On the curvature energy of Cartesian surfaces

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Abstract. We analyze the lower semicontinuous envelope of the curvature functional of Cartesian surfaces in codimension one. To this aim, following the approach by Anzellotti-Serapioni-Tamanini, we study the class of currents that naturally arise as weak limits of Gauss graphs of smooth functions. The curvature measures are then studied in the non-parametric case. Concerning homogeneous functions, some model examples are studied in detail. Finally, a new gap phenomenon is observed.

Introduction

Functionals depending on curvatures of curves and surfaces are quite natural objects, both from an analytical-geometrical and a physical point of view. After the work by J. Bernoulli and Euler on the problem of Elastica, they appear in the work of S. Germain as a reasonable model expressing the bending energy for an elastic plate, see the historical paper [26] and the references therein.

More recently, T. Willmore [27] proposed the study of compact immersed surfaces $\mathcal{M}$ in $\mathbb{R}^3$ which minimize the (Willmore) functional

$$W(\mathcal{M}) := \int_{\mathcal{M}} |\mathbf{H}|^2 \, d\mathcal{H}^2$$

which is given by the integral of the square of the *mean curvature* $\mathbf{H}$. Starting from that seminal paper, much work has been done by differential geometers concerning Willmore surfaces and critical points of more general functionals of the principal curvatures.

From the point of view of Direct Methods in the Calculus of Variations, we quote here the approach by J. Hutchinson [19] in terms of *curvature varifolds*, a class of generalized surfaces (defined by W. K. Allard [4]) having a p-summable weakly defined second fundamental form: see also [21], where C. Mantegazza studied the subclass of curvature varifolds with boundary.

A different approach was considered by G. Anzellotti, R. Serapioni, and I. Tamanini in [7], starting from the following observation: for a smooth n-dimensional surface $\mathcal{M}$ in $\mathbb{R}^{n+1}$, all the information about the curvatures are contained in the *graph*

$$\mathcal{GM} := \{(x, \nu(x)) \mid x \in \mathcal{M}\}$$

of the *Gauss map* $\nu : \mathcal{M} \to \mathbb{S}^n \subset \mathbb{R}^{n+1}$ of the surface. For example, see also [6], since the tangent plane to $\mathcal{GM}$ at a point $(x, \nu(x))$ is determined by the tangential derivatives of $\nu(x)$ at x, and hence by the second fundamental form to $\mathcal{M}$ at x, by the area formula when $n = 2$ it turns out that the area of the Gauss graph surface $\mathcal{GM}$ is linked to the principal curvatures of $\mathcal{M}$ by the relation:

$$\mathcal{H}^2(\mathcal{GM}) = \int_{\mathcal{M}} \Big(1 + ({\mathbf{k}_1}^2 + {\mathbf{k}_2}^2) + (\mathbf{k}_1\mathbf{k}_2)^2\Big)^{1/2} \, d\mathcal{H}^2 \, .$$

In order to study minimization problems in this framework, it is natural to consider the *current carried by the graph* of the Gauss map. The idea of considering the graph $\mathcal{G}_u$, instead of the map u, was already present in the approach to non-parametric minimal surfaces and more generally in the minimization problem of functionals $\int f(Du)\, dx$ with linear growth in the gradient of a scalar function, see [18, 10]. A crucial observation, which goes back to H. Federer's work [14], is that the boundary of the current carried by the subgraph $S\mathcal{G}_u := \{(x, z) \in \Omega \times \mathbb{R} \mid z < u(x)\}$ of a bounded and summable function $u : \Omega \to \mathbb{R}$ is an *integer multiplicity* (say i.m.) *rectifiable current* in $\Omega \times \mathbb{R}$ if and only if u is a *function of bounded variation.*

However, it is in the seminal paper by M. Giaquinta, G. Modica, and J. Souček [16] on nonlinear elasticity, where currents carried by the graph of vector-valued maps were firstly used to study polyconvex functionals with superlinear growth in the minors of the gradient matrix Du, see also [17]. Based on this approach, Anzellotti et al. studied in [7] limit points Σ of sequences $[\![\mathcal{GM}_h]\!]$ of currents carried by the Gauss graph of smooth surfaces with equibounded areas and e.g. prescribed boundary conditions. We recall here that the weak convergence as currents $[\![\mathcal{GM}_h]\!] \rightharpoonup \Sigma$ is defined by duality as the pointwise convergence $\langle [\![\mathcal{GM}_h]\!], \omega\rangle \to \langle \Sigma, \omega\rangle$ for any smooth and compactly supported n-form ω in $\mathbb{R}^{n+1} \times \mathbb{S}^n$. In fact, by Federer-Fleming's closure theorem [15], it turns out that any such weak limit point Σ is an n-dimensional i.m. rectifiable current in $\mathbb{R}^{n+1} \times \mathbb{S}^n$, with *finite mass*.

Most importantly, the weak convergence as currents preserving some geometric structures of Gauss graphs of smooth surfaces, the authors of [7] were able to equip Σ with a generalized second fundamental form, which extends the notion obtained by Hutchinson in terms of varifolds. As a consequence, they recovered in a non-smooth context a weak notion of Euler-Poincaré characteristic, and found a suitable definition of measures which are naturally associated to the elementary symmetric curvatures (i.e., to the Gauss curvature $\mathbf{K} = \mathbf{k}_1\mathbf{k}_2$ and mean curvature $2\,\mathbf{H} = \mathbf{k}_1 + \mathbf{k}_2$, in the 2-dimensional case).

Moreover, in order to discuss the existence of minimizers of functional depending on curvatures, Anzellotti et al. proposed to study the *relaxed energy functional*, see also the work by S. Delladio [13]. More precisely, they introduced the *curvature functional* of a smooth n-dimensional surface $\mathcal{M} \subset \mathbb{R}^{n+1}$, that for $n = 2$ reads as:

$$\|\mathcal{M}\| := \mathcal{H}^2(\mathcal{M}) + \int_{\mathcal{M}} \sqrt{\mathbf{k}_1^2 + \mathbf{k}_2^2}\, d\mathcal{H}^2 + \int_{\mathcal{M}} |\mathbf{k}_1\mathbf{k}_2|\, d\mathcal{H}^2 \,.$$

Under prescribed boundary conditions on $\partial\mathcal{M}$ and on the value of the Gauss map on $\partial\mathcal{M}$, the relaxation approach yields to consider for each weak limit current Σ as above the functional

$$\overline{F}(\Sigma) := \inf\{\liminf_{h\to\infty} \|\mathcal{M}_h\|\}$$

where the infimum is taken among all sequences $\{\mathcal{M}_h\}$ of smooth surfaces such that the currents $[\![\mathcal{GM}_h]\!]$ carried by their Gauss graphs weakly converge to Σ. However, even in dimension $n = 2$, it is an open problem to characterize the class of i.m. rectifiable currents Σ in $\mathbb{R}^3 \times \mathbb{S}^2$ for which the relaxed energy $\overline{F}(\Sigma)$ is finite. On the other hand, it would be desirable to have an explicit formula for the relaxed energy, another non-trivial open problem.

In this paper, we shall focus on the above mentioned relaxation problem in the non-parametric case. For this purpose, we restrict to the case $n = 2$ and try to analyze the class of currents which naturally arise as weak limits of Gauss graphs of smooth *Cartesian surfaces*. We refer to [11, 2, 3] for the analysis of the one-dimensional case of Cartesian curves.

Plan of the paper. In Sec. 1, we collect some notation from [6, 7] concerning Gauss graphs of codimension one surfaces. We shall then restrict to non-parametric surfaces $\mathcal{M} = \mathcal{G}_u$ given by the graph

$$\mathcal{G}_u := \{(x, u(x)) \mid x \in \Omega\}$$

of smooth bounded functions $u : \Omega \to \mathbb{R}$, where $\Omega \subset \mathbb{R}^2$ is a bounded domain. Therefore, in this case the Gauss map is naturally identified at each point of the graph by the outward unit normal

$$\nu_u(x) := \frac{1}{\sqrt{g_u}}\big(-\partial_1 u, -\partial_2 u, 1\big)\,, \qquad g_u := 1 + |\nabla u|^2\,, \qquad x \in \Omega$$

and hence the mean curvature $\mathbf{H}_u$ and Gauss curvature $\mathbf{K}_u$ at $(x, u(x))$ are written in terms of the first and second order partial derivatives of u, see (1.11) and (1.12). Moreover, the Gauss graph of the Cartesian surface $\mathcal{G}_u$ agrees with the non-parametric surface in $\mathbb{R}^6$

$$\mathcal{GG}_u = \{\Phi_u(x) \mid x \in \Omega\}$$

where $\Phi_u : \Omega \to (\Omega \times \mathbb{R}) \times \mathbb{S}^2$ is the smooth map

$$\Phi_u(x) = \big(\varphi_u(x), \nu_u(x)\big)\,, \quad \varphi_u(x) := (x, u(x))$$

and hence the tangent space at each point in the Gauss graph $\mathcal{GG}_u$ is oriented by the wedge product

$$\xi_u(x) := \partial_1\Phi_u(x) \wedge \partial_2\Phi_u(x) \,, \qquad x \in \Omega$$

so that by the area formula we get $\mathcal{H}^2(\mathcal{GG}_u) = \displaystyle\int_\Omega |\xi_u| \, d\mathcal{L}^2$.

In Sec. 2, we deal with the currents GG_u carried by the Gauss graph of smooth Cartesian surfaces. They are naturally defined through the formula $GG_u = \Phi_{u\#}[\![\,\Omega\,]\!]$, i.e., denoting $U := \Omega \times \mathbb{R}$, by:

$$\langle GG_u, \omega \rangle = \int_\Omega {\Phi_u}^\# \omega \qquad \forall\, \omega \in \mathcal{D}^2(U \times \mathbb{S}^2) \,.$$

The action of GG_u on qualitatively different 2-forms is explicitly computed in (2.1). We then analyze the curvature energy functional $\|\mathcal{M}\|$ when $\mathcal{M} = \mathcal{G}_u$, which becomes

$$\mathcal{E}(u) := \int_\Omega \big(|\xi_u^{(0)}| + |\xi_u^{(1)}| + |\xi_u^{(2)}| \big) \, d\mathcal{L}^2 \,.$$

In term of the *stratification* of the orienting 2-vector ξ_u, see (2.6), we in fact have:

$$|\xi_u^{(0)}| = \sqrt{g_u} \,, \quad |\xi_u^{(1)}| = \sqrt{g_u}\,\sqrt{(4\mathbf{H}_u^2 - 2\mathbf{K}_u)} \,, \quad |\xi_u^{(2)}| = \sqrt{g_u}\,|\mathbf{K}_u| \,.$$

The integration by parts formulas of the Gauss graph currents GG_u are then obtained.

In Sec. 3, we shall introduce the *relaxed curvature energy*, given for any function $u \in L^\infty(\Omega)$ by

$$\overline{\mathcal{E}}(u) := \inf\{ \liminf_{h\to\infty} \mathcal{E}(u_h) \mid \{u_h\}_h \subset C^2(\Omega) \,,\ u_h \to u \text{ strongly in } L^1(\Omega) \}$$

and discuss some general properties of functions u with finite relaxed energy. Since the Gauss map ν_u is a function of bounded variation, Theorem 3.1, we shall see that the current carried by the Gauss graph of u is well-defined as in the smooth case by $GG_u^a := \Phi_{u\#}[\![\,\Omega\,]\!]$, but this time the pull-back ${\Phi_u}^\#\omega$ of forms is computed by means of the approximate gradient of the BV-map Φ_u. Weak limits Σ of sequences of currents carried by the Gauss graph of smooth functions with equibounded curvature energies are also analyzed. We shall then introduce a *curvature functional* $\Sigma \mapsto \mathrm{E}(\Sigma)$ *on the currents* Σ, that agrees with the curvature energy $\mathcal{E}(u)$ when $\Sigma = GG_u$ for some smooth function u. Its relationship with the relaxed energy is finally outlined.

In Sec. 4, we shall discuss the notion of *generalized first and second symmetric curvatures* from [7] in our framework, by also giving an explicit computation in the case of two surfaces with line or point singularities: the union of two rectangles meeting at an edge, Example 4.3, and the lateral surface of a cone, Example 4.4. Referring to these two models, we shall then compare our definitions with the notion by J. M. Sullivan [24] of *mean and Gauss curvature for polyhedral surfaces* and 2-rectifiable sets. We shall then discuss the relation with the Gauss-Bonnet theorem. Finally, we shall introduce the distributional definition of weak and Gauss curvatures, given by

$$\widetilde{\mathbf{H}}_u := \frac{1}{2}\,\mathrm{Div}\Big[\frac{\nabla u}{\sqrt{1+|\nabla u|^2}} \Big] \,, \qquad \widetilde{\mathbf{K}}_u := \frac{1}{2}\,\mathrm{Div}\Big({\nu_u}^1 \partial_2 {\nu_u}^2 - {\nu_u}^2 \partial_2 {\nu_u}^1 , {\nu_u}^2 \partial_1 {\nu_u}^1 - {\nu_u}^1 \partial_1 {\nu_u}^2 \Big) \,.$$

Even if they are well-defined for any function u with finite relaxed energy, we shall see that they fail to be the good geometric objects.

From Sec. 5 to the end, we shall restrict to the subclass of 0-homogeneous functions defined in the open unit ball $\Omega = B^2$. In fact, beside the structure theorems 3.1 and 3.4, finding necessary and sufficient conditions to the membership of a bounded BV-function $u : \Omega \to \mathbb{R}$ to the class of functions with finite relaxed energy is a non-trivial open problem. We shall then assume that $u : B^2 \to \mathbb{R}$ satisfies $u(x) = u(x/|x|)$, and with an abuse of notation we shall write $u(x) = f(\theta)$ for some function $f : [0, 2\pi] \to \mathbb{R}$, where $x = (\rho\cos\theta, \rho\sin\theta)$. The explicit computation of the curvature energy functional $\mathcal{E}(u)$ in the case of homogeneous functions is postponed to Appendix A.

The *relaxation problem in the annulus* $\Omega_r := \{x \in B^2 \mid r < |x| < 1\}$ is then solved for any small radius $r > 0$. In the homogeneous case, in fact, the explicit formula for the relaxed energy in Ω_r is recovered from the one-dimensional results obtained for Cartesian curves in [2].

In Sec. 6, we shall compute the relaxed energy in the case of the homogeneous function

$$u(x) := \frac{x_1}{|x|}, \qquad x = (x_1, x_2).$$

The Gauss graph of u has a "hole" at the origin. More precisely, the current GG_u has an inner boundary which can be described by the 1-dimensional current Γ carried by a closed Lipschitz-continuous curve $\widetilde{\gamma}_0$ whose support is concentrated in $\{0_{\mathbb{R}^2}\} \times \mathbb{R} \times \mathbb{S}^2$, see (6.1) and (6.2). We shall prove that

$$\overline{\mathcal{E}}(u) = \mathcal{E}(u) + \mathrm{E}(S_1)$$

where S_1 is the minimal energy element among all the i.m. rectifiable currents that bound the integral 1-chain Γ and that satisfy the *geometric condition* inherited by the orthogonality of ν_{u_h} to the tangent space to $\mathcal{G}_{u_h}$ that holds true for the smooth approximating sequences.

In Sec. 7, we shall then consider the non-smooth homogeneous function

$$u(x) := \begin{cases} \pi/2 - \arctan(x_2/x_1) & \text{if} \quad x_1 < 0 \\ \pi & \text{if} \quad x_1 \geq 0 \quad \text{and} \quad x_2 > 0 \\ 0 & \text{if} \quad x_1 \geq 0 \quad \text{and} \quad x_2 < 0\,. \end{cases}$$

Roughly speaking, the boundary ∂SG_u of the subgraph of u is the surface given by two "floors", at level $z = 0$ and $z = \pi$, one "wall" of height π at the Jump set $J_u = (0, 1) \times \{0\}$, and a smooth "spiral staircase" connecting the two floors. Four horizontal edges appear at the boundary of the two floors, and a fifth vertical edge lives over the singular point $0_{\mathbb{R}^2}$, the edges meeting at the corner points $(0, 0, 0)$ or $(0, 0, \pi)$. Now, by applying results from [2] to the Gauss graph of the corresponding function f, given by

$$f(\theta) := \begin{cases} \pi & \text{if} \quad 0 < \theta < \pi/2 \\ 3\pi/2 - \theta & \text{if} \quad \pi/2 < \theta < 3\pi/2 \\ 0 & \text{if} \quad 3\pi/2 < \theta < 2\pi \end{cases}$$

it turns out that the (optimal) Gauss graph current $\widetilde{\Sigma}_u$ is well-defined by the formula

$$\widetilde{\Sigma}_u = GG_u^a + GG_u^C + GG_u^J + S_u^{Je} + S_u^e\,.$$

We have already seen that the *Absolute continuous* component GG_u^a is defined by $GG_u^a = \Phi_{u\#}[\![B^2]\!]$, through the approximate gradient of the BV-map Φ_u. The *Cantor* component $GG_u^C = 0$, as the distributional derivative of u has no Cantor part, $D^C u = 0$. The *Jump* component GG_u^J is the Gauss graph of the vertical wall at the discontinuity set of u. The *Jump-edge* component S_u^{Je} deals with the Jump of ν_u w.r.t. the outward normal to the wall surface at the upper and lower edges. Finally, the *Edge* component S_u^e deals with the Jump of ν_u where u is continuous.

We have thus obtained an i.m. rectifiable current $\widetilde{\Sigma}_u$ in $U \times \mathbb{S}^2$. The energy contribution of the several components in the above decomposition formula is computed in Appendix B. However, as in the previous example (where $\widetilde{\Sigma}_u$ reduces to GG_u^a) the null-boundary condition $\partial\widetilde{\Sigma}_u = 0$ is violated, as in general a "hole" appears at the origin $0_{\mathbb{R}^2}$, see (7.17), so that an extra energy contribution is expected in the relaxation process, yielding this time to the formula

$$\overline{\mathcal{E}}(u) = \mathcal{E}(u) + \mathrm{E}(GG_u^J) + \mathrm{E}(S_u^{Je}) + \mathrm{E}(S_u^e) + \mathrm{E}(S_1)\,,$$

where S_1 is the minimal energy element among all the i.m. rectifiable currents that "fill the hole" in the optimal current $\widetilde{\Sigma}_u$ and satisfy the above mentioned orthogonality condition.

In Sec. 8, we shall then describe a *gap phenomenon*, firstly discovered by G. Buttazzo and V. J. Mizel [9] in the relaxation process, that makes the above problem much more complicate. Consider in fact the piecewise constant and homogeneous BV-function u given by $u(x) = f(\theta)$, where

$$f(\theta) := \begin{cases} 1 & \text{if} \quad \theta \in (0, \pi/3) \cup (2\pi/3, \pi) \cup (4\pi/3, 5\pi/3) \\ 0 & \text{if} \quad \theta \in (\pi/3, 2\pi/3) \cup (\pi, 4\pi/3) \cup (5\pi/3, 2\pi)\,. \end{cases}$$

Roughly speaking, this time the boundary of the subgraph of u is given by a "cake" which has been divided into six equal parts, and where three non-consecutive slices have been removed. As before, one may define the (optimal) Gauss graph current $\widetilde{\Sigma}_u = GG_u^a + GG_u^J + S_u^{Je}$, as both the Cantor and Edge components GG_u^C and S_u^e are trivial. Now, by the symmetry of u it turns out that $\widetilde{\Sigma}_u$ has no homological boundary in $(B^2 \times \mathbb{R}) \times \mathbb{S}^2$. In fact, looking at the behavior of the outward unit normal (a "candle" moving on the "cake") on small circles around the origin, it turns out that the boundary of $\widetilde{\Sigma}_u$ is given by $\delta_{0_{\mathbb{R}^2}} \times \gamma_\# [\![0, 18]\!]$, where $\gamma : [0, 18] \to \mathbb{R} \times \mathbb{S}^2$ is the closed rectifiable arc parameterized by (8.4). It turns out that the closed arc γ is *homologically trivial*, as its support is parameterized twice and with opposite orientation. Therefore, in accordance with the previous examples, and by the optimality of the current $\widetilde{\Sigma}_u$, one expects that $\widetilde{\mathcal{E}}(u) = \mathrm{E}(\widetilde{\Sigma}_u)$.

However, we shall see that the loop γ is *topologically non-trivial* (both in $\mathbb{R} \times \mathbb{S}^2$ and in $\mathbb{R}^4$). As a consequence, we obtain the energy gap $\widetilde{\mathcal{E}}(u) > \mathrm{E}(\widetilde{\Sigma}_u)$. In terms of approximation in energy by smooth functions, *a topological obstruction* that cannot be treated by means of homological arguments occurs.

Finally, some ideas towards the direction of finding an explicit formula for the relaxed energy (a widely open problem even in the case of homogeneous functions) are collected in Sec. 9. Namely, in order to prove that (in accordance with our examples) the relaxed energy satisfies the *measure property*, one may consider the *localization* of the relaxed functional. Following E. Acerbi and G. Dal Maso [1], where the analogous feature concerning the relaxed area functional of vector-valued functions was firstly discovered, it is not clear if one could find a function u with finite relaxed energy such that the set function $A \mapsto \overline{\mathcal{E}}(u, A)$ fails to be subadditive, i.e., for which we can find open sets $A_1, A_2, A_3 \subset \Omega$ such that $A_3 \subset\subset A_1 \cup A_2$ but $\overline{\mathcal{E}}(u, A_3) > \overline{\mathcal{E}}(u, A_1) + \overline{\mathcal{E}}(u, A_2)$. In fact, due to the geometric constraints (the orthogonality condition), we expect that (at least for homogeneous functions u) the subadditivity property holds, and hence the set function $A \mapsto \overline{\mathcal{E}}(u, A)$ is a measure, as a consequence of the De Giorgi-Letta criterion [12].

In conclusion, we point out that the case of n-dimensional Cartesian surfaces may be analyzed by generalizing the ideas contained in this paper. On the other hand, a part from the 1-dimensional case studied in [2, 3], the analysis of the relaxed curvature energy functional of higher codimension Cartesian surfaces is certainly a much more complicate stuff, as the role played here by the Gauss map has to be replaced with more general geometric invariants.

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1 Gauss graphs of smooth Cartesian surfaces

In this section we report from Anzellotti et al. [7] some notation and properties concerning Gauss graphs of codimension one surfaces, see also [6]. We address to [23, 17, 20] for the main facts and notation on Geometric Measure Theory. We then focus on the case of Cartesian surfaces.

Gauss graphs. Following [6, 7], given a smooth (say C^2), bounded, and oriented surface $\mathcal{M} \subset \mathbb{R}^3$, the *Gauss map* $\nu : \mathcal{M} \to \mathbb{S}^2$ associates to each point x in $\mathcal{M}$ the unit normal $\nu(x) \in \mathbb{S}^2$, where

$$\mathbb{S}^2 := \{y \in \mathbb{R}^3 \,:\, |y| = 1\}$$

and the graph of the Gauss map (or *Gauss graph*) is the 2-dimensional surface in $\mathbb{R}^6$ given by

$$\mathcal{GM} := \{(x, \nu(x)) \mid x \in \mathcal{M}\} \subset \mathcal{M} \times \mathbb{S}^2 \subset \mathbb{R}^3_x \times \mathbb{R}^3_y .$$

The tangent 2-vector field $\tau : \mathcal{M} \to \Lambda^2 T\mathcal{M} \subset \Lambda^2 \mathbb{R}^3_x$ is given in terms of the Hodge operator by $\tau(x) = *\nu(x)$. Denoting by $\Phi : \mathcal{M} \to \mathbb{R}^3_x \times \mathbb{R}^3_y$ the graph map $\Phi(x) := (x, \nu(x))$, a continuous tangent 2-vector field $\xi : \mathcal{GM} \to \bigwedge^2 (\mathbb{R}^3_x \times \mathbb{R}^3_y)$ is given by $\xi(x, \nu(x)) := \bigwedge^2 d\Phi_x(\tau(x))$. Since $|\xi| \geq 1$ on $\mathcal{GM}$, the normalized 2-vector field $\overrightarrow{\zeta} := \xi / |\xi|$ determines an orientation to $\mathcal{GM}$. Therefore, the corresponding *integer multiplicity* (say *i.m.*) *rectifiable* 2-*current* $[\![\mathcal{GM}]\!]$ in $\mathcal{R}_2(\mathbb{R}^3_x \times \mathbb{R}^3_y)$ carried by the Gauss graph has multiplicity one and support contained in $\overline{\mathcal{M}}\times\mathbb{S}^2$. Its action on compactly supported smooth 2-forms ω in $\mathbb{R}^3_x\times\mathbb{R}^3_y$ is by integration:

$$\langle[\![\mathcal{GM}]\!],\omega\rangle=\int_{\mathcal{GM}}\langle\omega(x,y),\overrightarrow{\zeta}(x,y)\rangle\,d\mathcal{H}^2(x,y)\,,\qquad\omega\in\mathcal{D}^2(\mathbb{R}^3_x\times\mathbb{R}^3_y)\,.$$

By Stokes' theorem, the boundary current $\partial[\![\mathcal{GM}]\!]$ acts by integration of 1-forms on the naturally oriented boundary of $\mathcal{GM}$, so that $\partial[\![\mathcal{GM}]\!]=0$ if $\mathcal{M}$ is a closed smooth surface.

The *tangential Jacobian* of the graph map satisfies

$$J^{\mathcal{M}}_\Phi(x)=\Big(1+({\mathbf{k}_1}^2+{\mathbf{k}_2}^2)+(\mathbf{k}_1\mathbf{k}_2)^2\Big)^{1/2}\,,\qquad x\in\mathcal{M}$$

where $\mathbf{k}_1=\mathbf{k}_1(x)$ and $\mathbf{k}_2=\mathbf{k}_2(x)$ are the *principal curvatures* at $x\in\mathcal{M}$. In fact, we have $J^{\mathcal{M}}_\Phi(x)=|\xi(x,\nu(x))|$. Moreover, denoting by τ_1 and τ_2 the *principal directions*, and considering the obvious homomorphism $v\mapsto\widetilde{v}$ from $\mathbb{R}^3_x$ onto $\mathbb{R}^3_y$, one has

$$\xi(x,\nu(x))=\tau_1\wedge\tau_2+\Big(\mathbf{k}_2\tau_1\wedge\widetilde{\tau}_2-\mathbf{k}_1\tau_2\wedge\widetilde{\tau}_1\Big)+\mathbf{k}_1\mathbf{k}_2\,\widetilde{\tau}_1\wedge\widetilde{\tau}_2\,. \tag{1.1}$$

Also, denoting by $\mathbf{H}$ and $\mathbf{K}$ the *mean curvature* and *Gauss curvature*,

$$\mathbf{H}:=\frac{1}{2}(\mathbf{k}_1+\mathbf{k}_2)\,,\qquad\mathbf{K}:=\mathbf{k}_1\mathbf{k}_2$$

so that $\mathbf{k}_{1,2}=\mathbf{H}\pm\sqrt{\mathbf{H}^2-\mathbf{K}}$, we equivalently have

$$(J^{\mathcal{M}}_\Phi)^2=1+(2\mathbf{H})^2-2\mathbf{K}+\mathbf{K}^2=4\mathbf{H}^2+(1-\mathbf{K})^2\,.$$

Therefore, by the area formula, the area of the Gauss graph reads as

$$\mathcal{H}^2(\mathcal{GM})=\int_{\mathcal{M}}\Big(1+({\mathbf{k}_1}^2+{\mathbf{k}_2}^2)+(\mathbf{k}_1\mathbf{k}_2)^2\Big)^{1/2}d\mathcal{H}^2=\int_{\mathcal{M}}\sqrt{1+(4\mathbf{H}^2-2\mathbf{K})+\mathbf{K}^2}\,d\mathcal{H}^2 \tag{1.2}$$

and it agrees with the *mass* $\mathbf{M}([\![\mathcal{GM}]\!])$ of the current $[\![\mathcal{GM}]\!]$.

