Rational expectations equilibria in a Ramsey model of optimal growth with non-local spatial externalities

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Abstract

This work provides a rigorous treatment concerning the formation of rational expectations equilibria in a general class of spatial economic models under the effect of externalities, using techniques from calculus of variations and optimal control. Using detailed estimates for a parametric optimisation problem, the existence of rational expectations equilibria is proved via a fixed-point theorem, and they are characterised in terms of a nonlocal Euler-Lagrange equation.

1 Introduction

We consider in this work infinite horizon spatial optimal growth models, where capital accumulation and depreciation occur locally but output production is affected by general spatial interactions. In particular, eventhough there is no direct capital movement between various spatial locations in the model, there exist nonlocal externalities in the sense that local capital production is directly influenced by the distribution of capital in appropriate neighbourhoods, while it is also taken into account that local capital contributes in a self-consistent manner to the mean field distribution of capital. In these models a natural research question is the existence of a rational expectations equlibrium (REE) where a representative individual in each location, acting as a " local planner", maximises discounted utility of consumption by considering the spatial externality as exogenous. In this way he/she contributes to the mean field capital distribution, which in turn affects the local decision in terms of the nonlocal externalities. Consistency of the model requires the existence of such a mean field capital distribution which is compatible with the local decisions in the above manner, called hereafter an REE for the model.

In this paper we assume that spatial interactions among locations can be expressed as a spatial externality which in general attenuates with distance, as it is the case, for example, for knowledge spillover effects from one location to another. The main idea associated with knowledge spillovers is that innovation and new productive knowledge flows more easily among agents which are located within the same area (e.g. [12], [6]). Thus proximity is important in characterizing spatial spillovers ([3], [5]). We incorporate general spatial spillovers by interpreting the capital stock of each firm in a broad sense to include knowledge along with physical capital (e.g. [17]). As argued by [16] the effect of capital on each firm's output, at any given point in time, does not depend just on the accumulated

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stock by the firm up to this time, but on capital accumulated in nearby locations by other firms. Thus the spatial externality takes the form of a Romer ([17]) type externality where, by keeping all other factors in fixed supply, output is determined by own capital stock and by an appropriately defined aggregate of capital stocks of firms across the spatial domain. The capital stock aggregate is determined by a distance-response function (see [15] for an early use of distance-response functions) that measures the strength of the effect on the output of a firm in a certain location induced by the capital stock accumulated by a firm in another location.

A positive distance-response function that attenuates with distance can be interpreted as reflecting knowledge spillovers. A distance-response which is negative indicates a negative externality such as generalised congestion effects. Thus, by combining a distance-response function, centripetal and centrifugal responses can be introduced with the strength - positive or negative - of these forces diminishing with distance.

Modeling the spatial spillovers by a general nonlinear integral operator, we analyze the problem where forward-looking local planners maximise discounted utility by choosing a local consumption path, subject to local capital accumulation, by considering the spatial externality affecting their local production as parametric. A REE in the whole spatial domain is defined as the path of local capital stock and consumption which emerges when the capital stocks comprising the externality are determined endogenously through the optimality conditions in each location.

The aim of this paper is to provide a rigorous mathematical study of rational expectation equilibria emerging in the general growth model with spatial interactions, providing existence results using techniques from the calculus of variations in BV spaces and appropriate fixed point theorems. To the best of our knowledge such an analysis has not appeared in the literature before. Our choice of the functional space setting is the most general one may choose for the calculus of variations problem in consideration, and we will justify and discuss it better later on.

In this spirit, our detailed analysis highlights the effects of conditions on the primitives of the economy on the behaviour of such equilibria and clarifies aspects related to the fine qualitative properties of the optimal path and optimal policy. Furthermore, the characterisation of the rational expectations equilibria is obtained in terms of a nonlocal Euler-Lagrange equation that may be used for further study of the qualitative spatio-temporal properties of the optimal capital allocations. We think that the results of our paper shed some light towards a better understanding of the problem of spatial growth, which is receiving increasing attention in recent economic literature (see e.g. [7] or [4]).

On the other hand the problem of optimal growth in its temporal version has attracted the attention of the more mathematically-oriented literature in various forms. For example Ekeland has studied in detail and rigorously various aspects of the optimal growth problem, focusing on well-posedness as well as on the derivation of transversality conditions for the infinite horizon version of the model (see e.g. [10] and references therein). Techniques from the theory of calculus of variations in Sobolev spaces have been employed for the study of the temporal problem (see e.g. [11] or [13]; see also [18] for the case of recursive utilities) whereas techniques from the theory of Hamilton-Jacobi equation and viscosity solutions have been used for the study of stochastic effects in the Ramsey model (see e.g. [1] and [14]). However, to the best of our knowledge a rigorous mathematical treatment, with the aim of studying rational expectations equilibrium questions, of the problem of optimal growth in spatial economies has not been addressed so far.

It is the aim of this paper to make a few steps in this direction by presenting a rigorous treatment of a spatial economic growth model in the presence of spatial externalities, following up on previous work by Brock et al [7]. In particular, we present a proof of existence of rational expectations equilbria (REE) under rather general assumptions, which requires some detailed (and rather technical) results for a parametric individual optimisation problem in which the state of the externalities is assumed as a known functional parameter (which can be interesting in their own right.) Indeed, these parametric problems are exactly a non-autonomous optimal growth (Ramsey) problem (non-autonomous refers

to the fact that externalities can evolve in time, and are a time-dependent datum of the optimisation problem of each agent). To analyse it, it is preferable to bypass the usual way of treating the autonomous problem, which consists in building (or guessing) solutions to suitable sufficient optimality conditions (which are expressed in terms of differential equations and transversality conditions), as it is not always easy to solve them. Thus, one has to follow the direct method of calculus of variations, i.e. first proving existence in a very weak and wide functional space, which has to be chosen as the space where the optimisation problem itself provides compactness. In our case, we will see that we obtain bounds on the total mass of the derivative of the capital path, i.e. BV bounds, but a pr iori not more. Then, one has to prove regularity and properties about the optimisers, which also includes proving that they actually belong to better functional spaces, more natural for the economical interpretation. In our case, a crucial tool will also be the proof of lower bounds on the consumption: this is nowadays quite standard in the autonomous case, but requires more attention in the setting of our paper, on account of the explicit dependence on the spatial externalities, which are non-constant in time, and is one of the key technical parts of our work.

Once some desirable properties of the solution of the parametric individual problem have been established, the existence of REE is obtained by an application of the Schauder fixed point theorem in an appropriate functional space setting. The establishment of the necessary estimates for the individual problem as well as the derivation of the nonlocal Euler-Lagrange characterisation of the REE is based on a suitable application of a Pontryagin maximum principle. The Euler-Lagrange equation which characterises the REE can be used either for the numerical computation of the equilibrium, or for the derivation of qualitative aspects such as, for instance, pattern formation behaviour.

The paper is organised as follows. In Section 2, the model for optimal economic growth under spatial externalities that we consider is presented. In Section 3 a detailed study of the individual optimisation problem, where the externalities are taken as a given parameter, is performed. Using the results of Section 3, in Section 4 we study the problem of existence of RE and their characterisation in terms of the Euler-Lagrange equation and provide the desired existence result. The appendix details the proof of the most technical result of Section 3.

2 The model

We consider a spatial growth model with spatial externalities. In particular let D be a given geographic region (one can think at it as a compact subset of $\mathbb{R}^2$, but this is not crucial, and a more general metric space could also be used) and let the function $k : \mathbb{R}_+ \times D \to \mathbb{R}_+$ model the spatio-temporal distribution of capital stock in the region. We are in a one-all-purpose-commodity world where we produce output which can be used for consumption or investment, that is accumulation of capital. The production of output depends on spatial externalities, which display spatio-temporal variability as well, and are modeled in terms of the function $K : \mathbb{R}_+ \times D \to \mathbb{R}_+$. Both negative or positive externalities are included in the model in the sense that externalities may increase or decrease local production. Capital is assumed to be immobile but spatial effects are introduced through the varying level of externality effects K.

The capital accumulation satisfies the differential equation

$$k'(t,z) = f(k(t,z), K(t,z)) - c(t,z)$$

where $k'(t,z) = \frac{\partial k}{\partial t}(t,z)$, f is the net output given by the capital and the externalities, and $c : \mathbb{R}_+ \times D \to \mathbb{R}_+$ describes spatio-temporal consumption. By net output we mean the production, net of depreciation: we include capital depreciation in f, i.e., $f(k,K) = \varphi(k,K) - \delta k$. Here φ is a standard neoclassical production function and the parameter $\delta > 0$ gives capital loss due to depreciation. A possible example for the production function can be a Cobb-Douglas type production

function $\varphi(k, K) = Ak^\alpha K^\beta$ for appropriate values of α and β (with $\alpha + \beta < 1$), so that we would have $f(k, K) = Ak^\alpha K^\beta - \delta k$. To simplify the exposition we assume that in each location labour is fixed and fully employed, so the arguments in the production and net output functions can be regarded as per capita quantities.

