RIGIDITY AND TRACE PROPERTIES OF DIVERGENCE–MEASURE VECTOR FIELDS

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Abstract. We show some rigidity properties of divergence–free vector fields defined on half–spaces. As an application, we prove the existence of the classical trace for a bounded, divergence–measure vector field ξ defined on the Euclidean plane, at almost every point of a locally oriented rectifiable set S, under the assumption that its weak normal trace $[\xi \cdot \nu_S]$ attains a local maximum for the norm of ξ at the point.

Introduction

Fix $n \geq 2$, denote by $\mathbb{R}^n_+$ the upper half–space $\{x \in \mathbb{R}^n : \ x_n > 0\}$, and consider $z = (z_1, \dots, z_n)$ a bounded, smooth vector field defined on $\mathbb{R}^n_+$.

Question 0.1. We assume that

- $\operatorname{div} z = 0$ on $\mathbb{R}^n_+$,
- $z = 0$ on the hyperplane $\mathbb{R}^n_0 = \partial\mathbb{R}^n_+$
- $z_n \geq \varphi(|z|)$ for a continuous non-negative function φ such that $\varphi(t) = 0$ if and only if $t = 0$.

Can we conclude that the vector field z is identically zero on $\mathbb{R}^n_+$?

The structure of divergence–free vector fields is the object of an extensive research, with most applications ranging from conservation laws to fluid dynamics and electromagnetism (see for instance the comprehensive books [14] and [15], as well as the papers [13, 16, 17] and references therein). The determination of rigidity properties for such vector fields, like the one stated above, represents a fundamental task. For instance, in the case of the steady motion of an incompressible fluid, modeled by the Navier-Stokes system

$$\begin{cases} \nu\, \Delta v = v \cdot \nabla v + \nabla p \\ \operatorname{div} v = 0\,, \end{cases}$$

a classical conjecture, whose complete proof is still missing, says that if v additionally satisfies $\lim_{|x|\to\infty} v(x) = 0$ and $\int_{\mathbb{R}^3} |\nabla v|^2 < +\infty$, then necessarily $v = \nabla p = 0$ on $\mathbb{R}^3$. On this topic, one can read [6] and the very clear book [15]. Another example of rigidity result in the context of Navier–Stokes equations can be found in [18, Lemma 3.14].

From a different perspective, one sees that bounded, divergence–free vector fields in $\mathbb{R}^n$ arise as generic blow-up limits of vector fields belonging to a larger class, denoted as $\mathcal{DM}^\infty$, and containing all bounded vector fields on $\mathbb{R}^n$ whose distributional divergence is a finite Radon measure. The class $\mathcal{DM}^\infty$ has been first considered by Anzellotti [3, 4] with the aim of extending the Gauss-Green formula, and

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then by Chen and Frid [7] with a similar purpose and in view of applications to hyperbolic systems of conservation laws. We recall that, after the seminal work of De Giorgi and Federer, the Gauss-Green formula has been extended in several directions, including weakly differentiable (or divergence–measure) vector fields defined on weakly regular sets, with applications to transport equations, continuum and fluid mechanics, capillarity, etc. See [1, 7, 8, 9, 10], as well as the more recent works [11, 12, 19, 21].

A key step in the aforementioned papers is represented by the construction of the *weak normal trace* $[\xi \cdot \nu_S]$ of a vector field $\xi \in \mathcal{DM}^\infty$ on a given, oriented and $\mathcal{H}^{n-1}$-rectifiable set S, that typically coincides, at least locally, with the (reduced) boundary of a domain $\Omega \subset \mathbb{R}^n$ with finite perimeter. It is worth recalling that this weak notion of normal trace of a vector field, together with the construction of pairings between functions and vector fields, first appeared in [4] in the special case of Ω being an open bounded set with Lipschitz boundary. In view of generalizing the Gauss-Green formula to wider classes of domains and of vector fields, some particular care has to be taken when the distributional divergence of the considered vector field is a Radon measure only on Ω (and not a–priori on the whole space $\mathbb{R}^n$). This is the reason why some special properties of the topological boundary of Ω, and in particular that the measure $\mathcal{H}^{n-1}$ restricted to $\partial\Omega$ is locally finite and coincide with the perimeter measure, must be required (see [19] and [21]).

