TOWARDS GEOMETRIC INTEGRATION OF ROUGH DIFFERENTIAL FORMS

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Abstract. We provide a draft of a theory of geometric integration of "rough differential forms" which are generalizations of classical (smooth) differential forms to similar objects with very low regularity, for instance, involving Hölder continuous functions that may be nowhere differentiable. Borrowing ideas from the theory of rough paths, we show that such a geometric integration can be constructed substituting appropriately differentials with more general asymptotic expansions. This can be seen as the basis of geometric integration similar to that used in geometric measure theory, but without any underlying differentiable structure, thus allowing Lipschitz functions and rectifiable sets to be substituted by far less regular objects (e.g. Hölder functions and their images which may be purely unrectifiable). Our construction includes both the one-dimensional Young integral and multidimensional integrals introduced recently by R. Züst. To simplify the exposition, we limit ourselves to integration of forms of dimensions not exceeding two.

Contents

1. Introduction

In [15] and independently in [10] it has been shown that for Hölder continuous functions $f \in C^\alpha([0,1])$, $g \in C^\beta([0,1])$ with $\alpha + \beta > 1$ the Riemann-type integral

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$\int_0^1 f(x)\,\mathsf{d}g(x)$ may be defined (as usual, as a suitable limit of finite sums of the form $\sum_i f(x_i)(g(x_{i+1})-g(x_i)$ over a refining family of partitions $0=x_0\leq x_1\leq\ldots\leq x_m=1$ of the interval of integration, similarly to the classical case of smooth f and g). Much later, in the late 1980s, this construction which nowadays is commonly known as *Young integral*, has been understood as a particular case of *rough path* construction [11] via so-called *sewing lemma* [5, 2, 3]. The natural extension of the Young integral from one-dimensional to multidimensional case is even more recent. In fact, R. Züst showed in [16] that a k-dimensional Young-type integral (which we therefore will further call *Züst integral*)

$$\int_{[0,1]^k} f\,\mathsf{d}g^1\wedge\ldots\wedge\mathsf{d}g^k \tag{1.1}$$

can be defined provided that $f\in C^\alpha$, $g^i\in C^{\beta_i}$, and $\alpha+\sum_{i=1}^k\beta_i>k$. The definition provided in [16] is quite "manual" and uses dimension reduction via the integration by parts type procedure. In this paper we show that all the above-mentioned constructions, i.e. Young integrals, integrals of one-dimensional rough paths (of Hölder regularity larger than $1/3$) and multidimensional Züst integrals are particular cases of a general geometric construction which can be seen as integration of *"rough" differential forms*. For instance, we see the Young integral as an integral over an interval of a one-dimensional form $f\,\mathsf{d}g$, and Züst integral as an integral over a k-dimensional cube of a k-dimensional form $f\mathsf{d}g^1\wedge\ldots\wedge\mathsf{d}g^k$. Both are constructed from their "discrete versions", which can be viewed as "germs of rough forms". In particular, the one-dimensional form $f\,\mathsf{d}g$ and its integral are built from the one-dimensional germ, which is a function of two variables (or, equivalently, of a one-dimensional simplex) $(x,y)\mapsto f(x)(g(y)-g(x))$, while k-dimensional forms and their integrals are built from k-dimensional forms which are functions of $k+1$ variables (or, equivalently, of a k-dimensional simplex). The construction of forms and their integrals from germs is done by the "sewing" procedure, which in a certain sense "sews" local objects (germs) into global ones (forms and their integrals). The latter are provided by a general multidimensional *sewing lemma*, which is a natural extension of the one dimensional case of rough paths theory.

Let us point out that other approaches to integration of higher dimensional irregular objects (beyond measures) appear both in the stochastic and geometric literature. In the former setting, we mention [14, 4, 1] for integration of Young and Rough "sheets", [6] for paracontrolled calculus based on Fourier decomposition, and the reconstruction theorem in the celebrated theory of Regularity Structures [7]. To the authors' knowledge, even the definition of Züst integrals (1.1) for $k>1$ falls outside the scopes of these approaches, essentially because they miss a key "dimension reduction" feature which clearly appears when one uses the geometric language of simplexes and their boundaries. On the geometric side, besides [16], we mention also possible connections with the chainlet approach in [8], motivated by variational problems (Plateau's problem) and numerical approximations, with possibly very irregular integration domains: however, also in this case, already Züst integrals (1.1) seem to be outside the scope of the theory. It has to be noted that similar theories can be developed for more general germs, for instance $(x,y)\mapsto\left(⨍_{[x,y]} f\right)(g(y)-g(x))$, $⨍$ standing for the average. One dimensional integrals arising from such germs have been studied in [12] (see also [9]); here for simplicity we do not pursue this direction.

Aim of this paper is to provide a brief sketch of a possible general theory of integration of rough differential forms: to keep the exposition simple, we limit ourselves to 0, 1 and 2-dimensional forms. In Section 2 we introduce notation and basic facts about simplicial chains and "germs of forms". In Section 3 we state the main results, in particular the multidimensional *sewing lemma*, Theorem 3.19. In Section 4 we show how Young and Züst integrals fit into our framework, and in Section 5 we point out extensions to more irregular functions, borrowing ideas from rough paths theory. Most of the technical proofs are collected in the appendices.

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2. Simplices and germs

We introduce some basic notions of simplicial geometry, extending the point of view of [5, Section 2]. Let throughout the paper d, $\tilde{d}$, h, k, be nonnegative integers and $D \subseteq \mathbb{R}^d$, $\tilde{D} \subseteq \mathbb{R}^{\tilde{d}}$ be sets.

2.1. **Polyhedral chains.**

The basic object of our study are real polyhedral k-chains (which provide a discrete counterpart of k-currents). As already stated, our main results use these notions only for small values of k ($k \leq 3$), but at this point general definitions are more convenient.

Definition 2.1 (simplices and chains). *Any $S = [p_0 p_1 \ldots p_k] \in D^{k+1}$ such that $\operatorname{conv}(S) \subseteq D$ is called a k-simplex (in D). A (real polyhedral) k-chain (in D) is any element C belonging to the real vector space generated by k-simplices in D.*

We write $\operatorname{Simp}^k(D)$ for the set of k-simplices in D and $\operatorname{Chain}^k(D)$ for the linear space of k-chains in D. It is useful to define $\operatorname{Chain}^{-1}(D)$ as the set containing the singleton $[] = \emptyset$. We use throughout the paper the property that any map defined on $\operatorname{Simp}^k(D)$ with values on a vector space naturally extends to a linear map on $\operatorname{Chain}^k(D)$. In particular, this holds for maps taking values in $\operatorname{Chain}^h(D)$.

Remark 2.2 (representation of simplices and chains). A k-simplex can be thought as the "geometric" simplex $\operatorname{conv}(S)$ with a base point p_0 and the orientation given by the ordering of the points. Hence, 0-simplices are points, 1-simplices are oriented segments from the point p_0 to p_1 and 2-simplices are pointed oriented triangles, see Figure 2.1. Similarly, chains are weighted families of these objects (for simplicity we omit to specify unit weights in the drawings).

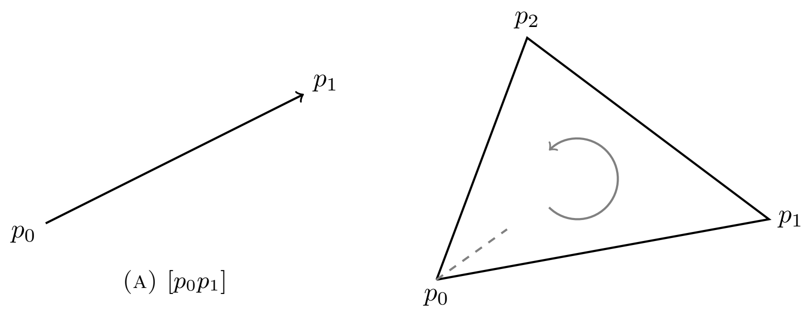
Push-forward. Any $\varphi : D \subseteq \mathbb{R}^d \to \mathbb{R}^{\tilde{d}}$ induces a *push-forward* operator (denoted with $\varphi_\natural$), mapping k-simplices in D to k-simplices in $\tilde{D} := \operatorname{conv}(\varphi(D))$,

$$\varphi : \operatorname{Simp}^k(D) \to \operatorname{Simp}^k(\tilde{D}), \qquad S = [p_0 p_1 \ldots p_k] \mapsto \varphi_\natural S := [\varphi(p_0)\varphi(p_1)\ldots\varphi(p_k)],$$

and extended by linearity to $\operatorname{Chain}^k(D)$. If $\varphi(x) = Ax + q$ is affine, we write $\varphi_\natural S = A_\natural S + q$, and if $\varphi(x) = \lambda x$, for $\lambda \in \mathbb{R}$, we write $\varphi_\natural S = \lambda_\natural S$.

Definition 2.3 (isometric simplices). *Given $S^i = [p_0^i p_1^i \ldots p_k^i] \in \operatorname{Simp}^k$, $i \in \{1, 2\}$, we say that S^1 and S^2 are* isometric *if there exists an isometry $\mathfrak{i} \colon \mathbb{R}^d \to \mathbb{R}^d$ such that $\mathfrak{i}_\natural S^1 = S^2$, i.e. $\mathfrak{i}(p_j^1) = p_j^2$, for every $j \in \{0, 1, \ldots, k\}$.*

Notice however that two k-simplices $S^i = [p_0^i p_1^i \ldots p_k^i]$, $i \in \{1, 2\}$, coincide if and only if $p_j^1 = p_j^2$ for $j \in \{0, 1, \ldots, k\}$.



(A) $[p_0p_1]$

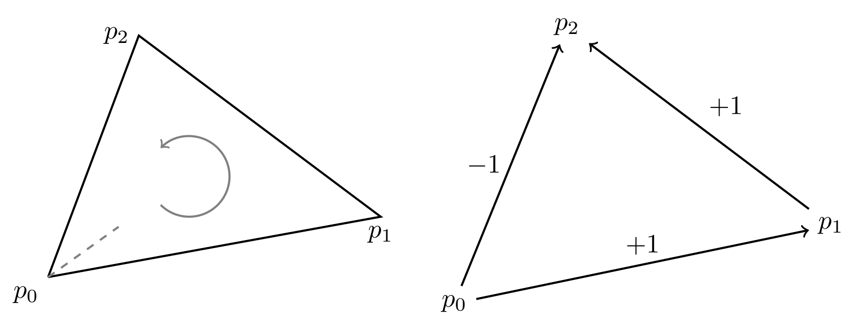
(B) $[p_0p_1p_2]$

Figure 2.1. Representations of 1-simplices and 2-simplices .

Boundary. The boundary operator is defined on k-simplexes as

$$\partial : \mathrm{Simp}^k(D) \to \mathrm{Chain}^{k-1}(D), \qquad \partial[p_0p_1\ldots, p_k] := \sum_{i=0}^{k} (-1)^i [p_0 \ldots \hat{p}_i \ldots p_k],$$

the notation $\hat{p}_i$ meaning that the element is removed from the list (Figure 2.1), and extended by linearity on k-chains.



(A) $[p_0p_1p_2]$

(B) $\partial[p_0p_1p_2] = [p_1p_2] - [p_0p_2] + [p_0p_1]$

Figure 2.2. A 2-simplex and its boundary.

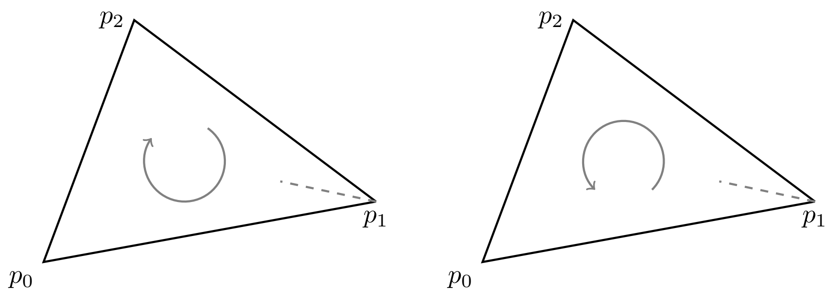
Clearly, $\partial\varphi_\natural = \varphi_\natural\partial$, for any $\varphi : D \to \mathbb{R}^{\tilde{d}}$. A simple argument gives $\partial\partial = 0$, for in the double summation that one obtains, for any $i < j$ the $(k-1)$-simplex obtained removing the elements $\{p_i, p_j\}$ appears once with sign $(-1)^{i+j}$ (when we remove first j and then i) and once with opposite sign (when we remove first i and then j).

Permutations. Any permutation σ on $\{0, 1, \ldots, k\}$, induces an operator (Figure 2.3)

$$\sigma : \mathrm{Simp}^k(D) \to \mathrm{Simp}^k(D), \qquad S = [p_0p_1 \ldots p_k] \mapsto \sigma S := [p_{\sigma^{-1}(0)}p_{\sigma^{-1}(1)} \ldots p_{\sigma^{1}(k)}].$$

One has $(\sigma\tau)S = \sigma(\tau S)$, for permutations σ, τ, and $\varphi_\natural\sigma = \sigma\varphi_\natural$, for $\varphi : D \to \mathbb{R}^{\tilde{d}}$.

2.2. **Germs of rough forms.** Function on simplices are counterparts of differential forms.



(A) $[p_1p_0p_2]$ (B) $[p_1p_2p_0]$

FIGURE 2.3. Permutations of the simplex $[p_0p_1p_2]$.

Definition 2.4 (germs and cochains)**.** *A* k-germ *(of a k-differential form in D) is a function* $\omega : \mathrm{Simp}^k(D) \to \mathbb{R}$*,*

$$S = [p_0p_1 \dots p_k] \mapsto \omega_S = \omega_{p_0p_1\dots p_k}.$$

A k-cochain *(in D) is a linear functional* $\omega : \mathrm{Chain}^k(D) \to \mathbb{R}$*,*

$$C \mapsto \langle C, \omega \rangle .$$

We write $\mathrm{Germ}^k(D)$ for the set of k-germs and $\mathrm{Cochain}^k(D)$ for the space of k-cochains in D. There is a natural correspondence between germs and cochains, and therefore we conveniently switch between the notations ω_S and $\langle S, \omega\rangle$, for $S \in \mathrm{Simp}^k(D)$, $\omega \in \mathrm{Germ}^k(D)$. For simplicity, we deal mainly with germs taking real values, but most of the notions and results extend verbatim to germs with values in a Banach space, and we then write $\mathrm{Germ}^k(D; B)$.

Examples 2.5. Since 0-simplices are points, 0-germs are functions $f(p_0) = f_{p_0} = \langle [p_0], f\rangle$. 1-germs are functions on oriented segments on D and 2-germs are functions on pointed oriented triangles.

Remark 2.6 (composition). The composition of $n \geq 1$ k-germs $(\omega^i)_{i=1}^n$ with a function $F : \mathbb{R}^n \to \mathbb{R}$ defines a k-germ $F \circ (\omega^i)_{i=1}^n : S \mapsto F\left(\left(\omega^i_S\right)_{i=1}^n\right)$. If F is not linear one should be careful when dealing with the corresponding k-cochain, because it is not given by the composition of F with the corresponding k-cochains. This notion however is useful e.g. to define the germ $|\omega|$, given by $|\omega|_S := |\omega_S|$.

Remark 2.7 (pointwise product). As another example of composition, we define the pointwise product of two k-germs as $(\omega^1\omega^2)_S := \omega^1_S\omega^2_S$, for $S \in \mathrm{Simp}^k(D)$.

Remark 2.8 (inequalities). Given k-germs ω, $\tilde{\omega}$, we write $\omega \leq \tilde{\omega}$ if, for every $S \in \mathrm{Simp}^k(D)$ one has $\omega_S \leq \tilde{\omega}_S$. This notion is useful to state bounds for germs such as $|\omega| \leq \mathsf{v}$, for some $\mathsf{v} \in \mathrm{Germ}^k(D)$.

Maps acting on simplices (or linear operators the space of chains) induce by duality maps on germs (or linear operators on cochains): in general, given $\tau\colon \mathrm{Simp}^k(D) \to \mathrm{Chain}^h(\tilde{D})$, let $\tau'\colon \mathrm{Cochain}^h(\tilde{D}) \to \mathrm{Cochain}^k(D)$ be the linear operator defined by

$$(\tau'\omega)_S = \langle S, \tau'\omega\rangle := \langle \tau S, \omega\rangle , \quad \text{for } S \in \mathrm{Simp}^k(D).$$

Let us consider the duals of the three maps introduced in the previous section.

(Algebraic) pull-back. For $\varphi : D \to \mathbb{R}^{\tilde{d}}$, $\tilde{D} = \mathrm{conv}(\tilde{D})$, the dual of $\varphi_\natural$ is the (algebraic) pull-back $\varphi^\natural := \varphi'_\natural$,

$$\varphi^\natural : \mathrm{Germ}^k(\tilde{D}) \to \mathrm{Germ}^k(D), \qquad \left\langle S, \varphi^\natural \omega \right\rangle := \left\langle \varphi_\natural S, \omega \right\rangle .$$

Exterior derivative. The dual operator $\delta := \partial'$,

$$\delta : \mathrm{Germ}^k(D) \to \mathrm{Cochain}^{k+1}(D), \qquad \langle S, \delta\omega \rangle := \langle \partial S, \omega \rangle ,$$

provide a provides a discrete counterpart to the exterior derivative.

Examples 2.9. If $\omega \in \mathrm{Germ}^0(D)$, then for $[p_0 p_1] \in \mathrm{Simp}^1(D)$,

$$\langle [p_0p_1], \delta f \rangle = \langle [p_1] - [p_0], f \rangle , \quad \text{or} \quad \delta f_{p_0 p_1} = f_{p_1} - f_{p_0}.$$

If $\omega \in \mathrm{Germ}^1(D)$, then for $[p_0p_1p_2] \in \mathrm{Simp}^2(D)$,

$$\langle [p_0p_1p_2], \delta\omega \rangle = \langle [p_1p_2] - [p_0p_2] + [p_0p_1], \omega \rangle$$

or, in a different notation, $\delta\omega_{p_0p_1p_2} = \omega_{p_1p_2} - \omega_{p_0p_2} + \omega_{p_0p_1}$.

Definition 2.10 (exact and closed germs)**.** *We say that a k-germ ω is* exact *if $\omega = \delta\eta$ for some $(k-1)$-germ η, and is* closed *if $\delta\omega = 0$.*

The identity $\partial\partial = 0$ gives $\delta\delta = 0$, i.e. every exact germ is closed.

Remark 2.11 (discrete Poincaré lemma). If D is convex, all closed germs are exact. In fact, fix a $\bar{p} \in D$, and for $\omega \in \mathrm{Germ}^k(D)$ closed, define $\eta \in \mathrm{Germ}^{k-1}(D)$ via

$$\langle [p_0p_1 \ldots p_{k-1}], \eta \rangle := \langle [\bar{p}p_0p_1 \ldots p_{k-1}], \omega \rangle ,$$

for $[p_0p_1 \ldots p_{k-1}] \in \mathrm{Simp}^{k-1}(D)$. Given $S = [p_0p_1 \ldots p_k] \in \mathrm{Simp}^k(D)$,

$$\begin{aligned} \langle S, \delta\eta \rangle &= \sum_{i=0}^{k} (-1)^i \langle [p_0p_1 \ldots \hat{p}_i \ldots p_k], \eta \rangle = \sum_{i=0}^{k} (-1)^i \langle [\bar{p}p_0p_1 \ldots \hat{p}_i \ldots p_k], \omega \rangle \\ &= \langle [p_0p_1 \ldots p_k], \omega \rangle - \langle [\bar{p}p_0 \ldots p_k], \delta\omega \rangle = \langle S, \omega \rangle . \end{aligned}$$

Definition 2.12. *We say that a k-germ is* alternating *if $\sigma'\omega = (-1)^\sigma \omega$, i.e.,*

$$\langle \sigma S, \omega \rangle = (-1)^\sigma \langle S, \omega \rangle , \quad \textit{for every } S \in \mathrm{Simp}^k(D),$$

for every permutation σ of $\{0, 1, \ldots, k\}$, where $(-1)^\sigma$ denotes its sign.

It is not difficult to check that $\delta\omega$ is alternating when so is ω. In particular, δf is alternating for every $f \in \mathrm{Germ}^0(D)$.

2.3. **Cup products.** We define the *cup product* between a k-germ ω and a h-germ $\tilde{\omega}$ as the $(k+h)$-germ $\omega \cup \tilde{\omega}$ acting on $[p_0p_1 \ldots p_kp_{k+1} \ldots p_{k+h}] \in \mathrm{Simp}^{k+h}(D)$ as

$$\langle [p_0p_1 \ldots p_kp_{k+1} \ldots p_{k+h}], \omega \cup \tilde{\omega} \rangle := \langle [p_0p_1 \ldots p_k], \omega \rangle \langle [p_kp_{k+1} \ldots p_{k+h}], \tilde{\omega} \rangle .$$

In [5] this operation is called external product, of which it can be also thought as a discrete counterpart. In general, the cup product is not commutative, although when $f \in \mathrm{Germ}^0(D)$ we may write with a slight abuse of notation $f\omega := f \cup \omega$. Indeed, for a 1-form ω, one has

$$\omega \cup f - f \cup \omega = (\delta f)\omega, \quad \text{i.e.} \quad \omega_{p_0p_1} f_{p_1} - f_{p_0}\omega_{p_0p_1} = (\delta f_{p_0p_1})\omega_{p_0p_1}. \tag{2.1}$$

The cup product is associative, and the following Leibniz rule holds: for $\omega \in \mathrm{Germ}^k(D)$, $\tilde{\omega} \in \mathrm{Germ}^h(D)$, one has

$$\delta(\omega \cup \tilde{\omega}) = (\delta\omega) \cup \tilde{\omega} + (-1)^k \omega \cup (\delta\tilde{\omega}). \tag{2.2}$$

Indeed, given $S=[p_0p_1\dots p_kp_{k+1}\dots p_{k+h}p_{k+h+1}]$, one has

$$\begin{aligned}\langle \partial S,\omega\cup\tilde\omega\rangle &= \sum_{i=0}^{k}(-1)^i\left\langle [p_0\dots\hat p_i\dots p_kp_{k+1}\dots p_{k+h+1}],\omega\cup\tilde\omega\right\rangle\\ &\quad+(-1)^k\sum_{i=1}^{h+1}(-1)^i\left\langle [p_0\dots\dots p_kp_{k+1}\dots\hat p_{k+i}\dots p_{k+h+1}],\omega\cup\tilde\omega\right\rangle\\ &=\sum_{i=0}^{k+1}(-1)^i\left\langle [p_0\dots\hat p_i\dots p_{k+1}],\omega\right\rangle\left\langle [p_{k+1}\dots p_{k+h+1}],\tilde\omega\right\rangle\\ &\quad+(-1)^k\sum_{i=0}^{h+1}(-1)^i\left\langle [p_0\dots\dots p_k],\omega\right\rangle\left\langle [p_k\dots\hat p_{k+i}\dots p_{k+h+1}],\tilde\omega\right\rangle\\ &=\left\langle S,(\delta\omega)\cup\tilde\omega+(-1)^k\omega\cup(\delta\tilde\omega)\right\rangle.\end{aligned}$$

2.4. **Continuity and convergence.** We say that

(i) $\omega\in\mathrm{Germ}^k(D)$ is *continuous* if $S\mapsto\omega_S$ is a continuous function of $S\in\mathrm{Simp}^k(D)=D^{k+1}$, i.e. given any sequence $S^n=[p_0^np_1^n\dots p_k^n]$ converging to $S=[p_0p_1\dots p_k]$, i.e., such that that $p_i^n\to p_i$, as $n\to\infty$, for $i\in\{0,1,\dots,k\}$, one has $\omega_{S^n}\to\omega_S$

(ii) a sequence $(\omega^n)_{n\geq1}\subseteq\mathrm{Germ}^k(D)$ converge pointwise to $\omega\in\mathrm{Germ}^k(D)$ if $\lim_{n\to+\infty}\omega^n_S=\omega_S$ for every $S\in\mathrm{Simp}^k(D)$.

Quantitative aspects of continuity and convergence follow by introducing suitable *gauges*, such as volumes.

Gauges. A k-gauge in D is a nonnegative k-germ $\mathsf{v}\in\mathrm{Simp}^k(D)$. We will call a k-gause *uniform* if

$$\langle S,\mathsf{u}\rangle=\langle S',\mathsf{u}\rangle,\qquad\text{whenever } S \text{ and } S' \text{ are isometric } k\text{-simplices in } D.$$

Given a k-germ ω and a k-gauge v, we write $[\omega]_{\mathsf{v}}\in[0,\infty]$ for the infimum over all constants c such that $|\omega|\leq\mathsf{c}\mathsf{v}$, i.e.

$$|\omega_S|\leq\mathsf{c}\,\langle S,\mathsf{v}\rangle\quad\text{for every } S\in\mathrm{Simp}^k(D).$$

In particular, we think of $[\omega]_{\mathsf{v}}<\infty$ as a condition on a "size" of ω, compared to v. Notice that, if $\mathsf{v}\leq\tilde{\mathsf{v}}$, then $[\cdot]_{\tilde{\mathsf{v}}}\leq[\cdot]_{\mathsf{v}}$ and that the triangle inequality holds:

$$[\omega+\tilde\omega]_{\mathsf{v}}\leq[\omega]_{\mathsf{v}}+[\tilde\omega]_{\mathsf{v}}.$$

In fact, $[\cdot]_{\mathsf{v}}$ defines a norm over the space of all $\omega\in\mathrm{Germ}^k(D)$ such that $[\omega]_{\mathsf{v}}<\infty$.

For $\{\omega,\tilde\omega\}\subset\mathrm{Germ}^k(D)$, and a k-gauge v, we write

$$\omega\approx_{\mathsf{v}}\tilde\omega\quad\text{if and only if}\quad[\omega-\tilde\omega]_{\mathsf{v}}<\infty.$$

The relation $\approx_{\mathsf{v}}$ is an equivalence relation. The properties of gauges imply that, if $\omega\approx_{\mathsf{v}}\tilde\omega$ and $\eta\approx_{\mathsf{w}}\eta'$, then $\omega+\eta\approx_{\max\{\mathsf{v},\mathsf{w}\}}\tilde\omega+\eta'$.

Remark 2.13 (gauges and pull-back). Given $\omega\in\mathrm{Germ}^k(\tilde D)$ and $\varphi:D\to\tilde D$, we have

$$[\varphi^\natural\omega]_{\varphi^\natural\mathsf{v}}\leq[\omega]_{\mathsf{v}}$$

for every k-gauge v in $\tilde D$. Indeed, for every $S\in\mathrm{Simp}^k(D)$ we have

$$\left|\left\langle S,\varphi^\natural\omega\right\rangle\right|=|\langle\varphi_\natural S,\omega\rangle|\leq[\omega]_{\mathsf{v}}\,\langle\varphi_\natural S,\mathsf{v}\rangle=[\omega]_{\mathsf{v}}\left\langle S,\varphi^\natural\mathsf{v}\right\rangle.$$

Volumes. Given $S = [p_0 \dots p_k] \in \mathrm{Simp}^k(D)$ its k-volume is

$$\mathrm{vol}_k(S) := \frac{1}{k!}\sqrt{\det(PP^\tau)}$$

where $P_{ij} := (p_i - p_0)^j$, $i \in \{1, \dots, k\}$. More generally, for $h \leq k$, we define the h-volume of S as

$$\mathrm{vol}_h(S) := \max_{S \supseteq S' \in \mathrm{Simp}^h(D)} \mathrm{vol}_h(S'),$$

the maximum in the above formula being taken among all possible choices of subsets S' of $h+1$ points from the $k+1$ points of S. We are particularly interested in the cases $h \in \{1, 2\}$. For $h = 1$, we have that

$$\mathrm{vol}_1(S) = \max_{0 \leq i,j \leq k} |p_i - p_j| =: \mathrm{diam}(S)$$

is just the diameter of S. Notice that vol_h (on k-simplices) defines a k-germ (in fact, a uniform k-gauge) for every $h \leq k$, so in what follows we write expressions such as $\omega \leq \mathrm{diam}$ or $\omega \leq \mathrm{vol}_h$ (or even functions of vol_h and diam) and $\varphi^\natural \,\mathrm{diam}$, $\varphi^\natural \,\mathrm{vol}_h$.

