AN ISOPERIMETRIC PROBLEM WITH A COULOMBIC REPULSION AND ATTRACTIVE TERM

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Abstract. We study an energy given by the sum of the perimeter of a set, a Coulomb repulsion term of the set with itself and an attraction term of the set to a point charge. We prove that there exists an optimal radius r_0 such that if $r < r_0$ the ball B_r is a local minimizer with respect to any other set with same measure. The global minimality of balls is also addressed.

1. Introduction

In this paper we study the minimizers under the volume constraint $|E| = m$ of the functional

$$I(E) = P(E) + V(E) - KR(E), \tag{1.1}$$

where

$$V(E) = \int_E \int_E \frac{1}{|x-y|^{n-2}} dxdy$$

is a Coulombic repulsive potential of the set with itself and

$$R(E) = \int_E \frac{1}{|x|^{n-2}} dx \tag{1.2}$$

is a repulsive term of the set with a point charge. Here $P(E)$ stands for the standard Euclidean perimeter in the De Giorgi sense and $K \geq 0$. In the three dimensional case this functional has been studied by Lu and Otto in [13]. In that paper they prove that if m is sufficiently large then the constrained minimum problem has no solutions. They also show that there exists a critical value m_c such that if $m < m_c$ the ball centered at the origin is the unique global minimizer. This result is obtained using a quantitative version of the isoperimetric inequality with a Coulombic term proved by Julin in [11] .

In case $K = 0$ functional (1.1) reduces to the Thomas-Fermi-Dirac-von Weizsäcker model and it has been studied for $n = 2, 3$ in [7], [16], [14] and in any dimension in [11], [12] and [6]. In particular, the latter paper shows that there exists a critical mass m_0 such that if $m < m_0$ balls are local minimizers. Moreover, in [6] it is also proved that there exists a critical value $0 < m_1 \leq m_0$ such that if the volume is smaller than m_1, balls are the unique global minimizers.

In this paper we extend the above mentioned results of [6] and [13] to the case $K > 0$ and $n \geq 3$. Precisely, we prove that there exists a critical radius $r_0 > 0$ such that if $r < r_0$ the ball B_r centered at the origin is a local minimizer of the constrained minimum problem and that this property fails when $r > r_0$. As in [6], we show also the global minimality of balls B_r when $r < r_1$, for some $0 < r_1 < r_0$. Note that both critical radii r_1 and r_0 tend to infinity as $K \to \infty$ and an argument provided in the last section, see Lemma 6.3, shows that the ratio r_0/r_1 stays bounded indipendently of K.

The paper is organized as follows. After a short section where we fix the notation and give some preliminary results, in Section 3 we provide a Fuglede type estimate for the functional I, see Theorem 3.1. Precisely, we prove that if $r < r_0$ the ball B_r is a local minimizer with respect to small C^1 variations.

In Section 4, after calculating the second variation of I, we show that the radius r_0 provided by Theorem 3.1 is indeed optimal since balls of radius $r > r_0$ are not local minimizers. Note that while the formula of the second variation of V can be obtained by more or less standard arguments, see for instance [6], the calculations leading to the second variation of the attractive term turn out to be more delicate due to the presence of a singularity in the integrand in (1.2).

In section 5 we pass from the local minimality of the ball B_r with respect to small C^1 variations to the full local minimality result. As usual in this framework, we follow a strategy first devised by Cicalese and Leonardi in [5], see also [1], based on the regularity theory for quasi minimizers of the perimeter. However, in our setting this approach turns out to be more complicated. Indeed, the main difficulty comes from the fact that, differently from most cases studied in the literature, see [5], [1], [10], [3], [6], [4], our functional is not translation invariant. Overcoming this difficulty requires a delicate estimate of the behavior of the repulsive term R on sets which are C^1 close to a ball centered at the origin.

2. Notation and preliminary results

In the following we shall denote by $B_r(x)$ the ball in $\mathbb{R}^n$ of radius r centered at x. If the center is the origin we shall simply write B_r, while the unit ball centered at 0 will be denoted by B. By ω_n we denote the Lebesgue measure of the unit ball in $\mathbb{R}^n$.

We recall some basic definitions of the theory of sets of finite perimeter. If E is any measurable subset of $\mathbb{R}^n$ and $\Omega \subset \mathbb{R}^n$ is an open set, the *perimeter of E in Ω* is defined by setting

$$P(E;\Omega) = \sup\left\{ \int_E \operatorname{div}\varphi\, dx : \varphi \in C_c^\infty(\Omega;\mathbb{R}^n),\ \|\varphi\|_\infty \le 1 \right\}.$$

The perimeter of E in $\mathbb{R}^n$ is denoted by $P(E)$. We say that E has *locally finite perimeter* if $P(E;\Omega) < \infty$ for all bounded open sets Ω. It is well known, see [2, Ch. 3], that E is a set of locally finite perimeter if and only its characteristic function χ_E has distributional derivative $D\chi_E$ which is a vector-valued measure in $\mathbb{R}^n$ with values in $\mathbb{R}^n$. Thus, from the above definition we have immediately that $P(E;\Omega) = |D\chi_E|(\Omega)$ for every open set Ω.

From Besicovitch derivation theorem we have that for $|D\chi_E|$-a.e. $x \in \mathbb{R}^n$ there exists

$$\nu_E(x) := -\lim_{r\to 0} \frac{D\chi_E(B_r(x))}{|D\chi_E|(B_r(x))} \quad \text{and} \quad |\nu_E(x)| = 1. \tag{2.1}$$

The set $\partial^* E$ where (2.1) holds is called the *reduced boundary* of E, while the vector $\nu^E(x)$ is the *generalized exterior normal* at x. For all the properties of sets of finite perimeter used herein we refer to the book [2].

A set $E \subset \mathbb{R}^n$ is said to be *nearly spherical* if there exist a ball B_r and a Lipschitz function $u : \mathbb{S}^{n-1} \to (-1/2, 1/2)$ such that

$$E = \{y = rx(1 + u(x)) : x \in B, \}, \tag{2.2}$$

where here and in the following we have tacitly assumed that the function u, originally defined only on the unit sphere, is extended to $\mathbb{R}^n \setminus \{0\}$ by setting $u(x) = u\big(\frac{x}{|x|}\big)$.

For any measurable set $E \subset \mathbb{R}^n$ we define the Coulombic potential $V(E)$ and the repulsive term $R(E)$ as follows

$$V(E) = \int_E \int_E \frac{1}{|x-y|^{n-2}}\, dxdy, \qquad R(E) = \int_E \frac{1}{|x|^{n-2}}\, dx.$$

We are interested in minimizing the nonlocal energy given by

$$I(E) = P(E) + V(E) - KR(E), \tag{2.3}$$

where K is a positive constant that will be fixed throughout the paper.

Note that in order to avoid trivial statements, *we shall assume throughout that the dimension of the ambient space* $\mathbb{R}^n$ *is greater than or equal to 3*, unless specified otherwise.

It is easily checked that if E is defined as in (2.2) then its measure and perimeter are given, respectively, by the following formulas

$$|E| = r^n \int_B (1+u(x))^n \, dx = \frac{r^n}{n} \int_{\mathbb{S}^{n-1}} (1+u(x))^n \, d\mathcal{H}^{n-1}, \tag{2.4}$$

$$P(E) = r^{n-1} \int_{\mathbb{S}^{n-1}} (1+u(x))^{n-1} \sqrt{1 + \frac{|D_\tau u(x)|^2}{(1+u(x))^2}} \, d\mathcal{H}^{n-1},$$

where $D_\tau u$ stands for the tangential gradient of u on $\mathbb{S}^{n-1}$ and $\mathcal{H}^{n-1}(\cdot)$ stands for the *Hausdorff* $(n-1)$-dimensional measure.

Similarly, $V(E)$ and $R(E)$ can be also represented as

$$\begin{aligned} V(E) &= \int_E \int_E \frac{1}{|x-y|^{n-2}} \, dxdy = r^{n+2} \int_B \int_B \frac{(1+u(x))^n (1+u(x))^n}{|x(1+u(x)) - y(1+u(y))|^{n-2}} \, dxdy \\ &= r^{n+2} \int_{\mathbb{S}^{n-1}} d\mathcal{H}_x^{n-1} \int_{\mathbb{S}^{n-1}} d\mathcal{H}_y^{n-1} \int_0^{1+u(x)} d\rho \int_0^{1+u(y)} \frac{\rho^{n-1}\sigma^{n-1}}{\big(|\rho-\sigma|^2 + \rho\sigma|x-y|^2\big)^{\frac{n-2}{2}}} \, d\sigma \\ \frac{R(E)}{r^2} &= \int_B \frac{(1+u(x))^2}{|x|^{n-2}} \, dx = \frac{1}{2} \int_{\mathbb{S}^{n-1}} (1+u(x))^2 \, d\mathcal{H}^{n-1}. \end{aligned}$$

For any integer $k \geq 0$, let us denote by $y_{k,i}$, $i = 1, \dots, G(n,k)$, the spherical harmonics of order k, i.e., the restrictions to $\mathbb{S}^{n-1}$ of the homogeneous harmonic polynomials of degree k, normalized so that $||y_{k,i}||_{L^2(\mathbb{S}^{n-1})} = 1$, for all $k \geq 0$ and $i \in \{1, \dots, G(n,k)\}$. The functions $y_{k,i}$ are eigenfunctions of the Laplace-Beltrami operator on $\mathbb{S}^{n-1}$ and for all k and i

$$-\Delta_{\mathbb{S}^{n-1}} y_{k,i} = \lambda_k y_{k,i} \, .$$

where $\lambda_k = k(k+n-2)$. Moreover if $u \in L^2(\mathbb{S}^{n-1})$ we have

$$u = \sum_{k=0}^{\infty} \sum_{i=1}^{G(n,k)} a_{k,i} y_{k,i}, \quad \text{where} \quad a_{k,i} := \int_{\mathbb{S}^{n-1}} u(x) y_{k,i}(x) \, d\mathcal{H}^{n-1}.$$

Therefore, for a function $u \in H^1(\mathbb{S}^{n-1})$ we have that

$$\|u\|^2_{L^2(\mathbb{S}^{n-1})} = \sum_{k=0}^{\infty} \sum_{i=1}^{G(n,k)} a_{k,i}^2, \qquad \|D_\tau u\|^2_{L^2(\mathbb{S}^{n-1})} = \sum_{k=1}^{\infty} \sum_{i=1}^{G(n,k)} \lambda_k a_{k,i}^2. \tag{2.5}$$

