Thin films with many small cracks

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Abstract We show with an example that the limiting theory as thickness goes to zero of a thin film with many small cracks can be three-dimensional rather than two-dimensional.

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1 Introduction

The use of asymptotic effective theories for thin structures like strings, rods, beams, films, membranes, plates and shells is a subject with a very long and rich history. In the development of these theories, one typically starts with an ansatz about the deformation: one assumes that the shear strain is constant across the thickness and that the longitudinal (normal) strain varies linearly across the thickness, or $\partial U/\partial X_3$ is independent of X_3 where U is the (three-dimensional vector-valued) displacement and X_3 is the coordinate along the thickness direction (see for example [1]). In the familiar Euler-Bernoulli beams or Kirchhoff plates one assumes that the shear is in fact zero, while the more general ansatz stated above is used for example in the theory of Cosserat membranes. This general ansatz has been found to be quite reliable, and has recently been justified rigorously in various settings: films made of isotropic materials (classical situation), anisotropic materials, phase-transforming materials and even in heterogeneous thin films in a suitable macroscopic sense (see for example [2, 9, 3, 5, 6, 10]). It is natural, then, to wonder under what the circumstances, if any, this ansatz fails. In this paper we present an example where this well-regarded wisdom *fails* due to the presence of a large number of cracks parallel to the plane of the film.

To be specific, let us consider a film $\Omega_\varepsilon = \omega \times (0, \varepsilon)$ of lateral extent ω (an open subset of $\mathbb{R}^2$) and thickness ϵ. The behavior of this film is obtained by minimizing the elastic energy

$$\int_{\Omega_\varepsilon} W(X, DU)\, dX,$$

over all possible displacements $U \in \mathrm{W}^{1,p}(\Omega_\varepsilon;\mathbb{R}^3)$ satisfying suitable boundary conditions. Here DU denotes the 3×3 matrix of partial derivatives of the displacement and W is the stored energy density which may depend explicitly on position X in heterogeneous films. We have used the formulation of finite elasticity, but this is not essential. We are interested in an asymptotic theory as $\varepsilon \to 0$. It is convenient to change variables $x_1 = X_1, x_2 = X_2, x_3 = \frac{1}{\varepsilon}X_3$, $u(x) = U(X(x))$ and rescale the energy by ε. We then obtain the problem of minimizing

$$\int_{\Omega} W\Big(x_1, x_2, \varepsilon x_3, D_1 u, D_2 u, \frac{1}{\varepsilon} D_3 u\Big)\, dx, \tag{1.1}$$

over all $u \in \mathrm{W}^{1,p}(\Omega;\mathbb{R}^3)$ satisfying suitable boundary conditions. Above, $\Omega = \omega \times (0,1)$ is independent of ε, $D_i u$ is the $3-$vector of partial derivatives of u with respect to x_i.

Notice that if W satisfies some growth conditions uniformly in x, this energy would blow up as $\varepsilon \to 0$ unless $D_3 u \to 0$. A slightly refined reasoning reveals that in the unscaled variables $\partial U/\partial X_3$ tends to a function independent of X_3 in a suitable sense. This motivates the classical ansatz that $\partial U/\partial X_3$ is independent of X_3. Furthermore, this allows one to minimize out the third direction and as $\varepsilon \to 0$ the functional in (1.1) tends to the two-dimensional functional

$$\int_{\omega} \widetilde{W}(x_\alpha, D_\alpha v)\, dx_\alpha.$$

Above $x_\alpha = (x_1, x_2)$ and $v \in \mathrm{W}^{1,p}(\omega;\mathbb{R}^3)$. Thus, we obtain the two-dimensional 'membrane energy' at order ε. This argument has been rigorously justified and $\widetilde{W}$ has been characterized in a variety of settings including finite elasticity, inhomogeneous materials, phase transforming materials and films with undulating surfaces [2, 9, 3, 5, 6, 10].

However, if W were suitably degenerate — i.e., if W did *not* satisfy a growth conditions uniformly in x — the classical ansatz could no longer be justified. The purpose of this paper is to present exactly such an example. Furthermore, since the ansatz does not hold, the limiting $\varepsilon-$order theory is three-dimensional, not two-dimensional. In other words, at $\varepsilon-$order we have to minimize a fully three-dimensional functional

$$\int_{\Omega} \widetilde{W}(Du)\, dx,$$

over $u \in \mathrm{W}^{1,p}(\Omega;\mathbb{R}^3)$ satisfying suitable boundary conditions, without the constraint $D_3 u = 0$!

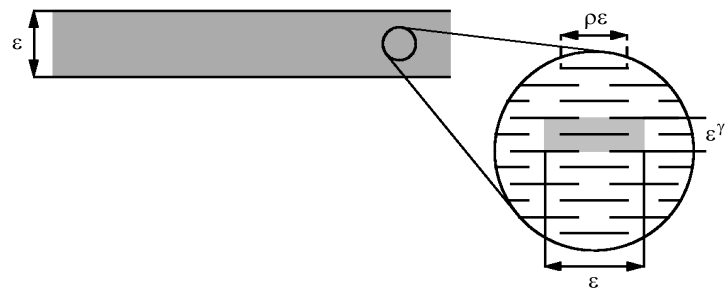

Figure 1: Thin-film with many small cracks

To understand this unusual problem, we change back to the original variables. At $\varepsilon-$order, we obtain the problem of minimizing

$$\int_{\Omega_\varepsilon} \widetilde{W}\Big(\frac{\partial U}{\partial X_1}, \frac{\partial U}{\partial X_2}, \varepsilon\frac{\partial U}{\partial X_3}\Big)\, dX,$$

over all possible displacements $U \in \mathrm{W}^{1,p}(\Omega_\varepsilon;\mathbb{R}^3)$. Notice that this is a three-dimensional problem in a thin domain where $\partial U/\partial X_3$ appears with coefficient ε. Therefore, $\partial U/\partial X_3$ can be quite large, and in particular we can *not* conclude that it is independent of X_3.

This may have interesting physical implications as well. In classical membranes since energetics forces $\partial U/\partial X_3$ to be independent of X_3 away from the boundaries, it is not very crucial how the boundary conditions are applied at the edges of the membrane; in particular it is not very crucial how the boundary conditions vary through the thickness. In constrast, it is crucially important in this example. Further, under compressive loads, such a membrane may bulge out in the thickness direction instead of buckling.

We now give a brief description of our example. Since our goal is to present an illustrative example, we will not consider general three dimensional elasticity but a much simpler situation. First we will consider only scalar-valued deformations u. Second we will consider the domain Ω to be in two dimensions, so we will study 2D to 1D asymptotics. Finally we will consider the energy to be $W(F) = |F|^2$. So one can think of this example as conduction in a strip. It will be clear from the discussion that this example can easily be generalized to the three dimensional elasticity setting.

