A LAGRANGIAN APPROACH FOR SCALAR MULTI-D CONSERVATION LAWS

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Abstract. We introduce a notion of Lagrangian representation for entropy solutions to scalar conservation laws in several space dimension

$$\begin{cases} \partial_t u + \mathrm{div}_x(\mathbf{f}(u)) = 0 & (t,x) \in (0,+\infty) \times \mathbb{R}^d, \\ u(0,x) = u_0 & t = 0. \end{cases}$$

The construction is based on the transport collapse method introduced by Brenier. As a first application we show that if the solution u is continuous, then it is hypograph is given by the set

$$\big\{(t,x,h) : h \leq u_0(x - \mathbf{f}(h)t)\big\},$$

i.e. it is the translation of each level set of u_0 by its characteristic speed.

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1. Introduction

In a series of papers [BM14, BY15, BM16], various notions of Lagrangian representation for the entropy solution u to a scalar conservation law in one space dimension

$$\partial_t u + \partial_x f(u) = 0$$

have been introduced. The basic idea is to use the wavefront tracking and observe that the wavefronts trajectories generates a flow $\mathtt{X}(t,y)$ which is Lipschitz in times and monotone in y: this compactness allows to pass to the limit as the initial data is BV, and using the notion of admissible boundary, even for L^∞ or measure valued entropy solutions [BM17]. A series of works culminating in [BM15] extends the Lagrangian representation also to systems of conservation laws.
An important application is the proof of the structure of L^∞ solutions, and as a consequence the fact that the entropy dissipation is concentrated (see [BM17]).

Aim of this note is to obtain a suitable notion of Lagrangian representation for the multidimensional scalar equation,

$$\partial_t u + \mathrm{div}_x \, \mathbf{f}(u) = 0, \qquad \mathbf{f} : \mathbb{R} \to \mathbb{R}^d \text{ smooth.} \tag{1.1}$$

The key step is always to find an a priori compactness estimate and an approximating scheme exploiting this compactness: in this situation, the transport collapse method introduced by Brenier [Bre84].

This approximation method is based on the interpretation of the evolution of the solution as the action of two operators:

Transport map: a translation of each level set of u by the transport map

$$\mathrm{hyp}\ u(t) := \big\{(x,h) : h \leq u(t,x)\big\} \mapsto \mathsf{Tr}(s, \mathrm{hyp}\ u(t)) := \big\{(x,h) : h \leq u(t, x - f(h)s)\big\};$$

Collapse operator: the monotone mapping of each x section of a generic set $E \subset [0,+\infty) \times \mathbb{R}^d$ into an interval with the same measure,

$$(E,x,h) \mapsto \mathsf{C}(E,x,h) := \big(x, \mathscr{H}^1((\{x\} \times [0,h]) \cap E)\big).$$

This interval is clearly an hypograph of a function.

The transport collapse method is then the standard operator splitting approximation applied to the two operators Tr, C: the solution $u(t)$ to (1.1) is the limit of the solutions

$$u_n(t) = \mathsf{Tr}\big(t - [2^n t]2^{-n}, \mathrm{hyp}\ u_n([2^n t]2^{-n})\big), \quad \mathrm{Graph}\, u_n([2^n t]2^{-n}) = \big(\mathsf{C}(\mathsf{Tr}(2^{-n}, \cdot), \|u\|_\infty)\big)^{[2^n t]} \mathrm{hyp}\ u_0, \tag{1.2}$$

where $[\cdot]$ is the integer part of a real number. The composition $\mathsf{C}(\mathsf{Tr}(2^{-n},\cdot), \|u\|_\infty)$ means that given a set, one first translates the level set according to the characteristic speed for a time 2^{-n}, and then find the total length on the vertical line at each pont $x \in \mathbb{R}^d$. Observe indeed that the projection operator C assign the new position of each point in a set $E \subset \mathbb{R}^{d+1}$, and does not just yields a function. A more detailed description is given in Section 3.3.

The natural compactness appears when interpreting the transport collapse method as a map acting on the whole hypograph of a function, i.e. assigning to every initial point $(x,h) \in \mathrm{hyp}\ u_0$ a trajectory

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$(\gamma^1(t), \gamma^2(t)) \in \mathbb{R}^{d+1}$. Indeed, by inspection of (1.2), the curve $t \mapsto \gamma^1(t)$ is uniformly Lipschitz, with Lipschitz constant bounded by $\|\mathbf{f}'\|_\infty$, while the second trajectory $t \mapsto \gamma^2(t)$ is decreasing in time.
The set of trajectories described above are clearly compact in the set of $L^1_{\rm loc}([0,+\infty), \mathbb{R}^{d+1})$ functions, so that one can apply standard compactness results to prove that there exists a bounded measure ω such that

(1) it is concentrated on the the solutions to the "characteristic" ODE
$$\dot\gamma^1 = \mathbf{f}'(\gamma^2), \quad \dot\gamma^2 \le 0,$$
(2) its push-forward $p_\sharp(\mathscr{L}^1 \times \omega)$ is the measure $\mathscr{L}^{d+2} \llcorner \operatorname{hyp} u$, where
$$p(t, \gamma^1, \gamma^2) = (t, \gamma^1(t), \gamma^2(t)). \tag{1.3}$$

We can think the measure ω as a continuous version of the transport collapse operator splitting method, and following the nomenclature used in the one dimensional case, we call the measure ω a *Lagrangian representation* of the entropy solution $u(t)$.

We now state the results and the structure of the note.

After introducing the notation (Section 1.1) and some preliminary results (Section 2), in Section 3 we give the precise definition of Lagrangian representation we introduce in this note:

Definition (Definition 3.1)**.** A *Lagrangian representation* of a solution u to (1.1) is a measure $\omega \in \mathscr{M}^+(\Gamma)$ such that:

(1) it holds
$$p_\sharp(\mathscr{L}^1 \times \omega) = \mathscr{L}^{d+2} \llcorner \operatorname{hyp} u,$$
where we recall p is the projection map defined in (1.3);
(2) ω is concentrated on the set of curves $\gamma = (\gamma^1, \gamma^2) \in \Gamma$ such that
$$\begin{cases} \dot\gamma^1(t) = \mathbf{f}'(\gamma^2(t)) & \mathscr{L}^1\text{-a.e. } t \in [0,+\infty), \\ \dot\gamma^2 \le 0 \text{ in the sense of distributions.} \end{cases}$$

A not-so-surprising fact is that u is an entropy solution, as it can be surmised by the transport collapse construction.

Proposition (Proposition 3.3)**.** *Let $\omega \in \mathscr{M}_+(\Gamma)$ be a non-negative measure on the space of curves and assume there exists a non-negative, bounded function $u\colon (0,+\infty) \times \mathbb{R}^d \to [0,+\infty)$ such that Conditions (1), (2) of the above definition hold. Then u is an entropy solution to* (1.1)*.*

Additional results yield that the measures $\dot\gamma^2$ are naturally associated to the dissipation, and they are concentrated on the essential boundary of the hypograph, and that at any time t the ω-measure of the curves γ^2 which have a downward jump is 0.

Next, in Section 3.2 we show the natural compactness enjoyed by the notion of Lagrangian representation.

Proposition (Proposition 3.6)**.** *Let $(\omega^n)_{n\in\mathbb{N}} \subset \mathscr{M}_+(\Gamma)$ be a sequence of bounded measures such that Condition (2) in the above definition. Assume that*
$$p_\sharp(\mathscr{L}^1 \times \omega^n) = \mathscr{L}^{d+2} \llcorner U^n$$
for some set $U^n \subset \mathbb{R}^{d+2}$ and assume that there exists $M > 0$ such that $U^n \subset (0,+\infty) \times \mathbb{R}^d \times [0,M]$ for every $n \in \mathbb{N}$. Assume furthermore that
$$\chi_{U^n} \to \chi_U \quad \text{in } L^1(\mathbb{R}^{d+2}),$$
for some set $U \subset \mathbb{R}^{d+2}$. Then $(\omega^n)_{n\in\mathbb{N}}$ is tight, every limit point ω satisfies Condition (2) in the above definition and it holds
$$p_\sharp\big(\mathscr{L}^1 \times \omega) = \mathscr{L}^{d+2} \llcorner U.$$

Using this compactness, first one shows the existence of a Lagrangian representation for a BV entropy solution (Proposition 3.11), and then the general case:

Theorem (Theorem 3.12)**.** *Let u be the entropy solution to the initial value problem* (1.1) *with $u(t=0) = u_0 \in L^\infty(\mathbb{R}^d)$. Then there exists a Lagrangian representation of u.*

The note is concluded with a first application of the above construction (Section 4).

Theorem (Theorem 4.3)**.** *Let u be a continuous bounded entropy solution in $[0,T) \times \mathbb{R}^d$ to* (1.1)*. Then for every $(t,x) \in [0,T) \times \mathbb{R}^d$, it holds*
$$u(t,x) = u_0(x - \mathbf{f}'(u(t,x))t).$$
Moreover for every $\eta : \mathbb{R} \to \mathbb{R}$, $\mathbf{q} : \mathbb{R} \to \mathbb{R}^d$ Lipschitz such that $\mathbf{q}' = \eta'\mathbf{f}'$ a.e. with respect to $\mathscr{L}^1$, it holds
$$\eta(u)_t + \operatorname{div}_x \mathbf{q}(u) = 0$$
in the sense of distributions.

