On calibrations for Lawson's cones

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Abstract

In this paper a calibration method is recalled and applied to Lawson's cones to prove their minimality. The original proof of Bombieri, De Giorgi and Giusti is reinterpreted and made simpler.

Keywords: Simons's cones, minimal surfaces, calibrations, foliation

1 Introduction

In 1968 Simons [13] established the regularity of area–minimizing hypersurfaces in $\mathbb{R}^n$ up to $n = 7$. Moreover he showed that the cones

$$C_{k,k} = \left\{ (x, y) \in \mathbb{R}^k \oplus \mathbb{R}^k \mid |x|^2 = |y|^2 \right\}$$

are locally stable for $k \geq 4$ and he raised the question whether they are a global minima of the area functional. This was proved in 1969 by Bombieri, De Giorgi and Giusti [4], thus providing a counterexample to regularity of minimal hypersurfaces in dimensions larger than 7.
After that many other examples of minimal cones have been added by different authors. These results can be summarized in the following

Theorem 1.1. *In $\mathbb{R}^n = \mathbb{R}^k \oplus \mathbb{R}^h$ let us consider the cone*

$$C_{k,h} = \left\{ (x, y) \in \mathbb{R}^k \oplus \mathbb{R}^h \;\middle|\; |x|^2 = \frac{k-1}{h-1} |y|^2 \right\}$$

with $k, h \geq 2$. Then

(i) *if $n > 8$, $C_{k,h}$ is of minimal area;*

(ii) *if $n = 8$, $C_{k,h}$ has mean curvature zero at every point except at the origin, wich is singular, and it is of minimal area if and only if $|k - h| \leq 2$.*

Point (i) of Theorem 1.1 was first proved in 1972 by H. B. Lawson [8]. In 1973 P. Simoes [12] used techniques related to those of Bombieri, De Giorgi and Giusti and completed the characterization of minimal cones by adding to Lawson's list the cone $C_{3,5}$ and by proving that the cone $C_{2,6}$ is of mean curvature zero, but not a global minimum for the area functional.
An elegant and simple proof of the minimality of Simons' s cones (i.e. the cones $C_{k,k}$ for $k \geq 4$) was given by M. Miranda [10] in 1977. In 1986 P. Concus and M. Miranda [5], by using a technique introduced in [9] and the MACSYMA computer programming system to perform the algebraic manipulations, were able to prove the minimality of Lawson's cones under the additional assumptions $h + 4 < 5k$ and $(h, k) \neq (3, 5)$ (and the corresponding symmetric ones). By using the same technique, the minimality of all Lawson's cones was reestablished in 1993 by D. Benarros and M. Miranda [3]. The software Mathematica was used to carry out the calculations.

The purpose of this paper is to give another proof of Theorem 1.1. Our proof is modeled on that of [4], which is reinterpreted in the framework of calibrations. We eventually reduce to the study of a first order ODE and prove with elementary calculus tools the existence of a suitable solution, thus simplifying the original proof, where more sophisticated techniques were used in the study of a corresponding differential system. We remark that our method is sufficiently general to apply to all minimal Lawson's cones, not only to Simons's ones. Moreover, computations could all be carried out by hand.

It will be convenient to consider a localized area functional of the form

$$\mathcal{A}(S, B) := \int_{B \cap S} a \, d\mathcal{H}^{n-1}, \tag{1}$$

where S and B are, respectively, an oriented Lipschitz hypersurface and a Borel set contained in an open set $\Omega \subset \mathbb{R}^n$ and $a : \Omega \to \mathbb{R}$ is a C^1 positive function. We will say that an oriented hypersurface S is *minimal* (for the functional $\mathcal{A}$) if

$$\mathcal{A}(S, \overline{U}) \leq \mathcal{A}(T, \overline{U})$$

for every oriented hypersurface T which is homologous to S and agrees with it outside some relatively compact open subset U of Ω (this means, roughly speaking, that the union of T and S coincides in U with the boundary of some open subset relatively compact in Ω). A necessary condition for $\mathcal{A}$-minimality is that the first variation of the area at S is null, that is

$$\frac{\mathrm{d}}{\mathrm{dt}} \mathcal{A}(F_t(S), U)\Big|_{t=0} = 0, \tag{2}$$

whenever $(F_t)_{t \in \mathbb{R}}$ is a smooth one-parameter family of diffeomorphisms of $\mathbb{R}^n$ such that $F_0 = \mathrm{Id}_{\mathbb{R}^n}$ and $\mathrm{spt}\,(F_t - \mathrm{Id}_{\mathbb{R}^n}) \subset U$ for some relatively compact, open set $U \subset \Omega$. We say that S is *critical* for the functional $\mathcal{A}$ (briefly, $\mathcal{A}$-*critical*) if (2) holds for any choice of the family $(F_t)_{t \in \mathbb{R}}$.