From now on, any bounded open set Ω with finite perimeter, such that $\mathcal{H}^{n-1}(\partial\Omega) = P(\Omega)$, will be said *weakly regular*. We remark that this definition is slightly more general than the one proposed in [19], where a further property on Ω, namely a Poincaré-trace inequality, is required also for guaranteeing the continuity of the trace operator from $BV(\Omega)$ to $L^1(\partial\Omega)$. Thus, if one assumes that $\xi \in \mathcal{DM}^\infty(\Omega)$ and Ω is weakly regular, then there exists a function $[\xi \cdot \nu_\Omega] \in L^\infty(\partial\Omega; \mathcal{H}^{n-1})$, called the *weak normal trace* of ξ on $\partial\Omega$, such that for all $u \in C^1_c(\mathbb{R}^n)$ one has

$$\int_\Omega u\, d\,\mathrm{div}\,\xi + \int_\Omega \xi \cdot \nabla u\, dx = \int_{\partial\Omega} u\, [\xi \cdot \nu_\Omega]\, d\mathcal{H}^{n-1}\,. \tag{1}$$

Moreover, by relying on (1) one can define the weak normal trace $[\xi \cdot \nu_S]$ when $\xi \in \mathcal{DM}^\infty(\Omega)$ and $S \subset \Omega$ is an oriented $\mathcal{H}^{n-1}$-rectifiable set with locally finite measure, as described in [1] (see also [12]).

As observed in the example presented at the very beginning of Section 3, the weak normal trace does not coincide in general with any classical or measure–theoretic limit of the scalar product of the vector field with the normal to S, even when S is of class C^∞ and the vector field is smooth and divergence–free in a neighborhood of S (minus S). However, there is some special situation in which we expect the existence of the classical trace, and precisely when the weak normal trace attains a local maximum for the modulus of the vector field. More specifically, we take a vector field $\xi \in \mathcal{DM}^\infty$ and an oriented $\mathcal{H}^{n-1}$-rectifiable set S with $\mathcal{H}^{n-1}(S) < +\infty$, such that $[\xi \cdot \nu_S]$ is *locally maximal* at a so-called *qualified point* $x_0 \in S$. By locally maximal we mean that there exists an open set U containing x_0 such that $[\xi \cdot \nu_S](x_0) \geq |\xi(x)|$ for almost every $x \in U$. Then we say that $x_0 \in S$ is qualified if the approximate tangent space to S at x_0 is well-defined, if x_0 is a Lebesgue point for the weak normal trace on S, and if the divergence of the vector field does not concentrate around x_0 (see conditions (a), (b) and (c) in page 8). In this case one expects that the vector field ξ satisfies

$$\xi(x_0) = [\xi \cdot \nu_S](x_0)\ \nu_S(x_0) \quad \text{in the trace sense}\,. \tag{2}$$

Basically, this should happen because the vector field ξ should not oscillate too wildly around x_0 if its "pointwise flow" at x_0 is maximal. Not too surprisingly, this fact turns out to be closely related to the rigidity property stated in Question 0.1. We are able to affirmatively answer Question 0.1, and consequently prove (2), but only in dimension $n = 2$, see Theorem 2.1 (a) and Theorem 3.3. For $n \geq 3$ we obtain an affirmative answer to Question 0.1 by making some extra assumptions on the vector field z, see Theorem 2.1 (b). The validity of (2), as well as of Theorem 3.3, in the case $n \geq 3$ remains conjectural.

Concerning the maximality assumption on the weak normal trace, we mention that it occurs for instance when $\xi \in \mathcal{DM}^\infty(\Omega)$ is defined as

$$\xi = Tu = \frac{\nabla u}{\sqrt{1 + |\nabla u|^2}}$$

and $u : \Omega \to \mathbb{R}$ is the solution (up to vertical translations) of the Prescribed Mean Curvature (PMC) equation

$$\operatorname{div} Tu = H \qquad \text{on } \Omega$$

in the so-called *extremal* case, that is, when Ω is a minimal Cheeger set and $H = \frac{P(\Omega)}{|\Omega|}$ (see [19] for more details). Thanks to (2), we are thus able to conclude that, roughly speaking, the 2-dimensional capillary surface (the graph of u) produced by a perfectly wetting fluid that partially fills a cylindrical container of cross-section Ω in a gravity–free environment, necessarily "meets the regular part of the boundary of the container in a classical, tangential way", even though Ω is only weakly regular. In this sense, (2) allows us to complete, in the physically relevant 2-dimensional case, the characterization of extremality for the PMC equation obtained in [19, Theorem 4.1] for weakly regular domains, see our final Remark 3.4.

