Homogenization of interacting media

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Abstract

We consider energies modelling the interaction of two media parameterized by the same reference set, such as those used to study interactions of a thin film with a stiff substrate, hybrid laminates, or skeletal muscles. Analytically, these energies consist of a (possibly non-convex) functional of hyperelastic type and a second functional of the same type such as those used in variational theories of brittle fracture, paired by an interaction term governing the strength of the interaction depending on a small parameter. The overall behaviour is described by letting this parameter tend to zero, using the terminology of Gamma-convergence and exhibiting an effective energy. Such energy depends on a single parameter and is of hyperelastic type. The form of its energy function highlights a microscopic optimization between microfracture and oscillations, mixing homogenization and high-contrast effects. It is interesting to note that the period and orientation of the microstructures giving rise to homogenization depend on the overall limit strain, and are not due to geometrical features of the media.

1 Introduction

The subject of this paper is the analysis of a model of interacting media governed by coupled energies. The energies that we consider depend on two functions u and v defined on a common reference configuration Ω in $\mathbb{R}^n$. The variable u may be thought as the deformation of a (possibly, brittle) hyperelasticelastic material governed by Griffith-type fracture energies, such as those used in recent analyses of crack propagation [7], of the form

$$\int_\Omega f(\nabla u)dx + \varepsilon \int_{S(u)} \varphi(\frac{u^+ - u^-}{\varepsilon}) d\mathcal{H}^{n-1},$$

where $S(u)$ is the fracture site of u and $\mathcal{H}^{n-1}$ denotes the $n-1$-dimensional Hausdorff surface measure. The small parameter ε models the possibility of "diffuse" micro-cracks. In the same reference configuration, we consider a second function v that we may interpret as the deformation of an hyperelastic material, with energy function g. The two materials are coupled by a simple integral penalizing u and v far apart, governed by a second small parameter δ. The complete form of the energy we are going to consider is then

$$F_{\delta,\varepsilon}(u,v) = \int_\Omega \Big(f(\nabla u) + g(\nabla v) + h\big(\frac{u-v}{\delta}\big)\Big)\,dx + \varepsilon \int_{S(u)} \varphi\big(\frac{u^+ - u^-}{\varepsilon}\big) d\mathcal{H}^{n-1}, \quad (1)$$

for some function h. In this expression the function v belongs to an appropriate Sobolev space, while the correct space for the variable u is the space of special functions with bounded variation $SBV(\Omega;\mathbb{R}^m)$. A possible simple microscopic derivation of such energies can be obtained considering a two-dimensional lattice model of "double-porosity" type with strong next-to-nearest neighbour interactions and weak nearest-neighbour interactions (see e.g. [11]).

In a one-dimensional setting, energies of the form (1) have been used for the description of different mechanical problems. For instance, Baldelli et al. ([6], see also previous work by Marigo and Truskinovsky [25]) have investigated fracture and debonding processes of a thin film on a stiff substrate with an elastic-brittle interface. In such a model, an additional dissipative energetic term is considered, representing the brittle fracture energy (delamination) of the interface. Instead, the pseudo-ductile response of thin-ply hybrid laminates has been captured by Alessi et al. [2, 3] by considering cohesive interface laws, possibly with a softening part, and an elastic-brittle behavior for both layers, with their corresponding additional energetic terms. In all theses cases, the relevant scaling for the energies $F_{\delta,\varepsilon}$ is indeed $\delta = \varepsilon$, and we will then consider only that case; i.e. energies

$$F_\varepsilon(u,v) = \int_\Omega \Big(f(\nabla u) + g(\nabla v) + h\big(\frac{u-v}{\varepsilon}\big)\Big)\,dx + \varepsilon \int_{S(u)} \varphi\big(\frac{u^+ - u^-}{\varepsilon}\big) d\mathcal{H}^{n-1} \quad (2)$$

(see Remark 6). For such energies we will describe the asymptotic behaviour as $\varepsilon \to 0$ The scaling $\delta = \varepsilon$ can also be derived from discrete models such as those analyzed in [27, 28].

Note that as a particular case we may consider the one where both media are elastic, in which case we consider, with a slight abuse of notation, energies as

$$F_\delta(u,v) = \int_\Omega \Big(f(\nabla u) + g(\nabla v) + h\big(\frac{u-v}{\delta}\big)\Big)\,dx, \quad (3)$$

thus ruling out the possibility of fracture for the medium described by u. These energies are defined on pairs of Sobolev spaces. If h blows up at infinity we expect the interaction term to force $u = v$ in the limit as δ tends to zero. However, for homogeneous energies (3) the effect of h is restricted to the fact that the effective energy may be described by some type of elastic energy governed by a single parameter v, and the resulting effective energy can be simply described by relaxation arguments (see Remark 5).

For the general energies F_ε in (2), it must be noted that superlinear growth conditions on g immediately imply that sequences $\{v_\varepsilon\}$ such that $F_\varepsilon(u_\varepsilon, v_\varepsilon)$ is equibounded are weakly precompact in some Sobolev space (up to the addition of constants), so that in that case we may assume that v_ε weakly converge to v in some $W^{1,p}$. Moreover, growth conditions on h give that $(u_\varepsilon - v_\varepsilon)/\varepsilon$ must be bounded in some Lebesgue space, so that actually also u_ε converge to the same v (e.g., in L^1). Note that in general u_ε may not weakly converge in any Sobolev space, and that the $\mathcal{H}^{n-1}$-measure of the sets $S(u_\varepsilon)$ may diverge; nevertheless, the limit of u_ε is a Sobolev function. This remark justifies the description of the limit by using only the variable v, integrating out the effects of u also in the case of coupling with a brittle medium.

Our first result is a general *homogenization theorem* that states that the Γ-limit of the energies above is a local functional that can be written as a usual hyperelastic energy

$$F_{\rm hom}(v) = \int_\Omega f_{\rm hom}(\nabla v)\,dx. \tag{4}$$

The energy density is characterized by the *asymptotic homogenization formula*

$$\begin{aligned} f_{\rm hom}(z) \quad = \quad & \lim_{T\to+\infty} \frac{1}{T^n} \inf\Big\{ \int_{(-\frac{T}{2},\frac{T}{2})^n} \Big(f(\nabla u) + g(\nabla v) + h(u-v)\Big)\,dx \\ & + \int_{S(u)\cap(-\frac{T}{2},\frac{T}{2})^n} \varphi(u^+ - u^-)d\mathcal{H}^{n-1} : u = v = zx \ \text{ in } \ \partial Q_T \Big\}, \end{aligned} \tag{5}$$

which highlights that minimizing behaviours are obtained by optimizing among microgeometries with interacting oscillations and discontinuities. Note that this formula mixes the asymptotic analysis typical of nonlinear periodic media with the interaction between terms depending on the gradient and ‘lower-order terms’ typical of double-porosity phenomena. An interesting remark is that this formula is optimal, in the sense that homogenization arguments must be used even though no periodicity is present in the original functional. Optimal configurations with average gradient z tend to be periodic with a period depending on z itself, so that if affine Dirichlet boundary values are enforced the optimal value for the infima above in general is achieved only as $T \to +\infty$. In the scalar case $m = 1$ (anti-plane case) and isotropic energies we show that optimal patterns are locally one-dimensional; i.e., optimal sequences have discontinuities and oscillations that arrange orthogonally to the direction of the gradient, and we may restrict to considering the homogenization formula for one-dimensional problems. Note that in this case the direction of optimal cracks or oscillations is locally determined by the orientation of the limit ∇v.

The analysis of a prototypical one-dimensional energy allows to highlight more in detail the effect of fractures and oscillations at the microscopic level. We concentrate on the effect of fracture by choosing f, g and h even and convex, and φ constant with value $k > 0$. Then $f_{\rm hom}(z)$ is obtained by minimizing

$$\min\Big\{\frac{1}{S}\int_0^S \big(f(u') + g(v') + h(u-v)\big)\,dx + \frac{k}{S} : \ v(0) = 0, \ v(S) = Sz\Big\} \tag{6}$$

for $S > 0$. The minimum in (6) is performed on u and v which are regular on $(0, S)$, and represents the average energy of periodic optimal arrangements with

u having consecutive discontinuities at distance S in the unscaled variables. The case $S=\infty$ corresponds to no fracture, for which the minimal u and v are equal and affine and the minimum is $f(z)+g(z)+\min h$. In the case of quadratic f, g and h we can compute $f_{\rm hom}(z)$ explicitly and highlight that

- $f_{\rm hom}(z)$ is strictly convex;
- there exists z_* such that $f_{\rm hom}(z)=f(z)+g(z)$ if $|z|\le z_*$ (no fracture);
- for large values of z, $f_{\rm hom}(z)-g(z)$ scales as $z^{2/3}$ and the optimal spacing of microfracture $S(z)$ scales as $z^{-2/3}$.

This example already shows interesting issues as the onset of microscopic fracture at a specific positive threshold z^* and the optimal arrangements of cracks following a scaling which is reminiscent of that appearing in the study of periodic minimizing sequences of singularly perturbed non-convex energies (see e.g. [26, 1]).

The plan of the paper is a follows. In Section 2 we prove the general homogenization Theorem 3, where we also include a possible highly oscillating periodic dependence in the energy densities. Note that this makes the limit process non trivial also when no possibility of fracture is taken into account. In Section 3 we consider isotropic energies, for which we show that optimal sequences have a one-dimensional structure. Section 4 contains the analysis of one-dimensional functionals in the case of Griffith fracture, and the explicit computation for quadratic energy densities hinted at above.

2 A general convergence result via homogenization

Before stating our convergence result, we briefly recall some notation. The letter capital C will denote a positive constant depending on the fixed parameters of the problem under consideration, which we will mention explicitly when relevant, and whose value may vary at each its appearance. We use standard notation for Sobolev spaces $W^{1,p}(\Omega;\mathbb{R}^m)$ of $\mathbb{R}^m$-valued maps defined on an open subset Ω of $\mathbb{R}^n$. We will also use the space of *special functions of bounded variation* $SBV^p(\Omega;\mathbb{R}^m)$ whose approximate gradient is p-integrable. For such a function u we denote by $S(u)$ the jump set of u, on which a measure-theoretical normal ν_u is defined $\mathcal{H}^{n-1}$-almost everywhere, where $\mathcal{H}^{n-1}$ denotes the $n-1$-dimensional Hausdorff surface measure, as well as the traces $u^\pm$ on both sides of $S(u)$. For a precise definition of all these quantities we refer to [8], and for an interpretation within the Griffith theory of brittle fracture to [5, 7].

