ON THE CODIMENSION OF THE ABNORMAL SET IN STEP TWO CARNOT GROUPS

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Abstract. In this article we prove that the codimension of the abnormal set of the endpoint map for certain classes of Carnot groups of step 2 is at least three. Our result applies to all step 2 Carnot groups of dimension up to 7 and is a generalisation of a previous analogous result for step 2 free nilpotent groups.

1. Introduction

Let G be a *Carnot group*, i.e., a connected and simply connected Lie group with *stratified* nilpotent Lie algebra $\mathfrak{g} = V_1 \oplus \cdots \oplus V_s$. Let End be the endpoint map

$$\begin{array}{rcl} \mathrm{End} : L^2([0,1], V_1) & \to & G \\ u & \mapsto & \gamma_u(1), \end{array}$$

where γ_u is the curve on G leaving from the identity $e \in G$ with $\dot\gamma_u(t) = (\mathrm{d}L_{\gamma_u(t)})_e u(t)$, L_g denoting left translation by g. The *abnormal set* is the subset $\mathrm{Abn}_G \subseteq G$ of all singular values of the endpoint map. Equivalently, Abn_G is the union of all *abnormal curves* passing through the origin. If the abnormal set has measure 0, then G is said to satisfy the *Sard Property.* Proving the Sard Property in the general context of sub-Riemannian manifolds is one of the major open problems in sub-Riemannian geometry, see the questions in [6, Sec. 10.2] and Problem III in [2]. See also [1], [3] and [7]. In [5], the authors of this note and others proved the Sard Property in a number of special cases, and they also obtained the following first result concerning the interesting problem of obtaining finer estimates on the size of the abnormal set.

Theorem ([5, Theorem 3.15])**.** *In any free nilpotent group of step* 2 *the abnormal set is an algebraic subvariety of codimension* 3*.*

In the present paper we discuss generalizations of the result above to some classes of step 2 Carnot groups that are not necessarily free. Our purpose is to present

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different possible approaches to the problem. As our examples will show, going from the free to the general case does not seem to be a trivial step. Before summarizing our contributions in the following statement, it is worth recalling that a free-nilpotent group of step 2 and *rank* r has dimension $\frac{r^2+r}{2}$

Theorem. *Let G be a step 2 Carnot group and let $\dim V_1 = r$. The abnormal set Abn_G is contained in an algebraic subvariety of codimension three (or more) in the following cases:*

(*i*) *G has dimension $r+1$ or $r+2$;*
(*ii*) *G has dimension $\frac{r^2+r}{2} - 1$ or $\frac{r^2+r}{2} - 2$;*
(*iii*) *$r = 4$ and G has dimension 7.*

After having established the notation and having recalled some known results in Section 2, we state and prove our contributions. In Section 3 we consider Carnot groups of step 2 and dimensions $r+1$ and $r+2$. This will be the content of Theorem 3.1 and Theorem 3.3. While the case $r+1$ is rather straightforward, in order to prove Theorem 3.3 we reason by induction on r and we need some fine observations on the bases of V_1. Already in dimension $r+3$ our technique apparently fails to being convenient. In Section 4, we consider dimensions $\frac{r^2+r}{2} - 1$ and $\frac{r^2+r}{2} - 2$ (Theorem 4.2 and Theorem 4.9). The idea here is to see the Carnot groups as quotients of free-nilpotent groups of step 2 by 1 and 2−dimensional ideals, respectively. Indeed, it turns out (see Proposition 2.16) that if $\pi : F \to G$ is a homomorphic projection of stratified groups, then the abnormal curves in G are the abnormal curves in F that remain abnormal under π. If the dimension of the kernel of π is 1 or 2 and F is free, we are able to study the abnormal curves of G using this method. However, when the dimension of the kernel is higher, the computations become more complicated. Finally, in Section 5 we prove Theorem 5.2, which concerns the case where $r = 4$ and the dimension is 7.

It can be easily checked that all stratified Lie groups up to dimension 7 fall in one of the cases (i), (ii) or (iii) above. In particular, we have the following consequence.

Corollary. *Let G be a step 2 Carnot group of topological dimension not greater than 7; then, the abnormal set Abn_G is contained in an algebraic subvariety of codimension at most three.*

2. Preliminaries

In this section we establish the notation we are going to use and we recall some preliminary facts that were proved in [5].

A *Carnot* (or *stratified*) *group* G is a connected, simply connected and nilpotent Lie group whose Lie algebra $\mathfrak{g}$ is *stratified*, i.e., it has a direct sum decomposition $\mathfrak{g} = V_1 \oplus \cdots \oplus V_s$ such that

$$V_{j+1} = [V_j, V_1] \;\forall\, j = 1, \ldots, s-1, \qquad V_s \neq \{0\} \qquad \text{and} \qquad [V_s, V_1] = \{0\}.$$

We refer to the integer s as the *step* of G and to $r := \dim V_1$ as its *rank*. The group identity will be denoted by e. We will indifferently view $\mathfrak{g}$ either as the tangent space to G at e or as the Lie algebra of left-invariant vector fields in G. Recall that in this case the exponential map $\exp : \mathfrak{g} \to G$ is a diffeomorphism; we write log for the inverse of exp. When we use log to identify $\mathfrak{g}$ with G, the group law on G becomes a polynomial map $\mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ with $0 \in \mathfrak{g}$ playing the role of the identity element $e \in G$. For all $g \in G$, denote by L_g and R_g the left and right multiplication by g, respectively. We write $\mathrm{Ad}_g := \mathrm{d}(R_{g^{-1}} \circ L_g)_e : \mathfrak{g} \to \mathfrak{g}$. For $X \in \mathfrak{g}$ we define $\mathrm{ad}_X : \mathfrak{g} \to \mathfrak{g}$ by $\mathrm{ad}_X(Y) := [X, Y]$.

Fix $u \in L^2([0,1], V_1)$ and denote by γ_u the curve in G solving

$$\frac{\mathrm{d}\gamma}{\mathrm{d}t}(t) = \left(\mathrm{d}L_{\gamma(t)}\right)_e u(t), \tag{2.1}$$

with initial condition $\gamma(0) = e$. Vice versa, if $\gamma : [0,1] \to G$ is an absolutely continuous curve that solves (2.1) for some $u \in L^2([0,1], V_1)$, then we say that γ is *horizontal* and that u is its *control*. The *endpoint map* End is

$$\begin{aligned} \mathrm{End} : L^2([0,1], V_1) &\to G \\ u &\mapsto \gamma_u(1). \end{aligned}$$

We will sometimes write End^G to underline the group G we are working with.

Let $\gamma : [0,1] \to G$ be a horizontal curve with control u and such that $\gamma(0) = e$. If $\mathrm{Im}(\mathrm{dEnd}_u) \subsetneq T_{\gamma(1)}G$ we say that γ is *abnormal*. In other words, a horizontal curve $\gamma : [0,1] \to G$ is abnormal if and only if $\gamma(1)$ is a critical value of End. The main goal of this paper is the study of the *abnormal set* Abn_G of G defined by

$$\mathrm{Abn}_G := \{\gamma(1) \ : \ \gamma : [0,1] \to G \text{ abnormal }, \gamma(0) = e\}. \tag{2.2}$$

The following result is proved in [5, Proposition 2.3].

Proposition 2.3. *If $\gamma : [0,1] \to G$ is a horizontal curve leaving from e with control u, then*

$$\mathrm{Im}(\mathrm{dEnd}_u) = (\mathrm{d}R_{\gamma(1)})_e(\mathrm{span}\{\mathrm{Ad}_{\gamma(t)} V_1 \ : \ t \in [0,1]\}). \tag{2.4}$$

It is clear from (2.4) that $\mathrm{Im}(\mathrm{dEnd}_u)$ depends only on γ and not on its parametrization (i.e., on the control u). Given a horizontal curve $\gamma : [0,1] \to G$ with $\gamma(0) = e$ we define

$$\mathscr{E}_\gamma := \mathrm{span}\{\mathrm{Ad}_{\gamma(t)} V_1 \ : \ t \in [0,1]\}. \tag{2.5}$$

By Proposition 2.3, γ is abnormal if and only if $\mathscr{E}_\gamma$ is not the whole Lie algebra $\mathfrak{g}$. In fact, $\mathscr{E}_\gamma \subset T_eG \equiv \mathfrak{g}$ is the image under the diffeomorphism $(\mathrm{d}R_{\gamma(1)})_e^{-1}$ of $\mathrm{Im}(\mathrm{dEnd}_u) \subset T_{\gamma(1)}G$ for any control u associated with γ. Evaluating (2.5) at $t = 0$ and $t = 1$ yields

$$V_1 + \mathrm{Ad}_{\gamma(1)} V_1 \subseteq \mathscr{E}_\gamma. \tag{2.6}$$

We will sometimes use the notation $\mathscr{E}_\gamma^G$ if we need to stress the group under consideration.