If $s \in (-1,1)$ and $u \in L^2(\mathbb{S}^{n-1})$, we set

$$[u]^2_{s,\mathbb{S}^{n-1}} := \int_{\mathbb{S}^{n-1}} \int_{\mathbb{S}^{n-1}} \frac{|u(x)-u(y)|^2}{|x-y|^{n-1+2s}} \, d\mathcal{H}_x^{n-1} d\mathcal{H}_y^{n-1}.$$

Also these seminorms can be represented using the Fourier coefficients of u and suitable sequences of eigenvalues. In particular, see formulas (7.12) and (7.5) in [6], we have

$$[u]^2_{-\frac{1}{2},\mathbb{S}^{n-1}} = \int_{\mathbb{S}^{n-1}} \int_{\mathbb{S}^{n-1}} \frac{|u(x)-u(y)|^2}{|x-y|^{n-2}} \, d_x\mathcal{H}^{n-1} \, d\mathcal{H}_y^{n-1} = \sum_{k=1}^{\infty} \sum_{i=1}^{G(n,k)} \mu_k a_{i,k}^2, \tag{2.6}$$

where the eigenvalues μ_k are given, for an integer $k \geq 0$, by the following expressions

$$\mu_k := \frac{4\pi^{\frac{n}{2}}}{\Gamma(\frac{n-2}{2})} \left(\frac{\Gamma(\frac{n-2}{2})}{\Gamma(\frac{n}{2})} - \frac{\Gamma(k+\frac{n-2}{2})}{\Gamma(k+\frac{n}{2})} \right). \tag{2.7}$$

It is easily checked that the above sequence is bounded and strictly increasing. Moreover, see [6, Prop. 7.5],

$$\mu_1 = 2(n+2)\frac{V(B)}{P(B)}, \qquad \mu_2 = \frac{2n}{n+2}\mu_1. \tag{2.8}$$

Finally, we recall the following useful estimate, proved in [6, Appendix C],

$$\frac{\lambda_k - \lambda_1}{\mu_k - \mu_1} \geq \frac{\lambda_2 - \lambda_1}{\mu_2 - \mu_1} \qquad \forall k \geq 2. \tag{2.9}$$

Let us now define a function $\mathcal{P} : [0, \infty) \to \mathbb{R}$ setting for $r \geq 0$

$$\mathcal{P}(r) := \inf_{k \geq 2} \left\{ \frac{\lambda_k - \lambda_1}{\mu_k - \mu_1} r^{n-3} - r^n + \frac{K(n-2)}{\mu_k - \mu_1} \right\}. \tag{2.10}$$

Lemma 2.1. *The function $\mathcal{P}$ defined in* (2.10) *is continuous. Moreover there exists $r_0 > 0$ such that*

$$\mathcal{P}(r_0) = 0, \tag{2.11}$$

$\mathcal{P}(r) > 0$ for $0 < r < r_0$ and $\mathcal{P}(r) < 0$ for $r > r_0$.

Proof. Observe that from (2.8) and (2.7)

$$\frac{2(n-2)V(B)}{P(B)} \leq \mu_k - \mu_1 \leq \frac{2(n-2)P(B)}{n} \qquad \forall k \geq 2. \tag{2.12}$$

From this inequality we have that $(\lambda_k - \lambda_1)/(\mu_k - \mu_1) \to \infty$ as $k \to \infty$, hence for any interval $a > 0$ there exists $k_a \geq 2$ such that

$$\mathcal{P}(r) = \inf_{2 \leq k \leq k_a} \left\{ \frac{\lambda_k - \lambda_1}{\mu_k - \mu_1} r^{n-3} - r^n + \frac{K(n-2)}{\mu_k - \mu_1} \right\} \qquad \text{for all } r \in [0, a].$$

This proves that $\mathcal{P}$ is continuous. Observe also that from (2.8), (2.9) and the second inequality in (2.12)

$$\mathcal{P}(r) \geq \frac{(n+1)P(B)}{2(n-2)V(B)} r^{n-3} - r^n + \frac{Kn}{2P(B)},$$

hence $\mathcal{P} > 0$ in a right neighborhood of the origin. Note also that $\mathcal{P}(r) \to -\infty$ as $r \to +\infty$.

Let us now set for any integer $k \geq 2$ and any $r \geq 0$

$$P_k(r) := \frac{\lambda_k - \lambda_1}{\mu_k - \mu_1} r^{n-3} - r^n + \frac{K(n-2)}{\mu_k - \mu_1}. \tag{2.13}$$

It is easily checked that P_k has exactly one zero $r_k > 0$ and that $P_k(r) < 0$ for $r > r_k$. Therefore, denoting by $r_0 > 0$ the first zero of $\mathcal{P}$ and by $k_0 \geq 2$ an integer such that $\mathcal{P}(r_0) = P_{k_0}(r_0) = 0$, we have that $\mathcal{P}(r) \leq P_{k_0}(r) < 0$ for all $r > r_0$. Hence, the proof follows. □

3. Nearly spherical sets

In this section we prove the local minimality of balls B_r with $r < r_0$ with respect to small variations in C^1.

Theorem 3.1. *Let $\sigma \in (0, r_0/2)$, where r_0 is defined as in* (2.11)*. There exist two positive constants ε_0 and c_0, depending only on n and σ, with the following property. If E is a nearly spherical set as in* (2.2)*, with $|E| = B_r$ and barycenter at the origin, $r \in (\sigma, r_0 - \sigma)$ and $\|u\|_{W^{1,\infty}(\mathbb{S}^{n-1})} \leq \varepsilon_0$, then*

$$I(E) - I(B_r) \geq c_0 \|u\|^2_{L^2(\mathbb{S}^{n-1})}. \tag{3.1}$$

Proof. We are going to prove (3.1) by an argument similar to the one introduced by Fuglede in [8]. To this end, it is convenient to rephrase the assumption replacing E by the set

$$E_t : \{y = r(1 + tu(x)) : x \in B\},$$

with $u \in W^{1,\infty}(\mathbb{S}^{n-1})$, $\|u\|_{W^{1,\infty}(\mathbb{S}^{n-1})} \le 1/2$,

$$|E_t| = |B_r|, \qquad \int_{E_t} x\,dx = 0,$$

$t \in (0, 2\varepsilon_0)$, where the constant $\varepsilon_0 < 1/2$ will be determined at the end of the proof. Thus, our assertion (3.1) becomes

$$I(E_t) - I(B_r) \ge c_0 t^2 \|u\|^2_{L^2(\mathbb{S}^{n-1})}, \tag{3.2}$$

for a suitable constant $c_0 > 0$ depending only on n and σ. In order to prove this inequality we estimate the differences between various quantities appearing in the definition (2.3) of I. We start by the perimeter term. In this case, see for instance the proof of Theorem 3.1 in [9], we have, provided ε_0 is sufficiently small,

$$\frac{P(E_t) - P(B_r)}{r^{n-1}} \ge \frac{t^2}{2}\left(\int_{\mathbb{S}^{n-1}} |D_\tau u|^2\,d\mathcal{H}^{n-1} - (n-1)\int_{\mathbb{S}^{n-1}} u^2\,d\mathcal{H}^{n-1}\right) - C(n)t^3\|u\|^2_{L^2}, \tag{3.3}$$

for some constant $C(n)$ depending only on n. The difference between the two potential terms is estimated in [6, (5.20)] as follows

$$\frac{V(E_t) - V(B_r)}{r^{n+2}} \ge \frac{t^2}{2}\Big(2(n+2)\frac{V(B)}{P(B)}\|u\|^2_{L^2} - [u]^2_{-\frac12}\Big) - C(n)t^3\big(\|u\|^2_{L^2} + [u]^2_{-\frac12}\big). \tag{3.4}$$

Let us now estimate the remaining difference.

$$\begin{aligned}\frac{R(E_t) - R(B_r)}{r^2} &= \frac12\int_{\mathbb{S}^{n-1}} \big((1+tu(x))^2 - 1\big)\,d\mathcal{H}^{n-1} = \frac12\int_{\mathbb{S}^{n-1}} 2tu + t^2u^2\,d\mathcal{H}^{n-1} \\ &= t\int_{\mathbb{S}^{n-1}} u\,d\mathcal{H}^{n-1} + \frac{t^2}{2}\int_{\mathbb{S}^{n-1}} u^2\,d\mathcal{H}^{n-1}.\end{aligned} \tag{3.5}$$

Using now the assumption $|E_t| = |B_r|$, from (2.4), after expanding $(1+tu)^n$ we obtain

$$n\int_{\mathbb{S}^{n-1}} tu\,d\mathcal{H}^{n-1} + \frac{n(n-1)}{2}\int_{\mathbb{S}^{n-1}} t^2u^2\,d\mathcal{H}^{n-1} + \sum_{k=3}^{n}\binom{n}{k}t^k\int_{\mathbb{S}^{n-1}} u^k d\mathcal{H}^{n-1} = 0. \tag{3.6}$$

Inserting in (3.5) the expression of the integral of u on $\mathbb{S}^{n-1}$ obtained from this identity, and recalling that $|u| < 1/2$ and $0 < t < 2\varepsilon_0 < 1$, we get

$$\frac{R(E_t) - R(B_r)}{r^2} \ge -\frac{n-2}{2}\int_{\mathbb{S}^{n-1}} t^2u^2 d\mathcal{H}^{n-1} - C(n)t^3\|u\|_{L^2}.$$

Collecting this inequality, (3.3) and (3.4), we have

$$\begin{aligned}I(E_t) - I(B) \ge{}& \frac{t^2r^{n-1}}{2}\big(\|D_\tau u\|^2_{L^2} - (n-1)\|u\|^2_{L^2}\big) + \frac{t^2r^{n+2}}{2}\Big(2(n+2)\frac{V(B)}{P(B)}\|u\|^2_{L^2} - [u]^2_{-\frac12}\Big) \\ &+ \frac{t^2r^2K(n-2)}{2}\|u\|^2_{L^2} - C(n)t^3\big(\|u\|^2_{L^2} + [u]^2_{-\frac12}\big).\end{aligned} \tag{3.7}$$