1. Preliminary notions and facts

We first introduce some basic notations. We fix $n \geq 2$ and denote by $\mathbb{R}^n$ the Euclidean n-space; by $\mathbb{R}^n_+$ the upper half–space $\{x \in \mathbb{R}^n : \ x_n > 0\}$; by $\mathbb{R}^n_0$ the boundary of $\mathbb{R}^n_+$. Let $E \subset \mathbb{R}^n$, then we denote by χ_E the characteristic function of E. For any $x \in \mathbb{R}^n$ and $r > 0$ we denote by $B_r(x)$ the Euclidean open ball of center x and radius r. Given a unit vector $v \in \mathbb{R}^n$ we set $B^v_r(z) = \{x \in B_r(z) \, : \, (x - z) \cdot v > 0\}$. We set $E_{x,r} = r^{-1}(E - x)$, where $E \subset \mathbb{R}^n$, $x \in \mathbb{R}^n$, and $r > 0$. Let $E \subset \Omega \subset \mathbb{R}^n$ with Ω open; we write $E \subset\subset \Omega$ whenever the topological closure of E, $\overline{E}$, is a compact subset of Ω. Given a Borel set E we denote by $|E|$ its n-dimensional Lebesgue measure. Whenever a measurable function, or vector field, f is defined on $\Omega \subset \mathbb{R}^n$, we denote by $\|f\|_\infty$ its L^∞-norm on Ω. Then, we denote by $\mathcal{DM}^\infty(\Omega)$ the space of bounded vector fields defined in Ω and whose divergence is a Radon measure. For brevity we shall write $\mathcal{DM}^\infty$ instead of $\mathcal{DM}^\infty(\mathbb{R}^n)$. It is convenient to consider the restriction to $\mathcal{DM}^\infty(\Omega)$ of the weak-$*$ topology of L^∞. Given a sequence $\{v_k\}_k$ of vector fields in $\mathcal{DM}^\infty(\Omega)$, we say that v_k converges to $v \in \mathcal{DM}^\infty(\Omega)$ in the weak-$*$ topology of L^∞, and write $v_k \rightharpoonup v$ in L^∞-w^*, if for every $f \in L^1(\Omega; \mathbb{R}^n)$ one has

$$\int_\Omega f \cdot v_k \, dx \to \int_\Omega f \cdot v \, dx, \qquad \text{as } k \to \infty\,.$$

We now recall some basic definitions and facts from Geometric Measure Theory and, in particular, from the theory of sets of locally finite perimeter.

Definition 1.1 (Points of density α)**.** Let E be a Borel set in $\mathbb{R}^n$, $x \in \mathbb{R}^n$. If the limit

$$\theta(E)(x) := \lim_{r \to 0^+} \frac{|E \cap B_r(x)|}{|B_r(x)|}$$

exists, it is called the density of E at x. In general $\theta(E)(x) \in [0, 1]$, hence, we define the set of points of density $\alpha \in [0, 1]$ for E as

$$E^{(\alpha)} := \{x \in \mathbb{R}^n \, : \, \theta(E)(x) = \alpha\}\ .$$

Definition 1.2 (One-sided approximate limit)**.** Let $\Omega \subset \mathbb{R}^n$ be an open set, and let $S \subset \Omega$ be an oriented $\mathcal{H}^{n-1}$-rectifiable set. Take a point $z \in S$ in which the "exterior normal" $\nu_S(z)$ exists, and choose a measurable function, or vector field, f defined on Ω. We write

$$\operatorname*{ap-lim}_{x \to z^-} f(x) = w$$

if for every $\alpha > 0$ the set $\{x \in \Omega \cap B_1^{-\nu_S}(z) :\ |f(x) - w| \geq \alpha\}$ has density 0 at z.

Definition 1.3 (Perimeter)**.** Let E be a Borel set in $\mathbb{R}^n$. We define the perimeter of E in an open set $\Omega \subset \mathbb{R}^n$ as

$$P(E;\Omega) := \sup\left\{ \int_\Omega \chi_E(x)\mathrm{div}g(x)\,dx\ : g \in C_c^1(\Omega;\mathbb{R}^n)\,, \|g\|_\infty \leq 1 \right\}.$$

We set $P(E) = P(E;\mathbb{R}^n)$. If $P(E;\Omega) < \infty$ we say that E is a set of finite perimeter in Ω. In this case (see [20]) one has that the perimeter of E coincides with the total variation $|D\chi_E|$ of the vector–valued Radon measure $D\chi_E$ (the distributional gradient of χ_E), which is defined for all Borel subsets of Ω thanks to Riesz Theorem.