The *curvature functional* of a smooth surface $\mathcal{M}\subset\mathbb{R}^3$ is defined in [7] by:

$$\|\mathcal{M}\|:=\mathcal{H}^2(\mathcal{M})+\int_{\mathcal{M}}\sqrt{\mathbf{k}_1^2+\mathbf{k}_2^2}\,d\mathcal{H}^2+\int_{\mathcal{M}}|\mathbf{k}_1\mathbf{k}_2|\,d\mathcal{H}^2 \tag{1.3}$$

i.e., equivalently,

$$\|\mathcal{M}\|:=\int_{\mathcal{M}}\big(1+\sqrt{4\mathbf{H}^2-2\mathbf{K}}+|\mathbf{K}|\big)\,d\mathcal{H}^2$$

so that by (1.2) one gets the bounds with the area of the Gauss graph:

$$\frac{1}{2}\,\|\mathcal{M}\|\leq\mathcal{H}^2(\mathcal{GM})\leq\|\mathcal{M}\|\,,\qquad\mathcal{H}^2(\mathcal{GM})=\mathbf{M}([\![\mathcal{GM}]\!])\,.$$

Also, in [7] two real measures on $\mathbb{R}^3_x\times\mathbb{R}^3_y$ are naturally associated to the mean and Gauss curvatures:

$$\chi^{\mathcal{M}}_1:=-\Phi_\#(\mathbf{H}\,\mathcal{H}^2\llcorner\mathcal{M})\,,\qquad\chi^{\mathcal{M}}_2:=\Phi_\#(\mathbf{K}\,\mathcal{H}^2\llcorner\mathcal{M}) \tag{1.4}$$

so that for any $\psi\in C_0(\mathbb{R}^3_x\times\mathbb{R}^3_y)$ we have

$$\langle\chi^{\mathcal{M}}_1,\psi\rangle=-\int_{\mathcal{M}}\mathbf{H}(x)\,\psi(x,\nu(x))\,d\mathcal{H}^2(x)\,,\qquad\langle\chi^{\mathcal{M}}_2,\psi\rangle=\int_{\mathcal{M}}\mathbf{K}(x)\,\psi(x,\nu(x))\,d\mathcal{H}^2(x)\,.$$

The above *curvature measures* are re-written in terms of the current $[\![\mathcal{GM}]\!]$ by the formulas

$$\langle\chi^{\mathcal{M}}_i,\psi\rangle=(-1)^i\langle[\![\mathcal{GM}]\!],\psi\,\Theta_i\rangle\qquad\forall\,\psi\in C^\infty_c(\mathbb{R}^3_x\times\mathbb{R}^3_y)\,,\qquad i=1,2\,.$$

The 2-forms $\Theta_i=\Theta_i(x,y)$ in $\mathbb{R}^3_x\times\mathbb{R}^3_y$ are defined in [7], and for 2-dimensional surfaces they become:

$$\begin{gathered}\Theta_1:=\frac{1}{2}\left(y^1(dx^2\wedge dy^3-dx^3\wedge dy^2)+y^2(dx^3\wedge dy^1-dx^1\wedge dy^3)+y^3(dx^1\wedge dy^2-dx^2\wedge dy^1)\right)\\ \Theta_2:=y^1dy^2\wedge dy^3+y^2dy^3\wedge dy^1+y^3dy^1\wedge dy^2\,.\end{gathered} \tag{1.5}$$

Remark 1.1 Since the mean curvature depends on the sign of the principal curvatures, we added a factor -1 in order that for Cartesian surfaces we recover the standard notation, see (4.2) below.

Finally, the weak limits Σ of sequences of currents carried by Gauss graphs of smooth surfaces $\{\mathcal{M}_h\}$ are studied in [7]. Assuming e.g. that each $\mathcal{M}_h$ is closed, supported in a given compact set $K \subset \mathbb{R}^3$, and $\sup_h \|\mathcal{M}_h\| < \infty$, it turns out that Σ is an i.m. rectifiable current in $\mathcal{R}_2(\mathbb{R}^3_x \times \mathbb{R}^3_y)$, with null boundary, $\partial\Sigma = 0$, and with support contained in $K \times \mathbb{S}^2$. Moreover, Σ satisfies the following structure properties:

Theorem 1.2 ([7]) *With the previous notation, one has:*

i) $\langle \Sigma, \eta \wedge \varphi \rangle = 0$ *for each* $\eta \in \mathcal{D}^1(\mathbb{R}^3_x \times \mathbb{R}^3_y)$ *;*

ii) $\langle \Sigma, \psi\, \varphi^* \rangle \geq 0$ *for each* $\psi \in C(\mathbb{R}^3_x \times \mathbb{R}^3_y)$ *such that* $\psi \geq 0$ *;*

where φ *and* φ^* *denote the* canonical 1-form *and* 2-form*, respectively:*

$$\varphi(x,y) := \sum_{j=1}^{3} y^j dx^j \,, \qquad \varphi^*(x,y) := y^1 dx^2 \wedge dx^3 + y^2 dx^3 \wedge dx^1 + y^3 dx^1 \wedge dx^2 \,. \tag{1.6}$$

Remark 1.3 We finally recall from [7] that property i) is equivalent to the orthogonality condition:

$$v \bullet (y, 0_{\mathbb{R}^3}) = 0 \quad \forall\, v \in T_{(x,y)}R$$

for $\mathcal{H}^2$-a.e. $(x,y) \in R$, where $\bullet$ denotes the scalar product, R is the 2-rectifiable set of positive multiplicity of Σ, and $T_{(x,y)}R$ is the approximate tangent space, see Remark 3.5 below.

Cartesian surfaces. We now restrict to smooth Cartesian surfaces, i.e., we assume that $\mathcal{M}$ is the graph of a smooth and bounded function $u : \Omega \to \mathbb{R}$, where $\Omega \subset \mathbb{R}^2$ is a given bounded domain.

We shall denote by (e_1, e_2, e_3) the canonical basis of $\Omega \times \mathbb{R}$, with $x = (x_1, x_2) \in \Omega$, $z = x_3 \in \mathbb{R}$, and hence (dx^1, dx^2, dz) is the dual basis. We shall also denote by Π_1 and Π_2 the orthogonal projections onto the first three and last three components, respectively, i.e., $\Pi_1((x,z),y) := (x,z)$, $\Pi_2((x,z),y) := y$. The canonical basis in $\mathbb{R}^3_y$ is $(\varepsilon_1, \varepsilon_2, \varepsilon_3)$, and its dual basis is (dy^1, dy^2, dy^3). Also, for a function $v : \Omega \to \mathbb{R}$, we shall always denote by ∇v the (approximate) gradient and by $\partial_i v$ and $\partial^2_{i,j} v$ the first and second order (approximate) partial derivatives, so that e.g. $\partial_i v(x) := \nabla v(x) \bullet e_i$ for $i = 1, 2$.

Assuming now $\mathcal{M} = \mathcal{G}_u$, where $\mathcal{G}_u$ is the graph of u

$$\mathcal{G}_u := \{(x, u(x)) \mid x \in \Omega\}$$

the Gauss map is naturally identified at each point of the graph by the outward unit normal

$$\nu_u(x) := \frac{1}{\sqrt{1 + |\nabla u|^2}} \big(-\partial_1 u, -\partial_2 u, 1\big) \,, \qquad x \in \Omega \tag{1.7}$$

and hence the Gauss graph of the Cartesian surface $\mathcal{G}_u$ agrees with

$$\mathcal{G}\mathcal{G}_u := \{\Phi_u(x) \mid x \in \Omega\} \tag{1.8}$$

where $\Phi_u : \Omega \to \Omega \times \mathbb{R}_z \times \mathbb{R}^3_y$ is the smooth map

$$\Phi_u(x) = \big(\varphi_u(x), \nu_u(x)\big) \,, \quad \varphi_u(x) := (x, u(x)) \,, \quad \nu_u(x) = ({\nu_u}^1(x), {\nu_u}^2(x), {\nu_u}^3(x)) \,. \tag{1.9}$$

The first fundamental form of $\mathcal{G}_u$ is identified by the symmetric matrix

$$I := \begin{pmatrix} E & F \\ F & G \end{pmatrix} = \begin{pmatrix} |\partial_1 \varphi_u|^2 & \partial_1 \varphi_u \bullet \partial_2 \varphi_u \\ \partial_1 \varphi_u \bullet \partial_2 \varphi_u & |\partial_2 \varphi_u|^2 \end{pmatrix} = \begin{pmatrix} 1 + (\partial_1 u)^2 & \partial_1 u\, \partial_2 u \\ \partial_1 u\, \partial_2 u & 1 + (\partial_2 u)^2 \end{pmatrix}$$

whose determinant is

$$g = g_u := EG - F^2 = (1 + (\partial_1 u)^2)(1 + (\partial_2 u)^2) - (\partial_1 u\, \partial_2 u)^2 = 1 + |\nabla u|^2 \tag{1.10}$$

whence the unit normal is $\nu_u = g_u^{-1/2}(-\partial_1 u, -\partial_2 u, 1)$. Therefore, by computing the second derivatives of the graph map φ_u, since $\partial^2_{i,j}\varphi_u = (0,0,\partial^2_{i,j}u)$ for $i,j=1,2$, it turns out that the second fundamental form is identified by the symmetric matrix

$$II := \begin{pmatrix} \ell & m \\ m & n \end{pmatrix} = \begin{pmatrix} \partial^2_{1,1}\varphi_u \bullet \nu_u & \partial^2_{1,2}\varphi_u \bullet \nu_u \\ \partial^2_{2,1}\varphi_u \bullet \nu_u & \partial^2_{2,2}\varphi_u \bullet \nu_u \end{pmatrix} = \frac{1}{\sqrt{g_u}} \begin{pmatrix} \partial^2_{1,1}u & \partial^2_{1,2}u \\ \partial^2_{2,1}u & \partial^2_{2,2}u \end{pmatrix}$$

where $\partial^2_{2,1}u = \partial^2_{1,2}u$. As a consequence, the mean curvature at $(x, u(x))$ becomes

$$\mathbf{H}_u = \frac{1}{2g}(En + G\ell - 2Fm) = \frac{1}{2}\frac{1}{{g_u}^{3/2}}\big((1+(\partial_1 u)^2)\partial^2_{2,2}u + (1+(\partial_2 u)^2)\partial^2_{1,1}u - 2\partial_1 u\,\partial_2 u\,\partial^2_{1,2}u\big) \tag{1.11}$$

and the Gauss curvature at $(x, u(x))$

$$\mathbf{K}_u = \frac{\ell n - m^2}{EG - F^2} = \frac{1}{{g_u}^2}\Big(\partial^2_{1,1}u\,\partial^2_{2,2}u - (\partial^2_{1,2}u)^2\Big). \tag{1.12}$$

We also recall the formulas:

$$\mathbf{H}_u = \frac{1}{2}\,\mathrm{div}\Big[\frac{\nabla u}{\sqrt{g_u}}\Big] = -\frac{1}{2}\,\mathrm{div}\big[({\nu_u}^1, {\nu_u}^2)\big], \qquad \mathbf{K}_u = \det\Big[\nabla\Big(\frac{\nabla u}{\sqrt{g_u}}\Big)\Big] = \det\big[\nabla({\nu_u}^1, {\nu_u}^2)\big]. \tag{1.13}$$

The tangent space at each point in the Gauss graph $\mathcal{GG}_u$ is oriented by the wedge product

$$\xi_u(x) := \partial_1\Phi_u(x) \wedge \partial_2\Phi_u(x), \qquad x \in \Omega.$$

We have

$$\partial_1\Phi_u = (1, 0, \partial_1 u, \partial_1{\nu_u}^1, \partial_1{\nu_u}^2, \partial_1{\nu_u}^3), \quad \partial_2\Phi_u = (0, 1, \partial_2 u, \partial_2{\nu_u}^1, \partial_2{\nu_u}^2, \partial_2{\nu_u}^3) \tag{1.14}$$

and hence, according to the number of ε_j-entries, we can write as in [7] the *stratification*

$$\xi_u = \xi_u^{(0)} + \xi_u^{(1)} + \xi_u^{(2)}$$

where, denoting by $|A|$ the determinant of a 2×2 matrix A, we obtain

$$\begin{aligned} \xi_u^{(0)} &= e_1\wedge e_2 + \partial_2 u\, e_1\wedge e_3 - \partial_1 u\, e_2 \wedge e_3 \\ \xi_u^{(1)} &= \sum_{j=1}^3 \partial_2{\nu_u}^j e_1\wedge\varepsilon_j - \sum_{j=1}^3 \partial_1{\nu_u}^j e_2\wedge\varepsilon_j + \sum_{j=1}^3 \begin{vmatrix} \partial_1 u & \partial_2 u \\ \partial_1{\nu_u}^j & \partial_2{\nu_u}^j \end{vmatrix} e_3\wedge\varepsilon_j \\ \xi_u^{(2)} &= \begin{vmatrix} \partial_1{\nu_u}^1 & \partial_2{\nu_u}^1 \\ \partial_1{\nu_u}^2 & \partial_2{\nu_u}^2 \end{vmatrix} \varepsilon_1\wedge\varepsilon_2 + \begin{vmatrix} \partial_1{\nu_u}^1 & \partial_2{\nu_u}^1 \\ \partial_1{\nu_u}^3 & \partial_2{\nu_u}^3 \end{vmatrix} \varepsilon_1\wedge\varepsilon_3 + \begin{vmatrix} \partial_1{\nu_u}^2 & \partial_2{\nu_u}^2 \\ \partial_1{\nu_u}^3 & \partial_2{\nu_u}^3 \end{vmatrix} \varepsilon_2\wedge\varepsilon_3 . \end{aligned} \tag{1.15}$$

Using (1.2), by the area formula we can write the area $\mathcal{H}^2(\mathcal{GG}_u)$ of the Gauss graph as

$$\int_\Omega \sqrt{g}\sqrt{1 + (4\mathbf{H}^2 - 2\mathbf{K}) + \mathbf{K}^2}\,dx = \int_{\mathcal{G}_u} \sqrt{1 + (4\mathbf{H}^2 - 2\mathbf{K}) + \mathbf{K}^2}\,d\mathcal{H}^2 = \int_\Omega |\xi_u|\,dx \tag{1.16}$$

where $g = g_u$, $\mathbf{H} = \mathbf{H}_u$, and $\mathbf{K} = \mathbf{K}_u$, which yields to the formula for the Jacobian of the map Φ_u

$$J_{\Phi_u} = |\xi_u| = \sqrt{g_u}\,\sqrt{1 + (4\mathbf{H}_u^2 - 2\mathbf{K}_u) + \mathbf{K}_u^2}\,. \tag{1.17}$$

Finally, by (1.1) we infer that

$$|\xi_u^{(0)}|^2 = g_u\,, \quad |\xi_u^{(1)}|^2 = g_u\,(4\mathbf{H}_u^2 - 2\mathbf{K}_u)\,, \quad |\xi_u^{(2)}|^2 = g_u\,\mathbf{K}_u^2 \tag{1.18}$$

whereas a unit 2-vector field orienting the tangent plane to $\mathcal{G}_u$ at $(x, u(x))$ is

$$\tau_u := *\nu_u = \frac{\xi_u^{(0)}}{|\xi_u^{(0)}|} = \frac{1}{\sqrt{1+|\nabla u|^2}}\big(e_1\wedge e_2 + \partial_2 u\, e_1\wedge e_3 - \partial_1 u\, e_2\wedge e_3\big)\,.$$

2 Currents carried by Gauss graphs

In this section we deal with the currents carried by the Gauss graph of smooth Cartesian surfaces, analyzing the curvature energy functional and the integration by parts formulas.

The current $[\![\, \mathcal{GG}_u \,]\!]$ in $\mathcal{R}_2(\mathbb{R}^6)$ of a smooth bounded function $u : \Omega \to \mathbb{R}$ is defined as above, with $\mathcal{M} = \mathcal{G}_u$. However, since the graph $\mathcal{G}_u$ is contained in the cylinder

$$U := \Omega \times \mathbb{R}$$

and its boundary in ∂U, we shall restrict to the action of compactly supported forms in $U \times \mathbb{S}^2$.

Currents carried by Gauss graphs. The i.m. rectifiable current GG_u in $\mathcal{R}_2(U \times \mathbb{S}^2)$ is naturally associated to the Gauss graph by integrating compactly supported smooth 2-forms ω in $U \times \mathbb{S}^2$ on the Gauss graph surface $\mathcal{GG}_u$ w.r.t. the natural orientation

$$\overrightarrow{\zeta_u}((x,z),y) := \frac{\xi_u}{|\xi_u|}(x,z)\,, \qquad ((x,z),y) \in \mathcal{GG}_u$$

so that $GG_u := [\![\, \mathcal{GG}_u, 1, \overrightarrow{\zeta_u} \,]\!]$, see Remark 3.3 below, with finite mass, $\mathbf{M}(GG_u) = \mathcal{H}^2(\mathcal{GG}_u) < \infty$. Therefore, by (1.8) we equivalently have $GG_u = \Phi_{u\#}[\![\, \Omega \,]\!]$, i.e.,

$$\langle GG_u, \omega \rangle = \int_\Omega {\Phi_u}^{\#}\omega \qquad \forall\, \omega \in \mathcal{D}^2(U \times \mathbb{S}^2)\,.$$

We compute for $i = 1,2$ and $j = 1,2,3$

$${\Phi_u}^{\#} dx^i = dx^i\,, \quad {\Phi_u}^{\#} dz = \partial_1 u\, dx^1 + \partial_2 u\, dx^2\,, \quad {\Phi_u}^{\#} dy^j = \partial_1 {\nu_u}^j\, dx^1 + \partial_2 {\nu_u}^j\, dx^2$$

and hence the pull-back of the basis of 2-forms in $\mathbb{R}^3_x \times \mathbb{R}^3_y$ gives the fifteen formulas

$${\Phi_u}^{\#}(dx^1 \wedge dx^2) = dx^1 \wedge dx^2\,, \quad {\Phi_u}^{\#}(dx^1 \wedge dz) = \partial_2 u\, dx^1 \wedge dx^2\,, \quad {\Phi_u}^{\#}(dx^2 \wedge dz) = -\partial_1 u\, dx^1 \wedge dx^2$$

$${\Phi_u}^{\#}(dx^1 \wedge dy^j) = \partial_2 {\nu_u}^j\, dx^1 \wedge dx^2\,, \quad {\Phi_u}^{\#}(dx^2 \wedge dy^j) = -\partial_1 {\nu_u}^j\, dx^1 \wedge dx^2\,, \quad j = 1,2,3$$

$${\Phi_u}^{\#}(dz \wedge dy^j) = \begin{vmatrix} \partial_1 u & \partial_2 u \\ \partial_1 {\nu_u}^j & \partial_2 {\nu_u}^j \end{vmatrix} dx^1 \wedge dx^2\,, \qquad j = 1,2,3$$

$${\Phi_u}^{\#}(dy^{j_1} \wedge dy^{j_2}) = \begin{vmatrix} \partial_1 {\nu_u}^{j_1} & \partial_2 {\nu_u}^{j_1} \\ \partial_1 {\nu_u}^{j_2} & \partial_2 {\nu_u}^{j_2} \end{vmatrix} dx^1 \wedge dx^2\,, \quad 1 \le j_1 < j_2 \le 3\,.$$

Therefore, for each compactly supported smooth function $\psi \in C^\infty_c(U \times \mathbb{S}^2)$ we have

$$\begin{aligned}
\langle GG_u, \psi\, dx^1 \wedge dx^2 \rangle &= \int_\Omega \psi(\Phi_u)\, d\mathcal{L}^2 \\
\langle GG_u, \psi\, dx^1 \wedge dz \rangle &= \int_\Omega \psi(\Phi_u)\, \partial_2 u\, d\mathcal{L}^2 \\
\langle GG_u, \psi\, dx^2 \wedge dz \rangle &= -\int_\Omega \psi(\Phi_u)\, \partial_1 u\, d\mathcal{L}^2 \\
\langle GG_u, \psi\, dx^1 \wedge dy^j \rangle &= \int_\Omega \psi(\Phi_u)\, \partial_2 {\nu_u}^j\, d\mathcal{L}^2\,, \quad j = 1,2,3 \\
\langle GG_u, \psi\, dx^2 \wedge dy^j \rangle &= -\int_\Omega \psi(\Phi_u)\, \partial_1 {\nu_u}^j\, d\mathcal{L}^2\,, \quad j = 1,2,3 \\
\langle GG_u, \psi\, dz \wedge dy^j \rangle &= \int_\Omega \psi(\Phi_u) \begin{vmatrix} \partial_1 u & \partial_2 u \\ \partial_1 {\nu_u}^j & \partial_2 {\nu_u}^j \end{vmatrix} d\mathcal{L}^2\,, \quad j = 1,2,3 \\
\langle GG_u, \psi\, dy^{j_1} \wedge dy^{j_2} \rangle &= \int_\Omega \psi(\Phi_u) \begin{vmatrix} \partial_1 {\nu_u}^{j_1} & \partial_2 {\nu_u}^{j_1} \\ \partial_1 {\nu_u}^{j_2} & \partial_2 {\nu_u}^{j_2} \end{vmatrix} d\mathcal{L}^2\,, \quad 1 \le j_1 < j_2 \le 3\,.
\end{aligned} \tag{2.1}$$

Remark 2.1 Notice that the (x,z)-projection $\Pi_{1\#}GG_u$ agrees with the i.m. rectifiable current G_u in $\mathcal{R}_2(\Omega\times\mathbb{R})$ carried by the graph $\mathcal{G}_u$ of u, i.e.,

$$\Pi_{1\#}\circ\Phi_{u\#}[\![\,\Omega\,]\!]=(\Pi_1\circ\Phi_u)_\#[\![\,\Omega\,]\!]=\varphi_{u\#}[\![\,\Omega\,]\!]=:G_u\,,\quad \varphi_u(x):=(x,u(x)) \tag{2.2}$$

whereas the y-projection agrees with the image current in $\mathcal{R}_2(\mathbb{S}^2)$ of the unit normal ν_u :

$$\Pi_{2\#}GG_u=(\Pi_2\circ\Phi_u)_\#[\![\,\Omega\,]\!]=\nu_{u\#}[\![\,\Omega\,]\!]\,.$$

Since moreover the third component of ν_u is non-negative, actually the support of $\Pi_{2\#}GG_u$ is contained into the upper half-sphere

$$\mathbb{S}^2_+:=\{y\in\mathbb{S}^2\mid y_3\geq 0\} \tag{2.3}$$

and hence the compact support of GG_u is contained in $\overline{U}\times\mathbb{S}^2_+$.

The curvature energy functional. Recalling (1.3), if $\mathcal{M}=\mathcal{G}_u$ for some smooth and bounded function $u:\Omega\to\mathbb{R}$, by the area formula we get $\|\mathcal{G}_u\|=\mathcal{E}(u)$, where we have set

$$\mathcal{E}(u):=\int_\Omega\sqrt{g_u}\,\big(1+\sqrt{4\mathbf{H}_u^2-2\mathbf{K}_u}+|\mathbf{K}_u|\big)\,d\mathcal{L}^2\,. \tag{2.4}$$

On account of (1.16), (1.17), and (1.18), we get

$$\mathcal{E}(u)=\int_\Omega\big(|\xi_u^{(0)}|+|\xi_u^{(1)}|+|\xi_u^{(2)}|\big)\,d\mathcal{L}^2\,,\quad \mathcal{H}^2(\mathcal{G}\mathcal{G}_u)=\int_\Omega|\xi_u|\,d\mathcal{L}^2 \tag{2.5}$$

where $|\xi_u|^2=|\xi_u^{(0)}|^2+|\xi_u^{(1)}|^2+|\xi_u^{(2)}|^2$ and more explicitly, by (1.15),

$$\begin{aligned}
|\xi_u^{(0)}|^2&=g_u=1+|\nabla u|^2\\
|\xi_u^{(1)}|^2&=g_u\,(4\mathbf{H}_u^2-2\mathbf{K}_u)=|\nabla\nu_u|^2+\sum_{j=1}^3\big(\partial_1 u\,\partial_2{\nu_u}^j-\partial_2 u\,\partial_1{\nu_u}^j\big)^2\\
|\xi_u^{(2)}|^2&=g_u\,\mathbf{K}_u^2=\sum_{1\leq j_1<j_2\leq 3}\big(\partial_1{\nu_u}^{j_1}\,\partial_2{\nu_u}^{j_2}-\partial_2{\nu_u}^{j_1}\,\partial_1{\nu_u}^{j_2}\big)^2\,.
\end{aligned} \tag{2.6}$$

Boundary and integration by parts. Stokes' theorem yields that for every compactly supported and smooth 1-form $\eta\in\mathcal{D}^1(U\times\mathbb{S}^2)$

$$\langle\partial GG_u,\eta\rangle:=\langle GG_u,d\eta\rangle=\int_{\mathcal{G}\mathcal{G}_u}d\eta=\int_{\partial\mathcal{G}\mathcal{G}_u}\eta=0\,.$$

Due to the previous computation, this yields to the following list of integration by parts formulas, where we shall use that for u smooth $d{\Phi_u}^\#\eta={\Phi_u}^\#d\eta$. Let $\psi\in C^\infty_c(U\times\mathbb{S}^2)$, where $\psi=\psi(x_1,x_2,z,y_1,y_2,y_3)$.

i) If $\eta=\psi\,dx^1$, then $-d\eta=\dfrac{\partial\psi}{\partial x_2}\,dx^1\wedge dx^2+\dfrac{\partial\psi}{\partial z}\,dx^1\wedge dz+\displaystyle\sum_{j=1}^3\dfrac{\partial\psi}{\partial y_j}\,dx^1\wedge dy^j$, whence

$$\begin{aligned}
d{\Phi_u}^\#(\psi\,dx^1)&=-\Big(\frac{\partial\psi}{\partial x_2}(\Phi_u)+\frac{\partial\psi}{\partial z}(\Phi_u)\,\partial_2 u+\sum_{j=1}^3\frac{\partial\psi}{\partial y_j}(\Phi_u)\,\partial_2{\nu_u}^j\Big)\,dx^1\wedge dx^2\\
&=-\partial_2[\psi(\Phi_u)]\,dx^1\wedge dx^2
\end{aligned}$$

which yields to

$$\langle\partial GG_u,\psi\,dx^1\rangle=-\int_\Omega\partial_2[\psi(\Phi_u)]\,d\mathcal{L}^2=0\,. \tag{2.7}$$

ii) If $\eta=\psi\,dx^2$, then $d\eta=\dfrac{\partial\psi}{\partial x_1}\,dx^1\wedge dx^2-\dfrac{\partial\psi}{\partial z}\,dx^2\wedge dz-\displaystyle\sum_{j=1}^3\dfrac{\partial\psi}{\partial y_j}\,dx^2\wedge dy^j$ and we similarly obtain:

$$\langle\partial GG_u,\psi\,dx^2\rangle=\int_\Omega\partial_1[\psi(\Phi_u)]\,d\mathcal{L}^2=0\,. \tag{2.8}$$

iii) If $\eta = \psi\, dz$, then $d\eta = \dfrac{\partial \psi}{\partial x_1}\, dx^1 \wedge dz + \dfrac{\partial \psi}{\partial x_2}\, dx^2 \wedge dz - \sum_{j=1}^{3} \dfrac{\partial \psi}{\partial y_j}\, dz \wedge dy^j$, whence

$$
\begin{aligned}
d\Phi_u{}^{\#}(\psi\, dz) &= \Big(\frac{\partial \psi}{\partial x_1}(\Phi_u)\, \partial_2 u - \frac{\partial \psi}{\partial x_2}(\Phi_u)\, \partial_1 u - \sum_{j=1}^{3} \frac{\partial \psi}{\partial y_j}(\Phi_u) \begin{vmatrix} \partial_1 u & \partial_2 u \\ \partial_1 \nu_u{}^j & \partial_2 \nu_u{}^j \end{vmatrix} \Big)\, dx^1 \wedge dx^2 \\
&= -\partial_1 u \Big(\frac{\partial \psi}{\partial x_2}(\Phi_u) + \sum_{j=1}^{3} \frac{\partial \psi}{\partial y_j}(\Phi_u)\, \partial_2 \nu_u{}^j \Big)\, dx^1 \wedge dx^2 \\
&\quad + \partial_2 u \Big(\frac{\partial \psi}{\partial x_1}(\Phi_u) + \sum_{j=1}^{3} \frac{\partial \psi}{\partial y_j}(\Phi_u)\, \partial_1 \nu_u{}^j \Big)\, dx^1 \wedge dx^2 \\
&= -\partial_1 u \Big(\partial_2[\psi(\Phi_u)] - \frac{\partial \psi}{\partial z}(\Phi_u)\, \partial_2 u \Big)\, dx^1 \wedge dx^2 \\
&\quad + \partial_2 u \Big(\partial_1[\psi(\Phi_u)] - \frac{\partial \psi}{\partial z}(\Phi_u)\, \partial_1 u \Big)\, dx^1 \wedge dx^2 \\
&= \Big(-\partial_1 u\, \partial_2[\psi(\Phi_u)] + \partial_2 u\, \partial_1[\psi(\Phi_u)] \Big)\, dx^1 \wedge dx^2
\end{aligned}
$$

which yields to

$$
\langle \partial GG_u, \psi\, dz \rangle = \int_\Omega \Big(-\partial_1 u\, \partial_2[\psi(\Phi_u)] + \partial_2 u\, \partial_1[\psi(\Phi_u)] \Big)\, d\mathcal{L}^2 = 0\,. \tag{2.9}
$$

iv) If $\eta = \psi\, dy^1$, then $d\eta = \sum_{i=1}^{2} \dfrac{\partial \psi}{\partial x_i}\, dx^i \wedge dy^1 + \dfrac{\partial \psi}{\partial z}\, dz \wedge dy^1 - \sum_{j=2}^{3} \dfrac{\partial \psi}{\partial y_j}\, dy^1 \wedge dy^j$, whence

$$
\begin{aligned}
d\Phi_u{}^{\#}(\psi\, dy^1) = {} & \Big(\frac{\partial \psi}{\partial x_1}(\Phi_u)\, \partial_2 \nu_u{}^1 - \frac{\partial \psi}{\partial x_2}(\Phi_u)\, \partial_1 \nu_u{}^1 \\
& + \frac{\partial \psi}{\partial z}(\Phi_u) \begin{vmatrix} \partial_1 u & \partial_2 u \\ \partial_1 \nu_u{}^1 & \partial_2 \nu_u{}^1 \end{vmatrix} - \sum_{j=2}^{3} \frac{\partial \psi}{\partial y_j}(\Phi_u) \begin{vmatrix} \partial_1 \nu_u{}^1 & \partial_2 \nu_u{}^1 \\ \partial_1 \nu_u{}^j & \partial_2 \nu_u{}^j \end{vmatrix} \Big)\, dx^1 \wedge dx^2 \\
= {} & -\partial_1 \nu_u{}^1 \Big(\frac{\partial \psi}{\partial x_2}(\Phi_u) + \frac{\partial \psi}{\partial z}(\Phi_u)\, \partial_2 u + \sum_{j=2}^{3} \frac{\partial \psi}{\partial y_j}(\Phi_u)\, \partial_2 \nu_u{}^j \Big)\, dx^1 \wedge dx^2 \\
& + \partial_2 \nu_u{}^1 \Big(\frac{\partial \psi}{\partial x_1}(\Phi_u) + \frac{\partial \psi}{\partial z}(\Phi_u)\, \partial_1 u + \sum_{j=2}^{3} \frac{\partial \psi}{\partial y_j}(\Phi_u)\, \partial_1 \nu_u{}^j \Big)\, dx^1 \wedge dx^2 \\
= {} & -\partial_1 \nu_u{}^1 \Big(\partial_2[\psi(\Phi_u)] - \frac{\partial \psi}{\partial y_1}(\Phi_u)\, \partial_2 \nu_u{}^1 \Big)\, dx^1 \wedge dx^2 \\
& + \partial_2 \nu_u{}^1 \Big(\partial_1[\psi(\Phi_u)] - \frac{\partial \psi}{\partial y_1}(\Phi_u)\, \partial_1 \nu_u{}^1 \Big)\, dx^1 \wedge dx^2 \\
= {} & \Big(-\partial_1 \nu_u{}^1\, \partial_2[\psi(\Phi_u)] + \partial_2 \nu_u{}^1\, \partial_1[\psi(\Phi_u)] \Big)\, dx^1 \wedge dx^2\,.
\end{aligned}
$$

Arguing in a similar way for $j = 1, 2, 3$, we get

$$
\langle \partial GG_u, \psi\, dy^j \rangle = \int_\Omega \Big(-\partial_1 \nu_u{}^j\, \partial_2[\psi(\Phi_u)] + \partial_2 \nu_u{}^j\, \partial_1[\psi(\Phi_u)] \Big)\, d\mathcal{L}^2 = 0\,. \tag{2.10}
$$

3 Relaxed energy and weak limit currents

In this section we introduce the relaxed curvature energy functional and discuss some general properties of functions u with finite relaxed energy. It turns out that the current carried by the Gauss graph of u is well-defined. We then analyze the weak limits Σ of sequences of currents carried by the Gauss graph of smooth functions with equibounded curvature energies. We shall see that the corresponding weak limit currents Σ retain some information from the L^1-limit function u. We then introduce a curvature functional on the currents Σ that agrees with the curvature energy $\mathcal{E}(u)$ when $\Sigma = GG_u$ for some smooth function u. Its relationship with the relaxed energy is finally discussed.