We now write all the norms in the previous inequality in terms of the Fourier coefficients $a_{k,i}$ of u. To this end, observe that from (3.6), using the fact that $|u| \le 1/2$ and $0 < t < 1$, we have in particular that

$$|a_0| \le C(n)t\|u\|^2_{L^2}. \tag{3.8}$$

From the condition that the barycenter of E_t is at the origin we have

$$\int_{\mathbb{S}^{n-1}} x(1+u(x))^n \, d\mathcal{H}^{n-1} = 0.$$

Therefore, arguing as in the proof of (3.6) we get

$$\sup_{i=1,\dots,n} |a_{1,i}| \le C(n)t\|u\|^2_{L^2}. \tag{3.9}$$

Finally, recalling that the eigenvalues μ_k are all bounded, from (2.6) we get that

$$[u]_{-\frac{1}{2}} \le C(n)\|u\|_{L^2}.$$

Using this inequality, recalling (2.5), (2.6) and that $\lambda_1 = n-1$ and $\mu_1 = 2(n+2)V(B)/P(B)$, see (2.8), from the estimate (3.7) we get

$$\begin{aligned} I(E_t) - I(B) &\ge \frac{t^2r^2}{2}\sum_{k=2}^{\infty}\sum_{i=1}^{G(k,n)} \big((\lambda_k-\lambda_1)r^{n-3} + (\mu_1-\mu_k)r^n + K(n-2)\big)\, a^2_{k,i} - C(n)t^3\|u\|^2_{L^2} \\ &\ge \frac{t^2r^2}{2}\sum_{k=2}^{\infty}\sum_{i=1}^{G(k,n)} (\mu_k-\mu_1)\Big(\frac{\lambda_k-\lambda_1}{\mu_k-\mu_1}r^{n-3} - r^n + \frac{K(n-2)}{\mu_k-\mu_1}\Big)a^2_{k,i} - C(n)t^3\|u\|^2_{L^2}. \end{aligned}$$

From this estimate, using (3.8), (3.9) and recalling that Lemma 2.1, we readily obtain

$$\begin{aligned} I(E_t) - I(B) &\ge \frac{(n-2)V(B)t^2r^2}{P(B)}\Big(\frac{(n+1)P(B)}{2(n-2)V(B)}r^{n-3} - r^n + \frac{Kn}{2P(B)}\Big)\sum_{k=2}^{\infty}\sum_{i=1}^{G(k,n)} a^2_{k,i} - C(n)t^3\|u\|^2_2 \\ &\ge c(n,\sigma)t^2\sum_{k=2}^{\infty}\sum_{i=1}^{G(k,n)} a^2_{k,i} - C(n)t^3\|u\|^2_{L^2} \ge c(n,\sigma)t^2\|u\|^2_{L^2} - C(n,\sigma)t^3\|u\|^2_{L^22}, \end{aligned}$$

for some suitable constants $c(n,\sigma), C(n,\sigma)$ depending only on n and σ.

From the inequality above, taking t, hence ε_0, sufficiently small we get (3.2). This proves the theorem. □

Observe that there exists a constant $C(n)$ depending only on n such that if E is a nearly spherical set as in (2.2) then

$$\frac{|E\Delta B_r|}{C(n)} \le \|u\|_{L^2(\mathbb{S}^{n-1})} \le C(n)|E\Delta B_r|.$$

In view of the above inequalities we may rewrite the previous theorem in the following equivalent way.

Theorem 3.2. *Let $\sigma \in (0, r_0/2)$, where r_0 is as in (2.11). There exist two positive constants ε_0 and c_1, depending only on n and σ, such that if E is a nearly spherical set satisfying the assumptions of Theorem 3.1, then*

$$I(E) - I(B_r) > c_1|E\Delta B_r|^2. \tag{3.10}$$

4. Second variation

In this section we will calculate the second variation of the functional $I(E)$. The resulting formula will be used to show that a ball B_r with $r > 0$ is never a local minimizer for the functional I with respect to L^1 variations.

First, we fix some notation. Given a vector field $X \in C_c^2(\mathbb{R}^n, \mathbb{R}^n)$, the *associated flow* is defined as the solution of the Cauchy problem

$$\begin{cases} \dfrac{\partial}{\partial t}\Phi(x,t) = X(\Phi(x,t)) \\ \Phi(x,0) = x. \end{cases} \tag{4.1}$$

In the following we shall always write Φ_t to denote the map $\Phi(\cdot, t)$. Note that for any given X there exists $\delta > 0$ such that for $t \in [-\delta, \delta]$, the map Φ_t is a diffeomorphism coinciding with the identity map outside a compact set.

If $E \subset \mathbb{R}^n$ is measurable, we set $E_t := \Phi_t(E)$. Denoting by $J\Phi_t$ the n-dimensional jacobian of $D\Phi_t$, the first and second derivatives $J\Phi_t$ are given by

$$\frac{\partial}{\partial t} J\Phi_t|_{t=0} = \mathrm{div}X, \qquad \frac{\partial^2}{\partial t^2} J\Phi_t|_{t=0} = \mathrm{div}((\mathrm{div}X)X). \tag{4.2}$$

From this formulas we have in particular that if E is a sufficiently smooth open set then

$$\frac{d}{dt}|E_t| = \int_{\partial E_t} X \cdot \nu_{E_t}\, d\mathcal{H}^{n-1}, \qquad \frac{d^2}{dt^2}|E_t| = \int_{\partial E_t} (X \cdot \nu_{E_t})\mathrm{div}X\, d\mathcal{H}^{n-1}.$$

If the flow is *volume preserving*, i.e., $|E_t| = |E|$ for all $t \in [-\delta, \delta]$, then in particular we have that for all $t \in [-\delta, \delta]$

$$\int_{\partial E_t} X \cdot \nu_{E_t}\, d\mathcal{H}^{n-1} = 0, \qquad \int_{\partial E_t} (X \cdot \nu_{E_t})\mathrm{div}X\, d\mathcal{H}^{n-1} = 0. \tag{4.3}$$

Finally, given a sufficiently smooth bounded open E and a vector field X we recall that the *first variation of the perimeter of* E at X is defined by setting

$$\delta P(E)[X] := \frac{d}{dt} P(\Phi_t(E))|_{t=0},$$

where Φ_t is the flow associated with X. The *second variation of the perimeter of* E at X is defined by

$$\delta^2 P(E)[X] := \frac{d^2}{dt^2} P(\Phi_t(E))|_{t=0}.$$

The first and second variations of the functionals R, V and I are defined accordingly.

If E is a C^2 open set we denote by $H_{\partial E}$ its *scalar mean curvature* of ∂E, i.e., the sum of the principal curvatures of ∂E. We denote by $B_{\partial E}$ the *second fundamental form of* ∂E and recall that the square $|B_{\partial E}|^2$ of its euclidean norm is equal to the sum of the squares of the principal curvatures of ∂E.

As we shall see below, the second variation of V involves some nonlocal variants of $H_{\partial E}$ and $|B_{\partial E}|^2$. To this end, if E is a bounded open set of class C^2, we set for every $x \in \partial E$

$$H^*_{\partial E}(x) := 2\int_E \frac{1}{|x-y|^{n-2}}\, dy.$$

The quantity $H^*_{\partial E}$ plays the role of $H_{\partial E}$, while the analogue of $|B_{\partial E}|^2$ is defined by setting for $x \in \partial E$

$$C^2_{\partial E}(x) := \int_{\partial E} \frac{|\nu_E(x) - \nu_E(y)|^2}{|x-y|^{n-2}}\, d\mathcal{H}^{n-1}_y. \tag{4.4}$$

We start by calculating the first and second variation of R.

Lemma 4.1. *Let $E \subset \mathbb{R}^n$ be a bounded open set of class C^2. Assume that $X \in C_c^2(\mathbb{R}^n; \mathbb{R}^n)$. Then*

$$\delta R(E)[X] = \int_{\partial E} \frac{X \cdot \nu_E}{|x|^{n-2}}\, d\mathcal{H}^{n-1}. \tag{4.5}$$

Moreover, if $0 \notin \partial E$,

$$\delta^2 R(E)[X] = \int_{\partial E} \left(\frac{(X\cdot \nu_E)\mathrm{div}X}{|x|^{n-2}} - (n-2)\frac{(X\cdot\nu_E)(X\cdot x)}{|x|^n} \right) d\mathcal{H}^{n-1}. \tag{4.6}$$

Proof. Given $X \in C^2_c(\mathbb{R}^n;\mathbb{R}^n)$, let Φ_t be the associated flow defined as in (4.1). Let $\delta > 0$ be such that the map Φ_t is a diffeomorphism for all $t \in [-\delta,\delta]$. As above, we set $E_t = \Phi_t(E)$ and denote by ∂_t and ∂_{tt} the first and second partial derivatives with respect to t, respectively.

In order to prove the formulas (4.5) and (4.6) we regularize R by setting for $\varepsilon > 0$

$$R_\varepsilon(E) = \int_E \frac{1}{|x|^{n-2}+\varepsilon}\,dx.$$

Since $\Phi_{t+s}(x) = \Phi_s(\Phi_t(x))$, changing variable, we have

$$\begin{aligned}\frac{d}{dt}R_\varepsilon(E_t) &= \frac{d}{ds}R_\varepsilon(E_{t+s})|_{s=0} = \frac{d}{ds}\Big(\int_{E_t}\frac{J\Phi_s}{|\Phi_s|^{n-2}+\varepsilon}\,dx\Big)|_{s=0}\\ &= \Big(\int_{E_t}\frac{\partial_s J\Phi_s}{|\Phi_s|^{n-2}+\varepsilon}\,dx - (n-2)\int_{E_t}\frac{|\Phi_s|^{n-4}\Phi_s\cdot\partial_s\Phi_s}{(|\Phi_s|^{n-2}+\varepsilon)^2}J\Phi_s\,dx\Big)|_{s=0}.\end{aligned}$$

Therefore, recalling the first identity in (4.2), we have

$$\frac{d}{dt}R_\varepsilon(E_t) = \int_{E_t}\left(\frac{\mathrm{div}X}{|x|^{n-2}+\varepsilon} - (n-2)\frac{(X\cdot x)|x|^{n-4}}{(|x|^{n-2}+\varepsilon)^2}\right)dx = \int_{\partial E_t}\frac{X\cdot\nu_{E_t}}{|x|^{n-2}+\varepsilon}\,d\mathcal{H}^{n-1}.$$

From this formula it follows that the functions $R_\varepsilon(t)$ converge uniformly in $[-\delta,\delta]$, together with their first derivatives, as $\varepsilon\to 0$. Thus, (4.5) follows immediately letting $\varepsilon \to 0$.

