AN EXTENSION OF THE PAIRING THEORY BETWEEN DIVERGENCE–MEASURE FIELDS AND BV FUNCTIONS

GRAZIANO CRASTA AND VIRGINIA DE CICCO

Abstract. In this paper we introduce a nonlinear version of the notion of Anzellotti's pairing between divergence–measure vector fields and functions of bounded variation, motivated by possible applications to evolutionary quasilinear problems. As a consequence of our analysis, we prove a generalized Gauss–Green formula.

1. Introduction

In recent years the pairing theory between divergence–measure vector fields and BV functions, initially developed by Anzellotti [6, 7], has been extended to more general situations (see e.g. [10–15, 18, 27, 29, 30] and the references therein). These extensions are motivated, among others, by applications to hyperbolic conservation laws and transport equations [2, 3, 10, 12–14, 19], problems involving the 1-Laplace operator [5, 26], the prescribed mean curvature problem [27, 28] and to lower semicontinuity problems in BV [8, 22, 23].

Another major related result concerns the Gauss–Green formula and its applications (see e.g. [6, 9, 15, 18, 27]).

Let us describe the problem in more details. Let $\mathcal{DM}^\infty$ denote the class of bounded divergence–measure vector fields $\boldsymbol{A}\colon \mathbb{R}^N \to \mathbb{R}^N$, i.e. the vector fields with the properties that $\boldsymbol{A}$ is bounded and $\operatorname{div} \boldsymbol{A}$ is a finite Radon measure. If $\boldsymbol{A} \in \mathcal{DM}^\infty$ and u is a function of bounded variation with precise representative u^*, then Chen and Frid [10] proved that

$$\operatorname{div}(u\boldsymbol{A}) = u^* \operatorname{div} \boldsymbol{A} + \mu,$$

where μ is a Radon measure, absolutely continuous with respect to $|Du|$. This measure μ has been denoted by Anzellotti [6] with the symbol $(\boldsymbol{A}, Du)$ and by Chen and Frid with $\overline{\boldsymbol{A} \cdot Du}$, and it is called the pairing between the divergence–measure field $\boldsymbol{A}$ and the gradient of the BV function u. The characterization of the decomposition of this measure into absolutely continuous, Cantor and jump parts has been studied in [10, 18]. In particular, the analysis of the jump part has been considered in [18] using the notion of weak normal traces of a divergence–measure vector field on oriented countably $\mathcal{H}^{N-1}$-rectifiable sets given in [2].

In view to applications to evolutionary quasilinear problems, our aim is to extend the pairing theory from the product $u\boldsymbol{A}$ to the mixed case $\boldsymbol{B}(x,u)$. Our main assumptions on $\boldsymbol{B}$ are that $\boldsymbol{B}(\cdot, w) \in \mathcal{DM}^\infty$ for every $w \in \mathbb{R}$, $\boldsymbol{B}(x,\cdot)$ is of class C^1 and the least upper bound

$$\sigma := \bigvee_{w\in\mathbb{R}} |\operatorname{div}_x \partial_w \boldsymbol{B}(\cdot, w)|$$

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is a Radon measure. (See Section 3.1 for the complete list of assumptions.) We remark that, in the case $\boldsymbol{B}(x,w) = w\,\boldsymbol{A}(x)$, with $\boldsymbol{A} \in \mathcal{DM}^\infty$, these assumptions are automatically satisfied with $\sigma = |\operatorname{div}\boldsymbol{A}|$. More precisely, we will prove that, if $u \in BV \cap L^\infty$, then the composite function $\mathbf{v}(x) := \boldsymbol{B}(x, u(x))$ belongs to $\mathcal{DM}^\infty$ and

$$\operatorname{div}[\boldsymbol{B}(x,u(x))] = \frac{1}{2}\left[F(x,u^+(x)) + F(x,u^-(x))\right]\,\sigma + \mu,$$

in the sense of measures. Here $F(\cdot,w)$ denotes the Radon–Nikodým derivative of the measure $\operatorname{div}_x \boldsymbol{B}(\cdot,w)$ with respect to σ, $u^\pm$ are the approximate limits of u and $\mu \equiv (\partial_w \boldsymbol{B}(\cdot,u), Du)$ is again a Radon measure, absolutely continuous with respect to $|Du|$ (see Theorem 4.3). We recall that the analogous chain rule for BV vector fields has been proved in [1]. Notice that, when $\boldsymbol{B}(x,w) = w\,\boldsymbol{A}(x)$, then $\mu = (\boldsymbol{A}, Du)$ is exactly the Anzellotti's pairing between $\boldsymbol{A}$ and Du. Even for general vector fields $\boldsymbol{B}(x,w)$ we can prove the following characterization of the decomposition $\mu = \mu^{ac} + \mu^c + \mu^j$ of this measure into absolutely continuous, Cantor and jump parts (see Theorem 4.6):

$$\begin{aligned}
\mu^{ac} &= \langle \partial_w \boldsymbol{B}(x,u(x))\,,\, \nabla u(x)\rangle\,\mathcal{L}^N,\\
\mu^c &= \left\langle \widetilde{\partial_w \boldsymbol{B}}(x,\widetilde{u}(x))\,,\, D^c u\right\rangle,\\
\mu^j &= [\beta^*(x,u^+(x)) - \beta^*(x,u^-(x))]\mathcal{H}^{N-1} \llcorner J_u,
\end{aligned}$$

where, for every $w \in \mathbb{R}$, $\beta^\pm(\cdot,w)$ are the normal traces of $\boldsymbol{B}(\cdot,w)$ on J_u and $\beta^*(\cdot,w) := [\beta^+(\cdot,w) + \beta^-(\cdot,w)]/2$, and μ^c is the Cantor part of the measure. We remark that, to prove the representation formula for μ^c, we need an additional technical assumption (see (31) in Theorem 4.6).

We recall that a similar characterization of $\operatorname{div}[\boldsymbol{B}(x,u(x))]$ has been proved in [19] under stronger assumptions, including the existence of the strong traces of $\boldsymbol{B}(\cdot,w)$.

As a consequence of our analysis we prove that, if $E \subset \mathbb{R}^N$ is a bounded set with finite perimeter, then the following Gauss–Green formula holds:

$$\int_{E^1} \frac{F(x,u^+(x)) + F(x,u^-(x))}{2}\,d\sigma(x) + \mu(E^1) = -\int_{\partial^* E} \beta^+(x,u^+(x))\,d\mathcal{H}^{N-1},$$

where E^1 is the measure theoretic interior of E, and $\partial^* E$ is the reduced boundary of E (see Theorem 6.1). In the particular case $\boldsymbol{B}(x,w) = w\,\boldsymbol{A}(x)$, the analogous formula has been proved in [18, Theorem 5.1]. We recall that the map $x \mapsto \beta^+(x,u^+(x))$ in the last integral coincides with the interior weak normal trace of the vector field $x \mapsto \boldsymbol{B}(x,u(x))$ on $\partial^* E$ (see Proposition 4.5).

The plan of the paper is the following. In Section 2 we recall some known results on functions of bounded variation, divergence–measure vector fields and their normal traces, and the Anzellotti's pairing.

In Section 3 we list the assumptions on $\boldsymbol{B}$ and we prove a number of its basic properties, including some regularity result of the weak normal traces. Finally, we prove that for every $u \in BV \cap L^\infty$, the composite function $x \mapsto \boldsymbol{B}(x,u(x))$ belongs to $\mathcal{DM}^\infty$.

In Section 4 we prove our main results on the pairing measure μ and its properties.

Finally, in Section 5 we describe some gluing construction and we prove an extension theorem that will be used in Section 6 to prove a Gauss–Green formula for weakly regular domains.

2. Preliminaries

In the following, Ω will always denote a nonempty open subset of $\mathbb{R}^N$.

Let $u \in L^1_{\rm loc}(\Omega)$. We say that u has an approximate limit at $x_0 \in \Omega$ if there exists $z \in \mathbb{R}$ such that

$$\lim_{r\to 0^+} \frac{1}{\mathcal{L}^N\left(B_r(x_0)\right)} \int_{B_r(x_0)} |u(x) - z|\, dx = 0. \tag{1}$$

The set $S_u \subset \Omega$ of points where this property does not hold is called the approximate discontinuity set of u. For every $x_0 \in \Omega \setminus S_u$, the number z, uniquely determined by (1), is called the approximate limit of u at x_0 and denoted by $\widetilde{u}(x_0)$.

We say that $x_0 \in \Omega$ is an approximate jump point of u if there exist $a, b \in \mathbb{R}$ and a unit vector $\nu \in \mathbb{R}^n$ such that $a \neq b$ and

$$\begin{gathered} \lim_{r\to 0^+} \frac{1}{\mathcal{L}^N(B_r^+(x_0))} \int_{B_r^+(x_0)} |u(y) - a|\, dy = 0, \\ \lim_{r\to 0^+} \frac{1}{\mathcal{L}^N(B_r^-(x_0))} \int_{B_r^-(x_0)} |u(y) - b|\, dy = 0, \end{gathered} \tag{2}$$

where $B_r^\pm(x_0) := \{y \in B_r(x_0) : \pm(y - x_0)\cdot \nu > 0\}$. The triplet (a, b, ν), uniquely determined by (2) up to a permutation of (a, b) and a change of sign of ν, is denoted by $(u^+(x_0), u^-(x_0), \nu_u(x_0))$. The set of approximate jump points of u will be denoted by J_u.

The notions of approximate discontinuity set, approximate limit and approximate jump point can be obviously extended to the vectorial case (see [4, §3.6]).

In the following we shall always extend the functions $u^\pm$ to $\Omega \setminus (S_u \setminus J_u)$ by setting

$$u^\pm \equiv \widetilde{u} \text{ in } \Omega \setminus S_u.$$

Definition 2.1 (Strong traces)**.** *Let $u \in L^\infty_{\rm loc}(\mathbb{R}^N)$ and let $\mathcal{J} \subset \mathbb{R}^N$ be a countably $\mathcal{H}^{N-1}$-rectifiable set oriented by a normal vector field ν. We say that two Borel functions $u^\pm \colon \mathcal{J} \to \mathbb{R}$ are the strong traces of u on $\mathcal{J}$ if for $\mathcal{H}^{N-1}$-almost every $x \in \mathcal{J}$ it holds*

$$\lim_{r\to 0^+} \int_{B_r^\pm(x)} |u(y) - u^\pm(x)|\, dy = 0,$$

where $B_r^\pm(x) := B_r(x) \cap \{y \in \mathbb{R}^N : \pm\langle y - x, \nu(x)\rangle \geq 0\}$.

Here and in the following we will denote by $\rho \in C_c^\infty(\mathbb{R}^N)$ a symmetric convolution kernel with support in the unit ball, and by $\rho_\varepsilon(x) := \varepsilon^{-N}\rho(x/\varepsilon)$.

In the sequel we will use often the following result.

Proposition 2.2. *Let $u \in L^1_{\rm loc}(\Omega)$ and define*

$$u_\varepsilon(x) = \rho_\varepsilon * u(x) := \int_\Omega \rho_\varepsilon(x - y)\, u(y)\, dy.$$

If $x_0 \in \Omega \setminus S_u$, then $u_\varepsilon(x_0) \to \widetilde{u}(x_0)$ as $\varepsilon \to 0^+$.

Proposition 2.3. *Let E be a Lebesgue measurable subset of $\mathbb{R}^N$ and G a Borel subset of $\mathbb{R}^M$. Let $g\colon E \times G \to \mathbb{R}$ be a Borel function such that for $\mathcal{L}^N$–a.e. $x \in E$ the function $g(x, \cdot)$ is continuous on G. Then there exists an $\mathcal{L}^N$-null set $\mathcal{M} \subset \mathbb{R}^N$ such that for every $t \in G$ the function $g(\cdot, t)$ is approximately continuous in $E \setminus \mathcal{M}$.*

Proof. (See the proof of Theorem 1.3, p. 539 in [24].) By the Scorza–Dragoni theorem (see [25, Theorem 6.35]), for every $i \in \mathbb{Z}^+$ there exists a closed set $K_i \subset E$, with $\mathcal{L}^N(E \setminus K_i) <$

$1/i$, such that the restriction of g to $K_i \times G$ is continuous. Let K_i^* be the set of points with density 1 of K_i, and define

$$\mathcal{E} := \bigcup_{i=1}^{\infty} (K_i \cap K_i^*).$$

Since $\mathcal{L}^N(K_i) = \mathcal{L}^N(K_i \cap K_i^*)$, it holds

$$\mathcal{L}^N(E \setminus \mathcal{E}) \leq \mathcal{L}^N(E \setminus K_i) \leq \frac{1}{i} \qquad \forall i \in \mathbb{Z}^+,$$

and so $\mathcal{L}^N(E \setminus \mathcal{E}) = 0$.