Consider the thin strip of thickness ϵ with cracks of length $\rho\epsilon$, and spacing ϵ longitudinally and ϵ^γ transversely shown in Figure 1. Specifically, for some given $\gamma > 0$ and $1/2 < \rho < 1$, consider the strip $\Omega_\varepsilon \cap \tilde{E}_\varepsilon$ where $\Omega_\varepsilon = (0, L) \times (0, \varepsilon)$, and

$$\tilde{E}_\varepsilon = \mathbb{R}^2 \setminus \bigcup_{i,j\in\mathbb{Z}} \Bigg(\Big((\varepsilon i, \varepsilon^\gamma j) + \frac{\varepsilon}{2} e_1 + \varepsilon K \Big) \cup \Big((\varepsilon i, \varepsilon^\gamma j) + \frac{\varepsilon^\gamma}{2} e_2 + \varepsilon K \Big) \Bigg)$$

with

$$K = \Big\{ (t,0) : -\frac{\rho}{2} \le t \le \frac{\rho}{2} \Big\},$$

$e_1 = (1,0)$ and $e_2 = (0,1)$. The elastic energy of the strip is

$$\tilde{J}_\varepsilon(U) = \int_{\Omega_\varepsilon \cap \tilde{E}_\varepsilon} \Big(\Big| \frac{\partial U}{\partial X} \Big|^2 + \Big| \frac{\partial U}{\partial Y} \Big|^2 \Big) \, dX \, dY$$

where $U \in \mathrm{H}^1(\Omega_\varepsilon \cap \tilde{E}_\varepsilon; \mathbb{R})$ is scalar-valued. We are interested in studying the asymptotic behavior of the strip at $\varepsilon-$order as $\varepsilon \to 0$.

It is convenient to make the following change of variables:

$$x = X, \quad y = Y/\varepsilon, \quad u(x,y) = U(X(x), Y(y)), \quad J_\varepsilon(u) = \frac{1}{\varepsilon} \tilde{J}_\varepsilon(U) \tag{1.2}$$

and work on the fixed domain $\Omega = (0,L) \times (0,1)$. We obtain, the problem of minimizing the functional

$$J_\varepsilon(u) = \int_{\Omega \cap E_\varepsilon} \Big(\Big| \frac{\partial u}{\partial x} \Big|^2 + \frac{1}{\varepsilon^2} \Big| \frac{\partial u}{\partial y} \Big|^2 \Big) \, dx \, dy, \tag{1.3}$$

on $\mathrm{H}^1(\Omega \cap E_\varepsilon; \mathbb{R})$ where

$$E_\varepsilon = \mathbb{R}^2 \setminus \bigcup_{i,j \in \mathbb{Z}} \Bigg(\Big((\varepsilon i, \varepsilon^{\gamma-1} j) + \frac{\varepsilon}{2} e_1 + \varepsilon K \Big) \cup \Big((\varepsilon i, \varepsilon^{\gamma-1} j) + \frac{\varepsilon^{\gamma-1}}{2} e_2 + \varepsilon K \Big) \Bigg).$$

We show that the behavior of the functional J_ε at $\varepsilon-$order can be described as follows.

- If $\gamma < 2$, then the classical ansatz $\partial u / \partial y = 0$ holds and the asymptotic behavior is described by the one-dimensional functional

  $$J(u) = \int_0^L \Big| \frac{\partial u}{\partial x} \Big|^2 \, dx$$

  where $u \in \mathrm{H}^1((0,L); \mathbb{R})$.

- If $\gamma = 2$, then the classical ansatz $\partial u / \partial y = 0$ does *not* hold and the asymptotic behavior is described by the two-dimensional functional

  $$J(u) = \int_\Omega \Big(\Big| \frac{\partial u}{\partial x} \Big|^2 + \frac{1}{2\rho - 1} \Big| \frac{\partial u}{\partial y} \Big|^2 \Big) \, dx \, dy$$

  where $\Omega = (0,L) \times (0,1)$ and $u \in \mathrm{H}^1(\Omega; \mathbb{R})$.

- If $\gamma > 2$, then the classical ansatz $\partial u / \partial y = 0$ does *not* hold and the asymptotic behavior is described by the degenerate two-dimensional functional

  $$J(u) = \int_\Omega \Big(\Big| \frac{\partial u}{\partial x} \Big|^2 \Big) \, dx \, dy$$

  where $\Omega = (0,L) \times (0,1)$ and $u \in \mathrm{H}^1(\Omega; \mathbb{R})$.

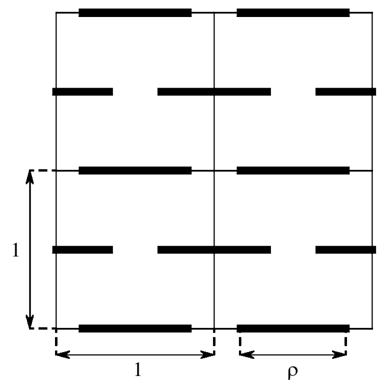

Figure 2: The set E

We now describe the heuristic idea behind these results. Note that we may regard (1.3) as the problem of finding the effective behavior of a medium made of materials with moduli 1 and $1/\varepsilon^2$, and with a periodic microstructure of cracks with period ε in the $x-$direction and $\varepsilon^{\gamma-1}$ in the $y-$direction. We are interested, in particular, to know if such a medium can sustain finite gradients of u in the $y-$direction with finite energy. We therefore look at an unit cell (of size $\varepsilon \times \varepsilon^{\gamma-1}$), and study the energy needed to change u by a quantity $\varepsilon^{\gamma-1}$ across it in the $y-$direction. It is convenient then to introduce a further change of variables:

$$x' = x/\varepsilon, \quad y' = y/\varepsilon^{\gamma-1}, \quad u'(x',y') = \frac{1}{\varepsilon^{\gamma-1}} u(x(x'), y(y')),$$

so that the microgeometry of cracks is described by the set

$$E = \mathbb{R}^2 \setminus \bigcup_{i,j\in\mathbb{Z}} \left(\Big((i,j) + \frac{1}{2}e_1 + K \Big) \cup \Big((i,j) + \frac{1}{2}e_2 + K \Big) \right).$$

shown in Figure 2. The energy per unit cell becomes

$$\int_{(0,1)^2\cap E} \left((\varepsilon^{\gamma-2})^2 \Big| \frac{\partial u'}{\partial x'} \Big|^2 + \frac{1}{\varepsilon^2} \Big| \frac{\partial u'}{\partial y'} \Big|^2 \right) dx'\, dy'.$$

We are interested in studying fields u' which take the value $-1/2$ along the lower edge of the unit cell away from the crack and value $1/2$ along the upper edge away from the crack.

If $\gamma < 2$, then both moduli increase as $\varepsilon \to 0$, and it is easy to see that any field of interest will necessarily have infinite energy in the limit. Thus we conclude that we can not sustain gradients in the $y-$direction with finite energy and hence infer the classical constraint $\partial u/\partial y = 0$ in this case.

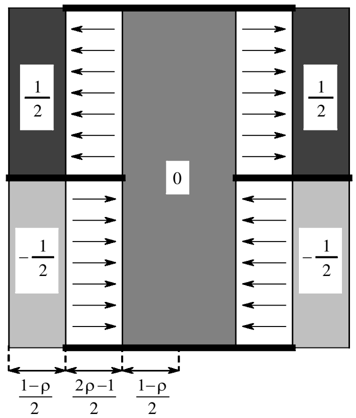

Figure 3: A very useful test field and the minimizer for $f(0,1)$.