Example 2.7. There are no abnormal curves in $\mathbb{R}^n$ (seen as a step 1 Carnot group). Indeed, by (2.6), $\mathfrak{g} = V_1 = \mathscr{E}_\gamma$ for any γ. In particular, $\mathrm{Abn}_{\mathbb{R}^n} = \emptyset$.

We restrict our analysis to a Carnot group G associated with a stratified Lie algebra $\mathfrak{g} = V_1 \oplus V_2$ of step 2. Denote by $\pi_{V_1} : \mathfrak{g} \to V_1$ the canonical projection; for any fixed $X \in \mathfrak{g}$ we recall the formula $\mathrm{Ad}_{\exp(X)} = e^{\mathrm{ad}\, X}$ to get

$$\mathrm{Ad}_{\exp(X)}(Y) = Y + [X, Y] = Y + [\pi_{V_1}(X), Y] \qquad \forall\, Y \in \mathfrak{g}. \tag{2.8}$$

We use this formula to compute more efficiently the linear space $\mathscr{E}_\gamma$ defined in (2.5).

Proposition 2.9. *Let G be a Carnot group associated with a stratified Lie algebra $\mathfrak{g} = V_1 \oplus V_2$ of step 2. Let γ be a horizontal curve in G with $\gamma(0) = e$ and define the linear space $P_\gamma \subseteq \mathfrak{g}$ by*

$$P_\gamma := \mathrm{span}\{\pi_{V_1}(\log \gamma(t)) \ : \ t \in [0, 1]\}, \tag{2.10}$$

where $\log : G \to \mathfrak{g}$ *is the inverse of* $\exp$. *Then*

$$\mathscr{E}_\gamma = V_1 \oplus [P_\gamma, V_1]$$

and, in particular, γ is abnormal if and only if $[P_\gamma, V_1] \neq V_2$.

Proof. Using (2.8) and the fact that $V_1 \subseteq \mathscr{E}_\gamma$ (see (2.6)), we obtain

$$\begin{aligned} \mathscr{E}_\gamma &= \mathrm{span}\{Y + [\pi_{V_1}(\log \gamma(t)), Y] \ : \ t \in [0, 1], Y \in V_1\} \\ &= V_1 \oplus \mathrm{span}\{[\pi_{V_1}(\log \gamma(t)), Y] \ : \ t \in [0, 1], Y \in V_1\} \\ &= V_1 \oplus [P_\gamma, V_1] \end{aligned}$$

as stated. □

Example 2.11. Given an integer $n \geq 1$, the n-th *Heisenberg group* H^n is the Carnot group associated with the $(2n + 1)$-dimensional stratified Lie algebra $V_1 \oplus V_2$ of step 2 where

$$V_1 := \mathrm{span}\{X_1, \dots, X_n, Y_1, \dots, Y_n\}, \qquad V_2 := \mathrm{span}\{T\}$$

and the only non-zero commutation relations between the generators are given by

$$[X_i, Y_i] = T \quad \text{ for any } i = 1, \dots, n.$$

In particular, for any horizontal curve γ with $\gamma(0) = e$ we have $[P_\gamma, V_1] = V_2$ unless $P_\gamma = \{0\}$. It follows that the only abnormal curve in H^n is the constant curve $\bar\gamma(t) \equiv e$ and, in particular, $\mathrm{Abn}_{H^n} = \{e\}$. We note here for future reference that $\mathscr{E}_{\bar\gamma} = V_1$.

Proposition 2.9 allows to give a completely algebraic description of Abn_G. Consider an abnormal curve γ in G leaving from the identity e and let P_γ be as in (2.10). Then

Im γ is contained in the subgroup of G associated to the Lie algebra generated by P_γ, i.e.,

$$\mathrm{Im}\,\gamma \subseteq \exp(P_\gamma \oplus [P_\gamma, P_\gamma]).$$

Assume for a moment that $\dim P_\gamma \in \{r, r-1\}$, i.e., that P_γ is either the whole horizontal layer V_1 or a hyperplane of V_1; in both cases one would have $[P_\gamma, V_1] = V_2$ and, by Proposition 2.9, γ would not be abnormal. This proves that

$$\mathrm{Abn}_G \subseteq \bigcup\{\exp(P \oplus [P,P]) \,:\, P \text{ linear subspace of } \mathfrak{g},\ \dim P \leq r-2, [P,P] \neq V_2\}.$$

The reverse inclusion holds as well. Indeed, if P is a linear subspace of $\mathfrak{g}$ such that $\dim P \leq r-2$ and $[P,P] \neq V_2$, then by Chow connectivity theorem any point in the subgroup $H := \exp(P \oplus [P,P])$ can be connected to e by a horizontal curve γ entirely contained in H, and such a γ must be abnormal by Proposition 2.9. We have therefore proved the following result (see also [5, Section 3.1]).

Proposition 2.12. *Let G be a Carnot group associated with a Lie algebra $\mathfrak{g} = V_1 \oplus V_2$ of step 2; then*

$$\mathrm{Abn}_G = \bigcup\{\exp(P \oplus [P,P]) \,:\, P \textit{ linear subspace of } \mathfrak{g},\ \dim P \leq r-2, [P,P] \neq V_2\}.$$

An immediate consequence of Proposition 2.12, proved in [5, Theorem 1.4], is the following result.

Theorem 2.13. *Let G be a Carnot group associated with a free Lie algebra of step 2; then Abn_G is contained in an affine algebraic subvariety of codimension 3.*

We conclude this section by proving two simple results that hold in general Carnot groups.

Proposition 2.14. *Let G and H be Carnot groups associated with stratified Lie algebras $\mathfrak{g}$ and $\mathfrak{h}$ (respectively). Let γ be a horizontal curve in the Carnot group $G \times H$ and write $\gamma = (\alpha, \beta)$ for unique horizontal curves α and β in G and H, respectively. Then, γ is abnormal in $G \times H$ if and only if either α is abnormal in G or β is abnormal in H; in particular*

$$\mathrm{Abn}_{G\times H} = (\mathrm{Abn}_G \times H) \cup (G \times \mathrm{Abn}_H).$$

Proof. Let V_1 and W_1 be the first layers in the stratifications of $\mathfrak{g}$ and $\mathfrak{h}$ respectively. If $u \in L^2([0,1], V_1)$ and $w \in L^2([0,1], W_1)$ are the controls associated to α and β respectively, then $(u,v) \in L^2([0,1], V_1 \times W_1)$ is a control of γ and

$$\mathrm{dEnd}^{G\times H}_{(u,v)} = \mathrm{dEnd}^G_u \otimes \mathrm{dEnd}^H_v.$$

In particular, $\mathrm{dEnd}^{G\times H}_{(u,v)}$ is surjective if and only if both dEnd^G_u and dEnd^H_v are, and this is enough to conclude. □

We need some terminology before stating our next result. Assume that $\mathfrak{f} = V_1 \oplus \cdots \oplus V_s$ is a nilpotent stratified Lie algebra and that $\mathfrak{I} \subset V_2 \oplus \cdots \oplus V_s$ is an ideal of $\mathfrak{f}$; consider the quotient Lie algebra $\mathfrak{g} = \mathfrak{f}/\mathfrak{I}$. Let $\pi : \mathfrak{f} \to \mathfrak{g}$ be the associated canonical projection and let F, G be the Carnot groups associated with $\mathfrak{f}, \mathfrak{g}$ respectively; we use the same symbol $\pi : F \to G$ to denote the canonical projection at the group level. It is well-known that any horizontal curve γ in G leaving from the identity (of G) admits a unique *lift* to F, i.e., a unique horizontal curve $\bar{\gamma}$ in F leaving from the identity (of F) and such that $\gamma = \pi \circ \bar{\gamma}$. Moreover, essentially by definition (see (2.5)) we have that for any horizontal curve c in F

$$\mathscr{E}_{\pi \circ c} = \pi(\mathscr{E}_c). \tag{2.15}$$

This proves the following result.