Example 2.14. Using inequalities between germs, we can restate α-Hölder continuity for a $f \in \mathrm{Germ}^0(D)$ by requiring that $|\delta f| \leq \mathsf{c}\,\mathrm{diam}^\alpha$, that is

$$|(\delta f)_{p_0p_1}| = |f_{p_1} - f_{p_0}| \leq \mathsf{c}\,|p_1 - p_0|^\alpha, \quad \text{for } [p_0p_1] \in \mathrm{Simp}^1(D),$$

where $\mathsf{c} \geq 0$ is some constant. The smallest such constant is denoted $\|\delta f\|_\alpha$.

Motivated by the previous example, when $\mathsf{v} = \mathrm{diam}^\gamma = \mathrm{vol}_1^\gamma$, for some $\gamma \in \mathbb{R}$, we simply write $\|\cdot\|_\gamma := [\cdot]_\mathsf{v}$ and $\approx_\gamma$ instead of $\approx_\mathsf{v}$, in particular, $\|\omega\|_0$ coincides with the sup norm of ω. Since, for some constant $\mathsf{c} > 0$, one has the isodiametric inequality (on triangles)

$$\mathrm{vol}_2 \leq \mathsf{c}\,\mathrm{diam}^2,$$

then for some (different) constant $\mathsf{c} = \mathsf{c}(\gamma_2) \geq 0$, one has $[\cdot]_{\mathrm{diam}^{\gamma_1+2\gamma_2}} \leq \mathsf{c}[\cdot]_{\mathrm{diam}^{\gamma_1}\,\mathrm{vol}_2^{\gamma_2}}$, for any $\gamma_2 \geq 0$.

Dini gauges. We introduce the following generalization of the "control functions" from [2].

Definition 2.15. *For an $r \geq 0$, a k-gauge $\mathsf{u} \in \mathrm{Germ}^k(D)$ is called* r-Dini *if it is uniform and for every $S \in \mathrm{Simp}^k(D)$,*

$$\lim_{n\to\infty} 2^{nr}\left\langle \left(2^{-n}\right)_\natural S, \mathsf{u}\right\rangle = 0.$$

If, in addition, u *is such that,*

$$\langle S, \tilde{\mathsf{u}}\rangle := \sum_{n=0}^{+\infty} 2^{nr}\left\langle \left(2^{-n}\right)_\natural S, \mathsf{u}\right\rangle < \infty, \quad \text{for every } S \in \mathrm{Simp}^k(D), \tag{2.3}$$

then u *is called* strong r-Dini*.*

Notice that, in general, $(2^{-n})_\natural S$ may not belong to $\mathrm{Simp}^k(D)$, but since u is the same on isometric simplices (because u is uniform), it is sufficient to consider any translated copy of it which belongs to $\mathrm{Simp}^k(D)$. The terminology "Dini" is reminiscent of the one, closely related, about moduli of continuity of functions.

Remark 2.16. The following immediate observations are worth being made.

(i) Every r-Dini gauge is strong h-Dini for every $h < r$.
(ii) For a k-gauge $\mathsf{u} \in \mathrm{Germ}^k(D)$ to be r-Dini (resp. strong r-Dini), it is sufficient that $\mathsf{u}(S) = o(\mathrm{diam}(S)^r)$ (resp. $\mathsf{u}(S) = o(\mathrm{diam}(S)^{r+\alpha})$ for some $\alpha > 0$) as $\mathrm{diam}\, S \to 0$, $S \in \mathrm{Germ}^k(D)$. For instance, if $h \in \{2, \ldots, k\}$, then the k-gauge vol_h is strong r-Dini for every $r \in (0, h)$.
(iii) Sums and pointwise maxima of a finite number of (strong) r-Dini gauges are (strong) r-Dini.
(iv) If u is strong r-Dini, then $\tilde{\mathsf{u}}$ in (2.3) is r-Dini, for

$$\begin{aligned}\lim_{n\to\infty} 2^{nr} \left\langle (2^{-n})_\natural S, \tilde{\mathsf{u}} \right\rangle &= \lim_{n\to\infty} \sum_{m=0}^{+\infty} 2^{mr} 2^{nr} \left\langle \left(2^{-m}\right)_\natural \left(2^{-n}\right)_\natural S, \mathsf{u} \right\rangle \\ &= \lim_{n\to\infty} \sum_{m=n}^{+\infty} 2^{mr} \left\langle \left(2^{-m}\right)_\natural S, \mathsf{u} \right\rangle = 0.\end{aligned}$$

(v) Moreover, if u is strong h-Dini for some $h > r$, then $\tilde{\mathsf{u}}$ in (2.3) defined using r (not h) is h-Dini as well, since it is smaller than $\tilde{\mathsf{u}}$ in (2.3) defined using h instead of r.

Example 2.17. The k-gauge $\mathsf{u} := \mathrm{diam}^{\gamma_1} \mathrm{vol}_2^{\gamma_2}$ is strong r-Dini for every $r < \gamma_1 + 2\gamma_2$, with

$$\begin{aligned}\langle S, \tilde{\mathsf{u}} \rangle = \sum_{n=0}^{+\infty} 2^{nr} \left\langle (2^{-n})_\natural S, \mathsf{u} \right\rangle &= \sum_{n=0}^{+\infty} 2^{nr} 2^{-n(\gamma_1+2\gamma_2)} \langle S, \mathsf{u} \rangle \\ = (1 - 2^{r-(\gamma_1+2\gamma_2)})^{-1} \langle S, \mathsf{u} \rangle\, ,&\end{aligned}$$

for every $S \in \mathrm{Simp}^k(D)$.

3. Sewable germs

In this section, we investigate k-germs that can be locally "integrated" along all affine k-planes, providing in particular a version of the so-called sewing lemma in case of 2-germs, Theorem 3.19. Its proof, together with that of the other stated theorems, is postponed in Appendix B.

3.1. Nonatomic and regular germs.

We further need the following notions.

Definition 3.1. *We call a k-germ $\omega \in \mathrm{Germ}^k(D)$*

(i) *nonatomic, if $\omega_S = 0$ for every $S \in \mathrm{Simp}^k(D)$ with $\mathrm{vol}_k\, S = 0$,*
(ii) *closed on k-planes, if for every $\varphi\colon I \subseteq \mathbb{R}^k \to D$ affine, the germ $\varphi^\natural \omega \in \mathrm{Germ}^k(I)$ is closed, i.e., $\delta \varphi^\natural \omega = 0$.*
(iii) *regular, if it is non-atomic and closed on k-planes.*

Clearly, each of the above conditions remain preserved by linear combinations of germs, and hence define the linear subspaces of germs. Notice also that the nonatomicity condition is void for $k = 0$, while for $k = 1$ it reduces just to $\omega_{pp} = 0$, for every $p \in D$. For $k \geq 2$, this condition becomes more relevant with respect to the possibility of "integrating" k-germs on k-planes. Finally, we mention that by Remark 2.11 one could replace closedness on k-planes with exactness on convex subsets of k-planes.

Remark 3.2. Every regular k-germ $\omega \in \mathrm{Germ}^k(D)$ is additive in the sense that if $S \in \mathrm{Simp}^k(D)$ and $\{S_i\} \subset \mathrm{Simp}^k(D)$ is a finite family such that $S = \sum_i S_i + N + \partial T$, where T is a $(k+1)$-chain with $\mathrm{vol}_{k+1}(T) = 0$, i.e., $\mathrm{conv}(T) \subseteq \pi$ for some k-plane π, and $N = \sum_j a_j N_j \in \mathrm{Chain}^k(D)$ with $\mathrm{vol}_k(N_j) = 0$ for every j, then then

$$\langle S, \omega \rangle = \sum_i \langle S_i, \omega \rangle ,$$

since $\langle N, \omega \rangle = 0$ and $\langle \partial T, \omega \rangle = \langle T, \delta\omega \rangle = 0$.

Proposition 3.3. *Every regular $\omega \in \mathrm{Germ}^k(D)$ is alternating in the sense of Definition 2.12.*

Proof. This follows from Lemma A.3 and Remark A.2 applied on the restriction of ω to a k-plane containing any gives $S \in \mathrm{Simp}^k(D)$. □

Proposition 3.4. *A k-germ $\omega \in \mathrm{Germ}^k(D)$ is regular if and only if both ω and $\delta\omega$ are nonatomic.*

Proof. If ω is regular, it suffices to show that $\delta\omega$ is nonatomic. The latter is true because if $T \in \mathrm{Simp}^{k+1}(D)$ with $\mathrm{vol}_{k+1}(T) = 0$, then T is supported over a k-plane, hence $\langle T, \delta\omega \rangle = 0$, because $\delta\omega$ is closed over k-planes.

On the contrary, if both ω and $\delta\omega$ are nonatomic, and $T \in \mathrm{Simp}^{k+1}(D)$ is supported over a k-plane, then $\langle T, \delta\omega \rangle = 0$ because $\mathrm{vol}_{k+1}(T) = 0$, so that ω is closed over k-planes and hence regular as claimed. □

Proposition 3.4 easily implies the following corollary, since non-atomicity is stable by pointwise convergence.

Corollary 3.5. *A limit of a pointwise converging sequence $(\omega^n)_{n\geq 1}$ of regular germs is regular.*

Remark 3.6. Note that $\delta\omega$ is nonatomic if and only if it is regular (since it is automatically closed). Thus δ sends regular germs in regular ones, in particular δf for $f \in \mathrm{Germ}^0(D)$ is always regular. Its restriction to regular germs will further be denoted by d, namely, for a regular $\omega \in \mathrm{Germ}^k(D)$ we write $\mathsf{d}\omega := \delta\omega$.

Example 3.7. Let x^i, $x^j : D \subseteq \mathbb{R}^d \to \mathbb{R}$ denote coordinate functions and consider the 1-germ $\omega^{ij} = \frac{1}{2}\left(x^i \cup (\delta x^j) + (\delta x^j) \cup x^i\right)$. It is regular because it is non-atomic and, using (2.2) we get

$$\delta\omega^{ij} = \frac{1}{2}\left(\delta x^i \cup \delta x^j - \delta x^j \cup \delta x^i\right),$$

which acts on $S = [pqr] \in \mathrm{Simp}^2(D)$ as the (oriented) area of the projection of S on the (x^i, x^j)-plane:

$$\left\langle [pqr], \delta(\omega^{ij}) \right\rangle = \frac{1}{2} \det \begin{pmatrix} q^i - p^i & r^i - q^i \\ q^j - p^j & r^j - q^j \end{pmatrix}. \tag{3.1}$$

Hence if $\mathrm{vol}_2([pqr]) = 0$, i.e., the three vertices are collinear, then $\delta\omega^{ij}_S = 0$ or in other words $\delta\omega^{ij}$ is non-atomic.

For regular germs $\omega \in \mathrm{Germ}^k(D)$, we use the integral notation, i.e. write

$$\int_S \omega := \langle S, \omega \rangle \quad \text{for } S \in \mathrm{Simp}^k(D).$$

This is justified by the following example.

Example 3.8. For any continuous f, $g \in \mathrm{Germ}^0(D)$ with $\|\delta g\|_1 < \infty$, i.e., g is Lipschitz continuous, the 1-germ $f\mathsf{d}g$, defined by

$$\langle [pq], f\mathsf{d}g\rangle = \int_{[pq]} f\mathsf{d}g := \int_0^1 f(p+t(q-p))\frac{\mathsf{d}}{\mathsf{d}t}g(p+t(q-p))\mathsf{d}t, \tag{3.2}$$

where the integral is intended in the Lebesgue sense (recall that $t \mapsto g(p+t(q-p))$ is Lipschitz continuous). This germ is clearly non-atomic and $\delta(f\mathsf{d}g)$ restricted to every line is zero, which is shown by a simple reparametrization and additivity of Lebesgue integral. Note that the germ ω^{ij} in Example 3.7 coincides with $x^i\mathsf{d}x^j$, by a direct computation of the integral in (3.2).

3.2. **Sewing as regularization.**

Definition 3.9 (sewable germs)**.** *We say that* $\omega \in \mathrm{Germ}^k(D)$ *is* v*-sewable for some* k*-Dini germ* $\mathsf{v} \in \mathrm{Germ}^k(D)$*, if there exists a regular germ* $\operatorname{sew}\omega \in \mathrm{Germ}^k(D)$ *such that*

$$\omega \approx_{\mathsf{v}} \operatorname{sew}\omega.$$

We will say that $\omega \in \mathrm{Germ}^k(D)$ *is sewable, if it is* v*-sewable for some* k*-Dini germ* $\mathsf{v} \in \mathrm{Germ}^k(D)$*.*

When, $\mathsf{v} = \mathrm{diam}^\alpha$, we say α-sewable instead of v-sewable. If $\omega \in \mathrm{Germ}^k(D)$ is sewable, then $\operatorname{sew}\omega$ is alternating (by Proposition 3.2) and additive (in the sense of Remark 3.3).

Sewable germs can be thought of as the linear subspace of germs "infinitesimally" near to regular germs. Any regular ω is v-sewable for any v, letting $\operatorname{sew}\omega := \omega$.

Example 3.10. For any f, $g \in \mathrm{Germ}^0(D)$ with f (uniformly) continuous and $\|\delta g\|_1 < \infty$, the 1-germ $f \cup \delta g$ is sewable with $\operatorname{sew}(f \cup \delta g) = f\mathsf{d}g$ introduced in Example 3.8. Indeed, letting η_f stand for the modulus of continuity of f, we have

$$\begin{aligned}\left|\int_{[pq]} f\mathsf{d}g - f_p \cup \delta g_{pq}\right| &= \left|\int_0^1 (f(p+t(q-p)) - f(p))\,\frac{\mathsf{d}}{\mathsf{d}t}g(p+t(q-p))\mathsf{d}t\right| \\ &\le \eta_f(|q-p|)[\delta g]_{\mathrm{diam}}|q-p| =: \mathsf{v}([pq]),\end{aligned}$$

which is a 1-Dini germ.

Example 3.11. The germ $\omega \in \mathrm{Germ}^1(\mathbb{R})$ defined by $\omega_{st} := |t-s|$ is not sewable. In fact, otherwise for some regular $\eta \in \mathrm{Germ}^1(\mathbb{R})$ one would have with $s_n := 2^{-n}$ the relationships

$$\begin{aligned} s_n &= \omega_{s_n 0} = \eta_{s_n 0} + o(s_n), \\ s_n &= \omega_{0 s_n} = \eta_{0 s_n} + o(s_n) = -\eta_{s_n 0} + o(s_n), \end{aligned}$$

so that $2s_n = o(s_n)$, a contradiction.

Note that the use of only k-Dini gauges in the Definition 3.9 of sewable k-germs is not casual. In fact, the germ $\omega \in \mathrm{Germ}^1(\mathbb{R})$ from the above Example 3.11 is v-near both the regular 1-germ $\eta \equiv 0$ and, say, the regular 1-germ ξ defined by $\xi_{st} := t-s$ with $\mathsf{v} := \mathrm{diam}^\alpha$ with every $\alpha \in [0,1)$. On the contrary, with Definition 3.9, the sewing map $\omega \mapsto \operatorname{sew}\omega$ is well-defined, i.e., $\operatorname{sew}\omega$ is uniquely determined, if it exists (i.e. when ω is sewable), as a corollary of the following result.

Theorem 3.12 (sewing, uniqueness)**.** *Let $k \in \{1,2\}$. If $\omega \in \operatorname{Germ}^k(D)$ is regular and $\omega \approx_{\mathsf{v}} 0$ for some k-Dini $\mathsf{v} \in \operatorname{Germ}^k(D)$, then $\omega = 0$.*

Corollary 3.13. *Let $\omega \in \operatorname{Germ}^k(D)$ be both v-sewable and $\tilde{\mathsf{v}}$-sewable, with associated regular k-germs $\operatorname{sew}\omega$ and $\widetilde{\operatorname{sew}}\omega$. Then $\operatorname{sew}\omega = \widetilde{\operatorname{sew}}\omega$.*

Proof. By definition, $\operatorname{sew}\omega \approx_{\mathsf{v}} \omega \approx_{\tilde{\mathsf{v}}} \widetilde{\operatorname{sew}}\omega$. Then, $\operatorname{sew}\omega \approx_{\mathsf{w}} \widetilde{\operatorname{sew}}\omega$, i.e., $\operatorname{sew}\omega - \widetilde{\operatorname{sew}}\omega \approx_{\mathsf{w}} 0$, with $\mathsf{w} := \max\{\mathsf{v}, \tilde{\mathsf{v}}\}$, which is k-Dini. Since the difference of regular germs is regular, by Theorem 3.12 we obtain the thesis. □

Another simple consequence of Theorem 3.12 is the following result.

Corollary 3.14. *Let $\omega, \tilde\omega \in \operatorname{Germ}^k(D)$ with ω v-sewable and $\omega \approx_{\mathsf{v}} \tilde\omega$. Then $\tilde\omega$ is v-sewable with $\operatorname{sew}\tilde\omega = \operatorname{sew}\omega$.*

Proof. One has $\tilde\omega \approx_{\mathsf{v}} \omega \approx_{\mathsf{v}} \operatorname{sew}\omega$. □

As an example, if $\omega \in \operatorname{Germ}^1(D)$ is sewable, also $\tilde\omega_{st} := \omega_{st} + |t-s|^p$ with $p > 1$ is sewable, and $\operatorname{sew}\tilde\omega = \operatorname{sew}\omega$.

As an additional consequence, the sewing map is linear: if ω, $\tilde\omega$ are sewable, then any linear combination $\lambda\omega + \tilde\lambda\tilde\omega$ is sewable as well, with

$$\operatorname{sew}(\lambda\omega + \tilde\lambda\tilde\omega) = \lambda \operatorname{sew}\omega + \tilde\lambda \operatorname{sew}\tilde\omega,$$

simply because $\lambda\omega + \tilde\lambda\tilde\omega \approx_{\mathsf{w}} \lambda \operatorname{sew}\omega + \tilde\lambda \operatorname{sew}\tilde\omega$, with $\mathsf{w} := \max\{\mathsf{v}, \tilde{\mathsf{v}}\}$, if we assume that ω is v-sewable and $\tilde\omega$ is $\tilde{\mathsf{v}}$-sewable.

Corollary 3.15 (locality)**.** *Let $\tilde D \subseteq D$ and $\omega \in \operatorname{Germ}^k(D)$ be sewable. Then, its restriction $\tilde\omega$ to $\operatorname{Simp}^k(\tilde D)$ is sewable and $\operatorname{sew}\tilde\omega$ is the restriction of $\operatorname{sew}\omega$ to $\operatorname{Simp}^k(\tilde D)$.*

Proof. The restriction of $\operatorname{sew}\omega$ to $\operatorname{Simp}^k(\tilde D)$ is regular, and $\tilde\omega \approx_{\tilde{\mathsf{v}}} \operatorname{sew}\omega$ with $\tilde{\mathsf{v}}$ being the restriction in $\tilde D$ of a k-Dini gauge $\mathsf{v} \in \operatorname{Germ}^k(D)$ such that ω is v-sewable. □

A further consequence is the following result. Recall that $\|\cdot\|_0$ denotes $[\cdot]_{\mathsf{v}}$ with $\mathsf{v} := \operatorname{diam}^0$, which gives the usual sup norm.

Corollary 3.16. *Let $f \in \operatorname{Germ}^0(D)$, $\omega \in \operatorname{Germ}^k(D)$ with ω v-sewable and $\|f\|_0 < \infty$. Then $f \cup \omega$ is v-sewable if and only if $f \cup \operatorname{sew}\omega$ is v-sewable, and in such a case one has*

$$\operatorname{sew}(f \cup \omega) = \operatorname{sew}\left(f \cup \operatorname{sew}\omega\right).$$

Proof. One has $[f \cup \omega - f \cup \operatorname{sew}\omega]_{\mathsf{v}} \le \|f\|_0 [\omega - \operatorname{sew}\omega]_{\mathsf{v}} < \infty$, hence $f \cup \omega \approx_{\mathsf{v}} f \cup \operatorname{sew}\omega$. □

At this level of generality, we may 'integral' of $\omega \in \operatorname{Germ}^k(\tilde D)$ along parametrizations $\varphi : I \to D$ by pull-back, i.e. as $\operatorname{sew}\varphi^\natural\omega$ (if it is sewable). As in the smooth setting, we have invariance with respect to the choice of φ, provided that the "reparametrization" is sufficiently nice.

Proposition 3.17. *Let $\omega \in \operatorname{Germ}^k(D)$, $\varphi : I \subseteq \mathbb{R}^n \to D$, $\psi : J \subseteq \mathbb{R}^m \to D$, $\varrho : I \to J$ with $\varphi = \psi \circ \varrho$. Let $\psi^\natural\omega$ be w-sewable, and $\varrho^\natural \mathsf{w} \le \mathsf{v}$, for some k-Dini $\mathsf{v} \in \operatorname{Germ}^k(I)$. Then, $\varphi^\natural\omega$ is v-sewable if and only if $\varrho^\natural \operatorname{sew}(\psi'\omega)$ is v-sewable, and in such a case one has*

$$\operatorname{sew}\left(\varrho^\natural \operatorname{sew}\left(\psi^\natural\omega\right)\right) = \operatorname{sew}\left(\varphi^\natural\omega\right).$$

Proof. We have

$$\begin{aligned}[\varrho^\natural \operatorname{sew}(\psi'\omega) - \varphi^\natural\omega]_{\mathsf{v}} &= [\varrho^\natural \left(\operatorname{sew}\psi^\natural\omega - \psi^\natural\omega\right)]_{\mathsf{v}} \quad \text{since } \varphi^\natural = \varrho^\natural\psi^\natural,\\ &\le [\varrho^\natural \left(\operatorname{sew}\psi^\natural\omega - \psi^\natural\omega\right)]_{\varrho^\natural \mathsf{w}} \quad \text{since } \varrho^\natural\mathsf{w} \le \mathsf{v},\\ &\le [\operatorname{sew}\psi^\natural\omega - \psi^\natural\omega]_{\mathsf{w}} \quad \text{by Remark 2.13},\\ &< \infty.\end{aligned}$$

Therefore, $\varrho^\natural \operatorname{sew}(\psi'\omega) \approx_{\mathsf{v}} \varphi^\natural\omega$ and the thesis follows. □

One could ask if for a sewable $(k-1)$-germ η one has that $\delta\eta$ is sewable with $\operatorname{sew}\delta\eta = \delta \operatorname{sew}\eta$, the problem being that one can easily prove $\delta\eta \approx_{\mathsf{v}} \delta\operatorname{sew}\eta$, but only for a $(k-1)$-Dini $\mathsf{v} \in \operatorname{Germ}^k(D)$. However, the answer is positive if $\delta\eta$ is sewable.

Theorem 3.18 (Stokes)**.** *Let $k \in \{1,2\}$, $\eta \in \operatorname{Germ}^{k-1}(D)$ be sewable with $\delta\eta$ sewable. Then $\delta \operatorname{sew}\eta = \operatorname{sew}\delta\eta$. In particular, if η is regular (so that $\mathsf{d}\eta := \delta\eta$ is regular by Proposition 3.4 and Remark 3.6), then*

$$\int_{\partial S} \eta = \int_S \mathsf{d}\eta \quad \textit{for every } S \in \operatorname{Simp}^k(D).$$

Sufficient conditions ensuring existence of $\operatorname{sew}\omega$ are provided by the following result.

Theorem 3.19 (sewing, existence)**.** *Let $k \in \{1,2\}$. For any continuous strong k-Dini $\mathsf{u} \in \operatorname{Germ}^{k+1}(D)$ such that the germ $\tilde{\mathsf{u}}$ in (2.3) is continuous, there exists a continuous k-Dini $\mathsf{v} \in \operatorname{Germ}^k(D)$ such that the following holds: if $\omega \in \operatorname{Germ}^k(D)$ is continuous, non-atomic and $\delta\omega \approx_{\mathsf{u}} 0$, then ω is v-sewable, with $\operatorname{sew}\omega$ continuous and*

$$[\omega - \operatorname{sew}\omega]_{\mathsf{v}} \le [\delta\omega]_{\mathsf{u}}. \tag{3.3}$$

Remark 3.20. If $k=1$, then the condition $\delta\omega \approx_{\mathsf{u}} 0$ already implies that ω is non-atomic, because $|\omega_{pp}| = |(\delta\omega)_{ppp}| \le [\delta\omega]_{\mathsf{u}}\mathsf{u}([ppp]) = 0$, for $p \in D$.

Remark 3.21. It is worth noting that this is the only theorem that requires continuity of the gauges u and $\tilde{\mathsf{u}}$. An inspection of its proof shows that the continuity conditions of u, $\tilde{\mathsf{u}}$ and ω can be replaced by continuity of their restrictions to every k-plane; in this case however also v and $\operatorname{sew}\omega$ are only guaranteed to be continuous on k-planes. Similarly, the condition $\delta\omega \approx_{\mathsf{u}} 0$ may be verified only over each k-plane, as long as it is satisfied uniformly over k-planes, then $[\delta\omega]_{\mathsf{u}}$ in (3.3) is replaced with its least upper bound over all k-planes.

Remark 3.22. The proof of Theorem 3.19 and Remark 2.16 (v) give that, if u is strong r-Dini (with $r \ge k$), then v is in fact r-Dini. For $k=1$, letting $\mathsf{u} := \operatorname{diam}^\gamma \in \operatorname{Germ}^2(D)$ (with $\gamma > 1$) the proof of Theorem 3.19 gives $\mathsf{v} \le \mathsf{c}(\gamma)\operatorname{diam}^\gamma$. For $k=2$, letting $\mathsf{u} := \operatorname{diam}^{\gamma_1}\operatorname{vol}_2^{\gamma_2} \in \operatorname{Germ}^{k+1}(D)$ (with $\gamma_1 + 2\gamma_2 > k$) the proof gives $\mathsf{v} \le \mathsf{c}(\gamma_1,\gamma_2)\operatorname{diam}^{\gamma_1}\operatorname{vol}_2^{\gamma_2}$.

We end this section with the following result on continuity of the sewing map.

Theorem 3.23 (sewing, continuity)**.** *Let $k \in \{1,2\}$, $\mathsf{v} \in \operatorname{Germ}^k(D)$ be k-Dini, $(\omega^j)_{j\ge1} \subseteq \operatorname{Germ}^k(D)$ be sewable with*

$$\omega^j \to \omega \in \operatorname{Germ}^k(D) \quad \textit{pointwise} \quad \textit{and} \quad \sup_{n\ge1}[\omega^j - \operatorname{sew}\omega^j]_{\mathsf{v}} < \infty.$$

Then ω is v-sewable and $\operatorname{sew}\omega^j \to \operatorname{sew}\omega$ *pointwise.*

In particular, in view of Theorem 3.19, the following result holds:

Corollary 3.24. *Let $k \in \{1,2\}$, $\mathsf{u} \in \operatorname{Germ}^{k+1}(D)$ be k-Dini, continuous with $\tilde{\mathsf{u}}$ in (2.3) continuous. Let $(\omega^j)_{j\ge 1} \subseteq \operatorname{Germ}^k(D)$ be continuous, non-atomic with*

$$\omega^j \to \omega \in \operatorname{Germ}^k(D) \quad \textit{pointwise} \quad \textit{and} \quad \sup_{j\ge 1}[\delta\omega]_{\mathsf{u}} < \infty.$$

Then $(\omega^j)_{j\ge 1}$ and ω are sewable and $\operatorname{sew}\omega^j \to \operatorname{sew}\omega$ *pointwise.*

4. Young and Züst integrals

In this section, we show how both Young [15] and Züst [16] integrals (in the case of 2-germs) can be recovered in the framework introduced above. For simplicity of exposition, we restrict ourselves to Dini gauges of the form $\operatorname{diam}^\gamma$ in case of 1-germs or $\operatorname{diam}^{\gamma_1}\operatorname{vol}_2^{\gamma_2}$ in case of 2-germs.