If $\gamma = 2$, then one modulus is finite while the other increases as $\varepsilon \to 0$. Thus we would like to increase u' in the $y'-$direction across the unit cell, but while keeping $\partial u'/\partial y' = 0$ everywhere. Figure 3 shows a test field that takes advantages of the cracks to do so: it is constant at the values marked in the shaded regions and increases linearly along the arrows. Thus, $\partial u'/\partial y' = 0$ everywhere and $|\partial u'/\partial x'| = 1/(2\rho - 1)$ in the strips marked by arrows. The energy per unit cell is given by $1/(2\rho - 1)$. We conclude that we can sustain macroscopic gradients in the $y'-$direction with finite energy, and this leads to the desired result.

Finally if $\gamma > 2$, then one modulus decreases while the other increases as $\varepsilon \to 0$. The test field in Figure 3 takes advantage of the decreasing modulus to increase u' across the unit cell in the $y'-$direction. The energy per unit cell for this test function goes to zero. Thus, we conclude that this case gives a degenerate energy in the limit.

We can note further that we require $1/2 < \rho < 1$ for the test-field in Figure 3 to be useful. For these values of ρ, $\frac{2\rho-1}{2} > 0$, and the projection of the cracks on the line $y' = 0$ overlap. Therefore we find strips that traverse the unit cell in the $y'-$direction and are broken up by cracks at every half-interval as in Figure 3 (the strips marked by arrows). The test field uses these strips to change the value of u' while keeping $\partial u'/\partial y' = 0$ everywhere by using the cracks to sustain discontinuities in u'. If ρ was outside this range, in particular if $0 < \rho < 1/2$, then we would not have these strips to change the field as desired. Instead we would have strips which traverse the unit cell in the $y'-$ direction completely unbroken by any cracks. Then the fact that $\partial u'/\partial y' = 0$ on these strips would immediately imply that there can be no macroscopic change in u' in the $y'-$direction. Therefore we conclude that a strip with $0 < \rho < 1/2$ would behave like a classical membrane even if $\gamma \geq 2$.

The rest of the paper makes rigorous the heuristic argument above for the critical case $\gamma = 2$. We shall use the framework of $\Gamma-$convergence, a notion of convergence for functionals introduced by Di Giorgi [8] (see also for example [4, 7]). This framework has been used extensively in the last few years to rigorously derive asymptotic theories for various thin films (see for example [2, 9, 3, 5, 6, 10]). The primary advantage of this method is that, unlike classical methods, it is not based on any a priori ansatz; instead, the argument itself suggests the right asymptotic form. This is a significant advantage for the problem at hand: the displacement u in this problem is very complicated and it would be very difficult to start with a reasonable guess.

$\Gamma-$convergence is a notion of convergence for functionals. Roughly, we say that the sequence of functionals $J_\varepsilon(u)$ $\Gamma-$converges to the functional $J(u)$ if the minimizers of J_ε converge to (or are approximated by) the minimizers of J. Note that the convergence is through the the minimizers, not the functionals directly. Therefore we can think of the limiting functional J as the 'effective theory' whose solution captures the relevant features of those of the $\varepsilon-$theory. Thus $\Gamma-$convergence is a natural framework for obtaining effective asymptotic theories. It turns out for technical reasons that the above definition is not quite applicable to every problem, and it is more convenient (and mathematically natural) to define separately upper and lower $\Gamma-$limits, and then say that we have convergence when these upper and lower limits coincide. One can also think of these upper and lower limits as bounds, and therefore these are also useful from the point of applications: one does not have to work with exact solutions to complicated problems, but only with test functions.

We begin in Section 2 by giving a precise definition of $\Gamma-$convergence, and recalling important results about it and homogenization. Section 3 deals with the critical case $\gamma = 2$, and the result is given in Theorem 3.1. Section 4 gives other examples.

2 Notation and Preliminaries

In what follows, $\mathbb{M}^{m\times n}$ stands for the space of $m \times n$ matrices. We use standard notation for Lebesgue and Sobolev spaces; in particular, we denote $H^1 = W^{1,2}$. The scalar product is denoted by $\langle \cdot, \cdot \rangle$, and, for fixed $z \in \mathbb{R}^n$, $\langle z, x \rangle$ denotes the linear function given by this product as well. The letter c denotes a strictly positive constant whose value may vary from line to line and does not depend on the parameters of the problem.

2.1 Γ-convergence

We recall the definition of De Giorgi's Γ-convergence in L^p spaces, $1 \leq p < +\infty$ (see [8], [7], [4] or [5]). Given a family of functionals $J_\varepsilon : \mathrm{L}^p(\Omega;\mathbb{R}^m) \to [0, +\infty]$,

$\varepsilon > 0$, for $u \in \mathrm{L}^p(\Omega; \mathbb{R}^m)$ we define

$$\Gamma\text{-}\liminf_{\varepsilon\to 0+} J_\varepsilon(u) := \inf\big\{\liminf_{\varepsilon\to 0+} J_\varepsilon(u_\varepsilon) \ : \ u_\varepsilon \to u \text{ in } \mathrm{L}^p(\Omega; \mathbb{R}^m)\big\},$$

and

$$\Gamma\text{-}\limsup_{\varepsilon\to 0+} J_\varepsilon(u) := \inf\big\{\limsup_{\varepsilon\to 0+} J_\varepsilon(u_\varepsilon) \ : \ u_\varepsilon \to u \text{ in } \mathrm{L}^p(\Omega; \mathbb{R}^m)\big\}.$$

If these two quantities coincide then their common value is called the Γ-*limit* of the sequence (J_ε) at u, and is denoted by $\Gamma\text{-}\lim_{\varepsilon\to 0+} J_\varepsilon(u)$. We recall that the Γ-upper and lower limits defined above are L^p-lower semicontinuous functions.

The importance of Γ-convergence in the calculus of variations lies in the following theorem about the convergence of minima, together with the stability properties of Γ-convergence by continuous perturbations. Its applications to thin films theory is described for example in [9].

Theorem 2.1 *Let $J = \Gamma\text{-}\lim_{\varepsilon\to 0} J_\varepsilon$, and suppose that there exists a family (u_ε) of J_ε which lies in a compact subset of $\mathrm{L}^p(\Omega; \mathbb{R}^m)$ such that $J_\varepsilon(u_\varepsilon) = \inf J_\varepsilon + o(1)$ as $\varepsilon \to 0$. Then the limit functional admits a minimum and*

$$\lim_{\varepsilon\to 0} \inf J_\varepsilon = \min J.$$

Moreover, if (ε_j) is a family of positive numbers converging to 0 such that (u_{ε_j}) converges to u in $\mathrm{L}^p(\Omega; \mathbb{R}^m)$ then u is a minimizer of J.

2.2 Homogenization

The following theorem can be found in [5] Theorem 14.8 and Example 14.11.