This is a corollary of the fact that the Lagrangian representation in this case is unique beacuse it satisfies $\dot{\gamma}^2 = 0$ (Proposition 4.2). In particular its graph is a bundle of characteristic curves as in the one dimensional case.

1.1. **Notations.** In the following, if $f\colon X \to [0, +\infty)$ is a non-negative function defined on some set X, we will denote its *hypograph* by

$$\operatorname{hyp} f := \{(x, h) \in X \times [0, +\infty) : 0 \leq h \leq f(x)\}.$$

Conversely, if $U \subset X \times [0, +\infty)$ we will use the notation

$$\operatorname{hyp}^{-1}(U) = f \tag{1.4}$$

to indicate that the set U is the hypograph of the function f. The power set of X will be denoted by $\mathcal{P}(X)$.

If X is a measurable space, the space of finite measures over X will be written as $\mathscr{M}(X)$ and as usual the total variation is defined for every measurable $E \subset X$ as

$$|\mu|(E) := \sup \left\{ \sum_{i=1}^{k} |\mu(E_i)| : \ E_i \cap E_j = \emptyset \text{ for } i \neq j, \ \bigcup_{i=1}^{k} E_i = E \right\}.$$

The norm of a measure $\mu \in \mathscr{M}(X)$ will be written as $\|\mu\|_{\mathscr{M}} := |\mu|(X)$. The space of non-negative measures over X will be written as $\mathscr{M}^+(X)$.

Often we will consider X to be the d-dimensional Euclidean space $\mathbb{R}^d$ or a suitable space of curves that will be denoted by Γ. In the former case, $\mathscr{L}^d$ will be the Lebesgue measure and $\mathscr{H}^{d-1}$ the $(d-1)$-dimensional Hausdorff measure; in the latter, elements of the space and measures will be generically denoted by greek letters, namely we will use γ for a generic curve and ω for a measure on the space of curves. Recall also that there are natural "projection" operators defined on the space of curves, namely the *evaluation map* at time $t > 0$

$$\begin{aligned} e_t \colon \Gamma &\to \mathbb{R}^d \\ \gamma &\mapsto \gamma(t) \end{aligned} \tag{1.5}$$

and

$$\begin{aligned} p \colon (0, +\infty) \times \Gamma &\to (0, +\infty) \times \mathbb{R}^{d+1} \\ (t, \gamma) &\mapsto \big(t, \gamma(t)\big). \end{aligned} \tag{1.6}$$

Usually, the curves we will consider are not necessarily continuous, but they enjoy BV regularity. Accordingly, we will use the symbols $\gamma(t\pm)$ for the right/left limits at t; for the derivative we will write

$$D_t \gamma = \widetilde{D}_t \gamma + D_t^{\mathrm{j}} \gamma \tag{1.7}$$

where $\widetilde{D}_t \gamma$ is the *continuous* (or diffuse) part and $D_t^{\mathrm{j}} \gamma$ is the *jump* part.

Finally, we will use the standard language of measure theory. In particular, a.e. (if not otherwise stated) refers to the Lebesgue measure. The Lebesgue spaces are denoted in the usual way L^p and the notation L^p_+ will be used for the space of non-negative functions with integrable p-power. The *essential interior* of a set $\Omega \subset \mathbb{R}^d$, $\operatorname{ess\,Int}(\Omega)$, is the set of points $x \in \mathbb{R}^d$ for which there exists a Lebesgue negligible set N such that $x \in \operatorname{Int}(\Omega \cup N)$, being Int the standard topological interior.

2. Preliminaries

Lemma 2.1. *Let $I = [a, b] \subset \mathbb{R}$ be a closed interval in $\mathbb{R}$. Let $(D_n)_n$ be an increasing sequence of finite sets $D_1 \subset D_2 \subset \ldots \subset I$ such that their union*

$$D := \bigcup_n D_n$$

is dense in I. Let moreover $(f_n)_{n \in \mathbb{N}}$ be a sequence of maps $f_n \colon I \to X$ where (X, d) is a complete metric space. Assume that:

(1) *$a \in D_1$;*
(2) *there exists a compact set $K \subset X$ such that for every $n, m \in \mathbb{N}$ with $n \leq m$ and for every $q \in D_n$, $f_m(q) \in K$;*
(3) *there exists a constant $C > 0$ such that for every $n, m \in \mathbb{N}$ with $n \leq m$, for every $q \in D_n$ and for every $x \in I$ with $q < x$ it holds*

$$d\big(f_m(q), f_m(x)\big) \leq C(x - q).$$

Then there exist a subsequence $(n_k)_k$ and a C-Lipschitz function $f\colon I \to X$ such that

$$f_{n_k} \to f \qquad \text{uniformly on } I \text{ as } k \to +\infty.$$

Proof. By condition (2) and the standard diagonal argument there exists a subsequence f_{n_k}, that we will denote by f_k, which converges pointwise in D. Therefore, for every $q \in D$, the sequence $(f_k(q))_{k\in\mathbb{N}}$ is a Cauchy sequence in X. Since D_n is finite for every $n \in \mathbb{N}$, the convergence is uniform on each D_n. In particular for every $n \in \mathbb{N}$, there exists $N_n : [0, +\infty) \to \mathbb{N}$ such that for every $\varepsilon > 0$, for every $l, m \geq N_n(\varepsilon)$ and for every $q \in D_n$, it holds $d(f_l(q), f_m(q)) \leq \varepsilon$.

Now we prove that actually the sequence $(f_k)_{k\in\mathbb{N}}$ is a Cauchy sequence with respect to the sup-norm. Fix $\varepsilon > 0$. Then by Condition (1), the monotonicity of the sequence $(D_n)_{n\in\mathbb{N}}$ and the density of $D \subset I$ there exists $\bar{n}$ such that for every $x \in I$ there exists $q \in D_{\bar{n}}$ such that $0 < x - q < \varepsilon$. Then for every $l, m \geq \bar{n} \vee N_{\bar{n}}(\varepsilon)$, it holds

$$\begin{aligned} d(f_l(x), f_m(x)) &\leq d(f_l(x), f_l(q)) + d(f_l(q), f_m(q)) + d(f_m(q), f_m(x)) \\ &\leq C(x-q) + \varepsilon + C(x-q) \\ &\leq (2C+1)\varepsilon. \end{aligned}$$

Therefore the sequence f_k converges uniformly to a function f. Now we check that f is C-Lipschitz. For every $x, y \in I$ with $x < y$ and for every $q \in D$ with $q < x$, it holds

$$\begin{aligned} d(f(x), f(y)) &\leq d(f(x), f(q)) + d(f(q), f(y)) \\ &\leq C(x - q + y - q). \end{aligned}$$

Letting $q \to x$ from below we get that f is C-Lipschitz and this concludes the proof. □

We will also need the following standard result in the theory of sets of finite perimeter.

Lemma 2.2. *Let $E \subset \mathbb{R}^d$ be a set of finite measure and of finite perimeter and let $v \in \mathbb{R}^d$ with $|v| = 1$. Then for every $\bar{t} \geq 0$ if $E_{\bar{t}v} := \{x + \bar{t}v : x \in E\}$ it holds*

$$\mathscr{L}^d(E \,\Delta\, E_{\bar{t}v}) \leq 2\bar{t}\,\mathrm{Per}(E).$$

Proof. By Anzellotti-Giaquinta Theorem [AFP00, Theorem 3.9] there exists a sequence $(u_n)_{n\in\mathbb{N}} \subset C^\infty \cap W^{1,1}(\mathbb{R}^d)$ such that $u_n \to \chi_E$ in $L^1(\mathbb{R}^d)$ and $Du^n \rightharpoonup D\chi_E$ in duality with continuous, bounded functions over $\mathbb{R}^d$ and $\|Du^n\| \to \|D\chi_E\|$. We want to compute

$$\mathscr{L}^d(E \,\Delta\, E_{tv}) = 2\int_{\mathbb{R}^d} (1 - \chi_E(x))\chi_{E_{tv}}(x)\,dx.$$

Now we set

$$g_n(t) := \int_{E^c} u^n(x - tv)\,dx, \qquad g(t) := \int_{E^c} \chi_{E_{tv}}(x)\,dx.$$

For $\phi \in C_c^\infty((0, +\infty))$ we have

$$-\langle D_t g_n, \phi\rangle = \int_0^{+\infty}\int_{E^c} u^n(x - tv)\,\phi'(t)\,dx\,dt = \int_{E^c}\int_0^{+\infty} \nabla u^n(x - tv)\cdot v\,\phi(t)\,dt\,dx.$$

This shows that

$$D_t g_n = -\int_{E^c} \nabla u^n(x - tv)\cdot v\,dx.$$

In particular,

$$|D_t g_n| \leq \int_{E^c} |\nabla u^n(x - tv)\cdot v|\,dx \leq \|Du^n\|.$$

We thus have

$$g_n(\bar{t}) - g_n(0) \leq \int_0^{\bar{t}} \|Du^n\| dt = \bar{t}\|Du^n\|.$$

By observing that $g_n \to g$ pointwise and using that $\|Du^n\| \to \|D\chi_E\| = \mathrm{Per}\,E$ we conclude the proof. □

3. Lagrangian representation

We consider *scalar multidimensional conservation laws*, i.e. first order partial differential equations of the form

$$\partial_t u + \mathrm{div}_x\,(\mathbf{f}(u)) = 0 \qquad \text{in } (0, +\infty)\times\mathbb{R}^d, \tag{3.1}$$

where $u \colon (0, +\infty)\times\mathbb{R}^d \to \mathbb{R}$ is a scalar function and $\mathbf{f} \colon \mathbb{R} \to \mathbb{R}^d$ is a smooth map, called the *flux function*.

