Quantitative minimality of strictly stable minimal submanifolds in a small flat neighbourhood

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Abstract. In this paper we extend the results of *A strong minimax property of nondegenerate minimal submanifolds* by White, where it is proved that any smooth, compact submanifold, which is a strictly stable critical point for an elliptic parametric functional, is the unique minimizer in a certain geodesic tubular neighbourhood. We prove a similar result, replacing the tubular neighbourhood with one induced by the flat distance and we provide quantitative estimates. Our proof is based on the introduction of a penalized minimization problem, in the spirit of *A selection principle for the sharp quantitative isoperimetric inequality* by Cicalese and Leoardi, which allows us to exploit the regularity theory for almost minimizers of elliptic parametric integrands.

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1. Introduction

It is well known that any strictly stable critical point of a smooth function $f : \mathbb{R}^n \to \mathbb{R}$ is locally its unique minimizer. In [8], B. White proves a statement of similar nature in a space of submanifolds of a Riemannian manifold, where the function f is replaced by an elliptic parametric functional. In his setting the term "locally" above should be intended with respect to the *strong* topology induced by the Riemannian distance. In the present paper we improve such result, replacing the strong topology with the one induced by the flat distance and providing also quantitative estimates. More precisely we prove the following result, where by $\mathbb{F}(T)$ we denote the flat norm of the integral current T.

1.1. Theorem. *Let M^m be a smooth, compact Riemannian manifold (or $M = \mathbb{R}^m$)*[1] *and suppose that $\Sigma^n \subset M^m$ is a smooth, embedded, compact, oriented submanifold with (possibly empty) boundary which is a strictly stable critical point for a smooth, elliptic, parametric functional F. Then there exist $\varepsilon > 0$ and $C > 0$ (depending on Σ and M) such that*

$$F(S) \geq F(\Sigma) + C(\mathbb{F}(S - \Sigma))^2, \tag{1.1}$$

whenever $0 < \mathbb{F}(S - \Sigma) \leq \varepsilon$ and S is an integral current on M, homologous to Σ.

1.2. **Idea of the proof.** Here is a sketch of the proof of Theorem 1.1. For simplicity, we replace (1.1) with the weaker (non quantitative) inequality $F(S) > F(\Sigma)$, which would imply that Σ is uniquely minimizing in the flat neighbourhood. The proof is by contradiction, and it is inspired by the technique used in [2]. We assume that for every $\delta > 0$ we can select S_δ, homologous to Σ, which satisfies $F(S_\delta) \leq F(\Sigma)$ and $0 < \mathbb{F}(S_\delta - \Sigma) < \delta$. We denote $\eta_\delta := \mathbb{F}(S_\delta - \Sigma)$ and define,

[1] Actually it suffices to require that M admits an embedding which has a tubular neighbourhood $\tau_r M$ and a Lipschitz projection $\pi : \tau_r M \to M$.

for $\lambda > 0$, a penalized functional $F_{\delta,\lambda}$ as

$$F_{\delta,\lambda}(T) := F(T) + \lambda|\mathbb{F}(T - \Sigma) - \eta_\delta| \,.$$

We then consider integral currents

$$R_{\delta,\lambda} \in \operatorname{argmin}\{F_{\delta,\lambda}(T) : T \text{ is homologous to } \Sigma\}.$$

By definition, we have

$$F(R_{\delta,\lambda}) \leq F_{\delta,\lambda}(R_{\delta,\lambda}) \leq F_{\delta,\lambda}(S_\delta) = F(S_\delta) \leq F(\Sigma),$$

and in particular $R_{\delta,\lambda} \neq \Sigma$. Moreover one can easily prove (Lemma 3.3) that, every R_λ such that $\mathbb{F}(R_{\delta_i,\lambda} - R_\lambda) \to 0$ for some $\delta_i \searrow 0$ is a minimizer of

$$F_{0,\lambda}(T) := F(T) + \lambda|\mathbb{F}(T - \Sigma)|.$$

We can also prove (Lemma 3.6) that for λ large enough, the only minimizer of $F_{0,\lambda}$ is Σ itself, hence, by standard compactness and lower semicontinuity properties, we can find a sequence $\delta_i \searrow 0$ such that the currents $R_{\delta_i,\lambda}$ converge to Σ. By the strict stability of Σ, the inequality $F(R_{\delta_i,\lambda}) \leq F(\Sigma)$ would immediately imply the contradiction that $R_{\delta_i,\lambda} = \Sigma$, for every i sufficiently large, if we could guarantee that $R_{\delta_i,\lambda}$ are globally parametrized as graphs of a regular maps on the normal bundle of Σ, converging to 0 strongly, up to the boundary. On the other hand, this is the case because every $R_{\delta_i,\lambda}$ is an almost minimizer for F (Lemma 3.8) and therefore its "graphicality" and the strong convergence are ensured by the regularity theory for almost minimizers.

1.3. **Comparison with results in the literature.** Extensions of White's result have been considered recently by other authors in several contests. In [6], the authors prove that any smooth, oriented hypersurface of a Riemannian manifold of dimension $m \leq 7$, which has constant mean curvature and positive second variation with respect to variations fixing the volume, is uniquely area minimizing among homological oriented hypersurfaces in a small L^1-neighbourhood. In [1], the authors prove a statment of similar nature for nonlocal isoperimetric problem, providing also quantitative estimates. Lastly, for a comparison on the quantitattive part of Theorem 1.1, we refer the reader to the paper [3] where the authors prove that, for uniquely regular area minimizing hypersurfaces, the validity of quadratic stability inequalities is equivalent to the uniform positivity of the second variation of the area.

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2. Notations and Preliminars

2.1. **Integral Currents.** A *rectifiable n-current* T on M is a continuous linear functional on the space of smooth differential n-forms on M admitting the following representation:

$$\langle T;\omega\rangle := \int_E \langle \omega(x), \tau(x)\rangle \theta(x)\, d\mathcal{H}^n(x), \quad \forall\, \omega \in \mathscr{C}_c^\infty(M, \Lambda^n(TM)), \tag{2.1}$$

where:

- E is a countably n-rectifiable set (see §11 of [7]) contained in M,
- $\mathcal{H}^n$ is the n-dimensional Hausdorff measure,
- $\tau(x) \in \Lambda_n(T_xM)$ is the *orientation* of T, i.e. a simple n-vectorfield with $|\tau(x)| = 1$ and spanning the approximate tangent space T_xE, for $\mathcal{H}^n$-a.e. $x \in E$,
- θ is a function in $L^1_{loc}(\mathcal{H}^n \llcorner E)$ which is called the *multiplicity* of T.