Let us now consider the following problem: we are explicitly given an oriented $\mathcal{A}$-critical hypersurface S and we want to prove it is minimal. How can we do that? It is sufficient to exhibit a vector field ω such that:

(i) $|\omega| \leq 1$ on Ω;

(ii) $\omega = \nu_S$ on S;

(iii) div $(a\,\omega) = 0$ on Ω,

where ν_S is the oriented unit normal along S. Such a vector field, if it exists, is called a *calibration* for S (see [11]). From the existence of a calibration we easily obtain the minimality of S. Indeed, take another oriented hypersurface T which is homologous to S in some relatively compact open set U. We can apply the divergence theorem to the open subset of U which is bounded by S and T and so, by taking into account assumption (iii), we get that the flux of $a\,\omega$ through the two hypersurfaces is equal, that is

$$\int_{T\cap\overline{U}} a\,\omega\cdot\nu_T\,d\mathcal{H}^{n-1} = \int_{S\cap\overline{U}} a\,\omega\cdot\nu_S\,d\mathcal{H}^{n-1}. \tag{3}$$

This relation, together with assumptions (i) and (ii), implies that

$$\mathcal{A}(T,\overline{U}) \geq \int_{T\cap\overline{U}} a\,\omega\cdot\nu_T\,d\mathcal{H}^{n-1} = \int_{S\cap\overline{U}} a\,\omega\cdot\nu_S\,d\mathcal{H}^{n-1} = \mathcal{A}(S,\overline{U}), \tag{4}$$

namely the minimality of S.

Yet, finding a calibration for a given surface is not in general an easy task, as there are no standard techniques. In this paper we describe a method to build a calibration and we explain how that can be applied to prove the minimality of the cones $C_{k,h}$. The idea is to find a family $(S_\lambda)_{\lambda\in\mathbb{R}}$ of pairwise disjoint oriented hypersurfaces such that $S_0 = S$ and their union is Ω. We call such a family a *foliation of* Ω. Once we have a foliation, we shall define ω by

$$\omega(x) := \nu_{S_\lambda}(x), \tag{5}$$

where ν_{S_λ} is the unitary normal to the unique hypersurface passing through x. We will prove that, if the hypersurfaces S_λ are all critical for the functional $\mathcal{A}$, then div$\,(a\,\omega) = 0$ and ω is therefore a calibration for S, as conditions (i) and (ii) are trivially satisfied. A foliation consisting of $\mathcal{A}$-critical hypersurfaces will be called *$\mathcal{A}$-critical.*

The method described above can be usefully applied to prove Theorem 1.1. The area functional in this case is simply defined as in (1) with $a = 1$, and we look for an $\mathcal{A}$-critical foliation of $\mathbb{R}^n$ which includes the cone $C_{k,h}$. The fact that $C_{k,h}$ is invariant under the action of the group $G := SO(k)\times SO(h)$ suggests to look for critical hypersurfaces with the same kind of simmetry. Observe that, for a regular G-invariant hypersurface Σ, we have

$$\mathcal{H}^{n-1}(\Sigma) = c\int_\Gamma u^{k-1}v^{h-1}\,d\mathcal{H}^1(u,v)$$

where c is some constant depending only on k and h, $u = |x|$, $v = |y|$ and Γ is the curve contained in the first quadrant of the (u,v)-plane generating Σ under the action of G. Obviously, if Σ is critical for $\mathcal{A}$, then Γ is critical for the length functional

$$\mathcal{L}(\Gamma) := \int_\Gamma u^{k-1} v^{h-1} \, d\mathcal{H}^1(u,v) \tag{6}$$

defined for all curves Γ contained in $\mathbb{R}^+ \oplus \mathbb{R}^+$. Consequently an $\mathcal{A}$-critical foliation of $\mathbb{R}^n$ yields an $\mathcal{L}$-critical foliation of $\mathbb{R}^+ \oplus \mathbb{R}^+$, and indeed also the converse is true (see Section 4). Therefore the problem reduces to that of finding an $\mathcal{L}$- critical foliation of $\mathbb{R}^+ \oplus \mathbb{R}^+$ which contains γ, where γ is the half-line generating the cone $C_{k,h}$ under the action of G. Let us see, then, which is the condition that a curve must satisfy to be critical. Let the curve be defined by the parametric equations

$$\begin{cases} u = \mathrm{e}^{z(t)} \cos t \\ v = \mathrm{e}^{z(t)} \sin t. \end{cases} \tag{7}$$