THE RELAXED ENERGY. We define for any $u \in L^\infty(\Omega)$

$$\overline{\mathcal{E}}(u) := \inf\{\liminf_{h\to\infty} \mathcal{E}(u_h) \mid \{u_h\}_h \subset C^2(\Omega)\,,\ u_h \to u \text{ strongly in } L^1(\Omega)\} \tag{3.1}$$

and we correspondingly denote by

$$\mathcal{E}(\Omega) := \{u \in L^\infty(\Omega) \mid \overline{\mathcal{E}}(u) < \infty\}$$

the class of L^∞-functions with finite relaxed energy. In the above definition, by a standard density argument one can restrict to consider uniformly bounded approximating sequences in $C^2_b(\Omega)$. Moreover, by lower-semicontinuity we clearly have:

$$\mathcal{E}(u) = \overline{\mathcal{E}}(u) \qquad \forall\, u \in C^2_b(\Omega)\,.$$

Notice that since for smooth functions $\mathcal{E}(u_h) \geq \int_\Omega \sqrt{1+|\nabla u_h|^2}\, d\mathcal{L}^2$, then it turns out that any function with finite relaxed energy has bounded variation, whence

$$\mathcal{E}(\Omega) \subset BV(\Omega)\,.$$

Therefore, the unit normal ν_u is well-defined a.e. by (1.7), but in terms of the approximate gradient ∇u, whose components will be denoted by $\partial_i u$. We refer to [5, 17] for the main properties of BV-functions.

By means of a slicing argument, and using results from [2], we also obtain the following.

Theorem 3.1 *Let $u \in \mathcal{E}(\Omega)$ and let $\{u_h\}_h \subset C^2_b(\Omega)$ be such that $u_h \to u$ strongly in $L^1(\Omega)$ and $\sup_h(\mathcal{E}(u_h)+\|u_h\|_\infty) < \infty$. Then $\Phi_{u_h}(x)$ converges to $\Phi_u(x)$ weakly in the BV-sense, and hence strongly in L^1. Therefore, the approximate unit normal ν_u is a function of bounded variation, $\nu_u \in BV(\Omega,\mathbb{S}^2)$.*

PROOF: We already know that $u_h \rightharpoonup u$ weakly in the BV-sense. We now claim that the gradient ∇u_h converges $\mathcal{L}^2 \llcorner \Omega$-a.e. to the approximate gradient ∇u. By the claim we deduce the $\mathcal{L}^2$-a.e. convergence of ν_{u_h} to ν_u. Since moreover $\sup_h \int_\Omega |\nabla \nu_{u_h}|\, dx < \infty$, by compactness we infer that ν_{u_h} converges to ν_u weakly in the BV-sense, whence $\nu_u \in BV(\Omega,\mathbb{S}^2)$.

In order to prove the claim, we denote by $p_i : \mathbb{R}^2_x \to \mathbb{R}$ the orthogonal projection $p_i(x) = x_i$. For any $x_1 \in p_1(\Omega)$, consider the slices $t \mapsto u_h(x_1,t)$ and the corresponding Cartesian curves

$$c_h(t) := (t, u_h(x_1,t))\,, \qquad t \in I_{x_1} := \{x_2 \in \mathbb{R} \mid (x_1,x_2) \in \Omega\}\,.$$

Since the principal curvatures $\mathbf{k}_1(h)$, $\mathbf{k}_2(h)$ of the graph surfaces $\mathcal{G}_{u_h}$ bound the curvature $\mathbf{k}_{c_h}$ of the Cartesian curve $c_h(t)$, along the curve c_h we have

$$\Big(1 + (\mathbf{k}_1(h)^2 + \mathbf{k}_2(h)^2) + (\mathbf{k}_1(h)\,\mathbf{k}_2(h))^2\Big)^{1/2} \geq \sqrt{1+|\mathbf{k}_{c_h}|^2}\,.$$

Therefore, by a slicing argument we deduce that

$$\sup_h \int_{c_h} (1+|\mathbf{k}_{c_h}|)\, d\mathcal{H}^1 < \infty \qquad \text{for } \mathcal{L}^1\text{-a.e. } x_1 \in p_1(\Omega)\,.$$

As a consequence, for $\mathcal{L}^1$-a.e. such x_1 the Cartesian curve $t \mapsto c(t) := (t, u(x_1,t))$ has finite relaxed energy in the sense of [2]. In particular, we infer that the derivatives $\dot c_h(t)$ converge a.e. in I_{x_1} to the approximative derivative $\dot c(t)$. This yields the convergence $\mathcal{L}^2$-a.e. in Ω of the partial derivative $\partial_1 u_h$ to the first component $\partial_1 u$ of the approximate gradient ∇u of the BV-function u. The same property holds true for the second derivative $\partial_2 u$, as required. $\square$

THE ABSOLUTELY CONTINUOUS COMPONENT. For $u \in \mathcal{E}(\Omega)$, we have just seen that the function $\Phi_u(x) := (x, u(x), \nu_u(x))$ has bounded variation. Therefore, similarly to the smooth case, we may define the current $GG^a_u \in \mathcal{D}_2(U \times \mathbb{S}^2)$ as

$$GG^a_u := \Phi_{u\#}[\![\, \Omega\,]\!] \tag{3.2}$$

where this time the pull-back of 2-forms is computed by means of the approximate gradient of Φ_u, given similarly to the smooth case by $\nabla\Phi_u = (\partial_1\Phi_u, \partial_2\Phi_u)$, see (1.9), so that (1.14) holds for a.e. $x \in \Omega$. Whence the formulas (2.1) continue to hold, with GG_u^a instead of GG_u, but this time in terms of the approximate gradient of Φ_u. In particular, as in Remark 2.1 we deduce that (2.2) holds, where $G_u := \varphi_{u\#}[\![\,\Omega\,]\!]$ is the current carried by the rectifiable graph of u in the sense of Giaquinta-Modica-Souček [17]. We also recall that due to the codimension one, there is a unique *Cartesian current* T_u in $\mathrm{cart}(\Omega\times\mathbb{R})$ that "fills the holes" of the rectifiable graph. It is given by the boundary of the (naturally oriented) subgraph of u, i.e.,

$$T_u := \partial[\![\,S\mathcal{G}_u\,]\!] \quad \text{on } \mathcal{D}^2(\Omega\times\mathbb{R}), \qquad S\mathcal{G}_u := \{(x,z)\in\Omega\times\mathbb{R} \mid z < u(x)\}\,. \tag{3.3}$$

It is well-known from Federer's work [14] that the current $\partial[\![\,S\mathcal{G}_u\,]\!]$ is i.m. rectifiable in $\mathcal{R}_2(\Omega\times\mathbb{R})$ if and only if $u \in BV(\Omega)$. Moreover, compare [17], the graph current G_u satisfies the null-boundary condition $(\partial G_u)\llcorner\Omega\times\mathbb{R} = 0$ if and only if $u \in W^{1,1}(\Omega)$. Therefore, for Sobolev functions we have $G_u = \partial[\![\,S\mathcal{G}_u\,]\!]$, whereas for general BV-functions we may decompose $T_u = G_u + T_u^s$, where the second component is determined by u and by the singular component $D^s u$ of its distributional derivative.

Weak limit currents. Let $\{u_h\}_h \subset C_b^2(\Omega)$ be a sequence satisfying $\sup_h \mathcal{E}(u_h) < \infty$ and $\sup_h \|u_h\|_\infty < \infty$. Then $\{GG_{u_h}\}_h$ is a sequence of i.m. rectifiable currents in $\mathcal{R}_2(U\times\mathbb{S}^2)$ with no inner boundary, $\partial GG_{u_h} = 0$ on $\mathcal{D}^1(U\times\mathbb{S}^2)$ for each h, and equibounded masses, $\sup_h \mathbf{M}(GG_{u_h}) < \infty$, as

$$\frac{1}{2}\,\mathcal{E}(u_h) \le \mathcal{H}^2(\mathcal{G}\mathcal{G}_{u_h}) = \mathbf{M}(GG_{u_h}) \le \mathcal{E}(u_h) \qquad \forall\, h\,.$$

Therefore, by Federer-Fleming's closure theorem [15], possibly passing to a subsequence the currents GG_{u_h} weakly converge in $\mathcal{D}_2(U\times\mathbb{S}^2)$ to some i.m. rectifiable current $\Sigma \in \mathcal{R}_2(U\times\mathbb{S}^2)$ with finite mass, $\mathbf{M}(\Sigma) < \infty$, and no inner boundary, $\partial\Sigma = 0$ on $\mathcal{D}^1(U\times\mathbb{S}^2)$. By Remark 2.1, we also deduce that the support of Σ is a compact set contained in $\overline{U}\times\mathbb{S}^2_+$, where $\mathbb{S}^2_+$ the upper half-sphere given by (2.3).

Since moreover $\Pi_{1\#}GG_{u_h} = G_{u_h}$, and $\sup_h \mathbf{M}(G_{u_h}) < \infty$, we deduce that $\{u_h\}_h$ weakly converges in the BV-sense to a function u with finite relaxed energy, $u \in \mathcal{E}(\Omega)$, and $\{G_{u_h}\}_h$ weakly converges in $\mathcal{D}_2(\Omega\times\mathbb{R})$ to the corresponding Cartesian current T_u, see (3.3). Therefore, we have:

$$\Pi_{1\#}\Sigma = T_u := \partial[\![\,S\mathcal{G}_u\,]\!] \in \mathrm{cart}(\Omega\times\mathbb{R})\,.$$

This yields that we may and do decompose

$$\Sigma = GG_u^a + \Sigma^s \tag{3.4}$$

where $u \in \mathcal{E}(\Omega)$ and Σ^s is an i.m. rectifiable current in $\mathcal{R}_2(U\times\mathbb{S}^2)$ satisfying:

$$\Pi_{1\#}GG_u^a = G_u\,, \quad \Pi_{1\#}\Sigma^s = T_u^s := T_u - G_u\,.$$

Since $(\partial\Sigma)\llcorner U\times\mathbb{S}^2 = 0$, the following null-boundary condition holds:

$$\langle \partial GG_u^a, \eta\rangle + \langle \partial\Sigma^s, \eta\rangle = 0 \qquad \forall\,\eta \in \mathcal{D}^1(U\times\mathbb{S}^2)\,. \tag{3.5}$$

Furthermore, the singular component Σ^s is "vertical" in the sense that

$$\Sigma^s(\psi\,dx^1\wedge dx^2) = 0 \qquad \forall\,\psi \in C_c^\infty(U\times\mathbb{S}^2)\,. \tag{3.6}$$

In fact, by the weak BV-convergence of Φ_{u_h} to Φ_u as $h\to\infty$, for any test function ψ we get:

$$\langle GG_{u_h}, \psi\,dx^1\wedge dx^2\rangle = \int_\Omega \psi(\Phi_{u_h})\,d\mathcal{L}^2 \to \int_\Omega \psi(\Phi_u)\,d\mathcal{L}^2 = \langle GG_u^a, \psi\,dx^1\wedge dx^2\rangle\,.$$

In a similar way, by (2.1) we recover for each $\psi \in C_c^\infty(U\times\mathbb{S}^2)$ the formulas

$$\begin{array}{ll} \langle \Sigma^s, \psi\,dx^1\wedge dz\rangle = \langle D_2^s u, \psi(\Phi_u)\rangle\,, & \langle \Sigma^s, \psi\,dx^2\wedge dz\rangle = -\langle D_1^s u, \psi(\Phi_u)\rangle\,, \\ \langle \Sigma^s, \psi\,dx^1\wedge dy^j\rangle = \langle D_2^s \nu_u{}^j, \psi(\Phi_u)\rangle\,, & \langle \Sigma^s, \psi\,dx^2\wedge dy^j\rangle = -\langle D_1^s \nu_u{}^j, \psi(\Phi_u)\rangle\,, \qquad j=1,2,3 \end{array} \tag{3.7}$$

which are to be intended by decomposing the singular part $D^s v$ of the weak derivative of a BV-function $v \in BV(\Omega)$ into the Jump and Cantor components: $D^s v = D^J v + D^C v$, and $D_i^s v = D^s v \bullet e_i$.

Remark 3.2 Due to the higher codimension, in general the action of the singular component Σ^s on forms of the type $\psi\, dz \wedge dy^j$ and $\psi\, dy^{j_1} \wedge dy^{j_2}$ is not identified by the function Φ_u or by the weak limits (in the sense of the measures) of the determinants appearing in the last formulas from (2.1).

We now recover the geometric properties from Theorem 1.2. They are inherited through the weak convergence as currents by the analogous ones for smooth functions. More precisely, if $u \in C^2_b(\Omega)$ we readily check:

i) $\langle GG_u, \eta \wedge \varphi \rangle = 0$ for each $\eta \in \mathcal{D}^1(U \times \mathbb{S}^2)$;

ii) $\langle GG_u, \psi\, \varphi^* \rangle \geq 0$ for each $\psi \in C(U \times \mathbb{S}^2)$ such that $\psi \geq 0$.

In fact, recalling the definition (1.6) of the canonical forms, we compute:

$$\Phi_u{}^{\#}\varphi = \Phi_u{}^{\#}(y^1 dx^1 + y^2 dx^2 + y^3 dz) = (\nu_u{}^1 dx^1 + \nu_u{}^2 dx^2 + \nu_u{}^3(\partial_1 u\, dx^1 + \partial_2 u\, dx^2)) = 0$$

so that i) holds. As to condition ii), we compute

$$\begin{aligned}\Phi_u{}^{\#}\varphi^* = {} & \Phi_u{}^{\#}(y^1 dx^2 \wedge dz + y^2 dz \wedge dx^1 + y^3 dx^1 \wedge dx^2) \\ = {} & \nu_u{}^1 dx^2 \wedge (\partial_1 u\, dx^1 + \partial_2 u\, dx^2) + \nu_u{}^2\, (\partial_1 u\, dx^1 + \partial_2 u\, dx^2) \wedge dx^1 + \nu_u{}^3 dx^1 \wedge dx^2 \\ = {} & g_u^{1/2} dx^1 \wedge dx^2\end{aligned}$$

so that we get

$$\langle GG_u, \psi\, \varphi^* \rangle = \int_\Omega \sqrt{g_u} \cdot \psi(\Phi_u)\, d\mathcal{L}^2 \qquad \forall\, \psi \in C(U \times \mathbb{S}^2)\,. \tag{3.8}$$

Remark 3.3 Since $\Sigma \in \mathcal{R}_2(U \times \mathbb{S}^2)$, there exist a 2-rectifiable set $R \subset U \times \mathbb{S}^2$, an $\mathcal{H}^2 \llcorner R$-summable multiplicity function $\theta : R \to \mathbb{N}^+$, and an $\mathcal{H}^2 \llcorner R$-measurable unit 2-vector field $\overrightarrow{\zeta} : R \to \bigwedge^2(\mathbb{R}^3_{(x,z)} \times \mathbb{R}^3_y)$ orienting the approximate tangent 2-space $T_{((x,z),y)}R$ at $\mathcal{H}^2$-a.e. point $((x,z),y) \in R$, such that

$$\langle \Sigma, \omega \rangle = \int_R \theta\, \langle \omega, \overrightarrow{\zeta} \rangle\, d\mathcal{H}^2 \qquad \forall\, \omega \in \mathcal{D}^2(U \times \mathbb{S}^2)\,.$$

In this case, we shall write $\Sigma = [\![\, R, \theta, \overrightarrow{\zeta}\,]\!]$. Moreover, similarly to (1.15), for future use we stratify the orienting unit 2-vector field as $\overrightarrow{\zeta} = \zeta^{(0)} + \zeta^{(1)} + \zeta^{(2)}$, whence

$$\zeta^{(0)} = \sum_{1 \leq i < j \leq 3} \zeta_{i,j}\, e_i \wedge e_j\,, \quad \zeta^{(1)} = \sum_{i,j=1}^{3} \zeta_i^j\, e_i \wedge \varepsilon_j\,, \quad \zeta^{(2)} = \sum_{1 \leq i < j \leq 3} \zeta^{i,j}\, \varepsilon_i \wedge \varepsilon_j\,. \tag{3.9}$$

If e.g. $\Sigma = GG_u$ for some $u \in C^2_b(\Omega)$, we have $\overrightarrow{\zeta} = \xi_u/|\xi_u|$, so that we get $\zeta_{1,1} = |\xi_u|^{-1} > 0$ for each $P = ((x,z),y) \in R$, as $R = \mathcal{GG}_u$. We thus denote

$$R_+ := \{P \in R \mid \zeta_{1,1}(P) > 0\}$$

and observe that if Σ is a weak limit current as above, so that (3.4) holds, then $\mathcal{H}^2$-a.e. point in the rectifiable set corresponding to the component GG^a_u clearly belongs to the set R_+. Furthermore, arguing in a way very similar to [17, Thm. 2, Sec. 4.2.3], it can be checked that $\Sigma \llcorner R_+ = GG^a_u$.

In conclusion, we may and do introduce the class

$$\begin{aligned}\mathrm{Gcart}(U \times \mathbb{S}^2) := {} & \{\Sigma \in \mathcal{D}_2(U \times \mathbb{S}^2) \mid \text{ there exists } \{u_h\} \subset C^2_b(\Omega) \text{ such that} \\ & GG_{u_h} \rightharpoonup \Sigma \text{ in } \mathcal{D}_2(U \times \mathbb{S}^2)\,,\ \sup_h(\, \mathbf{M}(GG_{u_h}) + \|u_h\|_\infty) < \infty\}\end{aligned} \tag{3.10}$$

for which the following structure properties have just been proved:

Theorem 3.4 *Let* $\Sigma \in \mathrm{Gcart}(U \times \mathbb{S}^2)$, *where* $U := \Omega \times \mathbb{R}$. *Then:*

i) Σ *is an i.m. rectifiable* 2*-current in* $\mathcal{R}_2(U \times \mathbb{S}^2)$ *, so that* $\Sigma = [\![\, R, \theta, \overrightarrow{\zeta}\,]\!]$ *, see Remark* 3.3 *;*

ii) *the current* Σ *has finite mass,* $\mathbf{M}(\Sigma)=\int_R \theta\, d\mathcal{H}^2<\infty$, *compact support* $\operatorname{spt}\subset \overline{U}\times\mathbb{S}^2_+$, *see* (2.3), *and no inner boundary,* $(\partial\Sigma)\llcorner U\times\mathbb{S}^2=0$ *;*

iii) $\Sigma=GG_u^a+\Sigma^s$ *for some* $u\in\mathcal{E}(\Omega)$, *where the absolute component is* $GG_u^a:=\Phi_{u\#}[\![\,\Omega\,]\!]$, *so that* (2.1) *holds (in the approximate sense, with* GG_u^a *for* GG_u *);*

iv) *on account of* (3.9), *we have:*

$$\Sigma\llcorner R_+=GG_u^a\,,\quad \textit{where}\quad R_+:=\{P\in R\mid \zeta_{1,1}(P)>0\}\,; \tag{3.11}$$

v) *the singular component* Σ^s *satisfies the "verticality" condition* (3.6) *and formulas* (3.7) *;*

vi) $\langle\Sigma,\eta\wedge\varphi\rangle=0$ *for each* $\eta\in\mathcal{D}^1(U\times\mathbb{S}^2)$, *see* (1.6) *;*

vii) $\langle\Sigma,\psi\,\varphi^*\rangle\geq 0$ *for each* $\psi\in C(U\times\mathbb{S}^2)$ *such that* $\psi\geq 0$ *.*

Remark 3.5 Arguing as in the smooth case, it turns out that both properties vi) and vii) hold true for the absolutely continuous component GG_u^a. Whence they are satisfied by the singular component Σ^s, too. Property vi), which is equivalent to the orthogonality condition described in Remark 1.3, makes sense only if the approximate tangent space $T_{((x,z),y)}R$ is not "completely vertical". More precisely, at points $((x,z),y)$ in the 2-rectifiable set R where $T_{((x,z),y)}R$ is orthogonal to the "horizontal" directions $e_1,\,e_2,e_3$, the orthogonality condition

$$v\bullet(y,0_{\mathbb{R}^3})=0\quad \forall\, v\in T_{((x,z),y)}R$$

is trivially satisfied and hence it gives no information on the geometry of the possible "completely vertical" components of Σ. Moreover, as to the positivity condition vii), by (3.8), and taking a good representative for the BV-map Φ_u, we get for every $\psi\in C(U\times\mathbb{S}^2)$

$$\langle GG_u^a,\psi\,\varphi^*\rangle=\int_\Omega\psi(\Phi_u)\sqrt{1+|\nabla u|^2}\,d\mathcal{L}^2\,,\qquad \langle\Sigma^s,\psi\,\varphi^*\rangle=\int_\Omega\psi(\Phi_u)\,d|D^su|\,.$$

Boundary of Gauss graphs. Let $\Sigma\in\operatorname{Gcart}(U\times\mathbb{S}^2)$, so that (3.4) holds for some $u\in\mathcal{E}(\Omega)$. Concerning the boundary of the absolutely continuous component GG_u^a, according to (2.7), (2.8), (2.9), and (2.10), by the definition (3.2) it turns out that for every test function $\psi\in C_c^\infty(U\times\mathbb{S}^2)$ the following six formulas hold:

$$\begin{aligned}
\langle\partial GG_u^a,\psi\,dx^1\rangle&=-\int_\Omega\partial_2[\psi(\Phi_u)]\,d\mathcal{L}^2\\
\langle\partial GG_u^a,\psi\,dx^2\rangle&=\int_\Omega\partial_1[\psi(\Phi_u)]\,d\mathcal{L}^2\\
\langle\partial GG_u^a,\psi\,dz\rangle&=\int_\Omega\Big(-\partial_1u\,\partial_2[\psi(\Phi_u)]+\partial_2u\,\partial_1[\psi(\Phi_u)]\Big)\,d\mathcal{L}^2\\
\langle\partial GG_u^a,\psi\,dy^j\rangle&=\int_\Omega\Big(-\partial_1{\nu_u}^j\,\partial_2[\psi(\Phi_u)]+\partial_2{\nu_u}^j\,\partial_1[\psi(\Phi_u)]\Big)\,d\mathcal{L}^2\,,\qquad j=1,2,3\,.
\end{aligned}\tag{3.12}$$

As we shall see in Sec. 6 below, in general the boundary current ∂GG_u^a is non-trivial, even if u is a Sobolev function in $W^{1,1}(\Omega)$. Therefore, by (3.5) we deduce that in general the singular component Σ^s is non-trivial, even if for Sobolev functions we have seen that $T_u=G_u$ and hence $\Pi_{1\#}\Sigma^s=T_u^s=0$.

The generalized curvature functional. Let now $\Sigma\in\mathcal{R}_2(U\times\mathbb{S}^2)$, so that we can write $\Sigma=[\![\,R,\theta,\overrightarrow{\zeta}\,]\!]$, see Remark 3.3. On account of the stratification (3.9) of the orienting unit 2-vector field $\overrightarrow{\zeta}=\zeta^{(0)}+\zeta^{(1)}+\zeta^{(2)}$, we define:

$$\mathrm{E}(\Sigma):=\mathrm{E}_1(\Sigma)+\mathrm{E}_2(\Sigma)+\mathrm{E}_3(\Sigma)\,,\qquad \mathrm{E}_i(\Sigma):=\int_R\theta\,|\zeta^{(i)}|\,d\mathcal{H}^2\,,\qquad i=0,1,2\,. \tag{3.13}$$

Since the mass of Σ is $\mathbf{M}(\Sigma)=\int_R\theta\,d\mathcal{H}^2$, we clearly have

$$\frac{1}{2}\,\mathrm{E}(\Sigma)\leq\mathbf{M}(\Sigma)\leq\mathrm{E}(\Sigma)<\infty\qquad\forall\,\Sigma\in\mathcal{R}_2(U\times\mathbb{S}^2)\,.$$

When considering currents in the subclass $\mathrm{Gcart}(U \times \mathbb{S}^2)$, the functional $\Sigma \mapsto \mathrm{E}(\Sigma)$ may be called the *generalized curvature functional*, as we actually have:

$$\mathrm{E}(GG_u) = \mathcal{E}(u) \qquad \forall\, u \in C^2_b(\Omega)\,. \tag{3.14}$$

In this case, in fact, with the above notation we can write $R = \mathcal{G}\mathcal{G}_u$, $\theta \equiv 1$, and $\overrightarrow{\zeta} = \overrightarrow{\zeta_u}$ where, we recall, $\overrightarrow{\zeta_u} = \xi_u/|\xi_u|$, with $|\xi_u| = J_{\Phi_u} = |\partial_{x_1}\Phi_u \wedge \partial_{x_2}\Phi_u|$. Therefore, by the area formula we get

$$\mathrm{E}_i(GG_u) := \int_{\mathcal{G}\mathcal{G}_u} |\eta_u^{(i)}|\, d\mathcal{H}^2 = \int_\Omega |\xi_u^{(i)}|\, d\mathcal{L}^2\,, \quad i = 0,1,2 \tag{3.15}$$

and hence formula (3.14) follows from (2.5).