We define the mass $\mathbf{M}_A(T)$ of the n-current T in the Borel set $A \subset M$ as the the measure $\theta\mathcal{H}^n \llcorner E(A)$. We will drop the subscript A when $A = \mathbb{R}^m$. The Radon measure $\theta\mathcal{H}^n \llcorner E$ will be often denoted simply by $\|T\|$, while the vectorfield τ will be also denoted $\vec{T}$. Clearly to a compact, orientable, smooth, n-dimensional submanifold Σ it is canonically associated a rectifiable n-current, whose mass coincides with the n-dimensional volume of Σ. The *boundary* of a rectifiable n-current T is the $(n-1)$-current ∂T defined by the relation

$$\langle \partial T, \phi\rangle = \langle T; d\phi\rangle \quad \forall\, \phi \in \mathscr{C}_c^\infty(M, \Lambda^{n-1}(TM)).$$

An *integral n-current* T is a rectifiable n-current with finite mass, such that the boundary ∂T is also a rectifiable $(n-1)$-current with finite mass and the multiplicity both in T and in ∂T takes only integer values.

The space of currents is naturally endowed with a notion of weak*- convergence. In some cases it is convenient to consider also the following notion of metric. The *flat norm* of an integral n-current T in the compact set K is the quantity

$$\mathbb{F}_K(T) = \inf\{\mathbf{M}_K(S) + \mathbf{M}_K(R) : T = S + \partial R, \quad S, R \text{ are integral currents}\}. \tag{2.2}$$

The flat norm $\mathbb{F}(T)$ of the current T is obtained by removing the subscript K in the previous formula.

2.2. **Parametric integrands.** A *parametric integrand* of degree n on $\mathbb{R}^m$ is a continuous map

$$F : \mathbb{R}^m \times \Lambda^n(\mathbb{R}^m) \to \mathbb{R}$$

which takes non-negative values, is positively homogeneous in the second variable and satisfies

$$\Lambda^{-1}\|\tau\| \le F(x,\tau) \le \Lambda\|\tau\|, \tag{2.3}$$

for some $\Lambda > 0$. The parametric integrand induces a functional (also denoted by F) on integral n-currents on $\mathbb{R}^m$ defined by

$$F(T) := \int_{\mathbb{R}^m} F(x, \vec{T}(x)) d\|T\|(x).$$

For fixed $x \in \mathbb{R}^m$ we define F_x to be the integrand obtained by "freezing" F at x, i.e.

$$F_x(y,\tau) := F(x,\tau).$$

Let F be a parametric integrand of degree n on $\mathbb{R}^m$. We call F *elliptic* if there is $C > 0$ such that for every $x \in \mathbb{R}^m$ it holds

$$F_x(T) - F_x(S) \geq C(\mathbf{M}(T) - \mathbf{M}(S)),$$

whenever S and T are compactly supported integral n-currents on $\mathbb{R}^m$, with $\partial S = \partial T$ and S is represented by a measurable subset of an n-dimensional, affine subspace.
Let F be an elliptic parametric functional on $\mathbb{R}^m$, T and S be two $n-$dimensional integral currents, and $A \subset \mathbb{R}^m$ a $\|T\|$-measurable set. We will use the following facts:

(1) $F(T) = F(T \llcorner A) + F(T \llcorner A^c)$,
(2) $F(T + S) \leq F(T) + F(S)$,
(3) F is lower semicontinuous with respect to the flat convergence of integral currents (cf Theorem 5.1.5 in [5]).

3. Proof of Theorem 1.1: Minimality of Σ

As discussed in the introduction of [8], by the results of [9, §8], it is sufficient to prove the theorem when $M = \mathbb{R}^m$. Throughout the paper we denote by T an n-dimensional integral current on $\mathbb{R}^m$.
The purpose of this section is to prove Proposition 3.1; the following weaker (non-quantitative) version of Theorem 1.1. We deduce from it that Σ is the unique minimizer in a small (flat) neighbourhood. The quantitative version will be proved in Section 4, exploiting this result.

3.1. Proposition. *Let M^m be a smooth Riemannian manifold and suppose that $\Sigma^n \subset M^m$ is a smooth, embedded, compact, oriented submanifold with (possibly empty) boundary, which is strictly stable for a smooth, elliptic parametric functional F. Then there exist $\varepsilon > 0$ (depending on Σ and M) such that*

$$F(S) > F(\Sigma), \tag{3.1}$$

whenever $0 < \mathbb{F}(S - \Sigma) \leq \varepsilon$ and S is an integral current on M, homologous to Σ.

3.2. **Proof of Proposition 3.1: Penalized functionals.** We start by assuming (by contradiction) that for any $\delta > 0$ there exists an integral current $S_\delta \neq \Sigma$ with the properties

(i) S_δ is homologuous to Σ;
(ii) $\mathbb{F}(S_\delta - \Sigma) < \delta$;
(iii) $F(S_\delta) \leq F(\Sigma)$.

If we could guarantee that S_δ were, in addition, a regular normal graph over Σ, then the stability of Σ would lead to the contradiction $\Sigma = S_\delta$ for small enough δ. However, this is not the case in general. For this reason we fix a parameter $\lambda > 0$ and replace each S_δ by a minimizer $R_{\delta,\lambda}$ of the penalized functional

$$F_{\delta,\lambda}(T) := F(T) + \lambda|\mathbb{F}(T - \Sigma) - \eta_\delta|\,, \tag{3.2}$$

where for brevity we denoted $\eta_\delta := \mathbb{F}(S_\delta - \Sigma)$. More precisely, we choose

$$R_{\delta,\lambda} \in \operatorname{argmin}\{F_{\delta,\lambda}(T) : T \text{ is homologous to } \Sigma\}\,, \tag{3.3}$$

which exists (although it may well be not unique), because of the usual compactness theorem for uniformly mass bounded (see (2.3)) integral currents and the lower semicontinuity of the functional F with respect to flat convergence.