Proposition 2.16. *Let $\mathfrak{f} = V_1 \oplus \cdots \oplus V_s$ be a nilpotent stratified Lie algebra, let $\mathfrak{I} \subset V_2 \oplus \cdots \oplus V_s$ be an ideal of $\mathfrak{f}$ and let $\mathfrak{g} = \mathfrak{f}/\mathfrak{I}$ be the quotient Lie algebra. Let $\pi : F \to G$ be the canonical projection between the Carnot groups F and G associated with $\mathfrak{f}$ and $\mathfrak{g}$, respectively. Then, for any abnormal curve γ in G, the lift $\bar{\gamma}$ is an abnormal curve in F; in particular,*

$$\mathrm{Abn}_G \subset \pi(\mathrm{Abn}_F).$$

3. Step 2, rank r, dimensions $r+1$ and $r+2$

In this section we show that if G is a Carnot group of step 2 such that $\dim V_1 = r$ and $\dim V_2 = 1$ or 2, then the abnormal set Abn has codimension at least 3. The case where $\dim V_2 = 1$ is somewhat elementary. Given an integer $\ell \geq 0$, we say that a vector $X \in V_1$ has rank ℓ if $\mathrm{rank}(\mathrm{ad}\, X) = \ell$. We denote by R_ℓ the set of vectors in V_1 of rank at most ℓ; notice that $R_0 = \mathfrak{z}(\mathfrak{g}) \cap V_1$.

Theorem 3.1. *Let G be the Carnot group associated with a stratified Lie algebra $\mathfrak{g} = V_1 \oplus V_2$ such that* $\dim V_1 = r$ *and* $\dim V_2 = 1$. *Then* $\dim \mathrm{Abn}_G \leq r - 2$.

Proof. By Proposition 2.9, a curve γ is abnormal if and only if $[P_\gamma, V_1] = \{0\}$, i.e., $P_\gamma \subseteq \mathfrak{z}(\mathfrak{g})$, the center of $\mathfrak{g}$. So the abnormal curves are contained in $\exp R_0$, which is a subgroup of dimension at most $r - 2$, for otherwise $\mathfrak{g}$ would be abelian. □

We also provide an alternative proof suggested to us by E. Le Donne.

Alternative proof of Theorem 3.1. It is an exercise left to the reader to show that G is isomorphic to the product $H^k \times \mathbb{R}^h$ for suitable integers $k \geq 1$ and $h \geq 0$. The conclusion follows from Proposition 2.14 and Examples 2.7 and 2.11 . □

The proof of the case $\dim V_2 = 2$ is less straightforward, and it requires some preliminary results.

Lemma 3.2. *Let $\mathfrak{g} = V_1 \oplus V_2$ be a step two stratified Lie algebra with $\dim V_1 = r \geq 3$ and $\dim V_2 = 2$. Then R_1 has codimension at least 1 in V_1.*

Proof. Let $X_1, \dots, X_r$ be a basis of V_1 and Z_1, Z_2 a basis of V_2; let $X = \sum_{i=1}^r x_i X_i \in V_1$. Asking X to have rank at most 1 amounts to an algebraic condition on $x_1, \dots, x_r$: in fact, X has rank at most 1 if and only if all the 2×2 minors of the matrix $M \in \mathbb{R}^{2\times r}$ defined by $[X_j, X] = M_{1j} Z_1 + M_{2j} Z_2$, $j = 1, \dots, r$ have null determinant. We notice that the entries of M are linear in $x_1, \dots, x_r$, hence all the determinants of the 2×2 minors are given by a (possibly zero) homogeneous second-order polynomial in $x_1, \dots, x_r$. It follows that R_1 is contained in an algebraic variety of V_1.

In order to show that this variety is not the whole V_1 (and hence it has positive codimension, as desired) it is enough to find a vector in V_1 of rank 2. This is trivial if $r = 3$. For $r \geq 4$ we reason by contradiction and assume that all vectors have rank 0 or 1. Fix $X_1, X_2 \in V_1$ such that $[X_1, X_2] = Z_1 \neq 0$. We can choose[1] a basis $\{X_1, X_2, X_3, \dots, X_r\}$ of V_1 so that $[X_j, X_1] = [X_j, X_2] = 0$ for every $j = 3, \dots, r$. Since $\dim V_2 = 2$, we may assume that Z_1 and $Z_2 := [X_3, X_4]$ are linearly independent. Then $X_1 + X_3$ has rank 2, and the proof is accomplished. □

Theorem 3.3. *Let G be the Carnot group associated with a stratified Lie algebra $\mathfrak{g} = V_1 \oplus V_2$ such that $\dim V_1 = r$ and $\dim V_2 = 2$. Then $\dim \mathrm{Abn}_G \leq r - 1$.*

Proof. We argue by induction on r. Since $\dim V_2 = 2$, the smallest possible dimension for V_1 is $r = 3$. If this is the case, then for every (non-constant) abnormal curve γ the space P_γ has dimension one (otherwise $[P_\gamma, V_1] = V_2$ and γ would not be abnormal). Therefore γ is a horizontal line, and $P_\gamma = \mathrm{span}\{X^\gamma\}$ for some $X^\gamma \in V_1$; actually, $X^\gamma \in R_1$, for otherwise $[P_\gamma, V_1] = V_2$. It follows that $\mathrm{Abn}_G \subset \exp(R_1)$; by Lemma 3.2, this has dimension at most 2, as stated.

Assume now that the thesis is true for $\dim V_1 \leq r - 1$ and let γ be an abnormal curve. Notice that if $\dim P_\gamma = 1$, then $\gamma \subseteq \exp(R_1)$. Therefore, if there is no abnormal curve γ such that $\dim P_\gamma \geq 2$, the conclusion follows from Lemma 3.2. Otherwise there is a curve γ which is abnormal (i.e., $\dim[P_\gamma, V_1] \leq 1$) and such that P_γ has dimension ≥ 2. If P_γ is abelian for every such γ, then the abnormal curves are all contained in the space $\exp(R_1)$, which has dimension at most $r - 1$ by Lemma 3.2.

Otherwise, suppose there is an abnormal curve γ such that $\dim P_\gamma \geq 2$, $\dim[P_\gamma, V_1] \leq 1$, and $[P_\gamma, P_\gamma] \neq 0$, so that $\dim[P_\gamma, P_\gamma] = \dim[P_\gamma, V_1] = 1$. Then there are $X_1, X_2 \in P_\gamma$ such that $[X_1, X_2] = Z_1 \neq 0$. We may complete[2] X_1 and X_2 to a basis $X_1, \dots, X_r$ of V_1 so that $[X_j, X_1] = [X_j, X_2] = 0$ for every $j = 3, \dots, r$. Denoting by $\mathfrak{g}$ the Lie algebra of G, we have two cases: either Z_1 is in the Lie algebra generated by $X_3, \dots, X_r$, or not. In the latter case, we may write $\mathfrak{g} = \mathfrak{h} \oplus \tilde{\mathfrak{g}}$, a Lie algebra direct sum where $\mathfrak{h} := \mathrm{span}\{X_1, X_2, Z_1\}$ is the three dimensional Heisenberg Lie algebra,

[1]Otherwise $[X_j, X_1] = aZ_1$ and $[X_j, X_2] = bZ_1$ and one could replace X_j with $X_j - bX_1 + aX_2$.
[2]See footnote 1.

$\tilde{\mathfrak{g}} := \mathrm{span}\{X_3, \ldots, X_r, Z_2\}$ is a subalgebra of $\mathfrak{g}$, and Z_2 is a vector in V_2 linearly independent from Z_1. In particular, by Proposition 2.14 the abnormal set satisfies

$$\mathrm{Abn}_G \subseteq \left(\mathrm{Abn}_H \times \tilde{G}\right) \cup (H \times \mathrm{Abn}_{\tilde{G}})$$

with H and $\tilde{G}$ denoting the subgroups of G with Lie algebras $\mathfrak{h}$ and $\tilde{\mathfrak{g}}$ respectively. One can then easily conclude using Theorem 3.1 (which applies to $\tilde{G}$) and Example 2.11.

