On uniqueness of weak solutions to transport equation with non-smooth velocity field

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Abstract Given a bounded, autonomous vector field $\mathbf{b}\colon \mathbb{R}^d \to \mathbb{R}^d$, we study the uniqueness of bounded solutions to the initial value problem for the associated transport equation

$$\partial_t u + \mathbf{b} \cdot \nabla u = 0. \tag{1}$$

This problem is related to a conjecture made by A. Bressan, raised studying the well-posedness of a class of hyperbolic conservation laws. Furthermore, from the Lagrangian point of view, this gives insights on the structure of the flow of non-smooth vector fields. In this work we will discuss the two dimensional case and we prove that, if $d = 2$, uniqueness of weak solutions for (1) holds under the assumptions that $\mathbf{b}$ is of class BV and it is *nearly incompressible*. Our proof is based on a splitting technique (introduced previously by Alberti, Bianchini and Crippa in [2]) that allows to reduce (1) to a family of 1-dimensional equations which can be solved explicitly, thus yielding uniqueness for the original problem.
This is joint work with S. Bianchini and N.A. Gusev [5].

KEYWORDS: transport equation, continuity equation, Lipschitz functions, Superposition Principle.

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1 Introduction

In this paper we consider the *continuity equation*

$$\partial_t u + \operatorname{div}(u\mathbf{b}) = 0 \tag{2}$$

and the *transport equation*

$$\partial_t u + \mathbf{b}\cdot\nabla u = 0, \tag{3}$$

where $u\colon I\times\Omega\to\mathbb{R}$ (we write $I=(0,T)$, $T>0$ and $\Omega\subset\mathbb{R}^2$ for an open set) is an unknown scalar function and $\mathbf{b}\colon I\times\Omega\to\mathbb{R}^2$ is a given vector field. We study the initial value problems for these equations with the initial condition

$$u(0,\cdot) = u_0(\cdot), \tag{4}$$

where $u_0\colon \Omega\to\mathbb{R}$ is a given scalar field.

Our aim is to investigate uniqueness of weak solutions to (2), (4) under weak regularity assumptions on the vector field $\mathbf{b}$. When $\mathbf{b}\in L^\infty(I\times\Omega)$ then (2) is understood in the standard sense of distributions: $u\in L^\infty(I\times\Omega)$ is called a *weak solution* of the continuity equation if (2) holds in $\mathscr{D}'(I\times\Omega)$. Concerning the initial condition, one can prove (see e.g. [8]) that, if u is a weak solution of (2), then there exists a map $\widetilde{u}\in L^\infty([0,T]\times\Omega)$ such that $u(t,\cdot)=\widetilde{u}(t,\cdot)$ for a.e. $t\in I$ and $t\mapsto\widetilde{u}(t,\cdot)$ is weakly* continuous from $[0,T]$ into $L^\infty(\Omega)$. Thus we can prescribe an initial condition (4) for a weak solution u of the continuity equation in the following sense: we say that $u(0,\cdot)=u_0(\cdot)$ holds if $\widetilde{u}(0,\cdot)=u_0(\cdot)$.

Definition of weak solutions of the transport equation (3) is slightly more delicate. If $\operatorname{div}\mathbf{b}\ll\mathscr{L}^2$, being $\mathscr{L}^2$ the Lebesgue measure in the plane, then (3) can be written as

$$\partial_t u + \operatorname{div}(u\mathbf{b}) - u\operatorname{div}\mathbf{b} = 0,$$

and the latter equation can be understood in the sense of distributions (see e.g. [9] for the details). However, we are here interested in the case when $\operatorname{div}\mathbf{b}$ is not absolutely continuous. In this case the notion of weak solution of (3) can be defined for the class of *nearly incompressible vector fields*.

Definition 1. A bounded, locally integrable vector field $\mathbf{b}\colon I\times\Omega\to\mathbb{R}^2$ is called *nearly incompressible* if there exists a function $\rho\colon I\times\Omega\to\mathbb{R}$ (called *density* of $\mathbf{b}$) and a constant $C>0$ such that $C^{-1}\le\rho(t,x)\le C$ for a.e. $(t,x)\in I\times\Omega$ and

$$\partial_t\rho + \operatorname{div}(\rho\mathbf{b}) = 0 \qquad \text{in } \mathscr{D}'(I\times\Omega). \tag{5}$$

It is easy to see that if $\operatorname{div}\mathbf{b}\in L^\infty(I\times\Omega)$ then $\mathbf{b}$ is nearly incompressible. The converse implication does not hold, so near incompressibility can be considered as a weaker version of the assumption $\operatorname{div}\mathbf{b}\in L^\infty(I\times\Omega)$.