Let us fix $t \in G$ and let us prove that the function $u(x) := g(x,t)$ is approximately continuous on $\mathcal{E}$. Let $\varepsilon > 0$ and $x_0 \in \mathcal{E}$. By definition of $\mathcal{E}$, there exists an index $i \in \mathbb{Z}^+$ such that $x_0 \in K_i \cap K_i^*$. Since the restriction of u to K_i is continuous, there exists $\delta > 0$ such that

$$|u(x) - u(x_0)| < \varepsilon \qquad \forall x \in B_\delta(x_0) \cap K_i.$$

As a consequence, for every $r \in (0, \delta)$, it holds

$$\mathcal{L}^N(B_r(x_0) \cap \{x \in E : \ |u(x) - u(x_0)| \geq \varepsilon\}) \leq \mathcal{L}^N(B_r(x_0) \setminus K_i),$$

hence the conclusion follows since x_0 is a Lebesgue point of K_i. □

Corollary 2.4. *Let E be a Lebesgue measurable subset of $\mathbb{R}^N$ and G a Borel subset of $\mathbb{R}^M$. Let $g\colon E \times G \to \mathbb{R}$ be a Borel function such that for $\mathcal{L}^N$–a.e. $x \in E$ the function $g(x, \cdot)$ is continuous on G. Let us define*

$$g_\varepsilon(x,t) := \int_{\mathbb{R}^N} \rho_\varepsilon(x-y)\, g(y,t)\, dy.$$

Then there exists an $\mathcal{L}^N$-null set $Z \subset \mathbb{R}^N$ such that for every $t \in G$ and for every $x \in E \setminus Z$ we have $g_\varepsilon(x_0, t) \to g(x_0, t)$, as $\varepsilon \to 0^+$.

2.1. **Functions of bounded variation and sets of finite perimeter.** We say that $u \in L^1(\Omega)$ is a function of bounded variation in Ω if the distributional derivative Du of u is a finite Radon measure in Ω. The vector space of all functions of bounded variation in Ω will be denoted by $BV(\Omega)$. Moreover, we will denote by $BV_{\mathrm{loc}}(\Omega)$ the set of functions $u \in L^1_{\mathrm{loc}}(\Omega)$ that belong to $BV(A)$ for every open set $A \Subset \Omega$ (i.e., the closure $\overline{A}$ of A is a compact subset of Ω).

If $u \in BV(\Omega)$, then Du can be decomposed as the sum of the absolutely continuous and the singular part with respect to the Lebesgue measure, i.e.

$$Du = D^a u + D^s u, \qquad D^a u = \nabla u\, \mathcal{L}^N,$$

where ∇u is the approximate gradient of u, defined $\mathcal{L}^N$-a.e. in Ω. On the other hand, the singular part $D^s u$ can be further decomposed as the sum of its Cantor and jump part, i.e.

$$D^s u = D^c u + D^j u, \qquad D^c u := D^s u \llcorner (\Omega \setminus S_u), \quad D^j u := D^s u \llcorner J_u,$$

where the symbol $\mu \llcorner B$ denotes the restriction of the measure μ to the set B. We will denote by $D^d u := D^a u + D^c u$ the diffuse part of the measure Du.

In the following, we will denote by $\theta_u\colon \Omega \to S^{N-1}$ the Radon–Nikodým derivative of Du with respect to $|Du|$, i.e. the unique function $\theta_u \in L^1(\Omega, |Du|)^N$ such that the polar decomposition $Du = \theta_u\, |Du|$ holds. Since all parts of the derivative of u are mutually singular, we have

$$D^a u = \theta_u\, |D^a u|, \quad D^j u = \theta_u\, |D^j u|, \quad D^c u = \theta_u\, |D^c u|$$

as well. In particular $\theta_u(x) = \nabla u(x)/|\nabla u(x)|$ for $\mathcal{L}^N$-a.e. $x \in \Omega$ such that $\nabla u(x) \neq 0$ and $\theta_u(x) = \mathrm{sign}(u^+(x) - u^-(x))\, \nu_u(x)$ for $\mathcal{H}^{N-1}$-a.e. $x \in J_u$.

Let E be an $\mathcal{L}^N$-measurable subset of $\mathbb{R}^N$. For every open set $\Omega \subset \mathbb{R}^N$ the perimeter $P(E, \Omega)$ is defined by

$$P(E,\Omega) := \sup \left\{ \int_E \operatorname{div} \varphi \, dx : \ \varphi \in C^1_c(\Omega, \mathbb{R}^N), \ \|\varphi\|_\infty \leq 1 \right\}.$$

We say that E is of finite perimeter in Ω if $P(E, \Omega) < +\infty$.

Denoting by χ_E the characteristic function of E, if E is a set of finite perimeter in Ω, then $D\chi_E$ is a finite Radon measure in Ω and $P(E, \Omega) = |D\chi_E|(\Omega)$.

If $\Omega \subset \mathbb{R}^N$ is the largest open set such that E is locally of finite perimeter in Ω, we call reduced boundary $\partial^* E$ of E the set of all points $x \in \Omega$ in the support of $|D\chi_E|$ such that the limit

$$\widetilde{\nu}_E(x) := \lim_{\rho \to 0^+} \frac{D\chi_E(B_\rho(x))}{|D\chi_E|(B_\rho(x))}$$

exists in $\mathbb{R}^N$ and satisfies $|\widetilde{\nu}_E(x)| = 1$. The function $\widetilde{\nu}_E \colon \partial^* E \to S^{N-1}$ is called the measure theoretic unit interior normal to E.

A fundamental result of De Giorgi (see [4, Theorem 3.59]) states that $\partial^* E$ is countably $(N-1)$-rectifiable and $|D\chi_E| = \mathcal{H}^{N-1} \llcorner \partial^* E$.

Let E be an $\mathcal{L}^N$-measurable subset of $\mathbb{R}^N$. For every $t \in [0, 1]$ we denote by E^t the set

$$E^t := \left\{ x \in \mathbb{R}^N : \ \lim_{\rho \to 0^+} \frac{\mathcal{L}^N(E \cap B_\rho(x))}{\mathcal{L}^N(B_\rho(x))} = t \right\}$$

of all points where E has density t. The sets E^0, E^1, $\partial^e E := \mathbb{R}^N \setminus (E^0 \cup E^1)$ are called respectively the measure theoretic exterior, the measure theoretic interior and the essential boundary of E. If E has finite perimeter in Ω, Federer's structure theorem states that $\partial^* E \cap \Omega \subset E^{1/2} \subset \partial^e E$ and $\mathcal{H}^{N-1}(\Omega \setminus (E^0 \cup \partial^e E \cup E^1)) = 0$ (see [4, Theorem 3.61]).

2.2. **Divergence–measure fields.** We will denote by $\mathcal{DM}^\infty(\Omega)$ the space of all vector fields $\boldsymbol{A} \in L^\infty(\Omega, \mathbb{R}^N)$ whose divergence in the sense of distribution is a bounded Radon measure in Ω. Similarly, $\mathcal{DM}^\infty_{\mathrm{loc}}(\Omega)$ will denote the space of all vector fields $\boldsymbol{A} \in L^\infty_{\mathrm{loc}}(\Omega, \mathbb{R}^N)$ whose divergence in the sense of distribution is a Radon measure in Ω. We set $\mathcal{DM}^\infty = \mathcal{DM}^\infty(\mathbb{R}^N)$.

We recall that, if $\boldsymbol{A} \in \mathcal{DM}^\infty_{\mathrm{loc}}(\Omega)$, then $|\operatorname{div} \boldsymbol{A}| \ll \mathcal{H}^{N-1}$ (see [10, Proposition 3.1]). As a consequence, the set

$$\Theta_{\boldsymbol{A}} := \left\{ x \in \Omega : \ \limsup_{r \to 0+} \frac{|\operatorname{div} \boldsymbol{A}|(B_r(x))}{r^{N-1}} > 0 \right\}, \tag{3}$$

is a Borel set, σ-finite with respect to $\mathcal{H}^{N-1}$, and the measure $\operatorname{div} \boldsymbol{A}$ can be decomposed as

$$\operatorname{div} \boldsymbol{A} = \operatorname{div}^a \boldsymbol{A} + \operatorname{div}^c \boldsymbol{A} + \operatorname{div}^j \boldsymbol{A},$$

where $\operatorname{div}^a \boldsymbol{A}$ is absolutely continuous with respect to $\mathcal{L}^N$, $\operatorname{div}^c \boldsymbol{A}(B) = 0$ for every set B with $\mathcal{H}^{N-1}(B) < +\infty$, and

$$\operatorname{div}^j \boldsymbol{A} = h \, \mathcal{H}^{N-1} \llcorner \Theta_{\boldsymbol{A}}$$

for some Borel function h (see [3, Proposition 2.5]).

2.3. **Anzellotti's pairing.** As in Anzellotti [6] (see also [10]), for every $\boldsymbol{A} \in \mathcal{DM}^\infty_{\rm loc}(\Omega)$ and $u \in BV_{\rm loc}(\Omega) \cap L^\infty_{\rm loc}(\Omega)$ we define the linear functional $(\boldsymbol{A}, Du)\colon C^\infty_0(\Omega) \to \mathbb{R}$ by

$$\langle (\boldsymbol{A}, Du)\, ,\, \varphi\rangle := -\int_\Omega u^* \varphi \, d \operatorname{div} \boldsymbol{A} - \int_\Omega u \, \boldsymbol{A} \cdot \nabla\varphi \, dx. \tag{4}$$

The distribution $(\boldsymbol{A}, Du)$ is a Radon measure in Ω, absolutely continuous with respect to $|Du|$ (see [6, Theorem 1.5] and [10, Theorem 3.2]), hence the equation

$$\operatorname{div}(u\boldsymbol{A}) = u^* \operatorname{div} \boldsymbol{A} + (\boldsymbol{A}, Du) \tag{5}$$

holds in the sense of measures in Ω. (We remark that, in [10], the measure $(\boldsymbol{A}, Du)$ is denoted by $\overline{\boldsymbol{A} \cdot Du}$.) Furthermore, Chen and Frid in [10] proved that the absolutely continuous part of this measure with respect to the Lebesgue measure is given by $(\boldsymbol{A}, Du)^a = \boldsymbol{A} \cdot \nabla u \, \mathcal{L}^N$.

3. Assumptions on the vector field and preliminary results

As we have explained in the Introduction, we are willing to compute the divergence of the composite function $\mathbf{v}(x) := \boldsymbol{B}(x, u(x))$ with $u \in BV$, where $\boldsymbol{B}(\cdot, t) \in \mathcal{DM}^\infty$ and $\boldsymbol{B}(x, \cdot) \in C^1$. Nevertheless, it will be convenient to state our assumptions on the vector field $\boldsymbol{b}(x, t) := \partial_t \boldsymbol{B}(x, t)$.

In Section 3.1 we list the assumptions on $\boldsymbol{b}$ and we prove a number of basic properties of $\boldsymbol{b}$ and $\boldsymbol{B}$.

Then, in Section 3.2 we prove some regularity result of the weak normal traces of $\boldsymbol{B}$.

Finally, in Section 3.3, we prove that $\mathbf{v} \in \mathcal{DM}^\infty$.

3.1. **Assumptions on the vector field $\boldsymbol{B}$.** In this section we list and comment all the assumptions on the vector field $\boldsymbol{B}(x, t)$.

Let $\Omega \subset \mathbb{R}^N$ be a non-empty open set. Let $\boldsymbol{b}\colon \Omega \times \mathbb{R} \to \mathbb{R}^N$ be a function satisfying the following assumptions:

(i) $\boldsymbol{b}$ is a locally bounded Borel function;
(ii) for $\mathcal{L}^N$–a.e. $x \in \Omega$, the function $\boldsymbol{b}(x, \cdot)$ is continuous in $\mathbb{R}$;
(iii) for every $t \in \mathbb{R}$, $\boldsymbol{b}(\cdot, t) \in \mathcal{DM}^\infty_{\rm loc}(\Omega)$;
(iv) the least upper bound

$$\sigma := \bigvee_{t\in\mathbb{R}} |\operatorname{div}_x \boldsymbol{b}(\cdot, t)|$$

is a Radon measure. (See [4, Def. 1.68] for the definition of least upper bound of measures.)

We remark that, since $\operatorname{div}_x \boldsymbol{b}(\cdot, t) \ll \mathcal{H}^{N-1}$ for every $t \in \mathbb{R}$, then also $\sigma \ll \mathcal{H}^{N-1}$.

From (i) and Proposition 2.3 it follows that there exists a set $Z_1 \subset \mathbb{R}^N$ such that $\mathcal{L}^N(Z_1) = 0$ and, for every $t \in \mathbb{R}$, the function $x \mapsto \boldsymbol{b}(x, t)$ is approximately continuous on $\mathbb{R}^N \setminus Z_1$.