We are left with the case where Z_1 is in the Lie algebra generated by $X_3, \ldots, X_r$. In this case $X_3, \ldots, X_r$ generate two independent vectors Z_1 and Z_2 in V_2. Define the Lie algebra $\mathfrak{g}' := \mathrm{span}\{X_3, \ldots, X_r, T, Z_2\}$ with Lie product given by declaring the map $\phi : \mathfrak{g} \to \mathfrak{g}'$, defined by $\phi(Z_1) = T$ and by the identity on the other basis vectors, to be an isomorphism. Then we can write

$$\mathfrak{g} = (\mathfrak{h} \oplus \mathfrak{g}')/\mathfrak{I},$$

where $\mathfrak{I} := \mathrm{span}\{T - Z_1\}$ and again $\mathfrak{h} := \mathrm{span}\{X_1, X_2, Z_1\}$. The homomorphic surjective projection $\pi : \mathfrak{h} \oplus \mathfrak{g}' \to \mathfrak{g}$ defines a natural projection (for which we use the same symbol π) at the Lie group level

$$\pi : H \times G' \to G,$$

where G' denotes the exponential group of $\mathfrak{g}'$. By Proposition 2.16 the abnormal curves γ in G are projections of abnormal curves (α, β) in $H \times G'$, with α and β horizontal curves in H and G' respectively. Namely, $\gamma = \pi(\alpha, \beta)$. By Proposition 2.14, (α, β) is abnormal in $H \times G'$ if and only if at least one of α and β is abnormal in H and G' respectively. So, the abnormal set Abn_G is contained in $\mathrm{A}_H \cup \mathrm{A}_{G'}$, where

$$\mathrm{A}_H := \{\gamma(1) : \gamma = \pi(\alpha, \beta) \text{ abnormal in } G, \gamma(0) = e, \alpha \text{ abnormal in } H\}$$

and

$$\mathrm{A}_{G'} := \{\gamma(1) : \gamma = \pi(\alpha, \beta) \text{ abnormal in } G, \gamma(0) = e, \beta \text{ abnormal in } G'\}$$

We first consider A_H. By Example 2.11, the only abnormal curves in H leaving from e is the constant curve $\bar{\alpha} \equiv e$; we claim that $\gamma = \pi(\bar{\alpha}, \beta)$ is abnormal in G if and only if β is abnormal in G'. Indeed, if this was not the case, then $\mathscr{E}_\beta^{G'} = \mathfrak{g}'$ and $\mathscr{E}_{\bar{\alpha}}^H = V_1^{\mathfrak{h}}$, where $V_1^{\mathfrak{h}} = \mathrm{span}\{X_1, X_2\}$. Recalling (2.15) we would have $\mathscr{E}_\gamma^G = \pi(\mathscr{E}_{\bar{\alpha}}^H \oplus \mathscr{E}_\beta^{G'}) = \mathfrak{g}$ and γ would not be abnormal in G. This proves that $A_H \subseteq \pi(\{e\} \times \mathrm{Abn}_{G'})$ and, in particular, that A_H has codimension at least 5.

Next, we consider $\mathrm{A}_{G'}$. If γ is abnormal in G and α is not constant, then $\mathscr{E}_\alpha^H = \mathfrak{h}$ by Example 2.11. Hence, in order to have $\mathscr{E}_\gamma^G \neq \mathfrak{g}$, it must be that

$$\mathscr{E}_\beta^{G'} \subseteq V_1^{\mathfrak{g}'} \oplus \mathbb{R}T,$$

with $V_1^{\mathfrak{g}'} = \mathrm{span}\{X_3, \ldots, X_r\}$. If $\mathscr{E}_\beta^{G'} = V_1^{\mathfrak{g}'}$, then $[P_\beta, V_1^{\mathfrak{g}'}] = \{0\}$. So $\beta \subseteq \exp(V_1^{\mathfrak{g}'} \cap \mathfrak{z}(\mathfrak{g}'))$, with $\mathfrak{z}(\mathfrak{g}')$ the center of $\mathfrak{g}'$. But $V_1^{\mathfrak{g}'} \cap \mathfrak{z}(\mathfrak{g}')$ has dimension at most $r - 5$, so we reach the conclusion in this case. Otherwise β is such that $\mathscr{E}_\beta^{G'} = V_1^{\mathfrak{g}'} \oplus \mathbb{R}T$, which implies that $[P_\beta, V_1^{\mathfrak{g}'}] = \mathbb{R}T$. Namely, $\beta \subseteq \exp(M \oplus \mathbb{R}T)$, where

$$M = \{X \in V_1^{\mathfrak{g}'} : \mathrm{Im}(\mathrm{ad}\, X) \subseteq \mathbb{R}T\}.$$

Notice that M is a linear subspace of $V_1^{\mathfrak{g}'}$ and that $\dim M \leq r-4$. Using the fact that $\pi(T - Z_1) = 0$ we obtain

$$\pi(\alpha, \beta) \subseteq \pi(H \times \exp(M \times \mathbb{R})) = \pi(H \times \exp(M))$$

and the set on the right hand side has dimension $r-1$. This concludes the proof.

□

4. Step 2, rank r, dimension $\frac{r^2+r}{2} - 1$ and $\frac{r^2+r}{2} - 2$.

Let $\mathfrak{g} = V_1 \oplus V_2$ be a step two Lie algebra with $\dim V_1 = r$. Let $\mathfrak{f}_{r,2}$ be the nilpotent free Lie algebra of rank r and step 2 and let $F_{r,2}$ be the Carnot group with $\mathfrak{f}_{r,2}$ as Lie algebra. Denote by $W_1 \oplus W_2$ a stratification of $\mathfrak{f}_{r,2}$, and recall that $\dim W_2 = \frac{r(r-1)}{2}$. Then the Lie algebra $\mathfrak{g}$ can be viewed as the quotient of $\mathfrak{f}_{r,2}$ by a subspace W of W_2. One possible strategy for studying the abnormal set of G is to study those abnormal curves in $F_{r,2}$ that project to abnormal curves on G. In the following two sections we use this idea to study the abnormal set of G when $\dim V_2 = \frac{r(r-1)}{2} - 1$ (i.e., $\dim W = 1$) and $\dim V_2 = \frac{r(r-1)}{2} - 2$ (i.e., $\dim W = 2$). As the dimension of the space W grows, the discussion becomes considerably more complicated and this strategy (apparently) ceases to be convenient. The following lemma will be useful for both cases

Lemma 4.1. *The set $\mathscr{A} \subseteq F_{r,2}$ defined by*

$$\mathscr{A} := \bigcup \{\mathit{Im}\ \gamma : \gamma \text{ is a curve in } F_{r,2} \text{ and } \dim P_\gamma \leq r-3\}$$

has dimension $\frac{r^2+r}{2} - 6$.

Proof. Denoting by $Gr(W_1, k)$ the Grassmannian of k-planes in W_1, we have

$$\mathscr{A} = \bigcup_{h=3}^{r-1} \mathscr{A}_h, \qquad \text{where} \qquad \mathscr{A}_h := \bigcup_{P \in Gr(W_1, r-h)} \exp(P \oplus [P, P]).$$

Each set $\mathscr{A}_h$ can (locally) be parametrized by a smooth map depending on a number of parameters equal to $\dim Gr(\mathbb{R}^r, r-h) + \dim F_{r-h,2}$, hence it has dimension

$$h(r-h) + \frac{(r-h)(r-h+2)}{2} = \frac{r^2+r}{2} - \frac{h(h+1)}{2} \leq \frac{r^2+r}{2} - 6$$

because $h \geq 3$. □

4.1. **Dimension $\frac{r^2+r}{2}-1$.** We prove the following.

Theorem 4.2. *Let G be the Carnot group associated with a step two stratified Lie algebra $\mathfrak{g} = V_1 \oplus V_2$ with $\dim V_1 = r$ and $\dim V_2 = \frac{r(r-1)}{2} - 1$. Then Abn_G has codimension at least* 3.

Proof. As we said, the algebra $\mathfrak{g}$ is isomorphic to the quotient of $\mathfrak{f}_{r,2}$ by a one dimensional subspace of W_2, say $\mathbb{R}Z$. From Proposition 2.16, every abnormal curve in G lifts to an abnormal curve in $F_{r,2}$. Denote by π the projection of $F_{r,2}$ onto G, and let γ be an abnormal curve in $F_{r,2}$. Then P_γ has dimension at most $r-2$ in W_1 by Proposition 2.12; the set Abn_G is thus contained in the union $A \cup B$, where

$$A := \{\pi \circ \gamma(1) : \gamma \text{ abnormal in } F_{r,2} \text{ and } \dim P_\gamma \le r-3\}$$

and

$$B := \{\pi \circ \gamma(1) : \gamma \text{ abnormal in } F_{r,2} \text{ and } \dim P_\gamma = r-2\}.$$

