n)| + |F(x_0,w_n)-F(x,w_n)| \\ &\quad + |F(x,w_n)-F(x,w)| \\ &\le |F(x_0,w_n)-F(x,w_n)| + 2\,|w-w_n|.\end{aligned}$$

By averaging over $B(x_0,r)$ and letting $r\to 0^+$ in the previous inequality we get

$$\limsup_{r\to 0^+} \frac{1}{\sigma\left(B\left(x_0,r\right)\right)} \int_{B(x_0,r)} |F(x_0,w)-F(x,w)|\,d\sigma \le 2\,|w-w_n|,$$

where we have used the fact that x_0 is a Lebesgue point of $F(\cdot,w_n)$ w.r.t. the measure σ. Since $w_n\to w$ letting $n\to\infty$, the previous inequality proves that every $x_0\in\Omega\setminus Z_2$ is a Lebesgue point of the function $F(\cdot,w)$ for all $w\in\mathbb{R}$. □

3.2. **Weak normal traces of $\boldsymbol{B}$.** By Lemma 3.1 it follows that, for almost every $t \in \mathbb{R}$, the traces of the normal component of the vector field $\boldsymbol{B}(\cdot, t)$ can be defined as distributions $\mathrm{Tr}^{\pm}(\boldsymbol{B}(\cdot, t), \Sigma)$ on every countably $\mathcal{H}^{N-1}$–rectifiable set $\Sigma \subset \Omega$ in the sense of Anzellotti (see [1, 5, 10]).

More precisely, let us briefly recall the construction given in [1] (see Proposition 3.4 and Remark 3.3). Since Σ is countably $\mathcal{H}^{N-1}$–rectifiable, we can find countably many *oriented* C^1 hypersurfaces Σ_i, with classical normal ν_{Σ_i}, and pairwise disjoint Borel sets $N_i \subseteq \Sigma_i$ such that $\mathcal{H}^{N-1}(\Sigma \setminus \bigcup_i N_i) = 0$.

Moreover, it is not restrictive to assume that, for every i, there exist two open bounded sets Ω_i, Ω_i' with C^1 boundary and outer normal vectors ν_{Ω_i} and $\nu_{\Omega_i'}$ respectively, such that $N_i \subseteq \partial\Omega_i \cap \partial\Omega_i'$ and

$$\nu_{\Sigma_i}(x) = \nu_{\Omega_i}(x) = -\nu_{\Omega_i'}(x) \qquad \forall x \in N_i.$$

At this point we choose, on Σ, the orientation given by $\nu_\Sigma(x) := \nu_{\Sigma_i}(x)$ $\mathcal{H}^{N-1}$-a.e. on N_i.

Using the localization property proved in [1, Proposition 3.2], for every $t \in \mathbb{R}$ we can define the normal traces of $\boldsymbol{B}(\cdot, t)$ on Σ by

$$\beta^-(\cdot, t) := \mathrm{Tr}(\boldsymbol{B}(\cdot, t), \partial\Omega_i), \quad \beta^+(\cdot, t) := -\mathrm{Tr}(\boldsymbol{B}(\cdot, t), \partial\Omega_i'), \qquad \mathcal{H}^{N-1} - \text{a.e. on } N_i. \tag{11}$$

These two normal traces belong to $L^\infty(\Sigma)$ (see [1, Proposition 3.2]) and

$$\mathrm{div}_x\, \boldsymbol{B}(\cdot, t) \llcorner \Sigma = \left[\beta^+(\cdot, t) - \beta^-(\cdot, t)\right] \mathcal{H}^{N-1} \llcorner \Sigma\,. \tag{12}$$

Lemma 3.3. *The maps $t \mapsto \beta^\pm(\cdot, t)$ are Lipschitz continuous from $\mathbb{R}$ to $L^\infty(\Sigma)$. More precisely*

$$\|\beta^\pm(\cdot, t) - \beta^\pm(\cdot, s)\|_{L^\infty(\Sigma)} \le \|\boldsymbol{b}\|_\infty\, |t - s| \qquad \forall t, s \in \mathbb{R}. \tag{13}$$

Proof. It is enough to observe that, for every i, the map $\boldsymbol{X} \mapsto \mathrm{Tr}(\boldsymbol{X}, \partial\Omega_i)$ is linear in $\mathcal{DM}^\infty$ and, by Proposition 3.2 in [1], it holds

$$\|\mathrm{Tr}(\boldsymbol{X}, \partial\Omega_i)\|_{L^\infty(\partial\Omega_i)} \le \|\boldsymbol{X}\|_{L^\infty(\Omega_i)} \qquad \forall \boldsymbol{X} \in \mathcal{DM}^\infty, \tag{14}$$

and the same inequality holds when Ω_i is replaced by Ω_i'. Hence (13) follows from Assumption (i). ☐

In the Step 6 of the proof of Theorem 4.3 we will use these normal traces on the set $\Sigma := J_u$, where J_u is the jump set of a BV function u. More precisely, we will use a slightly stronger property, stated in the following proposition.

Proposition 3.4. *There exists a set $\Sigma' \subseteq \Sigma$, with $\mathcal{H}^{N-1}(\Sigma \setminus \Sigma') = 0$, and representatives $\overline{\beta}^\pm$ of $\beta^\pm$ (in the sense that, for every $t \in \mathbb{R}$, $\overline{\beta}^\pm(\cdot, t) = \beta^\pm(\cdot, t)$ $\mathcal{H}^{N-1}$–a.e. in Σ) such that*

$$|\overline{\beta}^\pm(x, t) - \overline{\beta}^\pm(x, s)| \le \|\boldsymbol{b}\|_\infty\, |t - s| \qquad \forall x \in \Sigma',\ t, s \in \mathbb{R}. \tag{15}$$

Proof. Let $Q \subset \mathbb{R}$ be a countable dense subset of $\mathbb{R}$. There exists a set $\Sigma' \subseteq \Sigma$, with $\mathcal{H}^{N-1}(\Sigma \setminus \Sigma') = 0$, such that

$$|\beta^\pm(x, p) - \beta^\pm(x, q)| \le M|p - q| \qquad \forall x \in \Sigma',\ p, q \in Q, \tag{16}$$

where $M := \|\boldsymbol{b}\|_\infty$. Let us define the functions $\overline{\beta}^\pm$ in the following way. If $q \in Q$, we define $\overline{\beta}^\pm(x, q) = \beta^\pm(x, q)$ for every $x \in \Sigma$. If $t \in \mathbb{R} \setminus Q$, let $(q_j) \subset Q$ be a sequence converging to t, and define

$$\overline{\beta}^\pm(x, t) := \begin{cases} \lim\limits_{j \to +\infty} \beta^\pm(x, q_j), & \text{if } x \in \Sigma', \\ \beta^\pm(x, t), & \text{if } x \in \Sigma \setminus \Sigma'. \end{cases}$$

(We remark that, for every $x \in \Sigma'$, by (16) $(\beta^{\pm}(x, q_j))_j$ is a Cauchy sequence, hence it is convergent, in $\mathbb{R}$. Moreover, its limit is independent of the choice of the sequence $(q_j) \subset Q$ converging to t.) From Lemma 3.3 we have that

$$|\beta^{\pm}(x, q_j) - \beta^{\pm}(x, t)| \leq M|q_j - t| \qquad \text{for } \mathcal{H}^{N-1}\text{–a.e. } x \in \Sigma.$$

Passing to the limit as $j \to +\infty$, it follows that $\overline{\beta}^{\pm}(\cdot, t) = \beta^{\pm}(\cdot, t)$ $\mathcal{H}^{N-1}$–a.e. on Σ.

Let $t, s \in \mathbb{R}$, and let $(t_j), (s_j) \subset Q$ be two sequences in Q converging respectively to t and s. From (16) we have that

$$|\beta^{\pm}(x, t_j) - \beta^{\pm}(x, s_j)| \leq M|t_j - s_j| \qquad \forall x \in \Sigma',\ j \in \mathbb{N},$$

hence (15) follows passing to the limit as $j \to +\infty$. □

Remark 3.5. In what follows we will always denote by $\beta^{\pm}$ the representatives $\overline{\beta}^{\pm}$ of Proposition 3.4.

3.3. **Basic estimates on the composite function.** In this section we shall use a regularization argument to prove that the composite function $\mathbf{v}(x) := \boldsymbol{B}(x, u(x))$ belongs to $\mathcal{DM}^{\infty}_{\text{loc}}(\Omega)$.

Lemma 3.6. *Let $\boldsymbol{b}\colon \Omega \times \mathbb{R} \to \mathbb{R}^N$ satisfy conditions (i), (ii), (and extended to 0 in $(\mathbb{R}^N \setminus \Omega) \times \mathbb{R}$), and let $\boldsymbol{B}\colon \mathbb{R}^N \times \mathbb{R} \to \mathbb{R}^N$ be defined by* (8)*. For every $\varepsilon > 0$ and every $t \in \mathbb{R}$ let $\boldsymbol{b}_\varepsilon(\cdot, t) := \rho_\varepsilon * \boldsymbol{b}(\cdot, t)$ and $\boldsymbol{B}_\varepsilon(\cdot, t) := \rho_\varepsilon * \boldsymbol{B}(\cdot, t)$. Then it holds:*

(a) *there exists an $\mathcal{L}^N$–null set $Z \subset \Omega$ such that*

$$\lim_{\varepsilon \to 0^+} \boldsymbol{b}_\varepsilon(x, t) = \boldsymbol{b}(x, t)\,, \quad \lim_{\varepsilon \to 0^+} \boldsymbol{B}_\varepsilon(x, t) = \boldsymbol{B}(x, t)\,, \qquad \forall (x, t) \in (\Omega \setminus Z) \times \mathbb{R};$$

(b) *$\partial_t \boldsymbol{B}_\varepsilon(x, t) = \boldsymbol{b}_\varepsilon(x, t)$ for every $(x, t) \in \Omega_\varepsilon \times \mathbb{R}$, where $\Omega_\varepsilon := \{x \in \Omega :\ dist(x, \partial\Omega) > \varepsilon\}$.*

Proof. The conclusion (a) for $\boldsymbol{b}_\varepsilon$ follows directly from Corollary 2.4.