Now if λ is large enough we expect (ii) to hold also for $R_{\delta,\lambda}$, whereas (i) and (iii) follow from the definition:

$$F(R_{\delta,\lambda}) \leq F_{\delta,\lambda}(R_{\delta,\lambda}) \leq F_{\delta,\lambda}(S_\delta) = F(S_\delta) \leq F(\Sigma)\,.$$

The upshot is that the minimizers $R_{\delta,\lambda}$ are also "almost minimizers" for the functional F (see Lemma 3.8) and the desired graphicality is then a consequence of the regularity theory for almost minimizers.
Hence the task is to check that (ii) holds as well. More precisely, our aim is to find $\lambda > 0$ and a sequence of $\delta_i \searrow 0$ such that $\mathbb{F}(R_{\delta_i,\lambda} - \Sigma) \to 0$. The first step in this direction consists in proving that every subsequential limit of $R_{\delta_i,\lambda}$ is a minimizer for the functional

$$F_{0,\lambda}(T) := F(T) + \lambda\mathbb{F}(T - \Sigma).$$

3.3. Lemma. *Fix $\lambda > 0$. Let $\delta_i \searrow 0$ and let $\mathbb{F}(R_{\delta_i,\lambda} - R_\lambda) \to 0$. Then we have*

$$R_\lambda \in \operatorname{argmin}\{F_{0,\lambda}(T) : T \text{ is homologous to } \Sigma\}.$$

PROOF. Since $\mathbb{F}(R_{\delta_i,\lambda} - R_\lambda) \to 0$ and $\eta_{\delta_i} \to 0$ we have, by lower semicontinuity

$$\begin{aligned} F_{0,\lambda}(R_\lambda) &= F(R_\lambda) + \lambda\mathbb{F}(R_\lambda - \Sigma) \\ &\leq \liminf_{i\to\infty}\{F(R_{\delta_i,\lambda}) + \lambda|\mathbb{F}(R_{\delta_i,\lambda} - \Sigma) - \eta_{\delta_i}|\} \\ &= \liminf_{i\to\infty} F_{\delta_i,\lambda}(R_{\delta_i,\lambda}). \end{aligned}$$

Assume now by contradiction that there exist $\varepsilon > 0$ and an n-dimensional integral current S, homologous to Σ, such that

$$F_{0,\lambda}(S) < F_{0,\lambda}(R_\lambda) - 2\varepsilon.$$

Consider an integer N such that $\lambda\eta_{\delta_N} \leq \varepsilon$. For every $M \geq N$ it holds

$$\begin{aligned} F_{\delta_M,\lambda}(S) &= F(S) + \lambda|\mathbb{F}(S - \Sigma) - \eta_{\delta_M}| \leq F(S) + \lambda\mathbb{F}(S - \Sigma) + \lambda\eta_{\delta_M} \\ &\leq F(S) + \lambda\mathbb{F}(S - \Sigma) + \varepsilon = F_{0,\lambda}(S) + \varepsilon < F_{0,\lambda}(R_\lambda) - \varepsilon \\ &\leq \liminf_{i\to\infty} F_{\delta_i,\lambda}(R_{\delta_i,\lambda}) - \varepsilon, \leq \liminf_{i\to\infty} F_{\delta_i,\lambda}(S) - \varepsilon, \end{aligned}$$

which is a contradiction. □

The second step is to prove that, if λ is sufficiently large, then the only minimizer of $F_{0,\lambda}$ is Σ. This is achieved in Lemma 3.6. To prove it we need the estimate of Lemma 3.5, which, on the other hand, is based on the following general fact. For the sake of generality, only in the next lemma we consider the flat norm on the manifold M, i.e. we require that the currents R and S in (2.2) are supported on M.

3.4. Lemma. *Let M^m be a smooth, compact manifold in $\mathbb{R}^d$ (or $M = \mathbb{R}^m$). Then there exists $\varepsilon_0 = \varepsilon_0(M) > 0$ such that for every n-dimensional integral current T on M with $\partial T = 0$ and $\mathbb{F}(T) \leq \varepsilon_0$ there exists an integral $(n+1)$-current R on M satisfying $\partial R = T$ and*

$$\mathbf{M}(R) = \mathbb{F}(T).$$

PROOF. We will assume that M is a smooth, compact manifold in $\mathbb{R}^d$; the proof for $M = \mathbb{R}^m$ is identical. Fix $\delta > 0$ and let P and Q be integral currents in M satisfying $T = P + \partial Q$ and $\mathbf{M}(P) + \mathbf{M}(Q) \leq \mathbb{F}(T) + \delta$. Let S be an integral $(n+1)$-current in $\mathbb{R}^d$ minimizing the mass

among all integral currents with boundary equal to P (notice that $\partial P = 0$). For $\mathcal{H}^{n+1}$-a.e. $x \in \mathrm{spt}(S)$, the monotonicity formula (see formula (17.3) of [7]) yields

$$\mathbf{M}(S \llcorner B_r(x)) \geq \omega_{n+1} r^{n+1},$$

whenever $r < \mathrm{dist}(x, M)$. Moreover, the isoperimetric inequality (see Theorem 30.1 of [7]) gives

$$\mathbf{M}(S) \leq C_1 \mathbf{M}(P)^{\frac{n+1}{n}}.$$

This implies that, if $\mathbb{F}(T)$ is sufficiently small, the support of S is contained in a small tubular neighbourhood of M. In particular, by the assumptions on M, the closest point projection π on M is (uniquely defined and) Lipschitz on this neighbourhood. By Lemma 26.25 of [7], it follows that $\mathbf{M}(\pi_\sharp S) \leq C_2 \mathbf{M}(S)$, therefore if $\mathbb{F}(T)$ is sufficiently small, then $\mathbf{M}(\pi_\sharp S) \leq \mathbf{M}(P)$. Hence, setting $R_\delta = \pi_\sharp S + Q$, we have $T = \partial R_\delta$ and

$$\mathbf{M}(R_\delta) \leq \mathbf{M}(\pi_\sharp S) + \mathbf{M}(Q) \leq \mathbb{F}(T) + \delta.$$

By the usual compactness and lower semicontinuity, we can take R as a subsequential limit of any sequence R_{δ_i}, for $\delta_i \to 0$. Clearly R is supported on M. □

3.5. Lemma. *There exist $\varepsilon > 0$ and a constant $C = C(\Sigma)$ such that, for every n-dimensional integral current T on $\mathbb{R}^m$, homologous to Σ, such that $\mathbb{F}(\Sigma - T) \leq \varepsilon$, it holds*

$$F(\Sigma) - F(T) \leq C\mathbb{F}(\Sigma - T).$$

Proof. We present firstly a very simple proof of this fact, which is valid in the case $F(x, \tau) \equiv \|\tau\|$, i.e. $F(T) = \mathbf{M}(T)$.