Since A coincides with the projection $\pi(\mathscr{A})$ of the set $\mathscr{A}$ introduced in Lemma 4.1, A has codimension at least 5 in G. Next, we analyze the set B and consider an abnormal curve γ in $F_{r,2}$ such that P_γ is $(r-2)$-dimensional; in particular, $\dim[P_\gamma, W_1] = \frac{r(r-1)}{2} - 1 = \dim W_2 - 1$. It follows that the projection $\pi \circ \gamma$ is abnormal if and only if $Z \in [P_\gamma, W_1]$, which is an easy consequence of the equalities

$$\mathscr{E}_{\pi\circ\gamma} = \pi(\mathscr{E}_\gamma) = \pi(W_1 \oplus [P_\gamma, W_1]),$$

where we denoted by π also the projection $\mathfrak{f}_{r,2} \to \mathfrak{g}$. Let $\ell = \frac{r(r-1)}{2}$ and fix a basis $Z_1, \dots, Z_\ell$ of W_2. Without loss of generality, we may assume $Z_1 = Z$. Let $Z'_1, \dots, Z'_{\ell-1}$ be independent vectors of $[P_\gamma, W_1]$, that we can write as $Z'_j = \sum_{i=1}^{\ell} a_i^j Z_i$, $j = 1, \dots, \ell-1$. Then $Z \in [P_\gamma, W_1]$ if and only if the matrix

$$\begin{pmatrix} Z & Z'_1 & \cdots & Z'_{\ell-1} \end{pmatrix} = \begin{pmatrix} 1 & a_1^1 & \dots & a_{\ell-1}^{\ell} \\ 0 & \vdots & & \vdots \\ \vdots & \vdots & & \vdots \\ 0 & a_\ell^1 & \dots & a_\ell^{\ell-1} \end{pmatrix}$$

has determinant equal to zero. This gives a non-trivial algebraic condition on P_γ. Therefore, the space of all $(r-2)$-dimensional subspaces P_γ of W_1 such that $Z \in [P_\gamma, W_1]$ has dimension at most $\dim Gr(r, r-2) - 1 = 2(r-2) - 1$. Now, every $(r-2)$-dimensional plane in W_1 generates a Lie algebra of dimension $\frac{(r-2)(r-1)}{2}$. So B has dimension at most $2(r-2)-1+\frac{(r-2)(r-1)}{2} = \frac{r^2+r}{2} - 4$, that is, B has codimension at least 3 in G. We then conclude that Abn_G has codimension at least 3, as stated. $\square$

4.2. **Dimension $\frac{r^2+r}{2}-2$.** Let G be a Carnot group of step 2 whose algebra $\mathfrak{g} = V_1 \oplus V_2$ is such that $\dim V_1 = r$ and $\dim V_2 = \frac{r^2-r}{2} - 2$. Let $F = F_{r,2}$ be the free Carnot group of rank r and step 2; in this section it will be convenient to identify the Lie algebra $\mathfrak{f}$ of F with $V_1 \oplus \Lambda^2 V_1$; accordingly, we will indifferently use the symbols $[X, Y]$ and $X \wedge Y$ to denote the Lie bracket of $X, Y \in V_1$. There exists a 2-dimensional subspace $W \subseteq \Lambda^2 V_1$ such that

$$V_2 \equiv \Lambda^2 V_1 / W. \tag{4.3}$$

We use the same symbol π to denote the canonical projections $\pi : \Lambda^2 V_1 \to V_2$ and $\pi : F \to G$.

Before stating the main result of this section, some preparatory work is in order. Let $a \in W \setminus \{0\}$ be a 2-vector and let $k \geq 1$ be its rank; let $e_1, e_2, \ldots, e_{2k} \in V_1$ be such that

$$a = (e_1 \wedge e_2) + \cdots + (e_{2k-1} \wedge e_{2k}).$$

We complete $e_1, \ldots, e_{2k}$ to a basis $e_1, \ldots, e_r$ of V_1 and we endow V_1 of a scalar product making this basis orthonormal. This induces a canonical scalar product on $\Lambda^2 V_1$ making $(e_i \wedge e_j)_{1 \leq i < j \leq r}$ an orthonormal basis. Finally, we choose $b \in W$ in such a way that a, b is a basis of W; writing $b = \sum_{1 \leq i < j \leq r} c_{ij}\ e_i \wedge e_j$, up to replacing b with $b - c_{12} a$ we can assume that

$$c_{12} = 0. \tag{4.4}$$

Let us introduce the following skew-symmetric $r \times r$ matrices:

$$\Omega := \left(\begin{array}{cccccccc|c} 0 & 1 & & & & & & & \\ -1 & 0 & & & & & & & \\ & & 0 & 1 & & & & & \\ & & -1 & 0 & & & & & \mathbf{0} \\ & & & & \ddots & & & & \\ & & & & & & 0 & 1 & \\ & & & & & & -1 & 0 & \\ \hline & & & & \mathbf{0} & & & & \mathbf{0} \end{array}\right),$$

where k blocks of the form $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ appear, all not-shown entries are null and $\mathbf{0}$ denotes null matrices of proper sizes, and

$$C := \begin{pmatrix} 0 & c_{12} & c_{13} & c_{14} & \cdots \\ -c_{12} & 0 & c_{23} & c_{24} & \cdots \\ -c_{13} & -c_{23} & 0 & c_{34} & \cdots \\ \vdots & \vdots & \vdots & \ddots & \cdots \end{pmatrix}$$

These matrices have the following notable relations with a, b. If one considers $x, y \in V_1$ as column-vectors written in the basis $e_1, \dots, e_r$, then

$$(x \wedge y) \cdot a = x \cdot (\Omega y) = x^t \Omega y \quad \text{and} \quad (x \wedge y) \cdot b = x \cdot (Cy) = x^t C y, \tag{4.5}$$

where $\cdot$ denotes scalar product (either in V_1 or in $\Lambda^2 V_1$) and t denotes transposition. Notice also that, by skew-symmetry,

$$x \cdot (\Omega y) = -(\Omega x) \cdot y \quad \text{and} \quad x \cdot (Cy) = -(Cx) \cdot y. \tag{4.6}$$

We will later need the following special form of the matrix C in the particular case in which Ω and C commute.

Lemma 4.7. *Assume that $\Omega C = C\Omega$. Then, the basis $e_1, \dots, e_r$ of V_1 can be chosen in such a way that*

$$a = (e_1 \wedge e_2) + \cdots + (e_{2k-1} \wedge e_{2k})$$

and there exists a $(r-2k) \times (r-2k)$ skew-symmetric matrix D such that

$$C = \left(\begin{array}{cccccccc|c} 0 & c_{12} & & & & & & & \\ -c_{12} & 0 & & & & & & & \\ & & 0 & c_{34} & & & & & \\ & & -c_{34} & 0 & & & & & \mathbf{0} \\ & & & & \ddots & & & & \\ & & & & & & 0 & c_{2k-1,k} & \\ & & & & & & -c_{2k-1,k} & 0 & \\ \hline & & & & \mathbf{0} & & & & D \end{array} \right).$$

In particular, $c_{ij} = 0$ whenever $i = 1, \dots, 2k-1$ and $j \geq i+2$.

Proof. Since $-\Omega^2$ is the projection on span $\{e_1, \dots, e_{2k}\}$, for any i, j such that $i \leq 2k < j$ we have by skew-symmetry

$$\begin{aligned} 0 &= \Omega(e_i + e_j) \cdot C\Omega(e_i + e_j) = \Omega(e_i + e_j) \cdot \Omega C(e_i + e_j) \\ &= -\Omega^2(e_i + e_j) \cdot C(e_i + e_j) = e_i \cdot C(e_i + e_j) = c_{ij}. \end{aligned}$$

In particular, C takes the form

$$C = \left(\begin{array}{c|c} E & \mathbf{0} \\ \hline \mathbf{0} & D \end{array} \right)$$

for suitable $2k \times 2k$ and $(r-2k) \times (r-2k)$ matrices E, D.

Let now $i, j \in \{1, \dots, 2k\}$ be fixed with j odd. Upon agreeing that $c_{ii} = 0$ and $c_{ij} = -c_{ji}$ if $i > j$, one has

$$\begin{aligned} c_{ij} =& e_i \cdot Ce_j = e_i \cdot C\Omega e_{j+1} \\ =& e_i \cdot \Omega C e_{j+1} = -\Omega e_i \cdot C e_{j+1} = \begin{cases} -c_{i-1,j+1} & \text{if } i \text{ is even} \\ -c_{i+1,j+1} & \text{if } i \text{ is odd.} \end{cases} \end{aligned}$$

Hence the $2k \times 2k$ matrix E is composed of k^2 (2×2)-blocks of the form $\begin{pmatrix} a & b \\ -b & a \end{pmatrix}$, which in turn gives the following information: upon identifying $\mathbb{R}^{2k} \equiv \mathbb{C}^k$ in the standard way

$$(x_1, \dots, x_{2k}) \longleftrightarrow (x_1 + ix_2, \dots, x_{2k-1} + ix_{2k}),$$

C can be identified with a $\mathbb{C}$-linear endomorphism of $\mathbb{C}^k$ (notice that, with this identification, Ω is the scalar multiplication by $-i$ in $\mathbb{C}^k$). Since Ω and $\mathbb{C}$ commute, each of them preserve the other's eigenspaces; since $\mathbb{C}$ is algebraically closed, they can be simultaneously diagonalized in $\mathbb{C}^k$, which proves the Lemma. □

We will also need the following simple result

Lemma 4.8. *Let H be an hyperplane in V_1; then, the set $B \subseteq F$ defined by*

$$B := \bigcup_{U \in Gr(H,2)} \exp(U^\perp \oplus [U^\perp, U^\perp])$$

has dimension at most $\frac{r^2+r}{2} - 5$. In particular, the dimension of $\pi(B) \subseteq G$ is at most $\frac{r^2+r}{2} - 5$.