By the Fubini–Tonelli theorem we have that

$$\boldsymbol{B}_\varepsilon(x, t) = \int_0^t \boldsymbol{b}_\varepsilon(x, s)\, ds, \qquad \forall (x, t) \in \Omega_\varepsilon \times \mathbb{R}, \tag{17}$$

hence (b) follows. Moreover, for every $x \in \mathbb{R}^N \setminus Z$ and every $t \in \mathbb{R}$, the conclusion (a) for $\boldsymbol{B}_\varepsilon$ follows from the Dominated Convergence Theorem. □

Lemma 3.7. *Let $\boldsymbol{b}$ satisfy conditions (i)–(iv) and let $\boldsymbol{B}$ be defined by* (8)*. Then, for every $u \in BV_{loc}(\Omega) \cap L^{\infty}_{loc}(\Omega)$, the function $\mathbf{v}\colon \Omega \to \mathbb{R}^N$, defined by*

$$\mathbf{v}(x) := \boldsymbol{B}(x, u(x))\,,$$

belongs to $\mathcal{DM}^{\infty}_{loc}(\Omega)$ and

$$|\operatorname{div} \mathbf{v}|(K) \leq \|u\|_{L^\infty(K)} \sigma(K) + \|\boldsymbol{b}\|_\infty |Du|(K) \qquad \forall K \subset \Omega,\ K \text{ compact.} \tag{18}$$

Moreover, the functions $\mathbf{v}_\epsilon(x) := \boldsymbol{B}_\varepsilon(x, u(x))$ converge to $\mathbf{v}$ a.e. in Ω and in $L^1_{\text{loc}}(\Omega)$.

Remark 3.8. In the following we shall use the notation

$$\operatorname{div}_x \boldsymbol{B}(x, u(x)) := (\operatorname{div}_x \boldsymbol{B})(x, u(x)) = \operatorname{div}_x \boldsymbol{B}(x, t)|_{t=u(x)}\,.$$

The divergence of the composite function $x \mapsto \boldsymbol{B}(x, u(x))$ will be denoted by $\operatorname{div}[\boldsymbol{B}(x, u(x))]$.

Proof. Using the same notation of Lemma 3.6, one has

$$\operatorname{div}_x \boldsymbol{B}_\varepsilon(\cdot,t) = F(\cdot,t)\,\rho_\varepsilon * \sigma.$$

Since $F(x,t) = \int_0^t f(x,s)\,ds$ and $|f(x,s)| \le 1$, we have that, for every $t \in \mathbb{R}$,

$$|\operatorname{div}_x \boldsymbol{B}_\varepsilon(x,t)| \le |t|\,\rho_\varepsilon * \sigma(x), \qquad \text{for } \sigma\text{-a.e. } x \in \Omega_\varepsilon. \tag{19}$$

Let us define $\mathbf{v}_\varepsilon(x) := \boldsymbol{B}_\varepsilon(x,u(x))$. Consider first the case $u \in C^1(\Omega) \cap L^\infty(\Omega)$. We have that

$$\operatorname{div} \mathbf{v}_\varepsilon(x) = \operatorname{div}_x \boldsymbol{B}_\varepsilon(x,u(x)) + \langle \boldsymbol{b}_\varepsilon(x,u(x))\,,\,\nabla u(x)\rangle\,,$$

hence, from (19), for every compact subset K of Ω and for every $\varepsilon > 0$ small enough it holds

$$\int_K |\operatorname{div} \mathbf{v}_\varepsilon|\,dx \le \|u\|_{L^\infty(K)} \int_K \rho_\varepsilon * \sigma(x)\,dx + \|\boldsymbol{b}\|_\infty \int_K |\nabla u(x)|\,dx.$$

By an approximation argument (see [32, Theorem 5.3.3]), when $u \in BV_{\text{loc}}(\Omega) \cap L^\infty_{\text{loc}}(\Omega)$ we get

$$\int_K |\operatorname{div} \mathbf{v}_\varepsilon|\,dx \le \|u\|_{L^\infty(K)} \int_K \rho_\varepsilon * \sigma(x)\,dx + \|\boldsymbol{b}\|_\infty |Du|(K).$$

Finally, (18) follows observing that, by Lemma 3.6(a), $\boldsymbol{B}_\varepsilon(x,u(x)) \to \boldsymbol{B}(x,u(x))$ for $\mathcal{L}^N$-a.e. $x \in \Omega$, hence $\mathbf{v}_\varepsilon \to \mathbf{v}$ in $L^1_{\text{loc}}(\Omega)$. □

4. Main results

The main result of the paper is stated in Theorems 4.3 and 4.6. As a preliminary step, we will prove the theorem under the additional regularity assumption $u \in W^{1,1}$.

Theorem 4.1 ($\mathcal{DM}^\infty$-dependence and $u \in W^{1,1}$)**.** *Let $\boldsymbol{b}$ satisfy conditions (i)–(iv) and let $\boldsymbol{B}$ be defined by* (8)*. Then, for every $u \in W^{1,1}_{\text{loc}}(\Omega) \cap L^\infty_{\text{loc}}(\Omega)$, the function $\mathbf{v}\colon \Omega \to \mathbb{R}^N$, defined by*

$$\mathbf{v}(x) := \boldsymbol{B}(x,u(x))\,, \tag{20}$$

belongs to $\mathcal{DM}^\infty_{loc}(\Omega)$ and the following equality holds in the sense of measures:

$$\operatorname{div} \mathbf{v} = F(x,\widetilde{u}(x))\,\sigma + \langle \partial_t \boldsymbol{B}(x,u(x))\,,\,\nabla u(x)\rangle\,\mathcal{L}^N, \tag{21}$$

where $F(\cdot,t)$ is the Radon–Nikodým derivative of $\operatorname{div}_x \boldsymbol{B}(\cdot,t)$ *with respect to σ (see Section 3.1).*

Remark 4.2. With some abuse of notation, we will also write equation (21) as

$$\operatorname{div}\,\mathbf{v} = \operatorname{div}_x \boldsymbol{B}(x,t)|_{t=\widetilde{u}(x)} + \langle \partial_t \boldsymbol{B}(x,u(x))\,,\,\nabla u(x)\rangle\,\mathcal{L}^N\,. \tag{22}$$

Proof. By Lemma 3.7 the function $\mathbf{v}$ belongs to $\mathcal{DM}^\infty_{\text{loc}}(\Omega)$ and satisfies (18).

Since all results are of local nature in the space variables, it is not restrictive to assume that $\Omega = \mathbb{R}^N$, $\boldsymbol{b}$ is a bounded Borel function, and $\boldsymbol{b}(\cdot,t) \in \mathcal{DM}^\infty$ for every $t \in \mathbb{R}$.

We will use a regularization argument as in [23, Theorem 3.4]. More precisely, as in Lemma 3.7 let $\boldsymbol{B}_\varepsilon(\cdot,t) := \rho_\varepsilon * \boldsymbol{B}(\cdot,t)$ and $\mathbf{v}_\varepsilon(x) := \boldsymbol{B}_\varepsilon(x,u(x))$. Since $\boldsymbol{B}_\varepsilon$ is a Lipschitz function in (x,t), by using the chain rule formula of Ambrosio and Dal Maso (see [3, Theorem 3.101]) one has

$$\begin{aligned}\int_{\mathbb{R}^N} \langle \nabla\phi(x)\,,\,\mathbf{v}_\varepsilon(x)\rangle\,dx = &-\int_{\mathbb{R}^N} \phi(x)\operatorname{div}_x \boldsymbol{B}_\varepsilon(x,u(x))\,dx \\ &-\int_{\mathbb{R}^N} \phi(x)\,\langle \boldsymbol{b}_\varepsilon(x,u(x))\,,\,\nabla u(x)\rangle\,dx,\end{aligned} \tag{23}$$

and the claim will follow by passing to the limit as $\varepsilon \to 0^+$.

Namely, by Lemma 3.7, $\mathbf{v}_\varepsilon(x) \to \mathbf{v}(x)$ for $\mathcal{L}^N$ a.e. $x \in \mathbb{R}^N$. Then by Dominated Convergence Theorem we have

$$\lim_{\varepsilon\to 0^+} \int_{\mathbb{R}^N} \langle \nabla\phi(x)\,,\, \mathbf{v}_\varepsilon(x)\rangle\, dx = \int_{\mathbb{R}^N} \langle \nabla\phi(x)\,,\, \mathbf{v}(x)\rangle\, dx\,. \tag{24}$$

This is equivalent to say

$$\operatorname{div} \boldsymbol{B}_\varepsilon(x,u(x))\, \mathcal{L}^N \overset{*}{\rightharpoonup} \operatorname{div} \mathbf{v}(x)\,, \qquad \text{as } \varepsilon \to 0^+ \tag{25}$$

in the weak* sense of measures.