Let us introduce the following notation (extending that of Example 3.10): for a 0-germ f and a $(k-1)$-germ η, if $f \cup \delta\eta$ is sewable, we write $f\mathsf{d}\eta := \operatorname{sew}(f \cup \delta\eta)$.

4.1. Young integral.

As already noticed, every 1-germ δf is regular. Hence, Theorem 3.19 together with Remark 3.22 yields the following result.

Theorem 4.1 (Young integral)**.** *Let $\gamma > 1$, f, $g \in \operatorname{Germ}^0(D)$ be continuous with $\|\delta f \cup \delta g\|_\gamma < \infty$. Then, $f \cup \delta g$ is γ-sewable with*

$$\|f \cup \delta g - f\mathsf{d}g\|_\gamma \le \mathsf{c}\,\|\delta f \cup \delta g\|_\gamma\,, \tag{4.1}$$

where $\mathsf{c} = \mathsf{c}(\gamma)$.

In particular, the result applies if $\|\delta f\|_\alpha + \|\delta g\|_\beta < \infty$ and $\gamma := \alpha + \beta > 1$. The triangle inequality in (4.1) gives the bound, for any $\beta \le \gamma$,

$$\|f\mathsf{d}g\|_\beta \le \|f\|_0\,\|\delta g\|_\beta + \mathsf{c}\operatorname{diam}(D)^{\gamma-\beta}\,\|\delta f \cup \delta g\|_\gamma\,.$$

Remark 4.2 (continuity of Young integral)*.* By Corollary 3.24, we have that, if $\gamma > 1$, $(f^j)_{j\ge1}$, $(g^j)_{j\ge1} \subseteq \operatorname{Germ}^0(D)$ are continuous with $\sup_{j\ge1}\left\|\delta f^j \cup \delta g^j\right\|_\gamma < \infty$ and $f^j \to f \in \operatorname{Germ}^0$, $g^j \to g \in \operatorname{Germ}^0$ pointwise, then $f \cup \delta g$ is γ-sewable and $f^j\mathsf{d}g^j \to f\mathsf{d}g$ pointwise. Since $f\mathsf{d}g$, when $\|g\|_1 < \infty$, is given by classical Lebesgue integrals (Example 3.8), this would allow to transfer by approximation many results from classical differential forms to germs of Young type, e.g., the chain rule Proposition 4.5. However, we prefer to give alternative proofs allowing for self-contained exposition of the theory.

Remark 4.3 (locality)*.* Under the assumptions of Theorem 4.1, if g is constant in D, by uniqueness of sew we obtain $f\mathsf{d}g = 0$. By Corollary 3.15, we deduce that $f\mathsf{d}g$ is local, i.e., if g is constant on $\operatorname{conv}([p_0p_1]) \subseteq D$, then $\int_{[p_0p_1]} f\mathsf{d}g = 0$.

Corollary 4.4. *Let $\gamma > 1$, f, g, $h \in \operatorname{Germ}^0(D)$ be continuous with*

$$\|f\|_0 + \|h\|_0 < \infty \quad \textit{and} \quad \|\delta f \cup \delta g\|_\gamma + \|\delta h \cup \delta g\|_\gamma < \infty.$$

Then $f \cup (h\mathsf{d}g)$, $h \cup (f\mathsf{d}g)$ and $(hf) \cup \delta g$ are γ-sewable with

$$(fh)\mathsf{d}g = \operatorname{sew}((fh) \cup \delta g) = \operatorname{sew}(f \cup (h\mathsf{d}g)) = \operatorname{sew}(h \cup (f\mathsf{d}g))\,.$$

Proof. One has

$$\|\delta\left((hf)\cup\delta g\right)\|_\gamma \le \|f\|_0 \|\delta h\cup \delta g\|_\gamma + \|h\|_0 \|\delta f\cup\delta g\|_\gamma < \infty$$

hence $(hf)\cup\delta g$ is γ-sewable and the thesis follows from Corollary 3.16. □

We also have the following chain rule (which could be also proved by approximation with smooth functions, using Remark 4.2).

Proposition 4.5 (chain rule)**.** *Let D be bounded, α, β, $\sigma>0$ with $\min\{\alpha,\sigma\beta\}+\beta>1$ and let $D\subseteq\mathbb{R}^d$ be bounded. If $f\in \mathrm{Germ}^0(D)$, $g=(g^i)_{i=1}^\ell\in\mathrm{Germ}^0(D;\mathbb{R}^\ell)$, $\Psi\colon\mathbb{R}^\ell\to\mathbb{R}$ are such that $\|\delta f\|_\alpha<\infty$, $\left\|\delta g^i\right\|_\beta<\infty$, and Ψ is $C^{1,\sigma}$, i.e., the partial derivatives $\partial_i\Psi$ satisfy $\|\delta(\partial_i\Psi)\|_\sigma<\infty$ for every $i\in\{1,\dots,\ell\}$, then*

$$f\mathsf{d}(\Psi\circ g) = (f\nabla\Psi\circ g)\cdot\mathsf{d}g := \sum_{i=1}^{\ell}(f\partial_i\Psi\circ g)\mathsf{d}g^i. \tag{4.2}$$

Proof. Taylor formula for Ψ reads in our language as

$$\delta\Psi \approx_{1+\sigma} \sum_{i=1}^{\ell}\partial_i\Psi\cup\delta x^i,$$

where $(x^i)_{i=1}^\ell$ denote the coordinate functions on $\mathbb{R}^\ell$. By composition with g, using the assumption $\left\|\delta g^i\right\|_\beta<\infty$ for $i\in\{1,\dots,\ell\}$, we deduce that

$$\delta(\Psi\circ g)\approx_{\beta(1+\sigma)}\sum_{i=1}^{\ell}\partial_i\Psi\circ g\cup\delta g^i,$$

hence by Theorem 4.1 and uniqueness of sew, we conclude that

$$\mathsf{d}(\Psi\circ g)=\sum_{i=1}^{\ell}\partial_i\Psi\circ g\mathsf{d}g^i.$$

Applying $f\cup$ to both sides and using Corollary 4.4 we obtain the thesis. □

As usual, the chain rule implies Leibniz rule, for which we give a self-contained proof in a slightly more general assumption on $\left\|\delta g^1\cup\delta g^2\right\|_\gamma$ instead of $\left\|\delta g^1\right\|_\beta + \left\|\delta g^2\right\|_\beta$ as it follows from Proposition 4.5 (with $\beta=\gamma/2$).

Corollary 4.6 (Leibniz or integration by parts rule)**.** *Let g^1, $g^2\in\mathrm{Germ}^0(D)$ be continuous with $\left\|\delta g^1\cup\delta g^2\right\|_\gamma<\infty$ for some $\gamma>1$. Then $\delta(g^1g^2)=g^1\mathsf{d}g^2+g^2\mathsf{d}g^1$, i.e.,*

$$\int_{[pq]} g^1\mathsf{d}g^2 = \delta(g^1g^2)_{pq} - \int_{[pq]} g^1\mathsf{d}g^2, \quad \textit{for } [pq]\in\mathrm{Simp}^1(D).$$

In particular, if $g\in\mathrm{Germ}^0(D)$ is such that $\|\delta g\|_\alpha<\infty$ for $\alpha>1/2$, then $\delta g^2 = 2g\mathsf{d}g$.

Proof. We use the identity,

$$\delta(g^1g^2) = g^1\cup\delta g^2 + g^2\cup\delta g^1 + (\delta g^1)(\delta g^2).$$

For $[pq]\in\mathrm{Simp}^1(D)$, we have

$$\left\langle [pq], (\delta g^1)(\delta g^2)\right\rangle = (\delta g^1)_{pq}(\delta g^2)_{pq} = -\left\langle [pqp], \delta g^1\cup\delta g^2\right\rangle$$

hence, by the assumption $\left\|\delta g^1 \cup \delta g^2\right\|_\gamma < \infty$,

$$\delta(g^1 g^2) \approx_\gamma g^1 \cup \delta g^2 + g^2 \cup \delta g^1.$$

We conclude then as in Proposition 4.5, i.e., by Theorem 4.1 and uniqueness of sew. $\square$

4.2. **Züst integral.** To study the case of of integration of 2-germs of the type $f\mathsf{d}g^1 \wedge \mathsf{d}g^2$, for $f, g^1, g^2 \in \mathrm{Germ}^0(D)$, as introduced in R. Züst [16], the starting point is to notice that the naive germs $f \cup \delta g^1 \cup \delta g^2$ in general fail to be non-atomic, already when $D \subseteq \mathbb{R}$. Our construction (as well as Züst's) is then based on the identity $\delta g^1 \cup \delta g^2 = \delta\left(g^1 \cup \delta g^2\right)$, which suggests to apply Theorem 3.19 to the 2-germ $f \cup \delta\eta$, where $\eta = g^1 \mathsf{d}g^2$, i.e.,

$$\eta_{[pq]} = \int_{[pq]} g^1 \mathsf{d}g^2,$$

the integral being in the sense of Young as in the previous section. More explicitly,

$$\begin{aligned}\left\langle [p_0p_1p_2], \delta(g^1\mathsf{d}g^2)\right\rangle &= \left\langle \partial[p_0p_1p_2], g^1\mathsf{d}g^2\right\rangle \\ &= \int_{[p_1p_2]} g^1\mathsf{d}g^2 - \int_{[p_0p_2]} g^1\mathsf{d}g^2 + \int_{[p_0p_1]} g^1\mathsf{d}g^2,\end{aligned}$$

and

$$\left\langle [p_0p_1p_2], f \cup \delta(g^1\mathsf{d}g^2)\right\rangle = f_{p_0}\left\langle [p_0p_1p_2], \delta g^1\mathsf{d}g^2\right\rangle.$$

Example 4.7 (classical forms). When $g^1 = x^i$, $g^2 = x^j$ coordinate functions, in the notation of Example 3.7 we have that $\delta(x^i\mathsf{d}x^j)$ equals $\delta\omega^{ij}$ in (3.1), hence one can argue as in Example 3.10 and show that the 2-germ $f \cup \delta(x^i\mathsf{d}x^j)$ is sewable if f is (uniformly) continuous, with

$$\left\langle S, \mathrm{sew}(f \cup \delta(x^i\mathsf{d}x^j)\right\rangle = \int_S f\mathsf{d}x^i \wedge \mathsf{d}x^j,$$

the integral being in the classical sense of differential forms. Indeed, letting η_f stand for the modulus of continuity of f, we have

$$\begin{aligned}\left|\int_{[pqr]} f\mathsf{d}x^i \wedge \mathsf{d}x^j - f_p \cup \delta(x^i\mathsf{d}x^j)_{pqr}\right| &= \left|\int_{[pqr]} (f - f_p)\mathsf{d}x^i \wedge \mathsf{d}x^j\right| \\ &\le \eta_f(\mathrm{diam}[pqr])\,\mathrm{vol}_2([pqr]) =: \mathsf{v}([pq])\end{aligned}$$

which defines a 2-Dini gauge.

To argue with more general functions than coordinates, we rely on the following bound for $\delta(g^1\mathsf{d}g^2)$.

Proposition 4.8. *Let $\beta > 1$, $g^1, g^2 \in \mathrm{Germ}^0(D)$ be continuous with $\left\|\delta g^1 \cup \delta g^2\right\|_\beta < \infty$. For $\mathsf{c} = \mathsf{c}(\beta)$, one has then*

$$\left\|\delta(g^1\mathsf{d}g^2)\right\|_\beta \le \mathsf{c}\left\|\delta g^1 \cup \delta g^2\right\|_\beta. \tag{4.3}$$

Proof. For any $S \in \mathrm{Simp}^2(D)$, to prove that, for some constant $\mathsf{c} = \mathsf{c}(\beta)$, one has

$$\left|\left\langle S, \delta(g^1\mathsf{d}g^2)\right\rangle\right| \le \mathsf{c}\left\|\delta g^1 \cup \delta g^2\right\|_\beta \mathrm{diam}^\beta(S),$$

we add and subtract $\delta(g^1 \cup \delta g^2) = \delta g^1 \cup \delta g^2$, hence

$$\begin{aligned}\left|\left\langle S, \delta(g^1 \mathsf{d} g^2)\right\rangle\right| &\le \left|\left\langle S, \delta(g^1 \mathsf{d} g^2 - g^1 \cup \delta g^2)\right\rangle\right| + \left|\left\langle S, \delta g^1 \cup \delta g^2\right\rangle\right| \\ &\le \left|\left\langle \partial S, g^1 \mathsf{d} g^2 - g^1 \cup \delta g^2\right\rangle\right| + \left\|\delta g^1 \cup \delta g^2\right\|_\beta \operatorname{diam}^\beta(S).\end{aligned}$$

For the first term, since $\operatorname{diam}(S^i) \le \operatorname{diam}(S)$ for each of the three 1-simplices in $\partial S = \sum_{i=0}^2 (-1)^i S^i$, by (4.1) we have

$$\begin{aligned}\left|\left\langle \partial S, g^1 \mathsf{d} g^2 - g^1 \cup \delta g^2\right\rangle\right| &\le \sum_{i=0}^2 \left|\left\langle S^i, g^1 \mathsf{d} g^2 - g^1 \cup \delta g^2\right\rangle\right| \\ &\le \sum_{i=0}^2 \mathsf{c} \left\|\delta g^1 \cup \delta g^2\right\|_\beta \operatorname{diam}^\beta(S^i) \\ &\le 3\mathsf{c} \left\|\delta g^1 \cup \delta g^2\right\|_\beta \operatorname{diam}^\beta(S),\end{aligned}$$

hence the thesis. ◻

Remark 4.9 (bound improvement). By Lemma C.1, since $\delta(g^1 \mathsf{d} g^2)$ is regular, we can improve (4.3) to the inequality

$$[\delta(g^1 \mathsf{d} g^2)]_{\operatorname{diam}^{2-\beta} \operatorname{vol}_2^{\beta-1}} \le \mathsf{c}(\beta) \left\|\delta g^1 \cup \delta g^2\right\|_\beta . \tag{4.4}$$

Theorem 4.10 (Züst integral). *Let $\alpha > 0$, $\beta > 1$ with $\gamma := \alpha + \beta > 2$, let f, g^1, $g^2 \in \operatorname{Germ}^0(D)$ be continuous with $\|\delta f\|_\alpha + \left\|\delta g^1 \cup \delta g^2\right\|_\beta < \infty$. Then, $f \cup \delta(g^1 \mathsf{d} g^2)$ is γ-sewable with*

$$\left\|f \cup \delta(g^1 \mathsf{d} g^2) - f \mathsf{d}(g^1 \mathsf{d} g^2)\right\|_\gamma \le \mathsf{c} \|\delta f\|_\alpha \left\|\delta g^1 \cup \delta g^2\right\|_\beta , \tag{4.5}$$

where $\mathsf{c} = \mathsf{c}(\alpha, \beta)$.

Remark 4.11 (alternating property). The assumptions of Theorem 4.10 are symmetric in g^1, g^2, and by Corollary 4.6 we have $g^1 \mathsf{d} g^2 = \delta(g^1 g^2) - g^2 \mathsf{d} g^1$, hence $\delta(g^1 \mathsf{d} g^2) = -\delta(g^2 \mathsf{d} g^1)$. By uniqueness Theorem 3.12 we obtain that

$$f \mathsf{d}(g^1 \mathsf{d} g^2) = -f \mathsf{d}(g^2 \mathsf{d} g^1),$$

which justifies the introduction of the usual wedge notation

$$f \mathsf{d} g^1 \wedge \mathsf{d} g^2 := f \mathsf{d}(g^1 \mathsf{d} g^2).$$

Remark 4.12 (locality). Under the assumptions of Theorem 4.10, if g^1 or g^2 is constant in D, by uniqueness of sew we obtain $f \mathsf{d} g^1 \wedge \mathsf{d} g^2 = 0$. By Corollary 3.15, we deduce that $f \mathsf{d} g^1 \wedge \mathsf{d} g^2$ is local, i.e., if g^1 or g^2 is constant on the geometric simplex $\operatorname{conv}(S) \subseteq D$, then $\int_S f \mathsf{d} g^1 \wedge \mathsf{d} g^2 = 0$.

Remark 4.13 (continuity of Züst integral). As for Young type germs, by Corollary 3.24, we have that, if $\alpha > 0$, $\beta > 1$ with $\alpha + \beta > 2$, and $(f^j)_{j\ge1}$, $(g^j)_{j\ge1}$, $(\tilde{g}^j)_{j\ge1} \subseteq \operatorname{Germ}^0(D)$ are continuous with $\sup_{j\ge1} \left\|\delta f^j\right\|_\alpha + \left\|\delta g^j \cup \delta \tilde{g}^j\right\|_\beta < \infty$ and $f^j \to f \in \operatorname{Germ}^0$, $g^j \to g \in \operatorname{Germ}^0$, $\tilde{g}^j \to \tilde{g} \in \operatorname{Germ}^0(D)$ pointwise, then $f^j \mathsf{d} g^j \wedge \mathsf{d} \tilde{g}^j \to f \mathsf{d} g \wedge \mathsf{d} \tilde{g}$. Again, this would allow us to obtain calculus results by approximation from classical differential forms.

Remark 4.14 (bound improvement for integrals). From (4.4) we have that

$$[\delta f \cup \delta(g^1 \mathsf{d} g^2)]_{\operatorname{diam}^{\alpha+2-\beta} \operatorname{vol}_2^{\beta-1}} \le \|\delta f\|_\alpha [\delta(g^1 \mathsf{d} g^2)]_{\operatorname{diam}^{2-\beta} \operatorname{vol}_2^{\beta-1}} < \infty.$$

By Theorem 3.19 applied with the strong 2-Dini gauge $\mathrm{diam}^{\alpha+2-\beta}\mathrm{vol}_2^{\beta-2}$ (under the assumption $\alpha+\beta>2$) and Remark 3.22, we improve inequality (4.5) to

$$\left[f\cup\delta(g^1\mathsf{d}g^2)-f\mathsf{d}g^1\wedge\mathsf{d}g^2\right]_{\mathrm{diam}^{\alpha+2-\beta}\mathrm{vol}_2^{\beta-1}}\le \mathsf{c}\left\|\delta f\right\|_\alpha\left\|\delta g^1\cup\delta g^2\right\|_\beta,$$

from which, by the triangle inequality and (4.4) again, we deduce that

$$[f\mathsf{d}g^1\wedge\mathsf{d}g^2]_{\mathrm{diam}^{2-\beta}\mathrm{vol}_2^{\beta-1}}\le \mathsf{c}\left(\|f\|_0+\mathrm{diam}(D)^\alpha\|\delta f\|_\alpha\right)\left\|\delta g^1\cup\delta g^2\right\|_\beta. \tag{4.6}$$

This is analogous to [16, Corollary 3.4] (for $k=2$) with triangles instead of rectangles.

The following result is analogous to Corollary 4.4.

Corollary 4.15. *Let $\alpha>0$, $\beta>1$ with $\gamma:=\alpha+\beta>2$, let f, h, g^1, $g^2\in\mathrm{Germ}^0(D)$ be continuous with*

$$\|f\|_0+\|h\|_0<\infty\quad\textit{and}\quad\|\delta f\|_\alpha+\|\delta h\|_\alpha+\left\|\delta g^1\cup\delta g^2\right\|_\beta<\infty.$$

Then $f\cup(h\mathsf{d}g^1\wedge\mathsf{d}g^2)$, $h\cup(f\mathsf{d}g^1\wedge\mathsf{d}g^2)$ and $(fh)\cup(\mathsf{d}g^1\wedge\mathsf{d}g^2)$ are γ-sewable with

$$\begin{aligned}(fh)\mathsf{d}g^1\wedge\mathsf{d}g^2&=\mathrm{sew}((fh)\cup(\mathsf{d}g^1\wedge\mathsf{d}g^2))\\&=\mathrm{sew}(f\cup(h\mathsf{d}g^1\wedge\mathsf{d}g^2))=\mathrm{sew}(h\cup(f\mathsf{d}g^1\wedge\mathsf{d}g^2)).\end{aligned}$$

The chain rule also holds: in order to prove it we first establish the following Leibniz-type expansion.

Lemma 4.16. *Let α_1, α_2, $\beta>0$ with $\gamma:=\alpha_1+\alpha_2+\beta>2$. Let f^1, f^2, $g\in\mathrm{Germ}^0(D)$ with $\left\|\delta f^1\right\|_{\alpha_1}+\left\|\delta f^2\right\|_{\alpha_2}+\|\delta g\|_\beta<\infty$. Then*

$$\delta(f^1f^2\mathsf{d}g)\approx_\gamma f^1\cup\delta(f^2\mathsf{d}g)+f^2\cup\delta(f^1\mathsf{d}g).$$

Proof. Write $\omega:=\delta(f^1f^2\mathsf{d}g)-f^1\cup\delta(f^2\mathsf{d}g)-f^2\cup\delta(f^1\mathsf{d}g)\in\mathrm{Germ}^1(D)$. For any given $S=[p_0p_1p_2]$, we prove that, for some $\mathsf{c}=\mathsf{c}(\alpha_1,\alpha_2,\beta)$,

$$|\langle S,\omega\rangle|\le\mathsf{c}\left\|\delta f^1\right\|_{\alpha_1}\left\|\delta f^2\right\|_{\alpha_2}\|\delta g\|_\beta\,\mathrm{diam}^\gamma(S). \tag{4.7}$$

Without loss of generality, we assume that $D=S$ (since the constant c does not depend on D either). By linearity of Young integral, we have the identity

$$(f^1f^2\mathsf{d}g)-f^1_{p_0}(f^2\mathsf{d}g)+f^2_{p_0}(f^1\mathsf{d}g)=(f^1-f^1_{p_0})(f^2-f^2_{p_0})\mathsf{d}g-f^1_{p_0}f^2_{p_0}\delta g,$$

hence (with p_0 fixed)

$$\begin{aligned}\delta(f^1f^2\mathsf{d}g)-f^1_{p_0}\delta(f^2\mathsf{d}g)+f^2_{p_0}\delta(f^1\mathsf{d}g)&=\delta\left((f^1-f^1_{p_0})(f^2-f^2_{p_0})\mathsf{d}g\right)-f^1_{p_0}f^2_{p_0}\delta^2g\\&=\delta\left((f^1-f^1_{p_0})(f^2-f^2_{p_0})\mathsf{d}g\right),\end{aligned}$$

and evaluating at $S=[p_0p_1p_2]$ we obtain the identity

$$\langle S,\omega\rangle=\left\langle S,\delta\left((f^1-f^1_{p_0})(f^2-f^2_{p_0})\mathsf{d}g\right)\right\rangle.$$

The thesis follows by Proposition 4.8 applied with $\tilde\gamma:=\min\{\alpha_1,\alpha_2\}+\beta$ instead of γ, because

$$\left|\left\langle S,\delta\left((f^1-f^1_{p_0})(f^2-f^2_{p_0})\mathsf{d}g\right)\right\rangle\right|\le\mathsf{c}\left\|\delta\left((f^1-f^1_{p_0})(f^2-f^2_{p_0})\right)\cup\delta g\right\|_{\tilde\gamma}\mathrm{diam}(S)^{\tilde\gamma} \tag{4.8}$$

and

$$\begin{aligned}\|\delta\left((f^1-f^1_{p_0})(f^2-f^2_{p_0})\right)\cup\delta g\|_{\tilde\gamma} &\le \left\|\delta f^1\right\|_{\min\{\alpha_1,\alpha_2\}}\left\|f^2-f^2_{p_0}\right\|_0\|\delta g\|_\beta \\ &\quad+\left\|f^1-f^1_{p_0}\right\|_0\left\|\delta f^2\right\|_{\min\{\alpha_1,\alpha_2\}}\|\delta g\|_\beta \\ &\le 2\left\|\delta f^1\right\|_{\alpha_1}\left\|\delta f^2\right\|_{\alpha_2}\|\delta g\|_\beta\operatorname{diam}(S)^{\alpha_1+\alpha_2-\min\{\alpha_1,\alpha_2\}}, \\ &= 2\left\|\delta f^1\right\|_{\alpha_1}\left\|\delta f^2\right\|_{\alpha_2}\|\delta g\|_\beta\operatorname{diam}(S)^{\gamma-\tilde\gamma}\end{aligned}$$

hence (4.7) follows from (4.8). ∎

Proposition 4.17 (chain rule)**.** *Let $\alpha>0$, $\beta>0$, $\sigma\in(0,1]$ with $\min\{\alpha,\sigma\beta\}+2\beta>2$. If $f\in\mathrm{Germ}^0(D)$, $g=(g^i)_{i=1}^{\ell}\in\mathrm{Germ}^0(D;\mathbb{R}^\ell)$, $\Psi=(\Psi^1,\Psi^2)\colon\mathbb{R}^\ell\to\mathbb{R}^2$ are such that $\|\delta f\|_\alpha<\infty$, $\left\|\delta g^i\right\|_\beta<\infty$, and Ψ is $C^{1,\sigma}$, i.e., the partial derivatives $\partial_i\Psi^m$ satisfy $[\delta(\partial_i\Psi^m)]_\sigma<\infty$ for every $i\in\{1,\dots,\ell\}$, $m\in\{1,2\}$, then*

$$f\mathsf{d}(\Psi^1\circ g)\wedge\mathsf{d}(\Psi^2\circ g)=\sum_{\substack{i,j=1,\dots,\ell\\ i<j}}(f\,J^{ij}_\Psi\circ g)\mathsf{d}g^i\wedge\mathsf{d}g^j,\quad \textit{where}$$

$$J^{ij}_\Psi=\det\begin{pmatrix}\partial_i\Psi^1 & \partial_j\Psi^1\\ \partial_i\Psi^2 & \partial_j\Psi^2\end{pmatrix}.$$

Proof. Letting $\gamma:=\beta(\sigma+2)$, we prove that

$$\delta\left((\Psi^1\circ g)\mathsf{d}(\Psi^2\circ g)\right)\approx_\gamma\sum_{\substack{i,j=1,\dots,\ell\\ i<j}}(f\,J^{ij}_\Psi\circ g)\cup\delta(g^i\mathsf{d}g^j)\tag{4.9}$$

holds, hence equality must hold if we compose with sew both sides (the case with f follows from Corollary 4.15).