By the definition, it is readily checked that the above functional is lower semi-continuous along sequences of i.m. rectifiable currents in $\mathcal{R}_2(U \times \mathbb{S}^2)$ weakly converging to some current in $\mathcal{R}_2(U \times \mathbb{S}^2)$. In particular, if $\Sigma \in \mathrm{Gcart}(U \times \mathbb{S}^2)$ and $\{u_h\} \subset C^2_b(\Omega)$ satisfies $GG_{u_h} \rightharpoonup \Sigma$ in $\mathcal{D}_2(U \times \mathbb{S}^2)$, with $\sup_h(\mathbf{M}(GG_{u_h}) + \|u_h\|_\infty) < \infty$, we deduce that

$$\mathrm{E}(\Sigma) \le \liminf_{h\to\infty} \mathcal{E}(u_h) < \infty\,.$$

Remark 3.6 Finally, since $\Sigma = GG_u^a + \Sigma^s$ for some $u \in \mathcal{E}(\Omega)$, and (3.11) holds, we can decompose

$$\mathbf{M}(\Sigma) = \mathbf{M}(GG_u^a) + \mathbf{M}(\Sigma^s)\,, \qquad \mathrm{E}(\Sigma) = \mathrm{E}(GG_u^a) + \mathrm{E}(\Sigma^s)$$

where $\mathrm{E}(GG_u^a) = \mathcal{E}(u)$, the functional $\mathcal{E}(u)$ being defined as in (2.4), but in terms of the approximate derivatives of u. The energy contribution of the singular component Σ^s will be computed in Appendix B, referring to Example 7.1 below.

Further properties. On account of definition (3.10) and of the decomposition formula (3.4), we shall denote for any $u \in \mathcal{E}(\Omega)$

$$\mathrm{Gcart}_u := \{\Sigma \in \mathrm{Gcart}(U \times \mathbb{S}^2) \mid \Sigma = GG_u^a + \Sigma^s\} \tag{3.16}$$

the subclass of currents in Gcart with "underlying function" equal to u.

By the previous arguments, it turns out that Gcart_u is a non-empty class for each $u \in \mathcal{E}(\Omega)$. Moreover, by lower-semicontinuity we actually have:

$$\overline{\mathcal{E}}(u) \ge \inf\{\mathrm{E}(\Sigma) \mid \Sigma \in \mathrm{Gcart}_u\} \qquad \forall\, u \in \mathcal{E}(\Omega)\,. \tag{3.17}$$

However, we shall see that in general the equality is violated in (3.17) and hence a *gap phenomenon* holds. More precisely, in Example 8.2 below we shall find a piecewise constant BV-function $u \in \mathcal{E}(\Omega)$ such that

$$\overline{\mathcal{E}}(u) > \inf\{\mathrm{E}(\Sigma) \mid \Sigma \in \mathrm{Gcart}_u\}\,. \tag{3.18}$$

Remark 3.7 The relaxed energy is linked to a related relaxed energy on currents. More precisely, for each current $\Sigma \in \mathrm{Gcart}(U \times \mathbb{S}^2)$ we introduce the relaxed energy w.r.t. the weak convergence as currents

$$\overline{\mathrm{E}}(\Sigma) := \inf\{\liminf_{h\to\infty} \mathcal{E}(u_h) \mid \{u_h\}_h \subset C^2_b(\Omega)\,,\ GG_{u_h} \rightharpoonup \Sigma \text{ weakly in } \mathcal{D}_2(U \times \mathbb{S}^2)\}$$

so that by lower semicontinuity $\overline{\mathrm{E}}(\Sigma) \ge \mathrm{E}(\Sigma)$ for each Σ, and we readily check the following property:

Proposition 3.8 *For every* $u \in \mathcal{E}(\Omega)$ *we have*

$$\overline{\mathcal{E}}(u) = \inf\{\overline{\mathrm{E}}(\Sigma) \mid \Sigma \in \mathrm{Gcart}_u\}\,.$$

Proof: For any sequence $\{u_h\}_h \subset C^2(\Omega)$ converging to u in $L^1(\Omega)$ and satisfying $\sup_h(\mathcal{E}(u_h) + \|u_h\|_\infty) < \infty$, by closure-compactness, possibly passing to a subsequence the Gauss graphs GG_{u_h} weakly converge to some current $\Sigma \in \mathrm{Gcart}(U \times \mathbb{S}^2)$, whence $\overline{\mathrm{E}}(\Sigma) \le \liminf_h \mathcal{E}(u_h)$. Since actually $\Sigma \in \mathrm{Gcart}_u$, the inequality $\ge$ readily follows. On the other hand, if $GG_{u_h} \rightharpoonup \Sigma$ and $\Sigma \in \mathrm{Gcart}_u$, then $u_h \to u$ strongly in L^1, whence the opposite equality $\le$ holds, too. $\square$

4 Generalized curvature measures

In this section we discuss the notion of generalized first and second symmetric curvatures from [7] in the framework of weak limits of Gauss graphs of smooth Cartesian surfaces. We shall give an explicit computation in the case of two surfaces with line or point singularities: the union of two rectangles meeting at an edge, Example 4.3, and the lateral surface of a cone, Example 4.4. Referring to these two models, we shall then compare our definitions with the notion by Sullivan [24] of *mean and Gauss curvature for polyhedral surfaces* and 2-rectifiable sets. We then discuss the relation with the Gauss-Bonnet theorem. Finally, we introduce the distributional definition of weak and Gauss curvatures, showing that they fail to be good geometric objects.

GENERALIZED CURVATURE MEASURES. Following (1.4), if $\mathcal{M} = \mathcal{G}_u$ for some smooth function $u \in C^2_b(\Omega)$, we shall denote

$$\chi_1^{\mathcal{G}_u} := -\Phi_{u\#}(\mathbf{H}_u\, \mathcal{H}^2 \llcorner \mathcal{G}_u)\,, \qquad \chi_2^{\mathcal{G}_u} := \Phi_{u\#}(\mathbf{K}_u\, \mathcal{H}^2 \llcorner \mathcal{G}_u)\,. \tag{4.1}$$

In terms of the Gauss graph current GG_u, on account of (1.5) we thus get

$$\langle \chi_i^{\mathcal{G}_u}, \psi\rangle = (-1)^i \langle GG_u, \psi\, \Theta_i\rangle \qquad \forall\, \psi \in C^\infty_c(U\times\mathbb{S}^2)\,, \qquad i=1,2\,.$$

In fact, recalling that $GG_u = \Phi_{u\#}[\![\, \Omega\,]\!]$ we have:

Proposition 4.1 *If $u \in C^2_b(\Omega)$, then*

$$-{\Phi_u}^{\#}\Theta_1 = \sqrt{g_u}\, \mathbf{H}_u\, dx^1\wedge dx^2\,, \qquad {\Phi_u}^{\#}\Theta_2 = \sqrt{g_u}\, \mathbf{K}_u\, dx^1\wedge dx^2$$

where $\mathbf{H}_u$ and $\mathbf{K}_u$ are given by (1.11) *and* (1.12). *Therefore, for every $\psi \in C^\infty_c(U\times \mathbb{S}^2)$ we have*

$$\chi_1^{\mathcal{G}_u}(\psi) = \int_\Omega \psi(\Phi_u)\, \sqrt{g_u}\, \mathbf{H}_u\, d\mathcal{L}^2\,, \quad \chi_2^{\mathcal{G}_u}(\psi) = \int_\Omega \psi(\Phi_u)\, \sqrt{g_u}\, \mathbf{K}_u\, d\mathcal{L}^2\,. \tag{4.2}$$

PROOF: As to the first symmetric curvature, we compute:

$$\begin{aligned} 2\,{\Phi_u}^{\#}\Theta_1 = {} & {\nu_u}^1(dx^2\wedge d{\nu_u}^3 - du\wedge d{\nu_u}^2) \\ & + {\nu_u}^2(du\wedge d{\nu_u}^1 - dx^1\wedge d{\nu_u}^3) + {\nu_u}^3(dx^1\wedge d{\nu_u}^2 - dx^2\wedge d{\nu_u}^1) \\ = {} & \big({\nu_u}^1(-\partial_1{\nu_u}^3 - (\partial_1 u\, \partial_2{\nu_u}^2 - \partial_2 u\, \partial_1 {\nu_u}^2)) \\ & + {\nu_u}^2((\partial_1 u\, \partial_2 {\nu_u}^1 - \partial_2 u\, \partial_1{\nu_u}^1) - \partial_2{\nu_u}^3) + {\nu_u}^3(\partial_2{\nu_u}^2 + \partial_1{\nu_u}^1)\big)\, dx^1\wedge dx^2\,. \end{aligned}$$

Recalling (1.7) and (1.10), and using that for $i,j=1,2$

$$\partial_i{\nu_u}^j = \frac{1}{2\,g^{3/2}}\big(-2\,\partial^2_{j,i}u\, g + \partial_j u\, \partial_i g\big)\,, \quad \partial_i{\nu_u}^3 = -\frac{1}{2\,g^{3/2}}\,\partial_i g$$

where $g = g_u$, we get

$$\begin{aligned} 4\,g^2\,{\Phi_u}^{\#}\Theta_1 = {} & \big(-\partial_1 u\,\partial_1 g + (\partial_1 u)^2(-2\,\partial^2_{2,2}u\, g + \partial_2 u\,\partial_2 g) - \partial_1 u\,\partial_2 u\,(-2\,\partial^2_{2,1}u\, g + \partial_2 u\,\partial_1 g) \\ & -\partial_1 u\,\partial_2 u\,(-2\,\partial^2_{1,2}u\, g + \partial_1 u\,\partial_2 g) + (\partial_2 u)^2(-2\,\partial^2_{1,1}u\, g + \partial_1 u\,\partial_1 g) - \partial_2 u\,\partial_2 g \\ & -2\,\partial^2_{2,2}u\, g + \partial_2 u\,\partial_2 g - 2\,\partial^2_{1,1}u\, g + \partial_1 u\,\partial_1 g\big)\, dx^1\wedge dx^2 \\ = {} & -2\,g\,\big((1+(\partial_1 u)^2)\partial^2_{2,2}u + (1+(\partial_2 u)^2)\partial^2_{1,1}u - 2\partial_1 u\,\partial_2 u\,\partial^2_{1,2}u\big)\, dx^1\wedge dx^2 \end{aligned}$$

and hence the first formula follows from (1.11). As to the second symmetric curvature, we similarly have:

$${\Phi_u}^{\#}\Theta_2 = \left({\nu_u}^1 \begin{vmatrix} \partial_1{\nu_u}^2 & \partial_2{\nu_u}^2 \\ \partial_1{\nu_u}^3 & \partial_2{\nu_u}^3 \end{vmatrix} + {\nu_u}^2 \begin{vmatrix} \partial_1{\nu_u}^3 & \partial_2{\nu_u}^3 \\ \partial_1{\nu_u}^1 & \partial_2{\nu_u}^1 \end{vmatrix} + {\nu_u}^3 \begin{vmatrix} \partial_1{\nu_u}^1 & \partial_2{\nu_u}^1 \\ \partial_1{\nu_u}^2 & \partial_2{\nu_u}^2 \end{vmatrix}\right) dx^1\wedge dx^2$$

which yields

$$\begin{aligned} 4\,g^{7/2}\,{\Phi_u}^{\#}\Theta_2 = {} & \big(\partial_1 u\,\big[(-2\,\partial^2_{2,1}u\, g + \partial_2 u\,\partial_1 g)\,(-\partial_2 g) - (-2\,\partial^2_{2,2}u\, g + \partial_2 u\,\partial_2 g)\,\partial_1 g\big] \\ & + \partial_2 u\,\big[\partial_1 g\,(-2\,\partial^2_{1,2}u\, g + \partial_1 u\,\partial_2 g) - \partial_2 g\,(-2\,\partial^2_{1,1}u\, g + \partial_1 u\,\partial_1 g)\big] \\ & +(-2\,\partial^2_{1,1}u\, g + \partial_1 u\,\partial_1 g)\,(-2\,\partial^2_{2,2}u\, g + \partial_2 u\,\partial_2 g) \\ & -(-2\,\partial^2_{1,2}u\, g + \partial_1 u\,\partial_2 g)\,(-2\,\partial^2_{2,1}u\, g + \partial_2 u\,\partial_1 g)\big)\, dx^1\wedge dx^2 \\ = {} & 4\,g^2\,\big(\partial^2_{1,1}u\,\partial^2_{2,2}u - (\partial^2_{1,2}u)^2\big)\, dx^1\wedge dx^2 \end{aligned}$$

and hence the second formula follows from (1.12). Equations (4.2) readily follow. □

Remark 4.2 Since u is smooth and bounded, by a density argument and by the dominated convergence theorem it turns out that formulas (4.2) extend to test functions $\psi \in C(U \times \mathbb{S}^2)$.

Following [7], the *generalized principal curvatures* of a current $\Sigma \in \text{Gcart}(U \times \mathbb{S}^2)$ are the Radon measures defined by the formulas

$$\langle \mathfrak{c}_i^\Sigma, \psi \rangle := (-1)^i \langle \Sigma, \psi\, \Theta_i \rangle \qquad \forall\, \psi \in C(U \times \mathbb{S}^2)\,, \qquad i = 1, 2$$

so that $\mathfrak{c}_i^{GGu} = \chi_i^{\mathcal{G}u}$ if $\Sigma = GG_u$ for some $u \in C_b^2(\Omega)$. Moreover, if $\{u_h\} \subset C_b^2(\Omega)$ satisfies $GG_{u_h} \rightharpoonup \Sigma$ in $\mathcal{D}_2(U \times \mathbb{S}^2)$, we deduce that

$$\lim_{h\to\infty} \langle \chi_i^{\mathcal{G}_{u_h}}, \psi \rangle = \langle \mathfrak{c}_i^\Sigma, \psi \rangle \qquad \forall\, \psi \in C(U \times \mathbb{S}^2)\,, \qquad i = 1, 2\,.$$

Denoting by $\pi : \mathbb{R}^6 \to \mathbb{R}^2$ the orthogonal projection onto the first two components, $\pi((x,z),y) := x$, we may also consider the projected measures:

$$\langle \pi_\# \mathfrak{c}_i^\Sigma, \psi \rangle := \langle \mathfrak{c}_i^\Sigma, \psi \circ \pi \rangle\,, \qquad \psi \in C(\Omega)\,, \qquad i = 1, 2$$

which are signed Borel measures in Ω with finite total variation. Since $\Sigma \llcorner R^+ = GG_u^a$, see Remark 3.3, it turns out that the Radon-Nykodym derivative of $\pi_\# \mathfrak{c}_i^\Sigma$ w.r.t. the Lebesgue measure is the density of the measure corresponding to GG_u^a, and we thus can write for every $\psi \in C(\Omega)$

$$\begin{array}{ll} \langle (\pi_\# \mathfrak{c}_1^\Sigma)^a, \psi \rangle = \displaystyle\int_\Omega \psi(x)\, \sqrt{g_u}\, \mathbf{H}_u \, d\mathcal{L}^2 & \langle (\pi_\# \mathfrak{c}_2^\Sigma)^a, \psi \rangle = \displaystyle\int_\Omega \psi(x)\, \sqrt{g_u}\, \mathbf{K}_u \, d\mathcal{L}^2 \\ \langle (\pi_\# \mathfrak{c}_1^\Sigma)^s, \psi \rangle = -\langle \Sigma^s, \psi \circ \pi \cdot \Theta_1 \rangle & \langle (\pi_\# \mathfrak{c}_2^\Sigma)^s, \psi \rangle = \langle \Sigma^s, \psi \circ \pi \cdot \Theta_2 \rangle \end{array}$$

where $\mathbf{H}_u$ and $\mathbf{K}_u$ are defined $\mathcal{L}^2$-a.e. in Ω as in (1.11) and (1.12), but in terms of the approximate derivatives of the underlying function $u \in \mathcal{E}(\Omega)$.

Example 4.3 Let $\Omega = Q^2 :=]-1,1[^2$ and $u_m(x_1,x_2) := m\,|x_1|$, where $m > 0$, so that $\mathcal{G}_{u_m}$ is made of two rectangles meeting with an the exterior dihedral angle

$$\theta_{e_m} = \pi - 2 \arccos \frac{m}{\sqrt{1+m^2}} \tag{4.3}$$

along the edge $e_m = I \times \{0\}$, where $I := \{0\} \times]-1,1[$. It is readily checked that $u_m \in \mathcal{E}(Q^2)$, and $\nu_{u_m}(x) = (1+m^2)^{-1/2}(-\operatorname{sgn}(x_1)\, m, 0, 1)$ if $x \notin I$. We thus have $\mathbf{H}_{u_m} \equiv 0$ and $\mathbf{K}_{u_m} \equiv 0$ for $x \notin I$, whence we obtain $\mathcal{E}(u_m) = |Q^2|\,\sqrt{1+m^2}$. Moreover, we have

$$(\partial GG_{u_m}^a) \llcorner U \times \mathbb{S}^2 = [\![\, I \times \{0\} \,]\!] \times (\delta_{P_m^+} - \delta_{P_m^-})\,, \qquad U = Q^2 \times \mathbb{R}$$

where $P_m^\pm := (1+m^2)^{-1/2}(\pm m, 0, 1)$ are points in $\mathbb{S}^2$ with geodesic distance equal to θ_{e_m}. Now, there is an (optimal) i.m. rectifiable current $\Sigma_m \in \text{Gcart}(U \times \mathbb{S}^2)$ that "fills the fracture" in the Gauss graph $\mathcal{GG}_u$ at the edge $I \times \{0\}$, given by

$$\Sigma_m = GG_{u_m}^a + \Sigma_m^s\,, \qquad \Sigma_m^s := -[\![\, I \times \{0\} \,]\!] \times [\![\, \gamma_m \,]\!]$$

γ_m being the oriented curve in the half-sphere $\mathbb{S}^2_+$ parameterized by $\gamma_m(\theta) := (\cos\theta, 0, \sin\theta)$, where $\theta \in [\alpha_m, \pi - \alpha_m]$ with $\alpha_m := \arccos(m/\sqrt{1+m^2})$. In fact, we have $\partial [\![\, \gamma_m \,]\!] = \delta_{P_m^-} - \delta_{P_m^+}$ and hence

$$(\partial \Sigma_m^s) \llcorner U \times \mathbb{S}^2 = [\![\, I \times \{0\} \,]\!] \times \partial [\![\, \gamma_m \,]\!] = -(\partial GG_{u_m}^a) \llcorner U \times \mathbb{S}^2$$

which yields $(\partial \Sigma_m) \llcorner U \times \mathbb{S}^2 = 0$. The mass of the singular component Σ_m^s agrees with the area of the surface $I \times \{0\} \times \gamma_m$, whence, see (4.3),

$$\mathbf{M}(\Sigma_m^s) = \mathcal{H}^1(I) \cdot \mathcal{H}^1(\gamma_m)\,, \qquad \mathcal{H}^1(\gamma_m) = \theta_{e_m}\,.$$

Since moreover the unit tangent 2-vector $\overrightarrow{\zeta}$ at points of the surface $I \times \{0\} \times \gamma_m$ satisfies $\zeta^{(0)} = 0$ and $\zeta^{(2)} = 0$, whence $\overrightarrow{\zeta} = \zeta^{(1)}$, see (3.9), we find that $\mathrm{E}_0(\Sigma_m^s) = 0$, $\mathrm{E}_2(\Sigma_m^s) = 0$, and hence

$$\mathrm{E}(\Sigma_m^s) = \mathrm{E}_1(\Sigma_m^s) = \mathbf{M}(\Sigma_m^s) = \mathcal{H}^1(I) \cdot \mathcal{H}^1(\gamma_m)\,.$$

As a consequence, we compute:

$$\langle \mathfrak{c}_1^{\Sigma_m}, \psi \rangle = \langle \mathfrak{c}_1^{\Sigma_m^s}, \psi \rangle = \langle \Sigma_m^s, \psi\, \Theta_1 \rangle = \int_{I \times \{0\} \times \gamma_m} \psi((x,0),y)\, \langle \Theta_1, \zeta^{(1)} \rangle\, d\mathcal{H}^2$$

for every $\psi \in C(U \times \mathbb{S}^2)$, and the total variation of $\mathfrak{c}_1^{\Sigma_m}$ is

$$|\mathfrak{c}_1^{\Sigma_m}|(U \times \mathbb{S}^2) = \int_{I \times \gamma_m} \langle \Theta_1, \zeta^{(1)} \rangle\, d\mathcal{H}^2 = \frac{1}{2}\, \mathcal{H}^1(I) \cdot \mathcal{H}^1(\gamma_m) \tag{4.4}$$

whereas $\mathfrak{c}_2^{\Sigma_m} = 0$. Finally, the projected measure $\pi_\# \mathfrak{c}_1^{\Sigma_m}$ on Q^2 is singular w.r.t. the Lebesgue measure $\mathcal{L}^2 \llcorner Q$, and its singular component is concentrated at I,

$$(\pi_\# \mathfrak{c}_1^{\Sigma_m})^s = \pi_\# \mathfrak{c}_2^{\Sigma_m^s} = \frac{1}{2}\, \mathcal{H}^1(\gamma_m) \cdot \mathcal{H}^1 \llcorner I\,, \quad \mathcal{H}^1(\gamma_m) = \theta_{e_m}\,.$$

Example 4.4 Let $\Omega = B^2$, the unit disk of radius one, and $u_m(x) := m\,(1 - |x|)$, where $m > 0$, so that $\mathcal{G}_{u_m}$ is the lateral surface of the cone with basis $\partial B^2 \times \{0\}$ and vertex $P_m = (0,0,m)$. We again have $u_m \in \mathcal{E}(B^2)$, where $\nu_{u_m}(x) = (1+m^2)^{-1/2}(m\cos\theta, m\sin\theta, 1)$ if $x = \rho\,(\cos\theta, \sin\theta)$. We check

$$\mathbf{H}_{u_m}(x) = -\frac{m}{2\sqrt{1+m^2}}\, \frac{1}{|x|}\,, \qquad \mathbf{K}_{u_m}(x) = 0 \qquad \forall\, x \in B^2 \setminus \{0_{\mathbb{R}^2}\}$$

as the graph cone $\mathcal{G}_{u_m}$ is a developable surface. Whence we obtain:

$$\mathcal{E}(u_m) = \int_{B^2} \sqrt{1+m^2}\,(1 + |2\mathbf{H}_u|)\, d\mathcal{L}^2 = \pi\,\sqrt{1+m^2} + 2\,\pi\, m\,.$$

Now, arguing as in [17, Sec. 3.2.2], it turns out that

$$(\partial GG^a_{u_m}) \llcorner U \times \mathbb{S}^2 = -\delta_{P_m} \times [\![\, \Gamma_m \,]\!]\,, \qquad U = B^2 \times \mathbb{R}$$

Γ_m being the oriented circle in $\mathbb{S}^2$ parameterized by $\Gamma_m(\theta) := (1+m^2)^{-1/2}(m\cos\theta, m\sin\theta, 1)$, where $\theta \in [0, 2\pi]$. Furthermore, there is an optimal i.m. rectifiable current $\Sigma_m \in \mathrm{Gcart}(U \times \mathbb{S}^2)$ that "fills the hole" in the Gauss graph at the vertex P_m of the cone, given by

$$\Sigma_m = GG^a_{u_m} + \Sigma_m^s\,, \qquad \Sigma_m^s := \delta_{P_m} \times [\![\, S_m \,]\!] \tag{4.5}$$

S_m being the oriented surface parameterized by $S_m(\theta, \varphi) := (\cos\theta\sin\varphi, \sin\theta\sin\varphi, \cos\varphi)$, where $\theta \in [0, 2\pi]$ and $\varphi \in [0, \arccos(1/\sqrt{1+m^2})]$. In fact, we check $\partial [\![\, S_m \,]\!] = [\![\, \Gamma_m \,]\!]$ and hence $(\partial \Sigma_m) \llcorner U \times \mathbb{S}^2 = 0$. The mass of the singular component Σ_m^s agrees with the area of the surface S_m, whence

$$\mathbf{M}(\Sigma_m^s) = \mathcal{H}^2(S_m) = 2\pi\Big(1 - \frac{1}{\sqrt{1+m^2}}\Big)\,.$$

Also, this time the unit tangent 2-vector $\overrightarrow{\zeta}$ at points of the surface $\{P_m\} \times S_m$ satisfies $\zeta^{(0)} = 0$ and $\zeta^{(1)} = 0$, whence $\overrightarrow{\zeta} = \zeta^{(2)}$. We thus get $\mathrm{E}_0(\Sigma_m^s) = 0$, $\mathrm{E}_1(\Sigma_m^s) = 0$, and hence

$$\mathrm{E}(\Sigma_m^s) = \mathrm{E}_2(\Sigma_m^s) = \mathbf{M}(\Sigma_m^s) = 2\pi\Big(1 - \frac{1}{\sqrt{1+m^2}}\Big)\,.$$

As a consequence, we compute:

$$\begin{aligned} \langle \mathfrak{c}_1^{\Sigma_m}, \psi \rangle &= \langle \mathfrak{c}_1^{GG^a_{u_m}}, \psi \rangle = \int_{B^2} \psi(\Phi_{u_m})\, \sqrt{g_{u_m}}\, \mathbf{H}_{u_m}\, d\mathcal{L}^2 = -\frac{1}{2} \int_{B^2} \psi(\Phi_{u_m})\, \frac{m}{|x|}\, d\mathcal{L}^2 \\ \langle \mathfrak{c}_2^{\Sigma_m}, \psi \rangle &= \langle \mathfrak{c}_2^{\Sigma_m^s}, \psi \rangle = \langle \Sigma_m^s, \psi\, \Theta_2 \rangle = \int_{S_m} \psi(P_m, y)\, \langle \Theta_2, \eta^{(2)} \rangle\, d\mathcal{H}^2 \end{aligned}$$

for every $\psi \in C(U \times \mathbb{S}^2)$, whence we get the total variation of the generalized curvature measures:

$$\begin{aligned} |\mathfrak{c}_1^{\Sigma_m}|(U \times \mathbb{S}^2) &= \int_{B^2} \sqrt{g_{u_m}}\, |\mathbf{H}_{u_m}|\, d\mathcal{L}^2 = \pi\, m \\ |\mathfrak{c}_2^{\Sigma_m}|(U \times \mathbb{S}^2) &= \int_{S_m} \langle \Theta_2, \eta^{(2)} \rangle\, d\mathcal{H}^2 = \mathcal{H}^2(S_m) = 2\pi\Big(1 - \frac{1}{\sqrt{1+m^2}}\Big)\,. \end{aligned} \tag{4.6}$$

Finally, as to the projected measures in B^2, we infer that $\pi_\# \mathfrak{c}_1^{\Sigma_m} = -\dfrac{m}{2\,|x|} \cdot \mathcal{L}^2 \llcorner B^2$, whereas $\pi_\# \mathfrak{c}_2^{\Sigma_m}$ is singular w.r.t. the Lebesgue measure, and its singular part is concentrated at the origin,

$$\pi_\# \mathfrak{c}_2^{\Sigma_m} = (\pi_\# \mathfrak{c}_2^{\Sigma_m})^s = \mathcal{H}^2(S_m) \cdot \delta_{0_{\mathbb{R}^2}}\,.$$

Mean curvature of polyhedral surfaces. It was defined by Sullivan [24] in such a way that it is supported on the edges of a polyhedral surface $\mathcal{M}$ in $\mathbb{R}^3$. If e is an edge, then

$$|\mathbf{H}_e| = \mathcal{H}^1(e) \cdot 2\,\sin(\theta_e/2)$$

where θ_e is the exterior dihedral angle along the edge. Assuming e.g. $\mathcal{M} = \mathcal{G}_{u_m}$, where u_m is given by Example 4.3, and taking $e_m = I \times \{0\}$, then $\mathcal{H}^1(e_m) = 2$ and θ_{e_m} is given by (4.3). Therefore, (adding the factor $1/2$ as in our notation) on account of (4.4) it turns out that the definition of mean curvature measure by Sullivan [24] differs from the one of [7], that we adopted here for weak limit of Gauss graphs of Cartesian surfaces, as $2\sin(\theta_{e_m}/2) = 2\,m/\sqrt{1+m^2} < \theta_{e_m}$ when $m > 0$.

We recall that the same feature holds in the case of the curvature of polyhedral curves. In fact, following Sullivan [25], the *curvature force measure* of a piecewise smooth curve has a Dirac mass at each corner point with mass $2\sin(\theta/2)$, where θ is the *turning angle*, whereas the total curvature of a polyhedral curve is the sum of the turning angles, compare [2].