By definition of least upper bound of measures, we have that $\operatorname{div}_x \boldsymbol{b}(\cdot, t) \ll \sigma$ for every $t \in \mathbb{R}$. If we denote by $f(\cdot, t)$ the Radon–Nikodým derivative of $\operatorname{div}_x \boldsymbol{b}(\cdot, t)$ with respect to σ, we have

$$\operatorname{div}_x \boldsymbol{b}(\cdot, t) = f(\cdot, t)\, \sigma, \quad f(\cdot, t) \in L^1(\sigma), \qquad \forall t \in \mathbb{R}.$$

Moreover, since $|\operatorname{div}_x \boldsymbol{b}(\cdot, t)| \leq \sigma$, we have that

$$\forall t \in \mathbb{R}: \quad |f(x, t)| \leq 1 \quad \text{for } \sigma\text{-a.e. } x \in \Omega. \tag{6}$$

Let us extend $\boldsymbol{b}$ to 0 in $(\mathbb{R}^N \setminus \Omega) \times \mathbb{R}$, so that the vector field

$$\boldsymbol{B}(x,t) := \int_0^t \boldsymbol{b}(x,s)\, ds, \qquad x \in \mathbb{R}^N,\ t \in \mathbb{R}, \tag{7}$$

is defined for all $(x,t) \in \mathbb{R}^N \times \mathbb{R}$. Moreover $\boldsymbol{B}(x,0) = 0$ for every $x \in \mathbb{R}^N$ and, from (ii), for every $x \in \mathbb{R}^n$ one has $\boldsymbol{b}(x,t) = \partial_t \boldsymbol{B}(x,t)$ for every $t \in \mathbb{R}$.

Lemma 3.1. *For every $t \in \mathbb{R}$ it holds $\boldsymbol{B}(\cdot,t) \in \mathcal{DM}^\infty_{\mathrm{loc}}(\Omega)$ and $\operatorname{div}_x \boldsymbol{B}(\cdot,t) \ll \sigma$. If we denote by $F(\cdot,t)$ the Radon–Nikodým derivative of $\operatorname{div}_x \boldsymbol{B}(\cdot,t)$ with respect to σ, we have that*

$$F(x,t) = \int_0^t f(x,s)\, ds, \qquad \text{for } \sigma\text{–a.e. } x \in \Omega, \tag{8}$$

and, for every $t, s \in \mathbb{R}$,

$$|F(x,t) - F(x,s)| \le |t-s| \qquad \text{for } \sigma\text{–a.e. } x \in \Omega. \tag{9}$$

Proof. Let $\varphi \in C^1_c(\Omega)$. We have that

$$\begin{aligned}
\int_\Omega \nabla\varphi(x) \cdot \boldsymbol{B}(x,t)\, dx &= \int_\Omega \nabla\varphi(x) \cdot \int_0^t \boldsymbol{b}(x,s)\, ds\, dx \\
&= -\int_0^t \left(\int_\Omega \varphi(x)\, f(x,s)\, d\sigma(x) \right) ds \\
&= -\int_\Omega \varphi(x) \left(\int_0^t f(x,s)\, ds \right) d\sigma(x)\,.
\end{aligned}$$

Hence $\operatorname{div}_x \boldsymbol{B}(\cdot,t)$ is a Radon measure, it is absolutely continuous with respect to σ and its Radon–Nikodým derivative with respect to σ is $F(\cdot,t)$.

For every non-negative $\varphi \in C^1_c(\Omega)$ and for every $t, s \in \mathbb{R}$, taking into account that for every $s \in \mathbb{R}$, $|f(x,s)| \le 1$ for σ–a.e. $x \in \Omega$, an analogous integration gives

$$\begin{aligned}
\left| \int_\Omega \nabla\varphi(x) \cdot [\boldsymbol{B}(x,t) - \boldsymbol{B}(x,s)]\, dx \right| &= \left| \int_s^t \left(\int_\Omega \varphi(x)\, f(x,w)\, d\sigma(x) \right) dw \right| \\
&\le |t-s| \int_\Omega \varphi(x) d\sigma(x)\,,
\end{aligned}$$

so that

$$|\operatorname{div}_x \boldsymbol{B}(\cdot,t)(A) - \operatorname{div}_x \boldsymbol{B}(\cdot,s)(A)| \le |t-s|\, \sigma(A)$$

for every Borel set $A \Subset \Omega$, hence (9) follows. □

Lemma 3.2. *There exists a set $Z_2 \subset \Omega$, with $\sigma(Z_2) = 0$ and $\mathcal{H}^{N-1}(Z_2) = 0$, such that every $x_0 \in \Omega \setminus Z_2$ is a Lebesgue point of $F(\cdot,t)$ with respect to the measure σ for every $t \in \mathbb{R}$.*

Proof. Let $S \subset \mathbb{R}$ be a countable dense subset of $\mathbb{R}$ and, for every $s \in S$, let Ω_s be the set of the points $x_0 \in \Omega$ such that the Radon-Nikodým derivative $F(\cdot,s)$ of $\operatorname{div}_x \boldsymbol{B}(\cdot,s)$ exists in x_0 and x_0 is a Lebesgue point of the function $F(\cdot,s)$ with respect to the measure σ. Define

$$Z_2 := \mathbb{R}^N \setminus \bigcap_{s \in S} \Omega_s.$$

Clearly $\sigma(Z_2) = 0$ and $\mathcal{H}^{N-1}(Z_2) = 0$. We claim that every $x_0 \in \Omega \setminus Z_2$ is a Lebesgue point of the function $F(\cdot,w)$ for all $w \in \mathbb{R}$.

Fix $x_0 \in \Omega \setminus Z_2$ and $w \in \mathbb{R}$. Since the set S is dense in $\mathbb{R}$ we may find a sequence (w_n) of points in S such that $w_n \to w$. Then, by (9),

$$\begin{aligned}|F(x_0, w) - F(x, w)| &\leq |F(x_0, w) - F(x_0, w_n)| + |F(x_0, w_n) - F(x, w_n)| \\ &\quad + |F(x, w_n) - F(x, w)| \\ &\leq |F(x_0, w_n) - F(x, w_n)| + 2\,|w - w_n|.\end{aligned}$$

By averaging over $B_r(x_0)$ and letting $r \to 0^+$ in the previous inequality we get

$$\limsup_{r\to 0^+} \frac{1}{\sigma\left(B_r(x_0)\right)} \int_{B_r(x_0)} |F(x_0, w) - F(x, w)|\, d\sigma \leq 2\,|w - w_n|,$$

where we have used the fact that x_0 is a Lebesgue point of $F(\cdot, w_n)$ w.r.t. the measure σ. Since $w_n \to w$ letting $n \to \infty$, the previous inequality proves that every $x_0 \in \Omega \setminus Z_2$ is a Lebesgue point of the function $F(\cdot, w)$ for all $w \in \mathbb{R}$. □

3.2. **Weak normal traces of $\boldsymbol{B}$.** Let $\Sigma \subset \Omega$ be an oriented countably $\mathcal{H}^{N-1}$–rectifiable set. By Lemma 3.1, it follows that, for almost every $t \in \mathbb{R}$, the traces of the normal component of the vector field $\boldsymbol{B}(\cdot, t)$ can be defined as distributions $\mathrm{Tr}^{\pm}(\boldsymbol{B}(\cdot, t), \Sigma)$ in the sense of Anzellotti (see [2,6,10]). It turns out that these distributions are induced by L^∞ functions on Σ, still denoted by $\mathrm{Tr}^{\pm}(\boldsymbol{B}(\cdot, t), \Sigma)$, and

$$\|\mathrm{Tr}^{\pm}(\boldsymbol{B}(\cdot, t), \Sigma)\|_{L^\infty(\Sigma, \mathcal{H}^{N-1})} \leq \|\boldsymbol{B}(\cdot, t)\|_{L^\infty(\Omega)}.$$

More precisely, let us briefly recall the construction given in [2] (see Proposition 3.4 and Remark 3.3 therein). Since Σ is countably $\mathcal{H}^{N-1}$–rectifiable, we can find countably many *oriented* C^1 hypersurfaces Σ_i, with classical normal ν_{Σ_i}, and pairwise disjoint Borel sets $N_i \subseteq \Sigma_i$ such that $\mathcal{H}^{N-1}(\Sigma \setminus \bigcup_i N_i) = 0$.

Moreover, it is not restrictive to assume that, for every i, there exist two open bounded sets Ω_i, Ω_i' with C^1 boundary and outer normal vectors ν_{Ω_i} and $\nu_{\Omega_i'}$ respectively, such that $N_i \subseteq \partial\Omega_i \cap \partial\Omega_i'$ and

$$\nu_{\Sigma_i}(x) = \nu_{\Omega_i}(x) = -\nu_{\Omega_i'}(x) \qquad \forall x \in N_i.$$

At this point we choose, on Σ, the orientation given by $\nu_\Sigma(x) := \nu_{\Sigma_i}(x)$ $\mathcal{H}^{N-1}$-a.e. on N_i.

Using the localization property proved in [2, Proposition 3.2], for every $t \in \mathbb{R}$ we can define the normal traces of $\boldsymbol{B}(\cdot, t)$ on Σ by

$$\beta^-(\cdot, t) := \mathrm{Tr}(\boldsymbol{B}(\cdot, t), \partial\Omega_i), \quad \beta^+(\cdot, t) := -\mathrm{Tr}(\boldsymbol{B}(\cdot, t), \partial\Omega_i'), \qquad \mathcal{H}^{N-1} - \text{a.e. on } N_i. \tag{10}$$

These two normal traces belong to $L^\infty(\Sigma)$ (see [2, Proposition 3.2]) and

$$\mathrm{div}_x\, \boldsymbol{B}(\cdot, t) \llcorner \Sigma = \left[\beta^+(\cdot, t) - \beta^-(\cdot, t)\right] \mathcal{H}^{N-1} \llcorner \Sigma\,. \tag{11}$$

Lemma 3.3. *The maps $t \mapsto \beta^{\pm}(\cdot, t)$ are Lipschitz continuous from $\mathbb{R}$ to $L^\infty(\Sigma)$. More precisely*

$$\|\beta^{\pm}(\cdot, t) - \beta^{\pm}(\cdot, s)\|_{L^\infty(\Sigma)} \leq \|\boldsymbol{b}\|_\infty\, |t - s| \qquad \forall t, s \in \mathbb{R}. \tag{12}$$

Proof. It is enough to observe that, for every i, the map $\boldsymbol{X} \mapsto \mathrm{Tr}(\boldsymbol{X}, \partial\Omega_i)$ is linear in $\mathcal{DM}^\infty$ and, by Proposition 3.2 in [2], it holds

$$\|\mathrm{Tr}(\boldsymbol{X}, \partial\Omega_i)\|_{L^\infty(\partial\Omega_i)} \leq \|\boldsymbol{X}\|_{L^\infty(\Omega_i)} \qquad \forall \boldsymbol{X} \in \mathcal{DM}^\infty, \tag{13}$$

and the same inequality holds when Ω_i is replaced by Ω_i'. Hence (12) follows from Assumption (i). □

In the Step 6 of the proof of Theorem 4.3 we will use these normal traces on the set $\Sigma := J_u$, where J_u is the jump set of a BV function u. More precisely, we will use a slightly stronger property, stated in the following proposition.

Proposition 3.4. *There exists a set $\Sigma' \subseteq \Sigma$, with $\mathcal{H}^{N-1}(\Sigma \setminus \Sigma') = 0$, and representatives $\overline{\beta}^{\pm}$ of $\beta^{\pm}$ (in the sense that, for every $t \in \mathbb{R}$, $\overline{\beta}^{\pm}(\cdot, t) = \beta^{\pm}(\cdot, t)$ $\mathcal{H}^{N-1}$–a.e. in Σ) such that*

$$|\overline{\beta}^{\pm}(x,t) - \overline{\beta}^{\pm}(x,s)| \leq \|\boldsymbol{b}\|_\infty |t-s| \qquad \forall x \in \Sigma',\ t,s \in \mathbb{R}. \tag{14}$$

Proof. Let $Q \subset \mathbb{R}$ be a countable dense subset of $\mathbb{R}$. There exists a set $\Sigma' \subseteq \Sigma$, with $\mathcal{H}^{N-1}(\Sigma \setminus \Sigma') = 0$, such that

$$|\beta^{\pm}(x,p) - \beta^{\pm}(x,q)| \leq M|p-q| \qquad \forall x \in \Sigma',\ p,q \in Q, \tag{15}$$

where $M := \|\boldsymbol{b}\|_\infty$. Let us define the functions $\overline{\beta}^{\pm}$ in the following way. If $q \in Q$, we define $\overline{\beta}^{\pm}(x,q) = \beta^{\pm}(x,q)$ for every $x \in \Sigma$. If $t \in \mathbb{R} \setminus Q$, let $(q_j) \subset Q$ be a sequence converging to t, and define

$$\overline{\beta}^{\pm}(x,t) := \begin{cases} \lim\limits_{j \to +\infty} \beta^{\pm}(x,q_j), & \text{if } x \in \Sigma', \\ \beta^{\pm}(x,t), & \text{if } x \in \Sigma \setminus \Sigma'. \end{cases}$$

(We remark that, for every $x \in \Sigma'$, by (15) $(\beta^{\pm}(x,q_j))_j$ is a Cauchy sequence, hence it is convergent, in $\mathbb{R}$. Moreover, its limit is independent of the choice of the sequence $(q_j) \subset Q$ converging to t.) From Lemma 3.3 we have that

$$|\beta^{\pm}(x,q_j) - \beta^{\pm}(x,t)| \leq M|q_j - t| \qquad \text{for } \mathcal{H}^{N-1}\text{–a.e. } x \in \Sigma.$$

Passing to the limit as $j \to +\infty$, it follows that $\overline{\beta}^{\pm}(\cdot,t) = \beta^{\pm}(\cdot,t)$ $\mathcal{H}^{N-1}$–a.e. on Σ.