Proof. Each set of the form $\exp(U^\perp \oplus [U^\perp, U^\perp])$ is a subgroup of F isomorphic to $F_{r-2,2}$. Hence, the set B can (locally) be parametrized by a smooth map of a number of parameters equal to

$$\dim Gr(H,2) + \dim F_{r-2,2} = 2(\dim H - 2) + \frac{(r-2)(r-1)}{2} = \frac{r^2+r}{2} - 5.$$

□

We can now state the main result of this section.

Theorem 4.9. *Let G be a Carnot group associated with a step two stratified Lie algebra $\mathfrak{g} = V_1 \oplus V_2$ with $\dim V_1 = r$ and $\dim V_2 = \frac{r(r-1)}{2} - 2$. Then Abn_G has codimension at least 3, i.e., $\dim \mathrm{Abn}_G \leq \frac{r^2+r}{2} - 5$.*

Proof. As in the previous section, our strategy consists in studying the set of abnormal curves γ in F such that $\pi \circ \gamma$ is abnormal in G. By Lemma 4.1, if γ is such that $\dim P_\gamma \leq r - 3$, then $\pi \circ \gamma$ is contained in a subset of G with codimension at least 4. It is therefore enough to study those abnormal curves γ in F such that $\dim P_\gamma = r - 2$ and $\pi \circ \gamma$ is abnormal in G.

Let then such a γ be fixed; since $\dim[P_\gamma, V_1] = \frac{r^2-r}{2} - 1$, we have $\operatorname{codim} \mathscr{E}^F_\gamma = 1$ and, in particular,

$$\operatorname{codim} \mathscr{E}^G_{\pi\circ\gamma} = \operatorname{codim} \pi(\mathscr{E}^F_\gamma) \leq 1.$$

Since $\pi \circ \gamma$ is abnormal in G, we have therefore $\operatorname{codim} \pi(\mathscr{E}^F_\gamma) = 1$, which is equivalent to $W \subseteq \mathscr{E}^F_\gamma = V_1 \oplus [P_\gamma, V_1]$ (recall that W was introduced in (4.3)) and, in turn, to

$$W \subseteq [P_\gamma, V_1] = [P_\gamma^\perp, P_\gamma^\perp]^\perp. \tag{4.10}$$

We have therefore to study those planes $U := P_\gamma^\perp \in Gr(V_1, 2)$ such that $W \perp [U, U]$. By Lemma 4.8 it is enough to study the case in which $U \not\subseteq e_1^\perp$. Since U is 2-dimensional, there exists a unique (up to a sign) unit vector $x \in U \cap e_1^\perp$. Writing $x = (x_1, \dots, x_r)$ with respect to the basis $e_1, \dots, e_r$, we have $x_1^2 + \cdots + x_r^2 = 1$ and $x_1 = 0$. Let then $y \in U$ be the unique (up to a sign) unit vector such that $x \cdot y = 0$; notice that $y_1 \neq 0$, otherwise $U =\text{span}\{x, y\} \subseteq e_1^\perp$. Setting

$$\widetilde{M} := \{(x, y) \in \mathbb{R}^r \times \mathbb{R}^r : \|x\|^2 = \|y\|^2 = 1, x_1 = 0, x \cdot y = 0, y_1 \neq 0\},$$

the map

$$\widetilde{M} \ni (x, y) \longmapsto U_{x,y} := \text{span}\{x, y\} \in Gr(V_1, 2) \setminus Gr(e_1^\perp, 2)$$

is smooth, locally injective and surjective (actually, it is 4-to-1); moreover, $[U_{x,y}, U_{x,y}] =$ span $\{x \wedge y\}$. Therefore, the analysis of those planes $U \not\subseteq e_1^\perp$ such that $W \perp [U, U]$ can be reduced to the analysis of the couples $(x, y) \in \widetilde{M}$ such that $W =$ span $\{a, b\} \perp x \wedge y$, or equivalently (recall (4.5)) such that

$$x \cdot \Omega y = x \cdot C y = 0\,.$$

We divide our study in two cases according to whether Ω and C commute or not.

Case 1: $\Omega C \neq C\Omega$. In this case we have ker $(\Omega C - C\Omega) \neq V_1$, hence ker $(\Omega C - C\Omega)$ is contained in some hyperplane $H \subseteq V_1$. By Lemma 4.8 it is enough to consider those couples (x, y) such that $U_{x,y} \not\subseteq H$; notice that this condition implies that either $x \notin$ ker $(\Omega C - C\Omega)$ or $y \notin$ ker $(\Omega C - C\Omega)$. We will show that the set

$$M := \left\{ (x, y) \in \mathbb{R}^r \times \mathbb{R}^r : \begin{array}{l} \|x\|^2 = \|y\|^2 = 1, x_1 = 0, x \cdot y = 0, y_1 \neq 0, \\ x \cdot \Omega y = x \cdot C y = 0, \\ \|(\Omega C - C\Omega)x\|^2 + \|(\Omega C - C\Omega)y\|^2 > 0 \end{array} \right\} \tag{4.11}$$

is the union of two smooth manifolds M_1, M_2 of dimension $2r - 6$, and we claim that this will be enough to prove Theorem 4.9 in Case 1. Indeed, by the discussion above, in order to prove that Abn_G has dimension at most $\frac{r^2+r}{2} - 5$ it is enough to show that (the projection on G of) the set

$$\bigcup_{(x,y)\in M} \exp(U_{x,y}^\perp \oplus [U_{x,y}^\perp, U_{x,y}^\perp]) \;\cup \bigcup_{U \in Gr(e_1^\perp, 2) \cup Gr(H, 2)} \exp(U^\perp \oplus [U^\perp, U^\perp]) \tag{4.12}$$

has dimension at most $\frac{r^2+r}{2} - 5$. This would be true because the second set in the right hand side has dimension at most $\frac{r^2+r}{2} - 5$ by Lemma 4.8, while (provided $M = M_1 \cup M_2$ as claimed above) the first set can (locally) be parametrized by two smooth maps of $(2r - 6) + \dim F_{r-2,2} = \frac{r^2+r}{2} - 5$ parameters.

We start by showing that

$$M_1 := M \cap \{(x, y) \in \mathbb{R}^r \times \mathbb{R}^r : y \cdot (\Omega C - C\Omega)x \neq 0\}$$

is a smooth submanifold of dimension $2r - 6$. Defining the open set

$$\mathcal{O}_1 := \{(x, y) \in \mathbb{R}^r \times \mathbb{R}^r : y_1 \neq 0, \|(\Omega C - C\Omega)x\|^2 + \|(\Omega C - C\Omega)y\|^2 > 0, y \cdot (\Omega C - C\Omega)x \neq 0\}$$

we have

$$M_1 = \mathcal{O}_1 \cap \{(x,y) \in \mathbb{R}^r \times \mathbb{R}^r : F(x,y) = (1,1,0,0,0,0)\}$$

where $F(x,y) := (\|x\|^2, \|y\|^2, x \cdot y, x_1, x \cdot \Omega y, x \cdot Cy)$. We have to show that the rank of ∇F is 6 at all points of M_1; using (4.6) one can compute

$$\nabla F(x,y) = \left(\begin{array}{c|c|c|c|c|c} 2x & \mathbf{0} & y & \varepsilon_1 & \Omega y & Cy \\ \hline \mathbf{0} & 2y & x & \mathbf{0} & -\Omega x & -Cx \end{array} \right),$$

where $\varepsilon_1 = (1,0,\dots,0)^t \in \mathbb{R}^r$ and $\mathbf{0}$ is the null (column) vector in $\mathbb{R}^r$. Assume by contradiction that there exists $(x,y) \in M_1$ such that the six columns are linearly dependent, i.e., there exists $\lambda \in \mathbb{R}^6 \setminus \{0\}$ such that

$$\begin{cases} \lambda_1 x + \lambda_3 y + \lambda_4 \varepsilon_1 + \lambda_5 \Omega y + \lambda_6 C y = \mathbf{0} \\ \lambda_2 y + \lambda_3 x - \lambda_5 \Omega x - \lambda_6 C x = \mathbf{0} \end{cases} \tag{4.13}$$

Taking into account (4.6) and the fact that $(x,y) \in M$, by scalar multiplying by x and y the two equalities in (4.13) one obtains

$$\begin{cases} \lambda_1 + \lambda_4 x_1 = 0 \\ \lambda_3 = 0 \\ \lambda_3 + \lambda_4 y_1 = 0 \\ \lambda_2 = 0, \end{cases}$$

i.e., $\lambda_1 = \lambda_2 = \lambda_3 = \lambda_4 = 0$, because $x_1 = 0$ and $y_1 \neq 0$. Therefore $(\lambda_5, \lambda_6) \neq (0,0)$ are such that

$$\begin{cases} \lambda_5 \Omega y + \lambda_6 C y = \mathbf{0} \\ \lambda_5 \Omega x + \lambda_6 C x = \mathbf{0}, \end{cases}$$

i.e., $(\Omega x, \Omega y)$ and (Cx, Cy) are linearly dependent, hence

$$0 = -\Omega y \cdot Cx + Cy \cdot \Omega x = y \cdot (\Omega C - C\Omega)x,$$

a contradiction.