Similarly, from Lemma 3.6(a) we have that, for $\mathcal{L}^N$ a.e. $x \in \mathbb{R}^N$,

$$\lim_{\varepsilon\to 0^+} \boldsymbol{b}_\varepsilon(x,u(x)) = \boldsymbol{b}(x,u(x))\,.$$

Then by (ii) and the Dominated Convergence Theorem we have

$$\lim_{\varepsilon\to 0^+} \int_{\mathbb{R}^N} \langle \boldsymbol{b}_\varepsilon(x,u(x))\,,\, \nabla u(x)\rangle\, dx = \int_{\mathbb{R}^N} \langle \boldsymbol{b}(x,u(x))\,,\, \nabla u(x)\rangle\, dx\,. \tag{26}$$

It remains to prove that

$$I_\varepsilon := \int_{\mathbb{R}^N} \phi(x) \operatorname{div}_x \boldsymbol{B}_\varepsilon(x,u(x))\, dx \ \xrightarrow{\varepsilon\to 0}\ \int_{\mathbb{R}^N} \phi(x) F(x,\widetilde{u}(x))\, d\sigma(x). \tag{27}$$

Assume, for simplicity, that $u \geq 0$ and let $C > \|u\|_\infty$. Let us rewrite I_ε in the following way:

$$\begin{aligned} I_\varepsilon &= \int_{\mathbb{R}^N} \phi(x) \int_0^{u(x)} \operatorname{div}_x \boldsymbol{b}_\varepsilon(x,t)\, dt\, dx \\ &= \int_0^C dt \int_{\mathbb{R}^N} \phi(x)\chi_{\{u>t\}}(x) \int_{\mathbb{R}^N} \rho_\varepsilon(x-y)\, d\operatorname{div}_y \boldsymbol{b}(y,t) \\ &= \int_0^C dt \int_{\mathbb{R}^N} \rho_\varepsilon * (\phi\chi_{\{u>t\}})(y)\, d\operatorname{div}_y \boldsymbol{b}(y,t)\,. \end{aligned}$$

By Corollary 3.80 in [3] it holds

$$\rho_\varepsilon * (\phi\chi_{\{u>t\}})(x) \to \phi(x)\chi^*_{\{u>t\}}(x) = \phi(x)\chi_{\{\widetilde{u}>t\}}(x) \qquad \text{for } \mathcal{H}^{N-1}\text{-a.e. } x$$

(hence for $\operatorname{div}_x \boldsymbol{b}(\cdot,t)$-a.e. x). Passing to the limit as $\varepsilon \to 0$ we get

$$\begin{aligned} \lim_{\varepsilon\to 0} I_\varepsilon &= \int_0^C dt \int_{\mathbb{R}^N} \phi(x)\chi_{\{\widetilde{u}>t\}}(x)\, d\operatorname{div}_x \boldsymbol{b}(x,t) \\ &= \int_{\mathbb{R}^N} \phi(x) \int_0^{\widetilde{u}(x)} f(x,t)\, dt\, d\sigma(x) \\ &= \int_{\mathbb{R}^N} \phi(x) F(x,\widetilde{u}(x))\, d\sigma(x), \end{aligned}$$

hence (27) is proved and the proof is completed. □

We now state the main results of the paper; the proofs are collected at the end of the section.

Theorem 4.3 ($\mathcal{DM}^\infty$-dependence and $u \in BV$)**.** *Let $\boldsymbol{b}$ satisfy conditions (i)–(iv) and let $\boldsymbol{B}$ be defined by* (8)*. Then, for every $u \in BV_{loc}(\Omega) \cap L^\infty_{loc}(\Omega)$, the function $\mathbf{v}\colon \Omega \to \mathbb{R}^N$, defined by $\mathbf{v}(x) := \boldsymbol{B}(x, u(x))$, belongs to $\mathcal{DM}^\infty_{loc}(\Omega)$. Moreover, there exists a Radon measure $(\boldsymbol{b}(\cdot, u), Du)$ such that*

$$|(\boldsymbol{b}(\cdot,u), Du)|(E) \leq \|\boldsymbol{b}\|_\infty |Du|(E), \qquad \text{for every Borel set } E \subset \Omega, \tag{28}$$

and the following equality holds in the sense of measures:

$$\operatorname{div} \mathbf{v} = \frac{1}{2}\left[F(x, u^+(x)) + F(x, u^-(x))\right] \sigma + (\boldsymbol{b}(\cdot, u), Du). \tag{29}$$

Remark 4.4. The measure $(\boldsymbol{b}(\cdot, u), Du)$, defined by

$$\langle (\boldsymbol{b}(\cdot,u), Du), \varphi\rangle := -\int_\Omega \varphi \, d \operatorname{div}_x \boldsymbol{B}(x,t)|_{t=u(x)} - \int_\Omega \boldsymbol{B}(x,u)\cdot\nabla\varphi\, dx, \quad \forall \varphi \in C^\infty_c(\Omega), \tag{30}$$

extends the notion of pairing defined, in the case $\boldsymbol{b}(x, w) = \boldsymbol{A}(x)$, with $\boldsymbol{A} \in \mathcal{DM}^\infty$, by Anzellotti [5].

Proposition 4.5 (Traces of the composite function)**.** *Let the assumptions of Theorem 4.3 hold, and let $\Sigma \subset \Omega$ be a countably $\mathcal{H}^{N-1}$-rectifiable set, oriented as in Section 3.2. Then the normal traces on Σ of the composite function $\mathbf{v} \in \mathcal{DM}^\infty_{loc}(\Omega)$, defined at* (20)*, are given by*

$$\operatorname{Tr}^\pm(\mathbf{v}, \Sigma) = \begin{cases} \beta^\pm(x, u^\pm(x)), & \text{for } \mathcal{H}^{N-1}\text{-a.e. } x \in J_u, \\ \beta^\pm(x, \widetilde{u}(x)), & \text{for } \mathcal{H}^{N-1}\text{-a.e. } x \in \Sigma \setminus J_u, \end{cases} \tag{31}$$

where, for every $t \in \mathbb{R}$, $\beta^\pm(\cdot, t)$ are the normal traces of $\boldsymbol{B}(\cdot, t)$ on Σ (see (12)*). With our convention $u^\pm(x) = \widetilde{u}(x)$ if $x \in \Omega \setminus S_u$,* (31) *can be written as $\operatorname{Tr}^\pm(\mathbf{v}, \Sigma) = \beta^\pm(x, u^\pm(x))$ for $\mathcal{H}^{N-1}$-a.e. $x \in \Sigma$.*

Theorem 4.6 (Representation of the pairing measure)**.** *Let the assumptions of Theorem 4.3 hold, and consider the standard decomposition of $\mu := (\boldsymbol{b}(\cdot, u), Du)$*

$$\mu = \mu^{ac} + \mu^c + \mu^j, \quad \mu^d := \mu^{ac} + \mu^c.$$

Then

$$\mu^{ac} = \langle \boldsymbol{b}(x, \widetilde{u}(x)) \,,\, \nabla u(x)\rangle \, \mathcal{L}^N,$$
$$\mu^j = [\beta^*(x, u^+(x)) - \beta^*(x, u^-(x))]\mathcal{H}^{N-1} \llcorner J_u,$$

where, for every $t \in \mathbb{R}$, $\beta^\pm(\cdot,t)$ are the normal traces of $\boldsymbol{B}(\cdot, t)$ on J_u and $\beta^*(\cdot, t) = [\beta^+(\cdot,t) + \beta^-(\cdot,t)]/2$.

Moreover, if there exists a countable dense set $Q \subset \mathbb{R}$ such that

$$|D^c u|(S_{\boldsymbol{b}(\cdot,t)}) = 0 \qquad \forall t \in Q, \tag{32}$$

then

$$\mu^d = \left\langle \widetilde{\boldsymbol{b}}(x, \widetilde{u}(x)) \,,\, D^d u \right\rangle.$$

Therefore, under this additional assumption the following equality holds in the sense of measures:

$$\begin{aligned} \operatorname{div} \mathbf{v} = {} & \frac{1}{2}\left[F(x, u^+(x)) + F(x, u^-(x))\right] \sigma \\ & + \left\langle \widetilde{\boldsymbol{b}}(x, \widetilde{u}(x)) \,,\, \widetilde{D}u \right\rangle + [\beta^*(x, u^+) - \beta^*(x, u^-)]\mathcal{H}^{N-1} \llcorner J_u. \end{aligned} \tag{33}$$

Remark 4.7. Since $\mathcal{L}^N(S_{\boldsymbol{b}(\cdot,t)}) = 0$ for every $t \in \mathbb{R}$, assumption (32) is equivalent to $|D^d u|(S_{\boldsymbol{b}(\cdot,t)}) = 0$ for every $t \in Q$. In particular, it is satisfied, for example, if $S_{\boldsymbol{b}(\cdot,t)}$ is σ–finite with respect to $\mathcal{H}^{N-1}$, for every $t \in Q$ (see [3, Proposition 3.92(c)]). This is always the case if $\boldsymbol{b}(\cdot,t) \in BV_{\rm loc}(\Omega,\mathbb{R}^N) \cap L^\infty_{\rm loc}(\Omega,\mathbb{R}^N)$. Another relevant situation for which (32) holds happens when $D^c u = 0$, i.e. if u is a special function of bounded variation, e.g. if u is the characteristic function of a set of finite perimeter.