The externalities K are determined (in a self-consistent fashion, i.e. endogenously) by the capital allocation k. The endogeneity is modelled by assuming that we have $K = \mathcal{S}k$ where $\mathcal{S}$ is an operator modelling the positive or negative externality effects. As a particular example, we can consider ntegral operators, possibly composed with non-linearities:

$$(\mathcal{S}k)(t, z) = \psi\left(\int_D w(z, y)k(t, y)dy\right) \tag{1}$$

where $w : D \times D \to \mathbb{R}$ is a suitable kernel function and $\psi : \mathbb{R} \to I \subset \mathbb{R}_+$ is a given, smooth, nonlinearity, or even

$$(\mathcal{S}k)(t, z) = \psi\left(\int_D w(z, y)\psi_0(k(t, y))dy\right),$$

for a suitable function $\psi_0 : \mathbb{R}_+ \to \mathbb{R}$. We prefer to include nonlinearities in the model for the externalities because, for simplicity of exposition, we want a model where K is positive (in many cases, bounded from below by a strictly positive constant), but we also want to consider possible negative externalities.

More precisely we will consider an operator $\mathcal{S}$ satisfying the following condition

Assumption 2.1. Given an interval $\bar{I} \subset \mathbb{R}_+$, the operator $\mathcal{S} : L^\infty(D) \to C(D; \bar{I})$ is a (possibly nonlinear) Lipschitz map for the L^∞ topology and it maps bounded sets of $L^\infty(D)$ into compact subsets of $C(D; \bar{I})$, the space of continuous functions over D, taking values in the interval $\bar{I}$, endowed with the topology of uniform convergence.

The choice of the interval $\bar{I}$, which could be unbounded (but typically is bounded away from 0), will be made so that the net output function f and the externalities K satisfy some compatibility conditions which will be precised later on.

We will then use this operator for each fixed time, taking $K(t, \cdot) = \mathcal{S}(k(t, \cdot))$.

Consider now a local planner at location $z \in D$. The planner can only formulate expectations concerning the evolution of the externalities over the time horizon $(0, \infty)$, and his/her actions directly influence only the local variables taking the externality as parametric. Given his/her expectation of the exogenous temporal evolution of the effect of the externalities $K(\cdot, z)$ at location z, he/she maximises local intertemporal utility of consumption at location $z \in D$, i.e. solves the (parametric) maximisation problem

$$\begin{aligned} &\max \int_0^\infty e^{-rt}U(c(t, z))dt, \text{ subject to} \\ &k'(t, z) = f(k(t, z), K(t, z)) - c(t, z) \\ &k(t, z) \geq 0, \;\; c(t, z) \geq 0, \end{aligned} \tag{2}$$

where U is an appropriate utility function and $r > 0$ is a utility discount rate. The solution of problem (2) yields an optimal consumption rule $c^*(\cdot, z) = c^*(\cdot, z; K)$ and an optimal path $k^*(\cdot, z) = k^*(\cdot, z; K)$, where we include K in the notation to emphasise that this optimal path depends on the exogenous externality state K faced by the "representative" planner at location z. Each planner at location $z \in D$ solves a version of problem (2) (for the proper choice of z, thus obtaining a function

$k^*(\cdot,\cdot;K):\mathbb{R}_+\times D\times\mathbb{R}_+$, and a function $c^*(\cdot,\cdot;K):\mathbb{R}_+\times D\to\mathbb{R}_+$, such that $k^*(\cdot,z;K):\mathbb{R}_+\to\mathbb{R}_+$ and $c^*(\cdot,z;K):\mathbb{R}_+\to\mathbb{R}_+$ are the optimal path and the optimal and the optimal consumption of the "representative" planner at location $z\in D$, given the instantaneous state of the externalities $K:\mathbb{R}_+\times D\to\mathbb{R}$. Recall our assumption that externalities are determined endogenously in terms of the operator $\mathcal{S}$. This implies that if all individual agents, at any $z\in D$ are well informed concerning the state of the externalities K, and using this knowledge solve their individual problems (2) determining the optimal function $k^*)\cdot,\cdot,K)$, then consistency of the model imposes that $K=\mathcal{S}k^*$.

The above discussion leads to the definition of the operator $\mathcal{T}$ by $K\mapsto k^*(\cdot,\cdot,K)\mapsto\mathcal{S}k^*(\cdot,\cdot,K)$, in terms of which we may now define a rational expectations equilibrium for the spatial economy.

Definition 2.2. A fixed point of the operator $\mathcal{T}$, defined by $K\mapsto k^*(\cdot,\cdot,K)\mapsto\mathcal{S}k^*(\cdot,\cdot,K)$ is a rational expectations equilibrium for the spatial economy.

Remark 2.3. Our definition of a rational expectations equilibrium for the spatial economy is motivated by the interpretation of the above scheme as a best-response scheme. In particular given that the agent at location $z\in D$ anticipates that the externality effects at such a location are going to develop in the future as $K(\cdot,z)$ he/she designs the optimal path for the economy as the best response to this anticipated externality effects, which is the solution $k^*(\cdot,z;K(\cdot,z))$ to problem (2). This is true for any agent located at any other site $z'\in D$. Their joint best responses to the level of externalities they anticipate will clearly contribute to the formulation of the actual state of the externalities which will become $\mathcal{S}k^*(\cdot,\cdot;K(\cdot,\cdot))$. This actual state can be interpreted as an adjustment of the agents anticipation of the level of the externalities, hence the level of externalities where this adjustment procedure reac ertains the agents anticipation is called a spatial rational expectations equilibrium.

To this point we deliberately refrain from setting a detailed functional space setting for the problem, and we prefer to introduce this later on (see Sections 3 and 4) after the necessary technical estimates that clarify our choice are presented. Here, we only impose assumptions on the primitives of the problem and in particular to the production function and the utility function.

Assumption 2.4 (Assumptions on f)**.** The net output function f satisfies:

(i) f is continuous on $\mathbb{R}_+\times\mathbb{R}_+$ and $f(0,K)=0$ for every $K\in\bar{I}$ (where $\bar{I}$ is the interval in 2.1).

(ii) f is C^1 on $(0,+\infty)\times(0,+\infty)$ and by f_k, f_K we denote the two partial derivatives.

(iii) there exists a constant $\delta>0$ such that $f_k\geq-\delta$, and moreover f_k is bounded from above on any set of the form $[a,\infty)\times[0,c]$, for $a>0$ and $c>0$.

(iv) for every M there is $\bar{k}(M)$ such that $f(k,K)\leq 0$ for every $K\leq M$ and $k\geq\bar{k}(M)$; moreover, $\bar{k}(M)=o(M)$ as $M\to\infty$.

(v) there exists $k_1>0$ such that $f(k,K)>0$ for arbitrary $K\in\bar{I}$ and $k<k_1$.

(vi) $k\mapsto f(k,K)$ is strictly concave for every $K\in\bar{I}$.

These assumptions can be easily seen to be satisfied by the Cobb-Douglas net output function $f(k,K)=Ak^\alpha K^\beta-\delta k$ for $0<\alpha,\beta<1$ and $\alpha+\beta<1$ with the only important observation that point *(v)* requires $\inf\bar{I}>0$.

Assumption 2.5 (Assumptions on U)**.** The utility function $U:\mathbb{R}_+\to\mathbb{R}_+$ is a concave, strictly increasing function, C^1 on $\mathbb{R}_+\setminus\{0\}$, and such that $U'(0)=\infty$ and $U'(\infty)=0$.

Remark 2.6. Explicit dependence in z can be included both in the net output function f and/or in the the utility function U, provided sufficient regularity conditions are imposed on such a dependence, but we will not go into this generalisation.

3 The individual optimisation problem (2)

In this section we consider $K : \mathbb{R}_+ \to (0, \infty)$ (here the variable z will be considered as fixed and will not play any role, hence explicit dependence on z will be omitted) as a given bounded function and consider the parametric optimisation problem

$$\begin{aligned} &\max_c \int_0^\infty e^{-rt} U(c(t))dt, \text{ subject to} \\ &k'(t) = f(k(t), K(t)) - c(t) \\ &k(t) \geq 0, \;\; c(t) \geq 0, \quad a.e. \end{aligned} \tag{3}$$

From the economic point of view, c and k should be time-dependent functions, smooth enough (if possible, $k \in C^1$ and $c \in C^0$), but, as usual in mathematics, handling this strong regularity will not be feasible at first, and we will need to extend our problem to a weaker setting, proving later the regularity of the maximisers.

In particular, the choice of the exact functional space setting for the pairs (k, c) is motivated by some uniform bounds that we will obtain in a while.

We point out that, thanks to the assumptions on f, whenever K is bounded, then $t \mapsto f(k(t), K(t))$ is also bounded from above, by a constant that depends on K and will be denoted by $M(K) =: C_1$.

We need to introduce the following functional spaces. By $L^p(I)$, $p \in [1, \infty]$ we denote the standard Lebesgue spaces on $I \subset \mathbb{R}_+$ whereas $\mathcal{M}_{loc}(\mathbb{R}_+)$ is the space of real valued Radon measures on $\mathbb{R}_+$, $\mathcal{M}(I)$ is the space of real valued finite Radon measures on a compact interval $I \subset \mathbb{R}_+$, the latter equipped by the norm

$$\|\mu\|_{\mathcal{M}(I)} = |\mu|(I) = \sup \left\{ \int_I \varphi d\mu, \; ; \; \varphi \in C(\mathbb{R}_+), \; \|\varphi\|_\infty \leq 1 \right\}.$$

We also introduce the space $BV(I) = \{u : \mathbb{R}_+ \to \mathbb{R} \; : \; u \in L^1(I), \; Du \in \mathcal{M}(I)\}$, for every $I \subset \mathbb{R}_+$. It is a Banach space when equipped with the norm $\|u\|_{BV(I)} := \|u\|_{L^1(I)} + \|Du\|_{\mathcal{M}(I)}$ and may be considered as an extension of $W^{1,1}(I)$, in the sense that the distributional derivative Du is no longer an integrable function but rather a Radon measure, whereas $BV_{loc}(\mathbb{R}_+)$ is the space of functions that belong to $BV(I)$ for every $I \subset \mathbb{R}_+$, compact.