Let $f,g\colon\mathbb{R}^n\times\mathbb{R}^{n\times m}\to[0,+\infty)$, $h\colon\mathbb{R}^n\times\mathbb{R}^m\to[0,+\infty)$ and $\varphi\colon\mathbb{R}\times\mathbb{R}\times S^{n-1}\to[0,+\infty]$ be such that

(H1) $\frac{1}{c}(|z|^p-1)\le f(y,z),\ g(y,z),\ h(y,z)\le c(|z|^p+1)$

(H2) $\varphi(y,tw,\nu)\le c\,\varphi(y,w,\nu)$ for $|t|\le 1$

(H3) f,g,h,φ are Carathéodory functions, 1-periodic with respect to their first variable and continuous with respect to the other ones.

Given a bounded open set $\Omega \subset \mathbb{R}^n$ with Lipschitz boundary, $u \in SBV^p(\Omega;\mathbb{R}^m)$ and $v \in W^{1,p}(\Omega;\mathbb{R}^m)$, for $\varepsilon > 0$ we define:

$$\begin{aligned} F_\varepsilon(u,v;\Omega) = \int_\Omega \Big(f\big(\frac{x}{\varepsilon}, \nabla u\big) + g\big(\frac{x}{\varepsilon}, \nabla v\big) + h\big(\frac{x}{\varepsilon}, \frac{u-v}{\varepsilon}\big)\Big)\, dx \\ +\varepsilon \int_{S(u)\cap\Omega} \varphi\big(\frac{x}{\varepsilon}, \frac{u^+ - u^-}{\varepsilon}, \nu_u\big)\, d\mathcal{H}^{n-1}. \end{aligned} \tag{7}$$

Remark 1 (Energies with no jump part)**.** As a particular case, we can take $\varphi(y,w,\nu) = +\infty$ if $w \neq 0$ (and equal to 0 if $w = 0$ for completeness. Note that the value $w = 0$ is never taken into account). In this case, F_ε is finite only if $\mathcal{H}^{n-1}(S(u)\cap\Omega) = 0$ or, equivalently, $u \in W^{1,p}(\Omega;\mathbb{R}^m)$ and

$$F_\varepsilon(u,v;\Omega) = \int_\Omega \Big(f\big(\frac{x}{\varepsilon}, \nabla u\big) + g\big(\frac{x}{\varepsilon}, \nabla v\big) + h\big(\frac{x}{\varepsilon}, \frac{u-v}{\varepsilon}\big)\Big)\, dx.$$

Remark 2 (Compactness)**.** Let $u_\varepsilon, v_\varepsilon$ be such that $F_\varepsilon(u_\varepsilon, v_\varepsilon;\Omega) \leq C < +\infty$. Then in particular ∇v_ε is bounded in $L^p(\Omega;\mathbb{R}^{m\times n})$ so v_ε weakly converges to some v in $W^{1,p}(\Omega;\mathbb{R}^m)$ (up to subsequences and addition of constants). The growth conditions on h ensure that also u_ε converges to the same v in $L^p(\Omega;\mathbb{R}^m)$. This remark justifies the choice of the convergence $u_\varepsilon, v_\varepsilon \to v$ in L^p in the following result.

Theorem 3 (Homogenization)**.** *The functionals F_ε Γ-converge with respect to the convergence $u_\varepsilon, v_\varepsilon \to v$ in $L^p(\Omega;\mathbb{R}^m)$ to the functional F defined on $W^{1,p}(\Omega;\mathbb{R}^m)$ by*

$$F_{\rm hom}(v) = \int_\Omega f_{\rm hom}(\nabla v)\, dx \tag{8}$$

where

$$f_{\rm hom}(z) = \lim_{T\to+\infty} \frac{1}{T^n} \inf\Big\{ F_1(u,v;Q_T) : u = v = zx \ \ \text{in} \ \ \partial Q_T \Big\}, \tag{9}$$

where $Q_T = (-\frac{T}{2}, \frac{T}{2})^n$ and the boundary value of u and v are in the sense of inner traces.

Remark 4 (A generalization to $M+1$ interacting media)**.** Given an integer $M \geq 1$ we may consider more in general energies defined for $u_i \in SBV^p(\Omega;\mathbb{R}^m)$, $i \in \{1,\dots,M\}$ and $v \in W^{1,p}(\Omega;\mathbb{R}^m)$ by

$$\begin{aligned} F_\varepsilon(u,v;\Omega) = \sum_{i=1}^M \int_\Omega \Big(f_i\big(\frac{x}{\varepsilon}, \nabla u\big) + h_i\big(\frac{x}{\varepsilon}, \frac{u_i - v}{\varepsilon}\big)\Big)\, dx + \int_\Omega g\big(\frac{x}{\varepsilon}, \nabla v\big)\, dx \\ +\varepsilon \sum_{i=1}^M \int_{S(u_i)\cap\Omega} \varphi_i\big(\frac{x}{\varepsilon}, \frac{u_i^+ - u_i^-}{\varepsilon}, \nu_{u_i}\big)\, d\mathcal{H}^{n-1}, \end{aligned} \tag{10}$$

where $u = (u_1, \dots, u_M)$, and f_i, h_i and φ_i satisfy the same assumptions as f, h and φ, respectively. The result above can be then proved without major changes in the proof. Note that even more in general, we may also add a term of the form

$$\int_\Omega \sum_{i\neq j} h_{ij}\big(\frac{x}{\varepsilon}, \frac{u_i - u_j}{\varepsilon}\big)\, dx.$$

We will not discuss the various hypotheses that one can impose to h_i and h_{ij} so that the compactness arguments in Remark 2 still hold.

Remark 5 (Homogeneous energies with no jump part)**.** If we take into account homogeneous energies as in Remark 1, i.e. of the form

$$F_\varepsilon(u, v; \Omega) = \int_\Omega \Big(f(\nabla u) + g(\nabla v) + h\big(\frac{u - v}{\varepsilon}\big)\Big)\, dx,$$

then the homogenized energy density is simply given by

$$f_{\rm hom}(z) = Qf(z) + Qg(z) + \min h,$$

where Q denotes the quasiconvexification operator (see e.g. [12]). Indeed, the energy with density the right-hand side is clearly a lower bound (after taking into account Remark 2). To check that this is also an upper bound, by the integral representation in Theorem 3 it suffices to consider the case of a linear target function $v(x) = zx$. This can be seen by taking sequences $u_\varepsilon, v_\varepsilon$ converging to v such that

$$\int_\Omega f(\nabla u_\varepsilon)\, dx \to |\Omega|\ Qf(z) \quad \text{and} \quad \int_\Omega g(\nabla v_\varepsilon)\, dx \to |\Omega|\ Qg(z)$$

up to an arbitrarily small error, and with $u_\varepsilon - v_\varepsilon = \zeta_0 + o(\varepsilon)$, where $h(\zeta_0) = \min h$. This can be done remarking that we may take recovery sequences converging to v in L^∞ by a truncation argument in the target space (see e.g. [13]).

Remark 6 (Two-parameter scalings)**.** As remarked in the Introduction, more in general we may consider energies of the form (1) also depending on a parameter δ. We do not treat this general case since it would involve additional multi-scale arguments which are not central in our analysis (we refer to e.g. [24, 18] for similar multi-scale problems in different contexts). However, in the homogeneous case this analysis simplifies and we note the following.

1) if $\delta << \varepsilon$ then the interaction term forces $u - v = O(\delta) << \varepsilon$. This makes the introduction of jump points energetically non-favorable, so that the analysis of $F_{\delta,\varepsilon}$ simply reduces to that of

$$\int_\Omega \big(f(\nabla u) + g(\nabla v) + \min h\big)\, dx, \tag{11}$$

with $u = v + \delta\zeta_0 + o(\delta)$, where $h(\zeta_0) = \min h$. This energy can be analyzed by relaxation methods as in Remark 5;

2) if $\varepsilon << \delta$ then the condition $u - v = O(\delta) >> \varepsilon$ allows the minimization of the interaction term without influencing the rest of the energy, and the analysis of $F_{\delta,\varepsilon}$ simply reduces to that of the decoupled energies

$$\int_\Omega f(\nabla u)\,dx + \varepsilon \int_{S(u)} \varphi\Big(\frac{u^+ - u^-}{\varepsilon}\Big) d\mathcal{H}^{n-1} + \int_\Omega \Big(g(\nabla v) + \min h\Big)\,dx. \tag{12}$$

In this case, the part of the energy depending on u trivializes since we may approximate all u with piecewise-affine discontinuous functions jumping on a scale much larger than ε, and we reduce the analysis to that of

$$\int_\Omega \big(g(\nabla v) + \min h + \min f\big)\,dx. \tag{13}$$

In the following, we will use the notation

$$f^T_{\rm hom}(z) = \frac{1}{T^n} \inf\Big\{F_1(u, v; Q_T) : u = v = zx \ \text{ in } \ \partial Q_T\Big\}.$$

Remark 7. The existence of the limit in (9) can be proved by an usual argument of subadditivity (see [12, Prop. 14.4]). Following the same type of arguments, if f, g, h and φ do not depend on the spatial variable y, we can prove that

$$f_{\rm hom}(z) = \inf_{T>0} f^T_{\rm hom}(z). \tag{14}$$

Indeed, we fix $\delta > 0$; for any $T > 0$ let u_T, v_T such that $u_T = v_T = zx$ in ∂Q_T and

$$\frac{1}{T^n} F_1(u_T, v_T; Q_T) < f^T_{\rm hom}(z) + \delta.$$

For $S > T$ we consider the set of indices $\mathcal{I}_S = \{i \in \mathbb{Z}^n : Q_T + Ti \subset Q_S\}$ and define u and v in Q_S by setting

$$u(x) = u_T(x - Ti) + zTi, \quad v(x) = v_T(x - Ti) + zTi \quad \text{if } x \in Q_T + Ti,\ i \in \mathcal{I}_S$$

and $u(x) = v(x) = zx$ in $Q_S \setminus \bigcup_{i\in\mathcal{I}_S}(Q_T + Ti)$. Hence

$$\begin{aligned} \frac{1}{S^n} F_1(u, v; Q_S) &\le \frac{1}{S^n} \Big\lfloor \frac{S}{T} \Big\rfloor^n F_1(u_T, v_T; Q_T) + \frac{CT^n S^{n-1}}{S^n} \\ &\le f^T_{\rm hom}(z) + \delta + \frac{CT^n}{S}. \end{aligned}$$

Taking the limit for $S \to +\infty$ we get $f_{\rm hom}(z) \le f^T_{\rm hom}(z) + \delta$; this proves (14) since $\delta > 0$ is arbitrary.