Remark 4.8. For $u \in BV_{\rm loc}(\Omega)$ we introduce the following notation:

$$\operatorname{div}_x \boldsymbol{B}(\cdot,t)|_{t=u(x)} := \frac{1}{2}\left[\operatorname{div}_x \boldsymbol{B}(\cdot,u^+(x)) + \operatorname{div}_x \boldsymbol{B}(\cdot,u^-(x))\right]$$

(see also Remark 4.2). Then, with some abuse of notation, equation (33) can be written as

$$\begin{aligned}\operatorname{div} \mathbf{v} = {} & \operatorname{div}_x \boldsymbol{B}(x,t)|_{t=u(x)} \\ & + \left\langle \widetilde{\boldsymbol{b}}(x,\widetilde{u}(x)) \, , \, \widetilde{D}u \right\rangle + [\beta^*(x,u^+) - \beta^*(x,u^-)]\mathcal{H}^{N-1} \llcorner J_u.\end{aligned} \tag{34}$$

Remark 4.9 (Anzellotti's pairing). In the special case $\boldsymbol{B}(x,t) = t\,\boldsymbol{A}(x)$, with $\boldsymbol{A} \in \mathcal{DM}^\infty_{\rm loc}(\Omega)$, we have that

$$\boldsymbol{b}(x,t) = \boldsymbol{A}(x), \quad \sigma = |\operatorname{div} \boldsymbol{A}|, \quad f(x,t) = \frac{d \operatorname{div} \boldsymbol{A}}{d|\operatorname{div} \boldsymbol{A}|}\,, \quad F(x,t) = t\,\frac{d \operatorname{div} \boldsymbol{A}}{d|\operatorname{div} \boldsymbol{A}|}\,,$$

and formula (33) becomes

$$\operatorname{div}(u\boldsymbol{A}) = u^* \operatorname{div} \boldsymbol{A} + (\boldsymbol{A}, Du),$$

where $\mu = (\boldsymbol{A}, Du)$ is the Anzellotti's pairing.

Proof of Theorem 4.3. By Lemma 3.7 the function $\mathbf{v}$ belongs to $\mathcal{DM}^\infty_{\rm loc}(\Omega)$ and satisfies (18).

Since all results are of local nature in the space variables, it is not restrictive to assume that $\Omega = \mathbb{R}^N$, $\boldsymbol{b}$ is a bounded Borel function, and $\boldsymbol{b}(\cdot,t) \in \mathcal{DM}^\infty$ for every $t \in \mathbb{R}$.

We will use another regularization argument as in [10]. More precisely, let $u_\varepsilon := \rho_\varepsilon * u$ be the standard regularization of u, and $\mathbf{v}_\varepsilon(x) := \boldsymbol{B}(x,u_\varepsilon(x))$, then by Theorem 4.1, for any $\phi \in C^1_0(\mathbb{R}^N)$ we get

$$\begin{aligned}\int_{\mathbb{R}^N} \langle \nabla\phi(x)\, , \, \mathbf{v}_\varepsilon(x)\rangle \, dx = {} & -\int_{\mathbb{R}^N} \phi(x)\, F(x,u_\varepsilon(x))\, d\sigma(x) \\ & -\int_{\mathbb{R}^N} \phi(x)\, \langle \boldsymbol{b}(x,u_\varepsilon(x))\, , \, \nabla u_\varepsilon(x)\rangle \, dx\,.\end{aligned} \tag{35}$$

Now we will pass to the limit as $\varepsilon \to 0^+$ in each term.

STEP 1. Firstly, we note that

$$\lim_{\varepsilon\to 0^+} \int_{\mathbb{R}^N} \langle \nabla\phi(x)\, , \, \mathbf{v}_\varepsilon(x)\rangle \, dx = \int_{\mathbb{R}^N} \langle \nabla\phi(x)\, , \, \mathbf{v}(x)\rangle \, dx. \tag{36}$$

Indeed, $u_\varepsilon(x) \to u(x)$, as $\varepsilon \to 0^+$, for a.e. x, $\boldsymbol{B}(x,\cdot)$ is Lipschitz continuous with Lipschitz constant independent of x and $\boldsymbol{B}$ is locally bounded. Thus (36) holds by the Dominated Convergence Theorem.

STEP 2. We will prove that

$$\lim_{\varepsilon\to 0^+} \int_{\mathbb{R}^N} \phi(x)\, F(x,u_\varepsilon(x))\, d\sigma(x) = \int_{\mathbb{R}^N} \phi(x)\frac{1}{2}\left[F(x,u^+(x)) + F(x,u^-(x))\right] d\sigma(x). \tag{37}$$

From (9) it holds

$$\int_{\mathbb{R}^N} \phi(x)\, F(x,u_\varepsilon(x))\, d\sigma(x) = \int_{\mathbb{R}^N} \phi(x) \left[\int_0^{u_\varepsilon(x)} f(x,w)\, dw\right] d\sigma(x)\,. \tag{38}$$

Since $u_\varepsilon(x) \to u^*(x)$ for $\mathcal{H}^{N-1}$-a.e. x, and so also for σ-a.e. x (since $\sigma \ll \mathcal{H}^{N-1}$), passing to the limit in (38) we obtain

$$\lim_{\varepsilon\to 0^+} \int_{\mathbb{R}^N} \phi(x)\, F(x,u_\varepsilon(x))\, d\sigma(x) = \int_{\mathbb{R}^N} \phi(x) \left[\int_0^{u^*(x)} f(x,w)\, dw\right] d\sigma(x) =: I\,.$$

In the remaining part of the proof, for the sake of simplicity we assume $u \geq 0$. Let $C > \|u\|_{L^\infty(K)}$, where K is the support of ϕ. The integral I can be rewritten as

$$I = \int_0^C \left[\int_{\mathbb{R}^N} \phi(x)\chi_{\{u^*>w\}}(x) f(x,w)\, d\sigma(x)\right] dw\,.$$

On the other hand, for $\mathcal{L}^1$-a.e. $w \in \mathbb{R}$ we have that

$$\chi_{\{u^*>w\}} = \frac{1}{2}\left[\chi_{\{u^+>w\}} + \chi_{\{u^->w\}}\right], \qquad \mathcal{H}^{N-1}\text{-a.e. (hence } \sigma\text{-a.e.) in } \mathbb{R}^N$$

(see [23, Lemma 2.2]). Hence we get

$$\begin{aligned} I &= \int_0^C \left[\int_{\mathbb{R}^N} \phi(x)\frac{1}{2}[\chi_{\{u^+>w\}}(x) + \chi_{\{u^->w\}}(x)] f(x,w)\, d\sigma(x)\right] dw \\ &= \int_{\mathbb{R}^N} \phi(x) \left[\int_0^C \frac{1}{2}[\chi_{\{u^+>w\}}(x) + \chi_{\{u^->w\}}(x)] f(x,w)\, dw\right] d\sigma(x) \\ &= \int_{\mathbb{R}^N} \phi(x)\frac{1}{2}\left[F(x,u^+(x)) + F(x,u^-(x))\right] d\sigma(x)\,, \end{aligned}$$

so that (37) is proved.

STEP 3. We claim that there exists a Radon measure $\mu \equiv (\boldsymbol{b}(\cdot,u), Du)$, with locally bounded total variation such that $\mu \ll |Du|$, satisfying (28), and such that the equality (29) holds in the sense of measures.

Let us define μ in the following way:

$$\mu := \operatorname{div} \mathbf{v} - \frac{1}{2}\left[F(x,u^+(x)) + F(x,u^-(x))\right] \sigma.$$

By definition, μ is a Radon measure and formula (29) holds. Moreover, by (35), (36) and (37) we have that, for every $\phi \in C_c(\mathbb{R}^N)$,

$$\langle \mu, \phi\rangle = \lim_{\varepsilon\to 0^+} \int_{\mathbb{R}^N} \phi(x)\, \langle \boldsymbol{b}(x,u_\varepsilon(x))\,,\, \nabla u_\varepsilon(x)\rangle\, dx. \tag{39}$$

Let us prove that (28) holds. Namely, let $K \Subset U \subset \Omega$, with K compact and U open, and let $\phi \in C_c(\Omega)$ be a function with support contained in K. There exist $r_0 > 0$ such that $K_r := K + B_r(0) \subset U$ for every $r \in (0, r_0)$. Let $r \in (0, r_0)$ be such that $|Du|(\partial K_r) = 0$ (this property holds for almost every r). Then

$$|\langle \mu\,,\, \phi\rangle| \leq \|\phi\|_\infty \|\boldsymbol{b}\|_\infty \liminf_{\varepsilon\to 0} \int_{K_r} |\nabla u_\varepsilon|\, dx = \|\phi\|_\infty \|\boldsymbol{b}\|_\infty |Du|(K_r) \leq \|\phi\|_\infty \|\boldsymbol{b}\|_\infty |Du|(U),$$

hence

$$|\mu|(K) \leq \|\boldsymbol{b}\|_\infty |Du|(U),$$

so that (28) follows by the regularity of the Radon measures $|\mu|$ and $|Du|$. $\square$

Proof of Proposition 4.5. Since all results are of local nature in the space variables, it is not restrictive to assume that $\Omega = \mathbb{R}^N$, $\boldsymbol{b}$ is a bounded Borel function, and $\boldsymbol{b}(\cdot, t) \in \mathcal{DM}^\infty$ for every $t \in \mathbb{R}$.

We will use the same notations of Section 3.2. It is not restrictive to assume that J_u is oriented with ν_Σ on $J_u \cap \Sigma$.

Since, by Theorem 4.3, $\mathbf{v} \in \mathcal{DM}^\infty$, there exist the weak normal traces of $\mathbf{v}$ on Σ. Let us prove (31) for Tr^-.