If t ranges in the interval (a,b), the right hand side of (6) becomes

$$\int_a^b e^{(n-1)z} (\cos t)^{k-1} (\sin t)^{h-1} \sqrt{1+\dot{z}^2} \, dt$$

and the associated Eulero-Lagrange equation is

$$\ddot{z} = (1+\dot{z}^2) \left[(n-1) + \left(\frac{d-(n-2)\cos(2t)}{\sin(2t)} \right) \dot{z} \right] \tag{8}$$

where $d := k-h$. We now observe that, if $z(t)$ is a solution of (8), then $z(t)+\lambda$ is a solution too for every constant λ. That means that homothetic curves to a particular critical one of the form (7) are still critical. This remark suggests the following strategy to build the desired foliation: first, we look for a particular critical curve of the form (7), defined in $[0,t_0) \cup (t_0, \pi/2]$ and such that it tends asymptotically to the half-line γ when $t \to t_0$ (where t_0 is the angle formed by the u-axis with γ); then, we take the family of critical curves given by all its homothetics together with γ, which is critical too (see figure 1). It is easily seen that any point which does not lay on the half-line γ will belong to one and only one of the homothetic curves. Therefore such a family defines a foliation of $\mathbb{R}^+ \oplus \mathbb{R}^+$, which is $\mathcal{L}$-critical by construction.

We will see that such a solution of equation (8) exists if some relations between h and k are satisfied, relations which turn out to be precisely those stated in Theorem 1.1. By means of this method, the existence of a calibration for each minimal cone is thus proved.

This paper is organized as follows. In Section 2 we recall the main notations used in the paper. In Section 3 we show how to pass from foliations to calibrations. Section 4 contains the dimension reduction argument and in Section 5 we prove the existence of a suitable solution of (8).

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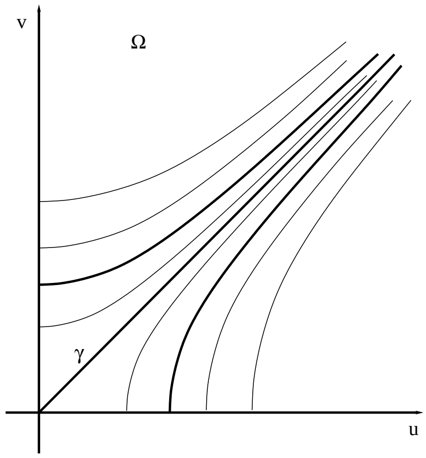

Figure 1: All the homothetics to a particular solution give a foliation of Ω.

2 Notations and preliminary results

We write here a list of symbols used throughout this paper.

Ω	open subset of $\mathbb{R}^n$ with Lipschitz boundary
$\mathrm{spt}f$	support of the function f
$\nu_\gamma(x)$	unitary normal to the curve γ in x
$\nu_S(x)$	unitary normal to the hypersurface S in x
$\mathcal{H}^n$	$n-$dimensional Hausdorff measure
$\overline{U}$	closure of the set U
$\mathrm{Int}(U)$	interior part of the set U
$\mathbb{R}^+$	set of non-negative real numbers

In the introduction we assumed that the area functional (1) was defined on Lipschitz hypersurfaces. We need to enlarge the class of admissible surfaces in order to include the cones. This can be done by considering boundaries of sets of locally finite perimeter. A set E is said to be of *locally finite perimeter* in Ω if its characteristic function χ_E belongs to $\mathrm{L}^1_{loc}(\Omega)$ and has distributional derivative $D\chi_E$ which is a Radon vector-valued measure in Ω. In this setting a hypersurface is seen as the boundary (in an appropriate measure theoretic sense) of a set E of locally finite perimeter and indeed the measure $D\chi_E$ is the $(n-1)$-dimensional Hausdorff measure restricted to that boundary times its unit inward normal. For the purpose of this paper we do not need to enter into the details of this theory (which can be found for instance in [6]). We just observe that we will deal with "nice" sets and for them this measure theoretic definition of boundary coincides with the usual one. Therefore in the sequel with the word hypersurface we will refer to the boundary (in the usual set theoretic sense) of a set of locally finite perimeter, in particular to a closed subset of Ω which is, up to an $\mathcal{H}^{n-1}$-negligible set, an oriented Lipschitz manifold of codimension one and with no boundary. The area functional given by (1) is assumed to be defined on this enlarged family of hypersurfaces. Moreover, we say that the hypersurface S (which is the boundary of some set E) is *minimal* for the functional $\mathcal{A}$ if