Proof of Theorem 3. *Lower bound.* We prove the lower inequality by using the blow up technique introduced by Fonseca and Müller (see [23, 16]).

Let $u_\varepsilon, v_\varepsilon$ be such that $F_\varepsilon(u_\varepsilon, v_\varepsilon; \Omega) \le C$ and $u_\varepsilon, v_\varepsilon \to v$ in L^p, and let $u_j = u_{\varepsilon_j}$ and $v_j = v_{\varepsilon_j}$ be subsequences such that $\liminf_{\varepsilon\to 0} F_\varepsilon(u_\varepsilon, v_\varepsilon; \Omega) =$

$\lim_{j\to+\infty} F_{\varepsilon_j}(u_j,v_j;\Omega)$. We define the measures μ_j by setting $\mu_j(A)=F_{\varepsilon_j}(u_j,v_j;A)$; since the family $\{\mu_j\}$ is equibounded, we can assume that $\mu_j \overset{*}{\rightharpoonup} \mu$ up to subsequences. The lower bound follows if we show that for almost all $x_0\in\Omega$

$$\frac{d\mu}{d\mathcal{L}^n}(x_0)\geq f_{\rm hom}(\nabla v(x_0)).$$

We first remark that, for almost every $x_0\in\Omega$, we have that

1) x_0 is a Lebesgue point for μ with respect to the Lebesgue measure, so that $\dfrac{d\mu}{d\mathcal{L}^n}(x_0)=\lim\limits_{\varrho\to 0}\dfrac{\mu(Q_\varrho(x_0))}{\varrho^n}$, where $Q_\varrho(x_0)=(x_0-\frac{\varrho}{2},x_0+\frac{\varrho}{2})^n$;

2) x_0 is such that $\Big(\dfrac{1}{\varrho^n}\displaystyle\int_{Q_\varrho(x_0)}|v(x)-v(x_0)-\nabla v(x_0)(x-x_0)|^p\,dx\Big)^{\frac{1}{p}}=o(\varrho)_{\varrho\to 0}$ since $v\in W^{1,p}(\Omega;\mathbb{R}^m)$ (see for instance [21, Th. 6.2]).

It is not restrictive to fix $x_0=0$ and $v(x_0)=0$. For every ϱ except for a countable set we have $\mu(Q_\varrho)=\lim_{j\to+\infty}F_{\varepsilon_j}(u_j,v_j;Q_\varrho)$, where $Q_\varrho=Q_\varrho(0)$. Note that in particular $\lim_j\frac{1}{\varrho^n}F_{\varepsilon_j}(u_j,v_j;Q_\varrho)$ is equibounded with respect to ϱ.

As a first step, following a classical method introduced by De Giorgi for matching boundary values (see [20] and [9, Sec. 4.2.1]), we show that, by modifying v_j and u_j near the boundary of Q_ϱ, it is not restrictive to assume that their boundary value is exactly $\nabla v(0)x$. Indeed, fixed $\delta\in(0,1)$ and $N\in\mathbb{N}$, for any $i=0,\dots N$ we define $Q^i=Q_{\varrho-\delta\varrho+i\frac{\delta\varrho}{N}}$. For $i\geq 1$, let ψ_i be a non-negative Lipschitz function such that

$$\psi_i(x)=0\ \text{ in }\ Q_\varrho\setminus Q^i,\quad \psi_i(x)=1\ \text{ in }\ Q^{i-1}\quad\text{and}\quad|\nabla\psi_i|\leq\frac{2N}{\delta\varrho}.$$

Setting $u_j^i=\psi_iu_j+(1-\psi_i)\nabla v(0)x$ and $v_j^i=\psi_iv_j+(1-\psi_i)\nabla v(0)x$, the growth hypotheses on f give

$$\begin{aligned}\int_{Q_\varrho}f\Big(\frac{x}{\varepsilon_j},\nabla u_j^i\Big)\,dx &= \int_{Q^{i-1}}f\Big(\frac{x}{\varepsilon_j},\nabla u_j^i\Big)\,dx+\int_{Q_\varrho\setminus Q^i}f\Big(\frac{x}{\varepsilon_j},\nabla u_j^i\Big)\,dx+\int_{Q^i\setminus Q^{i-1}}f\Big(\frac{x}{\varepsilon_j},\nabla u_j^i\Big)\,dx\\ &\leq \int_{Q_\varrho}f\Big(\frac{x}{\varepsilon_j},\nabla u_j\Big)\,dx+C(|\nabla v(0)|^p+1)\delta\varrho^n\\ &\quad+\frac{CN^p}{\delta^p\varrho^p}\int_{Q^i\setminus Q^{i-1}}|u_j-\nabla v(0)x|^p\,dx+C\int_{Q^i\setminus Q^{i-1}}f\Big(\frac{x}{\varepsilon_j},\nabla u_j\Big)\,dx\end{aligned}$$

where C denotes a positive constant depending only on p,n and c. Hence

$$\begin{aligned}\sum_{i=1}^N\int_{Q_\varrho}f\Big(\frac{x}{\varepsilon_j},\nabla u_j^i\Big)\,dx &\leq CN(|\nabla v(0)|^p+1)\delta\varrho^n+(N+C)\int_{Q_\varrho}f\Big(\frac{x}{\varepsilon_j},\nabla u_j\Big)\,dx\\ &\quad+\frac{CN^p}{\delta^p\varrho^p}\int_{Q_\varrho}|u_j-v|^p\,dx+\frac{CN^p}{\delta^p\varrho^p}\int_{Q_\varrho}|v-\nabla v(0)x|^p\,dx\end{aligned}$$

and correspondingly for $\sum_{i=1}^N\int_{Q_\varrho}g\big(\frac{x}{\varepsilon_j},\nabla v_j^i\big)\,dx$ thanks to the growth hypotheses on g. Moreover, the assumptions on h and φ give

$$\sum_{i=1}^{N}\int_{Q_\varrho} h\Big(\frac{x}{\varepsilon_j},\frac{u_j^i-v_j^i}{\varepsilon_j}\Big)\,dx = \sum_{i=1}^{N}\int_{Q_\varrho} h\Big(\frac{x}{\varepsilon_j},\psi_i\frac{u_j-v_j}{\varepsilon_j}\Big)\,dx$$
$$\leq CNh(0)\delta\varrho^n + N\int_{Q_\varrho} h\Big(\frac{x}{\varepsilon_j},\frac{u_j-v_j}{\varepsilon_j}\Big)\,dx + C\int_{Q_\varrho} h\Big(\frac{x}{\varepsilon_j},\frac{u_j-v_j}{\varepsilon_j}\Big)\,dx$$
$$\sum_{i=1}^{N}\int_{Q_\varrho\cap S(u_j^i)} \varphi\Big(\frac{x}{\varepsilon_j},\frac{(u_j^i)^+-(u_j^i)^-}{\varepsilon_j},\nu_{u_j^i}\Big)\,dx = \sum_{i=1}^{N}\int_{Q_\varrho\cap S(u_j^i)} \varphi\Big(\frac{x}{\varepsilon},\psi_i\frac{(u_j)^+-(u_j)^-}{\varepsilon_j},\nu_{u_j}\Big)\,dx$$
$$\leq N\int_{Q_\varrho\cap S(u_j)} \varphi\Big(\frac{x}{\varepsilon_j},\frac{u_j^+-u_j^-}{\varepsilon_j},\nu_{u_j}\Big)\,dx + C\int_{Q_\varrho\cap S(u_j)} \varphi\Big(\frac{x}{\varepsilon_j},\frac{u_j^+-u_j^-}{\varepsilon_j},\nu_{u_j}\Big)\,dx.$$

Therefore,

$$\begin{aligned}\frac{1}{N}\sum_{i=1}^{N}F_{\varepsilon_j}(u_j^i,v_j^i;Q_\varrho) \quad &\leq \quad F_{\varepsilon_j}(u_j,v_j;Q_\varrho) + \frac{C}{N}F_{\varepsilon_j}(u_j,v_j;Q_\varrho) + C\delta\varrho^n \\ &\quad + \frac{CN^p}{\delta^p\varrho^p}\int_{Q_\varrho}\Big(|u_j-u|^p + |v_j-u|^p + 2|u-\nabla u(0)x|^p\Big)\,dx\end{aligned}$$

where now C stands for a positive constant depending also on $\nabla v(0)$ and $h(0)$. We choose $\underline{i}$ such that $F_{\varepsilon_j}(u_j^{\underline{i}},v_j^{\underline{i}};Q_\varrho) \leq \frac{1}{N}\sum_{i=1}^{N}F_{\varepsilon_j}(u_j^i,v_j^i;Q_\varrho)$ and we set $\hat{u}_j = u_j^{\underline{i}}$ and $\hat{u}_j = v_j^{\underline{i}}$.

Since $\frac{1}{\varrho^n}F_{\varepsilon_j}(u_j,v_j;Q_\varrho)$ is equibounded, the convergence of $u_j,v_j \to u$ in L^p and the properties of the point $x_0 = 0$ ensure that for any $\delta \in (0,1)$ and $N \in \mathbb{N}$

$$\lim_{\varrho\to 0}\frac{1}{\varrho^n}\lim_{j\to+\infty}F_{\varepsilon_j}(\hat{u}_j,\hat{v}_j;Q_\varrho) \quad \leq \quad \lim_{\varrho\to 0}\frac{1}{\varrho^n}\lim_{j\to+\infty}F_{\varepsilon_j}(u_j,v_j;Q_\varrho) + \frac{\tilde{C}}{N} + C\delta.$$

This inequality allows us to assume that $u_j = v_j = \nabla v(0)x$ on ∂Q_ϱ.