Let $t,s \in \mathbb{R}$, and let $(t_j),(s_j) \subset Q$ be two sequences in Q converging respectively to t and s. From (15) we have that

$$|\beta^{\pm}(x,t_j) - \beta^{\pm}(x,s_j)| \leq M|t_j - s_j| \qquad \forall x \in \Sigma',\ j \in \mathbb{N},$$

hence (14) follows passing to the limit as $j \to +\infty$. □

Remark 3.5. In what follows we will always denote by $\beta^{\pm}$ the representatives $\overline{\beta}^{\pm}$ of Proposition 3.4.

3.3. **Basic estimates on the composite function.** In this section we shall use a regularization argument to prove that the composite function $\mathbf{v}(x) := \boldsymbol{B}(x, u(x))$ belongs to $\mathcal{DM}^\infty_{\mathrm{loc}}(\Omega)$.

Lemma 3.6. *Let $\boldsymbol{b}\colon \Omega \times \mathbb{R} \to \mathbb{R}^N$ satisfy assumptions (i), (ii), extended to 0 in $(\mathbb{R}^N \setminus \Omega) \times \mathbb{R}$, and let $\boldsymbol{B}\colon \mathbb{R}^N \times \mathbb{R} \to \mathbb{R}^N$ be defined by (7). For every $\varepsilon > 0$ and every $t \in \mathbb{R}$ let $\boldsymbol{b}_\varepsilon(\cdot,t) := \rho_\varepsilon * \boldsymbol{b}(\cdot,t)$ and $\boldsymbol{B}_\varepsilon(\cdot,t) := \rho_\varepsilon * \boldsymbol{B}(\cdot,t)$. Then it holds:*

(a) *there exists an $\mathcal{L}^N$–null set $Z \subset \Omega$ such that*

$$\lim_{\varepsilon \to 0^+} \boldsymbol{b}_\varepsilon(x,t) = \boldsymbol{b}(x,t)\,, \quad \lim_{\varepsilon \to 0^+} \boldsymbol{B}_\varepsilon(x,t) = \boldsymbol{B}(x,t)\,, \qquad \forall (x,t) \in (\Omega \setminus Z) \times \mathbb{R};$$

(b) *$\partial_t \boldsymbol{B}_\varepsilon(x,t) = \boldsymbol{b}_\varepsilon(x,t)$ for every $(x,t) \in \Omega_\varepsilon \times \mathbb{R}$, where $\Omega_\varepsilon := \{x \in \Omega :\ dist(x, \partial\Omega) > \varepsilon\}$.*

Proof. The conclusion (a) for $\boldsymbol{b}_\varepsilon$ follows directly from Corollary 2.4.

By the Fubini–Tonelli theorem we have that

$$\boldsymbol{B}_\varepsilon(x,t) = \int_0^t \boldsymbol{b}_\varepsilon(x,s)\,ds, \qquad \forall (x,t) \in \Omega_\varepsilon \times \mathbb{R}, \tag{16}$$

hence (b) follows. Moreover, for every $x \in \mathbb{R}^N \setminus Z$ and every $t \in \mathbb{R}$, the conclusion (a) for $\boldsymbol{B}_\varepsilon$ follows from the Dominated Convergence Theorem. □

Lemma 3.7. *Let $\boldsymbol{b}$ satisfy assumptions (i)–(iv) and let $\boldsymbol{B}$ be defined by (7). Then, for every $u \in BV_{\rm loc}(\Omega) \cap L^\infty_{\rm loc}(\Omega)$, the function $\mathbf{v}\colon \Omega \to \mathbb{R}^N$, defined by*

$$\mathbf{v}(x) := \boldsymbol{B}(x, u(x)), \qquad x \in \Omega,$$

belongs to $\mathcal{DM}^\infty_{\rm loc}(\Omega)$ and

$$|\operatorname{div} \mathbf{v}|(K) \le \|u\|_{L^\infty(K)} \sigma(K) + \|\boldsymbol{b}\|_\infty |Du|(K) \qquad \forall K \subset \Omega,\ K \text{ compact.} \tag{17}$$

Moreover, the functions $\mathbf{v}_\epsilon(x) := \boldsymbol{B}_\varepsilon(x, u(x))$ converge to $\mathbf{v}$ a.e. in Ω and in $L^1_{\rm loc}(\Omega)$.

Remark 3.8. In the following we shall use the notation

$$\operatorname{div}_x \boldsymbol{B}(x, u(x)) := (\operatorname{div}_x \boldsymbol{B})(x, u(x)) = \operatorname{div}_x \boldsymbol{B}(x,t)|_{t=u(x)}\,.$$

The divergence of the composite function $x \mapsto \boldsymbol{B}(x, u(x))$ will be denoted by $\operatorname{div}[\boldsymbol{B}(x, u(x))]$.

Proof. Using the same notation of Lemma 3.6, one has

$$\operatorname{div}_x \boldsymbol{B}_\varepsilon(\cdot, t) = F(\cdot, t)\, \rho_\varepsilon * \sigma.$$

Since $F(x,t) = \int_0^t f(x,s)\, ds$ and $|f(x,s)| \le 1$, we have that, for every $t \in \mathbb{R}$,

$$|\operatorname{div}_x \boldsymbol{B}_\varepsilon(x,t)| \le |t|\, \rho_\varepsilon * \sigma(x), \qquad \text{for } \sigma\text{-a.e. } x \in \Omega_\varepsilon. \tag{18}$$

Let us define $\mathbf{v}_\varepsilon(x) := \boldsymbol{B}_\varepsilon(x, u(x))$. Consider first the case $u \in C^1(\Omega) \cap L^\infty(\Omega)$. We have that

$$\operatorname{div} \mathbf{v}_\varepsilon(x) = \operatorname{div}_x \boldsymbol{B}_\varepsilon(x, u(x)) + \langle \boldsymbol{b}_\varepsilon(x, u(x))\,,\, \nabla u(x)\rangle\,,$$

hence, from (18), for every compact subset K of Ω and for every $\varepsilon > 0$ small enough it holds

$$\int_K |\operatorname{div} \mathbf{v}_\varepsilon|\, dx \le \|u\|_{L^\infty(K)} \int_K \rho_\varepsilon * \sigma(x)\, dx + \|\boldsymbol{b}\|_\infty \int_K |\nabla u(x)|\, dx.$$

By an approximation argument (see [31, Theorem 5.3.3]), when $u \in BV_{\rm loc}(\Omega) \cap L^\infty_{\rm loc}(\Omega)$ we get

$$\int_K |\operatorname{div} \mathbf{v}_\varepsilon|\, dx \le \|u\|_{L^\infty(K)} \int_K \rho_\varepsilon * \sigma(x)\, dx + \|\boldsymbol{b}\|_\infty |Du|(K).$$

Finally, (17) follows observing that, by Lemma 3.6(a), $\boldsymbol{B}_\varepsilon(x, u(x)) \to \boldsymbol{B}(x, u(x))$ for $\mathcal{L}^N$-a.e. $x \in \Omega$, hence $\mathbf{v}_\varepsilon \to \mathbf{v}$ in $L^1_{\rm loc}(\Omega)$. □

4. Main results

The main results of the paper are stated in Theorems 4.3 and 4.6. As a preliminary step, we will prove Theorems 4.3 under the additional regularity assumption $u \in W^{1,1}$.

Theorem 4.1 ($\mathcal{DM}^\infty$-dependence and $u \in W^{1,1}$)**.** *Let $\boldsymbol{b}$ satisfy assumptions (i)–(iv) and let $\boldsymbol{B}$ be defined by (7). Then, for every $u \in W^{1,1}_{\rm loc}(\Omega) \cap L^\infty_{\rm loc}(\Omega)$, the function $\mathbf{v}\colon \Omega \to \mathbb{R}^N$, defined by*

$$\mathbf{v}(x) := \boldsymbol{B}(x, u(x)), \qquad x \in \Omega, \tag{19}$$

belongs to $\mathcal{DM}^\infty_{\rm loc}(\Omega)$ and the following equality holds in the sense of measures:

$$\operatorname{div} \mathbf{v} = F(x, \widetilde{u}(x))\, \sigma + \langle \partial_t \boldsymbol{B}(x, u(x))\,,\, \nabla u(x)\rangle\, \mathcal{L}^N, \tag{20}$$

where $F(\cdot, t)$ is the Radon–Nikodým derivative of $\operatorname{div}_x \boldsymbol{B}(\cdot, t)$ with respect to σ (see Section 3.1).

Remark 4.2. With some abuse of notation, we will also write equation (20) as

$$\operatorname{div} \mathbf{v} = \operatorname{div}_x \boldsymbol{B}(x,t)|_{t=\widetilde{u}(x)} + \langle \partial_t \boldsymbol{B}(x,u(x)) \,,\, \nabla u(x) \rangle \, \mathcal{L}^N \,. \tag{21}$$

Proof. By Lemma 3.7 the function $\mathbf{v}$ belongs to $\mathcal{DM}^\infty_{\rm loc}(\Omega)$ and satisfies (17).

Since all results are of local nature in the space variables, it is not restrictive to assume that $\Omega = \mathbb{R}^N$, $\boldsymbol{b}$ is a bounded Borel function, and $\boldsymbol{b}(\cdot,t) \in \mathcal{DM}^\infty$ for every $t \in \mathbb{R}$.

We will use a regularization argument as in [22, Theorem 3.4]. More precisely, as in Lemma 3.7 let $\boldsymbol{B}_\varepsilon(\cdot,t) := \rho_\varepsilon * \boldsymbol{B}(\cdot,t)$ and $\mathbf{v}_\varepsilon(x) := \boldsymbol{B}_\varepsilon(x,u(x))$. Since $\boldsymbol{B}_\varepsilon$ is a Lipschitz function in (x,t), by using the chain rule formula of Ambrosio and Dal Maso (see [4, Theorem 3.101]) one has

$$\begin{aligned} \int_{\mathbb{R}^N} \langle \nabla\phi(x) \,,\, \mathbf{v}_\varepsilon(x) \rangle \, dx = & - \int_{\mathbb{R}^N} \phi(x) \operatorname{div}_x \boldsymbol{B}_\varepsilon(x,u(x)) \, dx \\ & - \int_{\mathbb{R}^N} \phi(x) \, \langle \boldsymbol{b}_\varepsilon(x,u(x)) \,,\, \nabla u(x) \rangle \, dx, \end{aligned} \tag{22}$$

and the claim will follow by passing to the limit as $\varepsilon \to 0^+$.

Namely, by Lemma 3.7, $\mathbf{v}_\varepsilon(x) \to \mathbf{v}(x)$ for $\mathcal{L}^N$–a.e. $x \in \mathbb{R}^N$. Then by Dominated Convergence Theorem we have

$$\lim_{\varepsilon \to 0^+} \int_{\mathbb{R}^N} \langle \nabla\phi(x) \,,\, \mathbf{v}_\varepsilon(x) \rangle \, dx = \int_{\mathbb{R}^N} \langle \nabla\phi(x) \,,\, \mathbf{v}(x) \rangle \, dx \,. \tag{23}$$

This is equivalent to say

$$\operatorname{div}[\boldsymbol{B}_\varepsilon(x,u(x))] \, \mathcal{L}^N \stackrel{*}{\rightharpoonup} \operatorname{div} \mathbf{v}(x) \,, \qquad \text{as } \varepsilon \to 0^+ \tag{24}$$

in the weak* sense of measures.