Let us differentiate $R(E_t)$ once again. Arguing as before we have

$$\begin{aligned}\frac{d^2}{dt^2}R(E_t) = \frac{d^2}{ds^2}R_\varepsilon(E_{t+s})|_{s=0} &= \Big(\int_{E_t}\frac{\partial_{ss}J\Phi_s}{|\Phi_s|^{n-2}+\varepsilon} - 2(n-2)\frac{|\Phi_s|^{n-4}\Phi_s\cdot\partial_s\Phi_s}{(|\Phi_s|^n+\varepsilon)^2}\partial_s J\Phi_s\,dx\Big)|_{s=0}\\ &\quad + \Big(\int_{E_t}\frac{\partial^2}{\partial s^2}\Big(\frac{1}{|\Phi_s|^{n-2}+\varepsilon}\Big)J\Phi_s\,dx\Big)|_{s=0} = J_1(t)+J_2(t).\end{aligned} \tag{4.7}$$

Recalling the identities (4.2), we have

$$\begin{aligned}J_1(t) &= \int_{E_t}\frac{\mathrm{div}(X\mathrm{div}X)}{|x|^{n-2}+\varepsilon}\,dx - 2(n-2)\int_{E_t}\frac{|x|^{n-4}(X\cdot x)\mathrm{div}X}{(|x|^{n-2}+\varepsilon)^2}\,dx\\ &= \int_{E_t}\mathrm{div}\Big(\frac{X\mathrm{div}X}{|x|^{n-2}+\varepsilon}\Big)\,dx - (n-2)\int_{E_t}\frac{|x|^{n-4}(X\cdot x)\mathrm{div}X}{(|x|^{n-2}+\varepsilon)^2}\\ &= \int_{\partial E_t}\frac{(X\cdot\nu_{E_t})\mathrm{div}X}{|x|^{n-2}+\varepsilon}\,d\mathcal{H}^{n-1} - (n-2)\int_{E_t}\frac{|x|^{n-4}(X\cdot x)\mathrm{div}X}{(|x|^{n-2}+\varepsilon)^2},\end{aligned} \tag{4.8}$$

where the last equality follows from the divergence theorem. Differentiating twice $1/(|\Phi_t|^{n-2}+\varepsilon)$ with respect to t, we have, using again the divergence theorem,

$$\begin{aligned}\frac{J_2(t)}{n-2} &= -\int_{E_t}\frac{|x|^{n-4}|X|^2 + |x|^{n-4}\langle DXX, x\rangle + (n-4)|x|^{n-6}(X\cdot x)^2}{(|x|^{n-2}+\varepsilon)^2}\,dx\\ &\qquad\qquad - 2(n-2)\int_{E_t}\frac{(|x|^{n-4}(X\cdot x))^2}{(|x|^{n-2}+\varepsilon)^3}\,dx\\ &= -\int_{E_t}\mathrm{div}\Big(\frac{|x|^{n-4}X(X\cdot x)}{(|x|^{n-2}+\varepsilon)^2}\Big)\,dx + \int_{E_t}\frac{|x|^{n-4}\mathrm{div}X(X\cdot x)}{(|x|^{n-2}+\varepsilon)^2}\,dx\\ &= -\int_{\partial E_t}\frac{|x|^{n-4}(X\cdot\nu_{E_t})(X\cdot x)}{(|x|^{n-2}+\varepsilon)^2}\,d\mathcal{H}^{n-1} + \int_{E_t}\frac{|x|^{n-4}\mathrm{div}X(X\cdot x)}{(|x|^{n-2}+\varepsilon)^2}\,dx.\end{aligned}$$

Then, from this last equality, (4.7) and (4.8), we have

$$\frac{d^2}{dt^2}R(E_t) = \int_{\partial E_t} \frac{(X\cdot\nu_{E_t})\mathrm{div}X}{|x|^{n-2}+\varepsilon}\,d\mathcal{H}^{n-1} - (n-2)\int_{\partial E_t}\frac{|x|^{n-4}(X\cdot\nu_E)(X\cdot x)}{(|x|^{n-2}+\varepsilon)^2}\,d\mathcal{H}^{n-1}.$$

As before the validity of (4.6) follows by observing that since $0\notin\partial E$ also the second derivatives of $R_\varepsilon(E_t)$ converge uniformly in a neighborhood of the origin as $\varepsilon\to 0$. $\square$

Let us now recall the first and second variation formulas for P and V. To this end, we shall denote by div_τ the tangential divergence and by X_τ the tangential component of the vector field X. For a proof of the next lemma we refer to [6, Sect. 6].

Lemma 4.2. *Let $E\subset\mathbb{R}^n$ be a bounded open set of class C^2 and $X\in C^2_c(\mathbb{R}^n;\mathbb{R}^n)$. Then*

$$\begin{aligned}\delta P(E)[X] &= \int_{\partial E} H_{\partial E}(X\cdot\nu_E)\,d\mathcal{H}^{n-1},\\ \delta^2 P(E)[X] &= \int_{\partial E}\left(|D_\tau(X\cdot\nu_E)|^2 - |B_{\partial E}|^2(X\cdot\nu_E)^2\right)d\mathcal{H}^{n-1}\\ &\quad+\int_{\partial E} H_{\partial E}(\mathrm{div}X(X\cdot\nu_E)-\mathrm{div}_\tau((X\cdot\nu_E)X_\tau).\end{aligned}\tag{4.9}$$

Moreover,

$$\begin{aligned}\delta V(E)[X] &= \int_{\partial E} H^*_{\partial E}(X\cdot\nu_E)\,d\mathcal{H}^{n-1},\\ \delta^2 V(E)[X] &= -\int_{\partial E}\int_{\partial E}\frac{|X\cdot\nu_E(x)-X\cdot\nu_E(y)|^2}{|x-y|^{n-2}}\,d\mathcal{H}^{n-1}_x\,d\mathcal{H}^{n-1}_y\\ &\quad+\int_{\partial E} C^2_{\partial E}(X\cdot\nu_E)^2 d\mathcal{H}^{n-1}+\int_{\partial E} H^*_{\partial E}(\mathrm{div}X(X\cdot\nu_E)-\mathrm{div}_\tau((X\cdot\nu_E)X_\tau))d\mathcal{H}^{n-1}.\end{aligned}\tag{4.10}$$

Definition 4.3. We say that a set of locally finite perimeter $E\subset\mathbb{R}^n$ is a *constrained, strict L^1-local minimizer* for the functional I if there exists $\delta>0$ such that whenever F is a set of locally finite perimeter such that $|F|=|E|$ and $0<|E\Delta F|\le\delta$, then

$$I(F)>I(E).$$

Using (4.5), (4.9) and (4.10), it is easily checked that if E is a C^2, bounded constrained local minimizer for I, there exists $\lambda\in\mathbb{R}$ such that

$$H_{\partial E}+H^*_{\partial E}-\frac{K}{|x|^{n-2}}=\lambda \qquad \text{on } \partial E.\tag{4.11}$$

Conversely, any C^2 bounded open set satisfying (4.11) will be called *a constrained critical set* for the functional I. Note that any ball B_r centered at the origin trivially satisfies (4.11), hence it is a constrained critical set for I. Moreover, if $0\notin\partial E$ and the flow associated with X is volume preserving, then, setting $\phi := X\cdot\nu_{B_r}$, we have, recalling (4.3),

$$\begin{aligned}\delta^2 I(B_r)[X] := \partial^2 I(B_r)[\phi] &= \int_{\partial B_r}\left(|D_\tau\phi|^2-\frac{n-1}{r^2}\phi^2\right)d\mathcal{H}^{n-1}+\frac{K(n-2)}{r^{n-1}}\int_{\partial B_r}\phi^2 d\mathcal{H}^{n-1}\\ &\quad-\int_{\partial B_r}\int_{\partial B_r}\frac{|\phi(x)-\phi(y)|^2}{|x-y|^{n-2}}d\mathcal{H}^{n-1}+rC^2_{\mathbb{S}^{n-1}}\int_{\partial B_r}\phi^2 d\mathcal{H}^{n-1}.\end{aligned}\tag{4.12}$$

Given a function $\phi\in H^1(\partial B_r)$ with $\int_{\partial B_r}\phi=0$, it is always possible to construct a sequence of vector fields $X_j\in C^\infty_c(\mathbb{R}^n;\mathbb{R}^n)$, such that div $X_j=0$ in a ball B_R with $R>r$ and such that $X_j\cdot\nu_E\to\phi$ in $H^1(\partial B_r)$, see for instance [1, Cor. 3.4]. Since the flows associated with the vector fields X_j are all volume preserving, from this approximation result and (4.12) it follows immediately that for any function $\phi \in H^1(\partial B_r)$ with $\int_{\partial B_r} \phi = 0$

$$\partial^2 I(B_r)[\phi] \geq 0. \tag{4.13}$$

Next result shows that for r sufficiently large the ball B_r is a never a constrained local minimizer for I.

Theorem 4.4. *Let $r_0 > 0$ be as in (2.11). If $r > r_0$ the ball B_r is not a constrained local minimizer of I.*

Proof. Fix $r > 0$. From lemma 2.1 it follows that there exists $k \geq 2$ such that

$$\frac{\lambda_k - \lambda_1}{\mu_k - \mu_1} r^{n-3} - r^n + \frac{K(n-2)}{\mu_k - \mu_1} < 0. \tag{4.14}$$

For every $x \in \partial B_r$ set $\phi(x) := y_k(x/r)$, where y_k is the restriction to $\mathbb{S}^{n-1}$ of a homogeneous harmonic polynomials of degree k, normalized so that $||y_k||_{L^2(\mathbb{S}^{n-1})} = 1$. Recalling that $\|D_\tau y_k\|_{L^2(\mathbb{S}^{n-1})} = \lambda_k$, from (4.13) we have

$$\partial^2 I(B_r)[\phi] = (\lambda_k - \lambda_1) r^{n-3} + K(n-2) - \int_{\partial B_r}\int_{\partial B_r} \frac{|\phi(x) - \phi(y)|^2}{|x-y|^{n-2}} d\mathcal{H}^{n-1} + r^n C^2_{\mathbb{S}^{n-1}}. \tag{4.15}$$

On the other hand from (2.6) we have

$$\int_{\partial B_r}\int_{\partial B_r} \frac{|\phi(x) - \phi(y)|^2}{|x-y|^{n-2}} d\mathcal{H}^{n-1} = \mu_k r^n \int_{\mathbb{S}^{n-1}} \phi^2 d\mathcal{H}^{n-1} = \mu_k r^n. \tag{4.16}$$

From the definition (4.4), using again (2.6) and recalling that the first order normalized spherical harmonic are the functions $x_i/\sqrt{\omega_n}$, we have

$$\begin{aligned} C^2_{\mathbb{S}^{n-1}} &= \frac{1}{n\omega_n} \int_{\mathbb{S}^{n-1}}\int_{\mathbb{S}^{n-1}} \frac{|x-y|^2}{|x-y|^{n-2}} \, d\mathcal{H}^{n-1}_x \, d\mathcal{H}^{n-1}_y = \frac{1}{n\omega_n} \sum_{i=1}^n \int_{\mathbb{S}^{n-1}}\int_{\mathbb{S}^{n-1}} \frac{|x_i - y_i|^2}{|x-y|^{n-2}} \, d\mathcal{H}^{n-1}_x \, d\mathcal{H}^{n-1}_y \\ &= \frac{1}{n\omega_n} \mu_1 \sum_{i=1}^n \left(\int_{\mathbb{S}^{n-1}} \frac{x_i^2}{\sqrt{\omega_n}} \, d\mathcal{H}^{n-1} \right)^2 = \mu_1. \end{aligned}$$

Therefore, from the equality above, (4.15), (4.16) and (4.14) we get that

$$\partial^2 I(B_r)[\phi] = (\lambda_k - \lambda_1) r^{n-3} + K(n-2) - (\mu_k - \mu_1) r^n < 0.$$

Hence, the result follows. □

5. L^1-LOCAL MINIMALITY

In this section we show the main result of the paper, i.e., the strict L^1-local minimality of balls centered at the origin with radius smaller than the radius r_0 defined in (2.11). This result will be proved using Theorem 3.1, following a strategy first introduced in this framework in [5] and later on improved in [1]. Our result goes as follows.