Now, setting $\overline{u}_j(x) = \frac{1}{\varepsilon_j}u_j(\varepsilon_j x)$ and $\overline{v}_j(x) = \frac{1}{\varepsilon_j}v_j(\varepsilon_j x)$, we get

$$\begin{aligned}\frac{1}{\varrho^n}F_{\varepsilon_j}(u_j,v_j;Q_\varrho) \quad &= \quad \frac{\varepsilon^n}{\varrho^n}\Big(\int_{Q_{\frac{\varrho}{\varepsilon}}} f(y,\nabla\overline{u}_j) + g(y,\nabla\overline{v}_j) + h(y,\overline{u}_j-\overline{v}_j)\,dy \\ &\qquad + \int_{Q_{\frac{\varrho}{\varepsilon}}\cap S(\overline{u}_j)}\varphi(y,\overline{v}_j^+-\overline{v}_j^-)\,d\mathcal{H}^{n-1}\Big) \\ &\geq \quad f_{\mathrm{hom}}^{\varrho/\varepsilon}(\nabla v(0)).\end{aligned}$$

The result then follows by taking the limit as $\varepsilon \to 0$.

Upper bound. We first prove that the function f_{hom} is continuous. We fix $z \in \mathbb{R}^{m\times n}$. For any $T > 0$ and $\delta \in (0,1)$ let $u_{T,\delta}, v_{T,\delta}$ be such that $u_{T,\delta} = v_{T,\delta} = zx$ in ∂Q_T and

$$\frac{1}{T^n}F_1(u_{T,\delta},v_{T,\delta};Q_T) \leq f_{\mathrm{hom}}^T(z) + \delta;$$

For any z' we extend $u_{T,\delta}$ and $v_{T,\delta}$ to $Q_{(1+\delta)T}$ by setting

$$\tilde{u}_{T,\delta}(x) = \tilde{v}_{T,\delta}(x) = \psi_{T,\delta}(x) z'x + (1-\psi_{T,\delta}(x))zx \quad \text{ in } \quad Q_{(1+\delta)T} \setminus Q_T,$$

where $\psi_{T,\delta}$ is a non-negative Lipschitz function such that $\psi_{T,\delta} = 0$ in Q_T, $\psi_{T,\delta} = 1$ in $\mathbb{R}^n \setminus Q_{(1+\delta)T}$ and $|\nabla \psi_{T,\delta}| \leq \frac{2}{T\delta}$. Hence, $\tilde{u}_{T,\delta}, \tilde{v}_{T,\delta}$ are test functions for the minimum problem for F_1 in $Q_{(1+\delta)T}$; note that $S(u_{T,\delta}) \cap Q_T = S(\tilde{u}_{T,\delta}) \cap Q_{(1+\delta)T}$. The growth conditions on f and g give

$$\begin{aligned}
f_{\text{hom}}^{(1+\delta)T}(z') &\leq \frac{1}{(1+\delta)^n T^n} F_1(\tilde{u}_{T,\delta}, \tilde{v}_{T,\delta}; [0,(1+\delta)T]^n) \\
&\leq \frac{1}{T^n} F_1(u_{T,\delta}, v_{T,\delta}; [0,T]^n) \\
&\quad + C\frac{\delta^{1-p}T^n}{(1+\delta)^n T^n}|z-z'|^p + C\frac{\delta T^n}{(1+\delta)^n T^n}\Big(|z-z'|^p + |z|^p + h(0)\Big) \\
&\leq f_{\text{hom}}^T(z) + \delta + C\frac{\delta^{1-p}}{(1+\delta)^n}|z-z'|^p + C\frac{\delta}{(1+\delta)^n}(|z|^p + h(0))
\end{aligned}$$

where the positive constant C depends only on n and on the growth of f e g. If $|z-z'| < \delta$ we get

$$\begin{aligned}
f_{\text{hom}}(z') &= \lim_{T\to+\infty} f_{\text{hom}}^{(1+\delta)T}(z') \leq \lim_{T\to+\infty} f_{\text{hom}}^T(z) + C\delta(1+|z|^p + h(0)) \\
&\leq f_{\text{hom}}(z) + C\delta(1+|z|^p + h(0))
\end{aligned}$$

By exchanging the roles of z and z' the argument above gives the inequality

$$f_{\text{hom}}(z) \leq f_{\text{hom}}(z') + C\delta(1+|z'|^p + h(0))$$

hence the continuity of f_{hom}.

Now, let $v \in W^{1,p}(\Omega;\mathbb{R}^m)$ and w_ε be piecewise-affine continuous functions such that $w_\varepsilon \to v$ as $\varepsilon \to 0$ in $W^{1,p}(\Omega;\mathbb{R}^m)$. The continuity of f_{hom} and the dominated convergence give

$$\lim_{\varepsilon\to 0} F_{\text{hom}}(w_\varepsilon) = F_{\text{hom}}(v).$$

Hence, it is sufficient to construct a recovery sequence for piecewise-affine continuous v. We start by considering the function $v(x) = zx$ on a n-dimensional open simplex S. We fix $\delta > 0$. Let $T_\varepsilon \in \mathbb{N}$ such that $T_\varepsilon \to +\infty$ and $\varepsilon T_\varepsilon \to 0$ as $\varepsilon \to 0$. Let $\widetilde{u}_\varepsilon, \widetilde{v}_\varepsilon$ be defined on Q_{T_ε} and such that

$$\frac{1}{T_\varepsilon^n} F_1(\tilde{u}_\varepsilon, \tilde{v}_\varepsilon; Q_{T_\varepsilon}) \leq f_{\text{hom}}^{T_\varepsilon}(z) + \delta.$$

For $i \in \varepsilon T_\varepsilon \mathbb{Z}^n$ we set $Q_\varepsilon^i = i + Q_{\varepsilon T_\varepsilon}$. Let $\mathcal{I}_\varepsilon$ be the set of the indices $i \in \varepsilon T_\varepsilon \mathbb{Z}^n$ such that $\overline{Q}_\varepsilon^i \subset S$ and define the recovery sequences by setting

$$u_\varepsilon(x) = \begin{cases} \varepsilon \tilde{u}_\varepsilon\big(\frac{x-i}{\varepsilon}\big) + zi & \text{if } x \in Q_\varepsilon^i,\ i \in \mathcal{I}_\varepsilon \\ zx & \text{if } x \in S \setminus \bigcup_{i\in\mathcal{I}_\varepsilon} Q_\varepsilon^i \end{cases}$$

and correspondingly v_ε. Note that $u_\varepsilon, v_\varepsilon \to v$ in $L^p(S;\mathbb{R}^m)$ and $S(u_\varepsilon)\cap S = S(u_\varepsilon)\cap\bigcup_{i\in\mathcal{I}_\varepsilon} Q^i_\varepsilon$. Since $|S\setminus\bigcup_{i\in\mathcal{I}_\varepsilon} Q^i_\varepsilon| \to 0$ as $\varepsilon\to 0$, recalling that f, g, h and φ are 1-periodic with respect to the first variable, and that $T_\varepsilon \in \mathbb{N}$, we get

$$\begin{aligned}
\limsup_{\varepsilon\to 0} F_\varepsilon(u_\varepsilon, v_\varepsilon; S) &\leq \limsup_{\varepsilon\to 0}\Big(\big|S\setminus\bigcup_{i\in\mathcal{I}_\varepsilon} Q^i_\varepsilon\big|(f(z)+g(z)) + F_\varepsilon(u_\varepsilon, v_\varepsilon; \bigcup_{i\in\mathcal{I}_\varepsilon} Q^i_\varepsilon)\Big)\\
&\leq \limsup_{\varepsilon\to 0} \#\mathcal{I}_\varepsilon \varepsilon^n F_1(\tilde u_\varepsilon, \tilde v_\varepsilon; Q_{T_\varepsilon})\\
&\leq \limsup_{\varepsilon\to 0} \big|\bigcup_{i\in\mathcal{I}_\varepsilon} Q^i_\varepsilon\big| \frac{1}{T^n_\varepsilon} F_1(\tilde u_\varepsilon, \tilde v_\varepsilon; Q_{T_\varepsilon})\\
&\leq |S| \lim_{\varepsilon\to 0} f^{T_\varepsilon}_{\rm hom}(z) + \delta|S|\\
&= |S| f_{\rm hom}(z) + \delta|S| \;=\; F_{\rm hom}(v) + \delta|S|.
\end{aligned}$$

Thanks to the arbitrariness of $\delta > 0$, $(u_\varepsilon, v_\varepsilon)$ is a recovery sequence for $v = zx$. Since $u_\varepsilon(x) = v_\varepsilon(x) = zx$ in a neighborhood of ∂S and since the functionals are defined up to translations, this inequality ensures that the upper estimate holds for a piecewise-affine and continuous v by repeating the construction in each simplex. □

3 One-dimensional behaviour of isotropic energies

In this section we consider a particular case of the Γ-convergence result of Theorem 3, with additional hypotheses of isotropy on f, g, h, φ ensuring that the limit is essentially locally one-dimensional. In particular, within this class fall energies that can be reduced to the examples contained in the next section.