We now show that $M_2 := M \setminus M_1$ is a $(2r-6)$-dimensional smooth manifold. Defining the open set

$$\mathcal{O}_2 := \{(x,y) \in \mathbb{R}^r \times \mathbb{R}^r : y_1 \neq 0, \|(\Omega C - C\Omega)x\|^2 + \|(\Omega C - C\Omega)y\|^2 > 0\}$$

we have

$$M_2 = \mathcal{O}_2 \cap \{(x,y) \in \mathbb{R}^r \times \mathbb{R}^r : G(x,y) = (1,1,0,0,0,0,0)\}$$

where $G(x,y) := (\|x\|^2, \|y\|^2, x \cdot y, x_1, x \cdot \Omega y, x \cdot Cy, y \cdot (\Omega C - C\Omega)x)$. We have to show that the rank of ∇G is 6 at all points of M_2; using $y \cdot (\Omega C - C\Omega)x = -x \cdot (\Omega C - C\Omega)y$ one can compute

$$\nabla G(x,y) = \left(\begin{array}{c|c|c|c|c|c|c} 2x & \mathbf{0} & y & \varepsilon_1 & \Omega y & Cy & -(\Omega C - C\Omega)y \\ \hline \mathbf{0} & 2y & x & \mathbf{0} & -\Omega x & -Cx & (\Omega C - C\Omega)x \end{array} \right).$$

Assume by contradiction that there exists $(x,y) \in M_2$ such that the rank of $\nabla G(x,y)$ is not 6; then, the first six columns are linearly dependent and, reasoning as above, one can show the existence of $(\lambda_5,\lambda_6) \neq (0,0)$ such that

$$\lambda_5(\Omega x, \Omega y) + \lambda_6(Cx, Cy) = 0.$$

Since $y_1 \neq 0$ we have $\Omega y \neq 0$, hence $\lambda_6 \neq 0$; in particular

$$(Cx, Cy) = \lambda(\Omega x, \Omega y) \qquad \text{for } \lambda := -\tfrac{\lambda_5}{\lambda_6}. \tag{4.14}$$

Since also the columns 1–5 and 7 of ∇G are linearly dependent, there exists $\mu \in \mathbb{R}^6 \setminus \{0\}$ such that

$$\begin{cases} \mu_1 x + \mu_3 y + \mu_4 \varepsilon_1 + \mu_5 \Omega y - \mu_6(\Omega C - C\Omega) y = \mathbf{0} \\ \mu_2 y + \mu_3 x - \mu_5 \Omega x + \mu_6 (\Omega C - C\Omega) x = \mathbf{0}. \end{cases}$$

After scalar multiplication by x and y, and taking into account that $v\cdot(\Omega C - C\Omega)v = 0$ for any $v \in V_1$, one gets

$$\begin{cases} \mu_1 + \mu_4 x_1 = 0 \\ \mu_3 = 0 \\ \mu_3 + \mu_4 y_1 = 0 \\ \mu_2 = 0, \end{cases}$$

and, as before, $\mu_1 = \mu_2 = \mu_3 = \mu_4 = 0$. We deduce that there exist $(\mu_5,\mu_6) \neq (0,0)$ such that $\mu_5 \Omega y - \mu_6(\Omega C - C\Omega) y = \mathbf{0}$. We scalar multiply this equation by Ωy (which is not null because $y_1 \neq 0$) to get

$$\begin{aligned} 0 &= \mu_5 \|\Omega y\|^2 - \mu_6 \Omega y \cdot (\Omega C - C\Omega) y \\ &\overset{(4.14)}{=} \mu_5 \|\Omega y\|^2 - \mu_6 \big(\Omega y \cdot \lambda \Omega \Omega y - \Omega y \cdot C \Omega y\big) = \mu_5 \|\Omega y\|^2, \end{aligned}$$

which contradicts the fact that $\Omega y \neq 0$. This concludes Case 1.

Case 2: $\Omega C = C\Omega$. Let C be as in Lemma 4.7; we can also assume (4.4). Reasoning as before (and, in particular, using Lemma 4.8 and the fact that $C \neq 0$, i.e., that ker C is contained in some hyperplane of $\mathbb{R}^r$), it will be enough to show that the variety

$$\begin{aligned} N := &\left\{ (x,y) \in \mathbb{R}^r \times \mathbb{R}^r : \begin{array}{l} \|x\|^2 = \|y\|^2 = 1, x_1 = 0, x\cdot y = 0, y_1 \neq 0, \\ x \cdot (\Omega y) = x \cdot (Cy) = 0 \text{ and } x, y \notin \ker C \end{array} \right\} \\ = &\{(x,y) \in \mathbb{R}^r \times \mathbb{R}^r : y_1 \neq 0, x, y \notin \ker C \text{ and } F(x,y) = (1,1,0,0,0,0)\} \end{aligned}$$

(where F is as above) is $(2r-6)$-dimensional, i.e., that for any $(x,y) \in N$ the columns of $\nabla F(x,y)$ are linearly independent. If this were not the case, reasoning as in Case 1 one would find $(x,y) \in N$ and $(\lambda_5,\lambda_6) \neq (0,0)$ such that

$$\lambda_5 \Omega x + \lambda_6 C x = \mathbf{0} \quad \text{and} \quad \lambda_5 \Omega y + \lambda_6 C y = \mathbf{0}.$$

By the particular forms of the matrices Ω and C one has

$$0 = (\lambda_5 \Omega y + \lambda_6 C y) \cdot (0,1,0,\ldots,0) = -\lambda_5 y_1$$

and, since $y_1 \neq 0$, we get $\lambda_5 = 0$ and $\lambda_6 \neq 0$. This implies that $x, y \in \ker C \neq \mathbb{R}^r$, a contradiction. $\square$

5. Step 2, rank 4, dimension 7

The cases discussed in the previous sections cover all Carnot groups of dimension up to 6 and, among Carnot groups of dimension 7, only those with rank 4 are left out. We discuss them in the following Theorem 5.2

Lemma 5.1. *Let $\mathfrak{g} = V_1 \oplus V_2$ be a step two stratified Lie algebra with $\dim V_1 = 4$ and $\dim V_2 = 3$; assume that there exists a linear subspace $P \subset V_1$ such that*

$$\dim P = 2, \quad \dim[P, V_1] = 2 \quad \textit{and} \quad \dim[P, P] = 1.$$

Then, there exist $\lambda \in \mathbb{R}$ and bases $X_1, \dots, X_4$ of V_1 and X_{21}, X_{31}, X_{32} of V_2 such that

$$[X_2, X_1] = X_{21}, \quad [X_3, X_1] = X_{31}, \quad [X_4, X_3] = X_{43}, \quad [X_4, X_2] = \lambda X_{31},$$
$$[X_4, X_1] = [X_3, X_2] = 0.$$

Proof. We can first fix a basis X_1, X_2 of P and a vector $X_3 \in V_1 \setminus P$ such that $X_{21} := [X_2, X_1]$ and $X_{31} := [X_3, X_1]$ are linearly independent. By assumption we will have

$$[X_3, X_2] = aX_{21} + bX_{31}$$

and, up to replacing X_3, X_2 with (respectively) $X_3 + aX_1, X_2 - bX_1$, we can assume $a = b = 0$. Complete X_1, X_2, X_3 to a basis of V_1 by choosing some $X_4 \in V_1$; we will have

$$[X_4, X_1] = cX_{21} + dX_{31}$$

and we can assume $c = d = 0$ up to replacing X_4 with $X_4 - cX_2 - dX_3$. At this point, we have

$$[X_4, X_2] = eX_{21} + \lambda X_{31}$$

and we can assume $e = 0$ up to replacing X_4 with $X_4 + eX_1$. The choice of $X_{43} := [X_4, X_3]$, which must necessarily be independent from X_{21}, X_{31}, completes the proof. $\square$

Theorem 5.2. *Let G be a Carnot group associated to a step two stratified Lie algebra $\mathfrak{g} = V_1 \oplus V_2$ with $\dim V_1 = 4$ and $\dim V_2 = 3$. Then Abn_G has codimension at least 3, i.e., $\dim \mathrm{Abn}_G \leq 4$.*

Proof. By Proposition 2.12, $\dim P_\gamma \leq 2$ for any abnormal curve γ in G. The union

$$\bigcup \Big\{ \mathrm{Im}\ \gamma : \gamma \text{ abnormal in } G \text{ and } \dim P_\gamma = 1 \Big\}$$

is contained in $\exp(V_1)$, whose codimension is 3. Similarly, if P_γ has dimension 2 but it is Abelian, then is contained in $\exp(V_1)$.