Let ω be a compactly supported n-form satisfying $\|\omega\|_\infty \leq 1$ and $\langle \omega; \tau_\Sigma \rangle = 1$ whenever τ_Σ is a tangent unit vector orienting Σ (which exists by smoothness of Σ). If $\varepsilon \leq \varepsilon_0$ in Lemma 3.4, we can find an integral $(n+1)$-current S satisfying

$$\partial S = \Sigma - T \quad \text{and} \quad \mathbf{M}(S) = \mathbb{F}(\Sigma - T). \tag{3.4}$$

Then we have

$$\begin{aligned} F(\Sigma) - F(T) &= \int_M 1 d\|\Sigma\| - \int_{\mathbb{R}^d} 1 d\|T\| \\ &\leq \int_{\mathbb{R}^d} \langle \omega(x); \tau_\Sigma(x) \rangle d\|\Sigma\|(x) - \int_{\mathbb{R}^d} \langle \omega(x); \vec{T}(x) \rangle d\|T\|(x) \\ &= \langle S; d\omega \rangle \leq \|d\omega\|_\infty \mathbf{M}(S) = \|d\omega\|_\infty \mathbb{F}(\Sigma - T). \end{aligned}$$

This completes the proof in the case $F(x, \tau) \equiv |\tau|$. In case F is a convex functional, it is easy to adapt the previous argument.

In the general case, the only proof we are able to devise is more involved. In particular we need to exploit the stability of Σ and we make use of Theorem 2 of [8].

Let $d(x) := \mathrm{dist}(x, \Sigma)$ and again choose $\varepsilon \leq \varepsilon_0$ in Lemma 3.4. Let S be an $(n+1)$-dimensional integral current such that $\partial S = T - \Sigma$ and $\mathbf{M}(S) = \mathbb{F}(T - \Sigma)$. Let $\varepsilon_1 < \varepsilon_0$ be such that the open tubular neighbourhood of radius ε_1 centered at Σ, i.e. the set

$$B_{\varepsilon_1}(\Sigma) := \{x \in \mathbb{R}^m : \mathrm{dist}(x, \Sigma) < \varepsilon_1\},$$

is contained in the open tubular neighbourhood U given by Theorem 2 of [8].
Denoting by $\langle S, d, t\rangle$ the "slices" of S according to the function d, we have by standard properties of the slicing (see Lemma 28.5 (1) and (2) of [7]) that there exists $t \in (\varepsilon_1/2, \varepsilon_1)$ such that

$$\langle S, d, t\rangle = \partial(S \llcorner B_t(\Sigma)) - (\partial S) \llcorner B_t(\Sigma) \quad \text{and} \quad \mathbf{M}(\langle S, d, t\rangle) \leq \frac{2\mathbf{M}(S)}{\varepsilon_1}. \tag{3.5}$$

Observe that $\tilde{T} := T \llcorner B_t(\Sigma) + \langle S, d, t\rangle$ is supported in $B_{\varepsilon_1}(\Sigma)$ and it is homologous to Σ, indeed

$$\begin{aligned}\tilde{T} - \Sigma &= T \llcorner B_t(\Sigma) + \partial(S \llcorner B_t(\Sigma)) - (\partial S) \llcorner B_t(\Sigma) - \Sigma \\ &= T \llcorner B_t(\Sigma) + \partial(S \llcorner B_t(\Sigma)) - (T - \Sigma) \llcorner B_t(\Sigma) - \Sigma \llcorner B_t(\Sigma) \\ &= \partial(S \llcorner B_t(\Sigma)).\end{aligned}$$

Eventually we compute:

$$\begin{aligned}F(\Sigma) - F(T) &= F(\Sigma) - (F(T \llcorner B_t(\Sigma)) + F(\langle S, d, t\rangle)) + F(\langle S, d, t\rangle) - F(T \llcorner B_t(\Sigma)^c) \\ &\leq F(\Sigma) - F(\tilde{T}) + F(\langle S, d, t\rangle),\end{aligned}$$

where we used the fact that parametric integrands are additive on currents supported on disjoint sets and subadditive for general rectifiable currents, and $F \geq 0$. Now, by Theorem 2 of [8], $F(\Sigma) - F(\tilde{T}) < 0$. So we can conclude that, whenever T is homologous to Σ and $F(T - \Sigma) \leq \varepsilon_1$, it holds:

$$F(\Sigma) - F(T) \leq F(\langle S, d, t\rangle) \overset{(2.3)}{\leq} \Lambda \mathbf{M}(\langle S, d, t\rangle) \overset{(3.5)}{\leq} \frac{2\Lambda \mathbf{M}(S)}{\varepsilon_1} \overset{(3.4)}{=} \frac{2\Lambda}{\varepsilon_1} \mathbb{F}(T - \Sigma). \quad \square$$

3.6. Lemma. *There exists $\lambda_0 > 0$ such that, for every $\lambda > \lambda_0$, there holds*

$$\operatorname{argmin}\{F_{0,\lambda}(T) : T \text{ is homologous to } \Sigma\} = \{\Sigma\}.$$

Proof. Assume by contradiction there exist $\lambda_i \to \infty$ and $S_i \neq \Sigma$ such that S_i is homologous to Σ and

$$S_i \in \operatorname{argmin}\{F_{0,\lambda_i}(T) : T \text{ is homologous to } \Sigma\}.$$

Notice that $F_{0,\lambda_i}(\Sigma) = F(\Sigma)$ for every i, therefore we have:

$$F(S_i) + \lambda_i \mathbb{F}(S_i - \Sigma) = F_{0,\lambda_i}(S_i) \leq F_{0,\lambda_i}(\Sigma) = F(\Sigma).$$