$$\mathcal{A}(S, \overline{U}) \leq \mathcal{A}(T, \overline{U})$$

for every hypersurface T which is the boundary of a set L that agrees with E outside some relatively compact open subset U of Ω. Observe that this means that the two hypersurfaces agree outside U and their union coincides in U with the boundary of the symmetric difference between E and L, and therefore they are, in some sense, "homologous" one to the other.
Last, we need to verify that in this new framework the existence of a calibration for a certain hypersurface is sufficient to guarantee its minimality. In the introduction this was proved only for globally Lipschitz surfaces, but one easily sees that all the statements hold in this setting too, provided we can apply the divergence theorem when two "homologous" hypersurfaces are compared. The following refined version of the the divergence theorem is all we need to conclude.

Lemma 2.1. *Let ϕ be a bounded vectorfield on Ω. Suppose there exist a closed set $N := N_0 \cup N_1$, where N_0 is an $\mathcal{H}^{n-1}$-negligible closed set and N_1 a (possibly disconnected) Lipschitz hypersurface, such that:*

(i) *ϕ is of class C^1 and $\operatorname{div}\phi = 0$ on $\Omega \setminus (N_0 \cup N_1)$;*

(ii) *ϕ is continuous on $\Omega \setminus N_0$.*

Then the identity $\operatorname{div}\phi = 0$ holds distributionally on Ω too. Moreover, let the hypersurfaces S and T be the perimeters of two sets which agree outside some open and relatively compact subset U of Ω. Then we have

$$\int_{S\cap\overline{U}} \phi \cdot \nu_S \, \mathcal{H}^{n-1} = \int_{T\cap\overline{U}} \phi \cdot \nu_T \, \mathcal{H}^{n-1}. \tag{9}$$

The previous lemma is a special case of Lemma 2.4 and Lemma 2.6 of [2], to which we refer for the proof. We remark that this lemma will be of great utility in the sequel too since it allows to work with calibrations and surfaces that are not necessarily regular.

Throughout this paper we will denote by G the group of rotations $SO(k) \times SO(h)$ on $\mathbb{R}^n = \mathbb{R}^k \oplus \mathbb{R}^h$ and by γ the half-line in $\mathbb{R}^+ \oplus \mathbb{R}^+$ which generates the cone $C_{k,h}$ under the action of G, namely

$$\gamma := \{(u, v) \,|\, u = \sqrt{(k-1)/(h-1)}\, v\, \}.$$

3 Calibrations through foliations

In this section we will denote by $(S_\lambda)_{\lambda\in\mathbb{R}}$ a foliation of Ω and by ω be the vector field defined by (5). We will clarify under which assumptions ω is a calibration for the hypersurfaces S_λ.

Proposition 3.1. *Let V be an open subset of Ω. Let $f \in C^2(V)$ such that $|Df(x)| \neq 0$ for all $x \in V$ and suppose that each S_λ agrees in V with the level set $\{x \in V \mid f(x) = \lambda\, \}$. If S_λ are critical for $\mathcal{A}$, then*

$$\operatorname{div}\left(a\frac{Df}{|Df|}\right) = 0 \quad \text{on } V.$$

Remark 3.2. Notice that, as Df is normal to the level sets of f, by the hypotheses of Proposition 3.1 the vector field $Df/|Df|$ coincides with ω on V. So, if the foliation of Ω is formed by $\mathcal{A}$-critical hypersurfaces that can be expressed (at least locally, since the condition that a vector field is divergence-free is local) as level sets of a non-zero gradient C^2 function, Proposition 3.1 guarantees that the vector field $a\,\omega$ is divergence-free and ω is therefore a calibration for each hypersurface of the foliation.

Remark 3.3. The previous proposition will be used together with Lemma 2.1 in the following situation: let N_0 and N_1 be as in Lemma 2.1 and let $(S_\lambda)_{\lambda\in\mathbb{R}}$ be an $\mathcal{A}$-critical foliation of Ω which satisfies the conditions of Proposition 3.1 with $V := \Omega \setminus (N_0 \cup N_1)$. Moreover, let us assume that the vector field ω, given by the unitary normals to the hypersurfaces S_λ, is continuous on $\Omega \setminus N_0$. Then ω is a calibration for each S_λ. In fact, by applying Proposition 3.1, we get that $\operatorname{div}(a\omega) = 0$ on $\Omega \setminus (N_0 \cup N_1)$. Hence we can apply Lemma 2.1 with $\phi := a\omega$, which guarantees that (3) and therefore (4) hold with $S := S_\lambda$, that is the claim.