We will now give a precise functional setting for Problem (3). The choice is motivated by the uniform bounds that we obtain below, in Lemma 3.1. Let us define, for fixed $k_0 \in \mathbb{R}_+$,

$$\mathcal{A}(K) := \left\{ (k, c) \; : \; k \in BV_{loc}(\mathbb{R}_+), \, c \in \mathcal{M}_{loc}(\mathbb{R}_+), \; \begin{array}{l} k \geq 0, \, k(0) = k_0, \, c \geq 0, \\ k' + c \leq f(k, K) \end{array} \right\},$$

where the last inequality is to be intended as an inequality between measures (the right-hand side being the measure on $\mathbb{R}_+$ with density $f(k(\cdot), K(\cdot))$). On the set of pairs (k, c) (and in particular on $\mathcal{A}(K)$), we consider the following notion of (weak) convergence: we say $(k_n, c_n) \rightharpoonup (k, c)$ if c_n and k_n' weakly-* converge to c and k', respectively, as measures on every compact interval $I \subset \mathbb{R}_+$. It is important to observe that, thanks to the fact that the initial value $k(0)$ is prescribed, the weak convergence as measures $k_n' \rightharpoonup k'$ also implies $k_n \to k$ a.e., as we have

$$k_n(t) = k_0 + \int_0^t k_n'(s)ds \to k_0 + \int_0^t k'(s)ds = k(t)$$

for every t which is not an atom for k' (i.e. every continuity point t for k). This condition is satisfied by all but a countable set of points t.

Lemma 3.1 (A priori bounds for $\mathcal{A}(K)$)**.** *Consider any pair $(k,c) \in \mathcal{A}(K)$. Then, $k \in L^\infty(\mathbb{R}_+)$, and for any compact subset $I \subset \mathbb{R}_+$, we have $\|c\|_{\mathcal{M}(I)} \leq C$ and $\|k'\|_{\mathcal{M}(I)} \leq C$, with the constant C depending on the choice of I but not on (k,c). In particular if $I = (0,t)$, we can choose $C = C(t) = \bar{C}_0 + \bar{C}_1 t$ for suitable $\bar{C}_0, \bar{C}_1$.*

Proof. Consider first the constraint

$$k' + c \leq f(k,K), \tag{4}$$

where the inequality is an inequality between measures. Since c is nonnegative and f is negative for large k (say for $k \geq \bar{k}$), then k is bounded from above by $\max\{k(0), \bar{k}\}$. Hence, $k \in L^\infty(\mathbb{R}_+)$.

We return to (4), and note that since the right hand side is positive and bounded above by C_1, we have that $k'(t) \leq C_1$, for any t. This provides upper L^∞ bounds for k', but we also need lower bounds if we want to bound k' in $\in \mathcal{M}_{loc}(\mathbb{R}_+)$. We break up k' into its positive and negative part as $k' = (k')^+ - (k')^-$. The positive part is bounded, by (4), and the upper bound is C_1. We proceed to obtain a bound for $(k')^-$. Clearly,

$$k(t) - k(0) = \int_0^t k' = \int_0^t (k')^+ - \int_0^t (k')^-,$$

which, using the fact that $k(t) \geq 0$ and the upper bound $(k')^+ \leq C_1$, leads to the estimate

$$-k(0) \leq k(t) - k(0) = \int_0^t (k')^+ - \int_0^t (k')^- \leq C_1\, t - \int_0^t (k')^-,$$

which upon rearrangement provides us with the estimate

$$\int_0^t (k')^- \leq k_0 + C_1\, t. \tag{5}$$

This allows us to estimate the total mass of $|k'| = (k')^+ + (k')^-$ as

$$\int_0^t |k'| = \int_0 (k')^+ + \int_0 (k')^- \leq C_1\, t + k_0 + C_1\, t = k_0 + 2C_1\, t, \tag{6}$$

where we used (5). This estimate implies that $\|k'\|_{\mathcal{M}(I)} < C$ for any compact $I = [0,t]$, where C depends on t.

Having obtained the desired bound for k' we return once more to (4) which yields, $0 \leq c \leq C_1 - k'$, from which the desired bound on c follows. □

We insist that the above lemma is the one which justifies the choice of the space BV as a functional setting. Indeed, Lemma 3.1 provides a suitable weak compactness in this space, while the same could not be obtained in stronger spaces such as $W^{1,p}$ (no L^p bound on the derivative k' or on c is possible to obtain just from the fact that (k,c) belongs to the admissible set $\mathcal{A}(K)$, not even for maximising sequences.

We now define the quantity which is optimised, in this functional setting, by every agent. Define

$$J(c) := \int_0^\infty e^{-rt} U(c^{ac}(t))dt,$$

where c^{ac} is the absolutely continuous part of the measure c (remember that every positive and locally finite measure c on $\mathbb{R}$ can be decomposed as a sum $c^{ac}(t)dt + c^{sing}$, where the first part has density

c^{ac} w.r.t. the Lebesgue measure on $\mathbb{R}$, and the second part is singular: it could include atoms or other measures concentrated on sets with zero Lebesgue measure).

Note that, from an economic point of view, the use of $c \in \mathcal{M}_{loc}(\mathbb{R}_+)$, and in particular the fact that J takes into account only the absolutely continuous part of c in the computation of the utility, implies that net capital formation could be reduced due to the singular part of the consumption, but the representative agent will not receive and additional utility from this singular part. We adopt this seemingly unrealistic – in terms of economics – definition of J at this stage of the problem because we want, for compactness reasons, to extend the problem to the space of measures. In this case standard lower semicontinuity results require that J is defined in this way. Since, as we show in Lemma 3.4, the optimal consumption c does not contain a singular part, our results are, in the end, meaningful in terms of economics.

For notational purposes, we also introduce, for $T \in \mathbb{R}_+ \cup \{+\infty\}$,the quantity $J_T : \mathcal{M}_{loc}(\mathbb{R}_+) \to [0, +\infty]$:

$$J_T(c) := \int_0^T e^{-rt} U(c^{ac}(t))dt.$$

Lemma 3.2 (Approximation and upper semicontinuity of J_∞)**.** *The following hold:*

(i) For any $\epsilon > 0$, there exists $T = T(\epsilon) < \infty$ such that $J_\infty(c) \leq J_T(c) + \epsilon$ for any $(k, c) \in \mathcal{A}(K)$ (in particular, $T(\epsilon)$ does not depend on c).

(ii) J_∞ is upper semicontinuous on $\mathcal{A}(K)$.

Proof. (i) We write

$$J_\infty(c) = J_T(c) + \int_T^\infty e^{-rt} U(c^{ac}(t))dt,$$

and we use the fact that U, as it is concave, is bounded above by an affine function $U(c) \leq A + B\,c$. The integral $\int_T^\infty e^{-rt} A dt$ can be made small as soon as T is chosen large enough. For the integral $\int_T^\infty e^{-rt} B c^{ac}(t)dt$ we use the bound in Lemma 3.1, applied to the intervals $[T+k, T+k+1]$. We have $\int_{T+k}^{T+k+1} c^{ac}(t)dt \leq \bar{C}_0 + \bar{C}_1(T+k+1)$ and hence

$$\int_{T+k}^{T+k+1} e^{-rt} B c^{ac}(t)dt \leq B(\bar{C}_0 + \bar{C}_1(T+k+1))e^{-k}e^{-T}.$$

The series of these quantities as k ranges from 0 to ∞ converges for every T, and can be also made small just by choosing T large enough.

(ii) Consider a sequence $\{c_n\} \in \mathcal{A}(K)$ such that $c_n \overset{*}{\rightharpoonup} c$ in $\mathcal{M}_{loc}(\mathbb{R}_+)$. By (i) for any $\epsilon > 0$, there exists $T = T(\epsilon)$ such that

$$J_\infty(c_n) \leq J_T(c_n) + \epsilon, \ \ \forall\, n \in \mathbb{N}. \tag{7}$$

Furthermore, using $U \geq 0$,

$$J_T(c) \leq J_\infty(c), \ \ \forall\, c \in \mathcal{M}_{loc}(\mathbb{R}_+). \tag{8}$$

The functional J_T is upper semicontinuous with respect to weak$-*$ convergence in $\mathcal{M}_{loc}(\mathbb{R}_+)$. Indeed, since U is concave, $-U$ is a convex function, and we can apply standard result on the semicontinuity of

$$c \mapsto \int h(c^{ac}(x))dx + \int h^\infty(c^{sing}),$$

where here $h = -U$ and $h^\infty = 0$ (see, for instance [2]).