Let $f, g, h\colon [0,+\infty)\to[0,+\infty)$ and $\varphi\colon [0,+\infty)\to[0,+\infty)$ be such that

(H1′) $\frac{1}{c}(|z|^p - 1) \leq f(z),\ g(z),\ h(z) \leq c(|z|^p+1)$

(H2′) $\varphi(tw) \leq c\,\varphi(w)$ for $0\leq t\leq 1$

(H3′) f, g are monotone not decreasing,

and consider the functionals

$$G_\varepsilon(u,v;\Omega) = \int_\Omega \Big(f(|\nabla u|) + g(|\nabla v|) + h\big(\frac{|u-v|}{\varepsilon}\big)\Big)\,dx + \varepsilon\int_{S(u)\cap\Omega} \varphi\big(\frac{|u^+-u^-|}{\varepsilon}\big)\,d\mathcal{H}^{n-1} \tag{15}$$

defined for $u\in SBV^p(\Omega)$ and $v\in W^{1,p}(\Omega)$. The hypotheses (H1′) and (H2′) on f, g, h, φ ensure that the functions $\tilde f(y,z) = f(|z|), \tilde g(y,z) = g(|z|), \tilde h(y,z) = h(|z|)$ and $\tilde\varphi(y,z,\nu) = \varphi(|z|)$ satisfy the hypotheses of Theorem 3, hence the Γ-limit of G_ε with respect to the convergence $u, v\to v$ in $L^p(\Omega)$ is given by

$$G_{\rm hom}(v) = \int_\Omega g_{\rm hom}(\nabla v)\,dx$$

where

$$g_{\rm hom}(z) = \lim_{T\to+\infty} \frac{1}{T^n} \inf\{G_1(u,v;Q_T) : u = v = zx \text{ in } \partial Q_T\}.$$

In the one-dimensional case; i.e., when $n = 1$, in order to highlight the dependence on the dimension, we denote $g_{\rm hom}$ as

$$\begin{aligned} g_{1,\rm hom}(z) = \lim_{T\to+\infty} \frac{1}{T} \inf\Big\{\int_0^T (f(|u'|) + g(|v'|) + h(|u-v|))\,dx + \sum_{S(u)\cap\Omega} \varphi(|u^+ - u^-|) \\ : u(0) = v(0) = 0, u(T) = v(T) = Tz\Big\}. \end{aligned} \tag{16}$$

Note that $g_{1,\rm hom}(z) = g_{1,\rm hom}(|z|)$. The following result holds.

Theorem 8. *If f, g, h, φ satisfy* (H1′)-(H3′)*, then*

$$G_{\rm hom}(v) = \int_\Omega g_{1,\rm hom}(|\nabla v|)\,dx.$$

Proof. Lower bound. We prove the lower bound by using a slicing argument (see [8, Sec. 4.1], [9, Sec. 3.4]). For each $\xi \in S^{n-1}$ we consider the orthogonal hyperplane passing through 0; that is, $\Pi_\xi = \{x \in \mathbb{R}^n : x\cdot\xi = 0\}$. Given a bounded subset A of Ω, for each $y \in \Pi_\xi$ we define the one-dimensional set $A_{\xi,y} = \{t \in \mathbb{R} : y + t\xi \in A\}$, and for w defined on Ω we denote by $w^{\xi,y}$ the one-dimensional function

$$w^{\xi,y}(t) = w(y + t\xi)$$

defined on $\Omega_{\xi,y}$. Note that if $v \in W^{1,p}(A)$ and $u \in SBV^p(A)$, then for any ξ the function $v^{\xi,y}$ belongs to $W^{1,p}(A_{\xi,y})$, the function $u^{\xi,y}$ belongs to $SBV^p(A_{\xi,y})$ for almost all $y \in \Pi_\xi$, and that $S(u_{\xi,y}) = \{t \in \mathbb{R} : y + t\xi \in S(u)\}$ (see for instance [8, Th. 4.1]).

Let I be a bounded open subset of $\mathbb{R}$; for $u \in SBV^p(I)$ and $v \in W^{1,p}(I)$ we define

$$G^{\xi,y}_\varepsilon(u,v;I) = \int_I \Big(f(|u'|) + g(|v'|) + h\big(\frac{|u-v|}{\varepsilon}\big)\Big)\,dt + \varepsilon \sum_{t\in S(u)\cap I} \varphi\big(\frac{|u^+ - u^-|}{\varepsilon}\big).$$

By Theorem 3 applied in dimension one, the Γ-limit of $G^{\xi,y}_\varepsilon$ is given by

$$G^{\xi,y}(v;I) = \int_I g_{1,\rm hom}(v')\,dt = \int_I g_{1,\rm hom}(|v'|)\,dt.$$

Now, setting for $v \in W^{1,p}(A)$ and $u \in SBV^p(A)$

$$G^\xi_\varepsilon(u,v;A) = \int_{\Pi_\xi} G^{\xi,y}_\varepsilon(u^{\xi,y}, v^{\xi,y}; A_{\xi,y})\,d\mathcal{H}^{n-1}(y)$$

by an application of Fubini's Theorem we get

$$\begin{aligned} G_\varepsilon^\xi(u,v;A) &= \int_A \Big(f(|\nabla u\cdot\xi|) + g(|\nabla v\cdot \xi|) + h\big(\frac{|u-v|}{\varepsilon}\big)\Big)\,dx \\ &\quad +\varepsilon\int_{S(u)\cap A} \varphi\big(\frac{|u^+-u^-|}{\varepsilon}\big)|\nu_u\cdot\xi|\,d\mathcal{H}^{n-1} \\ &\le \int_A \Big(f(|\nabla u|) + g(|\nabla v|) + h\big(\frac{|u-v|}{\varepsilon}\big)\Big)\,dx + \varepsilon\int_{S(u)\cap A}\varphi\big(\frac{|u^+-u^-|}{\varepsilon}\big)\,d\mathcal{H}^{n-1} \\ &= G_\varepsilon(u,v;A). \end{aligned}$$

thanks to the monotonicity of f and g. Now, let $u_\varepsilon, v_\varepsilon \to v$ as $\varepsilon\to 0$ in $L^p(\Omega)$. Note that, again by Fubini's Theorem, $u_\varepsilon^{\xi,y}, v_\varepsilon^{\xi,y} \to v^{\xi,y}$ in $L^p(\Omega_{\xi,y})$ for almost all $y\in\Pi_\xi$. Applying Fatou's Lemma we get

$$\begin{aligned} \liminf_{\varepsilon\to 0} G_\varepsilon(u_\varepsilon,v_\varepsilon;A) &\ge \liminf_{\varepsilon\to 0} G_\varepsilon^\xi(u_\varepsilon,v_\varepsilon;A) \\ &= \liminf_{\varepsilon\to 0}\int_{\Pi_\xi} G_\varepsilon^{\xi,y}(u_\varepsilon^{\xi,y}, v_\varepsilon^{\xi,y}; A_{\xi,y})\,d\mathcal{H}^{n-1}(y) \\ &\ge \int_{\Pi_\xi}\liminf_{\varepsilon\to 0} G_\varepsilon^{\xi,y}(u_\varepsilon^{\xi,y}, v_\varepsilon^{\xi,y}; A_{\xi,y})\,d\mathcal{H}^{n-1}(y) \\ &\ge \int_{\Pi_\xi}\int_{A_{\xi,y}} g_{1,\mathrm{hom}}(|v'_{\xi,y}|)\,dt\ d\mathcal{H}^{n-1}(y) \\ &= \int_A g_{1,\mathrm{hom}}(|\nabla v\cdot\xi|)\,dx. \end{aligned}$$

The functionals G_ε are local, hence the set function $\mu(A) = \Gamma\text{-}\liminf_{\varepsilon\to 0} G_\varepsilon(v;A)$ is super-additive on open sets with disjoint compact closure. Let $\{\xi_i\}$ be a countable dense subset of S^{n-1}. Since for any i

$$\mu(A) \ge \int_A \psi_i\,d\lambda$$

where $\lambda = \mathcal{L}^n$ and $\psi_i(x) = g_{1,\mathrm{hom}}(|\nabla v(x)\cdot\xi_i|)$, it follows that

$$\mu(A) \ge \int_A \sup_i g_{1,\mathrm{hom}}(|\nabla v\cdot\xi_i|)\,dx$$

(see [9, Lemma 3.1]). The continuity of $g_{1,\mathrm{hom}}$ gives

$$\Gamma\text{-}\liminf_{\varepsilon\to 0} G_\varepsilon(v;A) \ge \int_A g_{1,\mathrm{hom}}(|\nabla v|)\,dx.$$

Upper bound. Since Theorem 3 holds, it is sufficient to construct a recovery sequence for an affine function $v = \alpha\xi x$ in a n-dimensional open simplex S, with $\alpha\in\mathbb{R}$ and $\xi\in S^{n-1}$.

We fix $\delta > 0$. Let $T > 0$, $u \in SBV^p(0,T)$, and $v \in W^{1,p}(0,T)$ be such that $u(0) = v(0) = 0$, $u(T) = v(T) = |\alpha|T$, and

$$\frac{1}{T} G_1(u, v; (0,T)) < g_{1,\mathrm{hom}}(|\alpha|) + \delta.$$

Given a bounded interval $[a,b]$ we set $M_\varepsilon = \lfloor \frac{b-a}{\varepsilon T} \rfloor$ and consider the intervals $a + \varepsilon T[m, m+1)$ for any $m = 0, \dots, M_\varepsilon - 1$. We introduce the one-dimensional function $\tilde{u}_\varepsilon \colon [a,b] \to \mathbb{R}$ given by

$$\tilde{u}_\varepsilon(t) = \varepsilon u\big(\frac{t - \varepsilon T m - a}{\varepsilon}\big) + (a + \varepsilon T m)|\alpha| \quad \text{for } t \in a + \varepsilon T[m, m+1)$$

and $\tilde{u}_\varepsilon(t)$ affine in $[a+\varepsilon T M_\varepsilon, b]$ with $\tilde{u}_\varepsilon(a+\varepsilon T M_\varepsilon) = \varepsilon T M_\varepsilon + a|\alpha|$ and $\tilde{u}_\varepsilon(b) = b|\alpha|$. We define $\tilde{v}_\varepsilon$ correspondingly. We get

$$G_\varepsilon(\tilde{u}_\varepsilon, \tilde{v}_\varepsilon; (a,b)) \leq \varepsilon \sum_{m=0}^{M_\varepsilon - 1} G_1(u, v; (0,T)) + C\varepsilon \leq (b-a) g_{1,\mathrm{hom}}(|\alpha|) + \delta + C\varepsilon \quad (17)$$

where C depends only on α, a, b and $h(0)$. For a given $y \in \Pi_\xi$, if $\{t\xi + y : t \in \mathbb{R}\} \cap S \neq \emptyset$ then there exist $a(y)$, $b(y)$ with $a(y) < b(y)$ and such that $\{t\xi + y : t \in \mathbb{R}\} \cap S = \{t\xi + y : a(y) < t < b(y)\}$. Hence, setting