The proof that follows is just an adaptation to our case of the proof of Theorem 4.6 in [1].

Proof: Let us indicate with ϕ the vector field $a\,Df/|Df|$. Take an x_0 in V. As $|Df(x_0)| \neq 0$, by reordering the coordinates and by changing sign to f if necessary, we may as well suppose that $\frac{\partial f}{\partial x_n}(x_0) > 0$. Writing $x = (x', s) \in \mathbb{R}^{n-1} \times \mathbb{R}$, we have immediately that the map $(x', s) \mapsto (x', f(x', s))$ is a diffeomorphism in a neighborhood of x_0. Consequently, for suitable open sets $U \subset \mathbb{R}^n$, $A \subset \mathbb{R}^{n-1}$ and $I \subset \mathbb{R}$, it admits an inverse map

$$\begin{aligned} A \times I \quad &\to U \\ (x', \lambda) \quad &\mapsto (x', s(x', \lambda)) \end{aligned}$$

which is a diffeomorphism. By deriving the relation $s(x', f(x', s)) = s$, we immediately obtain

$$\begin{cases} \partial_\lambda s\, \partial_s f = 1 \\ D_{x'} s(x', f(x', s)) + \partial_\lambda s\, D_{x'} f(x', s) = 0. \end{cases} \tag{10}$$

Moreover, since $f(x', s(x', \lambda)) = \lambda$, we have, for fixed λ, that $s^\lambda(x') := s(x', \lambda)$ is a parametrization of the hypersurface S_λ in U, hence

$$\mathcal{A}(S_\lambda, U) = \int_A g(x', s^\lambda,\, D_{x'} s^\lambda)\, dx', \tag{11}$$

where $g(x', \xi, p) := a(x', \xi)\sqrt{1 + |p|^2}$. From the criticality of S_λ we have that s^λ satisfies the Eulero-Lagrange condition for (11), that is

$$\partial_\xi g(x', s^\lambda,\, D_{x'} s^\lambda) = \operatorname{div}_{x'}\left(\partial_p g(x', s^\lambda,\, D_{x'} s^\lambda)\right). \tag{12}$$

The vector field ϕ can be equivalently written in U as

$$\begin{cases} \phi^{x'}(x', s) = -\partial_p g(x', s^\lambda(x'),\, D_{x'} s^\lambda(x')) \\ \phi^{s}\ (x', s) = g(x', s^\lambda(x'),\, D_{x'} s^\lambda(x')) - \langle \partial_p g(x', s^\lambda(x'),\, D_{x'} s^\lambda(x')),\, D_{x'} s^\lambda(x') \rangle \end{cases}$$

where $\lambda = f(x', s)$. Let us compute now the divergence of the vector field ϕ at the point (x', s). All the following expressions are evaluated in x', $s = s^\lambda(x')$ and $p = D_{x'} s^\lambda(x')$, with $\lambda = f(x', s)$. We have

$$\mathrm{div}_{x'} \phi^{x'} = -\mathrm{div}_{x'} \Big(\partial_p g(x', s^\lambda,\, D_{x'} s^\lambda) \Big) - \langle \partial_{p\xi}\, g, \partial_\lambda s^\lambda \;\; D_{x'} f \rangle - \langle \partial_{pp}\, g\; \partial_\lambda\, D_{x'} s^\lambda,\, D_{x'} f \rangle$$

and

$$\begin{aligned} \partial_s \phi^s &= \partial_\xi g\, \partial_\lambda s^\lambda \partial_s f + \langle \partial_p g,\; \partial_\lambda\, D_{x'} s^\lambda \partial_s f \rangle - \langle \partial_p g,\; \partial_\lambda\, D_{x'} s^\lambda\, \partial_s f \rangle \\ &- \langle \partial_{p\xi} g,\; \partial_\lambda s^\lambda \partial_s f\, D_{x'} s^\lambda \rangle - \langle \partial_{pp} g\, \partial_\lambda\, D_{x'} s^\lambda,\; \partial_s f\, D_{x'} s^\lambda \rangle. \end{aligned}$$

When we sum up those terms, taking into account (10) and (12) everything simplifies and we eventually obtain that $\mathrm{div}\phi = \mathrm{div}_{x'} \phi^{x'} + \partial_s \phi^s = 0$. Since x_0 was arbitrarily choosen and the divergence is a local operator we get the claim. □

4 Dimension reduction

In this section we show how from a suitable $\mathcal{L}$-critical foliation of $\mathbb{R}^+ \oplus \mathbb{R}^+$ we can obtain a foliation of $\mathbb{R}^n$ which contains the cone $C_{k,h}$. Moreover, we show that the vector field given by the unitary normals to the hypersurfaces of the foliation defines a calibration for the cone.