Theorem 1.4 (De Giorgi Structure Theorem)**.** *Let E be a set of finite perimeter and let $\partial^* E$ be the reduced boundary of E defined as*

$$\partial^* E := \left\{ x \in \partial E\ :\ \lim_{r\to 0^+} \frac{D\chi_E(B_r(x))}{|D\chi_E|(B_r(x))} = -\nu_E(x) \in \mathbb{S}^{n-1} \right\}.$$

Then,

(i) *$\partial^* E$ is countably $\mathcal{H}^{n-1}$-rectifiable;*
(ii) *for all $x \in \partial^* E$, $\chi_{E_{x,r}} \to \chi_{H_{\nu_E(x)}}$ in $L^1_{loc}(\mathbb{R}^n)$ as $r \to 0^+$, where $H_{\nu_E(x)}$ denotes the half-space through 0 whose exterior normal is $\nu_E(x)$;*
(iii) *for any Borel set A, $P(E;A) = \mathcal{H}^{n-1}(A \cap \partial^* E)$, thus in particular $P(E) = \mathcal{H}^{n-1}(\partial^* E)$;*
(iv) *$\int_E \mathrm{div} g = \int_{\partial^* E} g \cdot \nu_E\, d\mathcal{H}^{n-1}$ for any $g \in C_c^1(\mathbb{R}^n;\mathbb{R}^n)$.*

For the sake of simplicity we recall a special version of the Gauss-Green formula, exploiting the pairing between $\mathcal{DM}^\infty$ and $BV(\Omega) \cap L^\infty(\Omega)$, where Ω is a bounded open set with finite perimeter (see [12] and references therein). We stress that replacing $\mathcal{DM}^\infty$ with the larger space $\mathcal{DM}^\infty(\Omega)$ would subordinate the validity of the Gauss-Green formula to some extra regularity property of Ω, like the weak regularity introduced in [19] (see also [21]).

Theorem 1.5 (Generalized Gauss-Green formula)**.** *Let $\Omega \subset \mathbb{R}^n$ be a weakly regular set. Then for any $\xi \in \mathcal{DM}^\infty(\Omega)$ and $u \in C_c^1(\mathbb{R}^n)$ one has*

$$\int_\Omega u\, d\,\mathrm{div}\,\xi + \int_\Omega \xi \cdot \nabla u\, dx = \int_{\partial^*\Omega} u\,[\xi\cdot\nu_\Omega]\, d\mathcal{H}^{n-1}\,,$$

where $[\xi \cdot \nu_\Omega]$ is the weak normal trace of ξ on $\partial^*\Omega$.

2. Rigidity results for divergence-free vector fields

Let z be a vector field in $L^\infty(\mathbb{R}^n_+;\mathbb{R}^n)$. Let $\varphi : [0,+\infty) \to [0,+\infty)$ be a continuous, strictly increasing and convex function, such that $\varphi(0) = 0$. Let $\omega : [0,+\infty)^2 \to [0,+\infty)$ be a continuous function, such that $\omega(r,t)$ is decreasing in r and increasing in t, and which satisfies

$$\Omega(t) := \int_0^{+\infty} r^{n-2}\omega(r,t)\,dr < +\infty, \qquad \lim_{t\to 0^+}\Omega(t) = 0\,. \tag{3}$$

Note that since $\omega(\cdot, t)$ is decreasing for every fixed $t \geq 0$, the finiteness of the integral $\Omega(t)$ implies that

$$\lim_{r\to+\infty} r^{n-2}\omega(r,t) = 0\,.$$

Indeed, one can assume for contradiction that $\omega(r_i, t) \geq c\, r_i^{2-n}$ for some positive constant c and for a diverging sequence $\{r_i\}_i$ such that $2r_i^{n-1} < r_{i+1}^{n-1}$, so that in particular $\omega(r,t) \geq c\, r_{i+1}^{2-n}$ for all $r \in (r_i, r_{i+1})$. Then

$$\infty > \int_0^{+\infty} r^{n-2}\omega(r,t)\,dr \geq \sum_i c\, r_{i+1}^{2-n} \int_{r_i}^{r_{i+1}} r^{n-2}\,dr = \frac{c}{n-1}\sum_i r_{i+1}^{2-n}(r_{i+1}^{n-1} - r_i^{n-1}) \geq \frac{c}{2(n-1)}\sum_i r_{i+1}$$

which is not possible as the last series is divergent.