To prove (4.9) notice first that Proposition 4.5 gives the identity

$$\delta(\Psi^2\circ g)=\sum_{i=1}^{\ell}(\partial_i\Psi^2\circ g)\mathsf{d}g^i\tag{4.10}$$

as well as (with $\Psi^1\circ g$ instead of f)

$$(\Psi^1\circ g)\mathsf{d}(\Psi^2\circ g)=\sum_{i=1}^{\ell}(\Psi^1\circ g\,\partial_i\Psi^2\circ g)\mathsf{d}g^i.$$

Therefore,

$$\begin{aligned}\delta(\Psi^1\circ g\mathsf{d}\Psi^2\circ g) &= \sum_{i=1}^{\ell}\delta\left((\Psi^1\circ g\partial_i\Psi^2\circ g)\mathsf{d}g^i\right)\\ &\approx_\gamma\sum_{i=1}^{\ell}\Psi^1\circ g\cup\delta(\partial_i\Psi^2\circ g\mathsf{d}g^i)+\partial_i\Psi^2\circ g\cup\delta(\Psi^1\circ g\mathsf{d}g^i)\\ &\qquad\text{by Lemma 4.16,}\\ &\approx_\gamma\sum_{i=1}^{\ell}\partial_i\Psi^2\circ g\cup\delta((\Psi^1\circ g)\mathsf{d}g^i),\end{aligned}\tag{4.11}$$

the sum of the first terms vanishing since, by (4.10),

$$\sum_{i=1}^{\ell}\Psi^1\circ g\cup\delta((\partial_i\Psi^2\circ g)\mathsf{d}g^i)=\Psi^1\circ g\cup\delta(\sum_{i=1}^{\ell}\partial_i\Psi^2\circ g\mathsf{d}g^i)=\Psi^1\circ g\cup\delta^2(\Psi^2\circ g)=0.$$

Integrating by parts, for every $i\in\{1,\dots,\ell\}$,

$$\delta(\Psi^1\circ g\mathsf{d}g^i)=-\delta^2(g^i\Psi^1\circ g)-\delta(g^i\mathsf{d}\Psi^1\circ g)=-\delta(g^i\mathsf{d}\Psi^1\circ g).$$

Using again Lemma 4.16, in a similar way as in (4.11), with Ψ^2 instead of Ψ^1 and g^i instead of $\Psi^1\circ g$, we obtain

$$\delta(g^i\mathsf{d}\Psi^1\circ g)\approx_\gamma\sum_{j=1}^{\ell}\partial_j\Psi^1\circ g\cup\delta(g^i\mathsf{d}g^j).$$

Using this relationship in each summand in the last line of (4.11), we have

$$\delta(\Psi^1\circ g\mathsf{d}\Psi^2\circ g)\approx_\gamma-\sum_{i,j=1}^{\ell}\left(\partial_i\Psi^2\circ g\,\partial_j\Psi^1\circ g\right)\cup\delta(g^i\mathsf{d}g^j),$$

which is equivalent to (4.9) by Remark 4.11. □

Remark 4.18 (classical chain rule). We notice that same arguments may be used to prove the classical chain rule for 2-germs of the type $f\mathsf{d}x^i\wedge\mathsf{d}x^j$ and maps Ψ that are differentiable with (uniformly) continuous derivatives, introducing the 2-Dini gauge $S\mapsto\eta(\operatorname{diam}(S))\operatorname{vol}_2(S)$, η standing for the modulus of continuity of $\nabla\Psi$.

Remark 4.19. The requirement $\Psi\in C^{1,\sigma}$ enters in the proof only to write (4.10) (and the analogous identity for Ψ^1). Therefore, substituting $\Psi\circ g$ with $h=(h^1,h^2)\in\operatorname{Germ}^0(D;\mathbb{R}^2)$ such that, for $m\in\{1,2\}$,

$$\delta h^m\approx_{\beta(1+\sigma)}\sum_{i=1}^{\ell}(\partial_{g^i}h^m)\cup\delta g^i \tag{4.12}$$

where $\partial_{g^i}h^m\in\operatorname{Germ}^0(D;\mathbb{R}^2)$ are such that $\left\|\delta(\partial_{g^i}h^m)\right\|_{\sigma\beta}<\infty$, we have that

$$f\mathsf{d}h^1\wedge\mathsf{d}h^2=\sum_{\substack{i,j=1,\dots,\ell\\ i<j}}f\,(J_gh)^{ij}\mathsf{d}g^i\wedge\mathsf{d}g^j,\quad\text{where}$$

$$(J_gh)^{ij}=\det\begin{pmatrix}\partial_{g^i}h^1 & \partial_{g^j}h^1\\ \partial_{g^i}h^2 & \partial_{g^j}h^2\end{pmatrix}.$$

The class of functions h satisfying (4.12) generalize that of controlled paths in [5] (which corresponds to the case $D\subseteq\mathbb{R}$) and it is tempting to call them g-*differentiable*, a concept that will be further investigated in [13]. Note that in particular when $\ell=1$ (i.e., $g^1:=g$) and h^1, h^2 are both g-differentiable, then $f\mathsf{d}h^1\wedge\mathsf{d}h^2=0$.

4.3. **Pull-back.** In this section, we study how germs of Young and Züst type introduced above behave via the pull-back of a map $\varphi : I \subseteq \mathbb{R}^k \to D$. More precisely, we compare the "algebraic" and "differential" pull-backs, in Young's case (i.e., $f\mathsf{d}g$) respectively given by

$$\varphi^\natural(f\mathsf{d}g) \quad \text{and} \quad \varphi^*(f\mathsf{d}g) := (f \circ \varphi)\mathsf{d}(g \circ \varphi), \tag{4.13}$$

and similarly in Züst's case (i.e., $f\mathsf{d}g^1 \wedge \mathsf{d}g^2$) by

$$\varphi^\natural(f\mathsf{d}g^1 \wedge \mathsf{d}g^2) \quad \text{and} \quad \varphi^*(f\mathsf{d}g^1 \wedge \mathsf{d}g^2) := (f \circ \varphi)\mathsf{d}(g^1 \circ \varphi) \wedge \mathsf{d}(g^2 \circ \varphi), \tag{4.14}$$

provided these are defined, e.g., if the assumptions of Theorem 4.1 (respectively Theorem 4.10) are met.

In the case of 1-germs of Young type, the two pull-backs in (4.13) are related to each other by a sewing procedure, as the following result shows.

Proposition 4.20 (integration over curves via pull-back)**.** *Let $\gamma > 1$, $\sigma > 1/\gamma$, $f, g \in \mathrm{Germ}^0(D)$ be continuous with $\|\delta f \cup \delta g\|_\gamma < \infty$ and let $\varphi\colon I \subseteq \mathbb{R} \to D$ with $\|\delta\varphi\|_\sigma < \infty$. Then $\varphi^\natural(f\mathsf{d}g)$ is $\sigma\gamma$-sewable with*

$$\mathrm{sew}\left(\varphi^\natural(f\mathsf{d}g)\right) = \mathrm{sew}(f \circ \varphi \cup \delta g \circ \varphi) = \varphi^*(f\mathsf{d}g)$$

The above proposition shows that when the latter regular germ is defined (using the conditions on γ and σ) this is a limit both of Riemann sums of increments $f_{p^i}\delta g_{p^ip^{i+1}}$ computed along the curve $\varphi(I)$ and of "integrated" increments $\int_{[p^ip^{i+1}]} f\mathsf{d}g$, even in cases where φ is not a rectifiable curve. In particular, this shows that the curvilinear integral $\int_\varphi f\mathsf{d}g := \int_I \varphi^*(f\mathsf{d}g)$ remains invariant by reparametrization of φ.

Proof. The inequality

$$\|\delta(f \circ \varphi) \cup \delta(g \circ \varphi)\|_{\sigma\gamma} \le \|\delta\varphi\|_\sigma \|\delta f \cup \delta g\|_\gamma < \infty$$

implies that $(f \circ \varphi) \cup \delta(g \circ \varphi)$ is $\sigma\gamma$-sewable by Theorem 4.1. Since $\varphi^\natural \operatorname{diam}^\gamma \le \|\delta\varphi\|_\sigma^\gamma \operatorname{diam}^{\sigma\gamma}$, we have

$$\left\|\varphi^\natural(f\mathsf{d}g) - \varphi^\natural(f \cup \delta g)\right\|_{\sigma\gamma} \le \mathsf{c}[\varphi^\natural(f\mathsf{d}g - f \cup \delta g)]_{\varphi^\natural \operatorname{diam}^\gamma} \le \mathsf{c}\,\|f\mathsf{d}g - f \cup \delta g\|_\gamma < \infty.$$

by Remark 2.13. Therefore,

$$\varphi^\natural(f\mathsf{d}g) \approx_{\sigma\gamma} \varphi^\natural(f \cup \delta g) = f \circ \varphi \cup \delta(g \circ \varphi) \approx_{\sigma\gamma} \mathrm{sew}((f \circ \varphi) \cup \delta(g \circ \varphi)) = (f \circ \varphi)\mathsf{d}(g \circ \varphi)$$

and the thesis follows. □

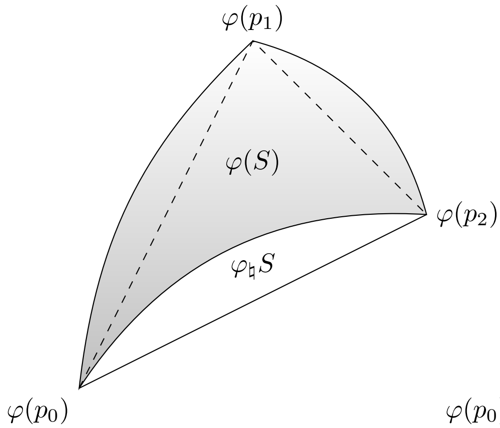
In case of 2-germs $f\mathsf{d}g^1 \wedge \mathsf{d}g^2$, the situation is not so simple, for additional boundary terms appear already in the classical (differentiable) case.

Example 4.21. Let $I \subseteq \mathbb{R}^2$, $D \subseteq \mathbb{R}^d$ be bounded, $g^1 = x^i$, $g^2 = x^j$ be coordinate functions and let $f \in \mathrm{Germ}^0(D)$ and $\varphi : I \subseteq \mathbb{R}^2 \to D$ be continuously differentiable. Then, by Example 4.7 and Remark 4.18,

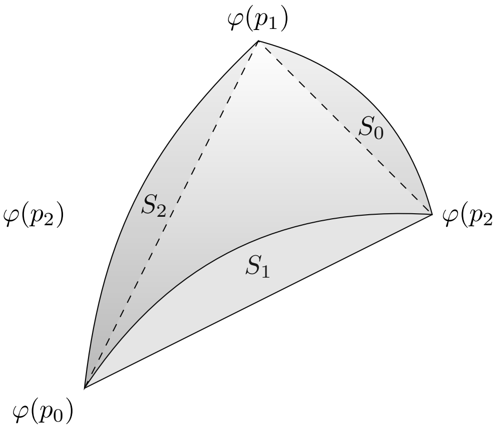
$$(f \circ \varphi)\mathsf{d}\varphi^i \wedge \mathsf{d}\varphi^j = (f \circ \varphi)J_\varphi \mathsf{d}s^1 \wedge \mathsf{d}s^2 \quad \text{with } J_\varphi = \det\begin{pmatrix} \partial_{s^1}\varphi^i & \partial_{s^2}\varphi^i \\ \partial_{s^1}\varphi^j & \partial_{s^2}\varphi^j \end{pmatrix}.$$

Given a simplex $S = [p_0p_1p_2]$, see Figure 4.1, the quantity

$$\left\langle S, \varphi^\natural(f\mathsf{d}x^i \wedge \mathsf{d}x^j)\right\rangle = \int_{[\varphi(p_0)\varphi(p_1)\varphi(p_1)]} f\mathsf{d}x^i \wedge \mathsf{d}x^j$$



(A) $\varphi_\natural S$ and $\varphi(S)$



(B) The "side faces" S_0, S_1, S_2

FIGURE 4.1

is computed on the "basis" $\varphi_\natural S = [\varphi(p_0)\varphi(p_1)\varphi(p_1)]$, while

$$\left\langle S, \varphi^*(f \mathsf{d}x^i \wedge \mathsf{d}x^j)\right\rangle = \int_S (f\circ\varphi) J_\varphi \mathsf{d}s^1 \wedge \mathsf{d}s^2$$

is computed over the surface $\varphi(S)$ (Figure 4.1a), hence their difference can be computed using Stokes theorem on a (piecewise C^1) domain U whose boundary consists of $\varphi_\natural S$, $\varphi(S)$ and the three suitably oriented "side faces" $(S_\ell)_{\ell=0}^2$ (Figure 4.1b). We have the identity

(4.15)

$$\begin{aligned}\left\langle S, \varphi^\natural(f \mathsf{d}x^i \wedge \mathsf{d}x^j) - \varphi^*(f \mathsf{d}x^i \wedge \mathsf{d}x^j)\right\rangle &= \int_{\varphi_\natural S} f \mathsf{d}x^i \wedge \mathsf{d}x^j - \int_{\varphi(S)} f \mathsf{d}x^i \wedge \mathsf{d}x^j \\ &= \sum_{\ell=0}^2 \int_{S_\ell} f \mathsf{d}x^i \wedge \mathsf{d}x^j + \int_{\partial U} f \mathsf{d}x^i \wedge \mathsf{d}x^j \\ &= \sum_{\ell=0}^2 \int_{S_\ell} f \mathsf{d}x^i \wedge \mathsf{d}x^j + \int_U \mathsf{d}f \wedge \mathsf{d}x^i \wedge \mathsf{d}x^j.\end{aligned}$$

Since the "orders" of the two terms in the last line are respectively $\operatorname{diam}^2(S)$ and $\operatorname{diam}^3(S)$, this shows that we cannot in general expect a result analogous to Proposition 4.20, i.e., that $\operatorname{sew}(\varphi^\natural(f \mathsf{d}x^i \wedge \mathsf{d}x^j)) = \varphi^*(f \mathsf{d}x^i \wedge \mathsf{d}x^j)$, unless some cancellations happen in the "side faces".

To generalize the above example to less regular germs and maps we suitably define the integrals along the "side faces" S_ℓ, $\ell \in \{0,1,2\}$. Ultimately, their appearance is due to the fact that the 1-germ $\omega := \varphi^\natural(f \mathsf{d}g^1 \wedge \mathsf{d}g^2)$ may fail to be non-atomic, hence we first study a more general construction of a "side compensator" for 2-germs.

Definition 4.22 (side compensator)**.** *Given $\omega \in \operatorname{Germ}^2(D)$, we say that $\eta \in \operatorname{Germ}^1(D)$ is a* side compensator *of ω if $\eta \approx_{\mathsf{u}} 0$ for some 1-Dini $\mathsf{u} \in \operatorname{Germ}^1(D)$*

and

$$\langle [prq], \omega - \delta\eta \rangle = 0 \quad \textit{for every } [prq] \in \mathrm{Simp}^1(D) \textit{ such that } r = \frac{p+q}{2}. \tag{4.16}$$

Example 4.23. If ω is nonatomic, then $\eta = 0$ is a side compensator for ω. If $\omega = \delta\eta$, with $\eta \approx_{\mathsf{u}} 0$ for some 1-Dini $\mathsf{u} \in \mathrm{Germ}^1(D)$ (η possibly not regular) η itself is a side compensator for ω.

Condition (4.16) is equivalent to

$$\eta_{pq} = \eta_{pr} + \eta_{rq} - \omega_{prq} \quad \text{for every } [prq] \in \mathrm{Simp}^1(D) \text{ such that } r = \frac{p+q}{2}. \tag{4.17}$$

Uniqueness of η is a consequence of the following result (applied to the difference of two side compensators of a given ω). Its proof is given in Appendix D.

Theorem 4.24 (side compensator, uniqueness)**.** *Let $\eta \in \mathrm{Germ}^1(D)$, $\eta \approx_{\mathsf{u}} 0$ for some 1-Dini $\mathsf{u} \in \mathrm{Germ}^1(D)$, and $\langle [prq], \delta\eta \rangle = 0$ for every $[prq] \in \mathrm{Simp}^1(D)$ such that $r = \frac{p+q}{2}$. Then $\eta = 0$.*

In what follows, we write $L(\omega) := \eta$ whenever the side compensator η of ω exists. With this notation, Example 4.23 gives

$$L(\omega) = 0 \quad \text{if } \omega \text{ is nonatomic, and} \tag{4.18}$$

$$L(\delta\eta) = \eta \quad \text{if } \eta \approx_{\mathsf{u}} 0 \text{ with } \mathsf{u} \text{ 1-Dini.} \tag{4.19}$$

The map $\omega \mapsto L(\omega)$ is linear on 2-germs for which it is defined and hence, by (4.19),

$$L(\omega - \delta L(\omega)) = L(\omega) - L(\delta L(\omega)) = L(\omega) - L(\omega) = 0. \tag{4.20}$$

Remark 4.25 (locality)*.* Arguing as in Corollary 3.15, we have that, if $\omega \in \mathrm{Germ}^2(D)$ is such that $L(\omega)$ exists, then, for every $\tilde{D} \subseteq D$, its restriction $\tilde{\omega}$ to $\mathrm{Simp}^k(\tilde{D})$ is such that $L(\tilde{\omega})$ exists, being simply the restriction of $L(\omega)$ to $\mathrm{Simp}^k(\tilde{D})$.

Remark 4.26. If ω is alternating, then $L(\omega)$ is also alternating, since if σ is the transposition $\sigma[pq] = [qp]$, then letting $\tilde{\sigma}[prq] := [qrp]$ one has $\tilde{\sigma}'\omega = -\omega$ and since $r = \frac{p+q}{2} = \frac{q+p}{2}$, we have

$$\begin{aligned} 0 &= \langle [qrp], \delta\eta - \omega \rangle = \langle \tilde{\sigma}[qrp], \delta\eta - \omega \rangle = \langle [prq], \tilde{\sigma}'(\delta\eta - \omega) \rangle \\ &= \langle [prq], \delta(\sigma'\eta) - \tilde{\sigma}'\omega \rangle \end{aligned}$$

yielding $\sigma'\eta = L(\tilde{\sigma}'\omega) = L(-\omega) = -L(\omega)$ (we also use the fact that $\sigma'\eta \approx_{\mathsf{u}} 0$ follows from $\eta \approx_{\mathsf{u}} 0$).

Sufficient conditions for existence of the side compensator are given in the following result, whose proof is postponed in Appendix D.

Theorem 4.27 (side compensator, existence)**.** *Let $\mathsf{u} \in \mathrm{Germ}^2(D)$ be strong 1-Dini. Then, there exists a 1-Dini $\mathsf{v} \in \mathrm{Germ}^1(D)$ such that, for every $\omega \in \mathrm{Germ}^2(D)$ with $\omega \approx_{\mathsf{u}} 0$, the side compensator $L(\omega)$ exists with $[L(\omega)]_{\mathsf{v}} \le [\omega]_{\mathsf{u}}$. Moreover, if $\mathsf{u} = \mathrm{diam}^\alpha$ then $\mathsf{v} = \mathsf{c}(\alpha)\,\mathrm{diam}^\alpha$.*

As in the case of Theorem 3.19, the proof of this result and Remark 2.16 (v) give that if u is strong r-Dini (with $r \ge 1$) it follows that v is r-Dini as well. Another useful result is the following ore.

Theorem 4.28. *Let $\omega \in \mathrm{Germ}^2(D)$ be closed on 2-planes, alternating and such that $\omega \approx_{\mathsf{u}} 0$ for some strong 2-Dini $\mathsf{u} \in \mathrm{Germ}^2(D)$. Then $\omega = \delta L(\omega)$.*

Using the notion of side compensator, we formulate the analogue of Proposition 4.20 for 2-germs.

Proposition 4.29. *Let $\sigma \in (0,1]$, $\alpha > 0$, $\beta > 1$ with $\gamma := \alpha + \beta > 2/\sigma$. Let $f, g^1, g^2 \in \mathrm{Germ}^0(D)$ be continuous with $\|\delta f\|_\alpha + \left\|\delta g^1 \cup \delta g^2\right\|_\beta < \infty$ and let $\varphi\colon I \subseteq \mathbb{R}^2 \to D$ with $\|\delta\varphi\|_\sigma < \infty$. Then*

$$\varphi^\natural(f \mathsf{d} g^1 \wedge \mathsf{d} g^2) - \varphi^*(f \mathsf{d} g^1 \wedge \mathsf{d} g^2) \approx_{\gamma\sigma} \delta\eta, \tag{4.21}$$

with $\eta = L(\varphi^\natural(f \mathsf{d} g^1 \wedge \mathsf{d} g^2))$.

Remark 4.30. By Theorem 4.10 and Example 4.7, letting x^0, x^1, x^2 coordinate functions in $D \subseteq \mathbb{R}^3$ bounded,

$$x^0 \mathsf{d} x^1 \wedge \mathsf{d} x^2 \approx_3 x^0 \cup \frac{1}{2}\left(\delta x^1 \cup \delta x^2 - \delta x^2 \cup \delta x^1\right).$$

Therefore, choosing $\varphi : I^2 \to D \subseteq \mathbb{R}^3$, $\varphi := (f, g^1, g^2)$ with $\|\delta\varphi\|_\alpha$ and $\alpha > 2/3$, we have

$$\varphi^\natural(x^0 \mathsf{d} x^1 \wedge \mathsf{d} x^2) \approx_{3\alpha} f \cup \frac{1}{2}\left(\delta g^1 \cup \delta g^2 - \delta g^2 \cup \delta g^1\right)$$

and $\varphi^*(x^0 \mathsf{d} x^1 \wedge \mathsf{d} x^2) = f \mathsf{d} g^1 \wedge \mathsf{d} g^2$, hence we obtain from (4.21) that

$$f \cup \frac{1}{2}\left(\delta g^1 \cup \delta g^2 - \delta g^2 \cup \delta g^1\right) - f \cup \delta(g^1 \mathsf{d} g^2) \approx_{3\alpha} \delta\eta,$$

with $\eta := L(\varphi^\natural(x^0 \mathsf{d} x^1 \wedge \mathsf{d} x^2))$. In particular,

$$\mathrm{sew}\left(f \cup \frac{1}{2}\left(\delta g^1 \cup \delta g^2 - \delta g^2 \cup \delta g^1\right) - \delta\eta\right) = f \mathsf{d} g^1 \wedge \mathsf{d} g^2.$$

Note also that, from Theorem 4.27, one has $\eta \approx_{3\alpha} L(f \cup \frac{1}{2}\left(\delta g^1 \cup \delta g^2 - \delta g^2 \cup \delta g^1\right))$.

Proof. Consider first the case $f = 1$. In this case, we claim that

$$L(\varphi^\natural(\mathsf{d} g^1 \wedge \mathsf{d} g^2)) = \varphi^\natural(g^1 \mathsf{d} g^2) - (g^1 \circ \varphi)\mathsf{d}(g^2 \circ \varphi) =: \eta, \tag{4.22}$$

so that (4.21) is an identity, i.e.,

$$\varphi^\natural(\mathsf{d} g^1 \wedge \mathsf{d} g^2) - \varphi^*(\mathsf{d} g^1 \wedge \mathsf{d} g^2) = \delta\eta. \tag{4.23}$$

Minding Example 4.21, this follows because the volume integral in the right hand side in the last line of (4.15) is identically zero. To prove (4.22), we notice that η in a 1-germ with $\eta \approx_{\beta\sigma} 0$, because of Proposition 4.20 (with β instead of γ) and if $[prq] \in \mathrm{Simp}^2(D)$ is such that $r = \frac{p+q}{2}$, then

$$\left\langle [prq], \delta\varphi^\natural(g^1 \mathsf{d} g^2) - \delta\eta \right\rangle = \left\langle [pqr], \delta(g^1 \circ \varphi)\mathsf{d}(g^2 \circ \varphi) \right\rangle = 0$$

since $(g^1 \circ \varphi)\mathsf{d}(g^2 \circ \varphi)$ is regular, hence $\delta(g^1 \circ \varphi)\mathsf{d}(g^2 \circ \varphi)$ is nonatomic.

Next, to obtain the thesis for a general f, we argue that

$$\begin{aligned}
&\varphi^\natural(f \mathsf{d} g^1 \wedge \mathsf{d} g^2) - \varphi^*(f \mathsf{d} g^1 \wedge \mathsf{d} g^2) \\
&\qquad \approx_{\gamma\sigma} \varphi^\natural\left(f \cup \delta(g^1 \mathsf{d} g^2)\right) - f \circ \varphi \cup \delta\varphi^*(g^1 \mathsf{d} g^2) \quad \text{by Theorem 4.10} \\
&\qquad = f \circ \varphi \cup \varphi^\natural\delta(g^1 \mathsf{d} g^2) - f \circ \varphi \cup \delta\varphi^*(g^1 \mathsf{d} g^2) \\
&\qquad = f \circ \varphi \cup \left(\varphi^\natural(\mathsf{d} g^1 \wedge \mathsf{d} g^2) - \varphi^*(\mathsf{d} g^1 \wedge \mathsf{d} g^2)\right) \\
&\qquad = f \circ \varphi \cup \delta L(\varphi^\natural(\mathsf{d} g^1 \wedge \mathsf{d} g^2)) \quad \text{by (4.23)} \\
&\qquad \approx_{\gamma\sigma} \delta L\left(\varphi^\natural(\mathsf{d} g^1 \wedge \mathsf{d} g^2)\right) \quad \text{(to be justified)},
\end{aligned}$$

which concludes the proof up to justifying the last passage. The latter justification follows from

$$f\circ\varphi\cup\delta L\left(\varphi^\natural(\mathsf{d}g^1\wedge\mathsf{d}g^2)\right) \quad\approx_{\gamma\sigma}\quad \delta L\left(f\circ\varphi\cup\varphi^\natural(\mathsf{d}g^1\wedge\mathsf{d}g^2)\right), \tag{4.24}$$

$$L\left(f\circ\varphi\cup\varphi^\natural(\mathsf{d}g^1\wedge\mathsf{d}g^2)\right) \quad\approx_{\gamma\sigma}\quad L\left(\varphi^\natural(f\mathsf{d}g^1\wedge\mathsf{d}g^2)\right). \tag{4.25}$$

To prove (4.25), we notice that, as a consequence of Theorem 4.27, if $\omega_1\approx_{\gamma\sigma}\omega_2$ and $L(\omega_1)$ and $L(\omega_2)$ exist, then $L(\omega_1)\approx_{\gamma\sigma}L(\omega_2)$ by linearity of L, and hence it is enough to apply this principle to $\omega_1 := f\circ\varphi\cup\delta\varphi^\natural(g^1\mathsf{d}g^2)$ and $\varphi^\natural(f\mathsf{d}g^1\wedge\mathsf{d}g^2)$ observing that $L(\omega_1)$ and $L(\omega_2)$ exist by Theorem 4.27 applied with $\mathsf{u}=\operatorname{diam}^{\beta\sigma}$, and by Theorem 4.10

$$f\mathsf{d}g^1\wedge\mathsf{d}g^2\approx_\gamma f\cup\delta(g^1\mathsf{d}g^2) \quad\text{hence}\quad \varphi^\natural(f\mathsf{d}g^1\wedge\mathsf{d}g^2)\approx_{\gamma\sigma} f\circ\varphi\cup\varphi^\natural(\delta g^1\mathsf{d}g^2)=\omega_2.$$

To prove (4.24), minding that it becomes an identity if $f\circ\varphi$ is constant, we fix $S=[p_0p_1p_2]$ and without loss of generality assume that $D=S$ and $f_{\varphi_{p_0}}=0$. In this case the left hand side of (4.24) evaluated at S is identically zero, and

$$\begin{aligned}\left|\left\langle S,\delta L\left(f\circ\varphi\cup\varphi^\natural(\mathsf{d}g^1\wedge\mathsf{d}g^2)\right)\right\rangle\right| &= \left|\left\langle \partial S, L\left(f\circ\varphi\cup\varphi^\natural(\mathsf{d}g^1\wedge\mathsf{d}g^2)\right)\right\rangle\right|\\ &\le 3\mathsf{c}(\beta\sigma)\left\|f\circ\varphi\cup\varphi^\natural(\mathsf{d}g^1\wedge\mathsf{d}g^2)\right\|_{\beta\sigma}\operatorname{diam}(S)^{\beta\sigma}\\ &\quad\text{by Theorem 4.27 with } \mathsf{u}:=\operatorname{diam}^{\beta\sigma}\\ &\le \mathsf{c}\left\|f\circ\varphi\right\|_0\left\|\varphi^\natural(\mathsf{d}g^1\wedge\mathsf{d}g^2)\right\|_{\beta\sigma}\operatorname{diam}(S)^{\beta\sigma}\\ &\le \mathsf{c}\left\|\delta f\circ\varphi\right\|_{\alpha\sigma}\left\|\varphi^\natural(\mathsf{d}g^1\wedge\mathsf{d}g^2)\right\|_{\beta\sigma}\operatorname{diam}(S)^{\gamma\sigma}\end{aligned}$$

using the assumption $f_{\varphi_{p_0}}=0$ in the latter inequality. ☐

Combining this result with Theorem 4.28, we deduce the following change of variables formula in top dimension (i.e., $d=2$), which in a sense improves [16, Proposition 3.3 (6)].