Hence

$$\lambda_i \mathbb{F}(S_i - \Sigma) \leq F(\Sigma) - F(S_i),$$

which, for i sufficiently large, contradicts Lemma 3.5. $\square$

3.7. **Proof of Proposition 3.1: Almost minimizers.** Now, fixing $\lambda > \lambda_0$, we have a sequence of integral currents $R_{\delta_i,\lambda}$ homologous to Σ such that $\mathbb{F}(R_{\delta_i,\lambda} - \Sigma) \to 0$ and $F(R_{\delta_i,\lambda}) \leq F(\Sigma)$, i.e. we managed to replace the original sequence S_{δ_i} (a sequence of minimizers of a functional under an additional constraint) by a sequence of global minimizers of a penalized functional. As already mentioned, being minimizers of the penalized functionals guarantees almost minimality properties, and by the regularity theory for such almost minimizers, one can deduce that the $R_{\delta_i,\lambda}$ are regular, normal graphs over Σ, for sufficiently large i. From this information, one would be able to conlcude the proof as in [8]. Actually, we do not need to repeat the final part of that proof, but we can recast the problem in White's setting and exploit his result, once we prove that the almost minimality of the $R_{\delta_i,\lambda}$'s implies that, for i sufficiently

large, they are supported in a small tubular neighbourhood of Σ (cf. Lemma 3.10).

We will adopt a special case of the notion of almost minimality introduced in [4]. Given an elliptic parametric functional F and $C > 0$, we say that an n-dimensional integral current S is *C-almost minimizing* for F, if

$$F(S) \leq F(S + X) + Cr\mathbf{M}(S \llcorner K + X)\,, \tag{3.6}$$

whenever X is a closed n-current with support in a compact set K, which is contained in a ball $B_r = B_r(x_0) \subset \mathbb{R}^m$ of radius r and center $x_0 \in \mathbb{R}^m$. For the regularity theory it is sufficient to have (3.6) only for small radii r and for all X satisfying in addition $\mathbf{M}(X) < 1$.
As we show in the next lemma, the currents $R_{\delta,\lambda}$ are C-almost minimizing. This follows almost directly from the following stronger property of $R_{\delta,\lambda}$. Fix any closed n-dimensional current X. Then $R_{\delta,\lambda} + X$ is a competitor in the optimization of $F_{\delta,\lambda}$ and consequently

$$F(R_{\delta,\lambda}) + \lambda|\mathbb{F}(R_{\delta,\lambda} - \Sigma) - \eta_\delta| \leq F(R_{\delta,\lambda} + X) + \lambda|\mathbb{F}(R_{\delta,\lambda} + X - \Sigma) - \eta_\delta|\,.$$

This implies that

$$F(R_{\delta,\lambda}) \leq F(R_{\delta,\lambda} + X) + \lambda\mathbb{F}(X)\,. \tag{3.7}$$

3.8. Lemma. *For every $\delta, \lambda > 0$, $R_{\delta,\lambda}$ are C-almost minimizing for $C = \frac{4\Lambda^2\lambda}{n+1}$.*

Proof. The first observation we make is that, due to the minimality of $R_{\delta,\lambda}$ with respect to $F_{\delta,\lambda}$, it's not possible write $R_{\delta,\lambda}$ inside a sufficiently small ball as the sum of a closed current and another current which has much smaller mass. To make this precise fix $r < r_0 := \frac{n+1}{4\lambda\Lambda}$. We claim that there exist no closed currents X supported in some compact $K \subset B_r \subset \mathbb{R}^m$ with $\mathbf{M}(X) < 1$ such that

$$4\Lambda^2\mathbf{M}(R_{\delta,\lambda} \llcorner K + X) \leq \mathbf{M}(X)\,. \tag{3.8}$$

Indeed, assume by contradiction that we can find a current X and set K with such property. By (3.7) we have

$$F(R_{\delta,\lambda}) \leq F(R_{\delta,\lambda} + X) + \lambda\mathbb{F}(X)\,.$$

Since F is additive on currents with disjoint supports and X is supported in K, we can subtract $F(R_{\delta,\lambda} \llcorner K^c)$ from both sides to get

$$F(R_{\delta,\lambda} \llcorner K) \leq F(R_{\delta,\lambda} \llcorner K + X) + \lambda\mathbb{F}(X)\,.$$

We estimate $\mathbb{F}(X)$ by the mass of an area-minimizing current with boundary X and use the isoperimetric inequality and the fact that $\mathbf{M}(X) < 1$ to get

$$\begin{aligned} F(R_{\delta,\lambda} \llcorner K) &\leq F(R_{\delta,\lambda} \llcorner K + X) + \lambda\frac{r}{n+1}\mathbf{M}(X) \\ &\leq \Lambda\mathbf{M}(R_{\delta,\lambda} \llcorner K + X) + \lambda\frac{r}{n+1}\mathbf{M}(X) \\ &< \frac{1}{2\Lambda}\mathbf{M}(X)\,, \end{aligned} \tag{3.9}$$

by (3.8) and the assumption on r. Observe that

$$4\Lambda^2\mathbf{M}(R_{\delta,\lambda} \llcorner K + X) \leq \mathbf{M}(X) \leq \mathbf{M}(X + R_{\delta,\lambda} \llcorner K) + \mathbf{M}(R_{\delta,\lambda} \llcorner K)\,,$$

and so

$$(4\Lambda^2 - 1)\mathbf{M}(R_{\delta,\lambda} \llcorner K + X) \leq \mathbf{M}(R_{\delta,\lambda} \llcorner K)\,.$$

Since we assume without loss of generality that $\Lambda > 1$, this implies

$$\mathbf{M}(X) \leq \mathbf{M}(R_{\delta,\lambda} \llcorner K + X) + \mathbf{M}(R_{\delta,\lambda} \llcorner K) \leq 2\mathbf{M}(R_{\delta,\lambda} \llcorner K) \leq 2\Lambda F(R_{\delta,\lambda} \llcorner K)\,.$$

Plugging into (3.9) yields the contradiction

$$F(R_{\delta,\lambda} \llcorner K) < F(R_{\delta,\lambda} \llcorner K)\,.$$

Consequently, when r is small enough, any integral current X as in the definition satisfies additionally

$$\mathbf{M}(X) \leq 4\Lambda^2 \mathbf{M}(R_{\delta,\lambda} \llcorner K + X)\,.$$