We consider the following properties:

(i) $\operatorname{div} z = 0$ on $\mathbb{R}^n_+$;
(ii) $[z\cdot e_n] = 0$ on $\mathbb{R}^n_0$;
(iii) $z_n \geq \varphi(|z|)$ almost everywhere on $\mathbb{R}^n_+$;
(iv) $|z_n(y,t)| \leq \omega(|y|, t)$ for all $t > 0$ and $y \in \mathbb{R}^{n-1}$.

We observe that property (iii) could have been equivalently stated using a generic, continuous function φ such that $\varphi \geq 0$ and $\varphi(t) = 0$ if and only if $t = 0$. Indeed it is always possible to replace φ by a smaller function which satisfies the same properties of φ on some interval $[0, T]$ and, additionally, is increasing and convex on that interval.

Next, we focus on the following question, closely related to Question 0.1: *are the properties (i)–(iv), or even just (i)–(iii), enough to force the vector field z to be identically zero on $\mathbb{R}^n_+$?* As we shall see later on, the answer is affirmative when $n = 2$ and (i)–(iii) hold true, no matter which function φ as above we choose. This gives a complete, positive answer to Question 0.1 in the 2-dimensional case. Then, we prove that the same affirmative answer holds when $n \geq 2$, (i)–(iv) are verified, and $\varphi(t) = ct$ for some constant $c > 0$. We stress that the case $n = 2$ and $\varphi(t) = ct^2$ will play a key role in the proof of the main result in the next section. Concerning the case $n \geq 2$, we note that requiring (iii) with $\varphi(t) = ct$ is equivalent to ask that the vector field z takes its values in a strictly convex cone having $\mathbb{R}^n_0$ as a supporting hyperplane; at the same time, property (iv) guarantees a suitable quantitative decay of $|z(y,t)|$ when $|y| \to +\infty$.

Let us start extending the vector field $z = 0$ on $\mathbb{R}^n \setminus \mathbb{R}^n_+$, so that $z \in L^\infty(\mathbb{R}^n;\mathbb{R}^n)$, $[z\cdot e_n] = 0$ on $\mathbb{R}^n_0$, and consequently $\operatorname{div} z = 0$ on $\mathbb{R}^n$. Fix a standard mollifier ρ_ε supported in B_ε, and define $z^\varepsilon = z * \rho_\varepsilon$. First, we notice that $z_n \geq \varphi(|z|)$ almost everywhere on $\mathbb{R}^n$ thanks to (iii), therefore by Jensen's inequality we get for all $x \in \mathbb{R}^n$ and $\varepsilon > 0$

$$\begin{aligned}\varphi(|z^\varepsilon(x)|) = \varphi\left(\left|\int_{B_\varepsilon} z(y)\rho_\varepsilon(x-y)\,dy\right|\right) &\leq \int_{B_\varepsilon} \varphi(|z(y)|)\rho_\varepsilon(x-y)\,dy \\ &\leq \int_{B_\varepsilon} z_n(y)\rho_\varepsilon(x-y)\,dy = z_n^\varepsilon(x)\,. \end{aligned} \tag{4}$$

We observe that z^ε is Lipschitz-continuous, as a consequence of the boundedness of z, and verifies $\operatorname{div} z^\varepsilon = 0$ on $\mathbb{R}^n$. Moreover, for every $x = (y,t)$ such that $t \leq -\varepsilon$ one has $z^\varepsilon(x) = 0$. Then, up to a translation in the variable x, we can assume $z^\varepsilon(x) = 0$ for all $x \in \mathbb{R}^n \setminus \mathbb{R}^n_+$. Fix $h > 0$, write $z = (\zeta, z_n)$ and $z^\varepsilon = (\zeta^\varepsilon, z_n^\varepsilon)$, then define the cylinder

$$\mathcal{Q}(r,h) = \{(y,t) \in \mathbb{R}^{n-1}\times\mathbb{R} :\ |y| < r,\ 0 < t < h\}\,.$$

By the divergence theorem we have

$$0=\int_{\mathcal{Q}(r,h)}\operatorname{div} z^\varepsilon=\int_{\partial\mathcal{Q}(r,h)} z^\varepsilon\cdot\nu_{\mathcal{Q}(r,h)}=\int_{|y|<r} z_n^\varepsilon(y,h)\,dy+\int_0^h\int_{|y|=r}\zeta^\varepsilon(y,t)\cdot\frac{y}{r}\,d\mathcal{H}^{n-2}(y)dt\,. \tag{5}$$

Now we split the discussion in the two cases presented before.