Let $x \in \Sigma$ satisfy:

(a) $x \in (\mathbb{R}^N \setminus S_u) \cup J_u$, $x \in N_i$ for some i, the set N_i has density 1 at x and x is a Lebesgue point of $\beta^-(\cdot, t)$, with respect to $\mathcal{H}^{N-1} \llcorner \partial\Omega_i$, for every $t \in \mathbb{R}$;
(b) $\sigma \llcorner \Omega_i(B_\varepsilon(x)) = o(\varepsilon^{N-1})$ as $\varepsilon \to 0$;
(c) $|\operatorname{div} \mathbf{v}| \llcorner \Omega_i(B_\varepsilon(x)) = o(\varepsilon^{N-1})$.

We remark that $\mathcal{H}^{N-1}$-a.e. $x \in \Sigma$ satisfies these conditions. In particular, (a) is satisfied thanks to Proposition 3.4, whereas (b) and (c) follow from [3, Theorem 2.56 and (2.41)].

In order to simplify the notation, in the following we set $u^-(x) := \widetilde{u}(x)$ if $x \in \Omega \setminus S_u$.

Let us choose a function $\varphi \in C_c^\infty(\mathbb{R}^N)$, with support contained in $B_1(0)$, such that $0 \leq \varphi \leq 1$. For every $\varepsilon > 0$ let $\varphi_\varepsilon(y) := \varphi\left(\frac{y-x}{\varepsilon}\right)$.

By the very definition of normal trace, the following equality holds for every $\varepsilon > 0$ small enough:

$$
\begin{aligned}
&\frac{1}{\varepsilon^{N-1}} \int_{\partial\Omega_i} [\mathrm{Tr}(\mathbf{v}, \partial\Omega_i) - \mathrm{Tr}(\boldsymbol{B}(\cdot, u^-(x)), \partial\Omega_i)]\, \varphi_\varepsilon(y)\, d\mathcal{H}^{N-1}(y) \\
&\quad = \frac{1}{\varepsilon^{N-1}} \int_{\Omega_i} \nabla\varphi_\varepsilon(y) \cdot [\mathbf{v}(y) - \boldsymbol{B}(y, u^-(x))]\, dy \\
&\qquad + \frac{1}{\varepsilon^{N-1}} \int_{\Omega_i} \varphi_\varepsilon(y)\, d[\operatorname{div} \mathbf{v} - \operatorname{div}_x \boldsymbol{B}(\cdot, u^-(x))](y)\,.
\end{aligned} \tag{40}
$$

Using the change of variable $z = (y - x)/\varepsilon$, as $\varepsilon \to 0$ the left hand side of this equality converges to

$$
[\mathrm{Tr}^-(\mathbf{v}, \Sigma)(x) - \beta^-(x, u^-(x))] \int_{\Pi_x} \varphi(z)\, d\mathcal{H}^{N-1}(z)\,,
$$

where Π_x is the tangent plane to Σ_i at x. Clearly φ can be chosen in such a way that $\int_{\Pi_x} \varphi\, d\mathcal{H}^{N-1} > 0$.

In order to prove (31) for Tr^- it is then enough to show that the two integrals $I_1(\varepsilon)$ and $I_2(\varepsilon)$ at the right hand side of (40) converge to 0 as $\varepsilon \to 0$.

With the change of variables $z = (y - x)/\varepsilon$ and by the very definition of $\mathbf{v}$ we have that

$$
I_1(\varepsilon) = \int_{\Omega_i^\varepsilon} \nabla\varphi(z) \cdot [\boldsymbol{B}(x + \varepsilon z, u(x + \varepsilon z)) - \boldsymbol{B}(x + \varepsilon z, u^-(x))]\, dz,
$$

where

$$
\Omega_i^\varepsilon := \frac{\Omega_i - x}{\varepsilon}.
$$

As $\varepsilon \to 0$, these sets locally converge to the half space $P_x := \{z \in \mathbb{R}^N : \ \langle z\,, \nu(x)\rangle < 0\}$, hence

$$
\lim_{\varepsilon\to 0} \int_{\Omega_i^\varepsilon \cap B_1} |u(x + \varepsilon z) - u^-(x)|\, dz = \lim_{\varepsilon\to 0} \int_{P_x \cap B_1} |u(x + \varepsilon z) - u^-(x)|\, dz = 0
$$

(see [3, Remark 3.85]) and, by (ii),

$$|I_1(\varepsilon)| \leq \|\boldsymbol{b}\|_\infty \, \|\nabla\varphi\|_\infty \int_{\Omega_i^\varepsilon \cap B_1} |u(x+\varepsilon z) - u^-(x)| \, dz \to 0.$$

From (b) we have that

$$\lim_{\varepsilon\to 0} \frac{1}{\varepsilon^{N-1}} \left| \int_{\Omega_i} \varphi_\varepsilon(y) \, d \operatorname{div}_x \boldsymbol{B}(\cdot, u^-(x))(y) \right| \leq \limsup_{\varepsilon\to 0} \frac{\sigma(B_\varepsilon(x))}{\varepsilon^{N-1}} = 0.$$

In a similar way, using (c), we get

$$\lim_{\varepsilon\to 0} \frac{1}{\varepsilon^{N-1}} \left| \int_{\Omega_i} \varphi_\varepsilon \, d \operatorname{div} \mathbf{v} \right| = 0,$$

so that $I_2(\varepsilon)$ vanishes as $\varepsilon \to 0$.

The proof of (31) for Tr^+ is entirely similar. □

Proof of Theorem 4.6. Since all results are of local nature in the space variables, it is not restrictive to assume that $\Omega = \mathbb{R}^N$, $\boldsymbol{b}$ is a bounded Borel function, and $\boldsymbol{b}(\cdot, t) \in \mathcal{DM}^\infty$ for every $t \in \mathbb{R}$.

We shall divide the proof into several steps.

STEP 1. We are going to prove that

$$\mu^{ac} = \langle \boldsymbol{b}(x, u(x)) \, , \, \nabla u(x) \rangle \; \mathcal{L}^N.$$

Let us choose $x \in \mathbb{R}^N$ such that

(a) there exists the limit $\displaystyle\lim_{r\to 0} \frac{\mu(B(x,r))}{r^N}$;

(b) $\displaystyle\lim_{r\to 0} \frac{|D^s u|(B(x,r))}{r^N} = 0$;

(c) $\displaystyle\lim_{r\to 0} \frac{1}{r^N} \int_{B(x,r)} |\, \langle \boldsymbol{b}(y, u(y)) \, , \, \nabla u(y) \rangle - \langle \boldsymbol{b}(x, u(x)) \, , \, \nabla u(x) \rangle \,| \, dx = 0.$

We remark that these conditions are satisfied for $\mathcal{L}^N$-a.e. $x \in \mathbb{R}^N$.

Let $r > 0$ be such that

$$|D^s u| \, (\partial B(x,r)) = 0. \tag{41}$$

Observe that

$$\nabla u_\varepsilon = \rho_\varepsilon * Du = \rho_\varepsilon * \nabla u + \rho_\varepsilon * D^s u.$$

Hence for every $\phi \in C_0(\mathbb{R}^N)$ with support in $B(x,r)$ it holds

$$\begin{aligned} &\left| \frac{1}{r^N} \int_{B(x,r)} \phi(y) [\langle \boldsymbol{b}(y, u_\varepsilon(y)) \, , \, (\rho_\varepsilon * Du)(y) \rangle - \langle \boldsymbol{b}(x, u(x)) \, , \, \nabla u(x) \rangle] \, dy \right| \\ &\leq \frac{1}{r^N} \int_{B(x,r)} \phi(y) \, |\langle \boldsymbol{b}(y, u_\varepsilon(y)) \, , \, (\rho_\varepsilon * \nabla u)(y) \rangle - \langle \boldsymbol{b}(x, u(x)) \, , \, \nabla u(x) \rangle| \; dy \\ &\quad + \frac{1}{r^N} \|\phi\|_\infty \|\boldsymbol{b}\|_\infty \int_{B(x,r)} \rho_\varepsilon * |D^s u| \, dy, \end{aligned} \tag{42}$$

where in the last inequality we use that $|\rho_\varepsilon * D^s u| \leq \rho_\varepsilon * |D^s u|$. We note that by (41)

$$\lim_{\varepsilon\to 0} \int_{B(x,r)} \rho_\varepsilon * |D^s u| \, dy = |D^s u|(B(x,r)).$$

Hence taking the limit as $\varepsilon \to 0$ we obtain

$$\left|\frac{1}{r^N}\int_{B(x,r)} \phi(y)\ d\mu(y) - \frac{1}{r^N}\int_{B(x,r)} \phi(y)\,\langle \boldsymbol{b}(x,u(x))\,,\,\nabla u(x)\rangle\ dy\right|$$
$$\leq \frac{1}{r^N}\int_{B(x,r)} \phi(y)\,|\langle \boldsymbol{b}(y,u(y))\,,\,\nabla u(y)\rangle - \langle \boldsymbol{b}(x,u(x))\,,\,\nabla u(x)\rangle|\ dy$$
$$+\frac{1}{r^N}\|\phi\|_\infty \|\boldsymbol{b}\|_\infty\,|D^s u|(B(x,r)).$$

When $\phi(y) \to 1$ in $B(x,r)$, with $0 \leq \phi \leq 1$, we get

$$\left|\frac{1}{\omega_N r^N}\mu(B(x,r)) - \langle \boldsymbol{b}(x,u(x))\,,\,\nabla u(x)\rangle\right|$$
$$\leq \frac{1}{\omega_N r^N}\int_{B(x,r)} |\langle \boldsymbol{b}(y,u(y))\,,\,\nabla u(y)\rangle - \langle \boldsymbol{b}(x,u(x))\,,\,\nabla u(x)\rangle|\ dy$$
$$+\frac{1}{\omega_N r^N}\|\boldsymbol{b}\|_\infty\,|D^s u|(B(x,r)).$$

Now the conclusion is achieved by taking the limit for $r \to 0$ and using (b) and (c) above.