Theorem 5.1. *Let $n \geq 3$, $\sigma \in (0, r_0/2)$, where r_0 is defined as in (2.11). There exist δ, γ, depending only on n, K, σ, such that if $E \subset \mathbb{R}^n$ is a measurable set such that $|E \Delta B_r| \leq \delta$ and $|E| = |B_r|$, then*

$$I(E) \geq I(B_r) + \gamma |E \Delta B_r|^2.$$

Before giving the proof, we recall some key definitions and results from the regularity theory for sets of finite perimeter.

Definition 5.2. Let $n \geq 2, \Lambda, r > 0$. We say that a set $E \subset \mathbb{R}^n$ of locally finite perimeter is a (Λ, r)-*almost minimizer of the perimeter* if for every ball $B_\varrho(x)$ with $\varrho < r$ and any set F such that $E\Delta F \subset\subset B_\varrho(x)$, then

$$P(E; B_\varrho(x)) \leq P(F; B_\varrho(x)) + \Lambda|E\Delta F|.$$

Next result is an important application of the regularity estimates for almost minimizers of the perimeter. For a proof, we refer to [15, Th. 26.6] and to [17]. To this end, we recall that a sequence of measurable sets $E_h \subset \mathbb{R}^n$ is said to *converge in measure* in an open set Ω to a measurable set $E \subset \mathbb{R}^n$ if

$$\lim_h \big|(E_h\Delta E) \cap \Omega\big| = 0,$$

that is if the characteristic functions χ_{E_h} converge to χ_E in $L^1(\Omega)$.

Theorem 5.3. *Let $n \geq 2$ and let $E_h \subset \mathbb{R}^n$ be a sequence of equibounded (Λ, r)-almost minimizers converging in measure to an open set E of class C^2. There exists h_0 such that, for $h \geq h_0$, ∂E_h is of class $C^{1,\frac{1}{2}}$ and*

$$\partial E_h = \{x + \psi_h(x)\nu_E(x) : \ x \in \partial E\}.$$

Moreover, $\psi_h \to 0$ in $C^{1,\alpha}$ for all $\alpha \in (0, \frac{1}{2})$.

We start with a simple lemma on the potential energy V.

Lemma 5.4. *Let $F, E \subset \mathbb{R}^n$ be measurable sets and $|F| < \infty$. Then*

$$V(F) - V(E) \leq \frac{n}{\omega_n}|F|^{\frac{2}{n}}|F \setminus E|. \tag{5.1}$$

Proof. Denote by r the radius of a ball with the same measure of F. Note that for every measurable set G with $|G| = |B_r|$

$$\int_G \frac{1}{|x|^{n-2}}\, dx \leq \int_{B_r} \frac{1}{|x|^{n-2}}\, dx.$$

Thus we have

$$\begin{aligned} V(F) - V(E) &\leq 2\int_{F\setminus E} dx \int_F \frac{1}{|x-y|^{n-2}}\, dy = 2\int_{F\setminus E} dx \int_{x-F} \frac{1}{|z|^{n-2}}\, dz \\ &\leq 2|F \setminus E| \int_{B_r} \frac{1}{|x|^{n-2}}\, dx = |F \setminus E| n\omega_n r^2. \end{aligned}$$

Hence, (5.1) follows. $\square$

Let us now state another simple lemma which we be useful to treat the perimeter term and the attraction term in the energy. In all the remaining part of this section we shall always assume $n \geq 3$.

Lemma 5.5. *Let $\sigma \in (0, r_0/2)$, where r_0 is defined as in (2.11). There exists Λ_0, depending on n, K, σ, such that if $\Lambda \geq \Lambda_0$ and $r \in [\sigma, r_0]$, the ball B_r is the unique minimizer of the functional*

$$P(E) - KR(E) + \Lambda\big||E| - |B_r|\big|$$

among all sets of finite measure.

Proof. Recall that for every set E of finite measure, we have

$$P(B_{r_E}) - KR(B_{r_E}) \leq P(E) - KR(E),$$

where B_{r_E} is the ball with the same volume of E. Therefore, to prove the lemma it is enough to show that if Λ is sufficiently large, then the function

$$f(\varrho) := n\varrho^{n-1} - \frac{Kn\varrho^2}{2} + \Lambda\big|\varrho^n - r^n\big|$$

has a unique minimum in $[0,\infty)$ at $\varrho = r$. Indeed if $0 \le \varrho \le r$

$$f'(\varrho) = n(n-1)\varrho^{n-2} - Kn\varrho - \Lambda n \varrho^{n-1} \le 0,$$

provided $\Lambda \ge (n-1)/r_0$. Similarly, if $\varrho \ge r$

$$f'(\varrho) = n(n-1)\varrho^{n-2} - Kn\varrho + \Lambda n \varrho^{n-1} \ge 0,$$

provided $\Lambda \ge K/\sigma^{n-2}$. Then the conclusion follows from the two previous estimates choosing $\Lambda \ge \max\{(n-1)/r_0, K/\sigma^{n-2}\}$. □

Lemma 5.6. *There exists $C_1 > 0$, depending only on n, such that, if $\eta \in (0,1)$ and $E \subset \mathbb{R}^n$ is a measurable set such that $|E \setminus B_r| < \eta$ for some $r > 0$, then we can find $r \le r_E \le r + C_1\eta^{\frac{1}{n}}$ such that*

$$P(E \cap B_{r_E}) \le P(E) - \frac{|E \setminus B_{r_E}|}{C_1 \eta^{\frac{1}{n}}} \tag{5.2}$$

Proof. For any $\varrho > 0$ we set $u(\varrho) := |E \setminus B_\varrho|$. By the area formula $u'(\varrho) = -\mathcal{H}^{n-1}(\partial B_\varrho \cap E)$ for $\mathcal{L}^1$-a.e. $\varrho > 0$. We set

$$C_1 := \frac{2n+1}{(n\omega_n)^{\frac{1}{n}}}. \tag{5.3}$$

If $u(r + C_1\eta^{\frac{1}{n}}) = 0$ then (5.2) trivially holds with $r_E = r + C_1\eta^{\frac{1}{n}}$.
If $u(r + C_1\eta^{\frac{1}{n}}) > 0$ we argue by contradiction assuming that for every $r \le \varrho \le r + C_1\eta^{\frac{1}{n}}$

$$-2u'(\varrho) - P(E \setminus B_\varrho) = P(E \cap B_\varrho) - P(E) > -\frac{u(\varrho)}{C_1\eta^{\frac{1}{n}}}.$$

Using the isoperimetric inequality we have that for all $\varrho \in \left(r, r + C_1\eta^{\frac{1}{n}}\right)$

$$-u'(\varrho) > \frac{1}{2}(n\omega_n)^{\frac{1}{n}}|E \setminus B_\varrho|^{\frac{n-1}{n}} - \frac{u(\varrho)}{2C_1\eta^{\frac{1}{n}}} = \frac{1}{2}(n\omega_n)^{\frac{1}{n}}u(\varrho)^{\frac{n-1}{n}} - \frac{u(\varrho)}{2C_1\eta^{\frac{1}{n}}}.$$

Since $u(\varrho) \le |E \setminus B_r| < \eta$, recalling the definition (5.3) of C_1, we get

$$-u'(\varrho) > \frac{n}{C_1}u(\varrho)^{\frac{n-1}{n}} \qquad \text{for all } r \le \varrho \le r + C_1\eta^{\frac{1}{n}}.$$

Integrating this inequality in $\left(r, r + C_1\eta^{\frac{1}{n}}\right)$ we obtain

$$u(r)^{\frac{1}{n}} - u(r + C_1\eta^{\frac{1}{n}})^{\frac{1}{n}} > \eta^{\frac{1}{n}},$$

which contradicts the assumption $\eta > |E \setminus B_r|$. Hence the result follows. □

The following lemma will be used in the proof of Theorem 5.1.

Lemma 5.7. *Let $\sigma \in (0, r_0/2)$, where r_0 is defined as in (2.11), and let $\Lambda_1 \ge 2n\omega_n^{\frac{2-n}{n}} r_0^2$, $\Lambda_2 \ge \Lambda_0$, with Λ_0 as in Lemma 5.5. There exists $\varepsilon_0 > 0$ such that if $0 \le \varepsilon \le \varepsilon_0$ and $r \in [\sigma, r_0 - \sigma]$, then the minimum problem*

$$\min\left\{I(E) + \Lambda_1\big||E\Delta B_r| - \varepsilon\big| + \Lambda_2\big||E| - |B_r|\big| : |E| < \infty\right\}$$

as at least a solution $F \subset B_R$, with $R = r_0 + C_1$, where C_1 is the constant in Lemma 5.6.