Definition 2. Let $\mathbf{b}$ be a nearly incompressible vector field with density ρ. We say that a function $u\in L^\infty(I\times\Omega)$ is a *(ρ–)weak solution* of (3) if

$$\partial_t(\rho u) + \operatorname{div}(\rho u \mathbf{b}) = 0 \quad \text{in } \mathscr{D}'(I \times \Omega).$$

Thanks to Definition 2 one can prescribe the initial condition for a ρ–weak solution of the transport equation similarly to the case of the continuity equation, which we mentioned above (see again [8] for further details).

Existence of weak solutions to the initial value problem for transport equation with a nearly incompressible vector field can be proved by a standard regularization argument [8].

The problem of *uniqueness* of weak solutions is much more delicate. The theory of uniqueness in the non-smooth framework started with the seminal paper of R.J. DiPerna and P.-L. Lions [9] where uniqueness was obtained as a corollary of the so-called *renormalization property* for vector fields with Sobolev regularity. This result was improved by Ambrosio in [3], where uniqueness is shown for vector fields of (locally) bounded variation in the space with absolutely continuous, bounded divergence. As we have pointed out, vector fields considered in [3] make up a proper subset of nearly incompressible ones, so it makes sense to wonder whether uniqueness holds in the latter, larger class. This is related to a conjecture, made by A. Bressan in [7], whose 2D statement is the following:

Conjecture 1 (Bressan). Uniqueness of weak solutions to (3), (4) holds for any nearly incompressible vector field $\mathbf{b} \in L^1(I; \mathrm{BV}_{\mathrm{loc}}(\mathbb{R}^2))$.

In the two dimensional, autonomous case the problem of uniqueness is addressed in the papers [1], [2] and [6]. Indeed, in two dimensions and for divergence-free autonomous vector fields, uniqueness results are available under milder assumptions on the regularity of $\mathbf{b}$, because of the underlying Hamiltonian structure. In [2], the authors characterize the autonomous, divergence-free vector fields $\mathbf{b}$ on the plane such that the Cauchy problem for the transport equation (3) admits a *unique* bounded weak solution for every bounded initial datum (4). The characterization they present relies on the so called *Weak Sard Property*, which is a (weaker) measure theoretic version of Sard's Lemma. Since the problem admits a Hamiltonian potential, uniqueness is proved following a strategy based on splitting the equation on the level sets of this function, reducing thus to a one-dimensional problem. This approach requires a preliminary study on the structure of level sets of Lipschitz maps defined on $\mathbb{R}^2$, which is carried out in the paper [1].

The main result of this paper is a partial answer to the Conjecture 1:

Theorem 1. *Suppose that* $\mathbf{b}\colon \mathbb{R}^2 \to \mathbb{R}^2$ *is a compactly supported, nearly incompressible BV vector field. Then uniqueness of weak solutions to* (1), (4) *holds.*

More precisely, we will show that for every bounded initial datum $u_0 \in L^\infty(\mathbb{R}^2)$ there exists a unique (ρ-)weak solution $u \in L^\infty(I \times \mathbb{R}^2)$ to the transport equation (3) with the initial condition $u(0,\cdot) = u_0(\cdot)$. The main idea behind the proof is to implement a sort of method of characteristics in a non-smooth setting: indeed, we will find a "large" (in a proper sense) family of integral curves γ of the vector field $\mathbf{b}$ such that, for any solution u, the function $t \mapsto u(t, \gamma(t))$ has to be constant: from this the uniqueness easily follows.

2 Preliminaries and useful results

We collect here some previously known results that will be used in the proof of Theorem 1.

2.1 Ambrosio's Superposition Principle

In [3], L. Ambrosio proved the *Superposition Principle*. Since we will use it later on, we present here the statement.