STEP 2. For the jump part of the measure μ it holds:

$$\mu^j = [\beta^*(x,u^+) - \beta^*(x,u^-)]\mathcal{H}^{N-1} \llcorner J_u \tag{43}$$

Namely, this is a direct consequence of Proposition 4.5 in the particular case $\Sigma = J_u$:

STEP 3. From now to the end of the proof, we shall assume that the additional assumption (32) holds.

Let $S := \bigcup_{q\in Q} S_{\boldsymbol{b}(\cdot,q)}$. By assumption (32) we have that $|D^c u|(S) = 0$.

We claim that, for every $x \in \mathbb{R}^N \setminus S$ and every $t \in \mathbb{R}$, there exists the approximate limit of $\boldsymbol{b}$ at (x,t) and

$$\widetilde{\boldsymbol{b}}(x,t) = \lim_j \widetilde{\boldsymbol{b}}(x,q_j), \qquad \forall (q_j) \subset Q,\ q_j \to t. \tag{44}$$

Namely, let us fix a point $x \in \mathbb{R}^N \setminus S$. By assumptions (i), (ii) and the Dominated Convergence Theorem, the map

$$\psi(t) := \fint_{B_r(x)} \boldsymbol{b}(y,t)\,dy, \qquad t \in \mathbb{R},$$

is continuous, and $\psi(q) = \widetilde{\boldsymbol{b}}(x,q)$ for every $q \in Q$. Hence the limit in (44) exists for every t and it is independent of the choice of the sequence $(q_j) \subset Q$ converging to t.

Let $t \in \mathbb{R}$ be fixed, let us denote by $c \in \mathbb{R}^N$ the value of the limit in (44) and let us prove that $c = \widetilde{\boldsymbol{b}}(x,t)$. We have that

$$\fint_{B_r(x)} |\boldsymbol{b}(y,t)-c|\,dy \leq \fint_{B_r(x)} |\boldsymbol{b}(y,t)-\boldsymbol{b}(y,q_j)|\,dy + \fint_{B_r(x)} |\boldsymbol{b}(y,q_j)-\widetilde{\boldsymbol{b}}(x,q_j)|\,dy + |\widetilde{\boldsymbol{b}}(x,q_j)-c|.$$

As $j \to +\infty$, the first integral at the r.h.s. converges to 0 by (i), (ii) and the Dominated Convergence Theorem. The second integral converges to 0 since $x \in \mathbb{R}^N \setminus S$ and $(q_j) \subset Q$. Finally, $\lim_j |\widetilde{\boldsymbol{b}}(x,q_j) - c| = 0$ by the very definition of c, so that the claim is proved.

STEP 4. We are going to prove that

$$\mu^d = \left\langle \widetilde{\boldsymbol{b}}(x,\widetilde{u}(x))\,,\,D^d u\right\rangle$$

in the sense of measures. We remark that, by Step 3, the approximate limit $\widetilde{\boldsymbol{b}}(x,t)$ exists for every $(x,t) \in (\mathbb{R}^N \setminus S) \times \mathbb{R}$, with $|D^c u|(S) = 0$. As a consequence, the function $x \mapsto \widetilde{\boldsymbol{b}}(x, \widetilde{u}(x))$ is well-defined for $|D^d u|$-a.e. $x \in \mathbb{R}^N$.

If we consider the polar decomposition $D^d u = \theta\, |D^d u|$, this equality is equivalent to

$$\frac{d\mu}{d|D^d u|}(x) = \frac{d\mu^d}{d|D^d u|}(x) = \left\langle \widetilde{\boldsymbol{b}}(x, \widetilde{u}(x))\,,\, \theta(x) \right\rangle$$

for $|D^d u|$-a.e. $x \in \mathbb{R}^N$. Let us choose $x \in \mathbb{R}^N$ such that

(a) x belongs to the support of $D^d u$, that is $|D^d u|(B_r(x)) > 0$ for every $r > 0$;

(b) there exists the limit $\displaystyle\lim_{r\to 0} \frac{\mu^d(B_r(x))}{|D^d u|(B_r(x))}$;

(c) $\displaystyle\lim_{r\to 0} \frac{|D^j u|(B_r(x))}{|Du|(B_r(x))} = 0$;

(d) $\displaystyle\lim_{r\to 0} \frac{1}{|D^d u|(B_r(x))} \int_{B_r(x)} \left| \langle \boldsymbol{b}(y, u(y))\,,\, \theta(y) \rangle - \left\langle \widetilde{\boldsymbol{b}}(x, \widetilde{u}(x))\,,\, \theta(x) \right\rangle \right| \, d|D^d u|(y) = 0.$

We remark that these conditions are satisfied for $|D^d u|$-a.e. $x \in \mathbb{R}^N$.

Let $r > 0$ be such that

$$|D^j u|\,(\partial B(x,r)) = 0. \tag{45}$$

Observe that $\nabla u_\varepsilon = \rho_\varepsilon * Du = \rho_\varepsilon * D^d u + \rho_\varepsilon * D^j u$. Hence for every $\phi \in C_0(\mathbb{R}^N)$ with support in $B(x,r)$ it holds

$$\begin{aligned} &\left| \frac{1}{|D^d u|(B(x,r))} \int_{B(x,r)} \phi(y)\, \langle \boldsymbol{b}(y, u_\varepsilon(y))\,,\, (\rho_\varepsilon * Du)(y) \rangle \, dy \right. \\ &\qquad \left. - \frac{1}{|D^d u|(B(x,r))} \int_{B(x,r)} \phi(y) \left\langle \widetilde{\boldsymbol{b}}(x, \widetilde{u}(x))\,,\, \theta(x) \right\rangle d|D^d u|(y) \right| \\ &\le \left| \frac{1}{|D^d u|(B(x,r))} \int_{B(x,r)} \phi(y) \left\langle \boldsymbol{b}(y, u_\varepsilon(y))\,,\, (\rho_\varepsilon * D^d u)(y) \right\rangle dy \right. \\ &\qquad \left. - \frac{1}{|D^d u|(B(x,r))} \int_{B(x,r)} \phi(y) \left\langle \widetilde{\boldsymbol{b}}(x, \widetilde{u}(x))\,,\, \theta(x) \right\rangle d|D^d u|(y) \right| \\ &\qquad + \frac{1}{|D^d u|(B(x,r))} \|\phi\|_\infty \|\boldsymbol{b}\|_\infty \int_{B(x,r)} \rho_\varepsilon * |D^j u| \, dy, \end{aligned} \tag{46}$$

where in the last inequality we use that $\left|\rho_\varepsilon * D^j u\right| \le \rho_\varepsilon * |D^j u|$. We note that by (45)

$$\lim_{\varepsilon\to 0} \int_{B(x,r)} \rho_\varepsilon * |D^j u| \, dy = |D^j u|(B(x,r)).$$

Hence by taking the limit as $\varepsilon \to 0$ in (46) we obtain

$$\begin{aligned}
&\left| \frac{1}{|D^d u|(B(x,r))} \int_{B(x,r)} \phi(y)\, d\mu(y) \right. \\
&\left. - \frac{1}{|D^d u|(B(x,r))} \int_{B(x,r)} \phi(y) \left\langle \widetilde{\boldsymbol{b}}(x, \widetilde{u}(x))\,,\, \theta(x) \right\rangle d|D^d u|(y) \right| \\
&\leq \frac{1}{|D^d u|(B(x,r))} \int_{B(x,r)} \phi(y) \left| \langle \boldsymbol{b}(y, u(y))\,,\, \theta(y)\rangle - \left\langle \widetilde{\boldsymbol{b}}(x, \widetilde{u}(x))\,,\, \theta(x) \right\rangle d|D^d u|(y) \right| \\
&\quad + \frac{1}{|D^d u|(B(x,r))} \|\phi\|_\infty \|\boldsymbol{b}\|_\infty \, |D^j u|(B(x,r)).
\end{aligned}$$

When $\phi(y) \to 1$ in $B(x,r)$, with $0 \leq \phi \leq 1$, we get

$$\begin{aligned}
&\left| \frac{1}{|D^d u|(B(x,r))} \mu(B(x,r)) - \left\langle \widetilde{\boldsymbol{b}}(x, \widetilde{u}(x))\,,\, \theta(x) \right\rangle \right| \\
&\leq \frac{1}{|D^d u|(B(x,r))} \int_{B(x,r)} \left| \langle \boldsymbol{b}(y, u(y))\,,\, \theta(y)\rangle - \left\langle \widetilde{\boldsymbol{b}}(x, \widetilde{u}(x))\,,\, \theta(x) \right\rangle \right| d|D^d u|(y) \\
&\quad + \frac{1}{|D^d u|(B(x,r))} \|\boldsymbol{b}\|_\infty \int_{B(x,r)} |D^j u|.
\end{aligned} \tag{47}$$

The conclusion is achieved now by taking $r \to 0$ and by using (c) and (d). □

5. Gluing constructions and extension theorems

A direct consequence of Theorems 4.3, 4.6 and [15, Theorems 5.1 and 5.3] are the following gluing constructions and extension theorems (see [14, Theorems 8.5 and 8.6] and [15, Theorems 5.1 and 5.3]).