3.1. **Definition and properties of the Lagrangian representation.** Since we only consider L^∞ solutions, up to a translation in the flux $\mathbf{f}$, we can assume $u \geq 0$. We denote by

$$\Gamma := \Big\{\gamma = (\gamma^1, \gamma^2)\colon (0,+\infty) \to \mathbb{R}^d \times [0,+\infty) : \gamma^1 \text{ is continuous and } \gamma^2 \text{ is decreasing}\Big\}$$

equipped with the product of the uniform convergence on compact sets topology and of the L^1_{loc}-topology.

Definition 3.1. A *Lagrangian representation* of a solution u to (3.1) is a measure $\omega \in \mathscr{M}^+(\Gamma)$ such that:

(1) it holds
$$p_\sharp(\mathscr{L}^1 \times \omega) = \mathscr{L}^{d+2} \llcorner \operatorname{hyp} u, \tag{3.2}$$
where we recall p is the projection map defined in (1.6);

(2) ω is concentrated on the set of curves $\gamma = (\gamma^1, \gamma^2) \in \Gamma$ such that
$$\begin{cases} \dot\gamma^1(t) = \mathbf{f}'(\gamma^2(t)) & \mathscr{L}^1\text{-a.e. } t \in [0,+\infty), \\ \dot\gamma^2 \leq 0 \text{ in the sense of distributions.} \end{cases} \tag{3.3}$$

The following lemma shows that the condition expressed in (3.2) is equivalent to its pointwise version.

Lemma 3.2. *Assume that $t \mapsto u(t)$ is strongly continuous in L^1. Then in Definition 3.1, Condition (1) can be replaced with the following:*

(1') *for every $t > 0$, it holds*
$$e_{t\sharp}\omega = \mathscr{L}^{d+1} \llcorner \operatorname{hyp} u(t), \tag{3.4}$$
where we recall e_t is the evaluation map defined in (1.5).

Proof. Condition (1') clearly implies (1). On the other hand, by Fubini, condition (1) gives that (3.4) for $\mathscr{L}^1$-a.e. t. By exploiting the L^1-continuity in time of entropy solutions u, we now show that (3.4) holds indeed for every $t \in [0,+\infty)$. To do this, we write $\gamma(t) = (\gamma^1(t), \gamma^2(t))$ and we fix $\bar t$; we take as test function the following

$$\varphi(t,x,h) = \phi(x,h)\psi_\delta(t)$$

where $\phi\colon \mathbb{R}^{d+1} \to \mathbb{R}$ is arbitrary, $\psi_\delta\colon [0,+\infty) \to \mathbb{R}$ is a non negative smooth function, with $\operatorname{supp}\psi_\delta \subset (\bar t, \bar t + \delta)$ and $\int_{\mathbb{R}^+} \psi_\delta = 1$. Taking the limit as $\delta \to 0^+$ of (3.2) tested against φ, we have

$$\int_{\mathbb{R}^{d+1}} \phi(x,h)\, d\mathscr{L}^{d+1} \llcorner \operatorname{hyp} u(\bar t) = \int_\Gamma \phi(\gamma(\bar t+))\, d\omega$$

where $\gamma(\bar t+)$ denotes the right limit (which exists because γ^1 is continuous and γ^2 is decreasing). Similarly, on the left side, we get

$$\int_{\mathbb{R}^{d+1}} \phi(x,h) d\mathscr{L}^{d+1} \llcorner \operatorname{hyp} u(\bar t) = \int_\Gamma \phi(\gamma(\bar t-))\, d\omega$$

thus, in particular,

$$0 = \int_\Gamma \phi\big(\gamma^1(\bar t), \gamma^2(\bar t-)\big) - \phi\big(\gamma^1(\bar t), \gamma^2(\bar t+)\big)\, d\omega.$$

Let us fix a compact set $K \subset \mathbb{R}^d$ and choose $\phi \in C^\infty_c(\mathbb{R}^{d+1})$ such that $\partial_h \phi \geq 1$ in $K \times (0, \|u\|_\infty)$ and $\partial_h\phi \geq 0$ in $\mathbb{R}^d \times (0, \|u\|_\infty)$: being γ^2 decreasing, we have

$$\begin{aligned} 0 &= \int_\Gamma \phi\big(\gamma^1(\bar t), \gamma^2(\bar t-)\big) - \phi\big(\gamma^1(\bar t), \gamma^2(\bar t+)\big)\, d\omega \\ &\geq \int_{\Gamma\setminus\Gamma_K} \phi\big(\gamma^1(\bar t), \gamma^2(\bar t-)\big) - \phi\big(\gamma^1(\bar t), \gamma^2(\bar t+)\big)\, d\omega + \int_{\Gamma_K} \big(\gamma^2(\bar t-) - \gamma^2(\bar t+)\big)\, d\omega \\ &\geq \int_{\Gamma_K} |\gamma^2(\bar t-) - \gamma^2(\bar t+)|\, d\omega, \end{aligned}$$

where $\Gamma_K \subset \Gamma$ is the set of curves such that $\gamma^1(\bar t) \in K$. This shows that for every $t \in (0,+\infty)$, ω-a.e. γ is continuous in t: in particular, we have $(e_t)_\sharp\omega = \mathscr{L}^{d+1} \llcorner \operatorname{hyp} u(t)$ for every t. □

We now present the following proposition, which says that Conditions (1), (2) in Definition (3.1) imply that u is an entropy solution to (3.1).

Proposition 3.3. *Let $\omega \in \mathscr{M}_+(\Gamma)$ be a non-negative measure on the space of curves and assume there exists a non-negative, bounded function $u\colon (0,+\infty) \times \mathbb{R}^d \to [0,+\infty)$ such that Conditions (1), (2) of Definition 3.1 hold. Then u is an entropy solution to (3.1).*

Proof. Let $(\eta, \mathbf{q})$ be an entropy-entropy flux pair with η convex (w.l.o.g. $\eta(0) = 0, \mathbf{q}(0) = 0$). Using the elementary identities

$$u(t,x) = \int_0^{+\infty} \chi_{[0,u(t,x)]}(h)\, dh$$

and

$$\eta(u(t,x)) = \int_0^{+\infty} \chi_{[0,u(t,x)]}(h)\eta'(h)\, dh, \qquad \mathbf{q}(u(t,x)) = \int_0^{+\infty} \chi_{[0,u(t,x)]}(h)\mathbf{q}'(h)\, dh$$

and recalling that $\mathbf{q}' = \eta'\mathbf{f}'$, we can write, for any non-negative test function $\phi \in C^1_c([0,+\infty) \times \mathbb{R}^d)$,

$$\begin{aligned}
-\langle \eta(u)_t + \mathrm{div}_x(\mathbf{q}(u)), \phi\rangle &= \int_{\mathbb{R}^d}\int_0^{+\infty} \eta(u(t,x))\phi_t(t,x) + \mathbf{q}(u(t,x))\cdot\nabla_x\phi(t,x)\, dt\, dx \\
&= \int_{\mathbb{R}^d}\int_0^{+\infty}\left[\int_0^{+\infty}\chi_{[0,u(t,x)]}(h)\eta'(h)\phi_t(t,x) + \mathbf{q}'(h)\cdot\nabla_x\phi(t,x)\, dh\right] dt\, dx \\
&= \int_{\mathbb{R}^d}\int_0^{+\infty}\int_0^{+\infty}\chi_{[0,u(t,x)]}(h)\eta'(h)\big(\phi_t(t,x) + \mathbf{f}'(h)\cdot\nabla_x\phi(t,x)\big)\, dh\, dt\, dx \\
&= \int_{\mathbb{R}^{d+2}} \eta'(h)\big(\phi_t(t,x) + \mathbf{f}'(h)\cdot\nabla_x\phi(t,x)\big)\, d\big(\mathscr{L}^{d+2}\llcorner \mathrm{hyp}\ u\big).
\end{aligned}$$

By Condition (1) we have $p_\sharp(\mathscr{L}^1 \times \omega) = \mathscr{L}^{d+2}\llcorner \mathrm{hyp}\ u$, so that

$$\begin{aligned}
-\langle \eta(u)_t + \mathrm{div}_x(\mathbf{q}(u)), \phi\rangle &= \int_{\mathbb{R}^{d+2}} \eta'(h)\big(\phi_t(t,x) + \mathbf{f}'(h)\cdot\nabla_x\phi(t,x)\big)\, d\big(\mathscr{L}^{d+2}\llcorner \mathrm{hyp}\ u\big) \\
&= \int_\Gamma\int_0^{+\infty} \eta'(\gamma^2(t))\Big(\phi_t(t,\gamma^1(t)) + \mathbf{f}'(\gamma^2(t))\cdot\nabla_x\phi(t,\gamma^1(t)\Big)\, dt\, d\omega.
\end{aligned}$$