Combining this with (3.7) we immediately get

$$\begin{aligned} F(R_{\delta,\lambda}) &\leq F(R_{\delta,\lambda} + X) + \lambda \mathbb{F}(X) \leq F(R_{\delta,\lambda} + X) + \lambda \frac{r}{n+1} \mathbf{M}(X) \\ &\leq F(R_{\delta,\lambda} + X) + \frac{4\Lambda^2 \lambda}{n+1} r \mathbf{M}(R_{\delta,\lambda} \llcorner K + X)\,, \end{aligned}$$

proving C-almost minimality with $C = \frac{4\Lambda^2\lambda}{n+1}$. □

3.9. Proof of Proposition 3.1: Hausdorff convergence and conclusion.

3.10. Lemma. *Let $\lambda > \lambda_0$ of Lemma 3.6 and let $\delta_i \searrow 0$ be such that $\mathbb{F}(R_{\delta_i,\lambda} - \Sigma) \to 0$. Then*

$$\lim_{i\to\infty} \sup_{x \in spt(R_{\delta_i,\lambda})} \{\mathrm{dist}(x, \Sigma)\} = 0.$$

PROOF. For notational convenience we set $R_i := R_{\delta_i,\lambda}$. Assume by contradiction, up to passing to a suitable subsequence, that there exists $r > 0$ such that for every $i \in \mathbb{N}$ there exists x_i in the support of R_i satisfying

$$\mathrm{dist}(x_i, \Sigma) > 2r.$$

By the density lower bound (see Lemma 2.1 in [4]) there exists $D > 0$ such that

$$F(R_i \llcorner B_r(x_i)) \geq D r^n, \tag{3.10}$$

for every $i \in \mathbb{N}$. This contradicts the almost minimality of R_i for i large enough, i.e. we can find closed n-dimensional integral currents X_i satisfying

$$F(R_i + X_i) \leq F(R_i) - \lambda \mathbb{F}(X_i)\,.$$

Indeed, since $\mathbb{F}(R_i - \Sigma) \to 0$, by Lemma 3.4 we can find integral $(n+1)-$currents S_i satisfying

$$\partial S_i = R_i - \Sigma \quad \text{and} \quad \mathbf{M}(S_i) = \mathbb{F}(R_i - \Sigma) \to 0\,.$$

As in the proof of Lemma 3.5, we consider the slices $\langle S_i, d, t_i\rangle$ of S_i with respect to the distance function $d(x) := \mathrm{dist}(x, \Sigma)$ for some $t_i \in (\frac{r}{2}, r)$ satisfying

$$\langle S_i, d, t_i\rangle = \partial(S_i \llcorner B_{t_i}(\Sigma)) - \partial S_i \llcorner B_{t_i}(\Sigma) \quad \text{and} \quad \mathbf{M}(\langle S_i, d, t_i\rangle) \leq \frac{2\mathbf{M}(S_i)}{r}\,.$$

Set $X_i = -\partial(S_i \llcorner (B_{t_i}(\Sigma))^c)$ and observe

$$\begin{aligned} R_i + X_i &= \partial S_i + \Sigma - \partial(S_i \llcorner (B_{t_i}(\Sigma))^c) = \partial(S_i \llcorner B_{t_i}(\Sigma)) + \Sigma = \langle S_i, d, t_i\rangle + \partial S_i \llcorner B_{t_i}(\Sigma) + \Sigma \\ &= \langle S_i, d, t_i\rangle + R_i \llcorner B_{t_i}(\Sigma)\,. \end{aligned}$$

Fix $I \in \mathbb{N}$ such that for every $i \geq I$ we have

$$\lambda \mathbb{F}(X_i) + F(\langle S_i, d, t_i \rangle) < \frac{1}{2\Lambda^2} D r^n \,.$$

Observe that $B_r(x_i) \subset (B_{t_i}(\Sigma))^c$ and hence with (3.10) we infer

$$\begin{aligned} F(R_i \llcorner (B_{t_i}(\Sigma))^c) &\geq \Lambda^{-1} \mathbf{M}(R_i \llcorner (B_{t_i}(\Sigma))^c) \geq \Lambda^{-1} \mathbf{M}(R_i \llcorner B_r(x_i)) \geq \Lambda^{-2} F(R_i \llcorner B_r(x_i)) \\ &\geq \Lambda^{-2} D r^n \,. \end{aligned}$$

Consequently for any $i \geq I$ we have

$$\begin{aligned} F(R_i + X_i) &\leq F(\langle S_i, d, t_i \rangle) + F(R_i \llcorner B_{t_i}(\Sigma)) = F(\langle S_i, d, t_i \rangle) + F(R_i) - F(R_i \llcorner (B_{t_i}(\Sigma))^c) \\ &\leq \frac{1}{2\Lambda^2} D r^n - \lambda \mathbb{F}(X_i) + F(R_i) - D \Lambda^{-2} r^2 < F(R_i) - \lambda \mathbb{F}(X_i) \,, \end{aligned}$$

contradicting (3.7). □

Note that at this point, we have a sequence $R_{\delta_i, \lambda}$ of integral currents which are supported in arbitrarily small tubular neighbourhoods of Σ and satisfy

$$F(R_{\delta_i, \lambda}) \leq F(\Sigma),$$

and this implies, by [8, Theorem 3], that $R_{\delta_i, \lambda} = \Sigma$, for i sufficiently large, which is the final contradiction.

4. Proof of Theorem 1.1: Quantitative Estimate

To prove the quantitative estimate (1.1) we will redo most of the proof in the previous section but slightly change the penalized functional $F_{\delta, \lambda}$. We again argue by contradiction and assume that for every $\delta > 0$ there exists $\tilde{S}_\delta \neq \Sigma$ which is homologous to Σ, has flat distance $\tilde{\eta}_\delta := \mathbb{F}(\tilde{S}_\delta - \Sigma) < \delta$ from Σ and satisfies

$$F(\tilde{S}_\delta) \leq F(\Sigma) + C(\mathbb{F}(\tilde{S}_\delta - \Sigma))^2 \,. \tag{4.1}$$

The constant C has to be thought as fixed for the moment and it will be chosen only at the end of the proof. Define the penalized functional $\tilde{F}_{\delta, \lambda}$ as

$$\tilde{F}_{\delta, \lambda}(T) := F(T) + \lambda(\mathbb{F}(T - \Sigma) - \tilde{\eta}_\delta)^2 \,. \tag{4.2}$$

The semicontinuity of $\tilde{F}_{\delta, \lambda}$ follows immediately from the semicontinuity of F. Therefore we can again consider minimizers $\tilde{R}_{\delta, \lambda}$ of the penalized functional $\tilde{F}_{\delta, \lambda}$ among all integral currents homologous to Σ. By repeating the steps of the previous section we can find a nice subsequence converging to Σ.