Similarly, from Lemma 3.6(a) we have that, for $\mathcal{L}^N$–a.e. $x \in \mathbb{R}^N$,

$$\lim_{\varepsilon \to 0^+} \boldsymbol{b}_\varepsilon(x,u(x)) = \boldsymbol{b}(x,u(x)) \,.$$

Then by (ii) and the Dominated Convergence Theorem we have

$$\lim_{\varepsilon \to 0^+} \int_{\mathbb{R}^N} \phi(x) \, \langle \boldsymbol{b}_\varepsilon(x,u(x)) \,,\, \nabla u(x) \rangle \, dx = \int_{\mathbb{R}^N} \phi(x) \, \langle \boldsymbol{b}(x,u(x)) \,,\, \nabla u(x) \rangle \, dx \,. \tag{25}$$

It remains to prove that

$$I_\varepsilon := \int_{\mathbb{R}^N} \phi(x) \operatorname{div}_x \boldsymbol{B}_\varepsilon(x,u(x)) \, dx \ \xrightarrow{\varepsilon \to 0} \ \int_{\mathbb{R}^N} \phi(x) F(x, \widetilde{u}(x)) \, d\sigma(x). \tag{26}$$

Assume first that $u \geq 0$ and let $C > \|u\|_\infty$. Let us rewrite I_ε in the following way:

$$\begin{aligned} I_\varepsilon &= \int_{\mathbb{R}^N} \phi(x) \int_0^{u(x)} \operatorname{div}_x \boldsymbol{b}_\varepsilon(x,t) \, dt \, dx \\ &= \int_0^C dt \int_{\mathbb{R}^N} \phi(x) \chi_{\{u>t\}}(x) \int_{\mathbb{R}^N} \rho_\varepsilon(x-y) \, d \operatorname{div}_y \boldsymbol{b}(y,t) \\ &= \int_0^C dt \int_{\mathbb{R}^N} \rho_\varepsilon * (\phi \chi_{\{u>t\}})(y) \, d \operatorname{div}_y \boldsymbol{b}(y,t) \,. \end{aligned}$$

By Corollary 3.80 in [4] it holds

$$\rho_\varepsilon * (\phi \chi_{\{u>t\}})(x) \to \phi(x) \chi^*_{\{u>t\}}(x) = \phi(x) \chi_{\{\widetilde{u}>t\}}(x) \qquad \text{for } \mathcal{H}^{N-1}\text{-a.e. } x$$

(hence for $\operatorname{div}_x \boldsymbol{b}(\cdot,t)$-a.e. x). Passing to the limit as $\varepsilon \to 0$ we get

$$\begin{aligned}\lim_{\varepsilon\to 0} I_\varepsilon &= \int_0^C dt \int_{\mathbb{R}^N} \phi(x)\chi_{\{\widetilde{u}>t\}}(x)\, d\operatorname{div}_x \boldsymbol{b}(x,t)\\ &= \int_{\mathbb{R}^N} \phi(x) \int_0^{\widetilde{u}(x)} f(x,t)\, dt\, d\sigma(x)\\ &= \int_{\mathbb{R}^N} \phi(x) F(x,\widetilde{u}(x))\, d\sigma(x),\end{aligned}$$

hence (26) is proved in the case $u \geq 0$.

The general case can be handled similarly. Namely, the integral I_ε can be written as

$$\begin{aligned}I_\varepsilon &= \int_{-C}^C dt \int_{\mathbb{R}^N} \phi(x)\chi_{u,t}(x) \int_{\mathbb{R}^N} \rho_\varepsilon(x-y)\, d\operatorname{div}_y \boldsymbol{b}(y,t)\\ &= \int_{-C}^C dt \int_{\mathbb{R}^N} \rho_\varepsilon * (\phi\chi_{u,t})(y)\, d\operatorname{div}_y \boldsymbol{b}(y,t)\,,\end{aligned}$$

where $\chi_{u,t}$ is the characteristic function of the set

$$\{x \in \mathbb{R}^N :\ t \text{ belongs to the segment of endpoints } 0 \text{ and } u(x)\},$$

and the limit as $\varepsilon \to 0$ can be computed exactly as in the previous case. □

We now state the main results of the paper; the proofs are collected at the end of the section.

Theorem 4.3 ($\mathcal{DM}^\infty$-dependence and $u \in BV$)**.** *Let $\boldsymbol{b}$ satisfy assumptions (i)–(iv), let $\boldsymbol{B}$ be defined by* (7)*, and let $u \in BV_{\rm loc}(\Omega) \cap L^\infty_{\rm loc}(\Omega)$. Then the distribution $(\boldsymbol{b}(\cdot,u), Du)$, defined by*

$$\begin{aligned}\langle(\boldsymbol{b}(\cdot,u),Du),\varphi\rangle := &-\frac{1}{2}\int_\Omega \left[F(x,u^+(x)) + F(x,u^-(x))\right]\,\varphi(x)\, d\sigma(x)\\ &-\int_\Omega \boldsymbol{B}(x,u(x))\cdot\nabla\varphi(x)\,dx, \qquad \forall\varphi\in C_c^\infty(\Omega),\end{aligned} \tag{27}$$

is a Radon measure in Ω, and satisfies

$$|(\boldsymbol{b}(\cdot,u),Du)|(E) \leq \|\boldsymbol{b}\|_\infty |Du|(E), \qquad \textit{for every Borel set } E\subset\Omega. \tag{28}$$

In other words, the composite function $\mathbf{v}\colon \Omega\to\mathbb{R}^N$, defined by $\mathbf{v}(x) := \boldsymbol{B}(x,u(x))$, belongs to $\mathcal{DM}^\infty_{\rm loc}(\Omega)$, and the following equality holds in the sense of measures:

$$\operatorname{div}\mathbf{v} = \frac{1}{2}\left[F(x,u^+(x)) + F(x,u^-(x))\right]\,\sigma + (\boldsymbol{b}(\cdot,u),Du). \tag{29}$$

Remark 4.4. The measure $(\boldsymbol{b}(\cdot,u),Du)$ extends the notion of pairing defined by Anzellotti [6], in the case $\boldsymbol{b}(x,w) = \boldsymbol{A}(x)$, with $\boldsymbol{A}\in\mathcal{DM}^\infty$.

Proposition 4.5 (Traces of the composite function)**.** *Let the assumptions of Theorem 4.3 hold, and let $\Sigma\subset\Omega$ be a countably $\mathcal{H}^{N-1}$-rectifiable set, oriented as in Section 3.2. Then the normal traces on Σ of the composite function $\mathbf{v}\in\mathcal{DM}^\infty_{\rm loc}(\Omega)$, defined at* (19)*, are given by*

$$\operatorname{Tr}^\pm(\mathbf{v},\Sigma) = \begin{cases}\beta^\pm(x,u^\pm(x)), & \textit{for } \mathcal{H}^{N-1}\textit{-a.e. } x\in J_u,\\ \beta^\pm(x,\widetilde{u}(x)), & \textit{for } \mathcal{H}^{N-1}\textit{-a.e. } x\in \Sigma\setminus J_u,\end{cases} \tag{30}$$

where, for every $t \in \mathbb{R}$, $\beta^{\pm}(\cdot, t)$ *are the normal traces of* $\boldsymbol{B}(\cdot, t)$ *on* Σ *(see* (11)*). With our convention* $u^{\pm}(x) = \widetilde{u}(x)$ *if* $x \in \Omega \setminus S_u$, (30) *can be written as* $\mathrm{Tr}^{\pm}(\mathbf{v}, \Sigma) = \beta^{\pm}(x, u^{\pm}(x))$ *for* $\mathcal{H}^{N-1}$*-a.e.* $x \in \Sigma$.

Theorem 4.6 (Representation of the pairing measure)**.** *Let the assumptions of Theorem 4.3 hold, and consider the standard decomposition of the measure* $\mu := (\boldsymbol{b}(\cdot, u), Du)$ *as*

$$\mu = \mu^{ac} + \mu^{c} + \mu^{j}, \qquad \mu^{d} := \mu^{ac} + \mu^{c}.$$

Then

$$\mu^{ac} = \langle \boldsymbol{b}(x, \widetilde{u}(x))\, ,\, \nabla u(x) \rangle\, \mathcal{L}^N,$$
$$\mu^{j} = [\beta^*(x, u^+(x)) - \beta^*(x, u^-(x))] \mathcal{H}^{N-1} \llcorner J_u,$$

where, for every $t \in \mathbb{R}$, $\beta^{\pm}(\cdot, t)$ *are the normal traces of* $\boldsymbol{B}(\cdot, t)$ *on* J_u *and* $\beta^*(\cdot, t) := [\beta^+(\cdot, t) + \beta^-(\cdot, t)]/2$.

Moreover, if there exists a countable dense set $Q \subset \mathbb{R}$ *such that*

$$|D^c u|(S_{\boldsymbol{b}(\cdot,t)}) = 0 \qquad \forall t \in Q, \tag{31}$$

then

$$\mu^d = \left\langle \widetilde{\boldsymbol{b}}(x, \widetilde{u}(x))\, ,\, D^d u \right\rangle.$$

Therefore, under this additional assumption the following equality holds in the sense of measures:

$$\begin{aligned} \operatorname{div} \mathbf{v} = {} & \frac{1}{2} \left[F(x, u^+(x)) + F(x, u^-(x)) \right] \sigma \\ & + \left\langle \widetilde{\boldsymbol{b}}(x, \widetilde{u}(x))\, ,\, D^d u \right\rangle + [\beta^*(x, u^+) - \beta^*(x, u^-)] \mathcal{H}^{N-1} \llcorner J_u. \end{aligned} \tag{32}$$

Remark 4.7. Since $\mathcal{L}^N(S_{\boldsymbol{b}(\cdot,t)}) = 0$ for every $t \in \mathbb{R}$, assumption (31) is equivalent to $|D^d u|(S_{\boldsymbol{b}(\cdot,t)}) = 0$ for every $t \in Q$. In particular, it is satisfied, for example, if $S_{\boldsymbol{b}(\cdot,t)}$ is σ–finite with respect to $\mathcal{H}^{N-1}$, for every $t \in Q$ (see [4, Proposition 3.92(c)]). This is always the case if $\boldsymbol{b}(\cdot, t) \in BV_{\mathrm{loc}}(\Omega, \mathbb{R}^N) \cap L^\infty_{\mathrm{loc}}(\Omega, \mathbb{R}^N)$. Another relevant situation for which (31) holds happens when $D^c u = 0$, i.e. if u is a special function of bounded variation, e.g. if u is the characteristic function of a set of finite perimeter.

Remark 4.8. For $u \in BV_{\mathrm{loc}}(\Omega)$ we introduce the following notation:

$$\begin{aligned} \operatorname{div}_x \boldsymbol{B}(\cdot, t)|_{t=u(x)} & := \frac{1}{2} \left[\operatorname{div}_x \boldsymbol{B}(\cdot, u^+(x)) + \operatorname{div}_x \boldsymbol{B}(\cdot, u^-(x)) \right] \\ & := \frac{1}{2} \left[F(x, u^+(x)) + F(x, u^-(x)) \right] \sigma \end{aligned}$$

(see also Remark 4.2). Then, with some abuse of notation, equation (32) can be written as

$$\begin{aligned} \operatorname{div} \mathbf{v} = {} & \operatorname{div}_x \boldsymbol{B}(x, t)|_{t=u(x)} \\ & + \left\langle \widetilde{\boldsymbol{b}}(x, \widetilde{u}(x))\, ,\, D^d u \right\rangle + [\beta^*(x, u^+) - \beta^*(x, u^-)] \mathcal{H}^{N-1} \llcorner J_u. \end{aligned} \tag{33}$$

Remark 4.9 (Anzellotti's pairing). In the special case $\boldsymbol{B}(x, t) = t\, \boldsymbol{A}(x)$, with $\boldsymbol{A} \in \mathcal{DM}^\infty_{\mathrm{loc}}(\Omega)$, we have that

$$\boldsymbol{b}(x, t) = \boldsymbol{A}(x), \quad \sigma = |\operatorname{div} \boldsymbol{A}|, \quad f(x, t) = \frac{d \operatorname{div} \boldsymbol{A}}{d|\operatorname{div} \boldsymbol{A}|}, \quad F(x, t) = t\, \frac{d \operatorname{div} \boldsymbol{A}}{d|\operatorname{div} \boldsymbol{A}|},$$

and formula (29) becomes

$$\operatorname{div}(u\boldsymbol{A}) = u^* \operatorname{div} \boldsymbol{A} + (\boldsymbol{A}, Du),$$

where $(\boldsymbol{A}, Du)$ is the Anzellotti's pairing.

The remaining of this section is devoted to the proofs of Theorem 4.3, Proposition 4.5 and Theorem 4.6.

Since all results are of local nature in the space variables, it is not restrictive to assume that $\Omega = \mathbb{R}^N$, $\boldsymbol{b}$ is a bounded Borel function, and $\boldsymbol{b}(\cdot, t) \in \mathcal{DM}^\infty$ for every $t \in \mathbb{R}$.

Proof of Theorem 4.3. By Lemma 3.7 the function $\mathbf{v}$ belongs to $\mathcal{DM}^\infty_{\rm loc}(\Omega)$ and satisfies (17).