Theorem 2 (Superposition Principle). *Let* $\mathbf{b}\colon [0,T]\times\mathbb{R}^d \to \mathbb{R}^d$ *be a bounded, Borel vector field and let* $[0,T]\ni t\mapsto \mu_t$ *be a positive, locally finite, measure-valued solution of the continuity equation*

$$\begin{cases} \partial_t \mu_t + \mathrm{div}(\mathbf{b}\mu_t) = 0 \\ \mu_0 = \overline{\mu} \end{cases} \qquad \text{in } \mathscr{D}'((0,T)\times\mathbb{R}^d). \tag{6}$$

Then there exists a non-negative measure η *on the space of continuous curves* $\Gamma := C([0,T];\mathbb{R}^d)$ *such that*

$$\mu_t = e_{t\#}\eta \qquad \text{for every } t\in[0,T], \tag{7}$$

being $e_t\colon \Gamma\to\mathbb{R}^d$ *the evaluation map* $\gamma\mapsto\gamma(t)$*. Furthermore, the measure* η *is concentrated on absolutely continuous solutions of the ordinary differential equation driven by* $\mathbf{b}$*.*

2.2 Level sets of Lipschitz functions and disintegration of Lebesgue measure

Suppose that $\Omega\subset\mathbb{R}^2$ is an open set and $H\colon \Omega\to\mathbb{R}$ is a compactly supported Lipschitz function. For any $h\in\mathbb{R}$, let $E_h := H^{-1}(h)$. We recall the following deep

Theorem 3 ([1, Theorem 2.5]). *For* $\mathscr{L}^1$*-a.e.* $h\in H(\Omega)$ *the following holds:*

(1) $\mathscr{H}^1(E_h) < +\infty$ *and* E_h *is countably* $\mathscr{H}^1$*-rectifiable;*
(2) *for* $\mathscr{H}^1$*-a.e.* $x\in E_h$ *the function* H *is differentiable at* x *with* $\nabla H(x)\neq 0$*;*
(3) *if* $\mathrm{Conn}^\star(E_h)$ *denotes the set of connected components* $\mathfrak{c}$ *of* E_h *with* $\mathscr{H}^1(\mathfrak{c})>0$*, then* $\mathrm{Conn}^\star(E_h)$ *is countable and every* $\mathfrak{c}\in\mathrm{Conn}^\star(E_h)$ *is a closed simple curve, which admits a parametrization* $\gamma_{\mathfrak{c}}$ *with the following properties:*

- $\gamma_{\mathfrak{c}}\colon I_{\mathfrak{c}}\to\mathbb{R}^2$ *is Lipschitz and injective (up to the end-points), where* $I_{\mathfrak{c}} = \mathbb{R}/\ell\mathbb{Z}$ *or* $I_{\mathfrak{c}}=[0,\ell]$ *for some* $\ell>0$*;*
- $\gamma_{\mathfrak{c}}'(s)$ *is orthogonal to* $\nabla H(\gamma_{\mathfrak{c}}(s))$ *for* $\mathscr{L}^1$*-a.e.* $s\in I_{\mathfrak{c}}$*.*

(4) *If $E_h^\star$ denotes the union of $\mathfrak{c} \in Conn^\star(E_h)$, then $\mathscr{H}^1(E_h \setminus E_h^\star) = 0$.*

In the following we will call *regular* the level sets corresponding to values of h satisfying Point (1) of Theorem 3.

Using Theorem 3 together with Disintegration Theorem [4, Thm. 2.28], we can characterize the disintegration of the Lebesgue measure restricted to an open set Ω. We have the following

Lemma 1 ([2, Lemma 2.8]). *There exist Borel families of measures $\{\sigma_h\}_{h\in\mathbb{R}}$ and $\{\kappa_h\}_{h\in\mathbb{R}}$, such that*

$$\mathscr{L}^2 \llcorner \Omega = \int \left(c_h \mathscr{H}^1 \llcorner E_h + \sigma_h\right) dh + \int \kappa_h \, d\zeta(h), \tag{8}$$

where

1. *$c_h \in L^1(\mathscr{H}^1 \llcorner E_h^\star)$ and, more precisely, by Coarea formula, we have $c_h = 1/|\nabla H|$ a.e. (w.r.t. $\mathscr{H}^1 \llcorner E_h^\star$);*
2. *σ_h, κ_h are concentrated on $E_h^\star \cap \{\nabla H = 0\}$;*
3. *$\zeta := H_\# \mathscr{L}^2 \llcorner (\Omega \setminus \bigcup_h E_h^\star)$ is singular w.r.t. $\mathscr{L}^1$.*