Gauss curvature of polyhedral surfaces. It was defined by Sullivan [24] in such a way that the Gauss-Bonnet theorem continues to hold. It is concentrated at the vertices, and in the case of a triangulated polyhedral surface $\mathcal{M}$, the Gauss curvature at a vertex P agrees with the *angle defect*, whence $\mathbf{K}_\mathcal{M}(P) := 2\pi - \sum_i \theta_i$, where θ_i is the angle of the i^{th}-triangle of $\mathcal{M}$ meeting at P.

Assume e.g. that $\mathcal{M}_m(n)$ is the lateral surface of the pyramid with vertex $P_m = (0, 0, m)$ and basis given by a regular polygon with n edges and inscribed at the boundary ∂B^2 of the unit disk. Each one of the n triangles of $\mathcal{M}_m(n)$ has angle θ_i at P_m equal to $2\,\arcsin\big(\sin(\pi/n)/\sqrt{1+m^2}\big)$. Whence, the Gauss curvature of $\mathcal{M}_m(n)$ at P_m (in the sense of Sullivan [24]) is

$$\mathbf{K}_{\Sigma_m(n)}(P_m) = 2\pi - 2\,n\,\arcsin\Big(\frac{\sin(\pi/n)}{\sqrt{1+m^2}}\Big) \qquad \forall\, n \in \mathbb{N}^+\,, \quad m > 0\,.$$

Now, choosing the right orientation, the currents $[\![\, \mathcal{M}_m(n)\,]\!]$ weakly converge to the graph current G_{u_m} as $n \to \infty$, where u_m is the function of Example 4.4. On the other hand, we find:

$$\lim_{n\to\infty} \mathbf{K}_{\Sigma_m(n)}(P_m) = 2\pi\Big(1 - \frac{1}{\sqrt{1+m^2}}\Big)\,.$$

Gauss-Bonnet theorem. On account of the second line from (4.6), the above example seems to imply (differently to what happens for the mean curvature) that the notion of Gauss curvature by Sullivan [24] agrees with the one of [7] for generalized surfaces in $\mathbb{R}^3$, and here re-adapted. This feature is coherent with the possible formulation of the Gauss-Bonnet theorem in this framework, compare [6, 7].

For this purpose, consider the 1-form

$$\omega = \frac{y_3}{|y|}\,\Big(\frac{y_2}{y_1^2 + y_2^2}\, dy^1 - \frac{y_1}{y_1^2 + y_2^2}\, dy^2\Big)\,.$$

One has $d\omega = \widetilde{\Theta}_2$ on the open set $A := \{y \in \mathbb{R}^3 \mid y_1^2 + y_2^2 > 0\}$, where

$$\widetilde{\Theta}_2 := \frac{1}{|y|^3}\,\big(y^1 dy^2 \wedge dy^3 + y^2 dy^3 \wedge dy^1 + y^3 dy^1 \wedge dy^2\big)$$

so that the 2-form $\widetilde{\Theta}_2$ is equal to Θ_2 on the holed 2-sphere $\mathbb{S}^2 \setminus \{(0,0,\pm 1)\}$, see (1.5).

Therefore, if e.g. $u : \overline{\Omega} \to \mathbb{R}$ is smooth, with Ω a smooth bounded domain in $\mathbb{R}^2$, and $\nabla u \neq 0_{\mathbb{R}^2}$ on $\overline{\Omega}$, using that $\nu_u{}^{\#} d\omega = d\nu_u{}^{\#}\omega$, by Stokes' theorem we get

$$\langle GG_u, \Theta_2 \rangle = \langle \Phi_{u\,\#}[\![\,\Omega\,]\!], \Theta_2 \rangle = \int_\Omega \Phi_u{}^{\#}\Theta_2 = \int_\Omega \nu_u{}^{\#}\widetilde{\Theta}_2 = \int_\Omega d\nu_u{}^{\#}\omega = \int_{\partial\Omega} \nu_u{}^{\#}\omega$$

and hence the integral $\langle GG_u, \Theta_2 \rangle$ only depends on the value of ν_u at the boundary $\partial\Omega$ of the domain through the pull-back

$$\begin{aligned} \nu_u{}^{\#}\omega = {} & \frac{1}{|\nabla u|^2}\big(\partial_1 u\, d\nu_u{}^1 - \partial_2 u\, d\nu_u{}^2\big) \\ = {} & \frac{1}{g_u{}^{1/2}|\nabla u|^2}\big[(\partial_2 u\,\partial^2_{1,1}u - \partial_1 u\,\partial^2_{1,2}u)\,dx^1 - (\partial_1 u\,\partial^2_{2,2}u - \partial_2 u\,\partial^2_{1,2}u)\,dx^2\big]\,. \end{aligned}$$

On the other hand, recalling that by Proposition 4.1 and by the area formula

$$\langle GG_u, \Theta_2 \rangle = \int_\Omega \Phi_u{}^{\#}\Theta_2 = \int_\Omega \sqrt{g_u}\,\mathbf{K}_u\,d\mathcal{L}^2 = \int_{\mathcal{G}_u} \mathbf{K}_u\,d\mathcal{H}^2$$

after integrating by parts, the Gauss-Bonnet theorem yields to the equation

$$\int_{\mathcal{G}_u} \mathbf{K}_u\,d\mathcal{H}^2 = -\int_{\partial\mathcal{G}_u} \mathbf{k}_g\,d\mathcal{H}^1 + 2\pi\cdot\chi(\mathcal{G}_u)$$

where $\mathbf{k}_g$ is the geodesic curvature of the (naturally oriented) boundary curve $\partial\mathcal{G}_u$ and the Euler-Poincaré characteristic $\chi(\mathcal{G}_u) = 1$, as the graph surface $\mathcal{G}_u$ is topologically equivalent to a disk.

THE DISTRIBUTIONAL MEAN CURVATURE. If $u \in \mathcal{E}(\Omega)$, the mean curvature is well-defined in the distributional sense by the formula

$$\widetilde{\mathbf{H}}_u := \frac{1}{2}\,\mathrm{Div}\Big[\frac{\nabla u}{\sqrt{1+|\nabla u|^2}}\Big] = -\frac{1}{2}\,\mathrm{Div}(\nu_u{}^1, \nu_u{}^2)\,.$$

In fact, if $\{u_h\}_h \subset C^2_b(\Omega)$ is such that $u_h \to u$ strongly in $L^1(\Omega)$ and $\sup_h(\mathcal{E}(u_h) + \|u_h\|_\infty) < \infty$, by Theorem 3.1, and using (1.13), for any test function $\varphi \in C^\infty_c(\Omega)$ we get as $h \to \infty$

$$\int_\Omega \mathrm{div}\Big[\frac{\nabla u_h}{\sqrt{1+|\nabla u_h|^2}}\Big]\,\varphi\,d\mathcal{L}^2 = -\int_\Omega \frac{\nabla u_h}{\sqrt{1+|\nabla u_h|^2}} \bullet D\varphi\,d\mathcal{L}^2 \to -\int_\Omega \frac{\nabla u}{\sqrt{1+|\nabla u|^2}} \bullet D\varphi\,d\mathcal{L}^2\,.$$

However, by slightly modifying the function from Example 4.3, letting $\Omega = Q^2 :=]-1,1[^2$ and $u(x_1,x_2) := m\,|x_1| + a\,x_2$, where $m, a > 0$, we readily obtain

$$\widetilde{\mathbf{H}}_u = \frac{m}{\sqrt{1+m^2+a^2}}\cdot\mathcal{H}^1 \llcorner I\,, \qquad I := \{0\}\times]-1,1[$$

and hence the total variation of such a measure does not return the definition of mean curvature measure by Sullivan [24], as it does not contain the information on the length of the edge in the graph $\mathcal{G}_u$.

THE DISTRIBUTIONAL GAUSS CURVATURE. Following the definition by J. M. Ball [8] of distributional determinant, for each $u \in \mathcal{E}(\Omega)$ one may similarly consider:

$$\widetilde{\mathbf{K}}_u := \frac{1}{2}\,\mathrm{Div}\Big(\nu_u{}^1\partial_2\nu_u{}^2 - \nu_u{}^2\partial_2\nu_u{}^1, \nu_u{}^2\partial_1\nu_u{}^1 - \nu_u{}^1\partial_1\nu_u{}^2\Big)\,.$$

Choosing the 1-form $\omega_1 := \dfrac{1}{2}\,(y^1dy^2 - y^2dy^1)$, for any test function $\varphi \in C^\infty_c(\Omega)$ we have:

$$\Phi_u{}^{\#}\omega_1 \wedge d\varphi = -\frac{1}{2}\,\Big(\nu_u{}^1\partial_2\nu_u{}^2 - \nu_u{}^2\partial_2\nu_u{}^1, \nu_u{}^2\partial_1\nu_u{}^1 - \nu_u{}^1\partial_1\nu_u{}^2\Big) \bullet \nabla\varphi\,dx^1\wedge dx^2$$

and hence we get

$$\langle \widetilde{\mathbf{K}}_u, \varphi\rangle = \int_\Omega {\Phi_u}^{\#}\omega_1 \wedge d\varphi = \langle GG_u^a, \omega_1 \wedge d\varphi\rangle \,.$$

Now, if u is smooth, by (1.13) we clearly have

$${\Phi_u}^{\#}(dy^1 \wedge dy^2) = \mathbf{K}_u \, dx^1 \wedge dx^2$$

whereas by Stokes' theorem we know that the current $GG_u^a = GG_u$ has no inner boundary in $U \times \mathbb{S}^2$, so that in particular $\langle GG_u, d(\varphi \wedge \omega_1)\rangle = 0$. Using that $d(\varphi \wedge \omega_1) = -\omega_1 \wedge d\varphi + \varphi \, dy^1 \wedge dy^2$, we thus obtain:

$$\langle \widetilde{\mathbf{K}}_u, \varphi\rangle = \langle GG_u, \varphi \, dy^1 \wedge dy^1\rangle = \int_\Omega \varphi \, {\Phi_u}^{\#}(dy^1 \wedge dy^2) = \int_\Omega \varphi \, \mathbf{K}_u \, dx \,.$$

In general, if $u \in \mathcal{E}(\Omega)$ we similarly obtain for every $\varphi \in C_c^\infty(\Omega)$

$$\langle \widetilde{\mathbf{K}}_u, \varphi\rangle = \langle GG_u^a, \varphi \, dy^1 \wedge dy^1\rangle - \langle \partial GG_u^a, \varphi \wedge \omega_1\rangle \,, \qquad \langle GG_u^a, \varphi \, dy^1 \wedge dy^1\rangle = \int_\Omega \varphi \, \mathbf{K}_u \, dx$$

where $\mathbf{K}_u$ is defined as in (1.13), but in terms of the approximate first and second derivatives of u.

If e.g. $u = u_m$ is given by Example 4.4, since $\mathbf{K}_{u_m} = 0$ on $B^2 \setminus \{0_{\mathbb{R}^2}\}$, we compute

$$\langle GG_{u_m}^a, \varphi \, dy^1 \wedge dy^1\rangle = \int_{B^2} \varphi \, \mathbf{K}_{u_m} \, dx = 0 \,.$$

Moreover, if $\Sigma = \Sigma_m$ is the corresponding Gauss graph current in (4.5), using this time that Σ_m has no inner boundary in $U \times \mathbb{S}^2$, the singular component of the distributional Gauss curvature gives:

$$-\langle \partial GG_u^a, \varphi \, d\omega_1\rangle = \langle \partial \Sigma_m^s, \varphi \, d\omega_1\rangle = \langle \delta_{P_m} \times [\![S_m]\!], \varphi \, d\omega_1\rangle = \varphi(0_{\mathbb{R}^2}) \cdot \langle [\![S_m]\!], d\omega_1\rangle \,.$$

Recalling that $\partial [\![S_m]\!] = [\![\Gamma_m]\!]$, we thus compute

$$\langle [\![S_m]\!], d\omega_1\rangle = \langle [\![\Gamma_m]\!], \omega_1\rangle = \int_0^{2\pi} {\Gamma_m}^{\#}\omega_1 = \frac{m^2}{1+m^2} \cdot 2\pi \,.$$

In conclusion, in accordance with the well-known counterexample by S. Müller [22] on the distributional determinant, we have obtained the formula:

$$\widetilde{\mathbf{K}}_{u_m} = \mathbf{K}_{u_m} \, \mathcal{L}^2 \llcorner B^2 + c_m \, \delta_{0_{\mathbb{R}^2}} \,, \qquad \mathbf{K}_{u_m} \equiv 0 \,, \quad c_m := \frac{m^2}{1+m^2} \cdot 2\pi \,.$$

However, since the weight c_m of the Dirac measure is different from the value $2\pi\big(1 - \frac{1}{\sqrt{1+m^2}}\big)$ of the Gauss curvature (in the sense of Sullivan [24]) at the vertex of the cone, we conclude that (similarly as to $\widetilde{\mathbf{H}}_u$) the distributional definition $\widetilde{\mathbf{K}}_u$ of Gauss curvature fails to be the right geometric object, too.

5 A first relaxation formula

Beside the structure theorems 3.1 and 3.4, and the previously described consequences, finding necessary and sufficient conditions to the membership of a bounded BV-function $u : \Omega \to \mathbb{R}$ to the class $\mathcal{E}(\Omega)$ of functions with finite relaxed energy is a non-trivial open problem. For this reason, in the sequel we shall restrict to the subclass of 0-homogeneous functions defined in the open unit ball $\Omega = B^2$.

More precisely, we shall assume that $u : B^2 \to \mathbb{R}$ satisfies $u(x) = u(x/|x|)$ for each $x \in B^2 \setminus \{0_{\mathbb{R}^2}\}$. Whence (with an abuse of notation) we shall identify $u(x) = f(\theta)$ for some function $f : [0, 2\pi] \to \mathbb{R}$, where $x = (\rho \cos\theta, \rho \sin\theta)$. Denoting for simplicity $s := \sin\theta$ and $c := \cos\theta$, we formally compute

$$\nabla u = \Big(-\frac{\dot f s}{\rho}, \frac{\dot f c}{\rho}\Big) \,, \quad \nu_u = \frac{1}{(\rho^2 + \dot f^2)^{1/2}} \, (\dot f s, -\dot f c, \rho) \qquad \text{on } B^2 \setminus \{0_{\mathbb{R}^2}\} \,.$$

Remark 5.1 We shall report in Appendix A the explicit computation of the curvature energy functional in the case of homogeneous functions. For future use, we only notice here that if $\dot f(\theta)\neq 0$ for some given θ, the corresponding radial limit at $0_{\mathbb{R}^2}$ of the outward unit normal exists:

$$\lim_{\rho\to 0^+}\nu_u(\rho\cos\theta,\rho\sin\theta)=\frac{\dot f(\theta)}{|\dot f(\theta)|}\big(\cos(\theta-\pi/2),\sin(\theta-\pi/2),0\big)\,.$$

In this section we consider the *relaxation problem in the annulus* $\Omega_r:=\{x\in B^2\mid r<\rho<1\}$ for any small radius $r>0$, where $\rho:=|x|$. More precisely, we denote

$$\overline{\mathcal{E}}(u,\Omega_r):=\inf\{\liminf_{h\to\infty}\mathcal{E}(u_h,\Omega_r)\mid\{u_h\}_h\subset C^2_b(\Omega_r)\,,\ u_h\to u\text{ strongly in }L^1(\Omega_r)\}\,.$$

In the case of homogeneous functions $u(x)=f(\theta)$, the explicit formula for the relaxed energy in Ω_r is recovered from the one-dimensional result obtained for Cartesian curves in [2], that we briefly recall here.

Cartesian curves. Consider a smooth function $f:I\to\mathbb{R}$, where for our purposes we let $I:=[0,2\pi]$, and the corresponding graph function $c_f(t):=(t,f(t))$. The one-dimensional analogous of the energy $\mathcal{E}(u)$ is

$$\mathcal{E}(f):=\mathcal{L}(c_f)+\int_{c_f}\mathbf{k}_f\,d\mathcal{H}^1\,,$$

where $\mathcal{L}(c_f)$ is the length of c_f and $\mathbf{k}_f$ is the curvature $\mathbf{k}_f(t):=\dfrac{|\ddot f(t)|}{(1+\dot f(t)^2)^{3/2}}$, $t\in I$, whence

$$\mathcal{E}(f)=\int_0^{2\pi}\Big(\sqrt{1+\dot f^2}+\frac{|\ddot f|}{1+\dot f^2}\Big)\,dt\,.$$

The corresponding relaxed energy of functions $f\in L^1(I)$ is:

$$\overline{\mathcal{E}}(f):=\inf\{\liminf_{h\to\infty}\mathcal{E}(f_h)\mid\{f_h\}_h\subset C^2(I)\,,\ f_h\to f\text{ strongly in }L^1(I)\}$$

On account of Remark A.1 below, for each radius $0<r<1$ we can find a positive real constant $c_r>0$ such that for any homogeneous functions $u(x)=f(\theta)$ we have

$$\frac{1}{c_r}\,\mathcal{E}(f)\le\mathcal{E}(u,\Omega_r)\le c_r\,\mathcal{E}(f)\,.$$

Therefore, it turns out that *in the homogeneous case, a function $u\in L^1(B^2)$ has finite relaxed energy $\overline{\mathcal{E}}(u,\Omega_r)$, for each $0<r<1$, if and only if the corresponding L^1-function f has finite relaxed energy $\overline{\mathcal{E}}(f)$.* In particular, the structure properties of f obtained in [11] in codimension one are verified.

Now, any function $f\in L^1(I)$ satisfying $\overline{\mathcal{E}}(f)<\infty$ clearly belongs to $BV(I)$. Following [2], where the explicit formula for the relaxed energy is extended to high codimension, we recall that a continuous function f has finite relaxed energy if and only if the Cartesian curve c_f has finite length and finite *total curvature* $\mathrm{TC}(c_f)$, and in this case the total curvature agrees with the total variation $|D\tau_f|(I)$ of the *Gauss map* $\tau_f=\frac{\dot c_f}{|\dot c_f|}$ that is defined a.e. in I by means of the approximate gradient $\dot f$. Notice that in codimension one we have $|D\tau_f|(I)=|D\arctan(\dot f)|(I)$. Furthermore, the explicit formula

$$\overline{\mathcal{E}}(f)=\mathcal{L}(c_f)+\mathrm{TC}(c_f)$$

holds. More generally, when f is a BV-function with finite relaxed energy and a non-trivial Jump set J_f we have, see [2]:

$$\overline{\mathcal{E}}(f)=\int_I|\dot c_f|(1+\mathbf{k}_f)\,dt+|D^Cf|(I)+|D^C\tau_f|(I)+\mathbf{M}(GG^J_f)+\mathbf{M}(S^c_f)+\mathbf{M}(S^{Jc}_f)\,. \qquad (5.1)$$

In order to explain the above formula, we first point out that a current GG_f in $\mathcal{D}_1((\mathring{I}\times\mathbb{R})\times\mathbb{S}^1)$ carried by the "Gauss graph" of f is well-defined, when $\overline{\mathcal{E}}(f)<\infty$. More precisely, the current GG_f decomposes as

$$GG_f = GG_f^a + GG_f^C + GG_f^J$$

into the so called *Absolutely continuous*, *Cantor*, and *Jump component*, respectively. The first component GG_f^a depends on the approximate first and second derivatives $\dot f$ and $\ddot f$. Moreover, it turns out that $GG_f^J=0$ if f has a continuous representative, and that also $GG_f^C=0$ if $f\in W^{1,1}(I)$. Since in general the current GG_f has a non-trivial boundary, an optimal "vertical" current S_f is defined in [2] in such a way that if $\Sigma_f:=GG_f+S_f$, then Σ_f is a i.m. rectifiable in $\mathcal{R}_1((I\times\mathbb{R})\times\mathbb{S}^1)$ with no inner boundary.

The current S_f lives upon the Jump set $J_f\cup J_{\dot f}$. It is given by two terms:

$$S_f = S_f^{Jc} + S_f^c$$

a *Jump-corner* component S_f^{Jc} that is concentrated upon the discontinuity set J_f, and a *Corner* component S_f^c that is concentrated upon the discontinuity points of the approximate gradient $\dot f$ where f is continuous, the so called "corner" points in $J_{\dot f}\setminus J_f$. Roughly speaking, the first component takes into account of the turning angles that appear when the "graph" of f meets a jump point, possibly giving rise to two corners at the points $(t,f_\pm(t))$, where one side of each corner is "vertical", since it follows the jump. The second component deals with the turning angles where f is continuous but $\dot f$ has a jump.

A DENSITY RESULT. Let us now turn back to the class of homogeneous functions $u(x)=f(\theta)$, $x\in B^2$. Denoting $U_r:=\Omega_r\times\mathbb{R}$, the optimal current $\Sigma_{u,r}\in\mathcal{R}_2(U_r\times\mathbb{S}^2)$ that "fills the holes" in the Gauss graph GG_u^a in $U_r\times\mathbb{S}^2$ is given by the radial extension of the current corresponding to Σ_f. More precisely, we let

$$\Sigma_{u,r}:=\Psi_\#([\![r,1]\!]\times\Sigma_f)\,,\qquad 0<r<1 \tag{5.2}$$

where $\Psi:(0,1)\times(I\times\mathbb{R}_z\times\mathbb{S}^1_w)\to(B^2\setminus\{0_{\mathbb{R}^2}\})\times\mathbb{R}_z\times\mathbb{S}^2_y$ is given by

$$\Psi(\rho,(\theta,z,w_1,w_2)):=\Big(\rho\cos\theta,\rho\sin\theta,z,\frac{1}{\sqrt{w_1^2+\rho^2w_2^2}}\,(-w_1\sin\theta,w_1\cos\theta,\rho\,|w_2|)\Big)\,. \tag{5.3}$$

The optimality of the current $\Sigma_{u,r}$ follows due to a symmetry argument, and on account of the results from [2], and by lower-semicontinuity, we get:

$$\overline{\mathcal{E}}(u,\Omega_r)=\mathcal{E}(\Sigma_{u,r})\,,\qquad 0<r<1\,. \tag{5.4}$$

In particular, if $u(x)=f(\theta)$, where $f\in L^1([0,2\pi])$ has finite relaxed energy, we have obtained:

Theorem 5.2 *Let $\{\widetilde{f}_h\}_h\subset C^2(\mathbb{R})$ be a smooth 2π-periodic sequence. Let $f_h:=\widetilde{f}_{h|I}$, where $I:=[0,2\pi]$. Assume that: 1) $f_h\to f$ in $L^1(I)$; 2) $\mathcal{E}(f_h)\to\overline{\mathcal{E}}(f)$; 3) $GG_{f_h}\rightharpoonup\Sigma_f$ weakly as currents. Let $u_h(x):=f_h(\theta)$. Then for each radius $0<r<1$ the smooth sequence $\{u_h\}\subset C^2_b(\Omega_r)$ converges in L^1 to $u_{|\Omega_r}$, the currents GG_{u_h} weakly converge to $\Sigma_{u,r}$ in $\mathcal{D}_2(U_r\times\mathbb{S}^2)$, and $\mathcal{E}(u_h,\Omega_r)\to\overline{\mathcal{E}}(u,\Omega_r)$ as $h\to\infty$.*

6 An explicit formula

In this section we compute the explicit formula of the relaxed energy (3.1) in the case of a homogeneous function that we now introduce.

Example 6.1 Let $u:B^2\to\mathbb{R}$ given by $u(x):=\dfrac{x_1}{|x|}$, $x=(x_1,x_2)$.

Then u is a Sobolev function in $W^{1,p}(B^2)$ for each $p<2$. We also have

$$\partial_1u(x)=\frac{x_2^2}{\rho^3}\,,\quad \partial_2u(x)=-\frac{x_1x_2}{\rho^3}\,,\quad \nu_u(x)=\frac{\rho}{\sqrt{\rho^4+x_2^2}}\Big(-\frac{x_2^2}{\rho^2},\frac{x_1x_2}{\rho^2},\rho\Big)\,,\quad x\neq0\,.$$

Therefore, the upper unit normal ν_u belongs to $W^{1,1}(B^2,\mathbb{S}^2)$. In this case we have $f(\theta)=\cos\theta$. With $s:=\sin\theta$ and $c:=\cos\theta$, according to Appendix A we thus obtain:

$$g_u=\frac{\rho^2+s^2}{\rho^2}\,,\quad |\nu_u|^2=\frac{\rho^4(1+s^2)+\rho^2s^2(1+2s^2)+s^6}{\rho^2(\rho^2+s^2)^3}$$

$$|\xi_u|=\frac{\big[(\rho^2+2s^2)^2+\rho^2(\rho^2+s^2)\,c^2\big]^{1/2}}{\rho\,(\rho^2+s^2)^{3/2}}\,,\quad \mathbf{K}_u=-\frac{s^2}{(\rho^2+s^2)^2}\,,\quad \mathbf{H}_u=-\frac{1}{2}\,\frac{\rho\,c}{(\rho^2+s^2)^{3/2}}\,.$$

Boundary. On account of Proposition 3.8, we first compute the boundary of the Gauss graph GG_u^a. It turns out that

$$(\partial GG_u^a)\llcorner U\times\mathbb{S}^2=-\widetilde{\gamma}_{0\#}[\![\,0,2\pi\,]\!] \tag{6.1}$$

where $\widetilde{\gamma}_0:[0,2\pi]\to U\times\mathbb{S}^2$ is the closed curve supported in $\{0_{\mathbb{R}^2}\}\times\mathbb{R}\times\mathbb{S}^2_+$ and with parameterization

$$\widetilde{\gamma}_0(\theta):=\begin{cases}\big(0,0,1,0,-\cos 2\theta,\sin 2\theta\big) & \text{if}\quad 0\le\theta\le\pi/2\\ \big(0,0,-\cos 2\theta,\sin 2\theta,-\cos 2\theta,0\big) & \text{if}\quad \pi/2\le\theta\le\pi\\ \big(0,0,-1,0,-\cos 2\theta,\sin 2\theta\big) & \text{if}\quad \pi\le\theta\le 3\pi/2\\ \big(0,0,\cos 2\theta,\sin 2\theta,-\cos 2\theta,0\big) & \text{if}\quad 3\pi/2\le\theta\le 2\pi\,.\end{cases} \tag{6.2}$$

In fact, for each $\varepsilon>0$ small we have $\Phi_{u\#}[\![\,\partial B_\varepsilon(0_{\mathbb{R}^2})\,]\!]=\widetilde{\gamma}_{\varepsilon\#}[\![\,0,2\pi\,]\!]$, where

$$\widetilde{\gamma}_\varepsilon(\theta):=\Big(\varepsilon\cos\theta,\varepsilon\sin\theta,\cos\theta,\frac{1}{\sqrt{\varepsilon^2+\sin^2\theta}}\big(-\sin^2\theta,\sin\theta\cos\theta,\varepsilon\big)\Big)\,,\qquad \theta\in[0,2\pi]\,.$$

Now, for each $K>0$ and for $\varepsilon>0$ small so that $k\varepsilon<1$, letting $P^k_\varepsilon:=\widetilde{\gamma}_\varepsilon(\arcsin(k\varepsilon))$, it turns out that

$$\lim_{\varepsilon\to 0}P^k_\varepsilon=\Big(0,0,1,\frac{1}{\sqrt{1+k^2}}\,(0,k,1)\Big)\,.$$

Therefore, by Remark 5.1 one checks that the currents $\Phi_{u\#}[\![\,\partial B_\varepsilon(0_{\mathbb{R}^2})\,]\!]$ weakly converge as $\varepsilon\to 0$ to the 1-current $S_0:=\widetilde{\gamma}_{0\#}[\![\,0,2\pi\,]\!]$, so that the above formula for the boundary of GG_u^a holds true.

Filling the hole at the origin. In principle, there are two qualitatively different ways to "fill the hole" at the origin. A first possibility is given by choosing S_1 as an energy minimizing current among all i.m. rectifiable currents S in $\mathcal{R}_2(U\times\mathbb{S}^2)$, with support $\operatorname{spt}S\subset\{0_{\mathbb{R}^2}\}\times\mathbb{R}\times\mathbb{S}^2_+$, and satisfying the boundary condition $\partial S=\widetilde{\gamma}_{0\#}[\![\,0,2\pi\,]\!]$. Alternatively, we may choose S_2 given by

$$S_2:=-F_\#([\![\,0,1\,]\!]\times\widetilde{\gamma}_{0\#}[\![\,0,2\pi\,]\!]) \tag{6.3}$$

where $F:[0,1]\times(\{0_{\mathbb{R}^2}\}\times\mathbb{R}_z\times\mathbb{R}^3_y)\to\mathbb{R}^2_x\times\mathbb{R}_z\times\mathbb{R}^3_y$ is the homotopy map $F(\lambda,(0_{\mathbb{R}^2},z,y)):=(\lambda\,\mathbf{v},z,y)$ for some fixed direction $\mathbf{v}\in\mathbb{S}^1$. In fact, by the homotopy formula [23, 26.22] we have

$$\partial S_2=F_\#((\delta_0-\delta_1)\times\widetilde{\gamma}_{0\#}[\![\,0,2\pi\,]\!])\,.$$

Setting then $\Sigma^i:=GG_u^a+S_i$, where $i=1,2$, in both cases the null-boundary condition $(\partial\Sigma^i)\llcorner U\times\mathbb{S}^2=0$ is satisfied, since $U=B^2\times\mathbb{R}$. However, see Proposition 3.8, the energy of the currents Σ_i is involved in the computation of the relaxed energy $\overline{\mathcal{E}}(u)$ only if $\Sigma^i\in\operatorname{Gcart}(U\times\mathbb{S}^2)$. Therefore, the geometric necessary condition given by property vi) in the structure theorem 3.4 has to be satisfied.