Then, taking the limsup on both sides of (7) yields

$$\limsup_n J_\infty(c_n) \le \limsup_n J_T(c_n) + \epsilon \le J_T(c) + \epsilon \le J_\infty(c) + \epsilon$$

where we used first the upper semicontinuity of J_T and then (8). Taking the limit as $\epsilon \to 0^+$ we obtain the upper semicontinuity of J_∞. □

Proposition 3.3. *Given any continuous and bounded function $K : \mathbb{R}_+ \to \mathbb{R}_+$, the parametric maximisation problem for the individuals* (3) *admits a maximiser $(k^*, c^*) \in (BV_{loc}(\mathbb{R}_+) \cap L^\infty(\mathbb{R}_+)) \times \mathcal{M}_{loc}(\mathbb{R}_+)$.*

Proof. Consider (k_n, c_n) a maximizing sequence for (3). Since $(k_n, c_n) \in \mathcal{A}(K)$ for any interval $I = [0, T]$, we conclude by Lemma 3.1 that k_n is bounded in $BV_{loc}(\mathbb{R}_+)$, and c_n in $\mathcal{M}_{loc}(\mathbb{R}_+)$. Using the weak$-*$ compactness properties of $\mathcal{M}_{loc}(\mathbb{R}_+)$, this guarantees the existence of $k \in BV_{loc}(\mathbb{R}_+)$ and $\zeta, c \in \mathcal{M}_{loc}(\mathbb{R}_+)$ such that up to subsequences

$$\begin{aligned} k_n &\to k && a.e. \\ k_n' &\overset{*}{\rightharpoonup} \zeta && \text{in } \mathcal{M}_{loc}(\mathbb{R}_+), \\ c_n &\overset{*}{\rightharpoonup} c && \text{in } \mathcal{M}_{loc}(\mathbb{R}_+). \end{aligned}$$

By standard arguments for distributional derivatives we can see that $\zeta = k'$.

Our first task is to show admissibility of the limit (k, c). It is straightforward to check that $c \ge 0$ and $k \ge 0$. It remains to check that $k' + c \le f(k, K)$. In order to see this, note first that since $(k_n, c_n) \in \mathcal{A}(K)$ for every $n \in \mathbb{N}$, we have that

$$k_n' + c_n \le f(k_n, K), \ \ \forall\, n \in \mathbb{N}. \tag{9}$$

We need to pass to the limit as $n \to \infty$ (along the converging subsequence) in the above. The left hand side, is linear and $k_n' + c_n \overset{*}{\rightharpoonup} k' + c$. Some care must be taken for the nonlinear right hand side. Since f is continuous and bounded from above, it is sufficient to use the pointwise convergence $k_n(t) \to k(t)$ a.e. in $t \in [0, T]$, guaranteed by the local BV bound on k_n.

To show that (k, c) is a maximiser to Problem (3), we use the upper semicontinuity result, $\limsup_n J_\infty(c_n) \le J_\infty(c)$ contained in Lemma 3.2. □

The above result guarantees the existence of a maximiser (k, c) where both k' and c are measures. Before going on with the analysis, we will show that, actually, they have not singular parts, i.e. they are functions. In terms of the state function k, this means $k \in W^{1,1}_{loc}$. This result is crucial both for the mathematical analysis of the optimisation problem and for its economic meaning.

Lemma 3.4. *Any optimiser (k, c) satifies $c \in L^1_{loc}(\mathbb{R}_+)$.*

Proof. Let us recall that, in general, any c such that $(k, c) \in \mathcal{A}(K)$ consists of an absolutely continuous and a singular part with respect to Lebesgue measure $c = c^{ac} + c^{sing}$.

For any Borel function h we will use the notation $\fint_a^b h(s)ds = \frac{1}{b-a}\int_a^b h(s)ds$ (if $a \ne b$, otherwise this is equal to $h(a)$).

Suppose (k, c) is optimal. Given an interval $I = [a, b] \subset \mathbb{R}_+$, we build $(\tilde{k}, \tilde{c})$ a new competitor in the following way: outside I, we take $\tilde{k} = k$ and $\tilde{c} = c$; $\tilde{k}$ is affine on $[a, b]$, connecting the values $k(a^-)$ to $k(b^+)$; $\tilde{c}$ restricted to $[a, b]$ is an absolutely continuous measure, with constant density. We have to choose the constant value for $\tilde{c}$ for the inequality $\tilde{k}' + \tilde{c} \le f(\tilde{k}(t), t)$ to be satisfied. Notice that on $[a, b]$ we have $\tilde{k}' = k'([a, b])/(b-a)$. If we set $\hat{c} = c([a, b])/(b-a)$ for sure we have $\tilde{k}' + \hat{c} \le \fint_a^b f(k(s), K(s))ds$.

Since we can assume f to be bounded (k only takes values in a bounded set, and $\tilde{k}$ as well, hence), we have $\fint_a^b f(k(s), K(s))ds \leq f(\tilde{k}(t), t) + M$. Hence, it is sufficient to set $\tilde{c} = \hat{c} - M$, provided $\hat{c} \geq M$.

Now, suppose c is not absolutely continuous. Then for sure there exists $I = [a, b]$ such that $c^{sing}([a, b]) > M(b-a)+c^{ac}([a, b])$ (otherwise $c^{sing} \leq M+c^{ac}$ and c^{sing} would be absolutely continuous). The interval $[a, b]$ may be taken as small as we want, which means that we can assume $(b - a)e^{-ra} \leq 2m_{a,b}$, where we set $m_{a,b} := \int_a^b e^{-rs}ds$. We choose such an interval and we build $\tilde{k}$ and $\tilde{c}$ as above, and we claim we find a contradiction.

Let us set $X = c^{ac}([b - a])/(b - a)$ and $Y = c^s([b - a])/(b - a)$. The optimality of (k, c) gives

$$\int_a^b e^{-rt}U(c^{ac}(t))dt \geq \int_a^b e^{-rt}U(\tilde{c}(t))dt.$$

Using Jensen's inequality, we get

$$\int_a^b e^{-rt}U(c^{ac}(t)) \leq m_{a,b}U\left(\frac{1}{m_{a,b}}\int_a^b c^{ac}(t)e^{-rt}dt\right) \leq m_{a,b}U\left(\frac{e^{-ra}}{m_{a,b}}\int_a^b c^{ac}(t)dt\right) \leq m_{a,b}U(2X)$$

and

$$\int_a^b e^{-rt}U(\tilde{c}(t))dt = m_{a,b}U(X + Y - M).$$

We deduce

$$U(X + Y - M) \leq U(2X),$$

but the interval $[a, b]$ was chosen so that $Y > X + M$, which gives a contradiction because of the strict monotonicity of U. □

We can use the above result to prove some useful properties of our maximisation problem.

Corollary 3.5. *For any optimiser (k, c), we have $k \in W^{1,1}_{loc}(\mathbb{R}_+)$ and saturation of the constraint $k' + c = f(k, K)$. Moreover, the optimal pair (k, c) is unique.*

Proof. First, note that, if $k \notin W^{1,1}_{loc}(\mathbb{R}_+)$, then this means that k' has a negative singular part $((k')^{sing})_-$. Yet the inequality $k' + c \leq f(k, K)$ stays true if we replace c with $c + ((k')^{sing})_-$, which would be another optimiser. But this contradicts Lemma 3.4, since every optimiser c should be absolutely continuous.

As soon as we know that both k' and c are absolutely continuous, it is clear that we must have $k' + c = f(k, K)$. Otherwise, we can replace c with $f(k, K) - k'$ and get a better result (we needed to prove the absolute continuity of k', since adding singular part to c does not improve the functional).

In what concerns uniqueness, we first stress that the functional we maximise is not strictly concave, for two reasons: on the one hand we did not assume U to be strictly concave, and on the other hand, anyway, there would be an issue as far as the singular part of c is involved. Yet, now that we know that optimisers are absolutely continuous and saturate the differential inequality constraint, uniqueness follows from the strict concavity of f.

Indeed, suppose that that (k_1, c_1) and (k_2, c_2) are two minimisers. Then, setting $c_3 = (c_1 + c_2)/2$ and $k_3 = (k_1+k_2)/2$, the pair (k_3, c_3) is also admissible (since f is concave in k) and optimal (since the cost is concave in c). Yet, strict concavity of f implies that (k_3, c_3) does not saturate the constraint, which is a contradiction with the first part of the statement, unless $k_1 = k_2$. As a consequence, we obtain uniqueness of k in the optimal pairs (k, c). But this, together with the relation $k'+c = f(k, K)$, which is now saturated, also provides uniqueness of c. □

We now establish uniform bounds on the optimal c, which will be useful for approximation issues in the next section. This is provided in the following technical proposition, whose proof is broken up into 4 lemmas, proven in the Appendix.

Proposition 3.6. *If $k_0 > 0$, then the optimal consumption c is bounded from above and below on every bounded interval, i.e., for every $T > 0$ there exist two constants $0 < c_-(T) < c^+(T) < \infty$, depending only on f, U, k_0 and T, such that $c(t) \in [c_-(T), c_+(T)]$ for every $t \in [0, T]$.*

Proof. The proof is complicated (a) by the non Lipschitz property of f at 0 (b) by the non-smoothness of U at 0 and (c) by the state constraint $k \geq 0$. Concerning the state constraint (c) one may resort to versions of the Pontryagin maximum principle (PMP) which take such constraints into account (see for instance [9]), however, as even this approach will still require smoothness of f and U here we choose a different strategy. In the appendix (see Lemmas A.1, A.2, A.3, A.4) we prove the claim in the case where f is Lipschitz, by approximating the utility function $U(c)$, which is itself non-smooth, with $U(c+h)$ for $h > 0$. One of the first points is to prove that the state constraint is not binding, so that the use of the standard PMP can be justified.