Theorem 2.2 *Let E be a 1-periodic set containing a unbounded connected component, let $W : \mathbb{M}^{m\times n} \to [0, +\infty)$ be a convex function satisfying a growth condition of the form*

$$c(-1 + |F|^p) \le W(F) \le C(1 + |F|^p).$$

If Ω is a bounded open set of $\mathbb{R}^n$ and we set for all $\varepsilon > 0$

$$J_\varepsilon(u) = \begin{cases} \displaystyle\int_{\Omega\cap\varepsilon E} W(Du(x))\,dx & \text{if } u \in \mathrm{W}^{1,p}(\Omega \cap \varepsilon E; \mathbb{R}^m) \\ +\infty & \text{otherwise in } \mathrm{L}^p(\Omega; \mathbb{R}^m), \end{cases} \tag{2.1}$$

then we have

$$\Gamma\text{-}\lim_{\varepsilon\to 0} J_\varepsilon(u) = \begin{cases} \displaystyle\int_{\Omega} W_\#(Du(x))\,dx & \text{if } u \in \mathrm{W}^{1,p}(\Omega; \mathbb{R}^m) \\ +\infty & \text{otherwise in } \mathrm{L}^p(\Omega; \mathbb{R}^m), \end{cases}$$

where $W_\#:\mathbb{M}^{m\times n}\to[0,+\infty)$ *is a convex function satisfying the homogenization formula*

$$W_\#(F)=\inf\Big\{\int_{(0,1)^n\cap E}W(F+Du(x))\,dx:\ u\in \mathrm{W}^{1,p}_\#((0,1)^n\cap E;\mathbb{R}^m)\Big\}\qquad(2.2)$$

for all $F\in\mathbb{M}^{m\times n}$*, where* $\mathrm{W}^{1,p}_\#((0,1)^n\cap E;\mathbb{R}^m)$ *is the set of* 1*-periodic functions belonging to* $\mathrm{W}^{1,p}_{\mathrm{loc}}(\mathbb{R}^n\cap E;\mathbb{R}^m)$*.*

3 The example

We limit our analysis to the critical case $\gamma=2$. Recall the strip $\Omega_\varepsilon\cap\tilde{E}_\varepsilon$ and the energy $\tilde{J}_\varepsilon(U)$ described earlier. After the change of variable (1.2) we obtain,

$$J_\varepsilon(u)=\int_{\Omega\cap\varepsilon E}\Big(\Big|\frac{\partial u}{\partial x}\Big|^2+\frac{1}{\varepsilon^2}\Big|\frac{\partial u}{\partial y}\Big|^2\Big)\,dx\,dy,$$

on $\mathrm{H}^1(\Omega\cap\varepsilon E;\mathbb{R})$ where

$$E=\mathbb{R}^2\setminus\bigcup_{i\in\mathbb{Z}^2}\Bigg(\Big(i+\frac{1}{2}e_1+K\Big)\cup\Big(i+\frac{1}{2}e_2+K\Big)\Bigg)$$

as before and shown in Figure 2. We extend $J_\varepsilon(u)$ to $L^2(\Omega;\mathbb{R})$ by setting $J_\varepsilon=+\infty$ on $\mathrm{L}^2(\Omega;\mathbb{R})\setminus\mathrm{H}^1(\Omega\cap\varepsilon E;\mathbb{R})$. We will prove the following result.

Theorem 3.1 *The functionals* J_ε *defined above* Γ*-converges with respect to the* $L^2(\Omega)$*-convergence to the functional* J *defined on* $\mathrm{L}^2(\Omega;\mathbb{R})$ *by*

$$J(u)=\begin{cases}\displaystyle\int_\Omega\Big(\Big|\frac{\partial u}{\partial x}\Big|^2+\frac{1}{2\rho-1}\Big|\frac{\partial u}{\partial y}\Big|^2\Big)\,dx\,dy & \text{if } u\in\mathrm{H}^1(\Omega;\mathbb{R})\\ +\infty & \text{otherwise.}\end{cases}$$

3.1 Lower bound

Proposition 3.2 *Let* $z\in\mathbb{R}^2$ *and let*

$$f(z)=\min\Big\{\int_{(0,1)^2\cap E}\Big|\frac{\partial u}{\partial x}\Big|^2\,dx\,dy:u\in\mathcal{A}(z),\ \frac{\partial u}{\partial y}=0\ a.e.\Big\}.\qquad(3.1)$$

where

$$\mathcal{A}(z)=\{u:u-\langle z,(x,y)\rangle\in\mathrm{H}^1_\#((0,1)^2\cap E;\mathbb{R})\}.$$

Then we have

$$f(z)=z_1^2+\frac{1}{2\rho-1}z_2^2.$$

PROOF. Clearly f is a quadratic form, and by the symmetry of E we easily infer that $f(z_1, z_2) = \alpha z_1^2 + \beta z_2^2$.

To compute $\alpha = f(1,0)$ we remark that for all $u \in \mathcal{A}(z)$ we have

$$\int_{(0,1)^2 \cap E} \left|\frac{\partial u}{\partial x}\right|^2 dx\, dy \geq \int_0^1 \left|\int_0^1 \frac{\partial u}{\partial x}\, dx\right|^2 dy = 1,$$

so that $f(1,0) \geq 1$. Since this value is achieved trivially on $u(x,y) = x$ we obtain $\alpha = 1$.

As for $\beta = f(0,1)$, note that $f(0,1) = \min_X \mathcal{J}$, where

$$\mathcal{J}(u) = \int_{(-1/2,1,2)^2 \cap E} \left|\frac{\partial u}{\partial x}\right|^2 dx\, dy,$$

and

$$\begin{aligned} X \quad = \quad & \Big\{ u : u(x,y) - y \in \mathrm{H}^1_{\#}((-1/2,1/2)^2 \cap E; \mathbb{R}), \\ & u(\pm 1/2, -1/2) = -1/2, u(\pm 1/2, 1/2) = 1/2, \frac{\partial u}{\partial y} = 0 \text{ a.e.} \Big\}. \end{aligned}$$

As $\mathcal{J}$ is lower semicontinuous and coercive on X there exists a minimum point u, which is unique up to constants since $\mathcal{J}$ is strictly convex in $\frac{\partial u}{\partial x}$. However, from the energy it is clear that if $u(x,y)$ is a minimizer, then $u(-x,y)$ and $-u(x,y)$ are also minimizers. Therefore we conclude that the unique minimizer satisfies

$$u(x,y) = u(-x,y) = -u(x,-y).$$

From these conditions, the periodicity assumption on u, and the condition $\frac{\partial u}{\partial y} = 0$ we deduce that

$$u(x,y) = 0 \text{ if } (x,y) \in \Big(-\frac{1-\rho}{2}, \frac{1-\rho}{2}\Big) \times \Big(-\frac{1}{2}, \frac{1}{2}\Big),$$

$$u(x,y) = -\frac{1}{2} \text{ if } (x,y) \in \Big(\Big(-\frac{1}{2}, -\frac{\rho}{2}\Big) \cup \Big(\frac{\rho}{2}, \frac{1}{2}\Big)\Big) \times \Big(-\frac{1}{2}, 0\Big),$$

and

$$u(x,y) = \frac{1}{2} \text{ if } (x,y) \in \Big(\Big(-\frac{1}{2}, -\frac{\rho}{2}\Big) \cup \Big(\frac{\rho}{2}, \frac{1}{2}\Big)\Big) \times \Big(0, \frac{1}{2}\Big)$$

as shown in Figure 3. As u minimizes $\mathcal{J}$ it must be affine in the remaining regions, and by the boundary values therein we obtain that

$$\frac{\partial u}{\partial x} = \pm \frac{1}{2\rho - 1},$$

and

$$f(0,1) = \frac{1}{2\rho - 1},$$

which gives β. $\square$

Proposition 3.3 *For all $u \in \mathrm{L}^2(\Omega;\mathbb{R})$ we have* $\Gamma\text{-}\liminf_{\varepsilon\to 0} J_\varepsilon(u) \geq J(u)$.