2.3 The Weak Sard Property

Let $H\colon \Omega \to \mathbb{R}$ be a Lipschitz function as in Section 2.2 and let S be the *critical set* of H, defined as the set of all $x \in \Omega$ where H is not differentiable or $\nabla H(x) = 0$. We will be interested in the following property:

the push-forward according to H of the restriction of $\mathscr{L}^2$ to S is singular with respect to $\mathscr{L}^1$, that is

$$H_\# \left(\mathscr{L}^2 \llcorner S\right) \perp \mathscr{L}^1.$$

This property clearly implies the following *Weak Sard Property*, which is used in [2, Section 2.13]:

$$H_\# \left(\mathscr{L}^2 \llcorner (S \cap E^\star)\right) \perp \mathscr{L}^1,$$

where the set $E^\star$ is the union over $h \in H(\Omega)$ of all connected components with positive length of E_h. The relevance of the Weak Sard Property in the framework of transport and continuity equation has been completely understood in [2, Theorem 4.7], to whom we refer the reader for further details.

Remark 1. Informally, the Weak Sard Property means that the “good” level sets of H do not intersect the critical set S, apart from a negligible set. In terms of the disintegration of the Lebesgue measure (8), it can be proved that H has the weak Sard property if and only if $\sigma_h = 0$ for a.e. h. This is actually the case when, for instance, $\nabla H \neq 0$ a.e. (see Point 2 of Lemma 1) or when $\nabla H \in \mathrm{BV}(\Omega; \mathbb{R}^2)$ (see [5]).

3 Proof I: the local argument

We can now start the proof of Main Theorem. We will split the proof in two steps: first we perform a *local* argument, finding a suitable covering of the plane in balls and disintegrating the equation inside each ball. Afterwards, we will "glue" together the local results in order to implement the weak method of characteristic. We start here presenting the local argument.

Let $\mathbf{b}\colon \mathbb{R}^2 \to \mathbb{R}^2$ be an *autonomous*, nearly incompressible vector field, with $\mathbf{b} \in \mathrm{BV}(\mathbb{R}^2) \cap L^\infty(\mathbb{R}^2)$; we assume $\mathbf{b}$ is compactly supported, defined everywhere and Borel.

3.1 Partition and curves

Let us consider the countable covering $\mathscr{B}$ of $\mathbb{R}^2$ given by

$$\mathscr{B} := \left\{ B(x,r) : x \in \mathbb{Q}^2, r \in \mathbb{Q}^+ \right\}.$$

where $B(x,r)$ is the euclidean ball of center x and radius r. For each ball $B \in \mathscr{B}$, we are interested to the trajectories of $\mathbf{b}$ which cross B and stay inside B for a positive amount of time. We therefore define, for every ball $B \in \mathscr{B}$ and for any rational numbers $s,t \in \mathbb{Q} \cap (0,T)$ such that $s < t$, the sets

$$\mathsf{T}_{B,s,t} := \left\{ \gamma \in \Gamma : \mathscr{L}^1(\{\tau \in [0,T] : \gamma(\tau) \in B\}) > 0,\, \gamma(s) \notin B,\, \gamma(t) \notin B \right\}.$$

In this first section we will work for simplicity with the sets $\mathsf{T}_B := \mathsf{T}_{B,0,T}$, where $B \in \mathscr{B}$ (and without any loss of generality we assume $T \in \mathbb{Q}$).

Remark 2. It is immediate to see that the union over $B \in \mathscr{B}$ of the sets T_B is equal to the set of all non-constant trajectories. The same is true also for the union over $B \in \mathscr{B}$ and $s,t \in \mathbb{Q} \cap [0,T]$ of the sets $\mathsf{T}_{B,s,t}$.