Corollary 4.31. *Under the assumptions of the above proposition, assume furthermore that $D\subseteq\mathbb{R}^2$. Then, for every $S=[p_0p_1p_2]\in\operatorname{Simp}^2(I)$,*

$$\int_{\varphi_\natural S} f\mathsf{d}g^1\wedge\mathsf{d}g^2=\int_S(f\circ\varphi)\mathsf{d}(g^1\circ\varphi)\wedge\mathsf{d}(g^2\circ\varphi)+\langle\partial S,\eta\rangle, \tag{4.26}$$

with $\eta=L(\varphi^\natural(f\mathsf{d}g^1\wedge\mathsf{d}g^2))$. In particular, if $\varphi(\operatorname{conv}([p_ip_j])\subseteq\operatorname{conv}([\varphi_{p_i}\varphi_{p_j}])$ for every $\{i,j\}\subseteq\{0,1,2\}$, then $\langle[p_ip_j],\eta\rangle=0$, and hence

$$\int_{\varphi_\natural S} f\mathsf{d}g^1\wedge\mathsf{d}g^2=\int_S(f\circ\varphi)\mathsf{d}(g^1\circ\varphi)\wedge\mathsf{d}(g^2\circ\varphi). \tag{4.27}$$

Proof. We apply Theorem 4.28 with $\omega:=\varphi^\natural(f\mathsf{d}g^1\wedge\mathsf{d}g^2)-\varphi^*(f\mathsf{d}g^1\wedge\mathsf{d}g^2)-\delta\eta$, $\eta:=L\left(\varphi^\natural(f\mathsf{d}g^1\wedge\mathsf{d}g^2)\right)$ and the strong 2-Dini $\mathsf{u}:=\operatorname{diam}^{\gamma\sigma}$. The condition $\omega\approx_{\mathsf{u}}0$ is (4.14), ω is alternating because so are $f\mathsf{d}g^1\wedge\mathsf{d}g^2$, $\varphi^*(f\mathsf{d}g^1\wedge\mathsf{d}g^2)$ and $\varphi^\natural$, L and δ preserve the alternating property (Remark 4.26). We have

$$\delta\omega=\varphi^\natural\left(\delta(f\mathsf{d}g^1\wedge\mathsf{d}g^2)\right)-\delta\varphi^*(f\mathsf{d}g^1\wedge\mathsf{d}g^2)=0,$$

the first term being zero because $\delta(f\mathsf{d}g^1\wedge\mathsf{d}g^2)=0$ by the top-dimensional assumption $D\subseteq\mathbb{R}^2$, the second term because it is a regular form. Finally,

$$\begin{aligned}L(\omega)&=L(\varphi^\natural(f\mathsf{d}g^1\wedge\mathsf{d}g^2)-\varphi^*(f\mathsf{d}g^1\wedge\mathsf{d}g^2)-\delta\eta)\\&=-L(\varphi^*(f\mathsf{d}g^1\wedge\mathsf{d}g^2))\quad\text{by (4.20) with } \varphi^\natural(f\mathsf{d}g^1\wedge\mathsf{d}g^2)\text{ instead of }\omega\\&=0\qquad\text{by (4.18), by nonatomicity of }\varphi^*(f\mathsf{d}g^1\wedge\mathsf{d}g^2),\end{aligned}$$

from which we deduce from Theorem 4.28 that $\omega=0$, i.e., (4.26) holds. To prove (4.27), it is sufficient to notice that the restriction of the 2-germ $\varphi^\natural(f\mathsf{d}g^1\wedge\mathsf{d}g^2)$ to $\operatorname{conv}([p_ip_j])$ is zero, because $f\mathsf{d}g^1\wedge\mathsf{d}g^2$ is nonatomic and $\operatorname{vol}_2(\varphi_\natural S)=0$ for every simplex S in $\operatorname{conv}([p_ip_j])$, hence by Remark 4.25 we deduce that $\langle[p_ip_j],\eta\rangle=0$. □

Other sufficient conditions implying that (4.27) holds, i.e., $\langle\partial S,\eta\rangle=0$, are for example $f\circ\varphi=0$ or $g^\ell\circ\varphi$ constant (for some $\ell=1,2$) on $\operatorname{conv}([p_ip_j])$, for every $\{i,j\}\subseteq\{0,1,2\}$, i.e. on the topological border of S.

5. More irregular germs and correctors

Theorem 3.19 and its Corollary 3.24 provide a linear continuous "regularizing" map $\omega\mapsto\operatorname{sew}\omega$ for a large class of k-germs, as discussed in Section 4. In this section we argue that the same result can be useful also in settings where it is not directly applicable, extending the scope of the sewing lemma [2, 5, 3] in rough paths theory to higher dimensions. To fix the ideas, consider the following example.

Example 5.1. Let $D\subseteq\mathbb{R}^d$ be bounded. For fixed $\xi\in\mathbb{R}^d$, we consider the sequence of 0-germs

$$f^n_p:=\frac{1}{\sqrt{n}}\cos(n\xi\cdot p)\quad g^n_p:=\frac{1}{\sqrt{n}}\sin(n\xi\cdot p),\quad\text{for } p\in D.$$

Although each f^n, g^n is smooth, pointwise converging to 0 as $j\to+\infty$, it is easy to calculate manually that

$$\lim_{n\to\infty}\int_{[pq]}f^n\mathsf{d}g^n\to\frac{1}{2}\xi\cdot(q-p).\tag{5.1}$$

In our terms, this means that although the germs $f^n\cup\delta g^n\to0$ pointwise, their regularizations $f^n\mathsf{d}g^n=\operatorname{sew}(f^n\delta g^n)\not\to0$, showing that we are not in the assumptions of Corollary 3.24. In particular, this shows that for any strong r-Dini gauge $\mathsf{u}\in\operatorname{Germ}^2(D)$ (satisfying the assumptions of Corollary 3.24) with $r\geq1$,

$$\sup_{n\geq1}\|\delta f^n\cup\delta g^n\|_{\mathsf{u}}=\infty.\tag{5.2}$$

Nevertheless, Example 5.1 may fit into our framework as follows. Given $f\in\operatorname{Germ}^0(D)$, $\eta\in\operatorname{Germ}^{k-1}(D)$ regular such that $\delta(f\cup\delta\eta)=(\delta f)\cup(\delta\eta)\not\approx_{\mathsf{u}}0$ for every strong k-Dini u (with u and $\tilde{\mathsf{u}}$ in (2.3) continuous), it may happen that we can find a non-atomic $\omega\in\operatorname{Germ}^k(D)$, called further *corrector*, such that

$$\delta(f\cup\delta\eta)\approx_{\mathsf{u}}\delta\omega,\tag{5.3}$$

hence Theorem 3.19 (under mild continuity assumptions) applies to $f\cup\delta\eta-\omega$ providing the regular k-germ

$$\operatorname{sew}(f\cup\delta\eta-\omega).$$

This scheme is advantageous if we are able to construct such a single corrector suitable for a reasonable class of f and η, since it allows for a "good" calculus in terms f, η and ω. Indeed, once a corrector ω is chosen for a single pair (f,η), one can generate correctors for pairs that are "approximately" close to (f,η). For example, if we consider the k-germ $\varphi(f)\cup\delta\eta$, where $\varphi:\mathbb{R}\to\mathbb{R}$ is smooth (with bounded derivative φ'), then we have

$$
\begin{aligned}
\delta\left(\varphi(f)\cup\delta\eta\right) &= \delta\varphi(f)\cup\delta\eta \\
&= \varphi'(f)\cup\delta f\cup\delta\eta + (\delta\varphi(f)-\varphi'(f)\cup\delta f)\cup\delta\eta \\
&\approx_{\mathsf{u}} \varphi'(f)\cup\delta\omega + (\delta\varphi(f)-\varphi'(f)\cup\delta f)\cup\delta\eta \\
&\qquad \text{by boundedness of } \varphi' \text{ and (5.3),} \\
&= \delta\left(\varphi'(f)\cup\omega\right) - \delta\varphi'(f)\cup\omega + (\delta\varphi(f)-\varphi'(f)\cup\delta f)\cup\delta\eta \\
&\approx_{\mathsf{u}} \delta\left(\varphi'(f)\cup\omega\right),
\end{aligned}
\tag{5.4}
$$

provided that

$$
-\delta\varphi'(f)\cup\omega + (\delta\varphi(f)-\varphi'(f)\cup\delta f)\cup\delta\eta \approx_{\mathsf{u}} 0. \tag{5.5}
$$

If the latter condition holds, then Theorem 3.19 provides existence of the regular k-germ

$$
\operatorname{sew}(\varphi(f)\cup\delta\eta-\varphi'(f)\cup\omega).
$$

Therefore, if we interpret $\operatorname{sew}(f\cup\delta\eta-\omega)+\omega$ as a "corrected" version of $f\mathsf{d}\eta$ (not necessarily regular, because so maybe ω), then the argument above shows how the "corrected" version of $\varphi(f)\mathsf{d}\eta$ can be consistently defined, i.e., $\operatorname{sew}(\varphi(f)\cup\delta\eta-\varphi'(f)\cup\omega)+\varphi'(f)\cup\omega$.

A second advantage of this scheme is a natural stability provided by Theorem 3.23 if a sequence $(f^j,\eta^j,\omega^j)_{j\geq 1}$ converges pointwise to (f,η,ω) satisfying (5.3) uniformly in j (i.e., $\sup_j[\delta f^j\cup\delta\eta^j-\omega^j]_{\mathsf{u}}<\infty$) then by Corollary 3.24 the corrected versions of $f^j\mathsf{d}\eta^j$ converge pointwise to that of $f\mathsf{d}\eta$.

Of course, a trivial choice for (5.3) to hold is simply $\omega := f\cup\delta\eta$ (note that it is non-atomic because η is assumed regular). On one side, this provides a trivial "correction" of $f\mathsf{d}g$, setting it equal to $f\cup\delta\eta$. On the other side, one faces several problems to extend the calculus, since to ensure that (5.5) hold, the first term therein is equal to

$$
\delta\varphi'(f)\cup\omega = (\delta\varphi'(f))\cup(f\cup\delta\eta)\approx_{\mathsf{u}}\delta\varphi'(f)\cup\delta\eta, \quad \text{(assuming that } f \text{ is bounded)}
$$

which is comparable (in terms of gauges) to $(\delta f)\cup(\delta\eta)$, because so is $\delta\varphi'(f)$ with δf, since φ' a Lipschitz function. Therefore, we cannot expect $\delta\varphi'(f)\cup(f\cup\delta\eta)\approx_{\mathsf{u}} 0$, while the second term in (5.5) is naturally expect to be $\approx_{\mathsf{u}} 0$ because of smoothness of φ (it contains the remainder of the first order Taylor expansion of φ).

To avoid such trivialities, in rough paths theory (dealing with the case $k=d=1$) one imposes that ω must be *more regular* than what $f\cup\delta\eta$ is expected to be, in the sense of Hölder (or p-variation) gauges, and the existence of ω in applications is usually shown by means of stochastic calculus.

Example 5.2 (Brownian motion as a Rough Path). In our terminology, if $D=[0,T]\subseteq\mathbb{R}$ is an interval and $f_t=\eta_t=B_t$ is a trajectory of Brownian motion, we may define a corrector ω (using Stratonovich integration) as a trajectory of

$$
\omega_{st} = B_s\delta B_{st} - \int_s^t B_r\circ\mathsf{d}B_r = -\frac{1}{2}(\delta B_{st})^2 \quad \text{for } s,\, t\in[0,T].
$$

Kolmogorov criterion gives that $\|\delta B\|_{1/2-\varepsilon} < \infty$ and $\|\omega\|_{1-2\varepsilon} < \infty$ for every $\varepsilon > 0$ (with full probability). As a consequence, one can prove that (5.5) holds with $\mathsf{u} = \operatorname{diam}^\gamma$ with $\gamma = (2+\alpha)(1/2-\varepsilon)$, hence if $\varphi : \mathbb{R} \to \mathbb{R}$ is differentiable with $[\delta\varphi']_\alpha < \infty$, (with $\alpha > 0$), one has that $\operatorname{sew}(\varphi(B) \cup \delta B - \varphi'(B) \cup \omega)_{st} = \int_s^t \varphi(B) \circ \mathsf{d}B$, obtaining the Rough Path (Stratonovich) integral.

Below we provide two deterministic examples of this scheme, in the cases $k \in \{1,2\}$ (the case $k = d = 1$ being that of rough paths – of regularity $\alpha > 1/3$), extending the well-known "pure-area" rough path construction [3, Exercise 2.17]. We point out that both examples could be also considered by elementary means or within the theory of Young measures, hence they should be just considered as proof-of-principles.

First, we show that we may use an appropriate corrector also in Example 5.1.

Example 5.3. In order to apply Corollary 3.24 in Example 5.1, for any $n \geq 1$, we choose the corrector

$$\omega^n_{pq} := f^n_p \delta g^n_{pq} - \int_{[pq]} f^n \mathsf{d}g^n. \tag{5.6}$$

A straightforward computation gives

$$\int_{[pq]} f^n \mathsf{d}g^n = \delta I^n_{pq}, \quad \text{where} \quad I^n_p = \frac{1}{2}\xi \cdot p + \frac{1}{2n}\sin(2n\xi \cdot p),$$

hence (5.3) is satisfied for every u (with f^n instead of f and g^n instead of η) since

$$\delta\omega^n = \delta f^n \cup \delta g^n - \delta(\delta I^n) = \delta f^n \cup \delta g^n.$$

In order to argue as in (5.4) we notice first that

$$\sup_{n\geq 1} \|\delta f^n\|_{1/2} + \|\delta g^n\|_{1/2} < \infty \quad \text{and also} \quad \sup_{n\geq 1} \|\omega^n\|_1 < \infty,$$

using the inequalities

$$|\cos(n\xi \cdot q) - \cos(n\xi \cdot p)| \leq \min\{2, n|\xi \cdot q - \xi \cdot p|\} \leq \sqrt{2n|\xi \cdot (q-p)|},$$

and similarly for the sine terms. If $\varphi : \mathbb{R} \to \mathbb{R}$ is differentiable with $\|\delta\varphi'\|_\alpha < \infty$, then

$$\|\delta\varphi'(f^n)\|_{\alpha/2} \leq \|\delta\varphi'\|_\alpha \|\delta f^n\|_{1/2}$$

and the remainder of the Taylor expansion (with $\operatorname{Id}(x) = x$) gives

$$\|\delta\varphi(f^n) - \varphi'(f^n) \cup \delta f^n\|_{(1+\alpha)/2} \leq \|\delta\varphi - \varphi' \cup \delta \operatorname{Id}\|_{1+\alpha} \|\delta f^n\|_{1/2},$$

hence the expression in (5.5) reads

$$-\delta\varphi'(f^n) \cup \omega^n + (\delta\varphi(f^n) - \varphi'(f^n) \cup \delta f^n) \cup \delta g^n \approx_{1+\frac{\alpha}{2}} 0,$$

uniformly as $n \to +\infty$. As a consequence, since $[\omega^n]_{\operatorname{diam}^2} < \infty$ for every $n \geq 1$ (though not uniformly in $n \geq 1$), one has

$$\operatorname{sew}(\varphi(f^n) \cup \delta g^n - \varphi'(f^n) \cup \omega^n) = \operatorname{sew}(\varphi(f^n) \cup \delta g^n) = \varphi(f^n)\mathsf{d}g^n$$

and we deduce convergence of the corrected versions of $\varphi(f^n)\mathsf{d}g^n$, in particular yielding the limit

$$\lim_{n\to+\infty} \int_{[pq]} \varphi(f^n)\mathsf{d}g^n = \frac{\varphi'(0)}{2}\xi \cdot (q-p).$$

given by (5.1).

Remark 5.4. We notice that another reasonable choice for a corrector in Example 5.3 would be $\tilde{\omega}^n := f^n \cup \delta g^n - \frac{1}{2}\xi \cdot \delta\,\mathrm{Id}$, neglecting the Lipschitz continuous summand in I^n above. With this choice, the only difference would be in the definition of corrected version of the integral of $\varphi(f^n)\mathsf{d}g^n$, where an additional integral would appear (still, negligible in the limit as $n \to +\infty$). However, when $\varphi = \mathrm{Id}$ this choice would give $\operatorname{sew}(f^n \cup \delta g^n - \tilde{\omega}^n) = \frac{1}{2}\xi \cdot \delta\,\mathrm{Id}$, this time independent of n.

To conclude, we give a similar example for 2-germs.

Example 5.5. Let $D \subseteq \mathbb{R}^2$ be bounded. Consider the sequence of 0-germs, for $p = (p_1, p_2) \in D$,

$$f^n_p := \frac{1}{n^{1-\varepsilon}} \cos(np_1)\cos(np_2), \quad g^n_p := \frac{1}{n^{\varepsilon+1/2}} \sin(np_1), \quad h^n_p := \frac{1}{\sqrt{n}} \sin(np_2),$$

for some $\varepsilon \in (0,1)$, so that in particular

$$\sup_{n\geq 1} \|\delta f^n\|_{1-\varepsilon} + \|\delta g^n\|_{\varepsilon+1/2} + \|\delta h^n\|_{1/2} < \infty.$$

We let $\eta^n := g^n \mathsf{d}h^n$, so that $\langle [pq], \eta^n \rangle = \int_{[pq]} f^n \mathsf{d}g^n$, and consider the (smooth) 2-germ $f^n \cup \delta\eta^n$, for which

(5.7)
$$\begin{aligned}
\langle [pqr], \operatorname{sew}(f^n \cup \delta\eta^n) \rangle &= \int_{[pqr]} f^n \mathsf{d}g^n \wedge \mathsf{d}h^n \\
&= \int_{[pqr]} f^n \det(\nabla g^n, \nabla h^n) \mathsf{d}x^1 \wedge \mathsf{d}x^2 \quad \text{by Proposition 4.17} \\
&= \int_{[pqr]} \cos^2(nx)\cos^2(ny) \mathsf{d}x^1 \wedge \mathsf{d}x^2 \\
&= \frac{1}{4}\int_{[pqr]} \left(1 - \cos(2nx^1)\right)\left(1 - \cos(2nx^2)\right) \mathsf{d}x^1 \wedge \mathsf{d}x^2 \\
&= \frac{1}{4}\int_{[pqr]} \mathsf{d}x^1 \wedge \mathsf{d}x^2 + o(1) \quad \text{as } n \to \infty.
\end{aligned}$$

Since $g^n \cup \delta h^n \to 0$ pointwise and $\sup_{n\geq 1} \|\delta g^n \cup \delta h^n\|_{1+\varepsilon} < \infty$, by Corollary 3.24 applied to $\omega^n := g^n \cup \delta h^n$, we obtain that $\eta^n = g^n \mathsf{d}h^n \to 0$ pointwise, hence also $f^n \cup \delta\eta^n \to 0$ pointwise. This fact, in combination with (5.7), shows that we are not in a position to apply Corollary 3.24 to the germs $f^n \cup \delta\eta^n$, in particular it must be $\sup_{n\geq 1} \|\delta f^n \cup \delta\eta^n\|_{\mathsf{u}} = \infty$ for every strong r-Dini gauge $\mathsf{u} \in \mathrm{Germ}^3(D)$ satisfying the assumptions in Corollary 3.24, with $r \geq 2$.

In this case we choose as a corrector,

$$\omega^n_{pqr} := f^n_p \delta\eta^n_{pqr} - \int_{[pqr]} f^n \mathsf{d}g^n \wedge \mathsf{d}h^n, \quad \text{for } [pqr] \in \mathrm{Simp}^2(D),$$

so that (5.3) becomes an identity $\delta\omega^n = \delta f^n \cup \delta\eta^n$, since the integral term is a regular top-dimensional form, hence closed. Moreover, by (5.7), we have $\omega^n \to \frac{1}{4}\mathsf{d}x \wedge \mathsf{d}y$ pointwise, where $\langle [pqr], \mathsf{d}x \wedge \mathsf{d}y \rangle = \det(q-p, r-p)$ is the "oriented area" 2-germ, and

$$\sup_{n\geq 1} [\omega^n]_{\mathrm{diam}^2} < \infty,$$

using also the inequalities (by Proposition 4.8 with $\gamma = 2$, g^n instead f and h^n instead of g)

$$[f^n \cup \delta\eta^n]_{\mathrm{diam}^2} \le \mathsf{c}\,\|f^n\|_0\,\|\delta g^n\|_1\,\|\delta h^n\|_1 \le \mathsf{c} n^{\varepsilon-1} n^{1/2-\varepsilon} n^{1/2} \le \mathsf{c}.$$

Finally, if $\varphi : \mathbb{R} \to \mathbb{R}$ is as in in Example 5.3, we obtain that the expression in (5.5) reads

$$-\delta\varphi'(f^n) \cup \omega^n + (\delta\varphi(f^n) - \varphi'(f^n) \cup \delta f^n) \cup \delta g^n \approx_{2+\alpha(1-\varepsilon)} 0,$$

uniformly as $n \to +\infty$. As a consequence, we have the convergence of the corrected versions of $\varphi(f^n)\mathsf{d}g^n$ given by (5.1), which in particular gives the limit

$$\lim_{n\to+\infty} \int_{[pqr]} \varphi(f^n)\mathsf{d}g^n \wedge \mathsf{d}h^n = \frac{1}{4}\varphi'(0)\det(q-p, r-p).$$

Remark 5.6. We notice that in Example 5.3 (respectively, in Example 5.5) we are exactly in the limit case of Young (respectively, Züst) non-integrability, that is, the sums of Hölder exponents of the integrands equal k, the dimension of the respective germs. One may also make similar examples with regularity below this threshold, for instance in Example 5.3 one could have, for $\varepsilon \in [0, 1/2)$,

$$g_p^n := \frac{1}{n^{1/2-\varepsilon}} \sin(n\xi \cdot p)\mathbb{1}_{A_n}(\xi \cdot p),$$

where $A_n \subseteq \mathbb{R}$ is a finite union of intervals with endpoints in the set $\{x\colon \sin(nx) = 0\}$ and, for every $s \le t \in \mathbb{R}$, $\lim_{n\to+\infty} n^\epsilon \int_{A_n \cap [s,t]} \mathsf{d}x = (t-s)$. With this choice, $\|g^n\|_{1/2-\varepsilon}$ is uniformly bounded but the resulting corrector defined as in (5.6) satisfies only $\sup_{n\ge 1} \|\omega^n\|_{1-\varepsilon} < \infty$. Estimating the expression (5.5) gives, for a $\varphi \in C^{1,\alpha}$,

$$-\delta\varphi'(f^n) \cup \omega^n + (\delta\varphi(f^n) - \varphi'(f^n) \cup \delta f^n) \cup \delta g^n \approx_{1+\frac{\alpha}{2}-\varepsilon} 0,$$

so that convergence of the corrected versions of $\varphi(f^n)\mathsf{d}g^n$ is guaranteed if $1+\frac{\alpha}{2}-\varepsilon > 1$, that is $\alpha > 2\varepsilon$.

Appendix A. Decomposition of simplices

Geometric maps. We say that a map $\tau : \mathrm{Simp}^k(D) \to \mathrm{Simp}^h(D)$, $S \mapsto \tau S$, is *geometric* if it is continuous and commutes with affine transformations, i.e., for every $\varphi : D \to D$ affine, one has $\tau\varphi_\natural = \varphi_\natural\tau$. We extend this definition to the linear space generated by geometric maps, i.e., we say that $\tau : \mathrm{Simp}^k(D) \to \mathrm{Chain}^h(D)$ is geometric if it is a finite linear combination of geometric maps $\tau^i : \mathrm{Simp}^k(D) \to \mathrm{Simp}^h(D)$, $\tau = \sum_{i=1}^\ell \lambda_i\tau_i$, with $\lambda_i \in \mathbb{R}$. We then define $|\tau|$ as the map $|\tau| := \sum_{i=1}^\ell \lambda_i\varrho_i$. We notice the following fact.

Lemma A.1 (geometric maps and Dini gauges)**.** *Let $\tau : \mathrm{Simp}^k(D) \to \mathrm{Chain}^h(D)$ be geometric and $\mathsf{u} \in \mathrm{Germ}^h(D)$. Then the following assertions are true.*

(i) If u is is r-Dini, then $|\tau|'\mathsf{u}$ is r-Dini.
(ii) If u is strong r-Dini then $|\tau|'\mathsf{u}$ is strong r-Dini with the germ as in (2.3) given by $|\tau|'\tilde{\mathsf{u}}$.
(iii) If u is continuous, then $|\tau|'\mathsf{u}$ is continuous.
(iv) For every $\omega \in \mathrm{Germ}^h(D)$, $[|\tau|'\omega]_{|\tau|'\mathsf{u}} \le [\omega]_{\mathsf{u}}$.

Proof. It is sufficient to argue in the case $\tau : \mathrm{Simp}^k(D) \to \mathrm{Simp}^h(D)$, the general case following since the properties are stable with respect to linear combinations. The only non-trivial property is then (i). This follows from the fact that if S is isometric to $S' \in \mathrm{Simp}^k(D)$ then τS is isometric to $\tau S'$, since geometric maps commute with isometries. Then, for every $n \geq 0$, one has

$$2^{nr} \left\langle (2^{-n})_\natural S, \tau' \mathsf{u} \right\rangle = 2^{nr} \left\langle (2^{-n})_\natural \tau S, \mathsf{u} \right\rangle \to 0.$$

as $n \to +\infty$. □

Decomposable maps. We say that $\tau : \mathrm{Simp}^k(D) \to \mathrm{Chain}^h(D)$ is *decomposable* if there exist geometric maps $\nu : \mathrm{Simp}^k(D) \to \mathrm{Chain}^h(D)$, $\varrho : \mathrm{Simp}^k(D) \to \mathrm{Chain}^{h+1}(D)$ such that

$$\nu = \sum_{i=1}^{\ell} \lambda_i \nu_i \quad \text{with } \mathrm{vol}_h \circ \nu_i = 0 \text{ for } i \in \{1, \ldots, \ell\} \tag{A.1}$$

and

$$\tau = \nu + \partial \varrho.$$

Given $\tau_1 : \mathrm{Simp}^k(D) \to \mathrm{Chain}^h(D)$, $\tau_2 : \mathrm{Simp}^k(D) \to \mathrm{Chain}^h(D)$, we write $\tau_1 \equiv \tau_2$ if the difference $\tau_1 - \tau_2$ is decomposable.

Remark A.2. If $\tau_1 \equiv \tau_2$, then for every $\omega \in \mathrm{Germ}^h(D)$ is closed and nonatomic one has $\tau_1' \omega = \tau_2' \omega$, since for $S \in \mathrm{Simp}^k(D)$, $\langle S, \tau_1'\omega - \tau_2'\omega \rangle = \langle \nu S, \omega \rangle + \langle \varrho S, \delta\omega \rangle = 0$.

We notice that, if $\tau_1 : \mathrm{Simp}^{k_1}(D) \to \mathrm{Chain}^{k_2}(D)$, $\tau_2 : \mathrm{Simp}^{k_2}(D) \to \mathrm{Chain}^{k_3}(D)$ are both geometric, then the composition $\tau_2 \circ \tau_1$ is geometric, and if moreover τ_2 is decomposable, then $\tau_2 \circ \tau_1$ is decomposable, for $\tau_2 \circ \tau_1 = \nu \circ \tau_1 + \partial(\varrho \circ \tau_1)$.

Permutations. A first example of decomposable maps is provided by permutations.