Proof. Let $\lambda > 0$ be fixed. For $\varepsilon \leq \lambda$ we have

$$J_\varepsilon(u) \geq J_\varepsilon^\lambda(u),$$

where

$$J_\varepsilon^\lambda(u) = \begin{cases} \displaystyle\int_{\Omega\cap\varepsilon E} \Big(\Big|\frac{\partial u}{\partial x}\Big|^2 + \frac{1}{\lambda^2}\Big|\frac{\partial u}{\partial y}\Big|^2\Big)\,dx\,dy & \text{if } u \in \mathrm{H}^1(\Omega\cap\varepsilon E;\mathbb{R}) \\ +\infty & \text{otherwise in } \mathrm{L}^2(\Omega;\mathbb{R}). \end{cases}$$

Hence,

$$\Gamma\text{-}\liminf_{\varepsilon\to 0} J_\varepsilon(u) \geq \Gamma\text{-}\lim_{\varepsilon\to 0} J_\varepsilon^\lambda(u).$$

However, by Theorem 2.2 we have

$$\Gamma\text{-}\lim_{\varepsilon\to 0} J_\varepsilon^\lambda(u) = J_\#^\lambda(u),$$

where

$$J_\#^\lambda(u) = \begin{cases} \displaystyle\int_\Omega f^\lambda(Du)\,dx\,dy & \text{if } u \in \mathrm{H}^1(\Omega;\mathbb{R}) \\ +\infty & \text{otherwise in } \mathrm{L}^2(\Omega;\mathbb{R}), \end{cases}$$

and $f^\lambda : \mathbb{R}^2 \to [0,+\infty)$ is the convex function satisfying the homogenization formula

$$f^\lambda(z) = \inf\Big\{\int_{(0,1)^2\cap E} \Big(\Big|\frac{\partial u}{\partial x}\Big|^2 + \frac{1}{\lambda^2}\Big|\frac{\partial u}{\partial y}\Big|^2\Big)\,dx\,dy :\ u \in \mathcal{A}(z)\Big\} \tag{3.2}$$

where $\mathcal{A}$ is defined below (3.1) for all $z \in \mathbb{R}^2$. Thus,

$$\Gamma\text{-}\liminf_{\varepsilon\to 0} J_\varepsilon(u) \geq \sup_\lambda J_\#^\lambda(u). \tag{3.3}$$

Let $z \in \mathbb{R}^2$ with $u \in \mathcal{A}(z)$. By the Monotone Convergence Theorem we have

$$\sup_\lambda \int_{(0,1)^2\cap E} \Big(\Big|\frac{\partial u}{\partial x}\Big|^2 + \frac{1}{\lambda^2}\Big|\frac{\partial u}{\partial y}\Big|^2\Big)\,dx\,dy = \int_{(0,1)^2\cap E} \Big|\frac{\partial u}{\partial x}\Big|^2\,dx\,dy$$

if $\frac{\partial u}{\partial y} = 0$, and $+\infty$ otherwise. It follows that

$$\begin{aligned}
\sup_\lambda f^\lambda(z) &= \sup_\lambda \inf_{u\in\mathcal{A}(z)} \int_{(0,1)^2\cap E} \Big(\Big|\frac{\partial u}{\partial x}\Big|^2 + \frac{1}{\lambda^2}\Big|\frac{\partial u}{\partial y}\Big|^2\Big)\,dx\,dy \\
&\leq \inf_{u\in\mathcal{A}(z)} \sup_\lambda \int_{(0,1)^2\cap E} \Big(\Big|\frac{\partial u}{\partial x}\Big|^2 + \frac{1}{\lambda^2}\Big|\frac{\partial u}{\partial y}\Big|^2\Big)\,dx\,dy \\
&\leq \inf_{u\in\mathcal{A}(z),\,\frac{\partial u}{\partial y}=0} \sup_\lambda \int_{(0,1)^2\cap E} \Big(\Big|\frac{\partial u}{\partial x}\Big|^2 + \frac{1}{\lambda^2}\Big|\frac{\partial u}{\partial y}\Big|^2\Big)\,dx\,dy \\
&= \inf_{u\in\mathcal{A}(z),\,\frac{\partial u}{\partial y}=0} \int_{(0,1)^2\cap E} \Big|\frac{\partial u}{\partial x}\Big|^2\,dx\,dy \\
&= f(z).
\end{aligned}$$

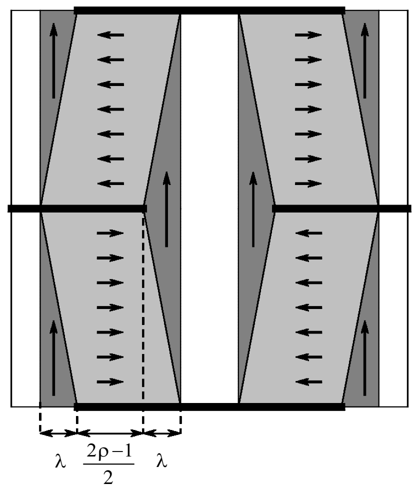

Figure 4: The test-field t for $\beta = f^\lambda(0,1)$.

However, for any $u \in \mathcal{A}(z)$,

$$\int_{(0,1)^2\cap E}\Big(\Big|\frac{\partial u}{\partial x}\Big|^2 + \frac{1}{\lambda^2}\Big|\frac{\partial u}{\partial y}\Big|^2\Big)\,dx\,dy \geq \int_{(0,1)^2\cap E}\Big|\frac{\partial u}{\partial x}\Big|^2\,dx\,dy,$$

from which it follows that $f^\lambda(z) \geq f(z)$. Thus,

$$\sup_\lambda f^\lambda(z) = f(z).$$

Therefore according to the Monotone Convergence Theorem, we have

$$\sup_\lambda \int_\Omega f^\lambda(Du)\,dx\,dy = \int_\Omega f(Du)\,dx\,dy$$

for any $u \in \mathrm{H}^1(\Omega;\mathbb{R})$. Thus, recalling the definitions of $J_\#^\lambda$ and J, the formula for f in (3.1) and the inequality (3.3), we obtain

$$\Gamma\text{-}\liminf_{\varepsilon\to 0} J_\varepsilon(u) \geq \sup_\lambda J_\#^\lambda(u) = J(u)$$

as desired. $\square$

We give an alternate proof which uses an estimate for f^λ instead of f.
Alternate Proof of Proposition 3.3 Recall f^λ defined in (3.2). Clearly f^λ is a quadratic form, and by the symmetry of E we easily infer that $f^\lambda(z_1, z_2) = \alpha z_1^2 + \beta z_2^2$. It is easily verified using Jensen's inequality that $\alpha = f^\lambda(1,0) = 1$. We

obtain a lower bound for $\beta = f^\lambda(0,1)$ using a dual variational formulation. Recall that for any $W : \mathbb{R}^2 \to \mathbb{R}$, we define the Fenchel transform as