Theorem 5.1 (Extension)**.** *Let $W \Subset int(E) \subset E \Subset U \subset \Omega$, where $\Omega, U, W \subset \mathbb{R}^N$ are open sets and E is a set of finite perimeter in Ω. Let*

$$\boldsymbol{b}_1 \colon U \times \mathbb{R} \to \mathbb{R}^N, \qquad \boldsymbol{b}_2 \colon (\Omega \setminus \overline{W}) \times \mathbb{R} \to \mathbb{R}^N$$

satisfy assumptions (i)–(iv) in $U \times \mathbb{R}$ and $\Omega \setminus \overline{W}$ respectively. Let $\boldsymbol{B}_1, \boldsymbol{B}_2$ be the corresponding integral functions with respect to the second variable. Given $u_1 \in BV_{loc}(U) \cap L^\infty_{loc}(U)$ and $u_2 \in BV_{loc}(\Omega \setminus \overline{W}) \cap L^\infty_{loc}(\Omega \setminus \overline{W})$, let $\mathbf{v}_i(x) := \boldsymbol{B}_i(x, u_i(x))$, $i = 1, 2$. Then the composite function

$$\mathbf{v}(x) := \begin{cases} \mathbf{v}_1(x), & \text{if } x \in E, \\ \mathbf{v}_2(x), & \text{if } x \in \Omega \setminus E, \end{cases}$$

belongs to $\mathcal{DM}^\infty_{loc}(\Omega)$ and

$$\operatorname{div} \mathbf{v} = \chi_{E^1} \operatorname{div} \mathbf{v}_1 + \chi_{E^0} \operatorname{div} \mathbf{v}_2 + [\operatorname{Tr}^+(\mathbf{v}_1, \partial^* E) - \operatorname{Tr}^-(\mathbf{v}_2, \partial^* E)]\, \mathcal{H}^{N-1} \llcorner \partial^* E.$$

Theorem 5.2 (Gluing)**.** *Let $U \Subset \Omega \subset \mathbb{R}^N$ be open sets with $\mathcal{H}^{N-1}(\partial U) < \infty$, and let*

$$\boldsymbol{b}_1 \colon U \times \mathbb{R} \to \mathbb{R}^N, \qquad \boldsymbol{b}_2 \colon (\Omega \setminus \overline{U}) \times \mathbb{R} \to \mathbb{R}^N$$

satisfy assumptions (i)–(iv) in $U \times \mathbb{R}$ and $\Omega \setminus \overline{U}$ respectively. Let $\boldsymbol{B}_1, \boldsymbol{B}_2$ be the corresponding integral functions with respect to the second variable. Given $u_1 \in BV(U) \cap L^\infty(U)$ and

$u_2 \in BV(\Omega \setminus \overline{U}) \cap L^\infty(\Omega \setminus \overline{U})$, *let*

$$\mathbf{v}_1(x) := \begin{cases} \boldsymbol{B}_1(x, u_1(x)), & \text{if } x \in U, \\ 0, & \text{if } x \in \Omega \setminus U, \end{cases} \qquad \mathbf{v}_2(x) := \begin{cases} 0, & \text{if } x \in \overline{U}, \\ \boldsymbol{B}_2(x, u_2(x)), & \text{if } x \in \Omega \setminus \overline{U}. \end{cases}$$

Then $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v} \in \mathcal{DM}^\infty(\Omega)$ *and*

$$\operatorname{div} \mathbf{v} = \chi_{U^1} \operatorname{div} \mathbf{v}_1 + \chi_{U^0} \operatorname{div} \mathbf{v}_2 + [\operatorname{Tr}^+(\mathbf{v}_1, \partial^* U) - \operatorname{Tr}^-(\mathbf{v}_2, \partial^* U)]\, \mathcal{H}^{N-1} \llcorner \partial^* U.$$

6. The Gauss–Green formula

Let $E \Subset \Omega$ be a set of finite perimeter. Using the conventions of Section 3.2, we will assume that the generalized normal vector on $\partial^* E$ coincides $\mathcal{H}^{N-1}$-a.e. on $\partial^* E$ with the measure–theoretic *interior* unit normal vector $\widetilde{\nu}_E$ to E.

We recall that, if $u \in BV_{\rm loc}(\Omega)$, then we will understand $u^\pm(x) = \widetilde{u}(x)$ for every $x \in \Omega \setminus S_u$.

The following result has been proved in [18] in the case $\boldsymbol{B}(x, w) = w\, \boldsymbol{A}(x)$ (see also [15, 28] for related results.)

Theorem 6.1 (Gauss–Green formula)**.** *Let* $\boldsymbol{b}$ *satisfy conditions (i)–(iv) and let* $\boldsymbol{B}$ *be defined by* (8)*. Let* $E \Subset \Omega$ *be a bounded set with finite perimeter and let* $u \in BV_{loc}(\Omega) \cap L^\infty_{loc}(\Omega)$*. Then the following Gauss–Green formulas hold:*

$$\int_{E^1} \frac{F(x, u^+(x)) + F(x, u^-(x))}{2}\, d\sigma(x) + \mu(E^1) = -\int_{\partial^* E} \beta^+(x, u^+(x))\, d\mathcal{H}^{N-1}, \tag{48}$$

$$\begin{aligned} &\int_{E^1 \cup \partial^* E} \frac{F(x, u^+(x)) + F(x, u^-(x))}{2}\, d\sigma(x) + \mu(E^1 \cup \partial^* E) \\ &\qquad\qquad = -\int_{\partial^* E} \beta^-(x, u^-(x))\, d\mathcal{H}^{N-1}, \end{aligned} \tag{49}$$

where E^1 *is the measure theoretic interior of* E*,* μ *is the Radon measure introduced in* (29)*, and* $\beta^\pm(\cdot, t) := \operatorname{Tr}^\pm(\boldsymbol{b}(\cdot, t), \partial^* E)$ *are the normal traces of* $\boldsymbol{b}(\cdot, t)$ *when* $\partial^* E$ *is oriented with respect to the interior unit normal vector.*

Proof. Since all results are of local nature in the space variables, it is not restrictive to assume that $\Omega = \mathbb{R}^N$, $\boldsymbol{b}$ is a bounded Borel function, and $\boldsymbol{b}(\cdot, t) \in \mathcal{DM}^\infty$ for every $t \in \mathbb{R}$.

Moreover, since E is bounded we can assume, without loss of generality, that $u \in BV(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$. By Theorem 4.3, the composite function $\mathbf{v}(x) := \boldsymbol{B}(x, u(x))$ belongs to $\mathcal{DM}^\infty$. Since E is a bounded set of finite perimeter, the characteristic function χ_E is a compactly supported BV function, so that $\operatorname{div}(\chi_E \mathbf{v})(\mathbb{R}^N) = 0$ (see [15, Lemma 3.1]).

We recall that, for every $w \in BV$ and every $\boldsymbol{A} \in \mathcal{DM}^\infty$, it holds

$$\operatorname{div}(w\boldsymbol{A}) = w^* \operatorname{div} \boldsymbol{A} + (\boldsymbol{A}, Dw),$$

where $(\boldsymbol{A}, Dw)$ is the Anzellotti pairing between the function w and the vector field $\boldsymbol{A}$ (see (6)). Hence, using the above formula with $w = \chi_E$ and $\boldsymbol{A} = \mathbf{v}$, it follows that

$$0 = \operatorname{div}(\chi_E \mathbf{v})(\mathbb{R}^N) = \int_{\mathbb{R}^N} \chi_E^*\, d \operatorname{div} \mathbf{v} + (\mathbf{v}, D\chi_E)(\mathbb{R}^N). \tag{50}$$

Since

$$(\mathbf{v}, D\chi_E) = [\operatorname{Tr}^+(\mathbf{v}, \partial^* E) - \operatorname{Tr}^-(\mathbf{v}, \partial^* E)]\mathcal{H}^{N-1} \llcorner \partial^* E,$$

from Proposition 4.5 we get

$$(\mathbf{v}, D\chi_E)(\mathbb{R}^N) = \int_{\partial^* E} [\beta^+(x, u^+(x)) - \beta^-(x, u^-(x))]\, d\mathcal{H}^{N-1}(x). \tag{51}$$

Since $\chi_{E^*} = \chi_{E^1} + \frac{1}{2}\chi_{\partial^* E}$, using again Proposition 4.5 and (29) it holds

$$\begin{aligned}\int_{\mathbb{R}^N} \chi_E^*\, d\operatorname{div}\mathbf{v} = {} & \operatorname{div}\mathbf{v}(E^1) + \frac{1}{2}\int_{\partial^* E} \frac{[\beta^+(x, u^+(x)) + \beta^-(x, u^-(x)}{2}\, d\mathcal{H}^{N-1}(x) \\ = {} & \int_{E^1} \frac{F(x, u^+(x)) + F(x, u^-(x))}{2}\, d\sigma(x) + \mu(E^1) \\ & + \frac{1}{2}\int_{\partial^* E} \frac{[\beta^+(x, u^+(x)) + \beta^-(x, u^-(x)}{2}\, d\mathcal{H}^{N-1}(x)\,.\end{aligned} \tag{52}$$

Formula (48) now follows from (50), (51) and (52).