Lemma A.3. *Let σ be a permutation on $\{0, 1, \ldots, k\}$. Then the induced map $\sigma : \mathrm{Simp}^k(D) \to \mathrm{Simp}^k(D)$ is geometric and $\sigma \equiv (-1)^\sigma \,\mathrm{Id}$.*

Proof. As already noticed in Section 2.1, σ commutes with push-fowards, hence in particular with affine transformations. To prove $\sigma \equiv (-1)^\sigma$ by composition, it is sufficient to assume that $\sigma[p_0 \ldots p_i p_{i+1} \ldots p_k] = [p_0 \ldots p_{i+1} p_i \ldots p_k]$ is a transposition with $i \in \{0, 1, \ldots, k-1\}$. For any $S \in \mathrm{Simp}^k(D)$, define the geometric maps

$$\varrho S := (-1)^i [p_0 p_1 \ldots p_i p_{i+1} p_i \ldots p_k] \in \mathrm{Simp}^{k+1}(D)$$

and

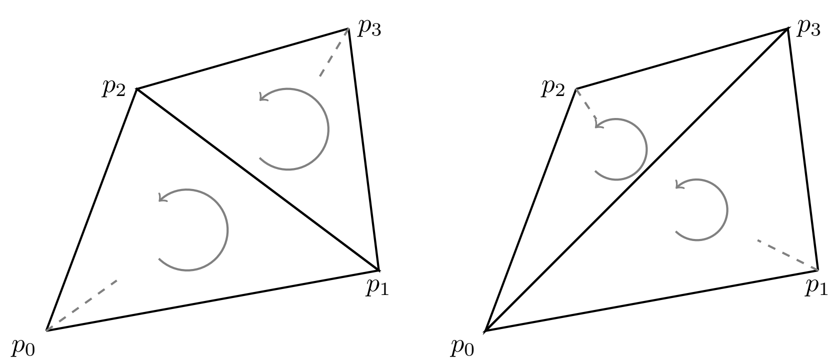
$$\begin{aligned} \nu S := -\sum_{j=0}^{i-1} (-1)^{i+j} [p_0 p_1 \ldots \hat{p}_j \ldots p_i p_{i+1} p_i \ldots p_k] + [p_0 p_1 \ldots p_i p_i \ldots p_k] \\ - \sum_{j=i+2}^{k} (-1)^{i+j} [p_0 p_1 \ldots p_i p_{i+1} p_i \ldots \hat{p}_j \ldots p_k] \in \mathrm{Chain}^k(D), \end{aligned}$$

which satisfies (A.1) p_i appears at least twice in every simplex. Then,

$$\begin{aligned}\partial \varrho S = &\sum_{j=0}^{i-1}(-1)^{i+j}[p_0p_1\dots\hat{p}_j\dots p_ip_{i+1}p_i\dots p_k] \\ &+[p_0p_1\dots p_{i-1}p_{i+1}p_i\dots p_k]-[p_0p_1\dots p_ip_i\dots p_k]+[p_0p_1\dots p_ip_{i+1}p_{i+2}\dots p_k] \\ &+\sum_{j=i+2}^{k}(-1)^{i+j}[p_0p_1\dots p_ip_{i+1}p_i\dots\hat{p}_j\dots p_k] \\ =& -\nu S+S+\sigma S.\end{aligned}$$

showing the equivalence $\sigma \equiv (-1)^\sigma$. □

Edge-flipping. A second example of decomposable maps is *edge-flipping* on 2-chains. In general, given $(p_i)_{i=0}^3$, a general edge-flipping consists of replacing $[p_0p_1p_2] + [p_3p_2p_1]$ with $[p_2p_0p_3] + [p_1p_3p_0]$ (Figure A.1).



(A) $[p_0p_1p_2] + [p_3p_2p_1]$ (B) $[p_1p_3p_0] + [p_2p_0p_3]$

FIGURE A.1. A general edge flipping operation.

However, to define it as a decomposable map on 2-simplices, we consider only the case when the points belong to a parallelogram, i.e.

$$p_0 + p_3 = p_1 + p_2. \tag{A.2}$$

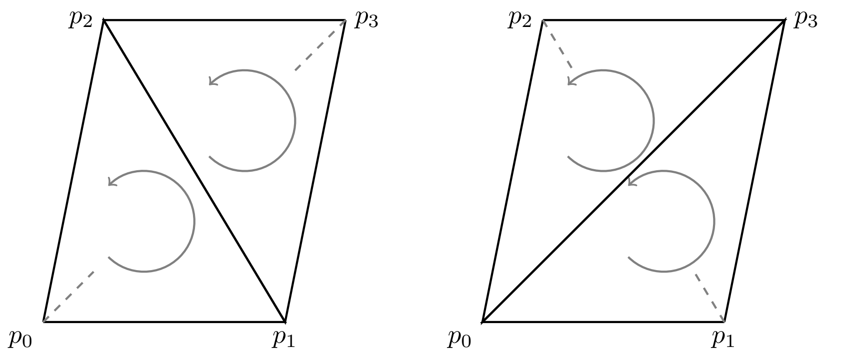
For $S = [p_0p_1p_2]$, we define the maps π, $\tilde{\pi} : \mathrm{Simp}^2(D) \to \mathrm{Chain}^2(D)$,

$$\pi S := [p_0p_1p_2] + [p_3p_2p_1], \quad \tilde{\pi} S := [p_1p_3p_0] + [p_2p_0p_3]$$

with p_3 such that (A.2) holds, and let $\operatorname{flip} S := \pi - \tilde{\pi}$.

Remark A.4. In fact, it may be that $[p_3p_2p_1]$, $[p_1p_3p_0]$ or $[p_2p_0p_3]$ do not belong to $\mathrm{Simp}^2(D)$, e.g. when $p_3 \notin D$. Hence, rigorously π, $\tilde{\pi}$ and flip are well-defined on a smaller domain than $\mathrm{Simp}^2(D)$, precisely those $S \in \mathrm{Simp}^2(D)$ such that the three simplices above still belong to $\mathrm{Simp}^2(D)$. In our applications, this will be always the case hence we avoid to add this specification.

Lemma A.5. *The maps π, $\tilde{\pi}$ are geometric and $\pi \equiv \tilde{\pi}$, i.e.* flip *is decomposable.*



(A) $\pi[p_0p_1p_2] = [p_0p_1p_2] + [p_3p_2p_1]$ (B) $\tilde{\pi}[p_0p_1p_2] = [p_1p_3p_0] + [p_2p_0p_3]$

FIGURE A.2. Parallelograms associated to $[p_0p_1p_2]$.

Proof. Given $S = [p_0p_1p_2]$, the maps $\tau_1 S := [p_1p_2p_3]$, $\tau_2 S := [p_1p_3p_0]$ and $\tau_3 S := [p_2p_0p_3]$ are clearly geometric entailing that $\pi = \mathrm{Id} + \tau_1$ and $\tilde{\pi} = \tau_2 + \tau_3$ are both geometric. To show that flip is decomposable, we notice that, introducing the geometric map $\varrho[p_0p_1p_2] := [p_3p_0p_1p_2]$, one has

$$\begin{aligned}\partial\varrho[p_0p_1p_2] &= [p_0p_1p_2] - [p_3p_1p_2] + [p_3p_0p_2] - [p_3p_0p_1] \\ &= \pi[p_0p_1p_2] - ([p_3p_1p_2] + [p_3p_2p_1]) \\ &\quad - \tilde{\pi}[p_0p_1p_2] - ([p_2p_0p_3] + [p_3p_0p_2]) + ([p_1p_3p_0] - [p_3p_0p_1])\end{aligned} \tag{A.3}$$

an identity entailing that $\partial\varrho \equiv \pi - \tilde{\pi} = \mathrm{flip}$. □

Remark A.6. From identity(A.3) it actually follows that if $\omega \in \mathrm{Germ}^2(D)$ is closed and alternating (but not necessarily nonatomic), then $\langle \mathrm{flip}[p_0p_1p_2], \omega\rangle = 0$.

Remark A.7 (change of base points). For technical reasons, we may require to perform slightly different edge-flippings: given $[p_0p_1p_2] \in \mathrm{Simp}^2(D)$, letting p_3 as above, we replace

$$\pi^\dagger[p_0p_1p_2] := [p_2p_0p_1] + [p_1p_3p_2] \quad \text{with} \quad \tilde{\pi}^\dagger[p_0p_1p_2] := [p_3p_0p_1] + [p_0p_3p_2],$$

(Figure A.3) letting $\mathrm{flip}^\dagger := \pi^\dagger - \tilde{\pi}^\dagger$. However, it is immediate to notice that $\pi^\dagger = \sigma\pi$, with $\sigma[p_0p_1p_2] := [p_2p_0p_1]$ and $\tilde{\pi}^\dagger = \tilde{\sigma}\tilde{\pi}^\dagger$ with $\tilde{\sigma}[p_0p_1p_2] := [p_1p_2p_0]$, hence $\mathrm{flip}^\dagger \equiv 0$.

Edge-cutting. As a third example, consider the following maps. For fixed $t \in [0,1]$, define

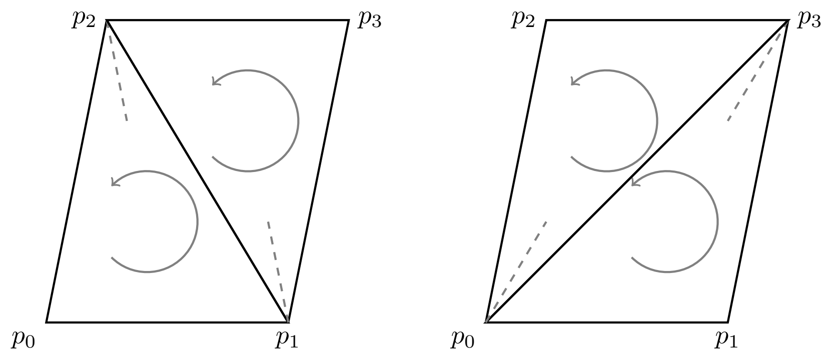
$$\mathrm{cut}_t : \mathrm{Simp}^1(D) \to \mathrm{Chain}^1(D), \quad \mathrm{cut}_t[p_0p_1] := [p_0p_t] + [p_tp_1]$$

where $p_t := (1-t)p_0 + tp_1$ (Figure A.4), and

$$\mathrm{cut}_t : \mathrm{Simp}^2(D) \to \mathrm{Chain}^2(D), \quad \mathrm{cut}_t[pp_0p_1] := [p_tp_1p] + [p_tpp_1],$$

where $p_t := (1-t)p_0 + tp_1$ (Figure A.5).

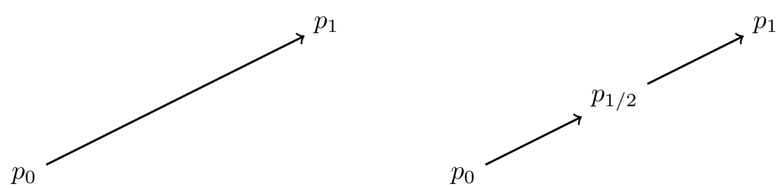
Lemma A.8. *The map* cut_t *is geometric and* $\mathrm{cut}_t \equiv \mathrm{Id}$.



(A) $\pi^\dagger[p_0p_1p_2] = [p_2p_0p_1] + [p_1p_3p_2]$ (B) $\tilde{\pi}^\dagger[p_0p_1p_2] = [p_3p_0p_1] + [p_0p_3p_2]$

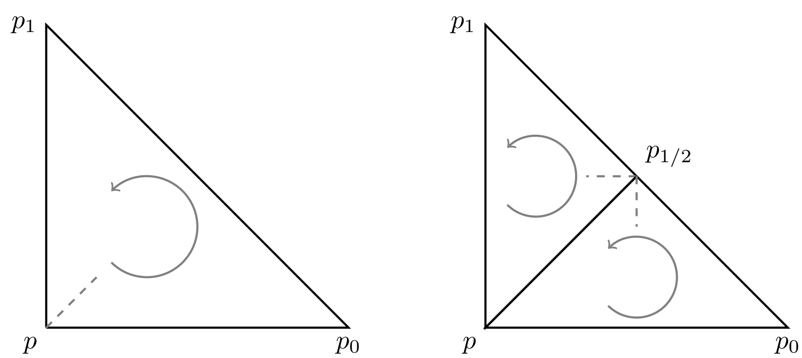
FIGURE A.3. The two parallelograms with changed base points.

Proof. Consider first the case of 1-simplices. Since $\tau_1[p_0p_1] := [p_0p_t]$, $\tau_2[p_0p_1] := [p_tp_1]$ are both geometric, so it is their sum cut_t. Moreover, letting $\varrho[p_0p_1] := [p_0p_tp_1]$, one has $\mathrm{cut}_t - \mathrm{Id} = \partial\varrho$.



(A) $[p_0p_1]$ (B) $[p_0p_{1/2}] + [p_{1/2}p_1]$

FIGURE A.4. $\mathrm{cut}_{1/2}[p_0p_1]$.



(A) $[pp_0p_1]$ (B) $[p_{1/2}p_1p] + [p_{1/2}pp_0]$

FIGURE A.5. $\mathrm{cut}_{1/2}[p_0p_1p_2]$.

In case of 2-simplices, the maps $\tau_1[pp_0p_1] := [p_tpp_0]$ and $\tau_2[pp_0p_1] := [p_tp_1p]$ are geometric and, letting $\varrho[pp_0p_1] := [p_tpp_0p_1]$, we notice that

$$\begin{aligned}\partial\varrho[pp_0p_1] &= [pp_0p_1] - [p_tp_0p_1] + [p_tpp_1] - [p_tpp_0]\\ &= (\mathrm{Id} - \mathrm{cut}_t)[pp_0p_1] - [p_tp_0p_1] + ([p_tpp_1] + [p_tp_1p])\\ &= (\mathrm{Id} - \mathrm{cut}_t)[pp_0p_1] - \nu[pp_0p_1] + (\mathrm{Id} - (-1)^\sigma)[p_tpp_1]\\ &\quad \text{with } \nu[pp_0p_1] := [p_tp_0p_1] \text{ and } \sigma[pqr] := [prq].\end{aligned} \tag{A.4}$$

Since $\mathrm{vol}_2(\nu[pp_0p_1]) = 0$, it follows that $\mathrm{Id} \equiv \mathrm{cut}_t$. □

Remark A.9. The following observation will be useful in the proof of Theorem 4.28. If $\omega \in \mathrm{Germ}^2(D)$ is closed (not necessarily nonatomic), alternating and $\langle [p_0p_tp_1], \omega\rangle = 0$ with $p_t = (1-t)p_1 + tp_2$, then by (A.4)

$$\langle [pp_0p_1], \omega\rangle = \langle \mathrm{cut}_t[pp_0p_1], \omega\rangle\,.$$

More generally, we need to cut one edge into n-equal parts. To this aim, we recursively define, for $n \geq 2$, the map (on 1-simplices)

$$\mathrm{cut}^n[p_0p_1] := \mathrm{cut}^{n-1}[p_0p_{1/n}] + [p_{1/n}p_1],$$

where $p_{1/n} := \left(1 - \frac{1}{n}\right)p_0 + \frac{1}{n}p_1$, and the map on 2-simplices

$$\mathrm{cut}^n[pp_0p_1] := \mathrm{cut}^{n-1}[pp_0p_{1/n}] + [pp_{1/n}p_1],$$

where $p_{1/n} := \left(1 - \frac{1}{n}\right)p_0 + \frac{1}{n}p_1$ (Figures A.6 and A.7). Notice that cut^2 is different from $\mathrm{cut}_{1/2}$ (but one can prove that $\mathrm{cut}^2 \equiv \mathrm{cut}_{1/2}$).

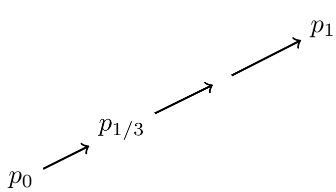


FIGURE A.6. $\mathrm{cut}^3[p_0p_1]$.

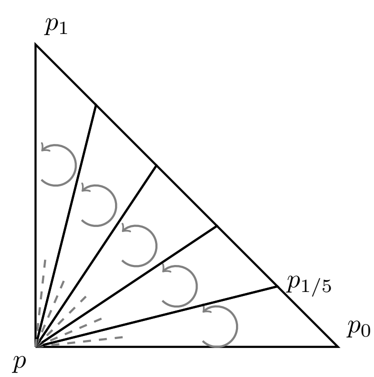


FIGURE A.7. $\mathrm{cut}^5[pp_0p_1]$.

Arguing inductively, one has the following result.

Lemma A.10. *For $n \geq 1$, the map cut^n is geometric and $\mathrm{cut}^n \equiv \mathrm{Id}$.*

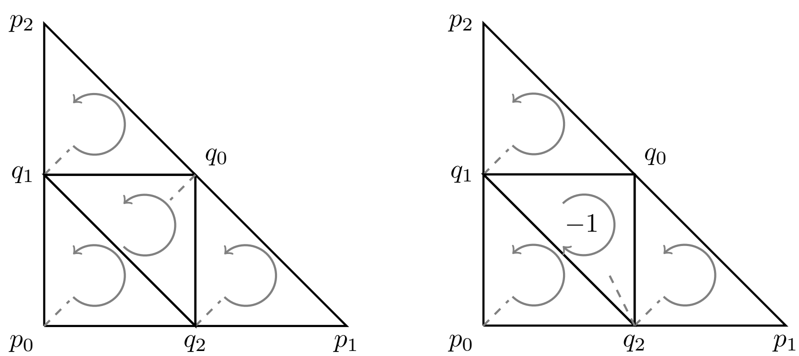
Dyadic decomposition. As a fourth operation, obtained by composition of the others, we introduce the following decomposition of $S \in \mathrm{Simp}^2(D)$ into four parts, each isometric to $(2^{-1})_\natural S$, hence the called *dyadic*. For $S = [p_0p_1p_2] \in \mathrm{Simp}^2(D)$, $i \in \{0,1,2\}$ let $q_i := (p_j + p_k)/2$, where $\{i,j,k\} = \{0,1,2\}$, and (Figure A.8a)

$$\mathrm{dya}^1 S := [q_0q_1q_2], \quad \mathrm{dya}^2 S := [q_1q_0p_2], \quad \mathrm{dya}^3 S := [q_2p_1q_0], \quad \mathrm{dya}^4 S := [p_0q_2q_1],$$

and finally $\operatorname{dya} := \sum_{i=1}^{4} \operatorname{dya}^i$. We introduce similarly a dyadic decomposition of 1-simplices, simply letting $\operatorname{dya}^1[p_0p_1] = [p_0q]$, $\operatorname{dya}^2[p_0p_1] = [qp_1]$, where $q = (p_0 + p_1)/2$ and $\operatorname{dya} = \operatorname{dya}^1 + \operatorname{dya}^2$, which actually coincides with cut^2 or $\operatorname{cut}_{1/2}$.

Remark A.11 (variants). Although the definition of dya on a 2-simplex appears quite natural, there are variants obtained by changing base points or orientation of the 4 simplices. A useful one is $\operatorname{dya}^\dagger := -\sigma \operatorname{dya}^1 + \operatorname{dya}^2 + \operatorname{dya}^3 + \operatorname{dya}^4$, where $\sigma[pqr] := [rqp]$ is a transposition (Figure A.8b), because the identity $\partial \operatorname{dya}^\dagger = \operatorname{dya} \partial$ holds (Figure A.9):

$$\begin{aligned}
\operatorname{dya} \partial[p_0p_1p_2] &= \operatorname{dya}\left([p_1p_2] - [p_0p_2] + [p_0p_1]\right)\\
&= [p_1q_0] + [q_0p_2] - [p_0q_1] - [q_1p_2] + [p_0q_2] + [q_2p_1]\\
&= -\left([q_1q_0] - [q_2q_0] + [q_2q_1]\right) + \left([p_1q_0] - [q_2q_0] + [q_2p_1]\right)\\
&\quad + \left([q_0p_2] - [q_1p_2] + [q_1q_0]\right) + \left([q_2q_1] - [p_0q_1] + [p_0q_2]\right)\\
&= \partial\left(-[q_2q_1q_0] + [q_2p_1q_0] + [q_1q_0p_2] + [p_0q_2q_1]\right) = \partial \operatorname{dya}^\dagger[p_0p_1p_2].
\end{aligned}$$



(A) $\operatorname{dya} S = \begin{array}{l}[q_0q_1q_2] + [q_1q_0p_2]\\ +[q_2p_1q_0] + [p_0q_2q_1]\end{array}$ (B) $\operatorname{dya}^\dagger S = \begin{array}{l}-[q_2q_1q_0] + [q_1q_0p_2]\\ +[q_2p_1q_0] + [p_0q_2q_1]\end{array}$

FIGURE A.8. Dyadic decompositions of $[p_0p_1p_2] \in \operatorname{Chain}^2(D)$.

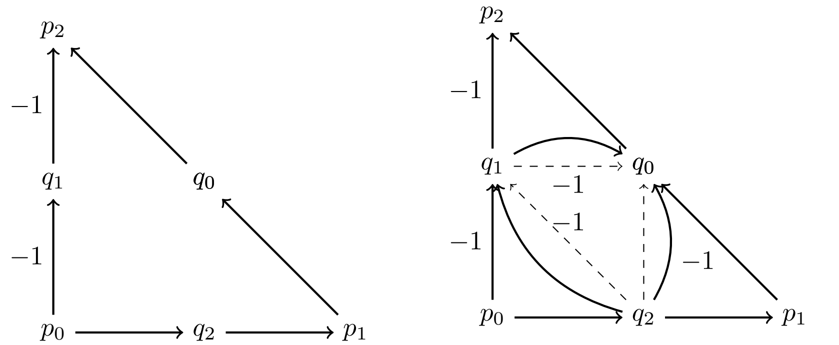
Lemma A.12. *For $k \in \{1, 2\}$, $i \in \{1, \dots, 2k\}$ the maps dya^i are geometric and* $\operatorname{dya} \equiv \operatorname{Id}$.

Proof. The case $k = 1$ is obvious since $\operatorname{dya} = \operatorname{cut}_{1/2}$. When $k = 2$, the maps dya^i are clearly geometric. By Lemma A.8 with $t = 1/2$, we have $\operatorname{Id} \equiv \operatorname{cut}_{1/2}$ (Figure A.10a) and, iterating, $\operatorname{Id} \equiv \operatorname{cut}_{1/2} \circ \operatorname{cut}_{1/2}$ (Figure A.10b). Finally, since $\operatorname{cut}_{1/2} \circ \operatorname{cut}_{1/2}$ differs from dya by a flip (applied to $[pq_2q_1]$), using Lemma A.5, we conclude that $\operatorname{cut}_{1/2} \circ \operatorname{cut}_{1/2} \equiv \operatorname{dya}$, hence $\operatorname{Id} \equiv \operatorname{dya}$. □

Remark A.13. By Remarks A.6 and A.9, in view of the above proof, it follows that if $\omega \in \operatorname{Germ}^2(D)$ is closed (not necessarily nonatomic), alternating and $\langle [prq], \omega \rangle = 0$ whenever $[prq] \in \operatorname{Simp}^2(D)$ with $r = \frac{p+q}{2}$, then $\omega = \operatorname{dya}' \omega$, i.e.,

$$\langle S, \omega \rangle = \langle \operatorname{dya} S, \omega \rangle, \quad \text{for every } S \in \operatorname{Simp}^2(D).$$

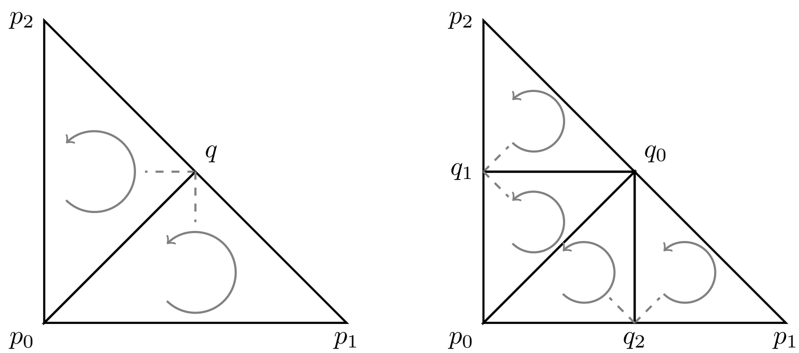
The dyadic decomposition is well-behaved with respect to the other maps.



(A) $\operatorname{dya}\partial[p_0p_1p_2]$

(B) $\partial\operatorname{dya}^\dagger[p_0p_1p_2]$, with dashed $-\partial[q_2q_1q_0]$

FIGURE A.9. The identity $\operatorname{dya}\partial = \partial\operatorname{dya}^\dagger$.



(A) $[q_0p_2p_0] + [q_0p_0p_1]$

(B) $[q_0p_2q_1]+[q_0q_1p_0]+[q_0p_0q_2]+[q_0q_2p_1]$

FIGURE A.10. $\operatorname{cut}_{1/2}[p_0p_1p_2]$ and $\operatorname{cut}_{1/2}\circ\operatorname{cut}_{1/2}[p_0p_1p_2]$.

Lemma A.14. *The following properties hold.*

(1) For every permutation σ of $\{0,1,2\}$, $\operatorname{dya}\sigma = \sigma\operatorname{dya}$.

(2) There are geometric maps $(\tau^i)_{i=1}^4$, $\tau^i : \operatorname{Simp}^2(D) \to \operatorname{Simp}^2(D)$ such that $\tau^i S$ is isometric to $(2^{-1})_\natural S$ for every $S \in \operatorname{Simp}^2(D)$ and, for $\tau = \sum_{i=1}^4$ one has

$$\operatorname{dya}\pi = \pi\tau, \quad \operatorname{dya}\tilde\pi = \tilde\pi\tau \quad \textit{hence} \quad \operatorname{dya}\operatorname{flip} = \operatorname{flip}\tau.$$

(3) For $k = 1$, $n \geq 1$, $S \in \operatorname{Simp}^1(D)$,

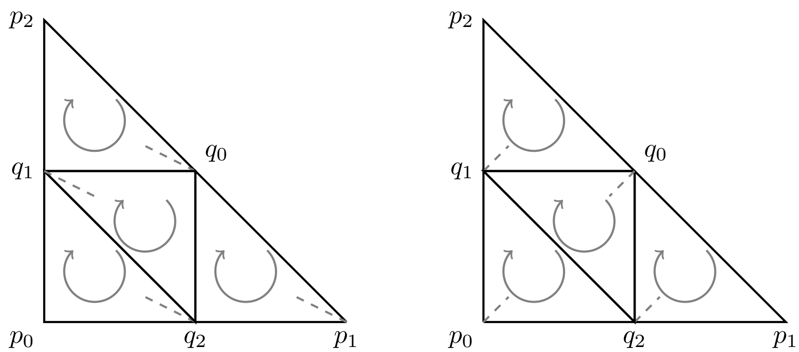
$$\operatorname{dya}\operatorname{cut}^n S = \operatorname{cut}^n \operatorname{dya} S.$$

(4) For $k = 2$, $n \geq 1$, $S \in \operatorname{Simp}^2(D)$, there exists $C \in \operatorname{Chain}^2(D)$ such that

$$\operatorname{dya}\operatorname{cut}^n S = \operatorname{cut}^n \operatorname{dya} S + \operatorname{flip}^\dagger C.$$

Proof. To prove *i)*, it is sufficient to consider transpositions. If $\sigma[p_0p_1p_2] = [p_1p_0p_2]$, then (Figure A.11a)

$$\begin{aligned}\operatorname{dya}\sigma[p_0p_1p_2] &= \operatorname{dya}[p_1p_0p_2] = [p_1q_2q_0] + [q_2p_0q_1] + [q_0q_1p_2] + [q_1q_0q_2]\\ &= \sigma\left([p_0q_2q_1] + [q_2p_1q_0] + [q_1q_0p_2] + [q_0q_1q_2]\right) = \sigma\operatorname{dya} C.\end{aligned}$$



(A) $\operatorname{dya}[p_1p_0p_2]$. (B) $\operatorname{dya}[p_0p_2p_1]$.

FIGURE A.11. Dyadic decompositions of permutations of $[p_0p_1p_2]$.