Example 6.2 Such an orthogonality condition (see Remark 3.5) can be imposed in the minimization problem, since it is preserved by the weak convergence as currents. If u is given by Example 6.1, it turns out that the optimal current S_1 is given by integration of 2-forms on the (suitably oriented) surface $\{0_{\mathbb{R}^2}\}\times\mathcal{M}$, where $\mathcal{M}$ is the surface supported in $[-1,1]\times\mathbb{S}^2$ and given by the union of four pieces:

i) the two quarters of the unit spheres $\{\pm 1\}\times\mathbb{S}^2_h$, where $\mathbb{S}^2_h:=\{y\in\mathbb{S}^2\mid y_1\le 0\,,\ y_3\ge 0\}$;

ii) the surface Σ_1 given by the union of the two pieces of the cylinder $[-1,1]\times\mathbb{S}^1\times\{0\}$ which are enclosed by the union of the two arcs $\widetilde{\gamma}_0([\pi/2,\pi])$, $\widetilde{\gamma}_0([3\pi/2,2\pi])$, where $\widetilde{\gamma}_0:[0,2\pi]\to U\times\mathbb{S}^2$ is the curve given by (6.2), and of the two half-circles $\{\pm 1\}\times\mathbb{S}^1_h$, where $\mathbb{S}^1_h:=\{y\in\mathbb{S}^2\mid y_1\le 0\,,\ y_3=0\}$.

We now observe that if the current S_2 is given by (6.3), for any choice of the direction $\mathbf{v}$ the above mentioned orthogonality condition is violated. More precisely, on account of definition (3.16) we have:

Proposition 6.3 *Any current Σ in Gcart_u is of the type $\Sigma=GG^a_u+\Sigma^s$, where the singular component Σ^s is supported in $\{0_{\mathbb{R}^2}\}\times\mathbb{R}\times\mathbb{S}^2_+$.*

PROOF: We shall make use of a slicing argument. Following [23, Sec. 28], and letting $d(x,z,y)=|x|$, for $(x,z,y)\in\mathbb{R}^2\times\mathbb{R}_z\times\mathbb{R}^3_y$, the sliced current $\langle\Sigma,d,r\rangle$ is i.m. rectifiable in $\mathcal{R}_1(\partial B^2_r\times\mathbb{R}\times\mathbb{S}^2)$, for $\mathcal{L}^1$-a.e. $r\in(0,1)$. Since moreover Σ has no boundary in $B^2\times\mathbb{R}\times\mathbb{S}^2$, the sliced current $\langle\Sigma,d,r\rangle$ has no boundary, and we actually have

$$\langle\Sigma,d,r\rangle=\langle GG^a_u,d,r\rangle+\langle\Sigma^s,d,r\rangle\,.$$

Moreover, by the verticality condition of Σ^s, and by the smoothness of u outside the origin, we deduce that the sliced current $\langle\Sigma^s,d,r\rangle$ is an integral 1-cycle that does not read 1-forms of the type $\varphi\,dx^i$, where $i=1,2$. Define now $\Psi_r:(I\times\mathbb{R}_z\times\mathbb{S}^1_w)\to\partial B^2_r\times\mathbb{R}_z\times\mathbb{S}^2_y$ by $\Psi_r(\theta,z,w_1,w_2):=\Psi(r,(\theta,z,w_1,w_2))$, where $I=[0,2\pi]$ and Ψ is given by (5.3). For a.e. $r\in(0,1)$ we thus have

$$\Psi^{-1}_{r\,\#}\langle\Sigma,d,r\rangle=GG_f+S_r\,,\quad S_r:=\Psi^{-1}_{r\,\#}\langle\Sigma^s,d,r\rangle$$

where $f(\theta)=\cos\theta$. Let now R_r denote the set of points with positive multiplicity of the integral cycle S_r. The geometric condition inherited from the current Σ, joined with the verticality condition, yields that at $\mathcal{H}^1$-a.e. point $P=(\theta,z,w_1,w_2)$ in R_r the approximate tangent 1-space is oriented by a vector $v=(v_1,v_2,v_3,v_4)$ in $\mathbb{R}^4$ satisfying $(v_1,v_2)\bullet(w_1,w_2)=0$, with $v_1=0$, whence $w_2=0$ if $v_2\neq 0$. Since $(w_1,w_2)\in\mathbb{S}^1$, this yields that $P=(\theta,z,\pm1,0)$, if $v_2\neq0$. We now observe that by (5.3)

$$\Psi_r(\theta,z,\pm1,0)=(r\cos\theta,r\sin\theta,z,\pm(\sin\theta,-\cos\theta,0))\,.$$

Now, by (6.1) we have $(\partial\Sigma^s)\llcorner U\times\mathbb{S}^2=\widetilde{\gamma}_{0\#}[\![\,0,2\pi\,]\!]$, where $\widetilde{\gamma}_0:[0,2\pi]\to U\times\mathbb{S}^2$ is the closed curve parameterized by (6.2). Therefore, the above considerations imply that $S_r=0$ for a.e. $r\in(0,1)$, and hence that the singular current Σ^s is supported in $\{0_{\mathbb{R}^2}\}\times\mathbb{R}\times\mathbb{S}^2_+$, as required. $\square$

A DENSITY RESULT. We shall now construct a smooth approximating sequence $\{u_h\}\subset C^2_b(B^2)$ such that GG_{u_h} weakly converges to $GG^a_u+S_1$, where S_1 is the minimizer defined in Example 6.2, and with energies $\mathcal{E}(u_h)$ converging to the energy $\mathcal{E}(u)+\mathrm{E}(S_1)$, as $h\to\infty$. As a consequence, on account of Proposition 6.3 and of the minimality of S_1, by Proposition 3.8 we conclude that

$$\overline{\mathcal{E}}(u)=\mathcal{E}(u)+\mathrm{E}(S_1)\,. \tag{6.4}$$

Theorem 6.4 *There exists a sequence $\{u_h\}\subset C^2_b(B^2)$ such that GG_{u_h} weakly converges to $GG^a_u+S_1$ and such that $\mathcal{E}(u_h)\to\mathcal{E}(u)+\mathrm{E}(S_1)$, as $h\to\infty$.*

PROOF: We first observe that it suffices to find a bounded approximating sequence in $W^{2,\infty}(B^2)$. In fact, if $u\in W^{2,\infty}(B^2)$ is bounded, by a standard argument (based e.g. on the convolution with a smooth kernel and on the dominated convergence theorem) we can find a strongly approximating smooth sequence. Since the weak convergence of currents with no boundary and with support contained in a compact set is metrizable, compare e.g. [23, Sec. 31], a diagonal argument will conclude the proof.

Let now $\phi:[0,+\infty)\to[0,1]$ be given by

$$\phi(\rho):=\begin{cases}\dfrac{4}{3}\,\rho & \text{if}\quad \rho\le 1/2\\ 1-\dfrac{4}{3}\,(1-\rho)^2 & \text{if}\quad 1/2\le\rho\le 1\\ 1 & \text{if}\quad \rho\ge 1\,.\end{cases}$$

We shall work with the function $v:\mathbb{R}^2\to\mathbb{R}$ given by $v(x):=\phi(|x|)\,x_1/|x|$, if $x\neq 0_{\mathbb{R}^2}$, and $v(0_{\mathbb{R}^2})=0$, so that in polar coordinates one has $v(x)=\phi(\rho)\,f(\theta)$, with $f(\theta):=\cos\theta$. For $0<\varepsilon<1$, let $v_\varepsilon:B^2\to\mathbb{R}$ be given by $v_\varepsilon(x):=v(x/\varepsilon)$, so that in polar coordinates $v_\varepsilon(x):=\phi(\rho/\varepsilon)\,f(\theta)$. Choosing $u_h:=v_{\varepsilon_h}$ for a sequence $\varepsilon_h\searrow 0$, it turns out that $\{u_h\}\subset W^{2,\infty}(B^2)$ and that the currents GG_{u_h} weakly converge to $GG_u^a+S_1$ as $h\to\infty$.

In fact, according to (1.9) one computes

$$\Phi_{v_\varepsilon}(x)=\Big(x,v_\varepsilon(x),\frac{1}{\sqrt{\varepsilon^2+|\nabla v(x/\varepsilon)|^2}}\,(-\partial_1 v(x/\varepsilon),-\partial_2 v(x/\varepsilon),\varepsilon)\Big)\,.$$

We have $v_\varepsilon(x)=3x_1/(4\varepsilon)$ for $|x|\le\varepsilon/2$ and $v_\varepsilon(x)=x_1/|x|$ for $\varepsilon\le|x|<1$, whereas in polar coordinates $x=\rho(c,s)$, with $c=\cos\theta$ and $s=\sin\theta$, and denoting $\sigma:=\rho/\varepsilon$, for $\varepsilon/2<|x|<\varepsilon$ we get

$$\Phi_{v_\varepsilon}(x)=\Big(\varepsilon\,\sigma\,c,\varepsilon\,\sigma\,s,\phi(\sigma)\,c,\frac{(-\dot\phi(\sigma)\,c^2-(\phi(\sigma)/\sigma)\,s^2,(\phi(\sigma)/\sigma-\dot\phi(\sigma))\,sc,\varepsilon)}{\sqrt{\varepsilon^2+\dot\phi^2(\sigma)\,c^2+(\phi(\sigma)/\sigma)^2\,s^2}}\Big)\,.$$

Letting $\varepsilon\to 0$, if $s\neq 0$ and $1/2<\sigma<1$, we get $\Phi_{v_\varepsilon}(x)\to(0,0,\Psi(\sigma,\theta),0)$, where we have set

$$\Psi(\sigma,\theta):=\Big(\phi(\sigma)\,c,\frac{(-\dot\phi(\sigma)\,c^2-(\phi(\sigma)/\sigma)\,s^2,(\phi(\sigma)/\sigma-\dot\phi(\sigma))\,sc)}{\sqrt{\dot\phi^2(\sigma)\,c^2+(\phi(\sigma)/\sigma)^2\,s^2}}\Big)\in\mathbb{R}\times\mathbb{S}^1\,. \tag{6.5}$$

One may then check the weak convergence of the sequence of Gauss graphs GG_{u_h} to the current $GG_u^a{+}S_1$.

In order to prove the energy convergence $\mathcal{E}(u_h)\to\mathcal{E}(u)+\mathrm{E}(S_1)$, it suffices to show that

$$\lim_{\varepsilon\to 0}\mathcal{E}(v_\varepsilon,B^2_\varepsilon\setminus B^2_{\varepsilon/2})=\mathrm{E}(S_1)\,.$$

In fact, one readily obtains:

$$\lim_{\varepsilon\to 0}\mathcal{E}(v_\varepsilon,B^2_{\varepsilon/2})=0\,,\qquad \lim_{\varepsilon\to 0}\mathcal{E}(v_\varepsilon,B^2\setminus B^2_\varepsilon)=\mathcal{E}(u)\,.$$

For this purpose, referring to Example 6.2, we have $\mathrm{E}(S_1)=\mathrm{E}_0(S_1)+E_1(S_1)+\mathrm{E}_2(S_1)$, where $\mathrm{E}_0(S_1)=0$, as the current S_1 is concentrated on $\{0_{\mathbb{R}^2}\}\times\mathbb{R}^4$. Moreover, the energy term $\mathrm{E}_2(S_1)$ is equal to the sum of the areas of the two quarters of the unit spheres $\{\pm1\}\times\mathbb{S}^2_h$, i.e., $\mathrm{E}_2(S_1)=2\pi$. Finally, the energy term $\mathrm{E}_1(S_1)$ is equal to the area $\mathcal{H}^2(\Sigma_1)$ of the surface Σ_1 previously described in ii), so that $\mathrm{E}_1(S_1)=4\pi-8$. Therefore, we have to show that

$$\lim_{\varepsilon\to 0}\mathcal{E}_0(v_\varepsilon,B^2_\varepsilon\setminus B^2_{\varepsilon/2})=0\,,\quad \lim_{\varepsilon\to 0}\mathcal{E}_2(v_\varepsilon,B^2_\varepsilon\setminus B^2_{\varepsilon/2})=2\pi\,,\quad \lim_{\varepsilon\to 0}\mathcal{E}_1(v_\varepsilon,B^2_\varepsilon\setminus B^2_{\varepsilon/2})=\mathcal{H}^2(\Sigma_1)\,. \tag{6.6}$$

The first limit is trivially checked. As to the second one, by (1.12) and (1.18), and by changing variable $\widehat{x}:=x/\varepsilon$, we first compute:

$$\mathcal{E}_2(v_\varepsilon,B^2_\varepsilon\setminus B^2_{\varepsilon/2})=\varepsilon\int_{B^2\setminus B^2_{1/2}}\frac{|\partial^2_{1,1}v\,\partial^2_{2,2}v-(\partial^2_{1,2}v)^2|}{(\varepsilon^2+|\nabla v|^2)^{3/2}}\,d\mathcal{L}^2$$

where in polar coordinates we have:

$$\partial^2_{1,1}v\,\partial^2_{2,2}v-(\partial^2_{1,2}v)^2=\ddot\phi(\rho)\Big(\frac{\dot\phi(\rho)}{\rho}-\frac{\phi(\rho)}{\rho^2}\Big)c^2-\Big(\frac{\dot\phi(\rho)}{\rho}-\frac{\phi^2(\rho)}{\rho^2}\Big)^2s^2=:\Delta(\rho,\theta)$$

and

$$\varepsilon^2+|\nabla v|^2=\varepsilon^2+\dot\phi^2(\rho)\,c^2+\frac{\phi^2(\rho)}{\rho^2}\,s^2\,.$$

Now, using that $\dot\phi(\rho)=8(1-\rho)/3$ and $\ddot\phi(\rho)=-8/3$, by changing variable $R=2(1-\rho)$ we obtain:

$$\mathcal{E}_2(v_\varepsilon,B^2_\varepsilon\setminus B^2_{\varepsilon/2})=4\varepsilon\int_0^1\int_0^{\pi/2}\frac{\frac{2(1-R)(3-R)}{3(2-R)}\,\Big|\frac{8}{3}\cos^2\theta-\frac{4(1-R)(3-R)}{3(2-R)^2}\,\sin^2\theta\Big|}{\Big(\varepsilon^2+\frac{16}{9}\,R^2\cos^2\theta+\frac{4}{9}\Big(\frac{3-R^2}{2-R}\Big)^2\sin^2\theta\Big)^{3/2}}\,\frac{1}{2}\,dR\,d\theta$$

so that by changing again variables $z = R/\varepsilon$ and $y = \theta/\varepsilon$ we get

$$\mathcal{E}_2(v_\varepsilon, B^2_\varepsilon \setminus B^2_{\varepsilon/2}) = 4 \int_0^{1/\varepsilon} \int_0^{\pi/(2\varepsilon)} \frac{\frac{(1-\varepsilon z)(3-\varepsilon z)}{3(2-\varepsilon z)} \left| \frac{8}{3} \cos^2(\varepsilon y) - \frac{4(1-\varepsilon z)(3-\varepsilon z)}{3(2-\varepsilon z)^2} \sin^2(\varepsilon y) \right|}{\left(1 + \frac{16}{9} z^2 \cos^2(\varepsilon y) + \frac{4}{9} \left(\frac{3-\varepsilon^2 z^2}{2-\varepsilon z}\right)^2 \frac{\sin^2(\varepsilon y)}{\varepsilon^2 y^2} y^2\right)^{3/2}} \, dz \, dy \,.$$

Therefore, by dominated convergence, and with $x = 4z/3$, we conclude that:

$$\lim_{\varepsilon \to 0} \mathcal{E}_2(v_\varepsilon, B^2_\varepsilon \setminus B^2_{\varepsilon/2}) = \int_{\mathbb{R}^2} \frac{1}{(1 + x^2 + y^2)^{3/2}} \, d\mathcal{L}^2 = 2\pi \,.$$

In order to prove the third limit in (6.6), denoting for simplicity $\nu_\varepsilon := \nu_{v_\varepsilon}$ we first observe that by dominated convergence

$$\lim_{\varepsilon \to 0} \int_{B^2_\varepsilon \setminus B^2_{\varepsilon/2}} |\partial_i {\nu_\varepsilon}^j| \, d\mathcal{L}^2 = 0$$

for $i = 1, 2$ and $j = 1, 2, 3$, whereas in a similar way one also checks:

$$\lim_{\varepsilon \to 0} \int_{B^2_\varepsilon \setminus B^2_{\varepsilon/2}} |\partial_1 v_\varepsilon \, \partial_2 {\nu_\varepsilon}^3 - \partial_2 v_\varepsilon \, \partial_1 {\nu_\varepsilon}^3| \, d\mathcal{L}^2 = 0 \,.$$

Therefore, on account of the second line in (1.15), we infer that

$$\lim_{\varepsilon \to 0} \mathcal{E}_1(v_\varepsilon, B^2_\varepsilon \setminus B^2_{\varepsilon/2}) = \lim_{\varepsilon \to 0} \int_{B^2_\varepsilon \setminus B^2_{\varepsilon/2}} \Big(\sum_{j=1}^{2} (\partial_1 v_\varepsilon \, \partial_2 {\nu_\varepsilon}^j - \partial_2 v_\varepsilon \, \partial_1 {\nu_\varepsilon}^j)^2 \Big)^{1/2} \, d\mathcal{L}^2 \,.$$

For each $x \in B^2 \setminus B^2_{1/2}$ such that $|\nabla v(x)| \neq 0$, i.e., $\mathcal{H}^2$-a.e. on $x \in B^2 \setminus B^2_{1/2}$, we have:

$$\lim_{\varepsilon \to 0} \varepsilon^{-2} (\partial_1 v_\varepsilon \, \partial_2 {\nu_\varepsilon}^1 - \partial_2 v_\varepsilon \, \partial_1 {\nu_\varepsilon}^1)(x/\varepsilon) = \partial_2 v(x) \, \frac{(\widetilde{\Delta} v)(x)}{|\nabla v(x)|^3}$$

and

$$\lim_{\varepsilon \to 0} \varepsilon^{-2} (\partial_1 v_\varepsilon \, \partial_2 {\nu_\varepsilon}^2 - \partial_2 v_\varepsilon \, \partial_1 {\nu_\varepsilon}^2)(x/\varepsilon) = -\partial_1 v(x) \, \frac{(\widetilde{\Delta} v)(x)}{|\nabla v(x)|^3}$$

where we have set

$$\widetilde{\Delta} v := (\partial_1 v)^2 \partial^2_{2,2} v + (\partial_2 v)^2 \partial^2_{1,1} v - 2 \, \partial_1 v \, \partial_2 v \, \partial^2_{1,2} v \,.$$

By dominated convergence we thus infer that

$$\lim_{\varepsilon \to 0} \mathcal{E}_1(v_\varepsilon, B^2_\varepsilon \setminus B^2_{\varepsilon/2}) = \int_{B^2 \setminus B^2_{1/2}} \frac{|(\widetilde{\Delta} v)(x)|}{|\nabla v(x)|^2} \, d\mathcal{L}^2 \,. \tag{6.7}$$

We now recall that when $|\nabla v(x)| \neq 0$ in $B^2 \setminus B^2_{1/2}$, in polar coordinates one has $\Phi_{v_\varepsilon}(x) \to (0, 0, \Psi(\sigma, \theta), 0)$ as $\varepsilon \to 0$, whereas by (6.5) in Euclidean coordinates we have

$$\Psi(\sigma, \theta) = \Big(v(x), \frac{-\partial_1 v(x)}{|\nabla v(x)|}, \frac{-\partial_1 v(x)}{|\nabla v(x)|} \Big) \,.$$

By computing the 2-dimensional Jacobian in Euclidean coordinates, it turns out that

$$J_2 \Psi(\sigma, \theta) = \frac{|(\widetilde{\Delta} v)(x)|}{|\nabla v(x)|^2} \,.$$

The function $\Psi(\sigma, \theta)$ being $\mathcal{H}^2$-a.e. injective on $(1/2, 1) \times (0, 2\pi)$, and with image equal to the surface Σ_1, by the area formula we get:

$$\int_{B^2 \setminus B^2_{1/2}} \frac{|(\widetilde{\Delta} v)(x)|}{|\nabla v(x)|^2} \, d\mathcal{L}^2 = \int_{B^2 \setminus B^2_{1/2}} J_2 \Psi \, d\mathcal{L}^2 = \mathcal{H}^2(\Sigma_1)$$

and hence by (6.7) we have proved the third limit in (6.6), as required. $\square$

7 Closing the Gauss graph

In Sec. 5, we have seen how the relaxation problem for homogeneous functions $u(x) = f(\theta)$ is solved outside the origin. In particular, we obtained formula (5.4). Now, choosing $r = 0$ in (5.2), or taking the limit as $r \to 0^+$, we find an optimal i.m. rectifiable current $\widetilde{\Sigma}_u := \Sigma_{u,0}$ in $\mathcal{R}_2(U \times \mathbb{S}^2)$.

Recalling that $\Sigma_f := GG_f + S_f$, where $GG_f = GG_f^a + GG_f^C + GG_f^J$ and $S_f = S_f^{Jc} + S_f^c$, using the map Ψ given by (5.3) we clearly have $GG_u^a = \Psi_\#([\![0,1]\!] \times GG_f^a)$. We thus may decompose

$$\widetilde{\Sigma}_u = GG_u^a + GG_u^C + GG_u^J + S_u^{Je} + S_u^e \tag{7.1}$$

where we have correspondingly set

$$\begin{array}{ll} GG_u^C := \Psi_\#([\![0,1]\!] \times GG_f^C)\,, & GG_u^J := \Psi_\#([\![0,1]\!] \times GG_f^J)\,, \\ S_u^{Je} := \Psi_\#([\![0,1]\!] \times S_f^{Jc})\,, & S_u^e := \Psi_\#([\![0,1]\!] \times S_f^c) \end{array} \tag{7.2}$$

which will be called the *Cantor*, *Jump*, *Jump-edge*, and *Edge* components, respectively.

We have obtained an i.m. rectifiable current $\widetilde{\Sigma}_u$ in $\mathcal{R}_2(U \times \mathbb{S}^2)$ supported in $\overline{U} \times \mathbb{S}^2_+$. However, the null-boundary condition $(\partial \widetilde{\Sigma}_u) \llcorner U \times \mathbb{S}^2 = 0$ is violated, as in general a "hole" may appear at the origin $0_{\mathbb{R}^2}$, the point singularity of a homogeneous functions $u(x) = f(\theta)$ with finite relaxed energy. An example is given by (6.1), (6.2). In this section we analyze in detail a non-smooth model example.

Example 7.1 Let $u : B^2 \to \mathbb{R}$ given by

$$u(x) := \begin{cases} \pi/2 - \arctan(x_2/x_1) & \text{if } x_1 < 0 \\ \pi & \text{if } x_1 \geq 0 \text{ and } x_2 > 0 \\ 0 & \text{if } x_1 \geq 0 \text{ and } x_2 < 0\,. \end{cases}$$

The boundary $\partial S\mathcal{G}_u$ of the subgraph of u is the surface given by two "floors", at level $z = 0$ and $z = \pi$, one "wall" of height π at the Jump set $J_u = (0,1) \times \{0\}$, and a smooth "spiral staircase" connecting the two floors. Four horizontal edges appear at the boundary of the two floors, and a fifth vertical edge lives over the singular point $0_{\mathbb{R}^2}$, the edges meeting at the corner points $(0,0,0)$ or $(0,0,\pi)$.

We have $u \in BV(B^2)$, with no Cantor part, $D^C u = 0$. Orienting the Jump set by $\mathbf{t} = (1,0)$, the unit normal is $\mathbf{n} = (0,1)$ and the one-sided limits are $u^+ \equiv \pi$ and $u^- \equiv 0$ on J_u. We also have

$$\nabla u(x) = \begin{cases} \left(x_2/\rho^2, -x_1/\rho^2\right) & \text{if } x_1 < 0 \\ (0,0) & \text{if } x_1 > 0 \text{ and } x_2 \neq 0 \end{cases}$$

which yields

$$\nu_u(x) = \begin{cases} \rho(\rho^2+1)^{-1/2}\left(-x_2/\rho^2, x_1/\rho^2, 1\right) & \text{if } x_1 < 0 \\ (0,0,1) & \text{if } x_1 > 0 \text{ and } x_2 \neq 0\,. \end{cases}$$

Therefore, the outward unit normal ν_u belongs to $BV(B^2, \mathbb{S}^2)$, it has no Cantor part, $D^C \nu_u = 0$, and its jump set is $J_{\nu_u} = \{x \in B^2 \mid x_1 = 0\}$. Furthermore, letting $\mathbf{n} = (-1,0)$ and $\mathbf{t} = (0,1)$ on J_{ν_u}, it turns out that the one-sided limits of ν_u are smooth on $J_{\nu_u} \setminus \{0_{\mathbb{R}^2}\}$ and we actually have

$$\nu_u{}^+(x) = \begin{cases} (1+x_2^2)^{-1/2}\left(-1,0,x_2\right) & \text{if } x_1 = 0 \text{ and } x_2 > 0 \\ (1+x_2^2)^{-1/2}\left(1,0,-x_2\right) & \text{if } x_1 = 0 \text{ and } x_2 < 0 \end{cases} \tag{7.3}$$

whereas $\nu_u{}^- \equiv (0,0,1)$ on $J_{\nu_u} \setminus \{0_{\mathbb{R}^2}\}$. Finally, we have $u(x) = f(\theta)$ with

$$f(\theta) := \begin{cases} \pi & \text{if } 0 < \theta < \pi/2 \\ 3\pi/2 - \theta & \text{if } \pi/2 < \theta < 3\pi/2 \\ 0 & \text{if } 3\pi/2 < \theta < 2\pi\,. \end{cases} \tag{7.4}$$

For $\theta \in (\pi/2, 3\pi/2)$, according to Appendix A we thus obtain

$$g_u = \frac{\rho^2+1}{\rho^2}\,, \quad |\nu_u|^2 = \frac{2\rho^4+3\rho^2+1}{\rho^2(\rho^2+1)^3}\,, \quad |\xi_u| = \frac{\rho^2+2}{\rho\,(\rho^2+1)^{3/2}}\,, \quad \mathbf{K}_u = -\frac{1}{(\rho^2+1)^2}\,, \quad \mathbf{H}_u = 0\,.$$

We now compute the various components (7.2) and the boundary of $\widetilde{\Sigma}_u$. To this aim, in the sequel of this section we shall denote $I := (0,1)$. The corresponding energy terms are computed in Appendix B.

Firstly, we see that the boundary of the *Absolutely continuous component* GG_u^a satisfies

$$(\partial GG_u^a) \llcorner U \times \mathbb{S}^2 = \sum_{k=0}^{2} \big(\Gamma^k_{+\#}[\![I]\!] - \Gamma^k_{-\#}[\![I]\!] \big) - S_0 \tag{7.5}$$

where for $s \in I$ we have set:

$$\begin{array}{ll} \Gamma^0_+(s) := \big(s,0,\pi,0,0,1\big) & \Gamma^0_-(s) := \big(s,0,0,0,0,1\big) \\ \Gamma^1_+(s) := \big(0,s,\pi,(1+s^2)^{-1/2}(-1,0,s)\big) & \Gamma^1_-(s) := \big(0,s,\pi,0,0,1\big) \\ \Gamma^2_+(s) := \big(0,s-1,0,(1+(1-s)^2)^{-1/2}(1,0,1-s)\big) & \Gamma^2_-(s) := \big(0,s-1,0,0,0,1\big)\,. \end{array}$$

Therefore, the terms $\Gamma^0_{\pm\#}[\![I]\!]$ are the upper and lower components of the boundary of the Gauss graph $\mathcal{GG}_u$ in correspondence to the discontinuity set of u, whereas for $k = 1,2$, the terms $\Gamma^k_{\pm\#}[\![I]\!]$ are the upper and lower component of the boundary in correspondence to the edges in the graph $\mathcal{G}_u$.