Moreover, let us define for a.e. $t \in (0,+\infty)$ and for ω-a.e. γ the function

$$g_\gamma(t) := \eta'\big(\gamma^2(t)\big). \tag{3.5}$$

Recall that η is convex and that for ω-a.e. γ the function γ^2 is decreasing by Condition (2); thus we have that g_γ is decreasing for ω-a.e. γ. Hence it holds $g'_\gamma \le 0$ in the sense of distributions. By Fubini Theorem, we finally have

$$\begin{aligned}
-\langle \eta(u)_t + \mathrm{div}_x(\mathbf{q}(u)), \phi\rangle &= \int_\Gamma\int_0^{+\infty} \eta'(\gamma^2(t))\Big(\phi_t\big(t,\gamma^1(t)\big) + \mathbf{f}'\big(\gamma^2(t)\big)\cdot\nabla_x\phi\big(t,\gamma^1(t)\big)\Big)\, dt\, d\omega \\
&= \int_\Gamma\int_0^{+\infty} \eta'(\gamma^2(t))\Big(\phi_t\big(t,\gamma^1(t)\big) + \dot\gamma^1(t)\cdot\nabla_x\phi\big(t,\gamma^1(t)\big)\Big)\, dt\, d\omega \\
&= \int_\Gamma\int_0^{+\infty} \eta'(\gamma^2(t))\frac{d}{dt}\phi(t,\gamma^1(t))\, dt\, d\omega \\
&= \int_\Gamma\int_0^{+\infty} g_\gamma(t)\phi'_\gamma(t)\, dt\, d\omega \ge 0
\end{aligned} \tag{3.6}$$

where the last inequality comes from the distributional definition of derivative for the function g_γ, being $\phi_\gamma(t) := \phi(t,\gamma^1(t))$ an admisible, non-negative test function. Thus we have established that, for any convex entropy η, it holds in the sense of distributions

$$\eta(u)_t + \mathrm{div}_x(\mathbf{q}(u)) \le 0. \tag{3.7}$$

In particular, by taking $\eta(s) = \pm s$ and repeating the computation above, we get

$$u_t + \mathrm{div}_x(\mathbf{f}(u)) = 0. \tag{3.8}$$

Having established the two conditions (3.7) and (3.8), we have that u is by definition an entropy solution to (3.1), hence the proof is complete. $\square$

This proof shows also how the dissipation measure can be decomposed along the characteristic curves. Since this fact will be useful, we fix some notation and explicit this decomposition.

Let η be a convex entropy and set

$$\mu^\eta_\gamma = (\mathbb{I},\gamma)_\sharp\Big((\eta'\circ\gamma^2)\widetilde{D}\gamma^2\Big) + \eta''(w)\mathscr{H}^1\llcorner\{(t,x,w) : \gamma^1(t) = x, w \in (\gamma^2(t+),\gamma^2(t-))\}.$$

Accordingly define

$$\nu^\eta := \int_\Gamma \mu^\eta_\gamma\, d\omega. \tag{3.9}$$

Lemma 3.4. *It holds*

$$(\pi_{t,x})_\sharp \nu^\eta = \mu^\eta,$$

where the map $\pi_{t,x} \colon \mathbb{R}^d \times [0,+\infty) \times [0,+\infty) \ni (t,x,h) \mapsto (t,x) \in \mathbb{R}^d \times [0,+\infty)$ *is the projection on the* t,x *variables.*

Proof. By definition we immediately get

$$(\pi_{t,x})_\sharp(\mu_\gamma) = (\mathbb{I},\gamma^1)_\sharp(D_t g_\gamma), \tag{3.10}$$

where g_γ is defined in (3.5). Including (3.10) in (3.6) we get

$$\begin{aligned}\langle \eta(u)_t + \mathrm{div}_x(\mathbf{q}(u)), \phi\rangle &= -\int_\Gamma \int_0^{+\infty} g_\gamma(t)\phi'_\gamma(t)\,dt\,d\omega \\ &= \int_\Gamma \int_{[0,+\infty)\times\mathbb{R}^d} \phi\, d((\pi_{t,x})_\sharp \mu_\gamma) d\omega \\ &= \int_{[0,+\infty)\times\mathbb{R}^d} \phi\, d((\pi_{t,x})_\sharp \nu^\eta),\end{aligned}$$

where in the last inequality we used the definition of ν (3.9) and the relation

$$\int_\Gamma (\pi_{t,x})_\sharp \mu^\eta_\gamma\, d\omega = (\pi_{t,x})_\sharp \left(\int_\Gamma \mu^\eta_\gamma\, d\omega\right). \qquad \square$$

Proposition 3.5. *The dissipation* ν *in the essential interior of* hyp u *is zero.*

Proof. Let $\psi : \mathbb{R}^d \times [0,+\infty) \to [0,+\infty)$ such that for every $t \in (t_1,t_2)$, $\mathrm{supp}\,\psi \subset \mathrm{ess\,Int}(\mathrm{hyp}\ u(t), \mathbb{R}^d \times [0,+\infty))$, then

$$t \mapsto \int \psi(x,w)\, d(e_t)_\sharp \omega$$

is constant. Take $(\bar t, \bar x, \bar w)$ in the essential interior of hyp u. Take $\psi(x,w) = \psi_1(x)\psi_2(w)$, where

$$\psi_1(x) = \sigma(|x-\bar x|), \qquad \partial_w \psi_2 < 0 \text{ in } [0,\bar w) \text{ and } \psi_2(w) = 0 \text{ for } w > \bar w,$$

where σ is smooth and nonnegative and $\sigma > 0$ in $[0,r)$, where $r \ll 1$. For every $\phi \in C^1_c((t_1,t_2))$, it holds

$$\begin{aligned}0 &= -\int_{t_1}^{t_2}\int \phi'(t)\psi(x,w) d(e_t)_\sharp \omega dt \\ &= \int\int \phi(t) d(D_t(\psi\circ\gamma)) d\omega \\ &= \int_\Gamma \int_{(t_1,t_2)} \phi(t)\nabla\psi(\gamma(t)) d(\widetilde{D}_t\gamma) + \int_\Gamma \sum_i \phi(t_i)\left(\psi(\gamma(t_i^+)) - \psi(\gamma(t_i^-))\right) d\omega,\end{aligned}$$

by Volpert chain rule, where $\widetilde{D}_t\gamma$ is the continuous part of the derivative defined in (1.7). For every $\phi \ge 0$, and using the assumptions on ψ

$$\begin{aligned}\int_\Gamma \int_{(t_1,t_2)} \phi(t)\nabla\psi(\gamma(t)) d(\widetilde{D}_t\gamma) = &\int_\Gamma \int_{t_1}^{t_2} \phi(t)\nabla_x\psi(\gamma(t)) \cdot \mathbf{f}'(\gamma^2(t)) dt d\omega \\ &+ \int_\Gamma \int_{t_1}^{t_2} \phi(t)\partial_w\psi(\gamma(t)) d(\widetilde{D}_t\gamma^2).\end{aligned}$$

by splitting horizontal and vertical components. We prove that the horizontal contribution is zero.

$$\begin{aligned}\int_\Gamma \int_{t_1}^{t_2} \phi(t)\nabla_x\psi(\gamma(t)) \cdot \mathbf{f}'(\gamma^2(t)) dt d\omega &= \int_{\mathbb{R}^{d+1}} \int_{t_1}^{t_2} \phi(t)\nabla_x\psi(x,w)\cdot \mathbf{f}'(w) dt d\mathscr{L}^{d+1} \llcorner (\mathrm{hyp}\ u(t)) \\ &= \int_{t_1}^{t_2} \phi(t) \int_0^{+\infty} \mathbf{f}'(w) \cdot \int_{B_r(\bar x)} \nabla_x \psi(x,w) d\mathscr{L}^d dw dt \\ &= 0.\end{aligned}$$

We conclude that

$$\begin{aligned}0 &= -\int_{t_1}^{t_2}\int \phi'(t)\psi(x,w)\, d(e_t)_\sharp \omega\, dt \\ &= \int_\Gamma \int_{t_1}^{t_2} \phi(t)\partial_w\psi(\gamma(t))\, d(\widetilde{D}_t\gamma^2) + \int_\Gamma \sum_i \phi(t_i)\left(\psi(\gamma(t_i^+)) - \psi(\gamma(t_i^-))\right) d\omega \\ &= \int_{\mathbb{R}^{d+2}} \phi(t)\partial_w\psi\, d\nu.\end{aligned}$$

By arbitrariness of ϕ,ψ or by using $\nu \le 0$ we get $\nu = 0$ in the interior of the hypograph. $\square$

3.2. **Compactness and stability of Lagrangian representations.** We now turn to analyze stability properties that, in particular, will be useful in the construction of Lagrangian representations. In the following proposition, we show how the compactness of approximate solutions translates into tightness of the corresponding Lagrangian measures and how conditions (1) and (2) pass to the limit.

Actually, we present the result in the more general framework in which the push forward of the measure $\mathscr{L}^1 \times \omega$ through the evaluation map p is merely the Lebesgue measure $\mathscr{L}^{d+2}$ restricted to a set U, and not necessarily an hypograph. This allows more freedom in the construction of approximate solutions (e.g. Brenier's Transport-Collapse scheme will fit in this setting).