Similarly, if $\sigma[p_0p_1p_2] = [p_2p_1p_0]$, then

$$\begin{aligned}\operatorname{dya}\sigma[p_0p_1p_2] &= \operatorname{dya}[p_2p_1p_0] = [p_2q_0q_1] + [q_1p_2p_0] + [q_1q_2p_0] + [q_2q_1q_0]\\ &= \sigma\left([p_0q_2q_1] + [q_2p_1q_0] + [q_1q_0p_2] + [q_0q_1q_2]\right) = \sigma \operatorname{dya}[p_0p_1p_2].\end{aligned}$$

Finally, if $\sigma[p_0p_1p_2] = [p_0p_2p_1]$, then (Figure A.11b)

$$\begin{aligned}\operatorname{dya}\sigma[p_0p_1p_2] &= \operatorname{dya}[p_0p_2p_1] = [p_0q_1q_2] + [q_2q_0p_1] + [q_1p_2q_0] + [q_0q_2q_1]\\ &= \sigma\left([p_0q_2q_1] + [q_2p_1q_0] + [q_1q_0p_2] + [q_0q_1q_2]\right) = \sigma[p_0p_1p_2].\end{aligned}$$

To show *ii)*, given $S = [p_0p_1p_2] \in \operatorname{Germ}^2(D)$, let $p_3 := p_1 + p_2 - p_0$ and $q = (p_1+p_2)/2$, $q_0 = (p_2+p_3)/2$, $q_1 = (p_2+p_0)/2$, $q_2 = (p_1+p_0)/2$, $q_3 = (p_1+p_3)/2$. Then (Figure A.12a),

$$\begin{aligned}\operatorname{dya}\pi S &= \operatorname{dya}[p_0p_1p_2] + \operatorname{dya}[p_3p_2p_1]\\ &= [p_0q_2q_1] + [q_2p_1q] + [q_1qp_2] + [qq_1q_2] + [p_3q_0q_3] + [q_0p_2q] + [q_3qp_1] + [qq_3q_0]\\ &= ([p_0q_2q_1] + [qq_1q_2]) + ([q_2p_1q] + [q_3qp_1]) + ([q_1qp_2] + [q_0p_2q]) + ([qq_3q_0] + [p_3q_0q_3])\\ &= \pi[p_0q_2q_1] + \pi[q_2p_1q] + \pi[q_1qp_2] + \pi[qq_3q_0].\end{aligned}$$

Similarly (see Figure A.12b),

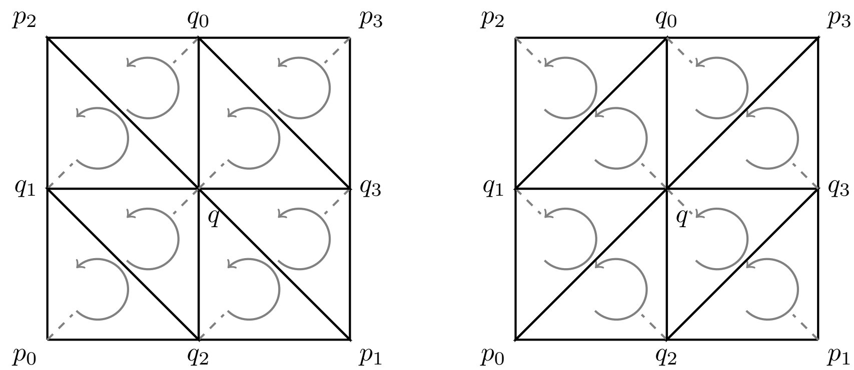
$$\begin{aligned}\operatorname{dya}\tilde\pi S &= \operatorname{dya}[p_1p_3p_0] + \operatorname{dya}[p_2p_0p_3]\\ &= [p_1q_3q_2] + [q_2qp_0] + [q_3p_3q] + [qq_2q_3] + [p_2q_1q_2] + [q_1p_0q] + [q_0qp_3] + [qq_0q_1]\\ &= ([q_2qp_0] + [q_1p_0q]) + ([p_1q_3q_2] + [qq_2q_3]) + ([qq_0q_1] + [p_2q_1q_2]) + ([q_3p_3q] + [q_0qp_3])\\ &= \tilde\pi[p_0q_2q_1] + \tilde\pi[q_2p_1q] + \tilde\pi[q_1qp_2] + \tilde\pi[qq_3q_0].\end{aligned}$$

Hence, the thesis follows defining the geometric maps

$$\tau^1 S := [p_0q_2q_1], \quad \tau^2 S := [q_2p_1q], \quad \tau^3 S := [q_1qp_2], \quad \text{and} \quad \tau^4 S := [qq_3q_0].$$

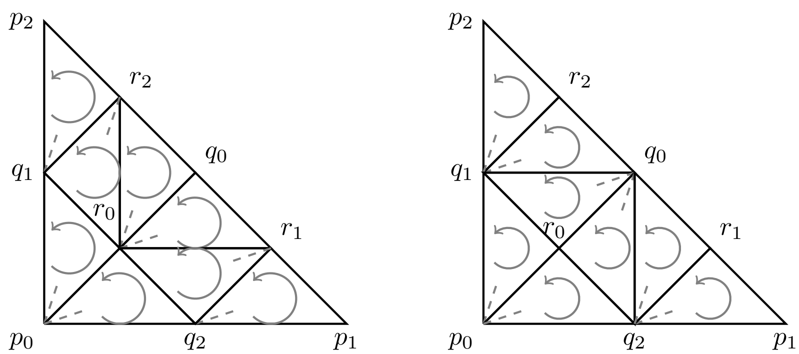
The validity of *iii)* is immediate by induction.

To show *iv)*, we also argue by induction, the case $n = 1$ being obvious and $n = 2$ being proved as follows. For $S = [p_0p_1p_2]$, let $q_i = (p_j + p_k)/2$, for $\{i, j, k\} =$



(A) $\operatorname{dya} \pi[p_0p_1p_2]$ (B) $\operatorname{dya} \pi'[p_0p_1p_2]$

FIGURE A.12. Dyadic decompositions of $\pi[p_0p_1p_2]$ and $\tilde{\pi}[p_0p_1p_2]$.



(A) $\operatorname{dya} \operatorname{cut}^2[p_0p_1p_2]$ (B) $\operatorname{cut}^2 \operatorname{dya}[p_0p_1p_2]$

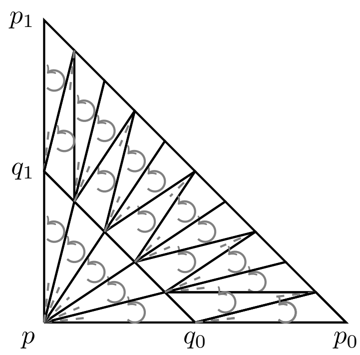
FIGURE A.13. Commutation between dya and cut^2 using edge flipping.

$\{0,1,2\}$, and $r_i = (p_i + q_0)/2$ for $i \in \{0,1,2\}$. Then, one has (Figure A.13)

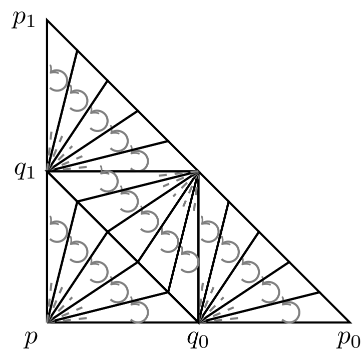
$$
\begin{aligned}
\operatorname{dya} \operatorname{cut}^2 S &= \operatorname{dya}[p_0p_1q_0] + \operatorname{dya}[p_0q_0p_2] \\
&= ([p_0q_2r_0] + [q_2p_1r_1] + [r_1r_0q_2] + [r_0r_1q_0]) \\
&\quad + ([p_0r_0q_1] + [q_1r_2p_2] + [r_0q_0r_2] + [r_2q_1r_0]) \\
&= ([p_0q_2r_0] + [p_0r_0q_1] + [q_2p_1r_1] + [q_1r_2p_2]) \\
&\quad + ([q_0r_0q_2] + [q_2r_1q_0]) + ([q_0q_1r_0] + [q_1q_0r_2]) + \operatorname{flip}^\dagger([r_0r_1q_0] + [r_2q_1r_0]) \\
&= \operatorname{cut}_{1/2} \operatorname{dya} S + \operatorname{flip}^\dagger([r_0r_1q_0] + [r_2q_1r_0])
\end{aligned}
$$

hence the thesis follows letting $C := [q_0r_2r_0] + [q_0r_0r_1]$.

The inductive step then goes from $n-2$ to n. Given $S = [pp_0p_1]$ (notice the slight change from notation of the previous case), define $q_0 = (p+p_0)/2$, $q_1 = (p+p_1)/2$, $p_t = (1-t)p_0 + tp_1$ and $q_t = (1-t)q_0 + tq_1$, for $t \in [0,1]$ (see Figures A.14a and A.14b, with $n = 5$).



(A) $\operatorname{dya}\operatorname{cut}^5[pp_0p_1]$

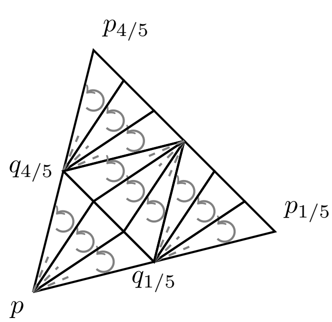


(B) $\operatorname{cut}^5\operatorname{dya}[pp_0p_1]$

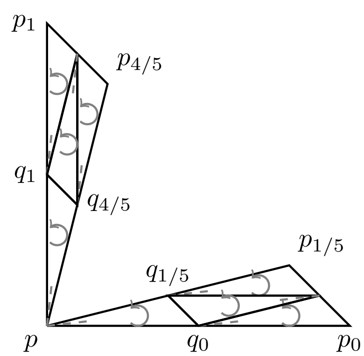
Since $\operatorname{cut}^n[pp_0p_1] = [pp_0p_{\frac{1}{n}}]+\operatorname{cut}^{n-2}[pp_{\frac{1}{n}}p_{\frac{n-1}{n}}]+[pp_{\frac{n-1}{n}}p_1]$, we have (Figures A.15a and A.15b)

$$\operatorname{dya}\operatorname{cut}^n[pp_0p_1] = \operatorname{dya}\left([pp_0p_{\frac{1}{n}}] + [pp_{\frac{n-1}{n}}p_1]\right) + \operatorname{cut}^{n-2}\operatorname{dya}[pp_{\frac{1}{n}}p_{\frac{n-1}{n}}] + \operatorname{flip}^\dagger C^{n-2},$$

where $C^{n-2} \in \operatorname{Chain}^2(D)$.



(A) $\operatorname{dya}\operatorname{cut}^3[pp_{1/5}p_{4/5}]$



(B) $\operatorname{dya}\left([pp_0p_{1/5}] + [pp_{4/5}p_1]\right)$

FIGURE A.15. Two chains in the inductive step with $n = 5$.

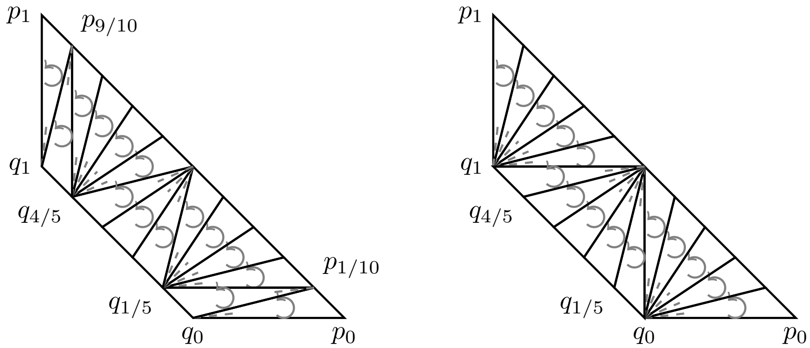
Collecting the chains $[pq_0q_{\frac{1}{n}}] + [pq_{\frac{n-1}{n}}q_1]$ and $\operatorname{cut}^{n-2}[pq_{\frac{1}{n}}q_{\frac{n-1}{n}}]$ which can be found in the expression above, we obtain $\operatorname{cut}^n[pq_0q_1]$, hence the thesis follows if we prove that the difference between the chain

$$\begin{aligned} R^1 :=&[q_0p_0p_{\frac{1}{2n}}] + [p_{\frac{1}{2n}}q_{\frac{1}{n}}q_0] + [q_{\frac{1}{n}}p_{\frac{1}{2n}}p_{\frac{1}{n}}] \\ &+ \operatorname{cut}^{n-2}\left([q_{\frac{1}{n}}p_{\frac{1}{n}}p_{\frac{1}{2}}] + [p_{\frac{1}{2}}q_{\frac{n-1}{n}}q_{\frac{1}{n}}] + [q_{\frac{n-1}{n}}p_{\frac{1}{2}}p_{\frac{n-1}{n}}]\right) \\ &+ [q_1p_{\frac{2n-1}{2n}}p_1] + [p_{\frac{2n-1}{2n}}q_{\frac{n-1}{n}}q_1] + [q_{\frac{n-1}{n}}p_{\frac{2n-1}{2n}}p_{\frac{n-1}{n}}] \end{aligned}$$

and the chain (Figure A.16)

$$R^n := \operatorname{cut}^n\left([q_0p_0p_{\frac{1}{2}}] + [p_{\frac{1}{2}}q_1q_0] + [q_1p_{\frac{1}{2}}p_1]\right)$$

can be written in the form $\operatorname{flip}^\dagger C$, for some $C \in \operatorname{Chain}^2(D)$.



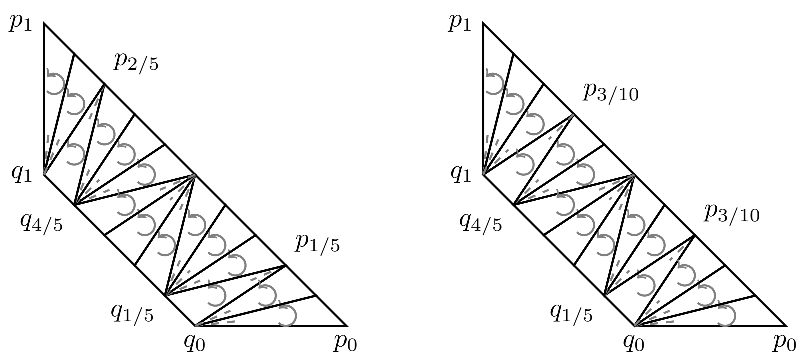
(A) R^1 (B) R^5

FIGURE A.16. The chains R^1 and R^5 in the case $n = 5$.

We achieve this by means of $(n-1)$ consecutive edge-flippings, i.e. building an "interpolating" sequence of chains $(R^k)_{k=1}^n$ such that $R^{k+1} - R^k = \mathrm{flip}\,\dagger C^k$, for some $C^k \in \mathrm{Chain}^2(D)$, for $k \in \{1, \dots, n-1\}$. Precisely, let (Figures A.16 and A.17)

$$\begin{aligned} R^k :=\, & \mathrm{cut}^k \left([q_0 p_0 p_{\frac{k}{2n}}] + [q_1 p_{\frac{2n-k}{2n}} p_1] \right) + \mathrm{cut}^{n-k-1} \left([q_{\frac{1}{n}} p_{\frac{k+1}{2n}} p_{\frac{1}{2}}] + [q_{\frac{n-1}{n}} p_{\frac{1}{2}} p_{\frac{2n-k-1}{2n}}] \right) \\ & + [p_{\frac{k}{2n}} q_{\frac{1}{n}} q_0] + [p_{\frac{2n-k}{2n}} q_1 q_{\frac{n-1}{n}}] + \mathrm{cut}^{n-2} \left([p_{\frac{1}{2}} q_{\frac{n-1}{n}} q_{\frac{1}{n}}] \right). \end{aligned}$$

and notice that R^1 and R^n coincide with the one above (just recollecting some terms).



(A) R^2 (B) R^3

FIGURE A.17. The chains R^2 and R^3 in the case $n = 5$.

Then, one has

$$\begin{aligned} R^{k+1} - R^k &= [p_{\frac{k+1}{2n}} q_{\frac{1}{n}} q_0] + [q_{\frac{1}{n}} p_{\frac{k}{2n}} p_{\frac{k+1}{2n}}] - [p_{\frac{k}{2n}} q_{\frac{1}{n}} q_0] - [q_{\frac{1}{n}} p_{\frac{k}{2n}} p_{\frac{k+1}{2n}}] \\ &\quad + [q_1 p_{\frac{2n-k-1}{2n}} p_{\frac{2n-k}{2n}}] + [p_{\frac{2n-k-1}{2n}} q_1 q_{\frac{n-1}{n}}] - [p_{\frac{2n-k}{2n}} q_1 q_{\frac{n-1}{n}}] - [q_{\frac{n-1}{n}} p_{\frac{2n-k-1}{2n}} p_{\frac{2n-k}{2n}}] \\ &= \mathrm{flip}^\dagger \left([q_{\frac{1}{n}} q_0 p_{\frac{k}{2n}}] + [q_1 q_{\frac{n-1}{n}} p_{\frac{2n-k}{2n}}] \right) \end{aligned}$$

and the thesis follows. $\square$

Appendix B. Proof of sewing results

In this section we collect the proofs of Theorem 3.12, Theorem 3.18 and Theorem 3.19, together with auxiliary results. We begin with a useful uniqueness result.

Lemma B.1 (uniqueness). *Let $\ell \geq 1$, $\lambda_i \in [-1,1]$, $\tau_i : \mathrm{Simp}^k(D) \to \mathrm{Simp}^k(D)$, $i \in \{1,\ldots,\ell\}$, be such that $\tau_i S$ is isometric to $(2^{-1})_\natural S$, for every $S \in \mathrm{Simp}^k(D)$ and let $\tau := \sum_{i=1}^{\ell} \lambda_i \tau_i$. Let $\omega \in \mathrm{Germ}^k(D)$ and v uniform k-gauge. Then, for every $n \geq 0$, $S \in \mathrm{Simp}^k(D)$,*

$$|\langle S, ((\tau)')^n \omega\rangle| \leq [\omega]_{\mathsf{v}} \ell^n \left\langle (2^{-n})_\natural S, \mathsf{v}\right\rangle .$$

If moreover v is $(\log_2 \ell)$-Dini and $\tau'\omega = \omega$, then $\omega = 0$.

Proof. For $n \geq 1$, $i = (i_1,\ldots,i_n) \in \{1,\ldots,\ell\}^n$, define

$$\lambda_i := \lambda_{i_1}\ldots\lambda_{i_n}, \quad \tau_i S := \tau_{i_n}\tau_{i_{n-1}}\ldots\tau_{i_1} S, \quad \text{for } S \in \mathrm{Simp}^k(D),$$

so that, by induction, $\tau_i S$ is isometric to $(2^{-n})_\natural S$ and $(\tau^n)'\omega = \sum_{i\in\{1,\ldots,\ell\}^n} \lambda_i \tau_i'\omega$. Since v is a uniform gauge, for any $S \in \mathrm{Simp}^k(D)$, one has

$$|\langle \tau^n S, \omega\rangle| \leq \sum_{i\in\{1,\ldots,\ell\}^n} |\lambda_i|\,|\langle \tau_i S, \omega\rangle| \leq [\omega]_{\mathsf{v}} \sum_{i\in\{1,\ldots,\ell\}^n} \left\langle (2^{-n})_\natural S, \mathsf{v}\right\rangle = \ell^n \left\langle (2^{-n})_\natural S, \mathsf{v}\right\rangle .$$

The last statement follows letting $n \to +\infty$, using $(\tau^n)'\omega = (\tau')^n\omega = \omega$ and the $(\log_2 \ell)$-Dini assumption on v. $\square$

Remark B.2 (convergence rates). In the situation of the result above, assume that $\omega \approx_{\mathsf{v}} \tilde\omega$ with $\tau'\tilde\omega = \tilde\omega$. Then, we obtain the inequality

$$|\langle S, \tilde\omega - (\tau^n)'\omega\rangle| \leq [\omega - \tilde\omega]_{\mathsf{v}} \ell^n \left\langle (2^{-n})_\natural S, \mathsf{v}\right\rangle .$$

and in particular if v is $\log_2 \ell$-Dini then $\lim_{n\to+\infty} \langle S, (\tau^n)'\omega\rangle = \langle S, \tilde\omega\rangle$.

B.1. **Proof of sewing (uniqueness) Theorem 3.12.** For $k \in \{1,2\}$, let $\omega \in \mathrm{Germ}^k(D)$ be regular, $\mathsf{v} \in \mathrm{Germ}^k(D)$ be k-Dini and $\omega \approx_{\mathsf{v}} 0$. To show that $\omega_S = 0$ for every $S \in \mathrm{Simp}^k(D)$, minding that ω is non-atomic, we may assume that $\mathrm{vol}_k(S) > 0$, hence we can find $\varphi : I \subseteq \mathbb{R}^k \to D$ affine injective such that $\varphi_\natural[0e_1e_2\ldots e_k] = S$. Moreover, being φ affine and injective, $\varphi^\natural\mathsf{v}$ is k-Dini and $\varphi^\natural\omega \approx_{\varphi^\natural\mathsf{v}} 0$ by Remark 2.13. Thus, without loss of generality, we can reduce ourselves to the case $D \subseteq \mathbb{R}^k$, $S \in \mathrm{Simp}^k(D)$, and $\omega \in \mathrm{Germ}^k(D)$ closed and non-atomic.

From the equivalence $\mathrm{dya} \equiv \mathrm{Id}$ (Lemma A.12), i.e.,

$$\mathrm{dya} - \mathrm{Id} = \nu + \partial\varrho,$$

we deduce $\omega = \mathrm{dya}'\,\omega$ by regularity of ω. Moreover, since $\mathrm{dya}^i\, S \equiv (2^{-1})_\natural S$, for $i \in \left\{1,\ldots,2^k\right\}$ and v is k-Dini, hence $\log_2(2^k) = k$-Dini we are in a position to apply Lemma B.1 with $\ell := 2^k$, $\lambda_i := 1$, $\tau_i := \mathrm{dya}^i$, $\tau := \mathrm{dya}$ and deduce $\omega = 0$.

B.2. **Proof of Stokes Theorem 3.18.** If $k=1$ first, i.e. $\eta = f$ is a function, then $\operatorname{sew} f = f$ and $\operatorname{sew} \delta f = \delta f$, because δf is regular.

If $k=2$, differently from the proof of Theorem 3.12, it is more convenient to use $\mathrm{dya}^\dagger$ (introduced in Remark A.11) instead of dya. Notice that, given a closed and non-atomic $\tilde\omega \in \mathrm{Germ}^k(D)$, from $\tilde\omega = \mathrm{dya}'\,\tilde\omega$ (consequence of $\mathrm{dya} \equiv \mathrm{Id}$) and the fact that $\tilde\omega$ is alternating, given that dya and $\mathrm{dya}^\dagger$ differ only for composition with $-\sigma$ (and $\sigma[p_0p_1p_2] := [p_2p_1p_0]$) on one of the four isometric simplices, we also have $\tilde\omega = (\mathrm{dya}^\dagger)'\tilde\omega$. Hence,

$$
\begin{aligned}
\operatorname{sew}\delta\eta &= \lim_{n\to+\infty} \left((\mathrm{dya}^\dagger)^n\right)' \delta\eta \qquad \text{by Remark B.2 with } \tilde\omega := \operatorname{sew}\delta\eta,\ \omega := \delta\eta\\
&= \lim_{n\to+\infty} \delta\,(\mathrm{dya}^n)'\,\eta \qquad \text{for } \partial\,\mathrm{dya}^\dagger = \mathrm{dya}\,\partial\\
&= \delta \lim_{n\to+\infty} (\mathrm{dya}^n)'\,\eta = \delta \operatorname{sew}\eta.
\end{aligned}
$$

B.3. **Proof of sewing (continuity) Theorem 3.23.** Given $S \in \mathrm{Simp}^k(D)$, by Remark B.2, with ω^j instead of ω, $\operatorname{sew}\omega^j$ instead of $\tilde\omega$, $\ell := 2^k$, $\lambda_i = 1$, $\tau_i := \mathrm{dya}^i$, $\tau := \mathrm{dya}$ and $n \geq 1$, we have

$$
\left|\left\langle S, \operatorname{sew}\omega^j - (\mathrm{dya}^n)'\omega^j \right\rangle\right| \leq [\omega^j - \operatorname{sew}\omega^j]_{\mathsf{v}} 2^{kn} \left\langle (2^{-n})_\natural S, \mathsf{v}\right\rangle .
$$

For any $\epsilon > 0$, we choose $\bar n \geq 1$ such that

$$
2^{k\bar n}\left\langle (2^{-\bar n})_\natural S, \mathsf{v}\right\rangle \leq \epsilon / \sup_{j\geq 1}[\omega^j - \operatorname{sew}\omega^j]_{\mathsf{v}}.
$$

By the pointwise convergence of $\omega^j \to \omega$, we have $(\mathrm{dya}^{\bar n})'\omega^j \to (\mathrm{dya}^{\bar n})'\omega$ pointwise as well, hence

$$
\begin{aligned}
\left|\left\langle S, \operatorname{sew}\omega^i - \operatorname{sew}\omega^j\right\rangle\right| &\leq \left|\left\langle S, \operatorname{sew}\omega^i - (\mathrm{dya}^{\bar n})'\omega^i\right\rangle\right| + \left|\left\langle S, \operatorname{sew}\omega^i - (\mathrm{dya}^{\bar n})'\omega^i\right\rangle\right|\\
&\quad + \left|\left\langle S, (\mathrm{dya}^{\bar n})'\omega^i - (\mathrm{dya}^{\bar n})'\omega^j\right\rangle\right|\\
&\leq 3\epsilon \quad \text{for for } i,\ j \text{ sufficiently large,}
\end{aligned}
$$

so that $(\operatorname{sew}\omega^j)_{j\geq 1}$ is pointwise Cauchy. Its limit $\tilde\omega$ is a regular germ by Corollary 3.5. Since

$$
\left|\left\langle S, \operatorname{sew}\omega^j - \omega^j\right\rangle\right| \leq \sup_{\ell\geq 1}[\omega^\ell - \operatorname{sew}\omega^\ell]_{\mathsf{v}} \left\langle S, \mathsf{v}\right\rangle ,
$$

letting $j \to +\infty$ in the above estimate, we obtain

$$
|\langle S, \tilde\omega - \omega\rangle| \leq \sup_{\ell\geq 1}[\omega^\ell - \operatorname{sew}\omega^\ell]_{\mathsf{v}} \langle S, \mathsf{v}\rangle
$$

hence $\tilde\omega \approx_{\mathsf{v}} \omega$ or in other words $\tilde\omega = \operatorname{sew}\omega$.