Moreover, the 1-current $S_0 \in \mathcal{R}_1(U \times \mathbb{S}^2)$ is supported in $\{0_{\mathbb{R}^2}\} \times \mathbb{R} \times \mathbb{S}^2$ and is given by the weak limit as $\varepsilon \to 0$ of the image currents $\Phi_{u\#}[\![\partial B_\varepsilon(0_{\mathbb{R}^2})]\!]$. We have $\Phi_{u\#}[\![\partial B_\varepsilon(0_{\mathbb{R}^2})]\!] = \widetilde{\gamma}_{\varepsilon\#}[\![0, 2\pi]\!]$, where

$$\widetilde{\gamma}_\varepsilon(\theta) := \left\{ \begin{array}{ll} \big(\varepsilon\cos\theta, \varepsilon\sin\theta, \pi, 0,0,1\big) & \text{if} \quad 0<\theta<\pi/2 \\ \big(\varepsilon\cos\theta, \varepsilon\sin\theta, 3\pi/2-\theta, (\varepsilon^2+1)^{-1/2}(-\sin\theta,\cos\theta,\varepsilon)\big) & \text{if} \quad \pi/2<\theta<3\pi/2 \\ \big(\varepsilon\cos\theta, \varepsilon\sin\theta, 0,0,0,1\big) & \text{if} \quad 3\pi/2<\theta<2\pi\,. \end{array} \right.$$

Therefore, this time the currents $\Phi_{u\#}[\![\partial B_\varepsilon(0_{\mathbb{R}^2})]\!]$ weakly converge to $S_0 := \widetilde{\gamma}_{0\#}[\![0, 2\pi]\!]$, where

$$\widetilde{\gamma}_0(\theta) := \left\{ \begin{array}{ll} \big(0,0,\pi,0,0,1\big) & \text{if} \quad 0<\theta<\pi/2 \\ \big(0,0,3\pi/2-\theta, \cos(\theta+\pi/2), \sin(\theta+\pi/2), 0\big) & \text{if} \quad \pi/2<\theta<3\pi/2 \\ \big(0,0,0,0,0,1\big) & \text{if} \quad 3\pi/2<\theta<2\pi\,. \end{array} \right. \tag{7.6}$$

Remark 7.2 Here and below, the parameter $s \in I$ refers to the horizontal directions e_1, e_2, whereas the parameter $\lambda \in I$ to the vertical directions e_3, ε_j, $j = 1,2,3$.

The *Cantor component* of $\widetilde{\Sigma}_u$ is trivial, $GG_u^C = 0$. Moreover, the *Jump component* is given by

$$GG_u^J = \Phi^J_\#[\![I \times I]\!]\,, \quad \Phi^J(s,\lambda) := \big(s,0,\lambda\,\pi,0,-1,0\big)\,, \quad (s,\lambda) \in I \times I\,. \tag{7.7}$$

Denoting by Q and P the end points of the Jump set, so that $Q = 0_{\mathbb{R}^2}$ and $P = (0,1) \in \partial B^2$, and computing the boundary through the formula

$$\partial GG_u^J = \Phi^J_\# \partial [\![I \times I]\!] = \Phi^J_\#\big((\delta_1 - \delta_0) \times [\![I]\!] - [\![I]\!] \times (\delta_1 - \delta_0) \big)$$

we obtain

$$(\partial GG_u^J) \llcorner U \times \mathbb{S}^2 = -\Gamma_{Q\#}[\![I]\!] - \big(\Gamma^J_{+\#}[\![I]\!] - \Gamma^J_{-\#}[\![I]\!] \big) \tag{7.8}$$

where we have set

$$\begin{array}{l} \Gamma_Q(\lambda) := \big(0,0,\lambda\,\pi,0,-1,0\big)\,, \quad \lambda \in I \\ \Gamma^J_+(s) := \big(s,0,\pi,0,-1,0\big)\,, \quad \Gamma^J_-(s) := \big(s,0,0,0,-1,0\big)\,, \quad s \in I\,. \end{array} \tag{7.9}$$

The *Jump-edge component* deals with the Jump of ν_u w.r.t. the outward normal to the "wall" surface given by $\Phi^J(I \times I)$ at the upper and lower edges. We have:

$$S_u^{Je} = \Phi^{Je}_{+\#}[\![I \times I]\!] - \Phi^{Je}_{-\#}[\![I \times I]\!] \tag{7.10}$$

where for $(s,\lambda) \in I \times I$ we have set

$$\begin{array}{ll} \Phi^{Je}_+(s,\lambda) := & \big(s,0,\pi,\phi^{Je}_+(s,\lambda)\big)\,, \\ \Phi^{Je}_-(s,\lambda) := & \big(s,0,0,\phi^{Je}_-(s,\lambda)\big)\,, \end{array} \quad \phi^{Je}_\pm(s,\lambda) := \big(0, \cos(\pi(2-\lambda)/2), \sin(\pi(2-\lambda)/2)\big)\,.$$

We thus compute as above

$$(\partial S_u^{Je}) \llcorner U \times \mathbb{S}^2 = -\big(\Gamma^Q_{+\#}[\![I]\!] - \Gamma^Q_{-\#}[\![I]\!]\big) - \big(\Gamma^0_{+\#}[\![I]\!] - \Gamma^J_{+\#}[\![I]\!]\big) + \big(\Gamma^0_{-\#}[\![I]\!] - \Gamma^J_{-\#}[\![I]\!]\big) \tag{7.11}$$

where for $\lambda \in I$ we have set

$$\begin{array}{rl} \Gamma^Q_+(\lambda) := & \big(0,0,\pi,0,\cos(\pi(2-\lambda)/2),\sin(\pi(2-\lambda)/2)\big) \\ \Gamma^Q_-(\lambda) := & \big(0,0,0,0,\cos(\pi(2-\lambda)/2),\sin(\pi(2-\lambda)/2)\big)\,. \end{array} \tag{7.12}$$

Finally, the *Edge component* deals with the Jump of ν_u where u is continuous, see (7.3). We have:

$$S_u^e = \sum_{k=1}^2 \Phi^e_{k\#}[\![I \times I]\!] \tag{7.13}$$

where the mappings $\Phi^e_k : I \times I \to U \times \mathbb{S}^2$ are defined by

$$\Phi^e_k(s,\lambda) := \big(\gamma^1_k(s), \gamma^2_k(s), u(\gamma_k(s)), \phi^e_k(s,\lambda)\big)\,, \quad (s,\lambda) \in I \times I\,.$$

In the above formula, we have set $\gamma_1(s) := (0,s)$ and $\gamma_2(s) := (0, s-1)$, so that $u(\gamma_1(s)) \equiv \pi$ and $u(\gamma_2(s)) \equiv 0$. Now, on account of (7.3), the one-sided limits of ν_u at the two edges of the graph $\mathcal{G}_u$ are:

$$\begin{array}{ll} {\nu_u}^+(\gamma_1(s)) = (1+s^2)^{-1/2}(-1,0,s) & {\nu_u}^-(\gamma_1(s)) = (0,0,1) \\ {\nu_u}^+(\gamma_2(s)) = (1+(1-s)^2)^{-1/2}(1,0,1-s) & {\nu_u}^-(\gamma_2(s)) = (0,0,1)\,. \end{array}$$

Therefore, for $k = 1,2$, the functions $\lambda \mapsto \phi^e_k(s,\lambda)$ parameterize with $\lambda \in I$ the oriented geodesic arc in $\mathbb{S}^2_+$ connecting ${\nu_u}^-(\gamma_k(s))$ to ${\nu_u}^+(\gamma_k(s))$, whence for $(s,\lambda) \in I \times I$ we have:

$$\begin{array}{ll} \phi^e_1(s,\lambda) := \big(\cos\theta_1(s,\lambda), 0, \sin\theta_1(s,\lambda)\big)\,, & \theta_1(s,\lambda) := (\pi/2 - \arctan s)\,\lambda + \pi/2\,, \\ \phi^e_2(s,\lambda) := \big(\cos\theta_2(s,\lambda), 0, \sin\theta_2(s,\lambda)\big)\,, & \theta_2(s,\lambda) := (\arctan(1-s) - \pi/2)\,\lambda + \pi/2\,. \end{array} \tag{7.14}$$

Denoting by Q_k and P_k the end points of the two segments γ_k, so that $P_1 = (0,1) \in \partial B^2$, $Q_1 = P_2 = 0_{\mathbb{R}^2}$, and $Q_2 = (0,-1) \in \partial B^2$, we thus compute:

$$(\partial S_u^e) \llcorner U \times \mathbb{S}^2 = \big(\Gamma_{P_2\#}[\![I]\!] - \Gamma_{Q_1\#}[\![I]\!]\big) - \sum_{k=1}^2 \big(\Gamma^k_{+\#}[\![J_k]\!] - \Gamma^k_{-\#}[\![J_k]\!]\big) \tag{7.15}$$

where we have set

$$\begin{array}{rl} \Gamma_{Q_1}(\lambda) := & \big(0,0,\pi,\cos(\pi(\lambda+1)/2),0,\sin(\pi(\lambda+1)/2)\big)\,, \\ \Gamma_{P_2}(\lambda) := & \big(0,0,0,\cos(\pi(1-\lambda)/2),0,\sin(\pi(1-\lambda)/2)\big)\,. \end{array} \tag{7.16}$$

Boundary. We now compute the boundary

$$\Gamma_{0_{\mathbb{R}^2}} := (\partial \widetilde{\Sigma}_u) \llcorner U \times \mathbb{S}^2\,.$$

Putting the boundary terms (7.5), (7.8), (7.11), and (7.15) together, after some simplification we get

$$\Gamma_{0_{\mathbb{R}^2}} = -\widetilde{\Gamma}_Q - \widetilde{\gamma}_{0\#}[\![(0,2\pi)]\!] + \Gamma_{P_2\#}[\![I]\!] - \Gamma_{Q_1\#}[\![I]\!]$$

where the first addendum

$$\widetilde{\Gamma}_Q := \Gamma_{Q\#}[\![I]\!] + \Gamma^Q_{+\#}[\![I]\!] - \Gamma^Q_{-\#}[\![I]\!]$$

is the sum of the lateral boundary of the Jump and Jump-edge components at the singular point $0_{\mathbb{R}^2}$, see (7.9) and (7.12). The other terms, see (7.6) and (7.16), are produced by the boundary of the Absolutely continuous and Edge components. Therefore, writing in order

$$\Gamma_{0_{\mathbb{R}^2}} = -\Gamma_{Q\#}[\![I]\!] + \Gamma^Q_{-\#}[\![I]\!] + \Gamma_{P_2\#}[\![I]\!] - \widetilde{\gamma}_{0\#}[\![0,2\pi]\!] - \Gamma_{Q_1\#}[\![I]\!] - \Gamma^Q_{+\#}[\![I]\!]$$

we infer that

$$(\partial \widetilde{\Sigma}_u) \llcorner U \times \mathbb{S}^2 = \delta_{0_{\mathbb{R}^2}} \times \gamma_{0_{\mathbb{R}^2}\#}[\![0,6]\!] \tag{7.17}$$

where $\gamma_{0_{\mathbb{R}^2}} : [0,6] \to \mathbb{R}_z \times \mathbb{S}^2$ is the closed Lipschitz curve

$$\gamma_{0_{\mathbb{R}^2}}(t) := \begin{cases} \big(\pi(1-t), 0, -1, 0\big) & \text{if } \ 0 \le t \le 1 \\ \big(0, 0, \cos(\pi(3-t)/2), \sin(\pi(3-t)/2)\big) & \text{if } \ 1 \le t \le 2 \\ \big(0, \cos(\pi(3-t)/2), 0, \sin(\pi(3-t)/2)\big) & \text{if } \ 2 \le t \le 3 \\ \big(\pi(t-3), \cos(\pi(3-t)), \sin(\pi(3-t)), 0\big) & \text{if } \ 3 \le t \le 4 \\ \big(\pi, \cos(\pi(6-t)/2), 0, \sin(\pi(6-t)/2)\big) & \text{if } \ 4 \le t \le 5 \\ \big(\pi, 0, \cos(\pi(t-4)/2), \sin(\pi(t-4)/2)\big) & \text{if } \ 5 \le t \le 6\,. \end{cases}$$

Notice in fact that

$$\gamma_{0_{\mathbb{R}^2}}(0) = \gamma_{0_{\mathbb{R}^2}}(6) = (\pi, 0, -1, 0)\,, \quad \gamma_{0_{\mathbb{R}^2}}(1) = (0, 0, -1, 0)\,, \quad \gamma_{0_{\mathbb{R}^2}}(2) = (0, 0, 0, 1)\,,$$
$$\gamma_{0_{\mathbb{R}^2}}(3) = (0, 1, 0, 0)\,, \quad \gamma_{0_{\mathbb{R}^2}}(4) = (\pi, -1, 0, 0)\,, \quad \gamma_{0_{\mathbb{R}^2}}(5) = (\pi, 0, 0, 1)\,.$$

Moreover, letting $\Gamma_i := \gamma_{0_{\mathbb{R}^2}\,\#}[\![\, i-1, i \,]\!]$, where $i = 1, \ldots, 6$, we have

$$\begin{array}{l} \Gamma_1 = -\Gamma_{Q\#}[\![\, I \,]\!]\,, \quad \Gamma_2 = \Gamma^Q_{-\#}[\![\, I \,]\!]\,, \quad \Gamma_3 = \Gamma_{P_2\#}[\![\, I \,]\!]\,, \\ \Gamma_4 = -\widetilde{\gamma}_{0\#}[\![\, 0, 2\pi \,]\!]\,, \quad \Gamma_5 = -\Gamma_{Q_1\#}[\![\, I \,]\!]\,, \quad \Gamma_6 = -\Gamma^Q_{+\#}[\![\, I \,]\!]\,. \end{array}$$

Remark 7.3 We finally point out that the Lipschitz-continuous curve $\gamma_{0_{\mathbb{R}^2}}$ of $\mathbb{R}_z \times \mathbb{R}^3_y$ is closed, symmetric w.r.t. to the plane $\Pi := \{z = 1/2,\, y_1 = 0\}$, and with a self-intersection at Π, as

$$\gamma_{0_{\mathbb{R}^2}}(1/2) = \gamma_{0_{\mathbb{R}^2}}(7/2) = O := \big(\pi/2, 0, -1, 0\big)\,.$$

8 Gap phenomenon

In the previous section we have seen how (for homogeneous functions, $u(x) = f(\theta)$) the current $\widetilde{\Sigma}_u$ depends on the relaxed energy of the function f through the formula (5.2), where $r = 0$. However, in general the current $\widetilde{\Sigma}_u$ has a non-trivial boundary in $U \times \mathbb{S}^2$, as a cavity may occur at the origin $0_{\mathbb{R}^2}$.

Following the example studied in detail in Sec. 6, one is then induced to look for an optimal current S^s_u that "fills the hole" at the origin, i.e., a *singular component* S^s_u supported in $\{0_{\mathbb{R}^2}\} \times \mathbb{R}_z \times \mathbb{S}^2$ and satisfying $\partial S^s_u = -\partial \widetilde{\Sigma}_u$ in $U \times \mathbb{S}^2$, in such a way that the current $\Sigma_u := \widetilde{\Sigma}_u + S^s_u$ belongs to the class $\mathrm{Gcart}(U \times \mathbb{S}^2)$. In fact, in the relaxation process an energy contribution given by a measure concentrated at the origin is expected. However, we now see that a *gap phenomenon* occurs:

Theorem 8.1 *There exists a piecewise constant and homogeneous BV-function $u \in \mathcal{E}(\Omega)$ such that the optimal current $\widetilde{\Sigma}_u$ has no boundary at the origin, and $\widetilde{\Sigma}_u \in \mathrm{Gcart}_u$, but $\widetilde{\mathcal{E}}(u) > \mathrm{E}(\widetilde{\Sigma}_u)$.*

Therefore, the strict inequality in (3.18) holds, and hence the equality in (3.17) is violated.

Example 8.2 Let $u : B^2 \to \mathbb{R}$ be the piecewise constant BV-function such that $u(x) = f(\theta)$, where

$$f(\theta) := \begin{cases} 1 & \text{if } \ \theta \in (0, \pi/3) \cup (2\pi/3, \pi) \cup (4\pi/3, 5\pi/3) \\ 0 & \text{if } \ \theta \in (\pi/3, 2\pi/3) \cup (\pi, 4\pi/3) \cup (5\pi/3, 2\pi)\,. \end{cases} \tag{8.1}$$

The jump set of u is given by the union of the six radial segments from the origin which are perpendicular to the unit vectors $\mathbf{v_i}$ in the (x_1, x_2)-plane with coordinates

$$\mathbf{v_1} := \big(-\sqrt{3}/2, 1/2\big)\,, \quad \mathbf{v_2} := \big(-\sqrt{3}/2, -1/2\big)\,, \quad \mathbf{v_3} := \big(0, -1\big)\,. \tag{8.2}$$

Moreover, we have $\nabla u = 0$ a.e. in B^n, and $D^C u = 0$.

Remark 8.3 Notice that according to the definition of the transformation Ψ from (5.3), for $i = 1, 2, 3$ we have $\mathbf{v_i} = (\cos(\theta_i + \pi/2), \sin(\theta_i + \pi/2)) = (-\sin\theta_i, \cos\theta_i)$, where $\theta_1 = \pi/3$, $\theta_2 = 2\pi/3$, $\theta_3 = \pi$ are the angles that correspond to the first three discontinuity points of the function f in (8.1).

Referring to the notation from Sec. 5, since the BV-function f has a derivative with no absolutely continuous and Cantor parts, we have $GG_f^a := \Phi_{f\#}[\![0,2\pi]\!]$, where $\Phi_f(\theta) := (\theta, f(\theta), 0, 1)$, and $GG_f^C = 0$. The Jump component is the Gauss graph of the vertical segments in the boundary of the subgraph of f, at the jump set, and actually in $[0,2\pi[\times\mathbb{R}\times\mathbb{S}^1$ we have:

$$GG_f^J = (\delta_0 + \delta_{2\pi/3} + \delta_{4\pi/3}) \times [\![0,1]\!] \times \delta_{(-1,0)} - (\delta_{\pi/3} + \delta_\pi + \delta_{5\pi/3}) \times [\![0,1]\!] \times \delta_{(1,0)} \,.$$

Since moreover $\dot f = 0$, no corner points occur, whence the Corner component $S_f^c = 0$. Also, two corners appear in the boundary of the subgraph of f at each discontinuity point, with turning angles equal to $\pi/2$. Therefore, the Jump-corner component S_f^{Jc} is given by:

$$S_f^{Jc} := (\delta_0 + \delta_{2\pi/3} + \delta_{4\pi/3}) \times (\delta_0 - \delta_1) \times [\![\gamma_-]\!] + (\delta_{\pi/3} + \delta_\pi + \delta_{5\pi/3}) \times (\delta_1 - \delta_0) \times [\![\gamma_+]\!]$$

where $\gamma_\pm$ is the oriented geodesic arc in $\mathbb{S}^1$ with initial point $(0,1)$ and final point $(\pm1,0)$.

As in (5.2), we let $\widetilde\Sigma_u := \Psi_\#([\![0,1]\!]\times\Sigma_f)$, where Ψ is the transformation in (5.3), so that this time

$$\widetilde\Sigma_u = GG_u^a + GG_u^J + S_u^{Je} \,,$$

according to (7.2). We now see that

$$\partial\widetilde\Sigma_u = 0 \quad \text{on} \quad \mathcal{D}^1(U\times\mathbb{S}^2) \tag{8.3}$$

so that the singular component is trivial, $S_u^s = 0$. In fact, on account of Remark 5.1 we deduce that

$$(\partial\widetilde\Sigma_u) \llcorner U\times\mathbb{S}^2 = \delta_{0_{\mathbb{R}^2}} \times \gamma_\#[\![0,18]\!]$$

where $\gamma : [0,18] \to \mathbb{R}_z \times \mathbb{S}^2_+$ is the closed rectifiable arc

$$\gamma(t) := \begin{cases}
\big(1, \sin(\pi t/2)\,\mathbf{v_1}, \cos(\pi t/2)\big) & \text{if} \quad t\in[0,1] \\
\big(2-t, \mathbf{v_1}, 0\big) & \text{if} \quad t\in[1,2] \\
\big(0, \cos(\pi(t-2)/2)\,\mathbf{v_1}, \sin(\pi(t-2)/2)\big) & \text{if} \quad t\in[2,3] \\
\big(0, \sin(\pi(t-3)/2)\,\mathbf{v_2}, \cos(\pi(t-3)/2)\big) & \text{if} \quad t\in[3,4] \\
\big(t-4, \mathbf{v_2}, 0\big) & \text{if} \quad t\in[4,5] \\
\big(1, \cos(\pi(t-5)/2)\,\mathbf{v_2}, \sin(\pi(t-5)/2)\big) & \text{if} \quad t\in[5,6] \\
\big(1, \sin(\pi(t-6)/2)\,\mathbf{v_3}, \cos(\pi(t-6)/2)\big) & \text{if} \quad t\in[6,7] \\
\big(8-t, \mathbf{v_3}, 0\big) & \text{if} \quad t\in[7,8] \\
\big(0, \cos(\pi(t-8)/2)\,\mathbf{v_3}, \sin(\pi(t-8)/2)\big) & \text{if} \quad t\in[8,9] \\
\big(0, \sin(\pi(t-9)/2)\,\mathbf{v_1}, \cos(\pi(t-9)/2)\big) & \text{if} \quad t\in[9,10] \\
\big(t-10, \mathbf{v_1}, 0\big) & \text{if} \quad t\in[10,11] \\
\big(1, \cos(\pi(t-11)/2)\,\mathbf{v_1}, \sin(\pi(t-11)/2)\big) & \text{if} \quad t\in[11,12] \\
\big(1, \sin(\pi(t-12)/2)\,\mathbf{v_2}, \cos(\pi(t-12)/2)\big) & \text{if} \quad t\in[12,13] \\
\big(14-t, \mathbf{v_2}, 0\big) & \text{if} \quad t\in[13,14] \\
\big(0, \cos(\pi(t-14)/2)\,\mathbf{v_2}, \sin(\pi(t-14)/2)\big) & \text{if} \quad t\in[14,15] \\
\big(0, \sin(\pi(t-15)/2)\,\mathbf{v_3}, \cos(\pi(t-15)/2)\big) & \text{if} \quad t\in[15,16] \\
\big(t-16, \mathbf{v_3}, 0\big) & \text{if} \quad t\in[16,17] \\
\big(1, \cos(\pi(t-17)/2)\,\mathbf{v_3}, \sin(\pi(t-17)/2)\big) & \text{if} \quad t\in[17,18]
\end{cases} \tag{8.4}$$

where we have denoted by $\mathbf{v_i}$ the unit vectors in the (y_1,y_2)-plane given by formulas (8.2).

Now, it turns out that the closed arc γ in $\mathbb{R}_z\times\mathbb{S}^2_+$ is *homologically trivial*, as its support is parameterized twice and with opposite orientation. More precisely, denoting with the letters A,B,C the oriented arcs $\gamma([0,3])$, $\gamma([3,6])$, and $\gamma([6,9])$, respectively, we have:

$$\begin{aligned}
[\![A]\!] &= \gamma_\#[\![0,3]\!] = -\gamma_\#[\![9,12]\!]\,, \\
[\![B]\!] &= \gamma_\#[\![3,6]\!] = -\gamma_\#[\![12,15]\!]\,, \\
[\![C]\!] &= \gamma_\#[\![6,9]\!] = -\gamma_\#[\![15,18]\!]\,.
\end{aligned}$$

Therefore, we get $\gamma_\#[\![0,18]\!] = 0$ and hence the null-boundary condition (8.3) holds.

However, the closed arc γ is *topologically non-trivial* (both in $\mathbb{R}^z \times \mathbb{S}^2_+$ and in $\mathbb{R}^4$). In fact, it turns out that the loop γ describes the sequence of letters $A\,B\,C\,A^{-1}B^{-1}C^{-1}$, hence it is not contractible: any homotopy map which deforms γ to a point has a nontrivial mapping area. More precisely, denoting by D the oriented line segment from $(0,0,0,1)$ to $(1,0,0,1)$, an area minimizing contraction has firstly to cover twice (with opposite orientation) the surface solution to the Plateau's problem with boundary given by the closed arc $C\,D$, reducing to the loop given by the sequence $A\,B\,D^{-1}A^{-1}B^{-1}D$, and, secondly, twice (with opposite orientation) the surface solution to the Plateau's problem with boundary given by the closed arc $B\,D^{-1}$, whence reducing to the contractible loop given by the sequence $A\,D\,D^{-1}\,A^{-1}D^{-1}D$.

Proof of Theorem 8.1: Let u be given by the previous example, so that (8.3) holds. It is not difficult to find a smooth sequence $\{u_h\} \subset C^2_b(B^2)$ such that $GG_{u_h} \rightharpoonup \widetilde{\Sigma}_u$ weakly in $\mathcal{D}_2(U \times \mathbb{S}^2)$ and $\sup_h \mathcal{E}(u_h) < \infty$, whence $\widetilde{\Sigma}_u \in \text{Gcart}(U \times \mathbb{S}^2)$, where $U := B^2 \times \mathbb{R}$, and actually $\widetilde{\Sigma}_u \in \text{Gcart}_u$. For any $r \in (0,1)$, let $f^{(r)}_h(\theta) := u_h(r\cos\theta, r\sin\theta)$, $\theta \in I := [0,2\pi]$. By a slicing argument, it turns out that for a.e. r the sequence $GG_{f^{(r)}_h}$ weakly converges in $\mathcal{D}_1(\mathring{I} \times \mathbb{R} \times \mathbb{S}^1)$ to the current $\Sigma_f := GG^a_f + GG^J_f + S^{Jc}_f$, where f is given by (8.1). Arguing as in Prop. 4.5 and Rmk. 4.6 from [2], we can find a smooth sequence $\Psi^{(r)}_h : I_L \to I \times \mathbb{R} \times \mathbb{S}^1$, where $I_L := [0,L]$ for a suitable $L > 0$, and a Lipschitz function $\Psi^{(r)} : I_L \to I \times \mathbb{R} \times \mathbb{S}^1$ such that the following properties hold:

i) the smooth curve $\Psi^{(r)}_h$ is one-to-one, has constant velocity, and $\Psi^{(r)}_h{}_\#[\![I_L]\!] = GG_{f^{(r)}_h}$;

ii) the Lipschitz curve $\Psi^{(r)}$ is one-to-one, has constant velocity, and $\Psi^{(r)}{}_\#[\![I_L]\!] = \Sigma_f$;

iii) the sequence $\{\Psi^{(r)}_h\}$ converges to $\Psi^{(r)}$ uniformly in I_L, and $\Psi^{(r)}_h{}_\#[\![I_L]\!] \rightharpoonup \Psi^{(r)}{}_\#[\![I_L]\!]$ weakly in $\mathcal{D}_1(I \times \mathbb{R} \times \mathbb{S}^1)$.

Moreover, denoting $\widetilde{\Sigma}_u = [\![R, \theta, \overrightarrow{\zeta}]\!]$, see Remark 3.3, we observe that for each $\varepsilon > 0$ we can find a small radius $r_\varepsilon > 0$ such that if D_{r_ε} denotes the cylinder $D_{r_\varepsilon} := B^2_{r_\varepsilon} \times \mathbb{R} \times \mathbb{S}^2$, then

$$\int_{R\cap D_{r_\varepsilon}} |\zeta^{(i)}|\,\theta\, d\mathcal{H}^2 < \varepsilon/4 \qquad \forall\, i = 0,1,2$$

and hence the curvature energy of $\widetilde{\Sigma}_u$ is small on D_{r_ε}.

Assume now by contradiction that $\mathcal{E}(u_h) \to \mathrm{E}(\widetilde{\Sigma}_u)$. Then, by lower semicontinuity, we can find h_ε such that for every $h \geq h_\varepsilon$

$$\int_{B^2_{r_\varepsilon}} \sqrt{g_{u_h}}\,\Big(1 + \sqrt{4\mathbf{H}^2_{u_h} - 2\mathbf{K}_{u_h}} + |\mathbf{K}_{u_h}|\Big)\, d\mathcal{L}^2 < \varepsilon\,.$$

As a consequence, we deduce that for a.e. $r \in (0, r_\varepsilon)$ and for every $h \geq h_\varepsilon$ the curve $\Psi^{(r)}_h$ can be deformed to a constant by means of a smooth homotopy $H^{(r)}_h$ with mapping area smaller than $c\,\varepsilon$, for some absolute constant $c > 0$. We now observe that the Lipschitz curves $\Psi^{(r)}$ uniformly converge to a suitable reparameterization of the closed arc $\gamma : [0,18] \to \mathbb{R}_z \times \mathbb{S}^2_+$, as $r \to 0$. The uniform convergence of $\Psi^{(r)}_h$ to the Lipschitz curve $\Psi^{(r)}$, and the non-triviality of the closed arc γ previously described, yield to a contradiction with the existence of the smooth homotopy map $H^{(r)}_h$ with small mapping area previously described. $\square$

9 Final remarks and open questions

In this final section we collect some ideas towards the direction of finding an explicit formula for the relaxed energy (3.1), a widely open problem even in the case of homogeneous functions.