Proof. Given a set of finite perimeter and finite measure E, we define for $\varepsilon > 0$

$$J_\varepsilon(E) := I(E) + \Lambda_1\big||E\Delta B_r| - \varepsilon\big| + \Lambda_2\big||E| - |B_r|\big|.$$

Let E_h be a minimizing sequence for J_ε such that

$$J_\varepsilon(E_h) \le \inf J_\varepsilon + \frac{1}{h}.$$

From this inequality, recalling Lemmas 5.4 and 5.5, we have,

$$J_\varepsilon(E_h) \le I(B_r) + \Lambda_1\varepsilon + \frac{1}{h} \le I(E_h) + V(B_r) - V(E_h) + \Lambda_1\varepsilon + \Lambda_2||E_h| - |B_r|| + \frac{1}{h}$$
$$\le I(E_h) + n\omega_n^{\frac{2-n}{n}} r^2|B_r \setminus E_h| + \Lambda_1\varepsilon + \Lambda_2||E_h| - |B_r|| + \frac{1}{h}.$$

Therefore, from this inequality, recalling that $\Lambda_1 \ge 2n\omega_n^{\frac{2-n}{n}} r_0^2$, we have

$$\Lambda_1||E_h\Delta B_r| - \varepsilon| \le \frac{\Lambda_1}{2}|B_r \setminus E_h| + \Lambda_1\varepsilon + \frac{1}{h},$$

hence

$$|E_h\Delta B_r| \le 4\varepsilon + \frac{2}{h\Lambda_1}.$$

Assume now that $\varepsilon \le \varepsilon_0 < 1/5$, with ε_0 to be chosen. Set $\eta := 5\varepsilon_0$. For h so large that $4\varepsilon + 2/(h\Lambda_1) < \eta$, denote by $r_h := r_{E_h} \in [r, r + C_1\eta^{\frac{1}{n}}]$ the radius provided by Lemma 5.6. Thus, recalling (5.2), we estimate for h large

$$J_\varepsilon(E_h \cap B_{r_h}) \le \Big(P(E_h) - \frac{|E_h \setminus B_{r_h}|}{C_1\eta^{\frac{1}{n}}}\Big) + V(E_h) - KR(E_h) + KR(E_h \setminus B_{r_h}) + \Lambda_1||E_h\Delta B_r| - \varepsilon|$$
$$+ \Lambda_1||(E_h \cap B_{r_h})\Delta B_r| - |E_h\Delta B_r|| + \Lambda_2||E_h| - |B_r|| + \Lambda_2|E_h \setminus B_{r_h}|$$
$$\le J_\varepsilon(E_h) + \Big(\frac{K}{r_h^{n-2}} + \Lambda_1 + \Lambda_2 - \frac{1}{C_1\eta^{\frac{1}{n}}}\Big)|E_h \setminus B_{r_h}| \le J_\varepsilon(E_h),$$

provided we choose η, hence ε_0, sufficiently small. Thus also $E_h \cap B_{r_h}$ is a minimizing sequence for J_ε. Since for h large the sets $E_h \cap B_{r_h} \subset B_R$ are equibounded and have equibounded perimeters, a standard argument shows that up to a not relabelled subsequence theyconverge in measure to a set $E \subset B_R$ who is an absolute minimizer for J_ε. □

In order to make the presentation clearer we split the proof of Theorem 5.1 in several lemmas. We will argue by contradiction.

Let r_0 be defined as in (2.11). Given $\sigma \in (0, r_0/2)$, we assume that there exists a sequence E_h of sets such that $|E_h| = |B_{r_h}|$, with $r_h \in [\sigma, r_0 - \sigma]$ and

$$\lim_h |E_h\Delta B_r| = 0, \qquad I(E_h) \le I(B_{r_h}) + C_0|E_h\Delta B_{r_h}|^2 \quad \text{for all} \quad h \in \mathbb{N}, \tag{5.4}$$

for some $C_0 > 0$ to be fixed later.

The idea of the proof is to replace the sets E_h with a sequence of sets still satisfying (5.4), possibly with a larger constant, and converging in C^1 to a ball B_r with $0 < r < r_0$. This convergence will then contradict the quantitative estimate (3.10), provided C_0 is sufficiently small.

To this end we consider the functionals

$$J_{\varepsilon_h}(E) := I(E) + \Lambda_1||E\Delta B_{r_h}| - \varepsilon_h| + \Lambda_2||E| - |B_{r_h}||, \tag{5.5}$$

where $\varepsilon_h := |E_h\Delta B_{r_h}|$, and Λ_1, Λ_2 satisfy the assumptions of Lemma 5.7. Thanks to this lemma we may conclude that for h sufficiently large the functional J_{ε_h} has an absolute minimizer F_h contained in B_R, where R is the radius provided by the lemma.

Next lemma shows that the above minimizers F_h converge in measure to a ball.

Lemma 5.8. *Let the sets F_h be defined as above. Then, up to a subsequence, they converge in measure to a ball B_r with $r \in [\sigma, r_0 - \sigma]$.*

Proof. Recall that for h large the sets F_h are equibounded. Moreover, still assuming h sufficiently large,

$$J_{\varepsilon_h}(F_h) \le J_{\varepsilon_h}(B_{r_h}) = I(B_{r_h}) + \Lambda_1\varepsilon_h \le C,$$

for some $C > 0$ independent of h. Thus, the F_h have equibounded perimeters. Therefore, up to a not relabeled subsequence, we may assume that they converge in measure to a set $F_\infty \subset B_R$ and that $r_h \to r \in [\sigma, r_0 - \sigma]$. It is easily checked that F_∞ is a minimizer of the functional

$$J(E) = I(E) + \Lambda_1|E\Delta B_r| + \Lambda_2||E| - |B_r||.$$

Let us now show that $F_\infty = B_r$. To this end we estimate, using Lemmas 5.5 and 5.4 ,

$$\begin{aligned}
J(F_\infty) &= P(F_\infty) + V(F_\infty) - KR(F_\infty) + \Lambda_1|F_\infty\Delta B_r| + \Lambda_2||F_\infty| - |B_r|| \\
&\geq P(B_r) + V(F_\infty) - KR(B_r) + \Lambda_1|F_\infty\Delta B_r| \\
&= J(B_r) + V(F_\infty) - V(B_r) + \Lambda_1|F_\infty\Delta B_r| \\
&\geq J(B_r) - n\omega_n^{\frac{2-n}{n}} r_0^2|B_r \setminus F_\infty| + \Lambda_1|F_\infty\Delta B_r|.
\end{aligned}$$

Then the conclusion follows by recalling that $\Lambda_1 \geq 2n\omega_n^{\frac{2-n}{n}} r_0^2$. □

The next lemma provides a density estimate for F_h. We give only a sketch of the proof since it follows quite closely a standard argument in the regularity theory of sets of finite perimeter.

Lemma 5.9. *There exist $\varrho_0 > 0$ and $\vartheta_0 > 0$ such that if $F_h \subset B_R$ is a minimizer of J_{ε_h} and $0 < \varrho \leq \varrho_0$ then for all $y \in \partial^* F_h$*

$$\frac{|F_h \cap B_\varrho(y)|}{|B_\varrho(y)|} \leq 1 - \vartheta_0. \tag{5.6}$$

Proof. From the minimality of F_h we have that $J_{\varepsilon_h}(F_h) \leq J_{\varepsilon_h}(F_h \cup B_\varrho(y))$ for all $\varrho \in (0,1)$. From this inequality we get that for $\mathcal{L}^1$-a.e. $\varrho \in (0,1)$

$$\begin{aligned}
P(F_h; B_\varrho(y)) &\leq \mathcal{H}^{n-1}(\partial B_\varrho(y) \cap F_h) + V(B_\varrho(y) \setminus F_h) - V(F_h) + KR(F_h) - KR(B_\varrho(y) \cup F_h) \\
&\quad + \Lambda_1||(B_\varrho(y) \cup F_h)\Delta B_{r_h}| - |F_h\Delta B_{r_h}|| + \Lambda_2||B_\varrho(y) \cup F_h| - |F_h|| \\
&\leq \mathcal{H}^{n-1}(\partial B_\varrho(y) \setminus F_h) + C|B_\varrho(y) \setminus F_h|,
\end{aligned}$$

where the constant C depends only on n, r_0, Λ_1 and Λ_2. Starting from this estimate, the conclusion then follows arguing exactly as in [15, Th. 16.14]. □

Lemma 5.10. *Let $\sigma \in (0, r_0/2)$, Λ_1, Λ_2 and ε_h be as above and let $F_h \subset B_R$ be a minimizer of the functional J_{ε_h} defined in* (5.5). *There exist $\Lambda, \overline{r} > 0$ and a not relabelled subsequence F_h such that every F_h is a $(\Lambda, \overline{r})$-almost minimizer of the perimeter.*

Proof. Observe that by Lemma 5.8 it follows that, passing possibly to a subsequence, we may assume that F_h converge in measure to a ball B_r with $r \in [\sigma, r_0 - \sigma]$. We set $\bar{\varrho} := \min\{\sigma/2, \varrho_0\}$, where ϱ_0 is the radius provided by Lemma 5.9. We claim that there exists h_0 such that

$$|B_{\bar{\varrho}} \setminus F_h| = 0 \qquad \text{for all } h \geq h_0. \tag{5.7}$$

Indeed, if the above claim were not true we could find a subsequence F_{h_k} such that $|B_{\bar{\varrho}} \setminus F_{h_k}| > 0$ for all k. Since F_h converges in measure to B_r and $r \geq 2\bar{\varrho}$, we may also assume that $|B_{\bar{\varrho}} \cap F_{h_k}| > 0$ for all k. Therefore, by the relative isoperimetric inequality we get that $P(F_{h_k}; B_{\bar{\varrho}}) > 0$. Hence, for all k there exists $y_k \in \partial^* F_{h_k} \cap B_{\bar{\varrho}}$. Passing possibly to another to a subsequence, we may assume that $y_k \to y \in \overline{B}_{\bar{\varrho}}$. By applying the estimate (5.6) we the get

$$|B_{\bar{\varrho}}(y)| = \lim_k |F_{h_k} \cap B_{\bar{\varrho}}(y_k)| \leq (1 - \vartheta_0) \lim_k |B_{\bar{\varrho}}(y_k)| = (1 - \vartheta_0)|B_{\bar{\varrho}}(y)|.$$

This contradiction proves (5.7).