We will use another regularization argument as in [10]. More precisely, let $u_\varepsilon := \rho_\varepsilon * u$ be the standard regularization of u, and $\mathbf{v}_\varepsilon(x) := \boldsymbol{B}(x, u_\varepsilon(x))$. Then, by Theorem 4.1, for any $\phi \in C^1_0(\mathbb{R}^N)$ we get

$$\begin{aligned}\int_{\mathbb{R}^N} \langle \nabla\phi(x)\,,\, \mathbf{v}_\varepsilon(x)\rangle\, dx = & -\int_{\mathbb{R}^N} \phi(x)\, F(x, u_\varepsilon(x))\, d\sigma(x) \\ & - \int_{\mathbb{R}^N} \phi(x)\, \langle \boldsymbol{b}(x, u_\varepsilon(x))\,,\, \nabla u_\varepsilon(x)\rangle\, dx\,.\end{aligned} \tag{34}$$

Now we will pass to the limit as $\varepsilon \to 0^+$ in each term.

STEP 1. Firstly, we note that

$$\lim_{\varepsilon\to 0^+} \int_{\mathbb{R}^N} \langle \nabla\phi(x)\,,\, \mathbf{v}_\varepsilon(x)\rangle\, dx = \int_{\mathbb{R}^N} \langle \nabla\phi(x)\,,\, \mathbf{v}(x)\rangle\, dx. \tag{35}$$

Indeed, $u_\varepsilon(x) \to u(x)$, as $\varepsilon \to 0^+$, for a.e. x, $\boldsymbol{B}(x, \cdot)$ is Lipschitz continuous with Lipschitz constant independent of x and $\boldsymbol{B}$ is locally bounded. Thus (35) holds by the Dominated Convergence Theorem.

STEP 2. We will prove that

$$\lim_{\varepsilon\to 0^+} \int_{\mathbb{R}^N} \phi(x)\, F(x, u_\varepsilon(x))\, d\sigma(x) = \int_{\mathbb{R}^N} \phi(x) \frac{1}{2} \left[F(x, u^+(x)) + F(x, u^-(x))\right]\, d\sigma(x). \tag{36}$$

From (8) it holds

$$\int_{\mathbb{R}^N} \phi(x)\, F(x, u_\varepsilon(x))\, d\sigma(x) = \int_{\mathbb{R}^N} \phi(x) \left[\int_0^{u_\varepsilon(x)} f(x, w)\, dw\right] d\sigma(x)\,. \tag{37}$$

Since $u_\varepsilon(x) \to u^*(x)$ for $\mathcal{H}^{N-1}$-a.e. x, and so also for σ-a.e. x (since $\sigma \ll \mathcal{H}^{N-1}$), passing to the limit in (37) we obtain

$$\lim_{\varepsilon\to 0^+} \int_{\mathbb{R}^N} \phi(x)\, F(x, u_\varepsilon(x))\, d\sigma(x) = \int_{\mathbb{R}^N} \phi(x) \left[\int_0^{u^*(x)} f(x, w)\, dw\right] d\sigma(x) =: I\,.$$

In the remaining part of the proof, for the sake of simplicity we assume $u \geq 0$. We remark that the general case can be handled as it has been illustrated at the end of the proof of Theorem 4.1.

Let $C > \|u\|_{L^\infty(K)}$, where K is the support of ϕ. The integral I can be rewritten as

$$I = \int_0^C \left[\int_{\mathbb{R}^N} \phi(x) \chi_{\{u^*>w\}}(x) f(x, w)\, d\sigma(x)\right] dw\,.$$

On the other hand, for $\mathcal{L}^1$-a.e. $w \in \mathbb{R}$ we have that

$$\chi_{\{u^*>w\}} = \frac{1}{2}\left[\chi_{\{u^+>w\}} + \chi_{\{u^->w\}}\right], \qquad \mathcal{H}^{N-1}\text{-a.e. (hence } \sigma\text{-a.e.) in } \mathbb{R}^N$$

(see [22, Lemma 2.2]). Hence we get

$$\begin{aligned} I &= \int_0^C \left[\int_{\mathbb{R}^N} \phi(x)\frac{1}{2}[\chi_{\{u^+>w\}}(x) + \chi_{\{u^->w\}}(x)] f(x,w)\, d\sigma(x)\right] dw \\ &= \int_{\mathbb{R}^N} \phi(x)\left[\int_0^C \frac{1}{2}[\chi_{\{u^+>w\}}(x) + \chi_{\{u^->w\}}(x)] f(x,w)\, dw\right] d\sigma(x) \\ &= \int_{\mathbb{R}^N} \phi(x)\frac{1}{2}\left[F(x,u^+(x)) + F(x,u^-(x))\right] d\sigma(x)\,, \end{aligned}$$

so that (36) is proved.

STEP 3. We claim that the distribution $(\boldsymbol{b}(\cdot,u), Du)$ defined at (27) is a Radon measure, satisfying (28) (and hence absolutely continuous with respect to $|Du|$).

For simplicity, let us denote by μ the distribution $(\boldsymbol{b}(\cdot,u), Du)$ defined at (27). Since

$$\mu = \operatorname{div}\mathbf{v} - \frac{1}{2}\left[F(x,u^+(x)) + F(x,u^-(x))\right]\sigma,$$

by Lemma 3.7 it is clear that μ is a Radon measure and (29) holds. Moreover, by (34), (35) and (36) we have that, for every $\phi \in C_c(\mathbb{R}^N)$,

$$\langle \mu, \phi\rangle = \lim_{\varepsilon\to 0^+} \int_{\mathbb{R}^N} \phi(x)\, \langle \boldsymbol{b}(x,u_\varepsilon(x))\,, \nabla u_\varepsilon(x)\rangle\, dx. \tag{38}$$

Let us prove that (28) holds. Namely, let $U \subset \mathbb{R}^N$ be an open set, let $K \Subset U$ be a compact set, and let $\phi \in C_c(\mathbb{R}^N)$ be a function with support contained in K. There exists $r_0 > 0$ such that $K_r := K + B_r(0) \subset U$ for every $r \in (0,r_0)$. Let $r \in (0,r_0)$ be such that $|Du|(\partial K_r) = 0$ (this property holds for almost every r). Then

$$|\langle \mu\,,\,\phi\rangle| \leq \|\phi\|_\infty \|\boldsymbol{b}\|_\infty \liminf_{\varepsilon\to 0}\int_{K_r} |\nabla u_\varepsilon|\, dx = \|\phi\|_\infty\|\boldsymbol{b}\|_\infty |Du|(K_r) \leq \|\phi\|_\infty\|\boldsymbol{b}\|_\infty|Du|(U),$$

hence

$$|\mu|(K) \leq \|\boldsymbol{b}\|_\infty |Du|(U),$$

so that (28) follows by the regularity of the Radon measures $|\mu|$ and $|Du|$. □

Proof of Proposition 4.5. We will use the same notations of Section 3.2. It is not restrictive to assume that J_u is oriented with ν_Σ on $J_u \cap \Sigma$.

Since, by Theorem 4.3, $\mathbf{v} \in \mathcal{DM}^\infty$, there exist the weak normal traces of $\mathbf{v}$ on Σ. Let us prove (30) for Tr^-.

Let $x \in \Sigma$ satisfy:

(a) $x \in (\mathbb{R}^N \setminus S_u) \cup J_u$, $x \in N_i$ for some i, the set N_i has density 1 at x and x is a Lebesgue point of $\beta^-(\cdot,t)$, with respect to $\mathcal{H}^{N-1} \llcorner \partial\Omega_i$, for every $t \in \mathbb{R}$;

(b) $\sigma \llcorner \Omega_i(B_\varepsilon(x)) = o(\varepsilon^{N-1})$ as $\varepsilon \to 0$;

(c) $|\operatorname{div}\mathbf{v}| \llcorner \Omega_i(B_\varepsilon(x)) = o(\varepsilon^{N-1})$.

We remark that $\mathcal{H}^{N-1}$-a.e. $x \in \Sigma$ satisfies these conditions. In particular, (a) is satisfied thanks to Proposition 3.4, whereas (b) and (c) follow from [4, Theorem 2.56 and (2.41)].

In order to simplify the notation, in the following we set $u^-(x) := \widetilde{u}(x)$ if $x \in \Omega \setminus S_u$.

Let us choose a function $\varphi \in C_c^\infty(\mathbb{R}^N)$, with support contained in $B_1(0)$, such that $0 \le \varphi \le 1$. For every $\varepsilon > 0$ let $\varphi_\varepsilon(y) := \varphi\left(\frac{y-x}{\varepsilon}\right)$.

By the very definition of normal trace, the following equality holds for every $\varepsilon > 0$ small enough:

$$\begin{aligned}
&\frac{1}{\varepsilon^{N-1}} \int_{\partial\Omega_i} [\mathrm{Tr}(\mathbf{v}, \partial\Omega_i) - \mathrm{Tr}(\boldsymbol{B}(\cdot, u^-(x)), \partial\Omega_i)]\, \varphi_\varepsilon(y)\, d\mathcal{H}^{N-1}(y) \\
&\quad = \frac{1}{\varepsilon^{N-1}} \int_{\Omega_i} \nabla\varphi_\varepsilon(y) \cdot [\mathbf{v}(y) - \boldsymbol{B}(y, u^-(x))]\, dy \\
&\quad\quad + \frac{1}{\varepsilon^{N-1}} \int_{\Omega_i} \varphi_\varepsilon(y)\, d[\operatorname{div} \mathbf{v} - \operatorname{div}_x \boldsymbol{B}(\cdot, u^-(x))](y)\,. \\
&\quad =: I_1(\varepsilon) + I_2(\varepsilon)\,.
\end{aligned} \tag{39}$$

Using the change of variable $z = (y-x)/\varepsilon$, as $\varepsilon \to 0$ the left hand side of this equality converges to

$$[\mathrm{Tr}^-(\mathbf{v}, \Sigma)(x) - \beta^-(x, u^-(x))] \int_{\Pi_x} \varphi(z)\, d\mathcal{H}^{N-1}(z)\,,$$

where Π_x is the tangent plane to Σ_i at x. Clearly φ can be chosen in such a way that $\int_{\Pi_x} \varphi\, d\mathcal{H}^{N-1} > 0$.

In order to prove (30) for Tr^- it is then enough to show that the two integrals $I_1(\varepsilon)$ and $I_2(\varepsilon)$ at the right hand side of (39) converge to 0 as $\varepsilon \to 0$.

With the change of variables $z = (y-x)/\varepsilon$ and by the very definition of $\mathbf{v}$ we have that

$$I_1(\varepsilon) = \int_{\Omega_i^\varepsilon} \nabla\varphi(z) \cdot [\boldsymbol{B}(x+\varepsilon z, u(x+\varepsilon z)) - \boldsymbol{B}(x+\varepsilon z, u^-(x))]\, dz,$$

where

$$\Omega_i^\varepsilon := \frac{\Omega_i - x}{\varepsilon}.$$

As $\varepsilon \to 0$, these sets locally converge to the half space $P_x := \{z \in \mathbb{R}^N :\ \langle z\,,\, \nu(x)\rangle < 0\}$, hence

$$\lim_{\varepsilon\to 0} \int_{\Omega_i^\varepsilon \cap B_1} |u(x+\varepsilon z) - u^-(x)|\, dz = \lim_{\varepsilon\to 0} \int_{P_x \cap B_1} |u(x+\varepsilon z) - u^-(x)|\, dz = 0$$

(see [4, Remark 3.85]) and, by (ii),

$$|I_1(\varepsilon)| \le \|\boldsymbol{b}\|_\infty \, \|\nabla\varphi\|_\infty \int_{\Omega_i^\varepsilon \cap B_1} |u(x+\varepsilon z) - u^-(x)|\, dz \to 0.$$

From (b) we have that

$$\lim_{\varepsilon\to 0} \frac{1}{\varepsilon^{N-1}} \left| \int_{\Omega_i} \varphi_\varepsilon(y)\, d \operatorname{div}_x \boldsymbol{B}(\cdot, u^-(x))(y) \right| \le \limsup_{\varepsilon\to 0} \frac{\sigma(B_\varepsilon(x))}{\varepsilon^{N-1}} = 0.$$

In a similar way, using (c), we get

$$\lim_{\varepsilon\to 0} \frac{1}{\varepsilon^{N-1}} \left| \int_{\Omega_i} \varphi_\varepsilon\, d \operatorname{div} \mathbf{v} \right| = 0,$$

so that $I_2(\varepsilon)$ vanishes as $\varepsilon \to 0$.

The proof of (30) for Tr^+ is entirely similar. □

Proof of Theorem 4.6. We shall divide the proof into several steps.