$$W^*(t) = \sup_{f\in\mathbb{R}^2} \{\langle t, f\rangle - W(f)\}.$$

It follows that for any $f \in \mathbb{R}^2$, $W(f) \geq \sup_{t\in\mathbb{R}^2}\{\langle t, f\rangle - W^*(t)\}$, and consequently for any $z \in \mathbb{R}^2$

$$\inf_{u\in\mathcal{A}(z)} \int_{(0,1)^2\cap E} W(Du)\,dx\,dy \geq \sup_{t\in\mathcal{B}} \int_{(0,1)^2\cap E} \big(\langle z, t\rangle - W^*(t)\big)\,dx\,dy$$

where

$$\mathcal{B} = \Big\{t \in L^2((0,1)^2;\mathbb{R}^2) : \int_{(0,1)^2\cap E} \langle \nabla\phi, t\rangle\,dx\,dy = 0 \;\forall\, \phi \in H^1_\#((0,1)^2\cap E;\mathbb{R}^2)\Big\}$$

is the space of all divergence-free 1-periodic L^2 functions with zero normal components on ∂E. Note that the last condition is meaningful since normal components of the trace are well-defined for divergence-free L^2 functions. Applying this inequality to the f^λ, we conclude that

$$f^\lambda(z) \geq \sup_{t\in\mathcal{B}} \int_{(0,1)^n\cap E} \Big(\langle z, t\rangle - \frac{1}{4}(t_1^2 + \lambda^2 t_2^2)\Big)\,dx\,dy$$

where the second term in the integrand is the Fenchel transform of the integrand in the definition of f^λ.

We construct a test function shown in Figure 4, first on $(0,1/2)^2$,

$$t = \begin{cases} \dfrac{1}{\lambda(2\rho-1)}(0,1) & \Big\{\frac{1-\rho}{2} - \lambda < x_1 < \frac{1-\rho}{2} - 2\lambda x_2\Big\} \\[2ex] \dfrac{1}{\lambda(2\rho-1)}(0,1) & \Big\{\frac{\rho}{2} + \lambda(1-2x_2) < x_1 < \frac{\rho}{2} + \lambda\Big\} \\[2ex] \dfrac{2}{2\rho-1}(1,0) & \Big\{\frac{1-\rho}{2} - 2\lambda x_2 < x_1 < \frac{\rho}{2} + \lambda(1-2x_2)\Big\} \\[2ex] 0 & \text{otherwise} \end{cases},$$

then extend it to $(0,1)^2$ using

$$t_1(x_1,x_2) = -t_1(x_1,1-x_2) = -t_1(1-x_1,x_2),$$
$$t_2(x_1,x_2) = t_2(x_1,1-x_2) = t_2(1-x_1,x_2),$$

and finally periodically to $\mathbb{R}^2$. It is easily verified that $t \in \mathcal{B}$. Using this test function, we conclude,

$$\begin{aligned} f^\lambda(0,1) &\geq \tfrac{4\lambda}{2}\Big(\tfrac{1}{\lambda(2\rho-1)} - \tfrac{\lambda^2}{4}\Big(\tfrac{1}{\lambda(2\rho-1)}\Big)^2\Big) + 4\Big(\tfrac{2\rho-1+2\lambda}{4}\Big)\Big(-\tfrac{1}{4}\Big(\tfrac{2}{2\rho-1}\Big)^2\Big) \\ &\geq \tfrac{1}{2\rho-1} - \lambda\Big(\tfrac{5}{2}\tfrac{1}{(2\rho-1)^2}\Big). \end{aligned}$$

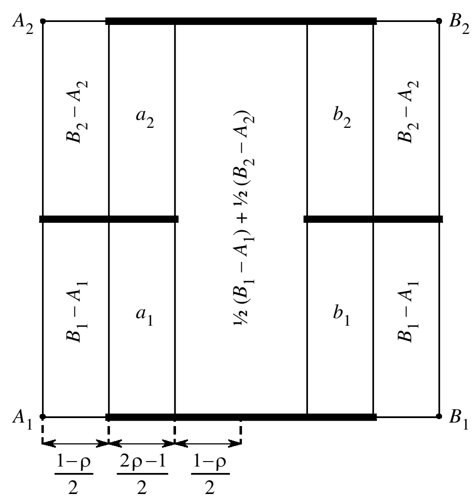

Figure 5: The piecewise affine test-field v takes the values A_1, A_2, B_1, B_2 at the corners while $\partial v/\partial x$ takes the values indicated in the different region.

Thus, for any $u \in \mathrm{H}^1(\Omega;\mathbb{R})$, we conclude that

$$J^{\lambda}_{\#}(u) = \int_{\Omega} f^{\lambda}(Du)\,dx\,dy \geq \int_{\Omega} \Big\{ \Big|\frac{\partial u}{\partial x}\Big|^2 + \Big(\frac{1}{2\rho - 1} - \frac{5\lambda}{2}\frac{1}{(2\rho-1)^2} \Big) \Big|\frac{\partial u}{\partial y}\Big|^2 \Big\}\,dx\,dy.$$

It follows again from the Monotone Convergence Theorem that

$$\Gamma\text{-}\liminf_{\varepsilon\to 0} J_{\varepsilon}(u) \geq \Gamma\text{-}\lim_{\varepsilon\to 0} J^{\lambda}_{\varepsilon}(u) = \sup_{\lambda} J^{\lambda}_{\#}(u) = J(u).$$

□

3.2 Upper bound

In order to estimate the upper bound we have to construct suitable test functions. Let $A_1, A_2, B_1, B_2 \in \mathbb{R}$. We define the function $v = v^{A_2,B_2}_{A_1,B_1}$ in $\mathrm{H}^1((-1/2,1/2)^2 \setminus E)$ as that satisfying (see Figure 5):

$$v\Big(-\frac{1}{2}, -\frac{1}{2}\Big) = A_1, \quad v\Big(\frac{1}{2}, -\frac{1}{2}\Big) = B_1, \quad v\Big(-\frac{1}{2}, \frac{1}{2}\Big) = A_2, \quad v\Big(\frac{1}{2}, \frac{1}{2}\Big) = B_2, \tag{3.4}$$

$$\frac{\partial v}{\partial y} = 0 \text{ a.e. on } \Big(-\frac{1}{2}, \frac{1}{2}\Big)^2 \tag{3.5}$$

$$\frac{\partial v}{\partial x} = \begin{cases} B_1 - A_1 & \text{on } \left(-\frac{1}{2}, -\frac{\rho}{2}\right) \times \left(-\frac{1}{2}, 0\right), \\ a_1 & \text{on } \left(-\frac{\rho}{2}, \frac{\rho-1}{2}\right) \times \left(-\frac{1}{2}, 0\right), \\ b_1 & \text{on } \left(\frac{1-\rho}{2}, \frac{\rho}{2}\right) \times \left(-\frac{1}{2}, 0\right), \\ B_1 - A_1 & \text{on } \left(\frac{\rho}{2}, \frac{1}{2}\right) \times \left(-\frac{1}{2}, 0\right), \\ \frac{1}{2}((B_2 - A_2) + (B_1 - A_1)) & \text{on } \left(-\frac{1-\rho}{2}, \frac{1-\rho}{2}\right) \times \left(-\frac{1}{2}, \frac{1}{2}\right) \\ B_2 - A_2 & \text{on } \left(-\frac{1}{2}, -\frac{\rho}{2}\right) \times \left(0, \frac{1}{2}\right), \\ a_2 & \text{on } \left(-\frac{\rho}{2}, \frac{\rho-1}{2}\right) \times \left(0, \frac{1}{2}\right), \\ b_2 & \text{on } \left(\frac{1-\rho}{2}, \frac{\rho}{2}\right) \times \left(0, \frac{1}{2}\right), \\ B_2 - A_2 & \text{on } \left(\frac{\rho}{2}, \frac{1}{2}\right) \times \left(0, \frac{1}{2}\right), \end{cases} \tag{3.6}$$