In order to prove the claim for non-Lipschitz functions f, it is enough to apply the results of the Appendix to an approximation f_n, under the condition that f_n is smooth, converges uniformly to f, and satisfies $f_n \geq f$. The passage to the limit as $n \to \infty$ can be done using the following Lemma 3.7, which also includes the possible approximation in U. The existence of a suitable approximation is proven in Lemma 3.8. □

Lemma 3.7. *Given a sequence of functions f_n, and two sequences $h_n, \varepsilon_n \to 0^+$, suppose $f_n \to f$ in the sense of uniform convergence as $n \to \infty$, $f_n(k + \varepsilon_n, K) \geq f(k, K)$ for all (k, K), and set $U_n(c) = U(c + h_n)$. Then we have*

$$\max\{J_\infty(c) \,:\, (k, c) \in \mathcal{A}(K), k(0) = k_0\} = \lim_n \max\{J^n_\infty(c) \,:\, (k, c) \in \mathcal{A}_n(K), k(0) = k_0 + \varepsilon_n\}$$

where the functional J^n is defined by replacing U with U_n and the admissibility set $\mathcal{A}_n$ is defined replacing f with f_n, and modifying the starting point. Moreover, we have $(k_n, c_n) \rightharpoonup (k, c)$ (in the sense of the convergence that we defined for pairs (k, c) in Section 2), where (k_n, c_n) and (k, c) are the optimisers in the above optimisation problems,.

Proof. Let us consider the minimisers (k_n, c_n): they satisfy the same uniform bounds in the space of measures that we saw in Lemma 3.1, and hence we can assume, up to subsequences, that we have $(k_n, c_n) \rightharpoonup (\tilde{k}, \tilde{c})$, but we do not know yet that the limit is the maximiser for the limit problem. However, we can pass to the limit the inequality $k^{(h)\prime} + c^{(h)} \leq f_n(k^{(h)}, K)$ because $(k_n, c_n) \rightharpoonup (\tilde{k}, \tilde{c})$ implies a.e. pointwise convergence for k_n. This, together with $k_n(0) = k_0 + \varepsilon_n \to k_0$, provides the admissibility of $(\tilde{k}, \tilde{c})$,. We can also apply the semicontinuity result of Lemma 3.2 to $c_n + h_n \rightharpoonup \tilde{c}$, thus getting

$$J_\infty(\tilde{c}) \geq \limsup_n J^{(n)}_\infty(c_n).$$

This proves that $\max J_\infty \geq J_\infty(\tilde{c}) \geq \limsup_n \max J^{(n)}_\infty$, and $(\tilde{k}, \tilde{c})$ is optimal if we can prove

$$\max J_\infty \leq \liminf_n \max J^{(n)}_\infty.$$

Let (k, c) be the (unique) maximiser for J_∞: if we define $k_n := k + \varepsilon_n$ then (k_n, c) is admissible in the problem with $J^{(n)}_\infty$, thanks to the assumption $f_n(k + \varepsilon_n, K) \geq f(k, K)$ and the shift in the initial datum $k_0 + \varepsilon_n$. Thus we get

$$\liminf_n \max J^{(n)}_\infty \geq \liminf_n J^{(n)}_\infty(c) = \liminf_n \int U(c + h_n)e^{-rt}dt \geq \int U(c)e^{-rt}dt = J_\infty(c) = \max J_\infty$$

(in the second inequality we used the monotonicity U). This proves that $\tilde{c}$ is optimal, hence $\tilde{c} = c$ by uniqueness and the whole sequence converges. □

Lemma 3.8. *Given a function f satisfying Assumptions 2.4 and a sequence of positive numbers $\varepsilon_n \to 0$, there exists a sequence f_n of smooth Lipschitz continuous functions also satisfying 2.4, converging uniformly to f, and such that $f_n(k+\varepsilon_n, K) \geq f(k, K)$ for all (k, K).*

Proof. Given ε_n, consider the function f extended to $k \in [-\varepsilon_n, 0]$ by the value 0, and define $\tilde{f}_n$ as the concave envelop in k of this extension. This function satisfies $f(-\varepsilon_n, K) = 0$, is Lipschitz continuous, and is bounded from below by f. We can then define $f_n(k, K) := \tilde{f}_n(k - \varepsilon_n, K)$, which satisfies all the required properties. □

Some corollaries of Proposition 3.6 are the following:

Corollary 3.9. *The optimal k is locally Lipschitz on each interval $[0, T]$, and its Lipschitz constant only depends on U, f, k_0 and T.*

Proof. The saturation of the constraint gives $k' = f(k, K) - c$. From the boundedness of k and K and the local upper bound on c obtained from Proposition 3.6, we obtain a local L^∞ bound on k', which proves the claim. □

Corollary 3.10. *The optimal k satisfies $k(t) > 0$ for every $t \in \mathbb{R}_+$. Moreover, the optimal (k, c) satisfies the differential equations*

$$\begin{cases} k' = f(k, K) - c \\ (U'(c))' = (r - f_k(k, K))U'(c) \end{cases}$$

Proof. The condition $k > 0$ is a consequence of the lower bound on c. Suppose that there exists an instant t_0 such that $k(t_0) = 0$. Take the lower bound $c_-(2t_0)$ for c on the interval $[0, 2t_0]$. Since $f(0, K) = 0$ and f and k are continuous, on a small neighbourhood $(t_0 - \epsilon, t_0 + \epsilon)$ of t_0 we have $f(k(t), K(t)) \leq c_-(2t_0)/2$. Then, on this interval we have $k'(t) \leq -c_-(2t_0)/2 < 0$. But this is impossible since $k \geq 0$ and $k(t_0) = 0$.

Once we know that neither c nor k vanish, then the constraints are not binding and standard Pontryagin principle (or standard Euler-Lagrange equations in calculus of variations), considering that now f and U are smooth, provide the desired optimality conditions in the form of the desired system of ODEs. Notice that, without the assumption $U'' \neq 0$, we cannot obtain regularity for c and write a rigorous ODE on it, which is why we only wrote an equation on $U'(c)$. □

4 Existence of rational expectations equilibria

We now turn our attention to the rational expectations equilibria. To this end, we recall that we defined the operator $\mathcal{T}$ as follows: Consider any externalities configuration K, solve the parametric individual optimisation problem (3) for any $z \in D$ to obtain the family of maximisers $\{k^*(\cdot, z; K)\}_{z \in D}$ and then act the operator $\mathcal{S}$ on the maximiser to obtain $\mathcal{S}k^*$, i.e.,

$$K \mapsto k^* \mapsto \mathcal{S}k^* =: K^\#, \quad \mathcal{T}K := K^\#.$$

A fixed point for the operator $\mathcal{T}$ is identified as a rational expectations equilibrium.

Our proof of existence is based upon the Schauder fixed point theorem and requires the continuity of the operator $\mathcal{T}$. Note that by defining in the above scheme first the operator $\mathcal{O}$ by $\mathcal{O}K := k^*$, we may decompose the operator $\mathcal{T}$ as $\mathcal{T} = \mathcal{S}\,\mathcal{O}$. Since $\mathcal{S}$ is a compact operator, we focus our attention on the properties of $\mathcal{O}$. As the operator $\mathcal{O}$ is defined by mapping the parameter K of the parametric optimisation problem (3) to the optimiser, in order to obtain the continuity of $\mathcal{O}$ we need to ensure the continuity of maximisers with respect to the parameter. This is essentially a Γ-convergence type

result (see e.g. [2] or [8]), but we will avoid using this theory, as we will directly consider optimisers. The argument is very similar to what we developed in Lemma 3.7.

Consider a sequence $\{K_n\} \subset L^\infty(\mathbb{R}_+)$ (uniformly bounded in $n \in \mathbb{N}$) such that $K_n \to K$, locally uniformly, and the corresponding sequence of constraint sets

$$\mathcal{A}(K_n) = \{(k, c) \ : \ k' + c \leq f(k, K_n)\},$$

and let (k_n^*, c_n^*) be the maximisers of the sequence of optimisation problems

$$\max_{(k,c)\in\mathcal{A}(K_n)} J_\infty(c).$$

We need to show that we have $(k_n^*, c_n^*) \to (k^*, c^*)$, where (k^*, c^*) is the maximiser of the optimisation problem

$$\max_{(k,c)\in\mathcal{A}(K)} J_\infty(c).$$

Note that the above maximiser is unique thanks to Corollary 3.5.

Proposition 4.1. *Suppose that $\{K_n\} \subset L^\infty(\mathbb{R}_+)$ is a uniformly bounded sequence such that $K_n \to K$ locally uniformly, and let (k_n^*, c_n^*) be the corresponding maximisers. Then we have $(k_n^*, c_n^*) \rightharpoonup (k^*, c^*)$, where (k^*, c^*) is the maximiser corresponding to K, and the convergence $k_n^* \to k^*$ is locally uniform.*

Proof. We follow a similar scheme as in Lemma 3.7.