STEP 1. We are going to prove that

$$\mu^{ac} = \langle \boldsymbol{b}(x,u(x))\,,\,\nabla u(x)\rangle\;\mathcal{L}^N.$$

Let us choose $x \in \mathbb{R}^N$ such that

(a) there exists the limit $\displaystyle\lim_{r\to 0}\frac{\mu(B_r(x))}{r^N}$;

(b) $\displaystyle\lim_{r\to 0}\frac{|D^s u|(B_r(x))}{r^N} = 0$;

(c) $\displaystyle\lim_{r\to 0}\frac{1}{r^N}\int_{B_r(x)} |\,\langle \boldsymbol{b}(y,u(y))\,,\,\nabla u(y)\rangle - \langle \boldsymbol{b}(x,u(x))\,,\,\nabla u(x)\rangle\,|\,dx = 0.$

We remark that these conditions are satisfied for $\mathcal{L}^N$-a.e. $x \in \mathbb{R}^N$.

Let $r > 0$ be such that

$$|D^s u|\,(\partial B_r(x)) = 0. \tag{40}$$

Observe that

$$\nabla u_\varepsilon = \rho_\varepsilon * Du = \rho_\varepsilon * \nabla u + \rho_\varepsilon * D^s u.$$

Hence for every $\phi \in C_0(\mathbb{R}^N)$ with support in $B_r(x)$ it holds

$$\begin{aligned}
&\left|\frac{1}{r^N}\int_{B_r(x)}\phi(y)[\langle \boldsymbol{b}(y,u_\varepsilon(y))\,,\,(\rho_\varepsilon * Du)(y)\rangle - \langle \boldsymbol{b}(x,u(x))\,,\,\nabla u(x)\rangle]\,dy\right| \\
&\le \frac{1}{r^N}\int_{B_r(x)}\phi(y)\,|\langle \boldsymbol{b}(y,u_\varepsilon(y))\,,\,(\rho_\varepsilon * \nabla u)(y)\rangle - \langle \boldsymbol{b}(x,u(x))\,,\,\nabla u(x)\rangle|\;dy \\
&\quad + \frac{1}{r^N}\|\phi\|_\infty \|\boldsymbol{b}\|_\infty \int_{B_r(x)} \rho_\varepsilon * |D^s u|\,dy,
\end{aligned} \tag{41}$$

where in the last inequality we use that $|\rho_\varepsilon * D^s u| \le \rho_\varepsilon * |D^s u|$. We note that by (40)

$$\lim_{\varepsilon\to 0}\int_{B_r(x)} \rho_\varepsilon * |D^s u|\,dy = |D^s u|(B_r(x)).$$

Hence taking the limit as $\varepsilon \to 0$ we obtain

$$\begin{aligned}
&\left|\frac{1}{r^N}\int_{B_r(x)}\phi(y)\;d\mu(y) - \frac{1}{r^N}\int_{B_r(x)}\phi(y)\,\langle \boldsymbol{b}(x,u(x))\,,\,\nabla u(x)\rangle\;dy\right| \\
&\le \frac{1}{r^N}\int_{B_r(x)}\phi(y)\,|\langle \boldsymbol{b}(y,u(y))\,,\,\nabla u(y)\rangle - \langle \boldsymbol{b}(x,u(x))\,,\,\nabla u(x)\rangle|\;dy \\
&\quad + \frac{1}{r^N}\|\phi\|_\infty \|\boldsymbol{b}\|_\infty\,|D^s u|(B_r(x)).
\end{aligned}$$

When $\phi(y) \to 1$ in $B_r(x)$, with $0 \le \phi \le 1$, we get

$$\begin{aligned}&\left|\frac{1}{\omega_N r^N}\mu(B_r(x)) - \langle \boldsymbol{b}(x,u(x))\,,\,\nabla u(x)\rangle\right| \\ &\le \frac{1}{\omega_N r^N}\int_{B_r(x)} |\langle \boldsymbol{b}(y,u(y))\,,\,\nabla u(y)\rangle - \langle \boldsymbol{b}(x,u(x))\,,\,\nabla u(x)\rangle|\,dy \\ &\quad + \frac{1}{\omega_N r^N}\|\boldsymbol{b}\|_\infty\,|D^s u|(B_r(x)).\end{aligned}$$

Now the conclusion is achieved by taking the limit for $r \to 0$ and using (b) and (c) above.

STEP 2. For the jump part of the measure μ it holds:

$$\mu^j = [\beta^*(x,u^+) - \beta^*(x,u^-)]\mathcal{H}^{N-1} \llcorner J_u \tag{42}$$

Namely, this is a direct consequence of Proposition 4.5 in the particular case $\Sigma = J_u$.

STEP 3. From now to the end of the proof, we shall assume that the additional assumption (31) holds.

Let $S := \bigcup_{q\in Q} S_{\boldsymbol{b}(\cdot,q)}$. By assumption (31) we have that $|D^c u|(S) = 0$.

We claim that, for every $x \in \mathbb{R}^N \setminus S$ and every $t \in \mathbb{R}$, there exists the approximate limit of $\boldsymbol{b}$ at (x,t) and

$$\widetilde{\boldsymbol{b}}(x,t) = \lim_j \widetilde{\boldsymbol{b}}(x,q_j), \qquad \forall (q_j) \subset Q,\ q_j \to t. \tag{43}$$

Namely, let us fix a point $x \in \mathbb{R}^N \setminus S$. By assumptions (i), (ii) and the Dominated Convergence Theorem, the map

$$\psi(t) := \lim_{r\to 0} \fint_{B_r(x)} \boldsymbol{b}(y,t)\,dy, \qquad t \in \mathbb{R},$$

is continuous, and $\psi(q) = \widetilde{\boldsymbol{b}}(x,q)$ for every $q \in Q$. Hence the limit in (43) exists for every t and it is independent of the choice of the sequence $(q_j) \subset Q$ converging to t.

Let $t \in \mathbb{R}$ be fixed, let us denote by $c \in \mathbb{R}^N$ the value of the limit in (43) and let us prove that $c = \widetilde{\boldsymbol{b}}(x,t)$. We have that

$$\fint_{B_r(x)} |\boldsymbol{b}(y,t) - c|\,dy \le \fint_{B_r(x)} |\boldsymbol{b}(y,t) - \boldsymbol{b}(y,q_j)|\,dy + \fint_{B_r(x)} |\boldsymbol{b}(y,q_j) - \widetilde{\boldsymbol{b}}(x,q_j)|\,dy + |\widetilde{\boldsymbol{b}}(x,q_j) - c|.$$

As $j \to +\infty$, the first integral at the r.h.s. converges to 0 by (i), (ii) and the Dominated Convergence Theorem. The second integral converges to 0 since $x \in \mathbb{R}^N \setminus S$ and $(q_j) \subset Q$. Finally, $\lim_j |\widetilde{\boldsymbol{b}}(x,q_j) - c| = 0$ by the very definition of c, so that the claim is proved.

STEP 4. We are going to prove that

$$\mu^d = \left\langle \widetilde{\boldsymbol{b}}(x,\widetilde{u}(x))\,,\,D^d u\right\rangle$$

in the sense of measures. We remark that, by Step 3, the approximate limit $\widetilde{\boldsymbol{b}}(x,t)$ exists for every $(x,t) \in (\mathbb{R}^N \setminus S) \times \mathbb{R}$, with $|D^c u|(S) = 0$. As a consequence, the function $x \mapsto \widetilde{\boldsymbol{b}}(x,\widetilde{u}(x))$ is well-defined for $|D^d u|$-a.e. $x \in \mathbb{R}^N$, and it belongs to $L^\infty(\mathbb{R}^N, |D^d u|)$.

If we consider the polar decomposition $D^d u = \theta\,|D^d u|$, this equality is equivalent to

$$\frac{d\mu}{d|D^d u|}(x) = \frac{d\mu^d}{d|D^d u|}(x) = \left\langle \widetilde{\boldsymbol{b}}(x,\widetilde{u}(x))\,,\,\theta(x)\right\rangle$$

for $|D^d u|$-a.e. $x \in \mathbb{R}^N$. Let us choose $x \in \mathbb{R}^N$ such that

(a) x belongs to the support of $D^d u$, that is, $|D^d u|(B_r(x)) > 0$ for every $r > 0$;

(b) there exists the limit $\displaystyle\lim_{r\to 0} \frac{\mu^d(B_r(x))}{|D^d u|(B_r(x))}$;

(c) $\displaystyle\lim_{r\to 0} \frac{|D^j u|(B_r(x))}{|Du|(B_r(x))} = 0$;

(d) $\displaystyle\lim_{r\to 0} \frac{1}{|D^d u|(B_r(x))} \int_{B_r(x)} \left| \left\langle \widetilde{\boldsymbol{b}}(y, \widetilde{u}(y)) \,,\, \theta(y) \right\rangle - \left\langle \widetilde{\boldsymbol{b}}(x, \widetilde{u}(x)) \,,\, \theta(x) \right\rangle \right| \, d|D^d u|(y) = 0.$

We remark that these conditions are satisfied for $|D^d u|$-a.e. $x \in \mathbb{R}^N$. In particular, (d) follows from the fact that the map $y \mapsto \widetilde{\boldsymbol{b}}(y, \widetilde{u}(y))$ belongs to $L^\infty(\mathbb{R}^N, |D^d u|)$.

Let $r > 0$ be such that

$$|D^j u| \left(\partial B_r(x)\right) = 0. \tag{44}$$

Observe that $\nabla u_\varepsilon = \rho_\varepsilon * Du = \rho_\varepsilon * D^d u + \rho_\varepsilon * D^j u$. Hence, for every $\phi \in C_c(\mathbb{R}^N)$ with support in $B_r(x)$, it holds

$$\begin{aligned}
&\Bigg| \frac{1}{|D^d u|(B_r(x))} \int_{B_r(x)} \phi(y) \left\langle \boldsymbol{b}(y, u_\varepsilon(y)) \,,\, (\rho_\varepsilon * Du)(y) \right\rangle \, dy \\
&\qquad - \frac{1}{|D^d u|(B_r(x))} \int_{B_r(x)} \phi(y) \left\langle \widetilde{\boldsymbol{b}}(x, \widetilde{u}(x)) \,,\, \theta(x) \right\rangle \, d|D^d u|(y) \Bigg| \\
&\quad \le \Bigg| \frac{1}{|D^d u|(B_r(x))} \int_{B_r(x)} \phi(y) \left\langle \boldsymbol{b}(y, u_\varepsilon(y)) \,,\, (\rho_\varepsilon * D^d u)(y) \right\rangle \, dy \\
&\qquad - \frac{1}{|D^d u|(B_r(x))} \int_{B_r(x)} \phi(y) \left\langle \widetilde{\boldsymbol{b}}(x, \widetilde{u}(x)) \,,\, \theta(x) \right\rangle \, d|D^d u|(y) \Bigg| \\
&\qquad + \frac{1}{|D^d u|(B_r(x))} \|\phi\|_\infty \|\boldsymbol{b}\|_\infty \int_{B_r(x)} \rho_\varepsilon * |D^j u| \, dy,
\end{aligned} \tag{45}$$

where in the last inequality we use that $\left|\rho_\varepsilon * D^j u\right| \le \rho_\varepsilon * |D^j u|$. We note that by (44)

$$\lim_{\varepsilon\to 0} \int_{B_r(x)} \rho_\varepsilon * |D^j u| \, dy = |D^j u|(B_r(x)).$$

Hence by taking the limit as $\varepsilon \to 0$ in (45) we obtain

$$\begin{aligned}
&\Bigg| \frac{1}{|D^d u|(B_r(x))} \int_{B_r(x)} \phi(y) \, d\mu(y) \\
&- \frac{1}{|D^d u|(B_r(x))} \int_{B_r(x)} \phi(y) \left\langle \widetilde{\boldsymbol{b}}(x, \widetilde{u}(x)) \,,\, \theta(x) \right\rangle \, d|D^d u|(y) \Bigg| \\
&\le \frac{1}{|D^d u|(B_r(x))} \int_{B_r(x)} \phi(y) \left| \left\langle \widetilde{\boldsymbol{b}}(y, \widetilde{u}(y)) \,,\, \theta(y) \right\rangle - \left\langle \widetilde{\boldsymbol{b}}(x, \widetilde{u}(x)) \,,\, \theta(x) \right\rangle \, d|D^d u|(y) \right| \\
&\quad + \frac{1}{|D^d u|(B_r(x))} \|\phi\|_\infty \|\boldsymbol{b}\|_\infty \, |D^j u|(B_r(x)).
\end{aligned}$$

When $\phi(y) \to 1$ in $B_r(x)$, with $0 \le \phi \le 1$, we get

$$\begin{aligned}
&\left| \frac{1}{|D^d u|(B_r(x))} \mu(B_r(x)) - \left\langle \widetilde{\boldsymbol{b}}(x, \widetilde{u}(x)) \, , \, \theta(x) \right\rangle \right| \\
&\le \frac{1}{|D^d u|(B_r(x))} \int_{B_r(x)} \left| \left\langle \widetilde{\boldsymbol{b}}(y, \widetilde{u}(y)) \, , \, \theta(y) \right\rangle - \left\langle \widetilde{\boldsymbol{b}}(x, \widetilde{u}(x)) \, , \, \theta(x) \right\rangle \right| \, d|D^d u|(y) \\
&\quad + \frac{1}{|D^d u|(B_r(x))} \|\boldsymbol{b}\|_\infty \, |D^j u|(B_r(x)).
\end{aligned} \tag{46}$$

The conclusion is achieved now by taking $r \to 0$ and by using (c) and (d). □

5. Gluing constructions and extension theorems

A direct consequence of Theorems 4.3, 4.6 and [15, Theorems 5.1 and 5.3] are the following gluing constructions and extension theorems (the proofs are entirely similar to that of [14, Theorems 8.5 and 8.6] and [15, Theorems 5.1 and 5.3]).