ENERGY ON GAUSS GRAPHS. We can write more explicitly the curvature energy on the Gauss graphs $\widetilde{\Sigma}_u$ of homogeneous functions $u \in \mathcal{E}(B^2)$. Since $\widetilde{\Sigma}_u$ has finite mass, on account of (7.2) it turns out that the related mass decomposition holds:

$$\mathbf{M}(\widetilde{\Sigma}_u) = \mathbf{M}(GG_u^a) + \mathbf{M}(GG_u^C) + \mathbf{M}(GG_u^J) + \mathbf{M}(S_u^{Je}) + \mathbf{M}(S_u^e) .$$

Therefore, by the definition of energy on currents (3.13), the corresponding decomposition formula holds:

$$\mathrm{E}(\widetilde{\Sigma}_u) = \mathrm{E}(GG_u^a) + \mathrm{E}(GG_u^C) + \mathrm{E}(GG_u^J) + \mathrm{E}(S_u^{Je}) + \mathrm{E}(S_u^e) .$$

We shall provide in Appendix B the explicit computation when u is given by Example 7.1.

EVENTUAL NON-LOCALITY. Consider the localization of the relaxed functional, defined for any bounded function $u \in \mathcal{E}(\Omega)$ and any open (or Borel) set $A \subset \Omega$ by:

$$\overline{\mathcal{E}}(u, A) := \inf\{\liminf_{h\to\infty} \mathcal{E}(u_h, A) \mid \{u_h\}_h \subset C_b^2(A) , \ u_h \to u \text{ strongly in } L^1(A)\}$$

where for smooth functions we have set

$$\mathcal{E}(u_h, A) := \int_A \left(|\xi_{u_h}^{(0)}| + |\xi_{u_h}^{(1)}| + |\xi_{u_h}^{(2)}| \right) d\mathcal{L}^2 .$$

It is not clear if one could find a function $u \in \mathcal{E}(\Omega)$ such that the set function $A \mapsto \overline{\mathcal{E}}(u, A)$ fails to be subadditive, i.e., for which we can find open sets $A_1, A_2, A_3 \subset \Omega$ such that $A_3 \subset\subset A_1 \cup A_2$ but $\overline{\mathcal{E}}(u, A_3) > \overline{\mathcal{E}}(u, A_1) + \overline{\mathcal{E}}(u, A_2)$. For this purpose, one could try to reproduce the argument used by Acerbi-Dal Maso [1] in order to show the *non-locality of the relaxed area functional.* Consider in fact the homogeneous map $u(x) := x/|x|$, where $x \in B^2$, so that (cf. [17, Sec. 3.2.2]) one has

$$(\partial G_u) \llcorner B^2 \times \mathbb{R}^2 = -\delta_{0_{\mathbb{R}^2}} \times [\![\mathbb{S}^1]\!] .$$

Roughly speaking, there are two qualitatively different ways to fill the hole in the graph of u: inserting a disk $\delta_{0_{\mathbb{R}^2}} \times [\![B^2]\!]$ or a cylinder $[\![I]\!] \times [\![\mathbb{S}^1]\!]$, where I is any oriented line segment connecting a point in the boundary ∂B^2 of the domain to the origin $0_{\mathbb{R}^2}$. This property implies the non-locality, see [1].

In our context, assume e.g. $\Omega = B^2$ and $u : B^2 \to \mathbb{R}$ homogeneous, $u(x) = f(\theta)$, so that in general with the notation (7.1) and (7.2) one has:

$$(\partial\widetilde{\Sigma}_u) \llcorner U \times \mathbb{S}^2 = -\delta_{0_{\mathbb{R}^2}} \times \Gamma$$

for some integral 1-cycle $\Gamma \in \mathcal{D}_1(\mathbb{R}_z \times \mathbb{S}^2)$, compare e.g. (7.17). As we already discussed in Sec. 6, where $\widetilde{\Sigma}_u = GG_u$, in principle there are two qualitatively different ways to "fill the hole" at the origin, namely:

i) choosing the energy minimizing current S_1 with support in $\{0_{\mathbb{R}^2}\} \times \mathbb{R}_z \times \mathbb{S}^2_+$ and satisfying the geometric constraint and the boundary condition $\partial S = \delta_{0_{\mathbb{R}^2}} \times \Gamma$;

ii) choosing e.g. $S_2 := -F_\#([\![0, 1]\!] \times (\delta_{0_{\mathbb{R}^2}} \times \Gamma))$, where F is the homotopy map $F(\lambda, (0_{\mathbb{R}^2}, z, y)) := (\lambda\, \mathbf{v}, z, y)$ for some direction $\mathbf{v} \in \mathbb{S}^1$.

MEASURE PROPERTY. However, Proposition 6.3 suggests that in general if we impose the above mentioned geometric condition, the only way to fill the hole of the current $\widetilde{\Sigma}_u$ is by means of 2-dimensional i.m. rectifiable currents with support in $\{0_{\mathbb{R}^2}\} \times \mathbb{R}_z \times \mathbb{S}^2_+$, so that the second alternative is excluded. Therefore, at least for homogeneous functions u we expect the set function $A \mapsto \mu_u(A) := \overline{\mathcal{E}}(u, A)$ to be subadditive, and hence a measure, as a consequence of the De Giorgi-Letta criterion [12]. This conjecture is motivated by the following examples.

Example 9.1 If u is given by Example 6.1, in Sec. 6 we have actually proved that set function μ_u is a finite Radon measure, with absolutely continuous component given by the integral $\mathcal{E}(u, \cdot)$, and singular part a Dirac measure at the origin and with weight equal to the energy of the optimal current S_1:

$$\mu_u = \mathcal{E}(u, \cdot) + c \cdot \delta_{0_{\mathbb{R}^2}} , \qquad c := \mathrm{E}(S_1)$$

where, we recall, $\mathrm{E}(S_1) = \mathrm{E}_0(S_1) + \mathrm{E}_1(S_1) + \mathrm{E}_2(S_2)$, with $\mathrm{E}_0(S_1) = 0$, $E_1(S_1) = 4\pi - 8$, $\mathrm{E}_2(S_1) = 2\pi$. The energy term $\mathcal{E}(u) = \mathcal{E}_1(u) + \mathcal{E}_1(u) + \mathcal{E}_2(u)$ can be computed according to the formulas from Remark A.1 below, where the energy density is obtained by taking with $f = \cos\theta$.

Example 9.2 In a similar way, if u is given by Example 7.1, on account of the density theorem 5.2, we expect the set function μ_u to be again a finite Radon measure with absolutely continuous component given by the integral $\mathcal{E}(u,\cdot)$. Since the distributional derivative of u has no Cantor component, $D^C u = 0$, there is no Cantor component GG_u^C and hence the measure μ_u has no other diffuse terms. This time, the singular component is given by two qualitatively different terms.

The first term is concentrated on a 1-dimensional set and depends on the energy of the other components of $\widetilde{\Sigma}_u$ in the formula (7.1). It is given by two components: the first one, say μ_1, lives on the jump set of u, i.e., on the interval $J_u = (0,1)\times\{0\}$, and it depends on the energies of the Jump component GG_u^J and of Jump-edge component S_u^{Je}. More precisely, on account of the definitions (7.7) and (7.10), and choosing $I_A \subset (0,1)$ through the formula $I_A \times \{0\} = A\cap J_u$, for any Borel set $A\subset B^2$ we get

$$\mu_1(A) = c_1 \cdot \mathcal{H}^1 \llcorner J_u(A) + \mathrm{E}({\Phi^{Je}_{+}}_{\#}[\![I_A\times(0,1)]\!]) + \mathrm{E}({\Phi^{Je}_{-}}_{\#}[\![I_A\times(0,1)]\!])$$

where the weight c_1 is given by the energy $\mathrm{E}(GG_u^J)$ of the Jump component.

The second component, say μ_2, lives on the jump set of the approximate gradient ∇u, i.e., on the interval $J_{\nabla u} = \{0\}\times(-1,1)$, and it depends on the energy of the Edge component S_u^e, compare (7.13). Choosing this time $I_A^1 \subset (0,1)$ and $I_A^2 \subset (-1,0)$ through the formulas $\{0\}\times I_A^1 = A_1 \cap J_{\nabla u}$ and $\{0\}\times I_A^2 = A_2\cap J_{\nabla u}$, respectively, where $A_1 := \{x\in A \mid x_2>0\}$ and $A_2 := \{x\in A\mid x_2<0\}$, for any Borel set $A\subset B^2$ we get

$$\mu_2(A) = \mathrm{E}(\Phi^e_{1\#}[\![I_A^1\times(0,1)]\!]) + \mathrm{E}(\Phi^e_{2\#}[\![(1+I_A^2)\times(0,1)]\!]\,.$$

Finally, a second term appears: as in the previous example, it is given by a Dirac measure centered at the origin and with weight equal to the energy of the optimal current S_1 that fills the hole in the current $\widetilde{\Sigma}_u$, according to formula (7.17), see also Remark 7.3. In conclusion, this time we get

$$\mu_u = \mathcal{E}(u,\cdot) + \mu_1 + \mu_2 + c\cdot\delta_{0_{\mathbb{R}^2}}\,,\quad c:=\mathrm{E}(S_1)$$

where $\mathrm{E}_0(S_1)=0$, $E_1(S_1) = 2\pi-4$, $\mathrm{E}_2(S_1)=\pi$. The energy term $\mathcal{E}(u) = \mathcal{E}_1(u)+\mathcal{E}_1(u)+\mathcal{E}_2(u)$ can be computed according to the formulas from Remark A.1 below, where the energy density is obtained by taking with f as in (7.4). The energy contribution of the different components GG_u^J, Su^J, S_u^{Je}, and S_u^e are computed in Appendix B. We omit any further detail.

A Homogeneous functions

Assume $u : B^2\to\mathbb{R}$ satisfies $u(x) = u(x/|x|)$ for each $x\in B^2\setminus\{0_{\mathbb{R}^2}\}$, whence we can write $u(x) = f(\theta)$ for some function $f:[0,2\pi]\to\mathbb{R}$, where $x = (\rho\cos\theta,\rho\sin\theta)$. Denoting for simplicity $s:=\sin\theta$ and $c:=\cos\theta$, we formally compute on $B^2\setminus\{0_{\mathbb{R}^2}\}$

$$\nabla u = \Big(-\frac{\dot f s}{\rho},\frac{\dot f c}{\rho}\Big)\,,\quad \nu_u = \frac{1}{(\rho^2+\dot f^2)^{1/2}}\,(\dot f s, -\dot f c, \rho)$$

and hence, setting $F:=\rho^2+\dot f^2$, we get

$$\begin{aligned}
\partial_1{\nu_u}^1 &= \frac{-1}{\rho\,F^{3/2}}\big[\rho^2(2\dot f s\,c + \ddot f s^2) + \dot f^3 s\,c\big]\,, & \partial_2{\nu_u}^1 &= \frac{1}{\rho\,F^{3/2}}\big[\rho^2(\dot f(c^2-s^2)+\ddot f s\,c) + \dot f^3 c^2\big]\,,\\
\partial_1{\nu_u}^2 &= \frac{1}{\rho\,F^{3/2}}\big[\rho^2(\dot f(c^2-s^2)+\ddot f s\,c) - \dot f^3 s^2\big]\,, & \partial_2{\nu_u}^2 &= \frac{1}{\rho\,F^{3/2}}\big[\rho^2(2\dot f s\,c - \ddot f c^2) + \dot f^3 s\,c\big]\,,\\
\partial_1{\nu_u}^3 &= \frac{1}{F^{3/2}}\,\dot f(\dot f c + \ddot f s)\,, & \partial_2{\nu_u}^3 &= \frac{1}{F^{3/2}}\,\dot f(\dot f s - \ddot f c)\,.
\end{aligned}$$

This yields to:

$$\begin{aligned}
|\nabla{\nu_u}^1|^2 =&\ \frac{1}{\rho^2F^3}\left[\rho^4(\dot f^2+\ddot f^2 s^2 + 2s\,c\,\dot f\ddot f) + 2\rho^2\dot f^3(\dot f c^2+\ddot f s\,c) + \dot f^6 c^2\right]\\
|\nabla{\nu_u}^2|^2 =&\ \frac{1}{\rho^2F^3}\left[\rho^4(\dot f^2+\ddot f^2 c^2 - 2s\,c\,\dot f\ddot f) + 2\rho^2\dot f^3(\dot f s^2-\ddot f s\,c) + \dot f^6 s^2\right]\\
|\nabla{\nu_u}^3|^2 =&\ \frac{1}{F^3}\,\dot f^2(\dot f^2+\ddot f^2)
\end{aligned}$$

and hence
$$|\nabla \nu_u|^2 = \frac{1}{\rho^2(\rho^2+\dot f^2)^3}\left[\rho^4(\ddot f^2+2\dot f^2)+\rho^2(3\dot f^4+\dot f^2\ddot f^2)+\dot f^6\right].$$

Setting moreover for simplicity $A_j := \begin{vmatrix} \partial_1 u & \partial_2 u \\ \partial_1 {\nu_u}^j & \partial_2 {\nu_u}^j \end{vmatrix}$, we also compute:

$$A_1 = \frac{\dot f^2 s}{F^{3/2}}\,, \quad A_2 = -\frac{\dot f^2 c}{F^{3/2}}\,, \quad A_3 = -\frac{\dot f^3}{\rho\, F^{3/2}}$$

so that
$$A_1^2 + A_2^2 + A_3^2 = \frac{\dot f^4}{\rho^2\,(\rho^2+\dot f^2)^2}\,.$$

Also, setting $B_{jk} := \begin{vmatrix} \partial_1 {\nu_u}^j & \partial_2 {\nu_u}^j \\ \partial_1 {\nu_u}^k & \partial_2 {\nu_u}^k \end{vmatrix}$, we compute

$$B_{12} = -\frac{\dot f^2}{F^2}\,, \quad B_{13} = -\frac{\dot f^3\, c}{\rho\, F^2}\,, \qquad B_{23} = -\frac{\dot f^3\, s}{\rho\, F^2}\,,$$

so that
$$B_{12}^2 + B_{13}^2 + B_{23}^2 = \frac{\dot f^4}{\rho^2\,(\rho^2+\dot f^2)^3}\,.$$

On account of (1.15) and (1.18), we thus obtain:

$$\begin{aligned}
|\xi_u^{(0)}|^2 = &\quad g_u = 1+|\nabla u|^2 = \frac{\rho^2+\dot f^2}{\rho^2} \\
|\xi_u^{(1)}|^2 = &\quad |\nabla \nu_u|^2 + A_1^2 + A_2^2 + A_3^2 = \frac{1}{\rho^2(\rho^2+\dot f^2)^2}\left[\rho^2\ddot f^2 + 2\,(\rho^2+\dot f^2)\dot f^2\right] \\
|\xi_u^{(2)}|^2 = &\quad B_{12}^2 + B_{13}^2 + B_{23}^2 = \frac{\dot f^4}{\rho^2\,(\rho^2+\dot f^2)^3}\,.
\end{aligned}$$

Now, by the formulas (1.11) and (1.12) we infer the explicit expressions of the Gauss and mean curvatures:

$$\mathbf{K}_u = -\frac{\dot f^2}{(\rho^2+\dot f^2)^2}\,, \qquad \mathbf{H}_u = \frac{1}{2}\,\frac{\rho\,\ddot f}{(\rho^2+\dot f^2)^{3/2}}$$

and hence we readily recover the expressions of the three terms $|\xi_u^{(i)}|$ given by the formulas (1.18). In particular, we get
$$\sqrt{g_u}\,\mathbf{K}_u = -\frac{\dot f^2}{\rho\,(\rho^2+\dot f^2)^{3/2}}\,, \qquad \sqrt{g_u}\,\mathbf{H}_u = \frac{1}{2}\,\frac{\ddot f}{\rho^2+\dot f^2}\,.$$

We also compute:

$$\begin{aligned}
|\xi_u|^2 = |\xi_u^{(0)}|^2 + |\xi_u^{(1)}|^2 + |\xi_u^{(2)}|^2 = &\quad \frac{g}{(\rho^2+\dot f^2)^4}\left[(\rho^2+2\dot f^2)^2 + \rho^2(\rho^2+\dot f^2)\,\ddot f^2\right] \\
= &\quad \frac{1}{\rho^2(\rho^2+\dot f^2)^3}\left[(\rho^2+2\dot f^2)^2 + \rho^2(\rho^2+\dot f^2)\,\ddot f^2\right].
\end{aligned}$$

Therefore, for f smooth, on account of (1.8) by the area formula we obtain

$$\mathcal{H}^2(\mathcal{G}\mathcal{G}_u) = \int_{B^2} |\xi_u|\,d\mathcal{L}^2 = \int_0^1\int_0^{2\pi} \frac{[(\rho^2+2\dot f^2)^2 + \rho^2(\rho^2+\dot f^2)\,\ddot f^2]^{1/2}}{(\rho^2+\dot f^2)^{3/2}}\,d\theta\,d\rho\,.$$

Remark A.1 Notice that $|\xi_u| \in L^1(B^2)$ if and only if $|\xi_u^{(i)}| \in L^1(B^2)$ for $i = 0, 1, 2$, and we get:

$$\begin{aligned}
\int_{B^2} |\xi_u^{(0)}|\, d\mathcal{L}^2 &= \int_0^1 \int_0^{2\pi} \sqrt{\rho^2 + \dot f^2}\, d\theta\, d\rho \\
\int_{B^2} |\xi_u^{(1)}|\, d\mathcal{L}^2 &= \int_0^1 \int_0^{2\pi} \frac{\left[\rho^2 \ddot f^2 + 2\,(\rho^2 + \dot f^2)\dot f^2\right]^{1/2}}{\rho^2 + \dot f^2}\, d\theta\, d\rho \\
\int_{B^2} |\xi_u^{(2)}|\, d\mathcal{L}^2 &= \int_0^1 \int_0^{2\pi} \frac{\dot f^2}{(\rho^2 + \dot f^2)^{3/2}}\, d\theta\, d\rho\,.
\end{aligned}$$

Moreover, since $|\xi_u| = \sqrt{g_u}\, \sqrt{(1 - \mathbf{K}_u)^2 + (2\,\mathbf{H}_u)^2}$ we have

$$\frac{1}{\sqrt{2}} \left(\sqrt{g_u}\, |1 - \mathbf{K}_u| + \sqrt{g_u}\, |2\mathbf{H}_u|\right) \le |\xi_u| \le \sqrt{g_u}\, |1 - \mathbf{K}_u| + \sqrt{g_u}\, |2\,\mathbf{H}_u|$$

where, we recall,

$$\sqrt{g_u} = \frac{\sqrt{\rho^2 + \dot f^2}}{\rho}\,, \quad |1 - \mathbf{K}_u| = 1 + \frac{\dot f^2}{(\rho^2 + \dot f^2)^2}\,, \quad |2\,\mathbf{H}_u| = \frac{\rho\, |\ddot f|}{(\rho^2 + \dot f^2)^{3/2}}$$

and hence we deduce that $|\xi_u| \in L^1(B^2)$ if and only if

$$\int_{B^2} \frac{(\rho^2 + \dot f^2)^{1/2}}{\rho}\, d\mathcal{L}^2 < \infty\,, \quad \int_{B^2} \frac{1}{\rho}\, \frac{\dot f^2}{(\rho^2 + \dot f^2)^{3/2}}\, d\mathcal{L}^2 < \infty\,, \quad \text{and} \quad \int_{B^2} \frac{|\ddot f|}{\rho^2 + \dot f^2}\, d\mathcal{L}^2 < \infty\,.$$

B An energy computation

We compute the energy of the various components of the current $\widetilde{\Sigma}_u$ in (7.1), referring to Example 7.1.

The Absolutely continuous term has energy $\mathrm{E}(GG_u^a) = \mathcal{E}(u)$, and it is given by the sum of the three integrals in Remark A.1, with f given by (7.4). The Cantor component GG_u^C is equal to zero.

The Jump component is $GG_u^J = \Phi^J_{\#}[\![\, I \times I \,]\!]$, see (7.7). Consider the 2-vector $\overrightarrow{\xi} := \partial_s \Phi^J \wedge \partial_\lambda \Phi^J$. The mass of GG_u^J agrees with the area of the Gauss graph surface $\Phi^J(I \times I)$. Moreover, the stratification $\overrightarrow{\xi} = \xi^{(0)} + \xi^{(1)} + \xi^{(2)}$ gives $\xi^{(2)} = 0$ and

$$\xi^{(0)} = [u(\gamma_0)]^{\pm} \big(\mathbf{t}_1\, e_1 \wedge e_3 + \mathbf{t}_2\, e_2 \wedge e_3\big)\,, \qquad \xi^{(1)} = -[u(\gamma_0)]^{\pm}\, \sigma_u\, \mathbf{k}\, \big(\mathbf{t}_1\, e_3 \wedge \varepsilon_1 + \mathbf{t}_2\, e_3 \wedge \varepsilon_2\big)$$

where $\gamma_0(s) := (s, 0)$, $\mathbf{t}(s) = (\mathbf{t}_1, \mathbf{t}_2)$ is the unit tangent vector to the jump set, $\sigma_u(s) := \mathrm{sgn}[u(\gamma_0(s))]^{\pm}$ is the sign of the Jump, and $\mathbf{k}(s)$ the *signed curvature*, whence $\mathbf{t}(s) \equiv (1, 0))$, $\sigma_u(s) \equiv 1$, $\mathbf{k}(s) \equiv 0$. We thus have $\mathrm{E}_2(GG_u^J) = 0$, whereas by the area formula we get:

$$\mathrm{E}_0(GG_u^J) = \int_{I \times I} |[u(\gamma_0(s))]^{\pm}|\, ds\, d\lambda = |Du^J|(B^2) = 1\,, \quad \mathrm{E}_1(GG_u^J) = \int_I |[u(\gamma_0(s))]^{\pm}|\, \mathbf{k}(s)\, ds = 0\,.$$

The Jump-edge component is $S_u^{Je} = \Phi^{Je}_{+\,\#}[\![\, I \times I \,]\!] - \Phi^{Je}_{-\,\#}[\![\, I \times I \,]\!]$, whence the mass decomposition

$$\mathbf{M}(S_u^{Je}) = \mathbf{M}(\Phi^{Je}_{+\,\#}[\![\, I \times I \,]\!]) + \mathbf{M}(\Phi^{Je}_{-\,\#}[\![\, I \times I \,]\!])$$

and a similar energy splitting hold. Furthermore, using the formulas after equation (7.10), we get

$$\partial_s \Phi^{Je}_{\pm} = \dot\gamma_0^1\, e_1 + \dot\gamma_0^2\, e_2 + \nabla u^{\pm}(\gamma_0) \bullet \dot\gamma_0\, e_3 + \sum_{j=1}^{3} \partial_s (\phi^{Je}_{\pm})^j\, \varepsilon_j\,, \quad \partial_\lambda \Phi^{Je}_{\pm} = \sum_{j=1}^{3} \partial_\lambda (\phi^{Je}_{\pm})^j\, \varepsilon_j$$

and hence we may decompose $\overrightarrow{\xi} := \partial_s \Phi^{Je}_{\pm} \wedge \partial_\lambda \Phi^{Je}_{\pm} = \xi^{(0)} + \xi^{(1)} + \xi^{(2)}$, where this time the first stratum $\xi^{(0)} = 0$ and

$$\xi^{(1)} = \sum_{j=1}^{3} \partial_\lambda (\phi^{Je}_{\pm})^j \Big(\sum_{i=1}^{2} \dot\gamma_0^i\, e_i \wedge \varepsilon_j + \nabla u^{\pm}(\gamma_0) \bullet \dot\gamma_0\, e_3 \wedge \varepsilon_j\Big)\,,$$

$$\xi^{(2)} = \sum_{1\le j_1<j_2\le 3} \big(\partial_s(\phi_\pm^{Je})^{j_1}\partial_\lambda(\phi_\pm^{Je})^{j_2} - \partial_\lambda(\phi_\pm^{Je})^{j_1}\partial_s(\phi_\pm^{Je})^{j_2}\big)\,\varepsilon_{j_1}\wedge\varepsilon_{j_2}\,.$$

We thus get

$$|\xi^{(1)}| = \sqrt{1+(\partial_{\mathbf{t}}u^\pm(\gamma_0))^2}\,|\partial_\lambda\phi_\pm^{Je}|\,, \quad |\xi^{(2)}| = |J_2\phi_\pm^{Je}|$$

and hence

$$\mathrm{E}(S_u^{Je}) = \mathrm{E}(\Phi^{Je}_{+\,\#}[\![\, I\times I\,]\!]) + \mathrm{E}(\Phi^{Je}_{-\,\#}[\![\, I\times I\,]\!])$$

where $\mathrm{E}_0(\Phi^{Je}_{\pm\,\#}[\![\, I\times I\,]\!])=0$ and, again by the area formula,

$$\mathrm{E}_1(\Phi^{Je}_{\pm\,\#}[\![\, I\times I\,]\!]) = \int_I \sqrt{1+(\partial_{\mathbf{t}}u^\pm(\gamma_0(s)))^2}\Big\{\int_I |\partial_\lambda\phi_\pm^{Je}(s,\lambda))|\,d\lambda\Big\}\,ds = \frac{\pi}{2}$$

where we used that $\partial_{\mathbf{t}}u^\pm(\gamma_0(s))\equiv 0$, whereas

$$\mathrm{E}_2(\Phi^{Je}_{\pm\,\#}[\![\, I\times I\,]\!]) = \mathcal{H}^2(\phi_\pm^{Je}(I\times I)) = \frac{\pi}{2}$$

and correspondingly

$$\mathbf{M}(\Phi^{Je}_{\pm\,\#}[\![\, I\times I\,]\!]) = \int_{I\times I}\Big((1+(\partial_{\mathbf{t}}u^\pm(\gamma_0))^2)\,(\partial_\lambda\phi_\pm^{Je})^2 + (J_2\phi_\pm^{Je})^2\Big)^{1/2}\,ds\,d\lambda = \frac{\sqrt{2}}{2}\,\pi\,.$$

Finally, the Edge component is $S_u^e=\sum_{k=1}^2\Phi^e_{k\#}[\![\, I\times I\,]\!]$, see (7.13). We similarly compute:

$$\mathbf{M}(S_u^e) = \sum_{k=1}^2 \mathbf{M}(\Phi^e_{k\#}[\![\, I\times I\,]\!])\,, \qquad \mathrm{E}(S_u^e) = \sum_{k=1}^2 \mathrm{E}(\Phi^e_{k\#}[\![\, I\times I\,]\!])$$

where for $k=1,2$ we have $\mathrm{E}_0(\Phi^e_{k\#}[\![\, I\times I\,]\!])=0$,

$$\mathrm{E}_1(\Phi^e_{k\#}[\![\, I\times I\,]\!]) = \int_I \sqrt{1+(\partial_{\mathbf{t}}u(\gamma_k(s)))^2}\Big\{\int_I |\partial_\lambda\phi_k^e(s,\lambda))|\,d\lambda\Big\}\,ds$$

with $\gamma_1(s):=(0,s)$ and $\gamma_2(s):=(0,s-1)$, and

$$\mathrm{E}_2(\Phi^e_{k\#}[\![\, I\times I\,]\!]) = \mathcal{H}^2(\phi_k^e(I\times I))$$

whereas concerning the mass we get:

$$\mathbf{M}(\Phi^e_{k\#}[\![\, I\times I\,]\!]) = \int_{I\times I}\Big((1+(\partial_{\mathbf{t}}u(\gamma_k))^2)\,(\partial_\lambda\phi_k^e)^2 + (J_2\phi_k^e)^2\Big)^{1/2}\,ds\,d\lambda\,.$$

By (7.14), we check $|\partial_\lambda\phi_1^e(s,\lambda))| = (\pi/2-\arctan s)$, $|\partial_\lambda\phi_2^e(s,\lambda))| = \pi/2-\arctan(1-s)$, and $J_2\phi_k^e\equiv 0$, whereas $\partial_{\mathbf{t}}u(\gamma_k(s))\equiv 0$. We thus conclude that for $k=1,2$:

$$\begin{gathered}\mathrm{E}_1(\Phi^e_{k\#}[\![\, I\times I\,]\!]) = \int_I\Big(\frac{\pi}{2}-\arctan s\Big)\,ds = \frac{\pi}{4}+\frac{1}{2}\,\log 2\\ \mathrm{E}_2(\Phi^e_{k\#}[\![\, I\times I\,]\!]) = 0\,, \qquad \mathbf{M}(\Phi^e_{k\#}[\![\, I\times I\,]\!]) = \mathrm{E}_1(\Phi^e_{k\#}[\![\, I\times I\,]\!])\,.\end{gathered}$$

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