The functions (k_n^*, c_n^*) satisfy the same uniform bounds in the space of measures that we saw in Lemma 3.1, and hence we can assume, up to subsequences, that they satisfy $(k_n^*, c_n^*) \rightharpoonup (\tilde{k}, \tilde{c})$ (in the sense of the convergence defined on pairs (k, c) in Section 2), but we do not know yet that the limit is the maximiser for the limit problem. Yet, we can pass to the limit the inequality $(k_n^*)' + c_n^* \leq f(k_n^*, K_n)$ and get the admissibility of $(\tilde{k}, \tilde{c})$. We can also apply the semicontinuity result of Lemma 3.2 to, thus getting

$$J_\infty(\tilde{c}) \geq \limsup_n J_\infty(c_n^*).$$

This proves that

$$\max\{J_\infty(c) \ : \ (k, c) \in \mathcal{A}(K)\} \geq J_\infty(\tilde{c}) \geq \limsup_n J_\infty(c_n^*) = \limsup_n \max\{J_\infty(c) \ : \ (k, c) \in \mathcal{A}(K_n)\}.$$

Then, $(\tilde{k}, \tilde{c})$ is optimal if we can prove $\max\{J_\infty(c) \ : \ (k, c) \in \mathcal{A}(K)\} \leq \limsup_n \max\{J_\infty(c) \ : \ (k, c) \in \mathcal{A}(K_n)\}$.

In order to do so, let us take the optimal pair (k^*, c^*), fix $\epsilon > 0$ and find T such that $J_\infty(c^*) \leq J_T(c^*) + \epsilon$. Then, let us define a sequence $(k_n, c_n) \in \mathcal{A}(K_n)$ as follows. Set $\epsilon_n := ||f(k^*, K) - f(k^*, K_n)||_{L^\infty([0,T])} \to 0$ and use

$$k_n(t) = k^*(t), \ c_n(t) := c^*(t) - \epsilon_n \quad \text{for } t \in [0, T];$$

for $t > T$, just use any admissible pair (k, c) satisfying $k(T) = k^*(T)$ and all the constraints. It is important to use Proposition 3.6 to guarantee that, at least for large n, the consumption c_n defined above is admissible (i.e. nonnegative).

Then, by monotone convergence we have

$$J_T(c^*) = \lim_n J_T(c_n) \leq \limsup_n J_\infty(c_n) \leq \limsup_n \max\{J_\infty(c) \ : \ (k, c) \in \mathcal{A}(K_n)\},$$

(where we use positivity of U, to pass from J_T to J_∞).

We deduce

$$J_\infty(c^*) \leq \limsup_n \max\{J_\infty(c) \,:\, (k,c) \in \mathcal{A}(K_n)\} + \epsilon$$

and the result is proven since ϵ is arbitrary.

The optimality of $(\tilde{k}, \tilde{c})$ together with the uniqueness of the maximer implies $(\tilde{k}, \tilde{c}) = (k^*, c^*)$ and the convergence of the full sequence without the need to extract a subsequence is standard.

We are only left to prove that the convergence $k_n^* \to k^*$ is locally uniform. To do this, it is enough to apply the uniform local Lipschitz bounds guaranteed by Corollary 3.9. □

We now define a space Y of continuous and locally bounded functions of the variables $(t, z) \in \mathbb{R}_+ \times D$ in the following way: set

$$||y||_Y := \sum_{m \geq 0} 2^{-m} ||y||_{L^\infty([0,m]\times D)},$$

and then use

$$Y = \{y \in L^\infty_{loc}(\mathbb{R}_+ \times D) \,:\, ||y||_Y < +\infty\},$$

endowed with the norm $||\cdot||_Y$. This is a Banach space, and we want to use Schauder's fixed point theorem in it. Whenever a sequence y_n is such that $\|y_n\|_\infty$ is bounded, then $y_n \to y$ in Y if and only if $y_n \to y$ uniformly on sets of the form $[0, m] \times D$, i.e. if and only if $y_n \to y$ uniformly in D and locally uniformly in time $t \in \mathbb{R}_+$.

The space Y is the space where externalities K reside, as we will prove an existence result involving uniformly bounded externalities (note that we have the inclusion $L^\infty(\mathbb{R}_+ \times D) \subset Y$). In particular, we define the operator $\mathcal{T}$ on $L^\infty(\mathbb{R}_+ \times D)$, and, as we said,we factorise it as $\mathcal{T} = \mathcal{S}\mathcal{O}$, where

$$(\mathcal{O}K)(\cdot, z) := \text{ the unique optimal curve } k^* \text{ associated with the externality } K(\cdot, z)$$

and $\mathcal{S} : (BV_{loc}(\mathbb{R}_+) \cap L^\infty_{loc}(\mathbb{R}_+)) \to Y$ is the integral operator defined in (1), which is essentially not affecting time but only space. Because of Proposition 4.1, the operator $\mathcal{O}$ is continuous, and hence is $\mathcal{T}$. We want to use Schauder's fixed point theorem on the operator $\mathcal{T}$, thus obtaining existence of a rational expectations equilibrium.

Theorem 4.2. *Under the standing assumptions of this paper, and the assumption* $\inf_z k_0(z) > 0$*, the operator* $\mathcal{T}$ *admits a fixed point in* Y*, and hence there exists a rational expectation equilibrium for the spatial economy.*

Proof. Consider $Y_0 \subset Y$ the set of continuous functions on $R_+ \times D$ bounded by a constant M such that

$$\mathrm{Lip}(\mathcal{S})|| \max\{\sup_z |k_0(z)|, \bar{k}(M)\} + ||\mathcal{S}(0)||_{L^\infty} < M$$

(here $\mathcal{S}(0)$ is the function obtained by applying $\mathcal{S}$ to the zero function, and we use the Lipschitz behaviour of $\mathcal{S}$ so that for every bounded function g we have $||\mathcal{S}(g)||_{L^\infty} \leq \mathrm{Lip}(\mathcal{S})||g||_{L^\infty}) + ||\mathcal{S}(0)||_{L^\infty})$. With this choice (which is always possible choosing M large enough, since we supposed $\bar{k}(M) = o(M)$), the operator $\mathcal{T}$ maps Y_0 into itself, since for every continuous function $K(\cdot)$ bounded by M the corresponding optimiser $k^*(\cdot)$ takes values in $[0, \max\{k_0, \bar{k}(M)\}]$.

The set Y_0 is a convex subset of Y and $\mathcal{T}$ is defined and continous on Y_0. Indeed, suppose $Y_0 \ni y_n \to y$, the convergence in Y being equivalent in this case to the local (in time) uniform convergence. Then, for every z we can apply Proposition 4.1 to $K_n := y_n(\cdot, z)$, and obtain the pointwise convergence $k_n^*(t, z) \to k^*(t, z)$, for every (t, z), together with uniform L^∞ bounds (the convergence is actually locally uniform in t for every z). For fixed t, composing with the operator $\mathcal{S}$ and using Assumption 2.1, the convergence of $\mathcal{S}k_n^*$ to $\mathcal{S}k^*$ is uniform in z. On the other hand,

for every $T > 0$, the functions $t \mapsto k_n^*(t,z)$ are Lipschitz for every z, and the Lipschitz constant is uniform both in z and in n. Since we assumed $\mathcal{S}$ to be Lipschitz for the L^∞ norm (Assumption 2.1), then the functions $\mathcal{S}k_n^*$ will be uniformly Lipschitz continuous on $[0,T] \times D$ in time and space, a nd the pointwise convergence $k_n^*(t,z) \to k^*(t,z)$ becomes uniform on $[0,T] \times D$. This, together with the uniform L^∞ bounds we already mentioned, provides the convergence of $\mathcal{SO}(y_n)$ to $\mathcal{SO}(y)$ in Y, and hence the continuity of $\mathcal{T}$.

Now, in order to apply Schauder's fixed point theorem we just need to check that $\mathcal{T}$ maps Y_0 into a compact subset of Y_0, but this also comes from the above considerations: because of our choice of convergence, we just need to guarantee equicontinuity of the functions of the form $\mathcal{T}(K)$ on each set $[0,T] \times D$. The continuity in time of each maximiser k_* is again guaranteed by Corollary 3.9, which provides Lipschitz bounds in time of k^*, combined with the Lipschitz behaviour of $\mathcal{S}$, and the continuity in space is guaranteed by the assumptions on $\mathcal{S}$.

As a consequence, $\mathcal{T}$ admits a fixed point. □

Theorem 4.3. *A triplet (k,c,K) represents a rational expectation equilibrium if and only if the following non-local Euler-Lagrange equation*

$$\begin{cases} k' = f(k, \mathcal{S}(k)) - c, \\ U''(c)c' = (r - f_k(k, \mathcal{S}(k)))U'(c), \\ K = \mathcal{S}(k) \end{cases}$$

is satisfied.