$$u_\varepsilon(x) = \tilde{u}_\varepsilon((x-y)\xi) \quad \text{and} \quad v_\varepsilon(x) = \tilde{v}_\varepsilon((x-y)\xi) \quad (18)$$

it follows that $u_\varepsilon, v_\varepsilon \to v$ in $L^p(S)$ and, thanks to (17)

$$\begin{aligned}
G_\varepsilon(u_\varepsilon, v_\varepsilon; S) &= \int_{S^\xi} \int_{a(y)}^{b(y)} f(|\xi \tilde{u}_\varepsilon'(t)|) + g(|\xi \tilde{u}_\varepsilon'(t)|) + h\Big(\frac{|\tilde{u}_\varepsilon - \tilde{v}_\varepsilon|}{\varepsilon}\Big)\, dt\, d\mathcal{H}^{n-1}(y) \\
&\quad + \int_{S^\xi} \varepsilon \sum_{t \in S(\tilde{u}_\varepsilon) \cap (a(y), b(y))} \varphi\Big(\frac{|\tilde{u}_\varepsilon^+ - \tilde{u}_\varepsilon^-|}{\varepsilon}\Big)\, d\mathcal{H}^{n-1}(y) \\
&= \int_{S^\xi} G_\varepsilon(\tilde{u}_\varepsilon, \tilde{v}_\varepsilon; (a(y), b(y)))\, d\mathcal{H}^{n-1}(y) \\
&\leq \int_{S^\xi} \Big((b(y) - a(y)) g_{1,\mathrm{hom}}(|\alpha|) + \delta + C\varepsilon\Big)\, d\mathcal{H}^{n-1}(y) \\
&= \int_S g_{1,\mathrm{hom}}(|\nabla v|)\, dx + |S|\delta + |S|C\varepsilon
\end{aligned}$$

where S^ξ denotes the set of $y \in \Pi_\xi$ such that $\{t\xi + y : t \in \mathbb{R}\} \cap S \neq \emptyset$. Thanks to the arbitrariness of $\delta > 0$, it follows that the sequence $(u_\varepsilon, v_\varepsilon)$ defined in (18) is a recovery sequence for the Γ-limit. □

Example 9 (Bi-stable energy density)**.** In the particular case $f(z) = (z-1)^2$, $g(z) = h(z) = z^2$ and $\varphi = +\infty$, thanks to Remark 5 we have

$$g_{1,\mathrm{hom}}(z) = \begin{cases} 2z^2 + 2z + 1 & \text{if } z \leq -1 \\ z^2 & \text{if } |z| \leq 1 \\ 2z^2 - 2z + 1 & \text{if } z \geq 1. \end{cases}$$

Theorem 8 highlights that recovery sequences exhibit oscillations at a scale much smaller than ε oriented orthogonally to the gradient of the target function v.

4 A prototypical one-dimensional example: free-discontinuity non-convexity

We analyze fracture-type non convex energies, for which the effective behaviour is determined by an optimal periodic arrangement of discontinuities.

We consider a particular case of the sequence of functional defined in (15) in the one-dimensional frame, with f, g, h not depending on the space variable, h even and strictly convex, f and g strictly convex and φ constant; that is,

$$G_\varepsilon(u, v; I) = \int_I \Big(f(u') + g(v') + h\big(\frac{u-v}{\varepsilon}\big)\Big)\, dx + k\varepsilon \# S(u). \tag{19}$$

Applying Theorem 3 we get that the Γ-limit of G_ε with respect to the convergence $u_\varepsilon, v_\varepsilon \to v$ in $L^p(I)$ is given by

$$G_{\hom}(v) = \int_I g_{\hom}(v')\, dx$$

where $g_{\hom}$ is defined in (16). In the sequel, we denote by $\widetilde{G}_\varepsilon(u, v; I)$ the functional

$$\widetilde{G}_\varepsilon(u, v; I) = \int_I \Big(f(u') + g(v') + h\big(\frac{u-v}{\varepsilon}\big)\Big)\, dx \tag{20}$$

again defined for $v \in W^{1,p}(I)$ and $u \in SBV^p(I)$. The following proposition holds.

Proposition 10. *If f, g, h are even and strictly convex and φ is the positive constant k, then*

$$g_{\hom}(z) = \inf_{S>0} \Big\{\psi(S, z) + \frac{k}{S}\Big\} \tag{21}$$

where

$$\psi(S, z) = \frac{1}{S} \min \Big\{\widetilde{G}_1(u, v; (0, S)) : u, v \in W^{1,p}(0, S),\ v(0) = 0,\ v(S) = Sz\Big\}. \tag{22}$$

Remark 11. A special case is when the minimum points of f and g coincide. Then, denoting by z^* the common minimum point, we have $\psi(S, z^*) = f(z^*) + g(z^*) + h(0)$, independent of S, so that $g_{\hom}(z) = f(z^*) + g(z^*) + h(0)$, and the infimum (21) is achieved for $S \to +\infty$.

Before giving the proof of Proposition 10 we state some preliminary properties.

Remark 12 (Symmetry properties of minimizers of $\psi(S, z)$)**.** Note that the minimum in (22) is achieved by the application of the direct method of the calculus of variations. The strict convexity of f, g and h gives the uniqueness of the minimum point. We observe that, by the strict convexity of f, g and h, if (u, v) realizes the minimum in (22) then

$$u\Big(\frac{S}{2}\Big) = v\Big(\frac{S}{2}\Big) = \frac{Sz}{2}. \tag{23}$$

Indeed, otherwise assume by contradiction that (u, v) solve the problem (22) and do not satisfy (23). We then set

$$\tilde{u}(t) = \frac{1}{2}\big(u(t) - u(S-t) + Sz\big), \quad \tilde{v}(t) = \frac{1}{2}\big(v(t) - v(S-t) + Sz\big)$$

so that $(\tilde{u}, \tilde{v})$ is admissible for (22) and satisfies (23). Note that $\tilde{u}(S) - \tilde{u}(0) = u(S) - u(0)$. Then, the convexity of f, g and h and the symmetry of h ensure that

$$\begin{aligned}
\widetilde{G}_1(\tilde{u}, \tilde{v}; (0, S)) &= \int_0^S \big(f(\tilde{u}') + g(\tilde{v}') + h(\tilde{u} - \tilde{v})\big)\, dt \\
&= \int_0^S f\Big(\frac{1}{2}(u'(t) + u'(S-t))\Big)\, dt + \int_0^S g\Big(\frac{1}{2}(v'(t) + v'(S-t))\Big)\, dt \\
&\quad + \int_0^S h\Big(\frac{1}{2}(u(t) - v(t)) + \frac{1}{2}(v(S-t) - u(S-t))\Big)\, dt \\
&\leq \int_0^S \big(f(u') + g(v') + h(u - v)\big)\, dt.
\end{aligned}$$

The uniqueness of the minimum point implies that $\tilde{u} = u$ and $\tilde{v} = v$, giving the contradiction.

Moreover, consider z not equal to the (possible) common minimum point of f and g, and (u, v) solving (22). Then $u(t) = v(t) = zt$ if and only if $t = \frac{S}{2}$. Indeed, if there exists $\sigma < \frac{S}{2}$ such that $u(\sigma) = v(\sigma) = \sigma z$, then by the minimality we deduce $u = v = zt$ in $[\sigma, \frac{S}{2}]$. Denoting by $[a, b]$ the larger interval containing $[\sigma, \frac{S}{2}]$ such that $u = v = zt$ in $[a, b]$, if $a > 0$ we construct a new pair $(\tilde{u}, \tilde{v})$ by setting

$$\tilde{u}(t) = \begin{cases} zt & \text{if } 0 \leq t \leq b - a \\ u(t - (b-a)) + z(b-a) & \text{if } b - a \leq t \leq b \\ u(t) & \text{if } b \leq t \leq S \end{cases}$$

and in the same way $\tilde{v}$. Since $\widetilde{G}_1(\tilde{u}, \tilde{v}; (0, S)) = \tilde{G}_1(u, v; (0, S))$, by uniqueness we deduce that $\tilde{u} = u$ and $\tilde{v} = v$. Hence, $u(t) = v(t) = zt$ in $[0, b]$ which gives a contradiction since $a > 0$ and $[a, b]$ is maximal. If $a = 0$ and $b < S$, a completely similar construction allows to prove that $u(t) = v(t) = zt$ in the whole interval $(0, S)$, which gives a contradiction since the affine and equal functions do not solve the minimum problem (22) where the boundary values are imposed only on v, and z does not coincide with the (possible) common minimum point of f and g.

We now prove a property of the minimum problem for $\widetilde{G}_1(u, v; (0, T))$ with fixed boundary values for v and a fixed number of jump points for u.

Lemma 13. *For any $T > 0$, $M \in \mathbb{N}$ and $z \in \mathbb{R}$ different from the (possible) common minimum point of f and g the following equality holds:*

$$\begin{aligned}
&\min\{\widetilde{G}_1(u,v;(0,T)) : v(0) = 0, v(T) = Tz, \#S(u) = M\} \\
&= \min\{\widetilde{G}_1(u,v;(0,T)) : (u,v) \in \mathcal{E}_M(T,z)\}
\end{aligned} \tag{24}$$

where $\mathcal{E}_M(T,z)$ is the set of pairs $(u,v) \in SBV^p(0,T) \times W^{1,p}(0,T)$ satisfying

$$S(u) = \Big\{\frac{iT}{M+1} : 1 \le i \le M\Big\},\ v\Big(\frac{iT}{M+1}\Big) = \frac{iTz}{M+1} \text{ for } 0 \le i \le M+1. \tag{25}$$

Proof. We consider the case $M \ge 1$ (noting that for $M = 0$ the thesis is obvious). We start by proving the existence of the first minimum in (24). Indeed, by semicontinuity and since the constraint is closed, there exists the minimum of $\widetilde{G}_1(u,v;(0,T))$ in

$$\{(u,v) :\ v(0) = 0,\ v(T) = Tz,\ \#S(u) \le M\}$$

where $v \in W^{1,p}(0,T)$ and $u \in SBV^p(0,T)$ as above (see e.g. [8]). Let $(\overline{u},\overline{v})$ realize the minimum.