B.4. **Proof of sewing (existence) Theorem 3.19.** *Step 1: definition of* $\operatorname{sew}\omega$. For any $S \in \mathrm{Simp}^k(D)$, we show that the sequence $(\langle \mathrm{dya}^n S, \omega\rangle)_{n\geq 0}$ is Cauchy, hence the limit

$$
\langle S, \operatorname{sew}\omega\rangle := \lim_{n\to+\infty} \langle \mathrm{dya}^n S, \omega\rangle
$$

is well-defined. Indeed, decomposing $\mathrm{dya} - \mathrm{Id} = \nu + \partial\varrho$, using the fact that ω is non-atomic, we have

$$
\langle (\mathrm{dya} - \mathrm{Id})S, \omega\rangle = \langle \varrho S, \delta\omega\rangle
$$

hence $[\mathrm{dya}'\,\omega-\omega]_{|\varrho^\natural|\mathsf{u}} \le [\delta\omega]_{\mathsf{u}} < \infty$. Lemma B.1 with $\tau^i = \mathrm{dya}^i$, $i \in \{1,\ldots,2^k\}$, $\tau = \mathrm{dya}$ and $\mathrm{dya}'\,\omega-\omega$ instead of ω entails, for $n \ge 0$,

$$\left|\left\langle \mathrm{dya}^{n+1} S, \omega\right\rangle - \langle \mathrm{dya}^n S, \omega\rangle\right| = \left|\left\langle \mathrm{dya}^n S, \mathrm{dya}'\,\omega-\omega\right\rangle\right| \le [\delta\omega]_{\mathsf{u}} 2^{kn} \left\langle (2^{-n})_\natural S, |\varrho|'\mathsf{u}\right\rangle,$$

which is summable, being $|\varrho|'\mathsf{u}$ strong k-Dini by Lemma A.1. Hence the sequence is Cauchy and we also obtain the bound

$$|\langle S, \omega - \mathrm{sew}\,\omega\rangle| \le [\delta\omega]_{\mathsf{u}} \sum_{n=0}^{\infty} 2^{kn}\left\langle (2^{-n})_\natural, |\varrho|'\mathsf{u}\right\rangle = [\delta\omega]_{\mathsf{u}} \langle S, \mathsf{v}\rangle \tag{B.1}$$

i.e., $\omega \approx_{\mathsf{v}} \mathrm{sew}\,\omega$, having defined (with $\tilde{\mathsf{u}}$ as in (2.3))

$$\langle S, \mathsf{v}\rangle := \sum_{n=0}^{\infty} 2^{kn}\left\langle (2^{-n})_\natural, |\varrho|'\mathsf{u}\right\rangle = \langle S, |\varrho|'\tilde{\mathsf{u}}\rangle\,,$$

Notice that, when $\mathsf{u} = \mathrm{vol}_1^{\gamma_1}\mathrm{vol}_2^{\gamma_2}$, one obtains that $\mathsf{v} \le \mathsf{c}(\gamma_1,\gamma_2)\,\mathrm{vol}_1^{\gamma_1}\mathrm{vol}_2^{\gamma_2}$ by Example 2.17, yielding Remark 3.22.

To show that $\mathrm{sew}\,\omega$ is continuous, let $\{S^m\}_{m\ge 0} \subseteq \mathrm{Simp}^k(D)$ converge to $S \in \mathrm{Simp}^k(D)$. For any $\varepsilon > 0$, one can find $\bar{n} \ge 0$ such that

$$\langle S, \mathsf{v}\rangle - \sum_{n=0}^{\bar{n}-1} 2^{kn}\left\langle (2^{-n})_\natural S, |\varrho|'\mathsf{u}\right\rangle = \sum_{n=\bar{n}}^{+\infty} 2^{kn}\left\langle (2^{-n})_\natural S, |\varrho|'\mathsf{u}\right\rangle < \varepsilon.$$

By continuity v and u we have that the same bound holds for any S^m instead of S, provided that m is sufficiently large. The k-germ $(\mathrm{dya}^{\bar{n}})'\omega$ is also continuous, hence

$$\left|\left\langle S^m - S, (\mathrm{dya}^{\bar{n}})'\omega\right\rangle\right| < \varepsilon$$

provided that m is sufficiently large. Hence, for every $m \ge \bar{m}$, one has

$$\begin{aligned} |\langle S^m - S, \mathrm{sew}\,\omega\rangle| &\le \left|\left\langle S^m - S, (\mathrm{dya}^{\bar{n}})'\omega\right\rangle\right| \\ &\quad + \left|\left\langle S^m, (\mathrm{dya}^{\bar{n}})'\omega - \mathrm{sew}\,\omega\right\rangle\right| + \left|\left\langle S, (\mathrm{dya}^{\bar{n}})'\omega - \mathrm{sew}\,\omega\right\rangle\right| \le 3\varepsilon. \end{aligned}$$

In the following steps, we prove that $\mathrm{sew}\,\omega$ is regular. Notice first that $\mathrm{sew}\,\omega$ is non-atomic and the identity $\mathrm{dya}'\,\mathrm{sew}\,\omega = \mathrm{sew}\,\omega$ is a straightforward consequence of our definition:

$$\mathrm{dya}'\,\mathrm{sew}\,\omega = \mathrm{dya}' \lim_{n\to\infty} (\mathrm{dya}^n)'\omega = \lim_{n\to\infty}(\mathrm{dya}^{n+1})'\omega = \mathrm{sew}\,\omega.$$

Step 2: $\mathrm{sew}\,\omega$ is alternating. Let σ be a permutation of $\{0,1,k\}$ and introduce the k-germ $(\sigma' + (-1)^\sigma)\,\mathrm{sew}\,\omega$. Since $\sigma \equiv (-1)^\sigma\,\mathrm{Id}$, we can decompose (for some geometric ν and ϱ different than above)

$$\sigma - (-1)^\sigma\,\mathrm{Id} = \nu + \partial\varrho, \quad \text{hence } \langle(\sigma - (-1)^\sigma\,\mathrm{Id})\,S, \omega\rangle = \langle \varrho S, \delta\omega\rangle\,,$$

hence $\sigma'\omega \approx_{|\varrho|'\mathsf{u}} (-1)^\sigma\omega$. On the other side, Lemma A.14 gives that $\mathrm{dya}\,\sigma = \sigma\,\mathrm{dya}$, so that

$$\mathrm{dya}\,\sigma'\,\mathrm{sew}\,\omega = \sigma'\,\mathrm{dya}\,\mathrm{sew}\,\omega = \sigma'\omega,$$

hence, letting $\tilde\omega := \sigma'\omega - (-1)^\sigma\omega$, one has $\mathrm{dya}\,\tilde\omega = \tilde\omega$. Finally, being σ geometric, one has $\sigma'\,\mathrm{sew}\,\omega \approx_{|\sigma'|\mathsf{v}} \sigma'\omega$ hence

$$(\sigma' - (-1)^\sigma)\,\mathrm{sew}\,\omega \approx_{\max\{|\sigma'|\mathsf{v},\mathsf{v}\}} (\sigma' - (-1)^\sigma)\omega \approx_{|\varrho|'\mathsf{u}} 0,$$

and Lemma B.1 implies $\sigma'\,\mathrm{sew}\,\omega = (-1)^\sigma\,\mathrm{sew}\,\omega$.

Step 3: $\mathrm{flip}' \operatorname{sew} \omega = 0$. This step is necessary (actually, defined) only for $k = 2$. Since $\mathrm{flip} \equiv 0$, we can decompose (for some geometric ν and ϱ different than above)

$$\mathrm{flip} = \nu + \partial \varrho, \quad \text{hence} \quad \langle \mathrm{flip}\, S, \omega \rangle = \langle \varrho S, \delta\omega \rangle ,$$

hence $\mathrm{flip}'\, \omega \approx_{|\varrho^\natural|\mathsf{u}} 0$. On the other side, Lemma A.14 provides geometric maps $(\tau_i)_{i=1}^4$ with $\tau_i S$ isometric to $(2^{-1})_\natural S$ such that $\mathrm{dya}\, \mathrm{flip} = \mathrm{flip}\, \tau$, hence

$$\tau' \,\mathrm{flip}' \operatorname{sew} \omega = \mathrm{flip}' \,\mathrm{dya}' \operatorname{sew} \omega = \mathrm{flip}'\, \omega.$$

Finally, since flip is geometric, one also has $\mathrm{flip}'\, \omega \approx_{|\mathrm{flip}'|\mathsf{v}} \operatorname{sew} \omega$. In conclusion we have $\mathrm{flip}' \operatorname{sew} \omega \approx_{|\mathrm{flip}'|\mathsf{v}} \mathrm{flip}'\, \omega \approx_{|\varrho^\natural|\mathsf{u}} 0$, so that Lemma B.2 (applied with $\tau = \sum_{i=1}^4 \tau_i$) gives $\mathrm{flip}' \operatorname{sew} \omega = 0$. In combination with the fact that $\operatorname{sew} \omega$ is alternating, we also obtain that $(\mathrm{flip}^\dagger)' \operatorname{sew} \omega = 0$.

Step 4: $\mathrm{cut}_t' \operatorname{sew} \omega = \operatorname{sew} \omega$. We argue in the case $k = 2$ only, the case $k = 1$ being completely analogous (using 1-simplexes instead of 2-simplexes) and even simpler because flip will not appear. First, we show that $(\mathrm{cut}^n)' \operatorname{sew} \omega = \operatorname{sew} \omega$, for every $n \geq 1$. For fixed $n \geq 1$, by Lemma A.10, since $\mathrm{cut}^n \equiv \mathrm{Id}$, arguing as above, we obtain, as in the previous two steps,

$$(\mathrm{cut}^n)' \omega \approx_{|\varrho|'\mathsf{u}} \omega.$$

On the other side, by Lemma A.14 one has for every $S \in \mathrm{Simp}^2(D)$,

$$\langle \mathrm{dya}\, \mathrm{cut}^n S, \operatorname{sew} \omega \rangle = \left\langle \mathrm{cut}^n \,\mathrm{dya}\, S - \mathrm{flip}^\dagger C, \operatorname{sew} \omega \right\rangle = \langle \mathrm{cut}^n S, \operatorname{sew} \omega \rangle ,$$

since $(\mathrm{flip}^\dagger)' \omega = 0$. Thus, $\mathrm{dya}(\mathrm{cut}^n)' \operatorname{sew} \omega = (\mathrm{cut}^n)' \operatorname{sew} \omega$. Finally, since cut^n is geometric, one has $(\mathrm{cut}^n)' \operatorname{sew} \omega \approx_{(\mathrm{cut}^n)'\mathsf{v}} (\mathrm{cut}^n)' \omega \approx_{|\varrho|'\mathsf{u}} \omega$, and again by Lemma B.1 we conclude that $(\mathrm{cut}^n)' \operatorname{sew} \omega = \operatorname{sew} \omega$.

Next, given $t = \frac{k}{n} \in [0,1]$ rational, $S = [pp_0p_1]$, $p_t := (1-t)p_0 + tp_1$, one has

$$\begin{aligned} \langle \mathrm{cut}_t [pp_0p_1], \operatorname{sew} \omega \rangle &= \langle [pp_0p_t], \operatorname{sew} \omega \rangle + \langle [pp_tp_1], \operatorname{sew} \omega \rangle \\ &= \left\langle \mathrm{cut}^{n-k} [pp_0p_t], \operatorname{sew} \omega \right\rangle + \left\langle \mathrm{cut}^{k} [pp_tp_1], \operatorname{sew} \omega \right\rangle \\ &= \langle \mathrm{cut}^n [pp_0p_1], \operatorname{sew} \omega \rangle = \langle [pp_0p_1], \operatorname{sew} \omega \rangle . \end{aligned}$$

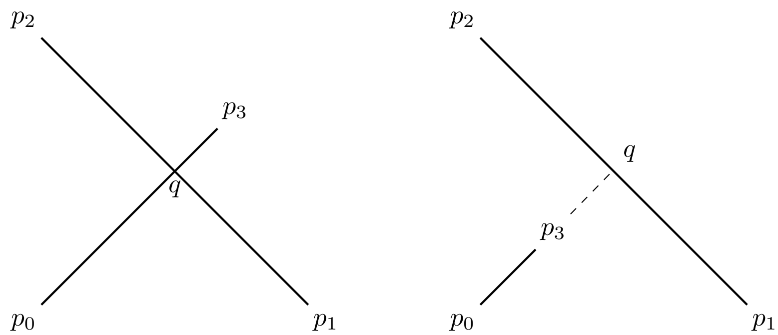
The general case $t \in [0,1]$ follows by approximation: choose $t^n \to t$ with $t^n \in [0,1]$ rational, and since $\operatorname{sew} \omega$ is continuous,

$$\begin{aligned} \langle [pp_0p_1], \operatorname{sew} \omega \rangle &= \lim_{n\to\infty} \langle [pp_0p_{t^n}], \operatorname{sew} \omega \rangle + \langle [pp_{t^n}p_1], \operatorname{sew} \omega \rangle \\ &= \langle [pp_0p_t], \operatorname{sew} \omega \rangle + \langle [pp_tp_1], \operatorname{sew} \omega \rangle . \end{aligned}$$

Step 5: $\operatorname{sew} \omega$ *is closed on k-planes.* Let $T \in \mathrm{Germ}^{k+1}(D)$ with $\mathrm{conv}(T)$ contained in an affine k-plane. To show that $\langle \partial T, \operatorname{sew} \omega \rangle = 0$, we argue separately for the case $k = 1$ and $k = 2$. When $k = 1$, up to permutations of $T = [p_0p_1p_2]$, we can assume that $p_1 = (1-t)p_0 + tp_2$, with $t \in [0,1]$, thus

$$\begin{aligned} \langle \partial [p_0p_1p_2], \operatorname{sew} \omega \rangle &= \langle [p_1p_2] - [p_0p_2] + [p_0p_1], \operatorname{sew} \omega \rangle \\ &= \langle \mathrm{cut}_t [p_0p_2], \operatorname{sew} \omega \rangle - \langle [p_0p_2], \operatorname{sew} \omega \rangle = 0. \end{aligned}$$

In case $k = 2$, since $\operatorname{sew} \omega$ is non-atomic, we can assume that $\mathrm{conv}(T)$ is not contained in a line. Up to permutations of $T = [p_0p_1p_2p_3]$, we can assume that the two straight lines, the one between p_0 and p_3 and the one between p_1 and p_2 intersect at a single point $q = (1-s)p_0 + sp_3 = (1-t)p_1 + tp_2$, s, $t \in \mathbb{R}$. Again, up to permutations, we are reduced to one of the two cases: s, $t \in [0,1]$, or $s \in [0,1]$, $t > 1$ (Figure B.1).



(A) $s,\, t \in [0,1]$ (B) $s \in [0,1]$, $t > 1$

FIGURE B.1. The two cases in the proof of Step 5.

In the first case (Figure B.2),

$$\begin{aligned}
\langle \partial T, \operatorname{sew}\omega\rangle &= \langle [p_1p_2p_3] - [p_0p_2p_3] + [p_0p_1p_3] - [p_0p_1p_2], \operatorname{sew}\omega\rangle \\
&= - \langle [p_0p_1p_2] + [p_3p_2p_1], \operatorname{sew}\omega\rangle \\
&\quad + \langle [p_1p_3p_0] + [p_2p_0p_3], \operatorname{sew}\omega\rangle \quad \text{since } \operatorname{sew}\omega \text{ is alternating} \\
&= - \langle [p_0p_1q] + [p_0qp_2] + [p_3p_2q] + [p_3qp_1], \operatorname{sew}\omega\rangle \\
&\quad + \langle [p_1p_3q] + [p_1qp_0] + [p_2p_0q] + [p_2qp_3], \operatorname{sew}\omega\rangle \quad \text{since } (\operatorname{cut}_t)' \operatorname{sew}\omega = \operatorname{sew}\omega, \\
&= - \langle [p_0p_1q] + [p_0qp_2] + [p_3p_2q] + [p_3qp_1], \operatorname{sew}\omega\rangle \\
&\quad + \langle [p_3qp_1] + [p_0p_1q] + [p_0qp_2] + [p_3p_2q], \operatorname{sew}\omega\rangle \quad \text{since } \operatorname{sew}\omega \text{ is alternating,} \\
&= 0.
\end{aligned}$$

Similarly, in the second case,

$$\begin{aligned}
\langle \partial T, \operatorname{sew}\omega\rangle &= \langle [p_1p_2p_3] - [p_0p_2p_3] + [p_0p_1p_3] - [p_0p_1p_2], \operatorname{sew}\omega\rangle \\
&= - \langle [p_0p_1p_2] + [p_3p_2p_1], \operatorname{sew}\omega\rangle \\
&\quad + \langle [p_1p_3p_0] + [p_2p_0p_3], \operatorname{sew}\omega\rangle \quad \text{since } \operatorname{sew}\omega \text{ is alternating} \\
&= - \langle [p_0p_1q] + [p_0qp_2] + [p_3p_2q] + [p_3qp_1], \operatorname{sew}\omega\rangle \\
&\quad + \langle [p_1qp_0] - [p_1qp_3] + [p_2qp_0] - [p_2qp_3], \operatorname{sew}\omega\rangle \quad \text{since } (\operatorname{cut}_t)' \operatorname{sew}\omega = \operatorname{sew}\omega, \\
&= - \langle [p_0p_1q] + [p_0qp_2] + [p_3p_2q] + [p_3qp_1], \operatorname{sew}\omega\rangle \\
&\quad + \langle [p_3qp_1] + [p_0p_1q] + [p_0qp_2] + [p_3p_2q], \operatorname{sew}\omega\rangle \quad \text{since } \operatorname{sew}\omega \text{ is alternating,} \\
&= 0.
\end{aligned}$$

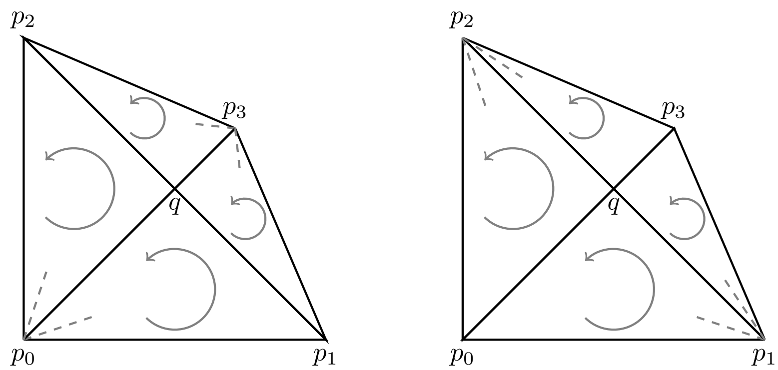
APPENDIX C. BOUND IMPROVEMENT FOR REGULAR GERMS

Aim of this section is to show the following result valid for regular 2-germs.

Lemma C.1. *For $\gamma \in [1,2]$ there exists $\mathsf{c} = \mathsf{c}(\gamma)$ such that if $\omega \in \mathrm{Germ}^2(D)$ is regular then*

$$\mathsf{c}^{-1} [\omega]_{\mathrm{diam}^\gamma} \le [\omega]_{\mathrm{diam}^{2-\gamma} \mathrm{vol}_2^{\gamma-1}} \le \mathsf{c}\, [\omega]_{\mathrm{diam}^\gamma}.$$

Proof. We already noticed in Section 2.4 that $[\omega]_{\mathrm{diam}^\gamma} \le \mathsf{c}[\omega]_{\mathrm{diam}^{2-\gamma} \mathrm{vol}_2^{\gamma-1}}$ always holds (even if ω is not regular). To prove the other inequality, we let $S = [pqr] \in$



(A) $[p_0p_1q] + [p_0qp_2] + [p_3p_2q] + [p_3qp_1]$ (B) $[p_1p_3q] + [p_1qp_0] + [p_2p_0q] + [p_2qp_3]$

FIGURE B.2. Decompositions in the proof of Step 5, first case.

$\mathrm{Simp}^2(D)$ and notice that it is sufficient to prove that, for some $\mathsf{c} = \mathsf{c}(\gamma)$,

$$\langle S, \omega\rangle \le \mathsf{c}\,\mathrm{diam}^{2-\gamma}(S)\,\mathrm{vol}_2^{\gamma-1}(S)[\omega]_{\mathrm{diam}^\gamma} \tag{C.1}$$

in the case that the triangle is rectangle at q. Indeed, to obtain the general case, it is sufficient to cut the longest side of S through its height, obtaining two rectangle triangles S^1 and S^2. Using the fact that ω is regular (so $\omega = \mathrm{cut}_t'\omega$ for any $t \in [0,1]$), we have

$$\begin{aligned}\langle S, \omega\rangle &= \left\langle S^1 + S^2, \omega\right\rangle \\ &\le 2\mathsf{c}\max\left\{\mathrm{diam}^{2-\gamma}(S^1)\,\mathrm{vol}_2^{\gamma-1}(S^1), \mathrm{diam}^{2-\gamma}(S^1)\,\mathrm{vol}_2^{\gamma-1}(S^2)\right\}[\omega]_{\mathrm{diam}^\gamma} \\ &\le 2\mathsf{c}\,\mathrm{diam}^{2-\gamma}(S)\,\mathrm{vol}_2^{\gamma-1}(S)[\omega]_{\mathrm{diam}^\gamma}.\end{aligned}$$

hence the thesis with $2\mathsf{c}$ instead of c.

To prove (C.1), we assume (without loss of generality, up to a permutation) that $\ell := |q-p| \le |q-r| =: L$ and decompose S into $2n+1$ triangles $(S^i)_{i=1}^{2n+1}$ such that $\mathrm{diam}(S^i) \le 4\ell$,

$$\langle S, \omega\rangle = \sum_{i=1}^{2n+1}\left\langle S^i, \omega\right\rangle,$$

and $n \le 2L/\ell$. Once this decomposition is performed, we obtain

$$\begin{aligned}|\langle S, \omega\rangle| &\le \sum_{i=1}^{2n+1}\left|\left\langle S^i, \omega\right\rangle\right| \le [\omega]_{\mathrm{diam}^\gamma}(2n+1)\ell^\gamma \\ &\le 2^5[\omega]_{\mathrm{diam}^\gamma}L\ell^{\gamma-1} \le 2[\omega]_{\mathrm{diam}^\gamma}L^{1-\gamma}(L\ell)^{\gamma-1} \\ &\le 8\cdot 2^{\gamma-1}[\omega]_{\mathrm{diam}^\gamma}\,\mathrm{diam}(S)^{1-\gamma}\,\mathrm{vol}_2(S)^{\gamma-1}\end{aligned}$$

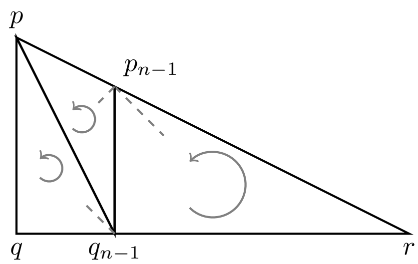
since $L\ell/2 = \mathrm{vol}_2(S)$.

Finally, to obtain such a decomposition, we argue recursively using the cut_t operation to produce a "Christmas tree" like decomposition of $[pqr]$ as in Figure C.1. Precisely, given any $n \ge 1$, for $k \in \{1, \ldots, n\}$, let $p_k := (kp + (n-k)r)/n$ and

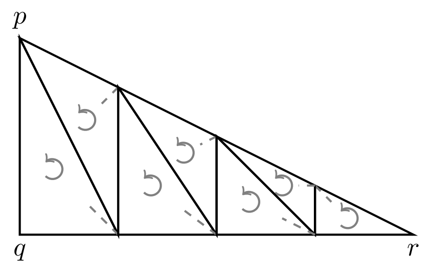
$q_k := (kq + (n-k)r)/n$. Then,

$$\begin{aligned}\langle [pqr], \omega\rangle &= \left\langle \mathrm{cut}_{1/n}[pqr], \omega\right\rangle \\ &= \langle [q_{n-1}pq] + [q_{n-1}rp], \omega\rangle \\ &= \left\langle [q_{n-1}pq] + \mathrm{cut}_{(n-1)/n}[q_{n-1}rp], \omega\right\rangle \\ &= \langle [q_{n-1}pq] + [p_{n-1}pq_{n-1}] + [p_{n-1}q_{n-1}r], \omega\rangle \quad \text{(see Figure C.1)} \\ &= \sum_{k=1}^{n} \langle [q_{k-1}p_k q_k] + [p_{k-1}p_k q_{k-1}], \omega\rangle + \langle [p_1 q_1 r], \omega\rangle\,,\end{aligned}$$

the last equality following by recursion (apply the decomposition with $n-1$ instead of n to the simplex $[p_{n-1}q_{n-1}r]$). By construction, the diameter of each of the simplexes $(S^i)_{i=1}^{2n+1}$ in the last line above is smaller than $\ell + 3L/n$. Therefore, letting n be the smallest integer larger than L/ℓ, so that in particular $L/\ell \le n \le 2L/\ell$, we conclude that $\mathrm{diam}(S^i) \le 4\ell$.



(A) First step of the construction.



(B) Decomposition with $n = 3$.

FIGURE C.1. Christmas tree decomposition.

□

APPENDIX D. PROOF OF RESULTS ON SIDE COMPENSATORS

D.1. **Proof of uniqueness of a side compensator Theorem 4.24.** If $\eta \in \mathrm{Germ}^1(D)$ is such that $\eta \approx_{\mathsf{u}} 0$ for some 1-Dini $\mathsf{u} \in \mathrm{Germ}^1(D)$ and $\delta\eta_{prq} = 0$ for every $[prq] \in \mathrm{Simp}^1(D)$ such that $r = \frac{p+q}{2}$, i.e.,

$$\eta_{pq} = \eta_{pr} + \eta_{rq},$$

then we are in a position to apply Lemma B.1 with $\ell = 2$, $\lambda_1 = \lambda_2 = 1$, $\tau_1[pq] := [pr]$ and $\tau_2[pq] := [rq]$, $\omega := \eta$ and $\mathsf{v} := \mathsf{u}$, so that $\eta = \tau\eta$ hence $\eta = 0$.

D.2. **Proof of existence of a side compensator Theorem 4.27.** We define recursively $L^0(\omega)_{pq} := \omega_{prq}$ where $[pq] \in \mathrm{Simp}^1(D)$ and $r := \frac{p+q}{2}$, and for $n \ge 0$,

$$L^{n+1}(\omega)_{pq} := L^n(\omega)_{pr} + L^n(\omega)_{rq} - \omega_{prq}.$$

Arguing by induction, we have, for every $n \ge 0$,

$$\left\langle [pq], L^{n+1}(\omega) - L^n(\omega)\right\rangle \le [\omega]_{\mathsf{u}} 2^n \left\langle (2^{-n})_\natural [prq], \mathsf{u}\right\rangle, \tag{D.1}$$

hence, being u strong 1-Dini, the sequence $\langle [pq], L^n(\omega)\rangle$ is Cauchy. The pointwise limit $\lim_{n\to+\infty} L^n(\omega) =: L(\omega)$ clearly satisfies (4.17), i.e.,

$$L(\omega)_{pq} = L(\omega)_{pr} + L(\omega)_{rq} - \omega_{prq}$$

as well as, by summing over $n \geq 0$ the inequalities (D.1),

$$|\langle [pq], L(\omega)\rangle| \leq [\omega]_{\mathsf{u}} \sum_{n=0}^{+\infty} 2^n \left\langle (2^{-n})_\natural [prq], \mathsf{u} \right\rangle = [\omega]_{\mathsf{u}} \left\langle [pq], \mathsf{v} \right\rangle,$$

where we introduce the 1-Dini gauge $\langle [pq], \mathsf{v}\rangle := \langle [prq], \tilde{\mathsf{u}}\rangle$, with $\tilde{\mathsf{u}}$ given as in (2.3). By Example 2.17, if $\mathsf{u} = \operatorname{diam}^\alpha$ (with $\alpha > 1$) we obtain that $\mathsf{v} = \mathsf{c}(\alpha) \operatorname{diam}^\alpha$, with $\mathsf{c}(\alpha) = (1 - 2^{1-\alpha})^{-1}$.

D.3. **Proof of Theorem 4.28.** First, we notice that Theorem 4.27 implies that $L(\omega) \in \mathrm{Germ}^1(D)$ is well-defined with $L(\omega) \approx_{\mathsf{v}} 0$ for some 2-Dini gauge v (by Remark 2.16, because u is assumed to be strong 2-Dini). We define $\tilde{\omega} := \omega - \delta L(\omega)$ which is closed on 2-planes, alternating, and $\tilde{\omega}_{prq} = 0$ whenever $r = \frac{p+q}{2}$ (by definition of L) hence by Remark A.13 (applied to the restriction of $\tilde{\omega}$ to a 2 plane contaning $S \in \mathrm{Simp}^2(D)$) we have

$$\langle \operatorname{dya} S, \tilde{\omega}\rangle = \langle S, \tilde{\omega}\rangle .$$

Using that $\tilde{\omega} \approx_{\mathsf{w}} 0$ with the 2-Dini gauge $\mathsf{w} := \max\{\mathsf{u}, \mathsf{v}\}$, by Lemma B.1 $m := 2^k$, $\tau_i := \operatorname{dya}^i$, $\tau := \operatorname{dya}$ we deduce $\tilde{\omega} = 0$.

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