By Definition 1, we have that there exists a function $\rho : [0,T] \times \mathbb{R}^2 \to \mathbb{R}$ which satisfies continuity equation (5) in $\mathscr{D}'((0,T) \times \mathbb{R}^2)$. Therefore, by Ambrosio's Superposition Principle (Theorem 2), there exists a measure η on Γ, concentrated on the set of integral curves of $\mathbf{b}$, such that

$$\rho(t,\cdot)\mathscr{L}^2 = (e_t)_\# \eta, \tag{9}$$

where we recall that $e_t\colon \Gamma \to \mathbb{R}^2$ is the evaluation map $\gamma \mapsto \gamma(t)$. For a fixed ball $B \in \mathscr{B}$, we consider the measure $\eta_B := \eta \llcorner \mathsf{T}_B$ and we define the density ρ_B by $\rho_B(t,\cdot)\mathscr{L}^2 = (e_t)_\# \eta_B$. We finally set

$$r_B(x) := \int_0^T \rho_B(t,x)\,dt, \qquad x \in B. \tag{10}$$

Now the following lemma is almost immediate.

Lemma 2. *It holds* $\operatorname{div}(r_B\mathbf{b}) = 0$ *in* $\mathscr{D}'(B)$.

3.2 Local disintegration of the equation div$(u\mathbf{b}) = \mu$

From Lemma 2, we deduce that (being every $B \in \mathscr{B}$ simply connected) there exists a Lipschitz map $H_B \colon B \to \mathbb{R}$ such that $r_B\mathbf{b} = \nabla^\perp H_B$, where $\nabla^\perp = (-\partial_2, \partial_1)$. The function H_B is well defined (being $\mathbf{b}$ compactly supported) and we will often refer to it as the *Hamiltonian* (relative to the ball B).

3.2.1 Reduction of the equation on the level sets

Fix now a ball $B \in \mathscr{B}$ and consider the equation

$$\operatorname{div}(u\mathbf{b}) = \mu, \qquad \text{in } \mathscr{D}'(B) \tag{11}$$

where $u \in L^\infty(\mathbb{R}^2)$ and μ is a Radon measure on $\mathbb{R}^2$.

The first step is the disintegration of the equation on the level sets of H:

Lemma 3. *The equation* (11) *is equivalent to:*

1. *the disintegration of* μ *with respect to* H *has the form*

$$\mu = \int \mu_h \, dh + \int \nu_h \, d\zeta(h), \tag{12}$$

   where ζ *is defined in Point* (3) *of Lemma 1;*

2. *for* $\mathscr{L}^1$*-a.e.* h

$$\operatorname{div}\left(uc_h\mathbf{b}\mathscr{H}^1 \llcorner E_h\right) + \operatorname{div}(u\mathbf{b}\sigma_h) = \mu_h; \tag{13}$$

3. *for* ζ*-a.e.* h

$$\operatorname{div}(u\mathbf{b}\kappa_h) = \nu_h. \tag{14}$$

3.2.2 Reduction on the connected components.

The next step is to reduce further the analysis of the equation (13) on the non-trivial (i.e. with positive length) connected components of the level sets.

Lemma 4. *The equation* (13) *holds if and only if*

- *for any non-trivial connected component* $\mathfrak{c}$ *of* E_h *it holds*

$$\operatorname{div}\left(uc_h\mathbf{b}\mathscr{H}^1 \llcorner \mathfrak{c}\right) = \mu_h \llcorner \mathfrak{c}, \tag{15a}$$

$$\operatorname{div}(u\mathbf{b}\sigma_h \llcorner \mathfrak{c}) = 0; \tag{15b}$$

- *it holds*
$$\operatorname{div}(u\mathbf{b}\sigma_h \llcorner (E_h \setminus E_h^\star)) = \mu_h \llcorner (E_h \setminus E_h^\star). \tag{16}$$

3.2.3 Reduction of the equation on connected components in parametric form

Finally, we would like to discuss the parametric version of the equation (15a). Let $\gamma_\mathfrak{c} \colon I_\mathfrak{c} \to \mathbb{R}^2$ be an injective Lipschitz parametrization of $\mathfrak{c}$, where $I_\mathfrak{c} = \mathbb{R}/\ell\mathbb{Z}$ or $I_\mathfrak{c} = (0,\ell)$ for some $\ell > 0$ is the domain of γ.

Lemma 5. *Equation* (15a) *holds iff*
$$\partial_s(\widehat{u}\widehat{c}_h|\widehat{\mathbf{b}}|) = \widehat{\mu}_h \tag{17}$$
where $(\gamma_\mathfrak{c})_\# \widehat{\mu}_h = \mu_h \llcorner \mathfrak{c}$, $\widehat{u} = u \circ \gamma_\mathfrak{c}$, $\widehat{c}_h = c_h \circ \gamma_\mathfrak{c}$ *and* $\widehat{\mathbf{b}} = \mathbf{b} \circ \gamma_\mathfrak{c}$.