We then have to consider only those abnormal curves γ such that $\dim P_\gamma = 2$, $[P_\gamma, V_1] \neq V_2$ and $\dim[P_\gamma, P_\gamma] = 1$. If there existed one such γ with $\dim[P_\gamma, V_1] = 1$, one could choose a basis $X_1, \ldots, X_4$ of V_1 with $X_1, X_2 \in P_\gamma$ such that

$$[X_i, X_j] = \begin{cases} X_{21} \neq 0 & \text{if } (i,j) = (2,1) \\ \in \mathbb{R}X_{21} & \text{for } i = 1,2 \text{ and } j = 3,4, \end{cases}$$

which would contradict the fact that $\dim V_2 = 3$. Therefore all abnormal curves γ we have to consider satisfy

$$\dim P_\gamma = 2, \quad \dim[P_\gamma, V_1] = 2 \quad \text{and} \quad \dim[P_\gamma, P_\gamma] = 1. \tag{5.3}$$

Assuming that one such γ exists (otherwise the proof would be concluded) we can use Lemma 5.1 to find bases $X_1, \ldots, X_4$ of V_1 and X_{21}, X_{31}, X_{32} of V_2 such that

$$\begin{aligned} &[X_2, X_1] = X_{21}, \quad [X_3, X_1] = X_{31}, \quad [X_4, X_3] = X_{43}, \quad [X_4, X_2] = \lambda X_{31}, \\ &[X_4, X_1] = [X_3, X_2] = 0 \end{aligned}$$

for a suitable $\lambda \in \mathbb{R}$. Let γ be an abnormal curve as in (5.3); the end point of γ will be the exponential $\exp(U)$ of a vector

$$U = X + Z = \sum_{i=1}^4 x_i X_i + x_{21}X_{21} + x_{31}X_{31} + x_{43}X_{43}$$

with $X = \sum_{i=1}^4 x_i X_i \in P_\gamma$. We can safely assume that $X \neq 0$, for otherwise the endpoint $\exp(U)$ of γ would be contained in a variety of dimension 3. Since $\dim[X, V_1] \leq 2$, we have to require that the matrix

$$M(X) := \begin{pmatrix} x_2 & -x_1 & 0 & 0 \\ x_3 & \lambda x_4 & -x_1 & -\lambda x_2 \\ 0 & 0 & x_4 & -x_3 \end{pmatrix},$$

has rank not greater than 2, where the columns of $M(X)$ represent (in the basis X_{21}, X_{31}, X_{43}) the vectors $[X, X_i]$ for $i = 1,2,3,4$. On computing the determinants of the four 3×3 minors we obtain

$$x_i(x_1x_3 + \lambda x_2x_4) = 0 \quad \forall i = 1, \ldots, 4$$

and, since $X \neq 0$, we deduce that

$$x_1x_3 + \lambda x_2x_4 = 0. \tag{5.4}$$

We also notice that, since $Z \in [P_\gamma, P_\gamma] \subset \operatorname{Im} \operatorname{ad}_X$, Z must be a linear combination of the columns of $M(X)$.

From now on we use exponential coordinates adapted to the basis $X_1, \ldots, X_{43}$ of $\mathfrak{g}$, i.e., we identify G and $\mathbb{R}^7$ by

$$\begin{aligned} &\mathbb{R}^7 \ni (x_1, x_2, x_3, x_4, x_{21}, x_{31}, x_{43}) \\ &\qquad \longleftrightarrow \exp(x_1X_1 + x_2X_2 + x_3X_3 + x_4X_4 + x_{21}X_{21} + x_{31}X_{31} + x_{43}X_{43}) \in G. \end{aligned}$$

Assume first that either $x_2 = x_3 = 0$ or $x_1 = x_4 = 0$; then the endpoint $\exp(U)$ of γ belongs to the variety $A \cup B$, where

$$A := \{(x_1, 0, 0, x_4, x_{21}, x_{31}, x_{43}) : (x_{21}, x_{31}, x_{43}) = a(-x_1, \lambda x_4, 0)+b(0, -x_1, x_4), x_1, x_4, a, b \in \mathbb{R}\}$$

and

$$B = \{(0, x_2, x_3, 0, x_{21}, x_{31}, x_{43}) : (x_{21}, x_{31}, x_{43}) = a(x_2, x_3, 0)+b(0, -\lambda x_2, -x_3), x_2, x_3, a, b \in \mathbb{R}\}.$$

This easily follows from the fact that Z is a linear combination of the columns of $M(X)$. The variety $A \cup B$ has dimension 4.

We can now assume that

$$(x_1, x_4) \neq (0, 0) \qquad \text{and} \qquad (x_2, x_3) \neq (0, 0). \tag{5.5}$$

Consider $Y = \sum_{i=1}^4 y_i X_i \in P_\gamma$ linearly independent from X: without loss of generality, we can also assume[3] that if $x_i \neq 0$ then $y_i \neq 0$. We consider three cases to conclude the proof.

(i) Suppose that $x_2 \neq 0$ and $x_4 \neq 0$. Then $\operatorname{Im} \operatorname{ad}_X$ contains the linearly independent vectors $(x_2, x_3, 0)$ and $(0, -x_1, x_4)$. Since $[P_\gamma, V_1]$ has dimension two, $\operatorname{Im} \operatorname{ad}_Y$ is also linearly generated by $(x_2, x_3, 0)$ and $(0, -x_1, x_4)$. In particular, our choice of Y yields

$$(y_2, y_3, 0) = a(x_2, x_3, 0) \quad \text{and} \quad (0, -y_1, y_4) = b(0, -x_1, x_4),$$

for $a, b \in \mathbb{R} \setminus \{0\}$. Since Z is a multiple of

$$\begin{aligned}[X, Y] &= (b - a)\big(x_1 x_2 X_{21} + (x_1 x_3 - \lambda x_2 x_4) X_{31} - x_3 x_4 X_{43}\big) \\ &= (b - a)\big(x_1 x_2 X_{21} + 2 x_1 x_3 X_{31} - x_3 x_4 X_{43}\big)\end{aligned}$$

by (5.4), the endpoint $\exp(U)$ of γ is in the four dimensional variety

$$\{(x_1, x_2, x_3, x_4, t x_1 x_2, 2 t x_1 x_3, -t x_3 x_4) : x_1, x_2, x_3, x_4, t \in \mathbb{R}, x_1 x_3 + \lambda x_2 x_4 = 0\}.$$

(ii) Suppose that $x_2 = 0$. By (5.5), $x_3 \neq 0$ and $M(x)$ has two linear independent columns $(0, x_3, 0)$ and $(0, 0, x_3)$, which therefore generate $\operatorname{Im} \operatorname{ad}_X$. By (5.4), $x_1 = 0$. Then the endpoint $\exp(U)$ of γ is in the four dimensional variety

$$\{(0, 0, x_3, x_4, 0, x_{31}, x_{43}) : x_3, x_4, x_{31}, x_{43} \in \mathbb{R}\},$$

because $(x_{21}, x_{31}, x_{43}) \in \operatorname{Im} \operatorname{ad}_X$.

(iii) Suppose that $x_4 = 0$. By (5.5), $x_1 \neq 0$ and the vectors $(1, 0, 0)$ and $(0, 1, 0)$ generate the column space of $M(x)$. By (5.4), $x_3 = 0$. As before, we conclude that the endpoint $\exp(U)$ of γ is in

$$\{(x_1, x_2, 0, 0, x_{21}, x_{31}, 0) : x_1, x_2, x_{21}, x_{31} \in \mathbb{R}\},$$

that has dimension 4.

The proof is accomplished. □

[3] Up to replacing Y with $Y + cX$ for a proper $c \in \mathbb{R}$.

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