Proposition 3.6 (Compactness and stability)**.** *Let $(\omega^n)_{n\in\mathbb{N}} \subset \mathscr{M}_+(\Gamma)$ be a sequence of bounded measures such that Condition (2) in Definition 3.1 holds. Assume that*

$$p_\sharp(\mathscr{L}^1 \times \omega^n) = \mathscr{L}^{d+2} \llcorner U^n$$

for some set $U^n \subset \mathbb{R}^{d+2}$ and assume that there exists $M > 0$ such that $U^n \subset (0, +\infty) \times \mathbb{R}^d \times [0, M]$ for every $n \in \mathbb{N}$. Assume furthermore that

$$\chi_{U^n} \to \chi_U \qquad \text{in } L^1(\mathbb{R}^{d+2}),$$

for some set $U \subset \mathbb{R}^{d+2}$. Then $(\omega^n)_{n\in\mathbb{N}}$ is tight, every limit point ω satisfies Condition (2) in Definition 3.1 and it holds

$$p_\sharp(\mathscr{L}^1 \times \omega) = \mathscr{L}^{d+2} \llcorner U.$$

Proof. Since ω^n satisfies Condition (2) in Definition 3.1, we have that

$$\operatorname{supp} \omega^n \subset \operatorname{Lip}((0, +\infty), \mathbb{R}^d) \times \mathcal{D}$$

with local uniform bounds, hence $(\omega^n)_n$ is locally tight. Using a diagonal argument, we construct a measure ω which is the limit of ω^n. We now show that

$$p_\sharp(\mathscr{L}^1 \times \omega) = \mathscr{L}^{d+2} \llcorner U.$$

where p is the evaluation map defined in (1.6). Indeed, let $\varphi = \varphi(t, x, h)$ be a test function; we get

$$\begin{aligned}
\int_{\mathbb{R}^+ \times \mathbb{R}^{d+1}} \varphi(t, x, h)\, dp_\sharp(\mathscr{L}^1 \times \omega)(t, x, h) &= \int_\Gamma \int_{\mathbb{R}^+} \varphi(t, \gamma(t))\, dt\, d\omega \\
&= \int_\Gamma \Phi(\gamma) d\omega(\gamma) \\
&= \lim_n \int_\Gamma \Phi(\gamma) d\omega^n(\gamma) \\
&= \lim_n \int_\Gamma \int_{\mathbb{R}^+} \varphi(t, \gamma(t))\, dt\, d\omega^n \\
&= \lim_n \int_{\mathbb{R}^+ \times \mathbb{R}^{d+1}} \varphi(t, x, h)\, dp_\sharp(\mathscr{L}^1 \times \omega^n) \\
&= \lim_n \int_{\mathbb{R}^+ \times \mathbb{R}^{d+1}} \varphi(t, x, h)\, d(\mathscr{L}^{d+2} \llcorner U^n) \\
&= \int_{\mathbb{R}^+ \times \mathbb{R}^{d+1}} \varphi(t, x, h)\, d(\mathscr{L}^{d+2} \llcorner U),
\end{aligned}$$

where we have used in the second line the continuous function

$$\Phi(\gamma) := \int_0^{+\infty} \phi(t, \gamma(t))\, dt.$$

□

We conclude this paragraph by pointing out the following corollary, whose proof can be obtained particularizing Proposition 3.6 in the case where U^n are hypographs of entropy solutions.

Corollary 3.7. *Let $(u^n)_{n\in\mathbb{N}}$ be a sequence of uniformly bounded entropy solutions to (3.1) and assume it is given a sequence $(\omega^n)_{n\in\mathbb{N}}$ of corresponding Lagrangian representations. If $u^n \to u$ locally in L^1, then $(\omega^n)_{n\in\mathbb{N}}$ is tight and every limit point ω is a Lagrangian representation of u.*

3.3. **Existence of Lagrangian representations for initial data in L^∞.** The compactness properties stated in Corollary 3.7 and standard approximation results imply that, in order to prove the existence of Lagrangian representations for solutions with initial data in L^∞, it is enough to construct them for solutions with bounded variation. In order to do this, we exploit a numerical scheme which was proposed by Brenier in [Bre84] and is called "transport-collapse". We consider the initial value problem

$$\begin{cases} \partial_t u + \operatorname{div}_x(\mathbf{f}(u)) = 0 & \text{in } (0, +\infty) \times \mathbb{R}^d, \\ u(0, \cdot) = u_0(\cdot) \end{cases} \tag{3.11}$$

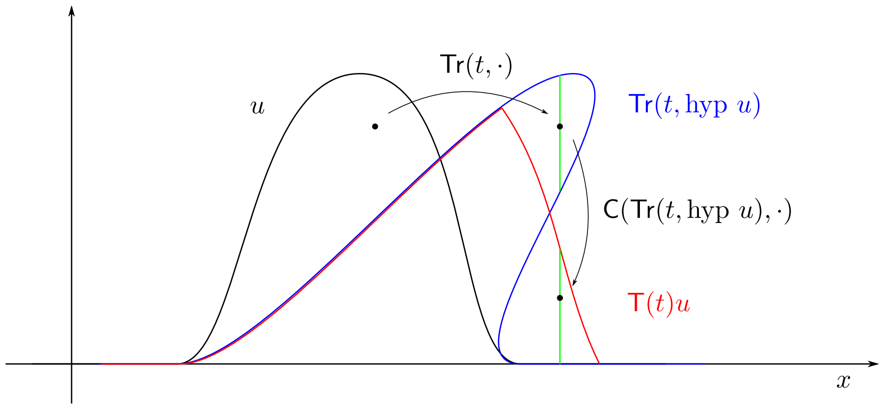

FIGURE 1. Picture of the transport collapse scheme.

with $u_0 \in L^\infty \cap \mathrm{BV}_{\mathrm{loc}}(\mathbb{R}^d)$ and we denote by u the entropy solution to (3.11). As before, we assume that $u \geq 0$.

We define the following *transport* map

$$\begin{aligned} \mathsf{Tr}\colon [0,+\infty) \times \mathbb{R}^d \times [0,+\infty) &\to \mathbb{R}^d \times [0,+\infty) \\ (t,x,h) &\mapsto (x + t\mathbf{f}'(h), h) \end{aligned}$$

which moves a point in $\mathbb{R}^d \times [0,+\infty)$ with the characteristic speed. Observe that, in general, if $v = v(x)$ is a function of x then, for $t > 0$, the image

$$\mathsf{Tr}(t, \text{hyp } v) := \bigcup_{(x,h)\in \text{hyp } v} \mathsf{Tr}(t,x,h) \subset \mathbb{R}^d \times [0,+\infty)$$

is not necessarily an hypograph.

Then we introduce the *collapse* operator: we first define the set

$$X := \left\{(E,x,h) \subset \mathcal{P}(\mathbb{R}^d \times [0,+\infty)) \times \mathbb{R}^d \times [0,+\infty) : (x,h) \in E\right\},$$

where we recall $\mathcal{P}$ denotes the power set and then

$$\begin{aligned} \mathsf{C}\colon X &\mapsto \mathbb{R}^d \times [0,+\infty) \\ (E,x,h) &\mapsto \big(x, \mathscr{H}^1((\{x\} \times [0,h]) \cap E)\big) \end{aligned}$$

where $\mathcal{H}^1$ is the (outer) 1-dimensional Hausdorff measure. The collapse operator moves points vertically in the negative direction. Moreover the image of a set is always an hypograph (possibly taking value $+\infty$) and $\mathsf{C}(E,\cdot,\cdot)$ is the identity if and only if E is an hypograph.

We now set

$$Y := \Big\{(v,x,h) \in L^\infty_+(\mathbb{R}^d) \times \mathbb{R}^d \times [0,+\infty) : (x,h) \in \text{hyp } v\Big\}.$$

We define the transport-collapse map at time $t > 0$ in the following way:

$$\begin{aligned} \mathsf{TC}_t\colon Y &\to \mathbb{R}^d \times [0,+\infty) \\ (v,x,h) &\mapsto \mathsf{C}(\mathsf{Tr}(t, \text{hyp } v), \mathsf{Tr}(t,x,h)) \end{aligned}$$

Remark 3.8. The contruction above is only a Lagrangian rephrase of the Transport-Collapse scheme proposed by Brenier in [Bre84]. There, the author defines the Transport-Collapse operator as the family of operators $\{\mathsf{T}(t)\}_{t>0}$ on $L^1(\mathbb{R}^d)$ whose restriction to the space of non-negative, integrable functions $L^1_+(\mathbb{R}^d)$ is

$$\begin{aligned} \mathsf{T}(t)\colon L^1_+(\mathbb{R}^d) &\to L^1_+(\mathbb{R}^d) \\ v &\mapsto (\mathsf{T}(t)v)(x) := \int_{\mathbb{R}} jv(x - t\mathbf{f}'(h), h)\, dh \end{aligned}$$

where

$$jv(x,h) := \chi_{\text{hyp } v}(x,h) = \begin{cases} 1 & \text{if } 0 < h < v(x), \\ 0 & \text{else.} \end{cases}$$

The link between the two formulations is the following:

$$\text{hyp } (\mathsf{T}(t)v) = \mathsf{TC}_t(v, \text{hyp } v).$$

On the other hand, the map TC_t chooses the image of *each* point in the hypograph and not only the image of the whole hypograph (see Figure 1) .

We are now in position to define an approximating sequence (TC^n_t) of the Kruzkov semigroup. We define first them inductively for $t \in 2^{-n}\mathbb{N}$:

$$\begin{cases} \mathsf{TC}^n_0(v,x,h) = (x,h) \\ \mathsf{TC}^n_{(k+1)\cdot 2^{-n}}(v,x,h) = \mathsf{TC}_{2^{-n}}\big(\text{hyp}^{-1}(\mathsf{TC}^n_{k\cdot 2^{-n}}(v, \text{hyp } v)), \mathsf{TC}^n_{k\cdot 2^{-n}}(v,x,h)\big) \end{cases}$$

where $\text{hyp}^{-1}(\cdot)$ is defined in (1.4).