3.3 Local disintegration of a balance law

We now pass to consider a general balance law associated to the Hamiltonian vector field $\mathbf{b}$, i.e.
$$\partial_t u + \operatorname{div}(u\mathbf{b}) = \nu, \qquad \text{in } \mathscr{D}'((0,T) \times B) \tag{18}$$
being $u \in L^\infty((0,T) \times \mathbb{R}^2)$ and ν a Radon measure on $(0,T) \times \mathbb{R}^2$. A reduction on the non-trivial connected components of the level sets of the Hamiltonian H_B can be still performed, similarly to what we have done for equation $\operatorname{div}(u\mathbf{b}) = \mu$ in Section 3.2. Roughly speaking, we present now the time-dependent version of Lemmas 3-4-5.

Lemma 6. *The equation* (18) *is equivalent to*

- *for $\mathscr{L}^1$-a.e. h, the function $\widehat{u}_h(t,s) := u(t,\gamma_\mathfrak{c}(s))$ solves*
$$\partial_t\big(\widehat{u}_h\widehat{c}_h|\widehat{\mathbf{b}}|\big) + \partial_s\big(\widehat{u}_h\widehat{c}_h|\widehat{\mathbf{b}}|\big) = \widehat{\nu}_h, \qquad \text{in } \mathscr{D}'((0,T) \times I_\mathfrak{c}) \tag{19}$$
where $\gamma_h \colon I_\mathfrak{c} \to \mathbb{R}^2$ is an injective, Lipschitz parametrization of a connected component $\mathfrak{c}$ of the level set E_h of the Hamiltonian H and $\widehat{\nu}_h$ is a measure such that $\widehat{\nu}_h = (\gamma_\mathfrak{c}^{-1})_\# \nu$;
- *it holds*
$$\operatorname{div}(u\mathbf{b}\sigma_h) = 0. \tag{20}$$

3.4 Matching Lemma and Weak Sard Property for H_B

Before passing to the global argument, we make sure that the Hamiltonians constructed in paragraph before are *matching*, in the sense given by the following definition:

Definition 3. Assume that $B_1, B_2 \in \mathscr{B}$ are such that $B_1 \cap B_2 \neq \emptyset$. The Hamiltonians $H_1 := H_{B_1}$ and $H_2 := H_{B_2}$ *match* in $B_1 \cap B_2$ if, denoting by C_x^i denotes the connected component of the level sets $H_i^{-1}(H_i(x))$ which contains x ($i = 1,2$), it holds $C_x^1 = C_x^2$ for $\mathscr{L}^2$-a.e. $x \in B_1 \cap B_2$ such that the level sets $H_i^{-1}(H_i(x))$ are regular.

It can be proved that in our setting matching property holds, as a consequence of the fact that $\nabla H_1 \parallel \nabla H_2$ (being both parallel to $\mathbf{b}$):

Lemma 7 (Matching lemma). *Let H_1, H_2 be defined as above. Then the Hamiltonians H_1 and H_2 match in $B_1 \cap B_2$.*

It is interesting to notice that an application of Lemma 7 yields also the following remarkable piece of information:

Lemma 8. *For every $B \in \mathscr{B}$, the Hamiltonian H_B has the Weak Sard Property.*

Remark 3. If we do not assume BV regularity of $\mathbf{b}$, but $\mathbf{b} \neq 0$ a.e., then the conclusion of Lemma 8 still holds.

After having established the validity of the Weak Sard Property for the Hamiltonians H_B, we can finally implement our method of characteristics locally.

Lemma 9. *Let $B \in \mathscr{B}$ be fixed and consider $u \in L^\infty([0,T] \times \mathbb{R}^2)$ a ρ-weak solution of the problem*

$$\begin{cases} \partial_t u + \mathbf{b} \cdot \nabla u = 0, \\ u(0,\cdot) = u_0(\cdot), \end{cases} \qquad \text{in } \mathscr{D}'((0,T) \times \mathbb{R}^2). \tag{21}$$

Then there exists a negligible set $Z = Z_B \subset \mathbb{R}$ such that for any $h \notin Z$ the level set $E_h = H_B^{-1}(h)$ is regular and for any non-trivial connected component $\mathfrak{c}$ of E_h with parametrization $\gamma_\mathfrak{c} \colon I_\mathfrak{c} \to \mathbb{R}^2$solving $\gamma_\mathfrak{c}'(s) = \nabla^\perp H(\gamma_\mathfrak{c}(s))$, it holds

$$u(t, \gamma_\mathfrak{c}(s+t)) = u_0(\gamma_\mathfrak{c}(s)) \tag{22}$$

for any $s \in I_\mathfrak{c}$ and for a.e. $t \in (0,T)$ such that $s + t \in I_\mathfrak{c}$.