Let us suppose we have found a foliation $(\gamma_\lambda)_{\lambda \in \mathbb{R}}$ of $\mathbb{R}^+ \oplus \mathbb{R}^+$ via $\mathcal{L}$-critical curves and a function $f : \mathbb{R}^+ \oplus \mathbb{R}^+ \to \mathbb{R}$ such that the following conditions are satisfied:

(i) $\gamma_\lambda \equiv f^{-1}(\lambda) := \{(u, v) \in \mathbb{R}^+ \oplus \mathbb{R}^+ \mid f(u, v) = \lambda\}$;

(ii) $\gamma_0 \equiv \gamma$;

(iii) f is of class C^2 and $|Df| \neq 0$ on $\mathrm{Int}(\mathbb{R}^+ \oplus \mathbb{R}^+ \setminus \gamma)$;

(iv) the vector field given by $Df/|Df|$ admits on $\mathrm{Int}(\mathbb{R}^+ \oplus \mathbb{R}^+)$ a continuous extension which agrees on γ with its unitary normal ν_γ. Let us call α such extension.

Then the hypersurfaces S_λ generated by γ_λ under the action of G provide the required foliation of $\mathbb{R}^n$. In fact we have the following

Proposition 4.1. *With the conditions and notations stated above, the vector field ω given by the unitary normal to the hypersurfaces S_λ is a calibration for each S_λ, in particular for S_0, namely $C_{k,h}$.*

Proof: First notice that, by Proposition 3.1 and the hypotheses above,

$$\mathrm{div} \left(u^{k-1} v^{h-1} \frac{Df}{|Df|} \right) = 0 \quad \text{on} \;\; \mathrm{Int}(\mathbb{R}^+ \oplus \mathbb{R}^+ \setminus \gamma).$$

Moreover, by Remark 3.3, the vector field α is a calibration for the curves γ_λ, in particular for γ (relatively to the functional $\mathcal{L}$).

Let us set $N_0 := \{(x, y) \in \mathbb{R}^k \oplus \mathbb{R}^h \;:\; |x|\,|y| = 0\}$ and $N_1 := C_{k,h} \setminus \{0\}$ and let ω be the vector field defined by (5). As ω may be not well defined on N_0, we set ω equal to zero on that set. Then it is sufficient to prove the following two facts:

(a) ω is of class C^1 and $\operatorname{div}\omega = 0$ on $\mathbb{R}^n \setminus (N_0 \cup N_1)$;

(b) ω is continuous on $\mathbb{R}^n \setminus N_0$.

In fact, $N_0 \cup N_1$ is closed and N_0 is $\mathcal{H}^{n-1}$-negligible, so, by Lemma 2.1 and Remark 3.3, properties (a) and (b) above give the claim of the proposition.
Let us define the function $F(x, y) := f(|x|, |y|)$, where $(x, y) \in \mathbb{R}^k \oplus \mathbb{R}^h$. We easily see that each hypersurface S_λ coincides with the λ-level set of F. Moreover, F is of class C^2 on $\mathbb{R}^n \setminus (N_0 \cup N_1)$ and on that set $\omega \equiv DF/|DF|$. Assertion (a) above will then follow from the next lemma. The proof is a straightforward check, and is omitted.

Lemma 4.2. *If* $\operatorname{div}\left(u^{k-1}v^{h-1}Df/|Df|\right) = 0$ *for* $uv \neq 0$, *then* $\operatorname{div}(DF/|DF|) = 0$ *for* $|x|\,|y| \neq 0$.

To prove assertion (b), we only need to show that the continuous extension of $DF/|DF|$ to $N_1 := C_{k,h} \setminus \{0\}$ coincide with its unitary normal $\nu_{C_{k,h}}$. But that easily follows from the definition of F and from assumption (iv).

□

5 Proof of Theorem 1.1

In this section we end the proof of Theorem 1.1 by showing the existence of an $\mathcal{L}$-critical foliation of $\mathbb{R}^+ \oplus \mathbb{R}^+$ and of a function $f : \mathbb{R}^+ \oplus \mathbb{R}^+ \to \mathbb{R}$ satisfying the conditions (i)-(iv) listed in Section 4.
We will denote by t_0 the angle formed by the u-axis with γ, namely $t_0 = \frac{1}{2}\arccos\left(\frac{d}{n-2}\right)$.
Let us go back to the differential equation (8), which, setting $w := \dot z$, can be rewritten in the following way

$$\dot w = (1 + w^2)\left[(n-1) + \left(\frac{d - (n-2)\cos(2t)}{\sin(2t)}\right) w\right], \tag{13}$$

where again $d := k - h$. The next lemma shows that, in order to get the desired critical foliation, it is enough to find a suitable solution of this differential equation.