Proof. It is easy to see, using Corollary 3.10, that if (k,c,K) is a rational expectation equilibrium, then it satisfies this system. On the other hand, if (k,c,K) satisfies the system, then we can solve the individual optimisation problems for given $K = \mathcal{S}(k)$ and find a solution k^* which is characterised (because of the concavity of the maximisation problem) by the equations in Corollary 3.10. This means that $k^* = k$, and hence $k = \mathcal{O}(K)$ and $K = \mathcal{S}(\mathcal{O}(K))$ is a fixed point. □

Remark 4.4. The arguments can be generalised in the case where the integral operator that models the effect of externalities, $\mathcal{S}$ acts on the temporal variable as well on the spatial variable, for instance $(\mathcal{S}k)(t,z) = \int_D \int_0^\infty w(z,y,t,s)k(y,s)dyds$, where now $\mathcal{S}$ can have some sort of regularisation effect on the temporal variable, or include memory effects or delays. This is particularly meaningful in order to model the effect of knowledge spillover in the generation of externalities, as we mentioned in the introduction. From the technical mathematical point of view, this requires minimal adaptations.

Remark 4.5. It is easy to see that, in case of lack of uniqueness for the maximisers of J_∞ in $\mathcal{A}(K)$, then the proof of Proposition 4.1 easily provides that, up to extracting a subsequence, we have convergence of (k_n^*, c_n^*) to one maximiser. This makes the operator $\mathcal{O}$ a multivalued map with closed graph. Moreover, if $f(\cdot, K)$ is concave, the set of maximisers is a convex set, which allows to obtain the existence of a fixed point by an easy application fo the Kakutani Theorem.

A Uniform local bounds on the optimal consumption

In this appendix we prove that the optimal path (k^*, c^*) of the individual parametrized optimization problem (2) satisifes upper and lower bound, for both k and c, on each bounded interval $[0,T]$. In this section we will suppose that the output function f, besides all the conditions of Assumption 2.4, is also globally Lipschitz continuous. Moreover, we set

$$a_0 := \frac{\min\{k_0, k_1\}}{2},$$

where k_0 is the initial value of k and k_1 is the constant appearing in Assumption 2.4-v, and we denote by $\bar{C}$ a constant such that $f_k(k, K) \leq \bar{C}$ for $k \geq a_0$. It will be important to obtain estimates which depend on $\bar{C}$ but not on the Lipschitz constant of f close to $k = 0$. Moreover, we extend f to $k < 0$ by defining setting $f(k, K) = -kf_k^+(0, K)$ (remember that we have $f(0, K) = 0$), and we still use the notation f for the extended function. In particular the extension also satisfies $f_k \geq -\delta$ and $f_k \leq \bar{C}$ for $k \geq a_0$.

Lemma A.1. *Suppose that on* $[0, T_0]$ *we have* $0 < \inf_{t\in[0,T_0]} k^*(t) < a_0$. *Then,* $c^* \geq c(T_0, \delta, \bar{C})$ *for some appropriate constant* c.

Proof. We first note that (k^*, c^*) also solves $\max J(c)$ for $k(0)$ and $k(T_0)$ fixed to the value $k^*(T_0)$ without imposing the constraint $k \geq 0$ (indeed, in a convex optimization problem whenever an optimiser, here k^*, satisifies a constraint in a non-binding way then it also optimises without the same constraint, since it satisfies the corresponding sufficient optimality conditions). To deal with the non smoothness of the utility function at 0, we consider next the modified problem $\max J^{(h)}(c)$, where $J^{(h)}(c) = \int_0^{T_0} U(c+h)e^{-rt}dt$, under the same boundary conditions and let (c_h, k_h) be the optimal path for this problem. This is a problem without state constraints with a smooth utility function so one may employ that Pontryagin principle in its standard form for control problems with fixed endpoints (which is just a re-writing of the Euler-Lagrange equations).

This provides the existence of a function θ_h satisfying

$$\theta_h'(t) = -\theta_h(t) f_k(k_h(t), K(t)), \tag{10}$$

while the optimal consumption satisfies

$$c_h \in \arg\max_{c\geq 0}\{e^{-rt}U(c + h) + \theta_h(t)(f(k_h(t), K(t)) - c)\}, \quad a.e.t. \tag{11}$$

The properties of U combined with (11) imply

$$\theta_h = U'(c_h + h)e^{-rt} \quad \text{if} \quad c_h > 0 \tag{12}$$

$$\theta_h \geq U'(h)e^{-rt} \quad \text{if} \quad c_h = 0 \tag{13}$$

Set $T_1 := \inf\{t : k_h(t) < a_0\}$, so that we have $k_h(T_1) = a_0$ while $k_h(t) > a_0$ for $t < T_1$. This implies that we cannot have $k_h' > 0$ a.e. in a left neighbourhood of T_1 (otherwise we would have $k_h < k_h(T_1) = a_0$ before T_1). Hence, for a sequence of times $t_n \to T_1^-$ we have $k_h'(t_n) \leq 0$: by the dynamic constraint we get $c_h(t_n) = f(k_h(t_n), K(t_n)) - k_h'(t_n) \geq f(k_h(t_n), K(t_n)) \geq C_0$ by the assumptions on f, which takes strictly positive values close to $k = a_0$, independently of K This lower bound on $c_h(t_n)$, together with the fact that $\theta_h = U'(h + c_h)e^{-rt}$ and the properties of U', leads to an upper bound of the form $\theta_h(T_1) \leq C_0$ for an appropriate constant C_0 (note that θ_h is a Lipschiz function, thanks to (10).

Since on $[0, T_1]$ we have $k_h(t) \geq a_0$, by definition we have $f_k \leq \bar{C}$, therefore, by (10) we see that $\theta_h' \geq -\bar{C}\theta_h$. Integrating backwards in time and utilizing the bound $\theta_h(T_1) \leq C_0$ we obtain a bound for θ_h over the whole interval $[0, T_1]$, i.e., there exists a C such that $\theta_h \leq C$ for all $t \in [0, T_1]$. Having obtained a bound for the initial value of θ_h and using $f_k \geq -\delta$ which is true everywhere, (10) implies that $\theta_h' \leq \delta\theta_h$, and by integration $\theta_h \leq C$ on $[0, T_0]$.

By (12) if $c_h(t) = 0$ for some $t \in [0, T_0]$ we have that $\theta_h(t) \geq U'(h)e^{-rt} \geq U'(h)e^{-rT_0}$ for any $t \in [0, T_0]$, and if h is chosen $h < h_0$ with h_0 such that $U'(h_0)e^{-rT_0} > C$, this contradicts $\theta_h(t) < C$ for any $t \in [0, T_0]$. Hence for $h < h_0$, we have $c_h(t) > 0$ for any $t \in [0, T_0]$. Then, $\theta_h(t) = U'(c_h(t) + h)e^{-rt} \geq U'(c_h(t) + h)e^{-rT_0}$ combined with $\theta_h(t) < C$ yields $U'(c_h + h) \leq Ce^{rT_0}$. This implies$c_h(t) + h \geq (U')^{-1}(Ce^{-rT_0}) =: C_1(T_0, \delta, \bar{C})$ where importantly the constant C_1 is independent of $h < h_0$. We conclude by letting $c_h \rightharpoonup c^*$ and using Lemma 3.7. □

Lemma A.2. *Under the standing assumptions we have $k^*(t) > 0$ for all $t \in \mathbb{R}$.*

Proof. If $T_0 = \inf\{t : k^*(t) = 0\}$, apply Lemma A.1 on $[0, T_0 - \epsilon]$ for every $\epsilon > 0$ to obtain $c^* \geq c_0$ on $[0, T_0]$.

On the other hand, on $[T_0, \infty[$ we have $k^* = c^* = 0$. Indeed, since f is Lipschitz and $f(0, K) = 0$, using also the positivity of c we have $(k^*)' \leq \mathrm{Lip}(f)k^*$ which implies $k^* = 0$ since $k^*(T_0) = 0$. This also implies $c^* = 0$.

Define a competitor $(\tilde{c}, \tilde{k})$ which coincides with (k^*, c^*) in $[0, T_0 - \epsilon]$ and $[T_0 + \epsilon, \infty[$, but is modified inside $(T_0 - \epsilon, T_0 + \epsilon)$ as follows:

$$\begin{aligned} \tilde{k}(t) &= k^*(T_0 - \epsilon) - \frac{k^*(T_0 - \epsilon)}{2\epsilon}(t - (T_0 - \epsilon)), \\ \tilde{c}(t) &= f(t, \tilde{k}) + \frac{k^*(T_0 - \epsilon)}{2\epsilon}. \end{aligned}$$

By optimality of (k^*, c^*), and using $c^* = 0$ on $[T_0, T_0 + \epsilon]$, we have

$$\int_{T_0-\epsilon}^{T_0+\epsilon} U(\tilde{c})e^{-rt}dt \leq \int_{T_0-\epsilon}^{T_0} U(c^*)e^{-rt}dt. \tag{14}$$

We estimate the right hand side of (14) using

$$\frac{1}{\epsilon}\int_{T_0-\epsilon}^{T_0} U(c^*)e^{-rt}dt \leq e^{-r(T_0-\epsilon)}\frac{1}{\epsilon}\int_{T_0-\epsilon}^{T_0} U(c^*)dt \leq e^{-r(T_0-\epsilon)}U\left(\frac{1}{\epsilon}\int_{T_0-\epsilon}^{T_0} c^*(t)dt\right), \tag{15}$$

where for the last inequality we used Jensen's inequality. Upon integration of $(k^*)' = f(k^*, K) - c^*$ on the interval $[T_0 - \epsilon, T_0]$ and keeping in mind that $k^*(T_0) = 0$ we get

$$\frac{1}{\epsilon}k^*(T_0 - \epsilon) = \frac{1}{\epsilon}\int_{T_0-\epsilon}^{T_0} c^*(t)dt - \frac{1}{\epsilon}\int_{T_0-\epsilon}^{T_0} f(k^*(t), K)dt$$

and keeping in mind that in the limit as $\epsilon \to 0$ it holds that $\frac{1}{\epsilon}\int_{T_0-\epsilon}^{T_0} f(k^*(t), K)dt \to f(k^*(T_0), K) = 0$ we conclude that as $\epsilon \to 0$,

$$\frac{k^*(T_0 - \epsilon)}{\epsilon} = \frac{1}{\epsilon}\int_{T_0-\epsilon}^{T_0} c^*(t)dt + o(1). \tag{16}$$