Case $n=2$, (i)–(iii). Since $z_2^\varepsilon\ge 0$, by (5) we find

$$\int_{-r}^{r}|z_2^\varepsilon(x_1,t)|\,dx_1=\int_{-r}^{r} z_2^\varepsilon(x_1,t)\,dx_1=\int_0^t z_1^\varepsilon(-r,x_2)-z_1^\varepsilon(r,x_2)\,dx_2\ \le 2t\|z\|_\infty\,.$$

Therefore $z^\varepsilon(\cdot,t)\in L^1(\mathbb{R};\mathbb{R}^2)$ for all $t>0$ (we do not need property (iv) in this case!) and

$$\|z_2^\varepsilon(\cdot,t)\|_1\le 2t\|z\|_\infty\,. \tag{6}$$

On one hand, by combining (4) with (6) and the fact that z_2^ε is Lipschitz, we infer that

$$\varphi\Big(|z_1^\varepsilon(x_1,t)|\Big)\le z_2^\varepsilon(x_1,t)\to 0 \qquad \text{as } x_1\to\pm\infty,$$

hence, owing to the properties of φ, we get for all $t>\varepsilon$

$$z_1^\varepsilon(x_1,t)\to 0\qquad \text{as } x_1\to\pm\infty\,. \tag{7}$$

On the other hand, since $\operatorname{div} z^\varepsilon=0$ we obtain $\partial_{x_1}z_1^\varepsilon(x_1,t)=-\partial_{x_2}z_2^\varepsilon(x_1,t)$ and thus

$$\int_{y_1}^{x_1}\partial_{x_2}z_2^\varepsilon(s,t)\,ds=z_1^\varepsilon(y_1,t)-z_1^\varepsilon(x_1,t)\,.$$

By virtue of (7), we can take the limits in the last equation as $x_1\to+\infty$ and $y_1\to-\infty$, and obtain

$$\int_{-\infty}^{+\infty}\partial_{x_2}z_2^\varepsilon(s,t)\,ds\ =0\qquad \text{for all } t>\varepsilon\,.$$

We now show that

$$\frac{d}{dt}\int_{-\infty}^{+\infty}z_2^\varepsilon(s,t)\,ds=\int_{-\infty}^{+\infty}\partial_{x_2}z_2^\varepsilon(s,t)\,ds\ =0 \tag{8}$$

for any $t>0$. To this aim we fix $t_1>0$, $t\in[0,t_1]$, and $s\in\mathbb{R}$, then observe that for $x=(s,t)$ we have

$$\begin{aligned}|\partial_{x_2}z_2^\varepsilon(x)|&=\left|\int_{B_\varepsilon(x)}z_2(y_1,y_2)\partial_{y_2}\rho_\varepsilon(s-y_1,t-y_2)\,dy_1\,dy_2\right|\\&\le\int_{B_\varepsilon(x)}z_2(y_1,y_2)|\partial_{y_2}\rho_\varepsilon(s-y_1,t-y_2)|\,dy_1\,dy_2\\&\le c_\varepsilon\int_{B_\varepsilon(x)}z_2(y_1,y_2)\,dy_1\,dy_2=c_\varepsilon\int_{t-\varepsilon}^{t+\varepsilon}\int_{s-\varepsilon}^{s+\varepsilon}z_2(y_1,y_2)\,dy_1\,dy_2\\&\le c_\varepsilon\int_{-\varepsilon}^{t_1+\varepsilon}\int_{-\varepsilon}^{\varepsilon}z_2(y_1+s,y_2)\,dy_1\,dy_2=:g_\varepsilon(s)\,.\end{aligned}$$

Since $z_2^\varepsilon(y_1,y_2)=0$ when $y_2<0$, we have

$$\int_{-\infty}^{+\infty}g_\varepsilon(s)\,ds=2\varepsilon c_\varepsilon\int_{-\varepsilon}^{t_1+\varepsilon}\|z_2^\varepsilon(\cdot,y_2)\|_1\,dy_2\le 4\varepsilon c_\varepsilon\|z\|_\infty\int_0^{t_1+\varepsilon}y_2\,dy_2<+\infty\,.$$