The proof of (49) is entirely similar. □

It is worth to mention a consequence of the gluing construction given in Theorem 5.2 and the Gauss–Green formula (48). To this end, following [29], any bounded open set $\Omega \subset \mathbb{R}^N$ with finite perimeter, such that $\mathcal{H}^{N-1}(\partial\Omega) = \mathcal{H}^{N-1}(\partial^*\Omega)$, will be called *weakly regular*. For weakly regular sets we have the following version of the Gauss–Green formula (see [15, Corollary 5.5] for a similar statement for autonomous vector fields).

Theorem 6.2 (Gauss–Green formula for weakly regular sets)**.** *Let $\Omega \subset \mathbb{R}^N$ be a weakly regular set. Let $\boldsymbol{b}$ satisfy conditions (i)–(iv), let $\boldsymbol{B}$ be defined by* (8) *and let $u \in BV(\Omega) \cap L^\infty(\Omega)$. Then the following Gauss–Green formula holds:*

$$\int_\Omega \frac{F(x, u^+(x)) + F(x, u^-(x))}{2}\, d\sigma(x) + \mu(\Omega) = -\int_{\partial\Omega} \beta^+(x, u^+(x))\; d\mathcal{H}^{N-1}\,. \tag{53}$$

Proof. From Federer's structure theorem (see [3, Theorem 3.61]) we have that $\mathbb{R}^N = \Omega^1 \cup \Omega^0 \cup \partial^*\Omega \cup Z$, with $\mathcal{H}^{N-1}(Z) = 0$. Since Ω is open, we also have $\Omega^1 \supseteq \Omega$, $\Omega^0 \supseteq \mathbb{R}^N \setminus \overline{\Omega}$, hence

$$\partial\Omega = \mathbb{R}^N \setminus [\Omega \cup (\mathbb{R}^N \setminus \overline{\Omega})] \supseteq \partial^*\Omega \cup Z.$$

Since Ω is weakly regular, this inclusion implies that $\mathcal{H}^{N-1}(\partial\Omega \setminus \partial^*\Omega) = 0$ and, in particular,

$$\mathcal{H}^{N-1} \llcorner \partial\Omega = \mathcal{H}^{N-1} \llcorner \partial^*\Omega, \quad \mathcal{H}^{N-1}(\Omega^1 \setminus \Omega) = 0, \quad \mathcal{H}^{N-1}(\Omega^0 \setminus (\mathbb{R}^N \setminus \overline{\Omega}) = 0. \tag{54}$$

Let us consider the vector field

$$\mathbf{v}(x) := \begin{cases} \mathbf{v}_1(x) := \boldsymbol{B}(x, u(x)), & \text{if } x \in \Omega, \\ 0, & \text{if } x \in \mathbb{R}^N \setminus \Omega\,. \end{cases}$$

By Theorem 5.2 and (54) we have that $\mathbf{v} \in \mathcal{DM}^\infty$ and

$$\operatorname{div}\mathbf{v} = \chi_\Omega \operatorname{div}\mathbf{v}_1 + \operatorname{Tr}^+(\mathbf{v}_1, \partial\Omega)\, \mathcal{H}^{N-1} \llcorner \partial\Omega,$$

hence (53) follows reasoning as in the proof of (48). □

References

[1] L. Ambrosio, G. Crippa, and S. Maniglia, *Traces and fine properties of a BD class of vector fields and applications*, Ann. Fac. Sci. Toulouse Math. (6) **14** (2005), no. 4, 527–561. MR2188582

[2] L. Ambrosio, C. De Lellis, and J. Malý, *On the chain rule for the divergence of BV-like vector fields: applications, partial results, open problems*, Perspectives in nonlinear partial differential equations, 2007, pp. 31–67. MR2373724

[3] L. Ambrosio, N. Fusco, and D. Pallara, *Functions of bounded variation and free discontinuity problems*, Oxford Mathematical Monographs, The Clarendon Press Oxford University Press, New York, 2000. MR1857292 (2003a:49002)

[4] F. Andreu-Vaillo, V. Caselles, and J.M. Mazón, *Parabolic quasilinear equations minimizing linear growth functionals*, Progress in Mathematics, vol. 223, Birkhäuser Verlag, Basel, 2004. MR2033382

[5] G. Anzellotti, *Pairings between measures and bounded functions and compensated compactness*, Ann. Mat. Pura Appl. (4) **135** (1983), 293–318 (1984). MR750538

[6] ———, *Traces of bounded vector–fields and the divergence theorem*, 1983. Unpublished preprint.

[7] G. Bouchitté and G. Dal Maso, *Integral representation and relaxation of convex local functionals on* BV(Ω), Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4) **20** (1993), no. 4, 483–533. MR1267597

[8] M. Carriero, G. Dal Maso, A. Leaci, and E. Pascali, *Relaxation of the nonparametric plateau problem with an obstacle*, J. Math. Pures Appl. (9) **67** (1988), no. 4, 359–396. MR978576

[9] V. Caselles, *On the entropy conditions for some flux limited diffusion equations*, J. Differential Equations **250** (2011), no. 8, 3311–3348. MR2772392

[10] G.-Q. Chen and H. Frid, *Divergence-measure fields and hyperbolic conservation laws*, Arch. Ration. Mech. Anal. **147** (1999), no. 2, 89–118. MR1702637

[11] ———, *Extended divergence-measure fields and the Euler equations for gas dynamics*, Comm. Math. Phys. **236** (2003), no. 2, 251–280. MR1981992

[12] G.-Q. Chen and M. Torres, *Divergence-measure fields, sets of finite perimeter, and conservation laws*, Arch. Ration. Mech. Anal. **175** (2005), no. 2, 245–267. MR2118477

[13] ———, *On the structure of solutions of nonlinear hyperbolic systems of conservation laws*, Commun. Pure Appl. Anal. **10** (2011), no. 4, 1011–1036. MR2787432 (2012c:35263)

[14] G.-Q. Chen, M. Torres, and W.P. Ziemer, *Gauss-Green theorem for weakly differentiable vector fields, sets of finite perimeter, and balance laws*, Comm. Pure Appl. Math. **62** (2009), no. 2, 242–304. MR2468610

[15] G.E. Comi and K.R. Payne, *On locally essentially bounded divergence measure fields and sets of locally finite perimeter*, 2017. Preprint.

[16] G.E. Comi and M. Torres, *One–sided approximation of sets of finite perimeter*, Atti Accad. Naz. Lincei Rend. Lincei Mat. Appl. **28** (2017), no. 1, 181–190.

[17] G. Crasta and V. De Cicco, *A chain rule formula in the space BV and applications to conservation laws*, SIAM J. Math. Anal. **43** (2011), no. 1, 430–456. MR2765698

[18] ———, *Anzellotti's pairing theory and the Gauss–Green theorem*, 2017. Preprint.

[19] ———, *On the chain rule formulas for divergences and applications to conservation laws*, Nonlinear Anal. **153** (2017), 275–293. MR3614672

[20] G. Crasta, V. De Cicco, and G. De Philippis, *Kinetic formulation and uniqueness for scalar conservation laws with discontinuous flux*, Comm. Partial Differential Equations **40** (2015), no. 4, 694–726. MR3299353

[21] G. Crasta, V. De Cicco, G. De Philippis, and F. Ghiraldin, *Structure of solutions of multidimensional conservation laws with discontinuous flux and applications to uniqueness*, Arch. Ration. Mech. Anal. **221** (2016), no. 2, 961–985. MR3488541

[22] V. De Cicco, N. Fusco, and A. Verde, *On L^1-lower semicontinuity in BV*, J. Convex Anal. **12** (2005), no. 1, 173–185. MR2135805 (2005m:49023)

[23] ———, *A chain rule formula in BV and application to lower semicontinuity*, Calc. Var. Partial Differential Equations **28** (2007), no. 4, 427–447. MR2293980 (2007j:49016)

[24] V. De Cicco and G. Leoni, *A chain rule in $L^1(\mathrm{div};\Omega)$ and its applications to lower semicontinuity*, Calc. Var. Partial Differential Equations **19** (2004), no. 1, 23–51. MR2027846 (2005c:49030)

[25] I. Fonseca and G. Leoni, *On lower semicontinuity and relaxation*, Proc. Roy. Soc. Edinburgh Sect. A **131** (2001), no. 3, 519–565. MR1838501

[26] ———, *Modern methods in the calculus of variations: L^p spaces*, Springer Monographs in Mathematics, Springer, New York, 2007. MR2341508

[27] B. Kawohl, *On a family of torsional creep problems*, J. Reine Angew. Math. **410** (1990), 1–22.
[28] G.P. Leonardi and G. Saracco, *The prescribed mean curvature equation in weakly regular domains*, 2017. Preprint.
[29] ———, *Rigidity and trace properties of divergence-measure vector fields*, 2017. Preprint.
[30] C. Scheven and T. Schmidt, *BV supersolutions to equations of 1-Laplace and minimal surface type*, J. Differential Equations **261** (2016), no. 3, 1904–1932. MR3501836
[31] ———, *An Anzellotti type pairing for divergence-measure fields and a notion of weakly super-1-harmonic functions*, 2017. Preprint.
[32] W.P. Ziemer, *Weakly differentiable functions*, Springer–Verlag, New York, 1989.

Dipartimento di Matematica "G. Castelnuovo", Univ. di Roma I, P.le A. Moro 2 – I-00185 Roma (Italy)
E-mail address: crasta@mat.uniroma1.it

Dipartimento di Scienze di Base e Applicate per l'Ingegneria, Univ. di Roma I, Via A. Scarpa 10 – I-00185 Roma (Italy)
E-mail address: virginia.decicco@sbai.uniroma1.it