Combining (15) and (16) we obtain

$$\frac{1}{\epsilon}\int_{T_0-\epsilon}^{T_0} U(c^*(t))e^{-rt}dt \leq e^{-r(T_0-\epsilon)}U\left(\frac{k^*(T_0 - \epsilon)}{\epsilon} + o(1)\right) \tag{17}$$

On the other hand, as $\epsilon \to 0$, using that, for small ϵ the values of $\tilde{k}$ on $[T_0 - \epsilon, T_0 + \epsilon]$, which are bounded by $k^*(T_0 - \epsilon)$, are small, we have $\tilde{c} \geq k^*(T_0 - \epsilon)/(2\epsilon)$ on the same interval, hence

$$\frac{1}{2\epsilon}\int_{T_0-\epsilon}^{T_0+\epsilon} U(\tilde{c}(t))e^{-rt}dt \geq e^{-r(T_0+\epsilon)}U\left(\frac{k^*(T_0 - \epsilon)}{2\epsilon}\right). \tag{18}$$

Combining (14), (17) and (18) we obtain

$$U\left(\frac{k^*(T_0 - \epsilon)}{2\epsilon}\right)2e^{-r(T_0+\epsilon)} \leq e^{-r(T_0-\epsilon)}U\left(\frac{k^*(T_0 - \epsilon)}{\epsilon} + o(1)\right). \tag{19}$$

We now set $\ell_\epsilon := k^*(T_0-\epsilon)/\epsilon$ and we use the fact that, for ϵ small, $\ell_\epsilon \geq c_0/2$, together with the fact that U is uniformly Lipschitz far from $c=0$, in order to simplfy the above inequality into

$$2U\left(\frac{\ell_\epsilon}{2}\right) \leq e^{2r\epsilon}(U(\ell_\epsilon)+o(1)). \tag{20}$$

We now use the following property

$$\inf_{\ell\geq c_0/2} 2U\Big(\frac{\ell}{2}\Big) - U(\ell) = \alpha > 0,$$

which can be easily proven by considering that the function $\ell \mapsto 2U(\ell/2)-U(\ell)$ is increasing, and its value in $\ell = c_0/2$ is strictly positive since U is concave and not linear close to 0 (as a consequence of $U'(0)=\infty$).

This allows to go on from (20) and obtain

$$U(\ell_\epsilon)+\alpha \leq (1+O(\epsilon))(U(\ell_\epsilon)+o(1)) = U(\ell_\epsilon)+O(\epsilon)U(\ell_\epsilon)+o(1).$$

This provides a contradiction for small ϵ: indeed, ℓ_ϵ could be bounded or not, but it is for sure at most $O(1/\epsilon)$, and since U is sublinear at infinity we have $O(\epsilon)U(\ell_\epsilon) = o(1)$, which contradicts $\alpha > 0$. □

Lemma A.3. *On* $[0,T_0]$ *we have* $\inf c^* = c_0(\delta, C, T_0) > 0$.

Proof. If $\inf_{[0,T_0]} k^* < a_0$ the result follows by applying first Lemma A.2 and then Lemma A.1.

If not, we can assume $k^* \geq a_0$ on $[0,T_0]$. We consider an approximation (k_h, c_h) by solving $J^{(h)}_{T_0} := \max_{c\geq 0}\int_0^{T_0} U(c+h)e^{-rt}dt$ with fixed endpoints on $[0,T_0]$, but without the state constraint $k\geq 0$ (since this state constraint is not binding in the limit). As in Lemma A.1 we may apply the standard Pontryagin maximum principle and in particular obtain the existence of a function θ_h, satisfying (10), (11) and (12). Note that by the assumptions on f, we have that $-\delta \leq f_k \leq C$ and hence $|\theta_h'| \leq C\theta_h$.

We want to give a lower bound on $\inf_{t\in[0,T_0]} c_h(t) \to 0$. Let $\bar J = \max J_\infty(c) = J_\infty(c^*) > 0$ be the value of the original problem: this value is strictly positive, since $c=0$ is not optimal. Using Lemma 3.2 suppose that T_0 is large enough (which is not restrictive) so that

$$\int_0^\infty U(c^*(t))dt \leq \int_0^{T_0} U(c^*(t))dt + \frac{1}{4}\bar J.$$

Using the monotone behaviour of U and $U(c+h)\geq U(c)$ for every c, we also obtain

$$\begin{aligned}\max\{\int_0^{T_0} U(c) \,:\, (k,c)\in\mathcal{A}(K),\, k(0)=k_0, k(T_0)=k^*(T_0)\} \\ \leq \max\{\int_0^{T_0} U(c+h) \,:\, (k,c)\in\mathcal{A}(K)\, k(0)=k_0, k(T_0)=k^*(T_0)\}\end{aligned} \tag{21}$$

and hence

$$\int_0^\infty U(c^*(t))dt \leq \int_0^{T_0} U(c_h(t)+h))dt + \frac{1}{4}\bar J. \tag{22}$$

Set ϵ_1 so that $U(\epsilon_1)\int_0^\infty e^{-rt}dt = \frac{1}{r}U(\epsilon_1) < \frac{1}{4}\bar J$ and ϵ_0 so that $U'(\epsilon_1) = U'(\epsilon_0)e^{-(C+r)T_0}$.

We want to prove that it is not possible to have $h+\inf c_h < \epsilon_0$. This will give a lower bound on c_h for small h, and hence on the optimal c^*.

Suppose by contradiction that we have $h + \inf c_h < \epsilon_0$. First, let us prove that this implies $\sup\{c_h(t) : t \in [0, T_0]\} \leq \epsilon_1$. Note that, for some $t^* \in [0, T_0]$, we have $c_h(t^*) + h < \epsilon_0$. By the properties of U' this means $U'(c_h(t^*) + h) > U'(\epsilon_0)$ and recalling (12) we have

$$\theta_h(t^*) > U'(\epsilon_0)e^{-rt^*}. \tag{23}$$

This lower bound on θ_h at t can be propagated forward and backward in time using $|\theta_h'| \leq C\theta_h$. This provides

$$\theta_h(t) > \theta_h(t^*)e^{-C|t-t^*|} \geq \theta_h(t^*)e^{-CT_0}. \tag{24}$$

Using again the formula for θ_h we obtain

$$U'(c_h(t) + h) > \theta_h(t)e^{rt} \geq U'(\epsilon_0)e^{-rt^*}e^{-CT_0}e^{rt} \geq U'(\epsilon_0)e^{-(C+r)T_0} = U'(\epsilon_1),$$

where the last equality comes from the choice of ϵ_0. Using once more the monotone decreasing behavior of U' we get $c_h + h \leq \epsilon_1$.

After proving this, using (22), we can go on and obtain

$$\bar{J} \leq \int_0^{T_0} U(h + c_h)e^{-rt}dt + \frac{1}{4}\bar{J} \leq U(\epsilon_1)\int_0^{T_0} e^{-rt}dt + \frac{1}{4}\bar{J} \leq U(\epsilon_1)\int_0^{\infty} e^{-rt}dt + \frac{1}{4}\bar{J} \leq \frac{1}{2}\bar{J}.$$

Since $c_h(t) \geq \epsilon_0$ for any $h > 0$, $t \in [0, T_0]$ passing to the limit as $h \to 0$ we conclude that $c^* \geq \epsilon_0$. □

Lemma A.4. *On every $[0, T_0]$ we have $c^* \leq C_0$ for a constant C_0 possibly depending on T_0.*

Proof. Since have already established that $k^* > 0$, $c^* > 0$, we may replace f and U with $\tilde{f}$ and $\tilde{U}$ which are smooth approximations about $k \geq 0$ and $c \geq 0$ and as above may use the standard PMP, to establish the existence of a function θ with properties similar to the ones provided in (10)-(12) and in particular $\theta' = -\tilde{f}_k\theta$ and $\theta(t) = \tilde{U}'(c)e^{-rt}$. Note that $\theta' \leq \delta\theta$.

Lemma 3.2 provides $\int_{T_0}^{T_0+1} c^*(t)dt \leq \bar{C}_0 + \bar{C}_1(T_0 + 1)$, which in turn provides $c^*(t^*)leqC(T_0)$ for a certain time $t^* \in [T_0, T_0 + 1]$. This implies a lower bound on $\theta(t^*)$. We then use the differential inequality $\theta' \leq \delta\theta$ to show that a similar lower bound (up to a factor $e^{\delta(T_0+1)}$ for θ is satisfied for all $t \in [0, T_0]$ and again by the definition of θ and the properties of U' we conclude that c is bounded from above on $[0, T]_0$. □

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