For the intermediate times $t = s + k\cdot 2^{-n}$, with $s \in (0, 2^{-n})$, we set

$$\mathsf{TC}^n_t := \mathsf{Tr}(s) \circ \big(\mathsf{TC}^n_{k\cdot 2^{-n}}\big).$$

Taking now $u_0 \in L^\infty(\mathbb{R}^d) \cap \mathrm{BV}(\mathbb{R}^d)$, we define accordingly for every $(x,h) \in \text{hyp } u_0$ and for every $t > 0$,

$$\gamma^n_{(x,h)}(t) := \mathsf{TC}^n_t(u_0, x, h)$$

and we set

$$\omega^n := \int_{\text{hyp } u_0} \delta_{\gamma^n_{(x,h)}}\, dx\, dh. \tag{3.12}$$

Since the transport collapse scheme is measure preserving, there exists $U^n \subset [0,+\infty) \times \mathbb{R}^d \times [0,+\infty)$ such that

$$(e_t)_\sharp \omega^n = \mathscr{L}^d \llcorner U^n(t), \tag{3.13}$$

where

$$U^n(t) := \big\{(x,h) \in \mathbb{R}^d \times [0,+\infty) : (t,x,h) \in U\big\}.$$

3.3.1. *Total variation along transport-collapse.* A crucial property in [Bre84] is that the total variation decreases along the transport-collapse scheme. This is indeed stated and proved in the following lemma and we present the proof for the sake of completeness.

Lemma 3.9. *For every $t \geq 0$ and $u \in L^1_+(\mathbb{R}^d)$ it holds*

$$\text{Tot.Var.}(\mathsf{T}(t)u) \leq \text{Tot.Var.}(u).$$

Proof. For every $t \geq 0$, for any test vector field $\Phi \in C^1_c(\mathbb{R}^d;\mathbb{R}^d)$, with $\|\Phi\|_\infty \leq 1$, we have

$$\begin{aligned} \int_{\mathbb{R}^d} (\mathsf{T}(t)u)(x) \operatorname{div}\Phi(x)\, dx &= \int_{\mathbb{R}^d}\int_0^{+\infty} ju(x - t\mathbf{f}'(h), h) \operatorname{div}\Phi(x)\, dh\, dx \\ &= \int_{\mathbb{R}^d}\int_0^{+\infty} ju(x,h) \operatorname{div}\Psi_h(x)\, dh\, dx \\ &\leq \int_0^{+\infty} \text{Tot.Var.}(ju(\cdot,h))\, dh \end{aligned}$$

where we have set $\Psi_h(x) = \Phi(x + t\mathbf{f}'(h))$ and the last inequality holds by definition of total variation (together with the trivial fact that $\|\Psi_h\|_\infty \leq 1$). Finally, by Coarea formula, we have

$$\int_0^{+\infty} \text{Tot.Var.}(ju(\cdot,h))dh = \text{Tot.Var.}(u).$$

Being Φ arbitrary, the proof is complete. □

3.3.2. *Passage to the limit of transport-collapse.* In this section we give an alternative proof of the fact that the iterated Transport-Collapse scheme converges to the Kruzkov semigroup, based on the Lagrangian representation. As a byproduct, we obtain the existence of Lagrangian representations for BV initial data and, as already noticed, this suffices for the general L^∞ case.

Let us also fix $D_n := \{\frac{k}{2^n} : k \in \mathbb{N}_{\geq 0}\}$ so that for every $\bar t \in D_n$ there exists $u^n(\bar t) \in L^\infty(\mathbb{R}^d)$ such that

$$U^n(\bar t) = \text{hyp } u^n(\bar t).$$

The key point to prove the compactness of the family $(U^n)_{n\in\mathbb{N}}$ is contained in the following lemma.

Lemma 3.10. *Let $\bar n \in \mathbb{N}$ and $\bar t \in D_{\bar n}$. Then for every $t > \bar t$ and for every $n \geq \bar n$ it holds*

$$\|(e_t)_\sharp\omega^n - (e_{\bar t})_\sharp\omega^n\|_{\mathscr{M}} = \mathscr{L}^{d+1}\big(U^n(t)\,\Delta\, U^n(\bar t)\big) \leq 2\|f'\|_\infty (t - \bar t)\text{Tot.Var.}(u_0). \tag{3.14}$$

Proof. Let us now write $t - \bar t = k\cdot 2^{-n} + s$ for $s \in [0, 2^{-n})$. For $j = 0, \dots, k-1$ set

$$I_j := [t_{j,n}, t_{j+1,n}], \qquad \text{where} \qquad t_{j,n} := \bar t + j2^{-n}.$$

Observe that it holds

$$\mathscr{L}^{d+1}\big(U^n(t)\,\Delta\, U^n(\bar t)\big) = 2\omega^n\left(\{\gamma : \gamma(\bar t) \in U^n(\bar t), \gamma(t) \notin U^n(\bar t)\}\right)$$

Being $U(\bar t)$ the hypograph of $u^n(\bar t)$, for every $j = 0, \dots, k-1$ and $\gamma \in \operatorname{supp}\omega^n$

$$\gamma(t_{j,n}-) \in U^n(\bar t) \qquad \Longrightarrow \qquad \gamma(t_{j,n}+) \in U^n(\bar t). \tag{3.15}$$

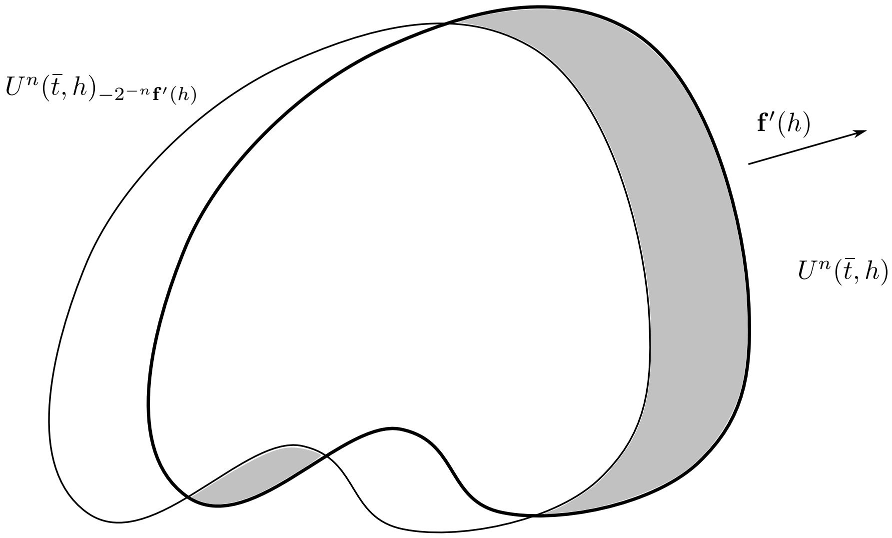

FIGURE 2. The set in grey is $U^n(\bar t,h)\cap\big(U^n(\bar t,h)_{-2^{-n}\mathbf{f}'(h)}\big)^c$.

For any $j=0,\dots,k-1$ we set

$$\mathcal{G}_{j,n}:=\Big\{\gamma\in\operatorname{supp}\omega^n:\gamma(t_{j,n}+)\in U^n(\bar t),\gamma(t_{j+1,n}-)\notin U^n(\bar t)\Big\}.$$

Finally, if $s=0$ we set $\mathcal{G}_k=\emptyset$ and if $s>0$,

$$\mathcal{G}_{k,n}:=\Big\{\gamma\in\operatorname{supp}\omega^n:\gamma(t_{k,n}+)\in U^n(\bar t),\gamma(t)\notin U^n(\bar t)\Big\}.$$

By (3.15), it holds

$$\{\gamma:\gamma(\bar t)\in U^n(\bar t),\gamma(t)\notin U^n(\bar t)\}\subset\bigcup_{j=0}^{k}\mathcal{G}_{j,n}.$$

Let us fix $j=0,\dots,k-1$. By (3.13) and definition of ω^n,

$$\begin{aligned}\omega^n(\mathcal{G}_{j,n})&=\mathscr{L}^{d+1}\left(\{(x,h)\in U^n(\bar t)\cap U^n(t_{j,n}):(x+\mathbf{f}'(h)2^{-n},h)\notin U^n(\bar t)\}\right)\\&=\int_0^{\|u_0\|_\infty}\mathscr{L}^d\left(\{x\in U^n(\bar t,h)\cap U^n(t_{j,n},h):x+\mathbf{f}'(h)2^{-n}\notin U^n(\bar t,h)\}\right)dh,\end{aligned}\tag{3.16}$$

where we have set $U(t,h):=\{x:(t,x,h)\in U\}$ and used Fubini theorem. Now we observe that

$$\Big\{x\in U^n(\bar t,h)\cap U^n(t_{j,n},h):x+\mathbf{f}'(h)2^{-n}\notin U^n(\bar t,h)\Big\}\subset U^n(\bar t,h)\cap(U^n(\bar t,h)_{-2^{-n}\mathbf{f}'(h)})^c,$$

where we recall that $E_{\mathbf{v}}:=E+\mathbf{v}$ (see Figure 2). Since

$$\mathscr{L}^d\left(U^n(\bar t,h)\cap(U^n(\bar t,h)_{-2^{-n}\mathbf{f}'(h)})^c\right)=\frac12\mathscr{L}^d\big(U^n(\bar t,h)\Delta(U^n(\bar t,h)_{-2^{-n}\mathbf{f}'(h)})\big),$$

by applying Lemma 2.2, we have

$$\mathscr{L}^d\big(U^n(\bar t,h)\Delta(U^n(\bar t,h)_{-2^{-n}\mathbf{f}'(h)})\big)\le 2\|f'\|_\infty 2^{-n}\operatorname{Per}(U^n(\bar t,h)).$$