Let us now set $\overline{r} = \bar{\varrho}/3$. Let $E \subset \mathbb{R}^n$ such that $E\Delta F_h \subset B_\varrho(y)$, with $\varrho < \overline{r}$ and $h \geq h_0$. If $|y| \leq 2\overline{r}/3$, then, since $B_\varrho(y) \cap F_h = B_\varrho(y)$ by (5.7), we have $P(F_h; B_\varrho(y) = 0$, hence, trivially

$$P(F_h; B_\varrho(y)) \leq P(E; B_\varrho(y)).$$

If instead $|y| > 2\overline{r}/3$, we are going to show that

$$P(F_h; B_\varrho(y)) \leq P(E; B_\varrho(y)) + \Lambda|E\Delta F_h|,$$

for some $\Lambda > 0$ that will be chosen below. From the minimality of F_h we get, recalling (5.1),

$$\begin{aligned} P(F_h; B_\varrho(y)) &\leq P(E; B_\varrho(y)) + V(E) - V(F_h) - KR(E) + KR(F_h) \\ &\qquad + \Lambda_1||F_h\Delta B_r| - |E_h\Delta B_r| + \Lambda_2||F_h|| - |E|| \\ &\leq P(E; B_\varrho(y)) + \frac{n}{\omega_n}|E|^{\frac{2}{n}}|E \setminus F_h| + K\int_{F_h\Delta E}\frac{1}{|x|^{n-2}}dx + (\Lambda_1 + \Lambda_2)|F_h\Delta E|. \end{aligned}$$

Since $|E| \leq |F_h| + |B_{\overline{r}}| \leq C(n, r_0)$ and $F_h\Delta E \subset \mathbb{R}^n \setminus B_{\overline{r}/3}$ from the above estimate we easily get that

$$P(F_h; B_\varrho(y)) \leq P(E; B_\varrho(y)) + C(n, r_0, \Lambda_1, \Lambda_2)|F_h\Delta E| + C(n)\overline{r}^{2-n}K|F_h\Delta E|,$$

for some postive constants $C(n)$ and $C(n, r_0, \Lambda_1, \Lambda_2)$. From this inequality the conclusion immediately follows by taking Λ sufficiently large. □

We are ready now to prove Theorem 5.1.

Proof of Theorem 5.1. **Step 1.** Given $\sigma \in (0, r_0/2)$, we argue by contradiction, assuming that there exists a sequence of sets of finite perimeter E_h satisfying (5.4). Then, we take $\Lambda_1 \geq 2n\omega_n^{\frac{2-n}{n}} r_0^2$ and $\Lambda_2 \geq \max\{2\Lambda_0, 4\Lambda_1\}$ and consider a sequence F_h of minimizers of the functionals (5.5), where $\varepsilon_h = |E_h\Delta B_{r_h}|$. Thanks to Lemma 5.7 and Lemma 5.8 we may assume, passing possibly to a subsequence, that $F_h \subset B_R$ for all h and that they converge in measure to the ball B_r for some $r \in [\sigma, r_0 - \sigma]$. Then by Lemma 5.10 we may also assume that the F_h are all $(\Lambda, \overline{r})$-almost minimizers of the perimeter for some $\Lambda, \overline{r} > 0$. Therefore, Theorem 5.3 yields that the sequence F_h converges in $C^{1,\alpha}$ to B_r. In particular, denoting by $\widetilde{r}_h$ the radius of the ball such that $|F_h| = |B_{\widetilde{r}_h}|$, we may assume that $\widetilde{r}_h \in [\sigma/2, r_0 - \sigma/2]$ for all h and that there exists a sequence $\psi_h \in C^1(\mathbb{S}^{n-1})$ converging in C^1 to 0 such that for all h

$$F_h = \{y = \widetilde{r}_h x(1 + \psi_h(x)) : x \in B\}. \tag{5.8}$$

By the minimality of the F_h, recalling Lemma 5.5 and Lemma 5.4 we have

$$\begin{aligned} I(F_h) + \Lambda_1\big||F_h\Delta B_{r_h}| - \varepsilon_h\big| + \Lambda_2||F_h| - |B_{r_h}|| &\leq I(E_h) \leq I(B_{r_h}) + C_0|E_h\Delta B_r|^2 \\ &\leq I(F_h) + V(B_{r_h}) - V(F_h) + \Lambda_0||F_h| - |B_{r_h}|| + C_0\varepsilon_h^2 \\ &\leq I(F_h) + n\omega_n^{\frac{2-n}{n}} r_0^n|B_{r_h} \setminus F_h| + \Lambda_0||F_h| - |B_{r_h}|| + C_0\varepsilon_h^2 \\ &\leq I(F_h) + \frac{\Lambda_1}{2}|B_{r_h}\Delta F_h| + \frac{\Lambda_2}{2}||F_h| - |B_{r_h}|| + C_0\varepsilon_h^2, \end{aligned} \tag{5.9}$$

where the last inequality follows from the choice of Λ_1 and Λ_2. From the above inequality we then get easily that

$$\varepsilon_h + \frac{\Lambda_2}{2\Lambda_1}||F_h| - |B_{r_h}|| \leq \frac{3}{2}|B_{r_h}\Delta F_h| + \frac{C_0}{\Lambda_1}\varepsilon_h^2.$$

Note that in particular we have that for h large $\varepsilon_h \le 2|B_{r_h}\Delta F_h|$. Therefore, passing possible to another subsequence if needed, we may assume without loss of generality that for all h

$$\begin{aligned}\varepsilon_h + \frac{\Lambda_2}{2\Lambda_1}||F_h|-|B_{r_h}|| &\le 2|B_{r_h}\Delta F_h| \le 2|F_h\Delta B_{\widetilde{r}_h}| + 2||B_{\widetilde{r}_h}|-|B_{r_h}|| \\ &= 2\ |F_h\Delta B_{\widetilde{r}_h}| + 2||F_h|-|B_{r_h}||.\end{aligned}$$

Recalling that we have chosen $\Lambda_2 \ge 4\Lambda_1$ from the above inequality we have that for all h

$$\varepsilon_h \le 2|F_h\Delta B_{\widetilde{r}_h}|. \tag{5.10}$$

Thus, using the second inequality in (5.9) we have that for all h

$$\begin{aligned}I(F_h) + \Lambda_1\big||F_h\Delta B_{r_h}| - \varepsilon_h\big| + \Lambda_2||F_h|-|B_{r_h}|| &\le I(B_{r_h}) + C_0\varepsilon_h^2 \\ &\le I(B_{\widetilde{r}_h}) + C(n,\sigma)|\widetilde{r}_h - r_h| + C_0\varepsilon_h^2,\end{aligned}$$

for a positive constant $C(n,\sigma)$ independent of h. Note however that there exists another constant $c(n,\sigma)$ still depending only on n and σ, such that

$$c(n,\sigma)|\widetilde{r}_h - r_h| \le ||F_h|-|B_{r_h}||.$$

Therefore, choosing $\Lambda_2 \ge C(n,\sigma)/c(n,\sigma)$, and Λ_1 accordingly, we have, recalling (5.10),

$$I(F_h) \le I(B_{\widetilde{r}_h}) + 4C_0|F_h\Delta B_{\widetilde{r}_h}|^2. \tag{5.11}$$

Let us now denote by x_h the barycenter of F_h and observe that

$$|x_h| = \frac{1}{|B_{\widetilde{r}_h}|}\left|\int_{F_h} x\,dx\right| \le \frac{1}{|B_{\widetilde{r}_h}|}\int_{F_h\Delta B_{\widetilde{r}_h}} |x|\ dx \to 0 \qquad \text{as } h\to\infty.$$

We set $G_h := F_h - x_h$. Since x_h is converging to 0, from (5.8) we deduce that there exists a sequence $\varphi_h \in C^1(\mathbb{S}^{n-1})$ converging in C^1 to 0 such that for all h

$$G_h = \{y = \widetilde{r}_h x(1+\varphi_h(x)) : x \in B\}. \tag{5.12}$$

Step 2. We now estimate $R(F_h) - R(G_h)$. To this end we use Lemma 4.1 (note that $0 \notin \partial G_h$), observing that $F_h = \Phi_1(G_h)$, where the flow is given by $\Phi_t(x) := x + tx_h$. Thus, recalling (4.5) and (4.6) we have

$$R(F_h) - R(G_h) = \int_{\partial G_h} \frac{x_h\cdot\nu_{G_h}}{|x|^{n-2}}\,d\mathcal{H}^{n-1} - \frac{n-2}{2}\int_{\partial G_{h,t_h}} \frac{(x_h\cdot\nu_{G_{h,t_h}})(x_h\cdot x)}{|x|^n}\,d\mathcal{H}^{n-1}, \tag{5.13}$$

where $G_{h,t_h} = G_h + t_h x_h$ for some $t_h \in (0,1)$. Observe now that

$$\begin{aligned}&\left|\int_{\partial G_{h,t_h}} \frac{(x_h\cdot\nu_{G_{h,t_h}})(x_h\cdot x)}{|x|^n}\,d\mathcal{H}^{n-1} - \int_{\partial G_h}\frac{(x_h\cdot\nu_{G_h})(x_h\cdot x)}{|x|^n}\,d\mathcal{H}^{n-1}\right| \\ &\qquad = \left|\int_{\partial G_h}(x_h\cdot\nu_{G_h}(x))\Big(\frac{x_h\cdot(x+t_hx_h)}{|x+t_hx_h|^n} - \frac{x_h\cdot x}{|x|^n}\Big)\,d\mathcal{H}^{n-1}\right| \le C|x_h|^3.\end{aligned}$$

Therefore, from (5.13) we have

$$R(F_h) - R(G_h) = \int_{\partial G_h}\frac{x_h\cdot\nu_{G_h}}{|x|^{n-2}}\,d\mathcal{H}^{n-1} - \frac{n-2}{2}\int_{\partial G_h}\frac{(x_h\cdot\nu_{G_h})(x_h\cdot x)}{|x|^n}\,d\mathcal{H}^{n-1} + o(|x_h|^2). \tag{5.14}$$

Recalling (5.12), we have that at the point $y = \widetilde{r}_h x(1+\varphi_h(x))$ with $x \in \mathbb{S}^{n-1}$,

$$\nu_{G_h}(z) = \frac{x(1+\varphi_h(x)) - D_\tau\varphi_h(x)}{\sqrt{(1+\varphi_h(x))^2 + |D_\tau\varphi_h(x)|^2}}.$$

Thus, denoting by div_τ the tangential divergence on the sphere and using the divergence theorem, we get

$$\begin{aligned}\int_{\partial G_h} \frac{x_h\cdot\nu_{G_h}}{|x|^{n-2}}\,d\mathcal{H}^{n-1} &= \widetilde{r}_h\int_{\mathbb{S}^{n-1}} x_h\cdot\big(x(1+\varphi_h(x)) - D_\tau\varphi_h(x)\big)\,d\mathcal{H}^{n-1}\\ &= \widetilde{r}_h\int_{\mathbb{S}^{n-1}}(x_h\cdot x)\varphi_h\,d\mathcal{H}^{n-1} - \widetilde{r}_h\int_{\mathbb{S}^{n-1}}\mathrm{div}_\tau(x_h\varphi_h)\,d\mathcal{H}^{n-1}\\ &= -(n-2)\widetilde{r}_h\int_{\mathbb{S}^{n-1}}(x_h\cdot x)\varphi_h\,d\mathcal{H}^{n-1}.\end{aligned}$$