4.1. Lemma. *There exists $\lambda_0 > 0$ such that for any $\lambda \geq \lambda_0$ there is a sequence $\delta_i \searrow 0$ such that $\mathbb{F}(\tilde{R}_{\delta_i, \lambda} - \Sigma) \to 0$ for $i \to +\infty$.*

Proof. The argument is again divided into two parts: firstly, any integral current $\tilde{R}_\lambda$ which is a subsequential limit of $\tilde{R}_{\delta, \lambda}$ is a minimizer of

$$\tilde{F}_{0, \lambda}(T) := F(T) + \lambda \mathbb{F}(T - \Sigma)^2 \,.$$

This part is entirely similar to the one in Lemma 3.3: let $\tilde{R}_\lambda, \tilde{R}_{\delta_i,\lambda}$ be such that $\mathbb{F}(\tilde{R}_{\delta_i,\lambda}-\tilde{R}_\lambda)\to 0$ and assume by contradiction that there exists $\varepsilon>0$ and a n-dimensional integral current S homologous to Σ such that

$$\tilde{F}_{0,\lambda}(S) < \tilde{F}_{0,\lambda}(\tilde{R}_\lambda) - 2\varepsilon. \tag{4.3}$$

Consider an integer N such that $\lambda\tilde{\eta}^2_{\delta_M}\le\varepsilon$ for every $M\ge N$. Since by the semicontinuity of $\tilde{F}_{\delta,\lambda}$ we have

$$\begin{aligned}\tilde{F}_{0,\lambda}(\tilde{R}_\lambda) &= F(\tilde{R}_\lambda) + \lambda\mathbb{F}(\tilde{R}_\lambda-\Sigma)^2\\ &\le \liminf_{i\to\infty}\{F(\tilde{R}_{\delta_i,\lambda}) + \lambda(\mathbb{F}(\tilde{R}_{\delta_i,\lambda}-\Sigma)-\tilde{\eta}_{\delta_i})^2\}\\ &= \liminf_{i\to\infty}\tilde{F}_{\delta_i,\lambda}(\tilde{R}_{\delta_i,\lambda})\,,\end{aligned}$$

we get for arbitrary $M\ge N$ the contradiction

$$\begin{aligned}\tilde{F}_{\delta_M,\lambda}(S) &= F(S) + \lambda(\mathbb{F}(S-\Sigma)-\tilde{\eta}_{\delta_M})^2\\ &= F(S) + \lambda(\mathbb{F}(S-\Sigma))^2 + \lambda\tilde{\eta}^2_{\delta_M} - 2\lambda\mathbb{F}(S-\Sigma)\tilde{\eta}_{\delta_M}\\ &\le F(S) + \lambda(\mathbb{F}(S-\Sigma))^2 + \varepsilon = \tilde{F}_{0,\lambda}(S)+\varepsilon \overset{(4.3)}{<} \tilde{F}_{0,\lambda}(\tilde{R}_\lambda)-\varepsilon\\ &\le \liminf_{i\to\infty}\tilde{F}_{\delta_i,\lambda}(\tilde{R}_{\delta_i,\lambda})-\varepsilon, \le \liminf_{i\to\infty}\tilde{F}_{\delta_i,\lambda}(S)-\varepsilon\,.\end{aligned}$$

Secondly, we claim that if $\lambda\ge\lambda_0$ is large enough then the only minimizer of $\tilde{F}_{0,\lambda}$ is Σ itself. Consider $\lambda>\frac{F(\Sigma)}{\varepsilon^2}$, where ε is the value defined in Proposition 3.1 and assume by contradiction that there exists $S_\lambda\neq\Sigma$ such that $\tilde{F}_{0,\lambda}(S_\lambda)\le\tilde{F}_{0,\lambda}(\Sigma)$. Then we have

$$\tilde{F}_{0,\lambda}(S_\lambda) = F(S_\lambda) + \lambda(\mathbb{F}(S_\lambda-\Sigma))^2 > F(\Sigma)\frac{(\mathbb{F}(S_\lambda-\Sigma))^2}{\varepsilon^2}. \tag{4.4}$$

From the last inequality, since $\tilde{F}_{0,\lambda}(S_\lambda)\le\tilde{F}_{0,\lambda}(\Sigma)=F(\Sigma)$, it follows that $\mathbb{F}(S_\lambda-\Sigma)<\varepsilon$. On the other hand, if $\mathbb{F}(S_\lambda-\Sigma)<\varepsilon$, by Proposition 3.1, it follows that $F(\Sigma)<F(S_\lambda)$, hence

$$\tilde{F}_{0,\lambda}(\Sigma) = F(\Sigma) < F(S_\lambda) < F(S_\lambda) + \lambda(\mathbb{F}(S_\lambda-\Sigma))^2 = \tilde{F}_{0,\lambda}(S_\lambda),$$

which is a contradiction. □

At this point, for every $\lambda>\lambda_0$ we have a sequence $(\tilde{R}_{\delta_i,\lambda})$ of integral currents which are homologous to Σ and converge flat to it. As before, the $(\tilde{R}_{\delta_i,\lambda})$ are almost minimizers if i is large enough: this follows as in Lemma 3.8, once we observe that for any closed n-current X with support in a compact $K\subset B_r(x_0)$ and with mass bound $\mathbf{M}(X)<1$ we have the following inequality

$$F(\tilde{R}_{\delta_i,\lambda}) \le F(\tilde{R}_{\delta_i,\lambda}+X) + 2\lambda\mathbb{F}(X)\,. \tag{4.5}$$