Taking into account (3.16), by coarea formula for functions of bounded variation

$$\begin{aligned}\omega^n(\mathcal{G}_{j,n})&\le\int_0^{\|u_0\|_\infty}\|f'\|_\infty 2^{-n}\operatorname{Per}(U^n(\bar t,h))\,dh\\&=2^{-n}\|f'\|_\infty\mathrm{Tot.Var.}(u^n(\bar t))\\&\le 2^{-n}\|f'\|_\infty\mathrm{Tot.Var.}(u_0),\end{aligned}$$

where the last inequality follows by Lemma 3.9. Similarly we can prove that

$$\omega^n(\mathcal{G}_{k,n})\le s\|f'\|_\infty\mathrm{Tot.Var.}(u_0),$$

therefore summing over $j=0,\dots,k$ we get

$$\begin{aligned}\mathscr{L}^{d+1}\big(U^n(t)\,\Delta\,U^n(\bar t)\big)&\le 2\sum_{j=0}^{k}\omega^n(\mathcal{G}_{j,n})\\&\le 2((2^{-n}k+s)\|f'\|_\infty\mathrm{Tot.Var.}(u_0)\\&=2(t-\bar t)\|f'\|_\infty\mathrm{Tot.Var.}(u_0).\end{aligned}$$

□

We now combine the estimate (3.14) together with Lemma 2.1 to deduce the existence of a Lagrangian representation for BV solutions.

Proposition 3.11. *The sequence $(\omega^n)_{n\in\mathbb{N}}$ constructed in (3.12) is tight and every limit point ω is a Lagrangian representation of the entropy solution to (3.11).*

Proof. As in the proof of Proposition 3.6, the tightness of the family follows from Condition (2) in Definition 3.1 together with uniform bounds. Let ω be any limit point.

We now want to apply Lemma 2.1: set $I=[0,T]$ and let $D_n := \{\frac{k}{2^n} : k = 0,\dots,2^nT\}$. Let then $X := L^1(\mathbb{R}^{d+1})$ and accordingly define

$$\begin{aligned} f_n \colon I &\to L^1(\mathbb{R}^{d+1}) \\ t &\mapsto \chi_{\mathrm{supp}\,(e_t)_\sharp\omega^n}(\cdot) \end{aligned}$$

Condition (1) is trivially satisfied; let us verify assumption (2). For any $n\in\mathbb{N}$, for every $t\in D_n$ and every $m>n$ we have $(e_t)_\sharp\omega^m$ is concentrated on the hypograph of some function $u^m(t)$. By Lemma 3.9 the functions $(u^m(t))_{m\geq n}$ have uniformly bounded total variation, hence they are compact in $L^1(\mathbb{R}^d)$ and therefore the hypographs are compact in $L^1(\mathbb{R}^{d+1})$. To verify condition (3), it is enough to apply Lemma 3.10.

Thus we obtain a Lipschitz function $f\colon I\to L^1(\mathbb{R}^{d+1})$; since $f(t)$ is the characteristic function of an hypograph for every $t\in D$, by continuity, there exists $u\in \mathrm{Lip}([0,T];\mathrm{BV}(\mathbb{R}^d))$ such that

$$f(t) = \chi_{\mathrm{hyp}\ u(t)}$$

for every $t\in[0,T]$.

Thanks to Proposition 3.6 we obtain that

$$(e_t)_\sharp\omega = \mathscr{L}^{d+1}\llcorner \mathrm{hyp}\ u(t)$$

for every $t\geq 0$. Finally, a direct application of Proposition 3.3 shows that the function u is the entropy solution to (3.11) and concludes the proof. □

The compactness and stability properties of Lagrangian representations stated in Corollary 3.7, together with standard approximation results, yield immediately the following

Theorem 3.12. *Let u be the entropy solution to the initial value problem (3.11) with $u_0\in L^\infty(\mathbb{R}^d)$. Then there exists a Lagrangian representation of u.*

4. The case of continuous solutions

In this section we prove that if u is a continuous entropy solution of (3.1) then for every entropy-entropy flux pair $(\eta,\mathbf{q})$ with $\eta\in C^1(\mathbb{R})$, the dissipation measure

$$\mu = \eta(u)_t + \mathrm{div}(\mathbf{q}(u)) = 0.$$

Denote by $\bar\nu$ the measure defined by (3.9) with $\eta(u)=u^2/2$ and consider its jump part defined by

$$\nu^j := \int_\Gamma \mu^j_\gamma d\omega, \qquad \text{where} \quad \mu^j_\gamma = \mathscr{H}^1\llcorner\{(t,x,h) : \gamma^1(t)=x, h\in(\gamma^2(t+),\gamma^2(t-))\}.$$

As an intermediate step we prove that $\nu^j=0$, which is equivalent, by definition, to the fact that ω is concentrated on continuous curves.

Lemma 4.1. *Let $u\colon[0,T)\times\mathbb{R}^d\to\mathbb{R}$ be a continuous solution of (3.1) and let ω be a Lagrangian representation of u. Then ω is concentrated on* continuous *characteristic curves.*

Proof. Since the solution u is continuous, for every $(t,x,h)\in[0,+\infty)\times\mathbb{R}^d\times(0,+\infty)$ such that $h<u(t,x)$, it holds $(t,x,h)\in \mathrm{Int}(\mathrm{hyp}\ u)$. Hence for every $\gamma\in\mathrm{supp}\,\omega$,

$$\mu^j_\gamma = \mu^j_\gamma\llcorner\mathrm{Int}(\mathrm{hyp}\ u).$$

Therefore

$$\nu^j = \nu^j\llcorner\mathrm{Int}(\mathrm{hyp}\ u) = 0,$$

by Proposition 3.5. This concludes the proof of this lemma. □

In the following proposition we show that in the continuous solutions the hypograph at time t is the translation of hyp u_0 along segments with characteristic speed.

Proposition 4.2. *Let $u\colon [0,T)\times\mathbb{R}^d\to\mathbb{R}$ be a continuous entropy solution of (3.1). Then*
$$\bar\omega = \int_{\mathrm{hyp}\ u_0} \delta_{\bar\gamma_{x,h}}\,dx\,dh,$$
where
$$\bar\gamma_{x,h}(t) = (x + t\mathbf{f}'(h), h), \qquad t\in[0,T)$$
is a Lagrangian representation of u.

Proof. To begin we notice that there exists a set E with $\mathscr{L}^{d+2}(\mathrm{hyp}\ u\setminus E)=0$ such that for every $z=(t,x,h)\in E$ there exists a curve $\gamma_z\colon[0,\bar t]\to\mathbb{R}^d\times[0,+\infty)$ with the following properties:

(1) $\gamma_z(t)=(x,w)$;
(2) γ_z is a continuous characteristic curve;
(3) $\gamma_z\big([0,\bar t]\big)\subset \mathrm{hyp}\ u$;
(4) γ^2_z is constant on the connected components of $\gamma_z^{-1}(\mathrm{Int}(\mathrm{hyp}\ u))$.

In fact, (1) follows from the definition of Lagrangian representation and (2) follows from Lemma 4.1. From the definition of Lagrangian representation ω is concentrated on curves that lie in hyp u for $\mathscr{L}^1$-a.e. $t\in[0,T]$. By continuity of u, we thus get (3). Finally (4) follows by Proposition 3.5.

Let $\bar t>0$ and for every $(x,h)\in \mathrm{hyp}\ u(\bar t)$ we consider the function
$$\begin{aligned}\sigma_{(x,h)}\colon[0,\bar t]&\to\mathbb{R}^d\times[0,+\infty)\\ t&\mapsto\big(x-(\bar t-t)\mathbf{f}'(h),h\big)\end{aligned}$$
We first prove that for every $(x,h)\in\mathrm{hyp}\ u(\bar t)$ the segments
$$\sigma_{(x,h)}\big([0,\bar t]\big)\subset \mathrm{hyp}\ u.$$
Fix $\varepsilon>0$ and let us construct by iteration a curve contained in the hypograph which approximates the segment. By uniform continuity of u there exists $\delta\in(0,1)$ such that
$$|(t,x)-(t',x')|\le\delta \quad\Rightarrow\quad |u(t,x)-u(t',x')|\le\varepsilon.$$
Let $\varepsilon'<\delta\varepsilon$ and fix $(t_1,x_1)\in[0,+\infty)\times\mathbb{R}$ and $\bar h>0$ such that $(t_1,x_1,\bar h)\in\mathrm{hyp}\ u$. For $k\ge1$ we define by recursion the points $\tilde z_k$, t_k and x_k in the following way:
$$\tilde z_k=(\tilde t_k,\tilde x_k,\tilde h_k)\in B_{\varepsilon'}((t_k,x_k,\bar h-\varepsilon))\cap E \tag{4.1}$$
with $\tilde t_k<t_k$ and
$$t_{k+1}:=\inf\{t\in[0,\tilde t_k]:\gamma_{\tilde z_k}(t)<\bar h+\varepsilon\},\qquad x_{k+1}:=\gamma^1_{\tilde z_k}(t_{k+1}+).$$
The procedure ends when $t_{k+1}=0$. The existence of points $\tilde z_k$ is ensured by the fact that E has full measure. We now prove that the procedure ends in finitely many steps. Since for every $k\ge0$, $\gamma^2_{\tilde z_k}$ is constant on each connected component of $\gamma^{-1}_{\tilde z_k}(\mathrm{Int}(\mathrm{hyp}\ u))$ and $\gamma^2_{\tilde z_k}(\tilde t_k)<u(\tilde t_k,\tilde x_k)-\varepsilon$, by the uniform continuity of u
$$\tilde t_k - t_{k+1}\ge\frac{\delta}{\|f'\|_\infty}\wedge\tilde t_k,$$
therefore the number of steps N after which the procedure ends is bounded by
$$N\le 1+\frac{\|f'\|_\infty\bar t}{\delta}. \tag{4.2}$$
We now prove the following claim, which states that $\gamma_{\tilde z_k}$ approximates $\sigma_{(\bar x,\bar h)}$ in $(t_{k+1},\tilde t_k)$.