where

$$a_1 = \tfrac{1}{2}((B_2 - A_2) + (B_1 - A_1)) + \tfrac{1}{2\rho-1}\Big(\tfrac{1-\rho}{2}(B_2 - B_1) + \tfrac{1+\rho}{2}(A_2 - A_1)\Big)$$
$$b_1 = \tfrac{1}{2}((B_2 - A_2) + (B_1 - A_1)) - \tfrac{1}{2\rho-1}\Big(\tfrac{1+\rho}{2}(B_2 - B_1) + \tfrac{1-\rho}{2}(A_2 - A_1)\Big)$$
$$a_2 = \tfrac{1}{2}((B_2 - A_2) + (B_1 - A_1)) - \tfrac{1}{2\rho-1}\Big(\tfrac{1-\rho}{2}(B_2 - B_1) + \tfrac{1+\rho}{2}(A_2 - A_1)\Big)$$
$$b_2 = \tfrac{1}{2}((B_2 - A_2) + (B_1 - A_1)) + \tfrac{1}{2\rho-1}\Big(\tfrac{1+\rho}{2}(B_2 - B_1) + \tfrac{1-\rho}{2}(A_2 - A_1)\Big)$$

Proposition 3.4 *The function v defined in (3.4–3.6) satisfies:*

(a) *the traces of v on $\partial(-1/2, 1/2)^2$ are:*

$$v(-1/2, y) = A_1, \quad v(1/2, y) = B_1 \qquad \text{if } y \in (-1/2, 0),$$
$$v(-1/2, y) = A_2, \quad v(1/2, y) = B_2 \qquad \text{if } y \in (0, 1/2),$$
$$v(x, -1/2) = A_1 + (B_1 - A_1)x \qquad \text{if } \frac{\rho}{2} < |x| < \frac{1}{2},$$
$$v(x, 1/2) = A_2 + (B_2 - A_2)x \qquad \text{if } \frac{\rho}{2} < |x| < \frac{1}{2},$$

(b) *we have $\dfrac{\partial v}{\partial y} = 0$ a.e., and*

$$\begin{aligned} \int_{(-1/2,1/2)^2 \cap E} \Big|\frac{\partial v}{\partial x}\Big|^2 dx\,dy \quad &= \quad \Big(\frac{(B_2 - A_2) + (B_1 - A_1)}{2}\Big)^2 \\ &\quad + \frac{1}{2\rho - 1}\Big(\frac{(B_2 - B_1) + (A_2 - A_1)}{2}\Big)^2 \\ &\quad + C(\rho)\Big((B_2 - A_2) - (B_1 - A_1)\Big)^2, \end{aligned}$$

where $C(\rho)$ is a constant depending only on ρ.

PROOF. The proposition can be easily checked by direct computation with

$$C(\rho) = \Big(\frac{-\rho^2 + 3\rho - 1}{4(2\rho - 1)}\Big)$$

in (b). □

Proposition 3.5 *Let $u \in C^2(\overline{\Omega})$. Then there exist functions $u_\varepsilon \in \mathrm{H}^1(\Omega \cap \varepsilon E)$ such that $\lim_{\varepsilon\to 0+} J_\varepsilon(u_\varepsilon) \le J(u)$.*

PROOF. We still denote by u a fixed extension of u in $C^2(\mathbb{R}^2)$.
With fixed $\varepsilon > 0$, for all $i \in \mathbb{Z}^2$ let

$$A_1(i) = u(\varepsilon(i + (-1/2, -1/2))), \qquad B_1(i) = u(\varepsilon(i + (1/2, -1/2))),$$

$$A_2(i) = u(\varepsilon(i + (-1/2, 1/2))), \qquad B_2(i) = u(\varepsilon(i + (1/2, 1/2))),$$

and let $v_i = v$ be defined as in (3.4–3.6) with $A_1 = A_1(i)$, $B_1 = B_1(i)$, $A_2 = A_2(i)$, $B_2 = B_2(i)$. We define u_ε piecewise by setting

$$u_\varepsilon(x) = v_i\Big(\frac{x - i}{\varepsilon}\Big) \qquad \text{for } x \in \varepsilon\Big(i + \Big(-\frac{1}{2}, \frac{1}{2}\Big)^2\Big).$$

By Proposition 3.4(a) the function u_ε thus defined belongs to $\mathrm{H}^1_{\mathrm{loc}}(\mathbb{R}^2 \setminus \varepsilon E)$ since the trace values of u_ε on the common boundaries of neighboring squares coincide. Moreover, if we set

$$I_\varepsilon = \{i \in \mathbb{Z}^2 : \ \Omega \cap \varepsilon(i + (-1/2, 1/2)^2) \neq \emptyset\},$$

by Proposition 3.4(b) we have

$$\begin{aligned} J_\varepsilon(u_\varepsilon) \ \le \ & \sum_{i\in I_\varepsilon} \varepsilon^2 \Bigg(\Big(\frac{(u(\varepsilon(i + (1/2, 1/2))) - u(\varepsilon(i + (-1/2, 1/2))))}{2\varepsilon} \\ & \quad + \frac{(u(\varepsilon(i + (1/2, -1/2))) - u(\varepsilon(i + (-1/2, -1/2))))}{2\varepsilon}\Big)^2 \\ & + \frac{1}{2\rho - 1}\Big(\frac{(u(\varepsilon(i + (1/2, 1/2))) - u(\varepsilon(i + (1/2, -1/2))))}{2\varepsilon} \\ & \quad + \frac{(u(\varepsilon(i + (-1/2, 1/2))) - u(\varepsilon(i + (-1/2, -1/2))))}{2\varepsilon}\Big)^2 \\ & + C(\rho)\Big(\frac{(u(\varepsilon(i + (1/2, 1/2))) - u(\varepsilon(i + (-1/2, 1/2))))}{\varepsilon} \\ & \quad - \frac{(u(\varepsilon(i + (1/2, -1/2))) - u(\varepsilon(i + (-1/2, -1/2))))}{\varepsilon}\Big)^2\Bigg) \\ \le \ & \sum_{i\in I_\varepsilon} \varepsilon^2 \Big(\Big|\frac{\partial u}{\partial x}(\varepsilon i)\Big|^2 + \frac{1}{2\rho - 1}\Big|\frac{\partial u}{\partial y}(\varepsilon i)\Big|^2 + c\varepsilon^2\Big), \end{aligned}$$

where we have used the Mean Value Theorem and the fact that $u \in C^2(\overline{\Omega})$ to get

$$\Big|\frac{u(\varepsilon(i+(1/2,\pm 1/2))) - u(\varepsilon(i+(-1/2,\pm 1/2)))}{\varepsilon}\Big|^2 \le \Big|\frac{\partial u}{\partial x}(\varepsilon i)\Big|^2 + c\varepsilon^2,$$

$$\Big|\frac{u(\varepsilon(i+(\pm 1/2,1/2))) - u(\varepsilon(i+(\pm 1/2,-1/2)))}{\varepsilon}\Big|^2 \le \Big|\frac{\partial u}{\partial y}(\varepsilon i)\Big|^2 + c\varepsilon^2,$$

and

$$\Big|\frac{(u(\varepsilon(i+(1/2,1/2))) - u(\varepsilon(i+(-1/2,1/2))))}{\varepsilon} - \frac{(u(\varepsilon(i+(1/2,-1/2))) - u(\varepsilon(i+(-1/2,-1/2))))}{\varepsilon}\Big|^2 \le c\varepsilon^2.$$