Theorem 5.1 (Extension)**.** *Let $W \Subset int(E) \subset E \Subset U \subset \Omega$, where $\Omega, U, W \subset \mathbb{R}^N$ are open sets and E is a set of finite perimeter in Ω. Let*

$$\boldsymbol{b}_1 \colon U \times \mathbb{R} \to \mathbb{R}^N, \qquad \boldsymbol{b}_2 \colon (\Omega \setminus \overline{W}) \times \mathbb{R} \to \mathbb{R}^N$$

satisfy assumptions (i)–(iv) in $U \times \mathbb{R}$ and $\Omega \setminus \overline{W}$ respectively. Let $\boldsymbol{B}_1, \boldsymbol{B}_2$ be the corresponding integral functions with respect to the second variable. Given $u_1 \in BV_{\rm loc}(U) \cap L^\infty_{\rm loc}(U)$ and $u_2 \in BV_{\rm loc}(\Omega \setminus \overline{W}) \cap L^\infty_{\rm loc}(\Omega \setminus \overline{W})$, let $\mathbf{v}_i(x) := \boldsymbol{B}_i(x, u_i(x))$, $i = 1, 2$. Then the function

$$\mathbf{v}(x) := \begin{cases} \mathbf{v}_1(x), & \text{if } x \in E, \\ \mathbf{v}_2(x), & \text{if } x \in \Omega \setminus E, \end{cases}$$

belongs to $\mathcal{DM}^\infty_{\rm loc}(\Omega)$ and

$$\operatorname{div} \mathbf{v} = \chi_{E^1} \operatorname{div} \mathbf{v}_1 + \chi_{E^0} \operatorname{div} \mathbf{v}_2 + [\operatorname{Tr}^+(\mathbf{v}_1, \partial^* E) - \operatorname{Tr}^-(\mathbf{v}_2, \partial^* E)] \, \mathcal{H}^{N-1} \llcorner \partial^* E.$$

Theorem 5.2 (Gluing)**.** *Let $U \Subset \Omega \subset \mathbb{R}^N$ be open sets with $\mathcal{H}^{N-1}(\partial U) < \infty$, and let*

$$\boldsymbol{b}_1 \colon U \times \mathbb{R} \to \mathbb{R}^N, \qquad \boldsymbol{b}_2 \colon (\Omega \setminus \overline{U}) \times \mathbb{R} \to \mathbb{R}^N$$

satisfy assumptions (i)–(iv) in $U \times \mathbb{R}$ and $\Omega \setminus \overline{U}$ respectively. Let $\boldsymbol{B}_1, \boldsymbol{B}_2$ be the corresponding integral functions with respect to the second variable. Given $u_1 \in BV(U) \cap L^\infty(U)$ and $u_2 \in BV(\Omega \setminus \overline{U}) \cap L^\infty(\Omega \setminus \overline{U})$, let

$$\mathbf{v}_1(x) := \begin{cases} \boldsymbol{B}_1(x, u_1(x)), & \text{if } x \in U, \\ 0, & \text{if } x \in \Omega \setminus U, \end{cases} \qquad \mathbf{v}_2(x) := \begin{cases} 0, & \text{if } x \in \overline{U}, \\ \boldsymbol{B}_2(x, u_2(x)), & \text{if } x \in \Omega \setminus \overline{U}. \end{cases}$$

Then $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v} \in \mathcal{DM}^\infty(\Omega)$ and

$$\operatorname{div} \mathbf{v} = \chi_{U^1} \operatorname{div} \mathbf{v}_1 + \chi_{U^0} \operatorname{div} \mathbf{v}_2 + [\operatorname{Tr}^+(\mathbf{v}_1, \partial^* U) - \operatorname{Tr}^-(\mathbf{v}_2, \partial^* U)] \, \mathcal{H}^{N-1} \llcorner \partial^* U.$$

6. The Gauss–Green formula

Let $E \Subset \Omega$ be a set of finite perimeter. Using the conventions of Section 3.2, we will assume that the generalized normal vector on $\partial^* E$ coincides $\mathcal{H}^{N-1}$-a.e. on $\partial^* E$ with the measure–theoretic *interior* unit normal vector $\widetilde{\nu}_E$ to E.

We recall that, if $u \in BV_{\rm loc}(\Omega)$, then we will understand $u^\pm(x) = \widetilde{u}(x)$ for every $x \in \Omega \setminus S_u$.

The following result has been proved in [18] in the case $\boldsymbol{B}(x,w) = w\,\boldsymbol{A}(x)$ (see also [15, 28] for related results). To simplify the notation, we will denote by $\mu := (\boldsymbol{b}(\cdot,u), Du)$ the Radon measure introduced in (27).

Theorem 6.1 (Gauss–Green formula)**.** *Let $\boldsymbol{b}$ satisfy assumptions (i)–(iv) and let $\boldsymbol{B}$ be defined by (7). Let $E \Subset \Omega$ be a bounded set with finite perimeter and let $u \in BV_{\rm loc}(\Omega) \cap L^\infty_{\rm loc}(\Omega)$. Then the following Gauss–Green formulas hold:*

$$\int_{E^1} \frac{F(x,u^+(x)) + F(x,u^-(x))}{2}\, d\sigma(x) + \mu(E^1) = -\int_{\partial^* E} \beta^+(x,u^+(x))\; d\mathcal{H}^{N-1}\,, \tag{47}$$

$$\begin{aligned} \int_{E^1\cup\partial^* E} & \frac{F(x,u^+(x)) + F(x,u^-(x))}{2}\, d\sigma(x) + \mu(E^1\cup\partial^* E) \\ & = -\int_{\partial^* E} \beta^-(x,u^-(x))\; d\mathcal{H}^{N-1}\,, \end{aligned} \tag{48}$$

where E^1 is the measure theoretic interior of E, and $\beta^\pm(\cdot,t) := \mathrm{Tr}^\pm(\boldsymbol{b}(\cdot,t), \partial^* E)$ are the normal traces of $\boldsymbol{b}(\cdot,t)$ when $\partial^* E$ is oriented with respect to the interior unit normal vector.

Proof. Since E is bounded we can assume, without loss of generality, that $u \in BV(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$. By Theorem 4.3, the composite function $\mathbf{v}(x) := \boldsymbol{B}(x,u(x))$ belongs to $\mathcal{DM}^\infty$. Since E is a bounded set of finite perimeter, the characteristic function χ_E is a compactly supported BV function, so that $\operatorname{div}(\chi_E \mathbf{v})(\mathbb{R}^N) = 0$ (see [15, Lemma 3.1]).

We recall that, for every $w \in BV \cap L^\infty$ and every $\boldsymbol{A} \in \mathcal{DM}^\infty$, it holds

$$\operatorname{div}(w\boldsymbol{A}) = w^* \operatorname{div}\boldsymbol{A} + (\boldsymbol{A}, Dw),$$

where $(\boldsymbol{A}, Dw)$ is the Anzellotti pairing between the function w and the vector field $\boldsymbol{A}$ (see (5)). Hence, using the above formula with $w = \chi_E$ and $\boldsymbol{A} = \mathbf{v}$, it follows that

$$0 = \operatorname{div}(\chi_E\mathbf{v})(\mathbb{R}^N) = \int_{\mathbb{R}^N} \chi_E^*\, d\operatorname{div}\mathbf{v} + (\mathbf{v}, D\chi_E)(\mathbb{R}^N). \tag{49}$$

Since

$$(\mathbf{v}, D\chi_E) = [\mathrm{Tr}^+(\mathbf{v},\partial^* E) - \mathrm{Tr}^-(\mathbf{v},\partial^* E)]\mathcal{H}^{N-1} \llcorner \partial^* E,$$

from Proposition 4.5 we get

$$(\mathbf{v}, D\chi_E)(\mathbb{R}^N) = \int_{\partial^* E} [\beta^+(x,u^+(x)) - \beta^-(x,u^-(x))]\, d\mathcal{H}^{N-1}(x). \tag{50}$$

Since $\chi_E^* = \chi_{E^1} + \frac{1}{2}\chi_{\partial^* E}$, using again Proposition 4.5 and (29) it holds

$$\begin{aligned} \int_{\mathbb{R}^N} \chi_E^*\, d\operatorname{div}\mathbf{v} = & \operatorname{div}\mathbf{v}(E^1) + \frac{1}{2}\int_{\partial^* E} \frac{[\beta^+(x,u^+(x)) + \beta^-(x,u^-(x)}{2}\, d\mathcal{H}^{N-1}(x) \\ = & \int_{E^1} \frac{F(x,u^+(x)) + F(x,u^-(x))}{2}\, d\sigma(x) + \mu(E^1) \\ & + \frac{1}{2}\int_{\partial^* E} \frac{[\beta^+(x,u^+(x)) + \beta^-(x,u^-(x)}{2}\, d\mathcal{H}^{N-1}(x)\,. \end{aligned} \tag{51}$$

Formula (47) now follows from (49), (50) and (51).

The proof of (48) is entirely similar. ☐

It is worth to mention a consequence of the gluing construction given in Theorem 5.2 and the Gauss–Green formula (47). To this end, following [27], any bounded open set $\Omega \subset \mathbb{R}^N$ with finite perimeter, such that $\mathcal{H}^{N-1}(\partial\Omega) = \mathcal{H}^{N-1}(\partial^*\Omega)$, will be called *weakly* *regular*. For weakly regular sets we have the following version of the Gauss–Green formula (see [15, Corollary 5.5] for a similar statement for autonomous vector fields).

Theorem 6.2 (Gauss–Green formula for weakly regular sets)**.** *Let $\Omega \subset \mathbb{R}^N$ be a weakly regular set. Let $\boldsymbol{b}$ satisfy assumptions (i)–(iv), let $\boldsymbol{B}$ be defined by* (7) *and let $u \in BV(\Omega) \cap L^\infty(\Omega)$. Then the following Gauss–Green formula holds:*

$$\int_\Omega \frac{F(x,u^+(x)) + F(x,u^-(x))}{2}\, d\sigma(x) + \mu(\Omega) = -\int_{\partial\Omega} \beta^+(x,u^+(x))\; d\mathcal{H}^{N-1}\,. \tag{52}$$

Proof. Since Ω is a set of finite perimeter, it holds $\partial^*\Omega \subseteq \partial\Omega$, hence the assumption $\mathcal{H}^{N-1}(\partial\Omega) = \mathcal{H}^{N-1}(\partial^*\Omega)$ of weak regularity implies that $\mathcal{H}^{N-1}(\partial\Omega \setminus \partial^*\Omega) = 0$. Consequently,

$$\mathcal{H}^{N-1} \llcorner \partial\Omega = \mathcal{H}^{N-1} \llcorner \partial^*\Omega, \quad \mathcal{H}^{N-1}(\Omega^1 \setminus \Omega) = 0, \quad \mathcal{H}^{N-1}(\Omega^0 \setminus (\mathbb{R}^N \setminus \overline{\Omega})) = 0. \tag{53}$$

Let us consider the vector field

$$\mathbf{v}(x) := \begin{cases} \mathbf{v}_1(x) := \boldsymbol{B}(x,u(x)), & \text{if } x \in \Omega, \\ 0, & \text{if } x \in \mathbb{R}^N \setminus \Omega\,. \end{cases}$$

By Theorem 5.2 and (53) we have that $\mathbf{v} \in \mathcal{DM}^\infty$ and

$$\operatorname{div} \mathbf{v} = \chi_\Omega \operatorname{div} \mathbf{v}_1 + \operatorname{Tr}^+(\mathbf{v}_1, \partial\Omega)\, \mathcal{H}^{N-1} \llcorner \partial\Omega,$$

hence (52) follows reasoning as in the proof of (47). □

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Dipartimento di Matematica "G. Castelnuovo", Univ. di Roma I, P.le A. Moro 2 – I-00185 Roma (Italy)

Email address: crasta@mat.uniroma1.it

Dipartimento di Scienze di Base e Applicate per l'Ingegneria, Univ. di Roma I, Via A. Scarpa 10 – I-00185 Roma (Italy)

Email address: virginia.decicco@sbai.uniroma1.it