We prove the thesis by showing that $\#S(\overline{u}) = M$. Set $K = \#S(\overline{u})$ and suppose by contradiction that $K < M$. Since z does not coincide with the (possible) common minimum point z^* of f and g, there exists at least one interval $(a,b) \subset (0,T)$ such that $a,b \in S(\overline{u}) \cup \{0,T\}$, $\overline{u} \in W^{1,p}(a,b)$ and $\overline{v}(b) - \overline{v}(a) = \overline{z}(b-a) \ne z^*(b-a)$. Recalling Remark 12, $\overline{u}(t) = \overline{v}(t) = \overline{v}(a) + \overline{z}(t-a)$ holds in (a,b) if and only if $t = \frac{a+b}{2}$. Denoting by $\tilde{u},\tilde{v}$ the unique solution of the minimum problem defining $\psi(\frac{b-a}{2},\overline{z})$, we set

$$u(t) = \begin{cases} \overline{u}(t) & t \in (0,T) \setminus (a,b) \\ \overline{u}(a) + \tilde{u}(t-a) & t \in (a, \frac{a+b}{2}) \\ \overline{u}(\frac{a+b}{2}) + \tilde{u}(t - \frac{a+b}{2}) & t \in (\frac{a+b}{2}, b) \end{cases}$$

and correspondingly we define v. Since $\overline{z}$ does not coincide with z^*, then $(u,v) \ne (\overline{u},\overline{v})$ in (a,b) and $\#S(u) = K+1$. The pair (u,v) is admissible as test function since $\#S(u) = K+1 \le M$, hence by the uniqueness of the solution of $\psi(\frac{b-a}{2},\overline{z})$ we get

$$\widetilde{G}_1(u,v;(0,T)) < \widetilde{G}_1(\overline{u},\overline{v};(0,T))$$

which gives a contradiction.

Now we show equality (24) for $M = 1$. Let (u,v) be such that $S(u) = \{\tau\}$, $v(0) = 0$ and $v(T) = Tz$. Setting $T_1 = \tau$, $T_2 = T - \tau$, $T_1 z_1 = v(\tau)$ and $T_2 z_2 = Tz - T_1 z_1$, for $i = 1,2$ we choose (u_i,v_i) solving the minimum problem for $\psi(T_i,z_i)$ and define

$$\tilde{u}(t) = \begin{cases} u_1(t) & \text{if } t < \frac{T_1}{2} \\ u_2\big(t + \frac{T_2 - T_1}{2}\big) + \frac{T_1 z_1 - T_2 z_2}{2} & \text{if } \frac{T_1}{2} \le t < \frac{T}{2} \\ u_2\big(t - \frac{T}{2}\big) + \frac{Tz}{2} & \text{if } \frac{T}{2} \le t < \frac{T}{2} + \frac{T_2}{2} \\ u_1(t - T_2) + T_2 z_2 & \text{if } \frac{T}{2} + \frac{T_2}{2} \le t \le T \end{cases}$$

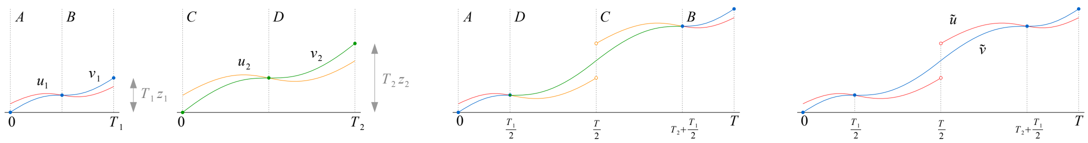

Figure 1: Construction of $\tilde{u}, \tilde{v}$.

In the same way we define $\tilde{v}$ (see Fig. 1).

Since (23) holds in $(0, T_i)$, we deduce that $S(\tilde{u}) = \{\frac{T}{2}\}$ and $\tilde{v}\big(\frac{T}{2}\big) = \frac{Tz}{2}$. By construction,

$$\widetilde{G}_1(\tilde{u}, \tilde{v}; (0, T)) = \widetilde{G}_1(u, v; (0, T))$$

concluding the proof of (25) for $M = 1$.

Note that if $\tau \neq \frac{T}{2}$ then $(\tilde{u}, \tilde{v})$ does not solve the minimum problem for $\psi(\frac{T}{2}, \frac{z}{2})$. Indeed, if it were not so, then, recalling Remark 12, both $\tilde{u}$ and $\tilde{v}$ would coincide with the affine function zt in $\frac{T}{4}$ and $\frac{T_1}{2}$. As noticed in Remark 12, the strict convexity of f, g, h would give a contradiction. The same holds if $\tau = \frac{T}{2}$ and $v(\tau) \neq \frac{Tz}{2}$, since the previous construction gives a function $\tilde{v}$ such that $\tilde{v}(\frac{T}{4}) \neq \frac{Tz}{4}$; hence also in this case $(\tilde{u}, \tilde{v})$ does not solve the minimum problem for $\psi(\frac{T}{2}, \frac{z}{2})$.

Then, if (u, v) does not belong to $\mathcal{E}_1(T, z)$, we can always find $\overline{u} \in SBV^p(0, T)$ with $S(u) = \{\frac{T}{2}\}$ and $\overline{v} \in W^{1,p}(0, T)$ with $\overline{v}(0) = 0$, $\overline{v}(\frac{T}{2}) = \frac{Tz}{2}$ and $\overline{v}(T) = Tz$ such that

$$\widetilde{G}_1(\overline{u}, \overline{v}; (0, T)) < \widetilde{G}_1(\tilde{u}, \tilde{v}; (0, T)).$$

If $M > 1$, let (u, v) be such that $v(0) = 0$ and $v(T) = Tz$ and $S(u) = \{\tau_i\}_{i=1}^M$ with $0 = \tau_0 < \tau_i < \tau_{i+1} < \tau_{M+1} = T$ for any $i = 1, \ldots, M-1$. We set $T_i = \tau_{i+1} - \tau_i$. If (u, v) does not satisfy (25) there exists j such that $T_j \neq T_{j-1}$ or $T_j = T_{j-1}$ and $v(\frac{\tau_{i-1}+\tau_{i+1}}{2}) \neq \frac{v(\tau_{i+1})+v(\tau_{i-1})}{2}$. Since the functionals are invariant by translations, we can suppose $j = 1$. The same argument of the case $M = 1$ allows to find $\overline{u} \in SBV^p(0, \tau_2)$ with $S(u) = \{\frac{\tau_2}{2}\}$ and $\overline{v} \in W^{1,p}(0, \tau_2)$ with $\overline{v}(0) = 0$, $\overline{v}(\frac{\tau_2}{2}) = \frac{v(\tau_2)}{2}$ and $\overline{v}(\tau_2) = v(\tau_2)$ such that

$$\widetilde{G}_1(\overline{u}, \overline{v}; (0, \tau_2)) < \widetilde{G}_1(u, v; (0, \tau_2)).$$

By defining $\tilde{u} = \overline{u}$ in $(0, \tau_2)$ and $\tilde{u}(t) = u(t)$ in (τ_2, T), and correspondingly $\tilde{v}$, we deduce

$$\widetilde{G}_1(\tilde{u}, \tilde{v}; (0, T)) < \widetilde{G}_1(u, v; (0, T))$$

concluding the proof. □

Proof of Proposition 10. For any fixed $N \in \mathbb{N}$ we consider the minimum problem

$$\frac{1}{T} \inf\{\widetilde{G}_1(u, v; (0, T)) : u(0) = v(0) = 0,\ u(T) = v(T) = Tz,\ \#S(u) = N\}. \quad (26)$$

Note that for $N=0$ the minimum in (26) is attained in $u(t)=v(t)=zt$ since f,g,h are even and convex, and

$$\frac{1}{T}\inf\{\widetilde{G}_1(u,v;(0,T)) : u(0)=v(0)=0,\ u(T)=v(T)=Tz,\ \#S(u)=0\} \\ = f(z)+g(z)+h(0).$$

If $N\geq 1$, we have

$$\frac{1}{T}\inf\{\widetilde{G}_1(u,v;(0,T)) : u(0)=v(0)=0,\ u(T)=v(T)=Tz,\ \#S(u)=N\} \\ \geq \frac{1}{T}\inf\{\widetilde{G}_1(u,v;(0,T)) : v(0)=0,\ v(T)=Tz,\ \#S(u)=N-1\}. \tag{27}$$

Indeed, let u,v be admissible functions for the minimum problem (26), and let $\tau\in(0,T)$ be such that $\tau\in S(u)$ and $u\in H^1(0,\tau)$. We consider a periodic extension of the problem and define $\tilde{u},\tilde{v}$ in $[0,T]$ by setting

$$\tilde{u}(x)=\begin{cases} u(x+\tau)-v(\tau) & \text{if } 0\leq x<T-\tau \\ u(x+\tau-T)-v(\tau)+Tz & \text{if } T-\tau\leq x\leq T\end{cases}$$

and $\tilde{v}$ correspondingly. Hence $\tilde{v}(0)=0$, $\tilde{v}(T)=Tz$ and $\#S(\tilde{u})=N-1$, and we get

$$\widetilde{G}_1(u,v;(0,T))=G_1(\tilde{u},\tilde{v};(0,T))$$

which proves (27).

Recalling that Lemma 13 ensures that for any $N\geq 1$

$$\frac{1}{T}\inf\{\widetilde{G}_1(u,v;(0,T)) : v(0)=0,\ v(T)=Tz,\ \#S(u)=N-1\}\geq \psi\Big(\frac{T}{N},z\Big),$$

it follows that

$$\begin{aligned} g_{\rm hom}(z) &= \inf_{T>0}\frac{1}{T}\{\widetilde{G}_1(u,v(0,T))+k\#S(u) : u(0)=v(0)=0,\ u(T)=v(T)=Tz\} \\ &\geq \min\Big\{f(z)+g(z)+h(0), \inf_{T>0}\inf_{N\geq 1}\Big\{\psi(\frac{T}{N},z)+k\frac{N}{T}\Big\}\Big\} \\ &\geq \min\Big\{f(z)+g(z)+h(0), \inf_{S>0}\Big\{\psi(S,z)+\frac{k}{S}\Big\}\Big\} \\ &\geq \inf_{S>0}\Big\{\psi(S,z)+\frac{k}{S}\Big\} \end{aligned}$$

since $f(z)+g(z)+h(0)\geq \inf_{S>0}\{\psi(S,z)+\frac{k}{S}\}$.