3.5 Level sets and trajectories

We now turn our attention to the link between the trajectories $\gamma \in \mathsf{T}_B$ and the level sets of the Hamiltonian H_B. As one may expect, the first result one can prove is that η-a.e. γ is contained in a level set. More precisely, we have the following

Lemma 10. *Up to a η_B negligible set $N \subset \Gamma$, the image of every $\gamma \in \mathsf{T}_B$ is contained in a connected component of a* regular *level set of H_B.*

Interlude: locality of the divergence on a class of L^∞ measure-divergence vector fields

Let $U \subset \mathbb{R}^d$ (for an integer $d \geq 1$) be an open set and let $\mathbf{E}\colon U \to \mathbb{R}^d$ be a vector field. Consider the following set

$$M := \left\{ x \in \mathbb{R}^d : \mathbf{E}(x) = 0, x \in \mathscr{D}_{\mathbf{E}} \text{ and } \nabla^{\mathrm{appr}}\mathbf{E}(x) = 0 \right\}, \tag{23}$$

where $\mathscr{D}_{\mathbf{E}}$ is the set of approximate differentiability points and $\nabla^{\mathrm{appr}}\mathbf{E}$ is the approximate differential, according to Definition [4, Def. 3.70]. In [5] we have proved the following

Proposition 1. *Assume that $\mathbf{E}$ is approximately differentiable a.e. and let $u \in L^\infty(U)$ be such that $\operatorname{div}(u\mathbf{E}) = \lambda$ in the sense of distributions, being λ a Radon measure on U. Then $|\lambda| \llcorner M = 0$.*

For shortness, we will refer to the property expressed in Proposition 1 as *locality of the divergence*. Notice that we do not assume any weak differentiability of u or $u\mathbf{E}$, so the conclusion of Proposition 1 does not follow immediately from the standard locality properties of the approximate derivative (see e.g. [4, Proposition 3.73]). Moreover, we point out that in [1] it can be found an example of a bounded vector field whose (distributional) divergence is in L^∞, is non-trivial but it is supported in the set where the vector field vanishes.

The property of locality of the divergence expressed in Proposition 1 holds true in our setting, being $\mathbf{b} \in \mathrm{BV}$, and this provides us with a better description of the relationship between the trajectories $\gamma \in \Gamma_B$ and the level sets of H_B, improving Lemma 10. Indeed, let $B \in \mathscr{B}$ be a fixed ball of the collection and, as usual, let H_B denote its Hamiltonian. Thanks to Lemma 10, there exists a η-negligible set N such that for every $\gamma \in \Gamma_B \setminus N$ the image $\gamma(0,T)$ is contained in a connected component $\mathfrak{c}$ of a regular level set of H_B.

We now ask the following question: which is the relationship between the trajectory $\gamma \in \Gamma_B \setminus N$ and the parametrization $\gamma_{\mathfrak{c}}$ of the corresponding connected component, given by Point 3 of Theorem 3? The answer is in the following

Proposition 2. *Let N be the set given by Lemma 10 and let $\gamma \in \mathsf{T}_B \setminus N$. Then (a suitable restriction of) γ coincides with $\gamma_{\mathfrak{c}}$ up to a translation in time.*

4 Proof II: the global argument

We now pass to analyze the second part of the proof of Theorem 1.