Claim. There exists $C>0$ independent of ε such that for every $t\in[0,\bar t]$ there exists $k=1,\dots,N$ and $s\in(t_{k+1},\tilde t_k)$ for which
$$|(s,\gamma_{\tilde z_k}(s))-(t,\sigma_{(\bar x,\bar h)}(t))|<C\varepsilon. \tag{4.3}$$
First we observe that for every $k=1,\dots,N$ and for every $s\in(t_{k+1},\tilde t_k)$ it holds
$$|\gamma^2_{\tilde z_k}(s)-\bar h|<2\varepsilon. \tag{4.4}$$
The estimate for the first components follows by (4.4) and (3.3): for every $k=1,\dots,N$,
$$\begin{aligned}|\gamma^1_{\tilde z_k}(t_{k+1})-\sigma^1_{(\bar x,\bar h)}(t_{k+1})| &= \left|\gamma^1_{\tilde z_k}(\tilde t_k)-\sigma^1_{(\bar x,\bar h)}(\tilde t_k)-\int_{t_{k+1}}^{\tilde t_k}\big(\dot\gamma^1_{\tilde z_k}(t)-\mathbf{f}'(\bar h)\big)\,dt\right|\\ &\le|\gamma^1_{\tilde z_k}(\tilde t_k)-\sigma^1_{(\bar x,\bar h)}(\tilde t_k)|+2\varepsilon(\tilde t_k-t_{k+1})\|\mathbf{f}''\|_\infty.\end{aligned} \tag{4.5}$$
Moreover, by (4.1),
$$\begin{aligned}|\gamma^1_{\tilde z_k}(\tilde t_k)-\sigma^1_{(\bar x,\bar h)}(\tilde t_k)| &\le|\gamma^1_{\tilde z_k}(\tilde t_k)-\gamma^1_{\tilde z_k}(t_k)|+|\gamma^1_{\tilde z_k}(t_k)-\sigma^1_{(\bar x,\bar h)}(t_k)|+|\sigma^1_{(\bar x,\bar h)}(t_k)-\sigma^1_{(\bar x,\bar h)}(\tilde t_k)|\\ &\le 2\|\mathbf{f}'\|_\infty\varepsilon'+|\gamma^1_{\tilde z_k}(t_k)-\sigma^1_{(\bar x,\bar h)}(t_k)|.\end{aligned} \tag{4.6}$$

By (4.5) and (4.6), it follows that for every $k = 1, \dots, N-1$ it holds

$$|\gamma^1_{\tilde{z}_k}(t_{k+1}) - \sigma^1_{(\bar{x},\bar{h})}(t_{k+1})| \le |\gamma^1_{\tilde{z}_k}(t_k) - \sigma^1_{(\bar{x},\bar{h})}(t_k)| + 2\varepsilon(\tilde{t}_k - t_{k+1})\|\mathbf{f}''\|_\infty + 2\|\mathbf{f}'\|_\infty \varepsilon'. \tag{4.7}$$

For every $t \in [0,\bar{t}]$ let $\bar{k} = 1, \dots, N-1$ and $s \in (t_{\bar{k}+1}, \tilde{t}_{\bar{k}})$ be such that $|s-t| < \varepsilon'$. Then, iterating (4.7) for $k = \bar{k}, \dots, N-1$ and by (4.2), we have

$$\begin{aligned}
|\gamma^1_{\tilde{z}_k}(s) - \sigma^1_{\bar{z}}(t)| &\le |\gamma^1_{\tilde{z}_k}(s) - \sigma^1_{\bar{z}}(s)| + |\sigma^1_{\bar{z}}(s) - \sigma^1_{\bar{z}}(t)| \\
&\le 2\varepsilon\|f''\|_\infty(\bar{t} - s) + 2(N-\bar{k})\varepsilon'\|f'\|_\infty + \|f'\|_\infty|t-s| \\
&\le 2\varepsilon\|f''\|_\infty T + 2\varepsilon'\|\mathbf{f}'\|_\infty + 2\varepsilon\|f'\|^2_\infty \bar{t} + \|f'\|_\infty \varepsilon' \\
&\le C\varepsilon,
\end{aligned} \tag{4.8}$$

where $C = 2\|f''\|_\infty T + 2\|\mathbf{f}'\|_\infty + 2\|f'\|^2_\infty T + \|f'\|_\infty$. The estimates (4.4) and (4.8) prove (4.3).

Since hyp u is closed, letting $\varepsilon \to 0$ we obtain that for every $(\bar{x},\bar{h}) \in \operatorname{hyp} u(\bar{t})$, the segment

$$\sigma_{(\bar{x},\bar{h})}([0,\bar{t}]) \subset \operatorname{hyp} u.$$

Let

$$\tilde{\omega} = \int_{\operatorname{hyp}\, u(\bar{t})} \delta_{\sigma_{x,h}} dx dh.$$

Since the translations are area-preserving, for every $t \in [0,\bar{t}]$, there exists $U(t) \subset [0,+\infty) \times \mathbb{R}^d$ such that

$$(e_t)_\sharp \tilde{\omega} = \mathscr{L}^{d+1} \llcorner U(t)$$

and

$$\mathscr{L}^{d+1}(U(t)) = \int_{\mathbb{R}^d} u(\bar{t},x) dx. \tag{4.9}$$

Since we proved that for every $t \in [0,\bar{t}]$ it holds $U(t) \subset \operatorname{hyp} u(t)$, (4.9) implies that $U(t) = \operatorname{hyp} u(t)$. This proves that $\tilde{\omega} = \bar{\omega}$ and it is a Lagrangian representation of u. □

Theorem 4.3. *Let u be a continuous bounded entropy solution in $[0,T) \times \mathbb{R}^d$ to (3.1). Then for every $(t,x) \in [0,T) \times \mathbb{R}^d$, it holds*

$$u(t,x) = u_0(x - \mathbf{f}'(u(t,x))t). \tag{4.10}$$

Moreover for every $\eta : \mathbb{R} \to \mathbb{R}$, $\mathbf{q} : \mathbb{R} \to \mathbb{R}^d$ Lipschitz such that $\mathbf{q}' = \eta' \mathbf{f}'$ a.e. with respect to $\mathscr{L}^1$, it holds

$$\eta(u)_t + \operatorname{div}_x \mathbf{q}(u) = 0 \tag{4.11}$$

in the sense of distributions.

Proof. The validity of (4.10) is an immediate consequence of Proposition (4.2). Concerning the second claim, if η is a convex C^2 entropy, then (4.11) follows by Lemma 3.4 and Proposition 4.2, since $\mu^\eta_\gamma = 0$ for every $\gamma \in \operatorname{supp}\omega$. If η is C^2, then there exist η_1, η_2 of class C^2 and convex such that $\eta = \eta_1 - \eta_2$ and thus it is enough to apply the previous result to both η_1 and η_2. Finally, in order to prove (4.11) holds for Lipschitz $(\eta, \mathbf{q})$, we consider a sequence $(\eta^n)_{n\in\mathbb{N}}$ such that $\eta^n \to \eta$ uniformly on $\mathbb{R}$ and $(\eta^n)' \to \eta'$ in $L^1_{\mathrm{loc}}(\mathbb{R})$ with the associated $\mathbf{q}^n$ such that $\mathbf{q}^n(0) = \mathbf{q}(0)$. We have that $\mathbf{q}^n \to \mathbf{q}$ in $L^1_{\mathrm{loc}}(\mathbb{R}^d)$ and hence, for every test function $\phi \in C^\infty_c([0,T) \times \mathbb{R}^d)$,

$$\begin{aligned}
-\langle \eta(u)_t + \operatorname{div}_x \mathbf{q}(u), \phi \rangle &= \int_0^T \int_{\mathbb{R}^d} \phi_t \eta(u) + \mathbf{q}(u) \cdot \nabla\phi \, dx \, dt \\
&= \lim_{n\to+\infty} \int_0^T \int_{\mathbb{R}^d} \phi_t \eta^n(u) + \mathbf{q}^n(u) \cdot \nabla\phi \, dx \, dt = 0,
\end{aligned}$$

and this completes the proof. □

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