Now we have to show the opposite inequality. We fix $\delta>0$; let S_δ be such that

$$\psi(S_\delta,z)+\frac{k}{S_\delta}<\inf_{S>0}\Big\{\psi(S,z)+\frac{k}{S}\Big\}+\delta$$

and $\overline{u},\overline{v}\in W^{1,p}(0,S_\delta)$ solving the minimum problem (22). We set

$$\tilde{u}(x)=\begin{cases} u(x+\frac{S_\delta}{2})-v(\frac{S_\delta}{2}) & \text{if } 0\leq x<\frac{S_\delta}{2} \\ u(x-\frac{S_\delta}{2})-v(\frac{S_\delta}{2})+S_\delta z & \text{if } \frac{S_\delta}{2}\leq x\leq S_\delta\end{cases}$$

and correspondingly $\tilde{v}$. Since (23) holds, then $\tilde{u}(0) = \tilde{v}(0) = 0$, $\tilde{u}(S_\delta) = \tilde{v}(S_\delta) = S_\delta z$, and $\#S(\tilde{u}) \leq 1$. We get

$$\begin{aligned}
\psi(S_\delta, z) + \frac{k}{S_\delta} &= \frac{1}{S_\delta}\widetilde{G}_1(\tilde{u}, \tilde{v}; (0, S_\delta)) + \frac{k}{S_\delta} \\
&\geq \frac{1}{S_\delta}\inf\{G_1(u, v; (0, S_\delta)) : u(0) = v(0) = 0, u(S_\delta) = v(S_\delta) = S_\delta z\} \\
&\geq g_{\text{hom}}(z).
\end{aligned}$$

Hence

$$\inf_{S>0}\Big\{\psi(S, z) + \frac{k}{S}\Big\} + \delta \geq g_{\text{hom}}(z).$$

Since $\delta > 0$ is arbitrary, this concludes the proof of (21). □

Example 14. We now compute the homogenized energy density for an explicit example, which has also been analyzed in [6]. This example will allow us to highlight the behaviour of g_{hom} close to 0 and at infinity. A similar discrete energy has been studied in [27]; for a possible future comparison with that paper, we use the same notation therein: we take $f(z) = \frac{1}{2}z^2$, $g(z) = \frac{a}{2}z^2$, $h(z) = \frac{b}{2}z^2$, and $k = \frac{\eta}{2}$. With this choice

$$E_\varepsilon(u, v; I) = \int_I \Big(\frac{1}{2}(u')^2 + \frac{a}{2}(v')^2 + \frac{b}{2}\big(\frac{u-v}{\varepsilon}\big)^2\Big)\,dt + \frac{\eta}{2}\varepsilon\#S(u) \tag{28}$$

where I is a bounded interval, $v \in H^1(I)$ and $u \in SBV(I)$. We can apply Proposition 10, obtaining that the Γ-limit of E_ε with respect to the convergence $u_\varepsilon, v_\varepsilon \to v$ in $L^2(I)$ is given by

$$E_{\text{hom}}(w) = \int_I e_{\text{hom}}(v')\,dt$$

with e_{hom} given by Proposition 10.

Proposition 15 (Explicit representation of e_{hom})**.** *The following equality holds*

$$e_{\text{hom}}(z) = \inf_{S>0}\Big\{\frac{1}{2}\frac{(a+1)\frac{\omega S}{2}}{\frac{\omega S}{2} + \frac{1}{a}\tanh(\frac{\omega S}{2})}z^2 + \frac{\eta}{2S}\Big\} \tag{29}$$

where $\omega^2 = \frac{(a+1)b}{a}$. Moreover, the function $e_{\text{hom}}(z)$ is strictly convex, and

1. $e_{\text{hom}}(z) = \dfrac{a+1}{2}z^2$ *in $[0, z_c]$, where* $z_c = \sqrt{\frac{\eta\omega a}{2(a+1)}}$*;*
2. *for $z \to +\infty$ $e_{\text{hom}}(z) \sim \dfrac{a}{2}z^2 + Cz^{2/3}$, where $C > 0$ depends only on a, b, η.*

Remark 16. In the proof of the proposition above, we show that for $z \leq z_c$ the inf in (29) is given by the limit for $S \to +\infty$, and the minimum is attained in a unique $S(z) > 0$ otherwise. Hence, we can introduce the function

$$S(z) = \begin{cases} +\infty & \text{if } z \leq z_c \\ \arg\min\Big\{\frac{1}{2}\frac{(a+1)\frac{\omega S}{2}}{\frac{\omega S}{2} + \frac{1}{a}\tanh(\frac{\omega S}{2})}z^2 + \frac{\eta}{2S} : S > 0\Big\} & \text{if } z > z_c \end{cases}$$

representing the optimal distance between fracture points at a given strain. In Fig. 2 we picture the graph of the "effective" energy $e_{\rm hom}(z) - \frac{a}{2}z^2$ (given by the difference between the homogenized density energy and the energy of the elastic substrate) and the density of fracture $\frac{1}{S(z)}$ (see the corresponding pictures in [6, Fig. 5]).

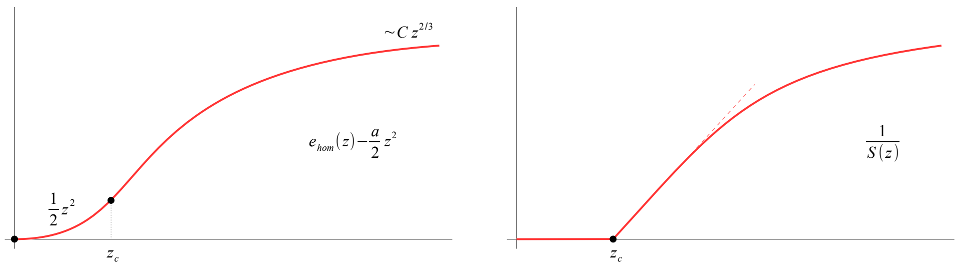

Figure 2: The energy $e_{\rm hom}(z) - \frac{a}{2}z^2$ and the density of fracture $\frac{1}{S(z)}$.

Proof of Proposition 15. By solving the Euler-Lagrange equations for

$$\widetilde{E}_1(u, v; (0, S)) = \frac{1}{2}\int_0^S ((u')^2 + a(v')^2 + b(u-v)^2)\,dt$$

and by minimizing on the boundary values of u we get the explicit expression for ψ in (22) as

$$\psi(S, z) = \frac{1}{2}\frac{(a+1)\frac{\omega S}{2}}{\frac{\omega S}{2} + \frac{1}{a}\tanh(\frac{\omega S}{2})}z^2$$

where $\omega^2 = \frac{(a+1)b}{a}$, proving (29). In order to complete the proof of the proposition, we simplify the expression of $e_{\rm hom}(z)$ by writing

$$e_{\rm hom}(z) = \frac{\eta\omega}{4}\inf_{x>0}\Big\{\frac{2(a+1)}{\eta\omega}z^2\frac{ax}{ax+\tanh(x)} + \frac{1}{x}\Big\} = \frac{\eta\omega}{4}\inf_{x>0} H\Big(x, z\sqrt{\frac{2(a+1)}{\eta\omega}}\Big)$$

where $H(x, y) = \frac{ax}{ax+\tanh(x)}y^2 + \frac{1}{x}$. Since for $y \le \sqrt{a}$

$$\frac{x^2}{y^2}\frac{\partial H}{\partial x}(x, y) = \frac{a\tanh(x) - ax(1-\tanh^2(x))}{(ax+\tanh(x))^2}x^2 - \frac{1}{y^2} < 0,$$

the infimum coincides with the limit for $x \to +\infty$, and we get that

$$e_{\rm hom}(z) = \frac{a+1}{2}z^2 \quad \text{for} \quad z \le z_c = \sqrt{\frac{\eta\omega a}{2(a+1)}}. \tag{30}$$

Now we consider the case $z > z_c$, corresponding to $y > \sqrt{a}$. In this case $\inf_{x>0} H(x,y) = H(x(y),y)$, where $x = x(y)$ is implicitly defined by

$$\frac{a\tanh x - ax(1-\tanh^2 x)}{(ax+\tanh x)^2}y^2 = \frac{1}{x^2}. \tag{31}$$

We deduce that $x(y)$ is strictly decreasing, and tends to 0 as $y \to +\infty$. Moreover, again by (31) we get that $x(y) \sim \sqrt[3]{\frac{3(a+1)^2}{2a}}y^{-\frac{2}{3}}$ for $y \to +\infty$. Hence,

$$\inf_{x>0} H(x,y) = H(x(y),y) \sim \frac{a}{a+1}y^2 + \sqrt[3]{\frac{2a}{3(a+1)^2}}y^{2/3} \quad \text{for} \quad y \to +\infty$$

and consequently $e_{\hom}(z) \sim \dfrac{a}{2}z^2 + \sqrt[3]{\dfrac{4a}{3(a+1)\eta\omega}}z^{2/3}$ for $z \to +\infty$.

As for the strict convexity of $e_{\hom}(z)$, we prove that $(\inf_{x>0} H(x,y))'$ is strictly increasing. By (31) we get

$$(\inf_{x>0} H(x,y))' = \frac{d}{dy}H(x(y),y) = 2y\frac{ax(y)}{ax(y)+\tanh x(y)}$$

which is strictly positive, then it is strictly increasing if and only if $\big((\inf_{x>0} H(x,y))'\big)^2$ is strictly increasing. Again by (31) we deduce

$$\Big(2y\frac{x(y)}{ax(y)+\tanh x(y)}\Big)^2 = \frac{4}{a\tanh x(y) - ax(y)(1-\tanh^2 x(y))},$$

and $\Big(\dfrac{4}{a\tanh x(y) - ax(y)(1-\tanh^2 x(y))}\Big)' > 0$ since $x'(y) < 0$, thus concluding the proof. □

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This paper directly stems from discussions with Lev Truskinovsky on some discrete models introduced in two recent companion papers [27, 28], to which the models in this paper are freely inspired.

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