4.1 Covering property of the regular level sets

Let us recall that for each ball $B \in \mathscr{B}$ and for any rational numbers $s,t \in \mathbb{Q} \cap (0,T)$ with $s < t$ we have set

$$\mathsf{T}_{B,s,t} := \left\{\gamma \in \Gamma : \mathscr{L}^1(\{\tau \in [0,T] : \gamma(\tau) \in B\}) > 0,\, \gamma(s) \notin B,\, \gamma(t) \notin B\right\}$$

and we have seen in Remark 2 that the union over $B \in \mathscr{B}$ and $s,t \in \mathbb{Q} \cap [0,T]$ of the sets $\mathsf{T}_{B,s,t}$ concides with the set of non-constant trajectories. For each $B \in \mathscr{B}$, $s \in \mathbb{Q} \cap (0,T)$, $t \in \mathbb{Q} \cap (s,T)$ restricting η to $\mathsf{T}_{B,s,t}$, we can construct the local Hamiltonian $H_{B,s,t}$ as in Section 3.1. Setting now

$$E^\star := \bigcup_{\substack{B \in \mathscr{B} \\ s,t \in \mathbb{Q} \cap [0,T]}} \bigcup_{h \in H_{B,s,t}(\mathbb{R}^2)} E_h^\star \tag{24}$$

we can prove the following covering property, which shows that a.e. point where $\mathbf{b}$ is non zero is contained in a regular level set of one Hamiltonian.

Lemma 11. *It holds* $E^\star = \{\mathbf{b} \neq 0\} \mod \mathscr{L}^2$.

4.2 Selection of appropriate trajectories

We now present the following

Lemma 12. *Up to an η-negligible set, any non-constant, integral curve γ of the vector field $\mathbf{b}$ has the following properties:*

1. *for any $B \in \mathscr{B}$, if $\gamma \in \mathsf{T}_{B,s,t}$ then each connected component of $\gamma([s,t]) \cap B$ is contained in a regular level set of H_B;*
2. *for any $\tau \in (0,T)$ there exist a ball $B \in \mathscr{B}$, $s \in \mathbb{Q} \cap (0,\tau)$ and $t \in \mathbb{Q} \cap (\tau,T)$ such that $\gamma \in \mathsf{T}_{B,s,t}$.*

In view of Lemma 12, for η-a.e. non constant γ we can find a ball of the collection $\mathscr{B}$ such that a piece of γ is covered by a regular level set of H_B. Now we would apply Lemma 9 to solve equation locally in this ball. Before doing that, we need the following technical

Lemma 13. *Let $Z_{B,s,t}$ denote negligible set given by Lemma 9. Then η-a.e. $\gamma \in \Gamma$ which is non-constant satisfies*

$$H_{B,s,t}(\gamma([0,T])) \cap Z_{B,s,t} = \emptyset. \tag{25}$$

Furthermore, the endpoints $\gamma(0)$ and $\gamma(T)$ are contained in regular level sets of some Hamiltonians

$$\gamma(0) \in E^\star \qquad \text{and} \qquad \gamma(T) \in E^\star. \tag{26}$$

Using Lemma 12 and Lemma 13, we have been removing η-negligible sets of trajectories of $\mathbf{b}$. Let us summarize in the following proposition some properties of the remaining ones:

Proposition 3. *For a.e. non-constant $\gamma \in \Gamma$ and any $\tau \in [0,T]$ there exists $\delta > 0$ and a constant w such that the function $\xi \mapsto u(\xi, \gamma(\xi))$ is equal to w for a.e. $\xi \in (\tau - \delta, \tau + \delta) \cap [0,T]$. Moreover, if $\tau = 0$ then the constant w is equal to $u_0(\gamma(0))$.*

4.3 Conclusion: solutions are constant along η-a.e. trajectory

Now we are in a position to recover the method of characteristics in our weak setting. The following is the global analog of Lemma 9 and is an immediate consequence of Proposition 3 and the compactness of $[0,T]$:

Lemma 14. *Let $u \in L^\infty([0,T] \times \mathbb{R}^2)$ be a ρ-weak solution of the Problem 21. Then for η-a.e. $\gamma \in \Gamma$ for a.e. $t \in [0,T]$ it holds that*

$$u(t, \gamma(t)) = u_0(\gamma(0)).$$

Proof. Call N the set given by Proposition 3. Let $\gamma \in \Gamma \setminus N$ be a non-constant trajectory. For any $\tau \in [0,T]$ there exists $\delta > 0$ such that the function $t \mapsto u(t, \gamma(t))$ is equal to some constant w_τ for a.e. $t \in (\tau - \delta, \tau + \delta) \cap [0,T]$. Moreover, if $\tau = 0$ then $w_\tau = u_0(\gamma(0))$. It remains to extract a finite covering of $[0,T]$. □

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