Using the fact that $u \in C^2(\overline{\Omega})$ again, we obtain

$$\sum_{i\in I_\varepsilon} \varepsilon^2\Big(\Big|\frac{\partial u}{\partial x}(\varepsilon i)\Big|^2 + \frac{1}{2\rho-1}\Big|\frac{\partial u}{\partial y}(\varepsilon i)\Big|^2\Big) \le \int_\Omega \Big(\Big|\frac{\partial u}{\partial x}\Big|^2 + \frac{1}{2\rho-1}\Big|\frac{\partial u}{\partial y}\Big|^2\Big)\,dx\,dy + c\varepsilon^2,$$

and the proposition follows taking the limit as $\varepsilon \to 0+$. □

The following proposition concludes the proof of Theorem 3.1.

Proposition 3.6 *We have* $\Gamma\text{-}\limsup_{\varepsilon\to 0+} J_\varepsilon(u) \le J(u)$ *for all* $u \in \mathrm{L}^2(\Omega;\mathbb{R})$.

Proof. Let $J''(u) = \Gamma\text{-}\limsup_{\varepsilon\to 0+} J_\varepsilon(u)$. With fixed u choose $u_j \in C^2(\overline{\Omega})$ such that $u_j \to u$ in $H^1(\Omega;\mathbb{R})$, and $\lim_j J(u_j) = J(u)$. Proposition 3.5 shows that $J''(u_j) \le J(u_j)$ for all j. Hence, by the lower semicontinuity of J'' we have

$$J''(u) \le \liminf_j J''(u_j) \le \liminf_j J(u_j) = J(u),$$

as required. □

4 Other geometries

The procedure outlined above can be repeated for other geometries of crack arrays. We include two remarks on arrays obtained by a slight variation of the example described above. We again confine the analysis to the critical case $\gamma = 2$.

Remark 4.1 If the set E is defined as in Section 3, but with $0 \le \rho \le 1/2$ then the result of Theorem 3.1 is clearly not valid. The Γ-limit is then given by the expression

$$J(u) = \begin{cases} \displaystyle\int_\Omega \Big|\frac{\partial u}{\partial x}\Big|^2\,dx\,dy & \text{if } u \in \mathrm{H}^1(\Omega) \text{ and } \dfrac{\partial u}{\partial y} = 0 \text{ a.e.} \\ +\infty & \text{otherwise,} \end{cases}$$

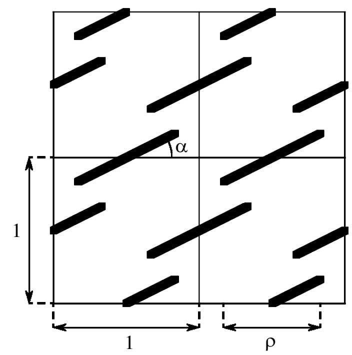

Figure 6: An array of asymmetric cracks

i.e. the effect of the cracks is not felt: the limit is 1-dimensional and its form does not depend on ρ. The upper bound is trivial, while the lower bound is obtained by comparison with the functionals with $\rho > 1/2$ considered before.

Remark 4.2 Asymmetric arrays of cracks may give rise to anisotropic limit energies. For example, we can consider a periodic array of cracks with an inclination of an angle α with respect to the x-axis.

Namely, let $-\frac{\pi}{4} < \alpha < \frac{\pi}{4}$, $\frac{1}{2} < \rho < 1$,

$$K = \Big\{ t\rho(1, \tan\alpha) : \ -\frac{1}{2} \le t \le \frac{1}{2} \Big\},$$

and define

$$E = \mathbb{R}^2 \setminus \bigcup_{i\in\mathbb{Z}^2} \bigg(\Big(i + \frac{1}{2}e_1 + K \Big) \cup \Big(i + \frac{1}{2}e_2 + K \Big) \bigg).$$

Note that E coincides with that defined in Section 3 if $\alpha = 0$.

Let J_ε be defined as in Section 3 with E as above. Then, following the reasoning of the proof of Theorem 3.1 it can be easily proved that the Γ-limit of the functionals J_ε as $\varepsilon \to 0$ is given by

$$J(u) = \begin{cases} \displaystyle\int_\Omega f(Du)\, dx\, dy & \text{if } u \in \mathrm{H}^1(\Omega) \\ +\infty & \text{otherwise,} \end{cases}$$

where $f(z)$ is given by the minimum problem (3.1) in Proposition 3.2.

To compute f note that the Euler-Lagrange equations for the problems defining each $f(z)$ are linear. Hence, it suffices to deal with the case $z = (1, 0)$ and $z = (0, 1)$. In the first case, a minimizer is again $u_1(x, y) = x$, while a minimizer

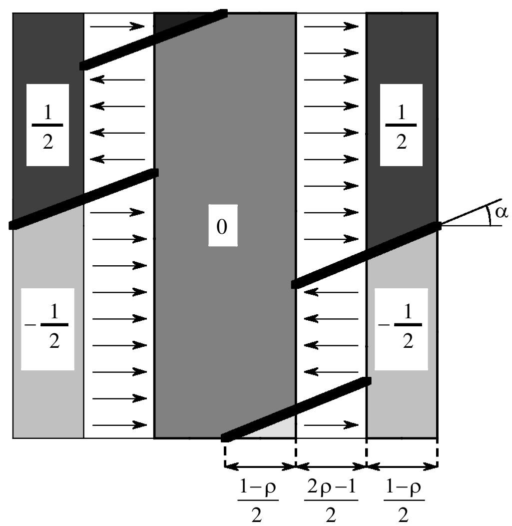

Figure 7: The minimizer for $f(0,1)$

u_2 giving $f(0,1)$ is obtained repeating the reasoning of the proof of Proposition 3.2, and is described in Figure 7.

Finally, we easily compute

$$f(z) = \int_{(-1/2,1/2)^2} \left| z_1 \frac{\partial u_1}{\partial x} + z_2 \frac{\partial u_2}{\partial x} \right|^2 dx\, dy = z_1^2 + \frac{1}{2\rho - 1} z_2^2 + 2 z_1 z_2 \tan \alpha \,,$$

recovering the result of Theorem 3.1 when $\alpha = 0$.

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