Lemma 5.1. *Let* $\overline{w}(t)$ *be a solution of (13) defined in* $[0, t_0) \cup (t_0, \pi/2]$ *and such that* $\lim_{t\to t_0} |\overline{w}(t)| = +\infty$. *Then there exists an* $\mathcal{L}$*-critical foliation* $(\gamma_\lambda)_{\lambda\in\mathbb{R}}$ *of* $\mathbb{R}^+ \oplus \mathbb{R}^+$ *and a function* $f : \mathbb{R}^+ \oplus \mathbb{R}^+ \to \mathbb{R}$ *satisfying the conditions (i)-(iv) listed in Section 4.*

Proof: Let us consider a curve of the form (7) such that $\dot z(t) = \overline{w}(t)$. This curve is defined on $[0, t_0) \cup (t_0, \pi/2]$ and is $\mathcal{L}$-critical by definition. By classical results on Cauchy problem, the solution $\overline{w}(t)$ is easily seen to be at least of class C^1 away from $t = 0$ and $t = \pi/2$, so that $z(t)$ is at least of class C^2. We want to show that the curve tends asymptotically to the half line γ when t goes to t_0, i.e. that $\rho(t)$ and γ have no intersection points, where we have setted $\rho(t) := \mathrm{e}^{z(t)}$. If that were not the case, we would have, by the hypothesis on $\overline{w}(t)$, that $\dot\rho(t_0) = \lim_{t\to t_0} |\dot\rho(t)| = +\infty$. That would mean that $\rho(t)$ and γ have the same derivative in their intersection point and

that is impossible as they are both $\mathcal{L}$-critical, i.e. solutions of the same second order Cauchy problem.
Let us see how this special solution can be used to build the desired foliation. Let us denote by γ_1 and γ_{-1} the $\mathcal{L}$-critical curves corresponding to $\rho(t)$ respectively for $t < t_0$ and $t > t_0$. As explained in the introduction, the homothetics to this particular two curves together with the half-line γ give us an $\mathcal{L}$-critical foliation of $\mathbb{R}^+ \oplus \mathbb{R}^+$. To build a function having those curves as level sets, let us define

$$\varphi(t,\rho) := \begin{cases} \rho/\rho(t) & \text{if } 0 \le t < t_0 \\ 0 & \text{if } t = t_0 \\ -\rho/\rho(t) & \text{if } t_0 < t \le \frac{\pi}{2} \end{cases}$$

and set $f(u,v) := \varphi\left(\arctan(v/u),\, \sqrt{u^2+v^2}\right)$. Taking into account that $\rho(t)$ is of class C^2, it is immediate that such an f satisfies conditions from (i) to (iii) of Section 4. Finally, by using the fact that $\lim\limits_{t\to t_0} |\overline{w}(t)| = +\infty$, a straightforward check shows that $Df/|Df|$ has a continuous extension to γ, where it coincides with its unitary normal ν_γ, and (iv) is thus satisfied too. $\square$

We prove now the existence of this particular solution $\overline{w}(t)$.

Lemma 5.2. *For h and k satisfying the conditions of Theorem 1.1, there exists a solution $\overline{w}(t)$ of (13) defined on $[0,t_0) \cup (t_0, \pi/2\,]$ such that $\lim\limits_{t\to t_0} |\overline{w}(t)| = +\infty$.*

Proof: Let us denote with $H(t,w)$ the right-hand side of equation (13) and set

$$g(t) := (n-1)\frac{\sin(2t)}{(n-2)\cos(2t) - d}.$$

Notice that $H(t, g(t)) = 0$ for every t, and then g is a super-solution of (13). If in addition we could find an $\alpha \in (0,1)$ such that $\alpha g(t)$ is a sub-solution, that is

$$H(t, \alpha g(t)) \ge \alpha \dot{g}(t) \qquad \text{for every } t \in (0,t_0) \cup (t_0, \pi/2)\,, \tag{14}$$

then a standard argument gives the existence of a solution $\overline{w}(t)$ to equation (13) defined on $(0,t_0) \cup (t_0,\pi/2)$ and such that $\alpha|g(t)| \le |\overline{w}(t)| \le |g(t)|$. By extending $\overline{w}(t)$ continuously to $[0,t_0) \cup (t_0,\pi/2\,]$ we get the claim.