The latter inequality on the other hand is a consequence of

$$\begin{aligned}F(\tilde{R}_{\delta_i,\lambda}) + \lambda(\mathbb{F}(\tilde{R}_{\delta_i,\lambda}-\Sigma)-\tilde{\eta}_{\delta_i})^2 &= \tilde{F}_{\delta_i,\lambda}(\tilde{R}_{\delta_i,\lambda}) \le \tilde{F}_{\delta_i,\lambda}(\tilde{R}_{\delta_i,\lambda}+X)\\ &= F(\tilde{R}_{\delta_i,\lambda}+X) + \lambda(\mathbb{F}(\tilde{R}_{\delta_i,\lambda}+X-\Sigma)-\tilde{\eta}_{\delta_i})^2\,,\end{aligned}$$

since this implies

$$\begin{aligned}F(\tilde{R}_{\delta_i,\lambda}) \leq & F(\tilde{R}_{\delta_i,\lambda}+X) - 2\lambda\tilde{\eta}_{\delta_i}\left(\mathbb{F}(\tilde{R}_{\delta_i,\lambda_0}+X-\Sigma)-\mathbb{F}(\tilde{R}_{\delta_i,\lambda_0}-\Sigma)\right)\\ &+\lambda\left(\mathbb{F}(\tilde{R}_{\delta_i,\lambda}+X-\Sigma)+\mathbb{F}(\tilde{R}_{\delta_i,\lambda}-\Sigma)\right)\left(\mathbb{F}(\tilde{R}_{\delta_i,\lambda_0}+X-\Sigma)-\mathbb{F}(\tilde{R}_{\delta_i,\lambda_0}-\Sigma)\right)\\ \leq & F(\tilde{R}_{\delta_i,\lambda}+X) - 2\lambda\tilde{\eta}_{\delta_i}\left(\mathbb{F}(\tilde{R}_{\delta_i,\lambda_0}+X-\Sigma)-\mathbb{F}(\tilde{R}_{\delta_i,\lambda_0}-\Sigma)\right)\\ &+\lambda\mathbb{F}(X)\left(\mathbb{F}(\tilde{R}_{\delta_i,\lambda}+X-\Sigma)+\mathbb{F}(\tilde{R}_{\delta_i,\lambda}-\Sigma)\right)\\ \leq & F(\tilde{R}_{\delta_i,\lambda_0}+X)+\lambda\mathbb{F}(X)\left(\mathbf{M}(X)+2\tilde{\eta}_{\delta_i}+2\mathbb{F}(\tilde{R}_{\delta_i,\lambda}-\Sigma)\right),\end{aligned}$$

which yields (4.5) for i large enough.
The almost minimality of the members of the sequence and the stability of Σ now imply that $\tilde{R}_{\delta_i,\lambda}=\Sigma$ for i large enough. This is a consequence of the regularity of almost minimizers. In particular we will use the well known fact that, for i sufficiently large, we can write $\tilde{R}_{\delta_i,\lambda}$ as a graph of normal vectorfield u_i over Σ, converging to 0 in $C^{1,\beta}$, up to the boundary (see [4]). In turn, this fact implies that, for i sufficiently large, it holds

$$\|u_i\|^2_{W^{1,2}} \geq c_0\|u_i\|^2_{L^1} \geq c_1\left(\mathbb{F}(\tilde{R}_{\delta_i,\lambda}-\Sigma)\right)^2,$$

for some $c_1>0$ (depending on Σ). Indeed the first inequality is just Hölder's inequality and the second is due to the bound on the C^1-norm of u_i. Now we compute

$$\tilde{F}_{\delta_i,\lambda}(\tilde{R}_{\delta_i,\lambda}) \leq \tilde{F}_{\delta_i,\lambda}(\tilde{S}_{\delta_i}) \overset{(4.2)}{=} F(\tilde{S}_{\delta_i}) \overset{(4.1)}{\leq} F(\Sigma)+C(\mathbb{F}(\tilde{S}_{\delta_i}-\Sigma))^2,$$

and by the stability of Σ, denoting $2\eta_0$ the (strictly positive) minimal eigenvalue of the Jacobi operator of F for Σ, we get, for every $\lambda>\lambda_0$ and for i sufficiently large,

$$F(\tilde{R}_{\delta_i,\lambda}) \geq F(\Sigma)+\eta_0\|u_i\|^2_{W^{1,2}} \geq F(\Sigma)+\eta_0 c_1(\mathbb{F}(\tilde{R}_{\delta_i,\lambda}-\Sigma))^2.$$

We deduce that, for i sufficiently large,

$$\begin{aligned}F(\Sigma)+\eta_0 c_1(\mathbb{F}(\tilde{R}_{\delta_i,\lambda}-\Sigma))^2+\lambda(\mathbb{F}(\tilde{R}_{\delta_i,\lambda}-\Sigma)-\tilde{\eta}_{\delta_i})^2 &\leq F(\tilde{R}_{\delta_i,\lambda})+\lambda(\mathbb{F}(\tilde{R}_{\delta_i,\lambda}-\Sigma)-\tilde{\eta}_{\delta_i})^2\\ &= \tilde{F}_{\delta_i,\lambda}(\tilde{R}_{\delta_i,\lambda}) \leq F(\Sigma)+C\tilde{\eta}^2_{\delta_i}.\end{aligned}$$

Since $\tilde{\eta}^2_{\delta_i} \leq 2(\tilde{\eta}_{\delta_i}-\mathbb{F}(\tilde{R}_{\delta_i,\lambda}-\Sigma))^2+2(\mathbb{F}(\tilde{R}_{\delta_i,\lambda}-\Sigma))^2$ we infer

$$\eta_0 c_1(\mathbb{F}(\tilde{R}_{\delta_i,\lambda}-\Sigma))^2+\lambda(\mathbb{F}(\tilde{R}_{\delta_i,\lambda}-\Sigma)-\tilde{\eta}_{\delta_i})^2 \leq 2C((\tilde{\eta}_{\delta_i}-\mathbb{F}(\tilde{R}_{\delta_i,\lambda}-\Sigma))^2+(\mathbb{F}(\tilde{R}_{\delta_i,\lambda}-\Sigma))^2),$$

or equivalently

$$(\eta_0 c_1-2C)(\mathbb{F}(\tilde{R}_{\delta_i,\lambda}-\Sigma))^2 \leq (2C-\lambda)(\tilde{\eta}_{\delta_i}-\mathbb{F}(\tilde{R}_{\delta_i,\lambda}-\Sigma))^2,$$

which is not possible for $C=C(\eta_0)$ small enough and λ large enough.

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