Since G_h has barycenter at the origin, arguing as in the proof of (3.9) we have that for h large

$$\left|\int_{\partial G_h}\frac{x_h\cdot\nu_{G_h}}{|x|^{n-2}}\,d\mathcal{H}^{n-1}\right| = (n-2)\widetilde{r}_h\left|x_h\cdot\int_{\partial\mathbb{S}^{n-1}} x\varphi(x)\,d\mathcal{H}^{n-1}\right| \le C(n)|x_h|\|\varphi_h\|^2_{L^2(\mathbb{S}^{n-1})}. \tag{5.15}$$

Let us now estimate the second integral on the right hand side of (5.14). To this end we estimate

$$\begin{aligned}\int_{\partial G_h}\frac{(x_h\cdot\nu_{G_h})(x_h\cdot x)}{|x|^n}\,d\mathcal{H}^{n-1} &= \int_{\mathbb{S}^{n-1}}\frac{\big(x_h\cdot(x(1+\varphi_h)-D_\tau\varphi_h)\big)(x_h\cdot x)}{1+\varphi_h(x)}\,d\mathcal{H}^{n-1}\\ &\ge \int_{\mathbb{S}^{n-1}}\big(x_h\cdot(x(1+\varphi_h)-D_\tau\varphi_h)\big)(x_h\cdot x)\,d\mathcal{H}^{n-1} - C(n)|x_h|^2\|\varphi_h\|_{H^1(\mathbb{S}^{n-1})}\\ &\ge \int_{\mathbb{S}^{n-1}}|x_h\cdot x|^2\,d\mathcal{H}^{n-1} - C(n)|x_h|^2\|\varphi_h\|_{H^1(\mathbb{S}^{n-1})} = \omega_n|x_h|^2 - C(n)|x_h|^2\|\varphi_h\|_{H^1(\mathbb{S}^{n-1})}.\end{aligned}$$

From this estimate and from (5.15) we finally obtain, recalling (5.14), that for h large

$$\begin{aligned}R(F_h)-R(G_h) &\le C(n)|x_h|\|\varphi_h\|^2_{L^2(\mathbb{S}^{n-1})} - \frac{n-2}{2}\omega_n|x_h|^2 + C(n)|x_h|^2\|\varphi_h\|_{H^1(\mathbb{S}^{n-1})}\\ &\le C(n)|x_h|\|\varphi_h\|^2_{L^2(\mathbb{S}^{n-1})} - \frac{n-2}{3}\omega_n|x_h|^2.\end{aligned}$$

Therefore, for h large, we have

$$I(G_h) = I(F_h) + KR(F_h) - KR(G_h) \le I(F_h) + C(n)K|x_h|\|\varphi_h\|^2_{L^2(\mathbb{S}^{n-1})} - \frac{K(n-2)}{3}\omega_n|x_h|^2.$$

Therefore, using (5.11), from the above inequality we have for h large, recalling that $x_h\to 0$,

$$\begin{aligned}I(G_h) &\le I(B_{\widetilde{r}_h}) + 4C_0|F_h\Delta B_{\widetilde{r}_h}|^2 + C(n)K|x_h|\|\varphi_h\|^2_{L^2(\mathbb{S}^{n-1})} - \frac{K(n-2)}{3}\omega_n|x_h|^2\\ &\le I(B_{\widetilde{r}_h}) + 8C_0(|G_h\Delta B_{\widetilde{r}_h}|^2 + |G_h\Delta F_h|^2) + C(n)K|x_h|\|\varphi_h\|^2_{L^2(\mathbb{S}^{n-1})} - \frac{K(n-2)}{3}\omega_n|x_h|^2\\ &\le I(B_{\widetilde{r}_h}) + 9C_0|G_h\Delta B_{\widetilde{r}_h}|^2 + 8C_0|G_h\Delta F_h|^2 - \frac{K(n-2)}{3}\omega_n|x_h|^2\\ &\le I(B_{\widetilde{r}_h}) + 9C_0|G_h\Delta B_{\widetilde{r}_h}|^2,\end{aligned}$$

where the last inequality follows by observing that

$$8C_0|G_h\Delta F_h|^2 - \frac{K(n-2)}{3}\omega_n|x_h|^2 \le C(n)8C_0|x_h|^2 - \frac{K(n-2)}{3}\omega_n|x_h|^2 < 0,$$

provided C_0 is sufficiently small. In conclusion we have shown that for h large

$$I(G_h) \le I(B_{\widetilde{r}_h}) + 9C_0|G_h\Delta B_{\widetilde{r}_h}|^2$$

and this inequality contradicts (3.10) if we assume also $C_0 < c_1/9$. □

6. Global minimality

In this last section we prove the existence of a critical radius $r_1 \leq r_0$ such that if $r < r_1$, the ball centered at the origin with radius r is the unique global minimizer of I among all sets of prescribed measure. We start with a simple lemma.

Lemma 6.1. *Let $n \geq 3$. There exists a constant $C(n) > 0$ such that if $E \subset \mathbb{R}^n$ is a Borel set with $|E| = |B_r|$ then*

$$\frac{R(B_r) - R(E)}{r^2} > C(n) \left(\frac{|E \Delta B_r|}{r^n} \right)^2. \tag{6.1}$$

Proof. Since both quantities in (6.1) are scaling invariant we may assume $r = 1$. Thus, let E be a Borel set with $|E| = |B|$ with $|E\Delta B| > 0$ and let us decompose it as $E = (E \cap B) \cup (E \setminus B)$. Let $0 < \varrho < 1 < r$ such that $|B_\varrho| = |B \setminus E|$, $|B_r \setminus B| = |E \setminus B|$ and set

$$E^* := B_{r_1} \cup (B_{r_2} \setminus B).$$

Clearly, we have that

$$|E\Delta B| = |E^* \Delta B|, \qquad R(E^*) > R(E).$$

At this point we can easily evaluate the left handside of (6.1)

$$R(B) - R(E) \geq R(B) - R(E^*) = \frac{n\omega_n}{2}(2 - (\varrho^2 + r^2)).$$

Since $r^n = 2 - \varrho^n$, from the inequality above we have

$$R(B) - R(E) \geq \frac{n\omega_n}{2}(2 - \varrho^2 - (2 - \varrho^n)^{\frac{2}{n}}) := f(\varrho).$$

The conclusion then follows by observing that

$$\lim_{\varrho \to 1} \frac{f(\varrho)}{(1-\varrho)^2} = c(n) > 0.$$

□

Before stating the main result of this section, let us define

$$r_1 := \left(\frac{K}{2\omega_n} \right)^{\frac{1}{n}}. \tag{6.2}$$

Theorem 6.2. *Let $n \geq 3$. If $r < r_1$, where r_1 is defined as in (6.2), the ball centered at the origin is the only gobal minimizer of I among all sets $E \subset \mathbb{R}^n$ with prescribed volume $|E| = |B_r|$. Moreover,*

$$I(E) - I(B_r) \geq c|E\Delta B_r|^2,$$

for some positive constant c depending only on n and r.

Proof. We start by observing that

$$V(B_r) - V(E) = \int_{B_r}\int_{B_r} \frac{1}{|x-y|^{n-2}} dxdy - \int_E \int_E \frac{1}{|x-y|^{n-2}} dxdy \tag{6.3}$$

$$= 2\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{\chi_{B_r}(x)(\chi_{B_r}(y) - \chi_E(y))}{|x-y|^{n-2}} dxdy - \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{(\chi_{B_r}(y) - \chi_E(y))(\chi_{B_r}(x) - \chi_E(x))}{|x-y|^{n-2}} dxdy.$$

set for all $x \in \mathbb{R}^n$

$$u(x) := \int_{\mathbb{R}^n} \frac{(\chi_{B_r}(y) - \chi_E(y))}{|x-y|^{n-2}}. \tag{6.4}$$

Then

$$-\Delta u = c_n(\chi_{B_r} - \chi_E),$$

for some constant $c(n) > 0$. Therefore, integrating by parts,

$$\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{(\chi_{B_r}(y) - \chi_E(y))(\chi_{B_r}(x) - \chi_E(x))}{|x-y|^{n-2}} dxdy = -\int_{\mathbb{R}^n} u(y)\Delta u(y)dy = c_n \int_{\mathbb{R}^n} |Du|^2\, dx. \tag{6.5}$$

Combining (6.3), (6.5) and the fact that u is superharmonic in B_r, we get

$$V(B_r) - V(E) \leq 2\int_{B_r} u(y)dy \leq 2|B_r|u(0) = 2|B_r|\,(R(B_r) - R(E))$$

Thus using the isoperimetric inequality, the above lemma and that $r < r_1$ we can conclude that

$$I(E) - I(B_r) \geq P(E) - P(B_r) + (K - 2|B_r|)\big(R(B_r) - R(E)\big) \geq c(n,r)(K - 2|B_r|)|E\Delta B_r|^2.$$

□

From definitions (2.11) and (6.2) it is clear that both r_0 and r_1 tend to ∞ as $K \to \infty$. However the ratio r_0/r_1 stays bounded.

Lemma 6.3. *Let $n \geq 3$. Then*

$$\limsup_{K\to+\infty} \frac{r_0}{r_1} \leq \Big(\frac{n\omega_n^2}{V(B)}\Big)^{\frac{1}{n}}.$$

Proof. Let P_2 be defined as in (2.13) and denote by $r_2 \geq r_0$ the unique zero of P_2. From the second equation in (2.8), we have that

$$P_2(r) = a(n)r^{n-3} - r^n + \frac{K(n+2)}{\mu_1},$$

where, using the first equation in (2.8), $a_n = (n+1)P(B)/[2(n-2)V(B)]$. Recalling the first equation in (2.8), (2.12) and the definition (6.2) of r_1 we have at once that

$$\frac{K(n+2)}{\mu_1} = \gamma_n^n r_1^n, \qquad \text{where } \gamma_n = \Big(\frac{n\omega_n^2}{V(B)}\Big)^{\frac{1}{n}} > 1.$$

Fix now $\varepsilon > 0$. Then

$$P_2(\gamma_n(1+\varepsilon)r_1) = \gamma_n^n r_1^n\bigg(\frac{a_n(1+\varepsilon)^{n-3}}{r_1^3} - (1+\varepsilon)^n + 1\bigg) < 0,$$

provided r_1, hence K, is large enough. Therefore we may conclude that for K sufficiently large

$$r_0 \leq r_2 < \gamma_n(1+\varepsilon)r_1.$$

Hence, the result follows. □

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