Let us see then for which integers k and h there exists an $\alpha \in (0,1)$ for which (14) is satisfied (notice that H depends on k and h). A computation will lead to

$$(1-\alpha) + (1-\alpha)\alpha^2 (g(t))^2 - 2\alpha \frac{(n-2) - d\cos(2t)}{[(n-2)\cos(2t) - d]^2} \ge 0 \tag{15}$$

and we want this inequality to be satisfied on $(0,t_0) \cup (t_0,\pi/2)$. Simple computations show that this is true if

$$\begin{aligned} G(t) \;&:=\; (1-\alpha)\alpha^2(n-1)^2(\sin 2t)^2 - 2\alpha(n-2-d\cos 2t) + \\ &+\; (1-\alpha)[(n-2)\cos 2t - d]^2 \ge 0 \qquad \text{on } [0,\pi/2]. \end{aligned} \tag{16}$$

The function $G(t)$ has a minimum in the point $\theta_0 \in (0, \pi/2)$, where

$$\cos(2\theta_0) = \frac{d}{(1-\alpha)((1+\alpha)n - (2+\alpha))},$$

hence (16) holds if $G(\theta_0) \geq 0$. Such condition is equivalent to

$$[\alpha^2(n-1) - \alpha n + 2]d^2 + (1-\alpha)[\alpha(n-1) + (n-2)][\alpha(1-\alpha)(n-1)^2 - 2(n-2)] \geq 0$$

which can be rewritten in the following form

$$d^2 \leq \frac{(1-\alpha)[\alpha(n-1) + (n-2)][\alpha(1-\alpha)(n-1)^2 - 2(n-2)]}{\alpha n - \alpha^2(n-1) - 2} \tag{17}$$

provided the denominator at the right-hand side is positive.
For $n = 8$ and $\alpha = 1/2$ we have $|d| \leq 2$, and so the lemma is proved under the conditions of statement (ii) of Theorem 1.1.
To get the claim under the conditions of statement (i) of Theorem 1.1, we have to show that for every $n > 8$ it is possible to choose α such that (17) is verified for the maximal value of d, that is $n-4$ (remember that $k, h \geq 2$ by hypothesis). To this purpose, we replace d with $n-4$ in (17) and we try the value $\alpha = 1/\sqrt{n-1}$. After some computation we get

$$3(\sqrt{n-1})^4 - 9(\sqrt{n-1})^3 - 6(\sqrt{n-1})^2 + 23(\sqrt{n-1}) - 7 \geq 0,$$

which, by setting $x := \sqrt{n-1}$, can be rewritten in the following form

$$3x^4 - 9x^3 - 6x^2 + 23x - 7 \geq 0.$$

Since $5x > 7$ it will be enough to show that

$$3x^4 - 9x^3 - 6x^2 + 18x = 3x(x^2-2)(x-3) \geq 0,$$

which is obviously true when $n > 9$. In the case $n = 9$, inequality (17) with $d = n-4$ is still verified for some admissible value of α (for example, take $\alpha = 2/5$). $\square$

Remark 5.3. We want to stress that also the conditions $k, h \geq 2$ and $n \geq 8$ arise naturally from the study of inequality (15), that is they are necessary for the existence of an admissible α. In fact, if we want this inequality to be satisfied for $t \to 0^+$ and for $t \to \pi/2^-$ it must be $\alpha \leq \min(1 - 1/h,\ 1 - 1/k)$ and therefore $k, h \geq 2$. By the study of inequality (15) when $t \to t_0$ we obtain the second condition, that is $n \geq 8$.

Proof of Theorem 1.1: the claim now follows by combining the assertions stated in the previous sections . In fact, let h and k satisfy the hypotheses of the theorem. Then, by Lemma 5.2 and Lemma 5.1, there exists an $\mathcal{L}$-critical foliation $(\gamma_\lambda)_{\lambda\in\mathbb{R}}$ of $\mathbb{R}^+ \oplus \mathbb{R}^+$. Let S_λ be the hypersurface of $\mathbb{R}^n = \mathbb{R}^k \oplus \mathbb{R}^h$ generated by γ_λ under the action of $G := SO(k) \times SO(h)$. As the family of curves $(\gamma_\lambda)_{\lambda\in\mathbb{R}}$ contains the half-line γ, the family of hypersurfaces $(S_\lambda)_{\lambda\in\mathbb{R}}$ is a foliation of $\mathbb{R}^n$ which contains the cone $C_{k,h}$. Let ω be the vector field defined by (5). By applying Proposition 4.1 we conclude that ω is a calibration for the cone $C_{k,h}$, which is therefore minimal.

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