Therefore we can differentiate under the integral sign and obtain (8) for $t\in(0,t_1)$. Since the choice of $t_1>0$ is arbitrary, the function $t\mapsto\|z_2^\varepsilon(\cdot,t)\|_1$ turns out to be constant when $t>0$. On the other hand, by (6) we necessarily conclude that $\|z_2^\varepsilon(\cdot,t)\|_1=0$ for all $t>0$, and thus that $z^\varepsilon=0$ on $\mathbb{R}^2$.

Case $n\ge 2$, (i)–(iv), $\varphi(t)=ct$. Thanks to $\operatorname{div}_y\zeta^\varepsilon(y,t)=-\partial_{x_n}z_n^\varepsilon(y,t)$ we have for all $t\in\mathbb{R}$

$$F_\varepsilon(r,t):=\int_{|y|<r}\partial_{x_n}z_n^\varepsilon(y,t)\,dy=-\int_{|y|<r}\operatorname{div}_y\zeta^\varepsilon(y,t)\,dy=-\int_{|y|=r}\zeta^\varepsilon(y,t)\cdot\frac{y}{r}\,d\mathcal{H}^{n-2}(y)\,. \tag{9}$$

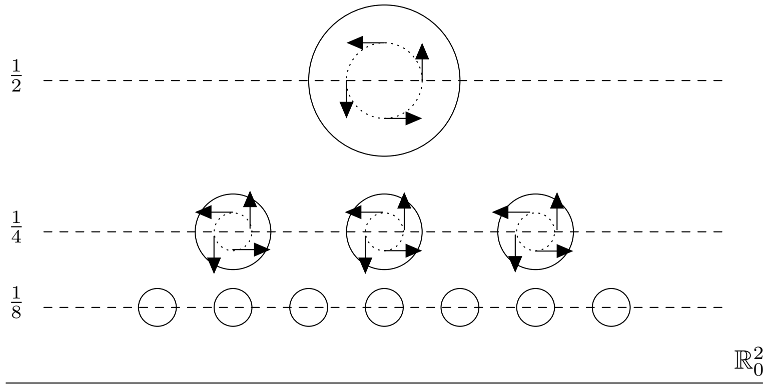

FIGURE 1. A twisting vector field defined on $\mathbb{R}^2_+$.

We now combine (iii), the choice $\varphi(t) = ct$, and (9), obtaining

$$|F_\varepsilon(r,t)| \le \int_{|y|=r} |\zeta^\varepsilon(y,t)|\, d\mathcal{H}^{n-2}(y) \le c^{-1} \int_{|y|=r} z_n^\varepsilon(y,t)\, d\mathcal{H}^{n-2}(y)\,. \tag{10}$$

Now using (iv) and the properties of $\omega(r,t)$ we conclude that the right-hand side of (10) tends to zero as $r \to +\infty$, hence that

$$\text{p.v.} \int_{y\in\mathbb{R}^{n-1}} \partial_{x_n} z_n^\varepsilon(y,t)\, dy = 0, \qquad \text{for all } t > 0\,.$$

By a similar argument as in the previous case (here we take advantage of the function $\omega(r,t)$ and its properties) we can differentiate under the integral sign and obtain

$$\frac{d}{dt}\int_{y\in\mathbb{R}^{n-1}} z_n^\varepsilon(y,t)\, dy = \text{ p.v.} \int_{y\in\mathbb{R}^{n-1}} \partial_{x_n} z_n^\varepsilon(y,t)\, dy = 0, \qquad \text{for all } t > 0\,. \tag{11}$$

We finally conclude as before on observing that $\|z_2^\varepsilon(\cdot,t)\|_1$ is constant in $t > 0$ and, at the same time, satisfies $\|z_2^\varepsilon(\cdot,t)\|_1 \le \Omega(t) \to 0$ as $t \to 0^+$. In conclusion we have proved the following result.

Theorem 2.1 (Rigidity)**.** *Let $n \ge 2$ and $z \in L^\infty(\mathbb{R}^n_+;\mathbb{R}^n)$. Assume that at least one of the following cases occurs:*

(a) *$n = 2$ and (i)–(iii) hold for a general choice of $\varphi(t)$;*
(b) *$n \ge 2$ and (i)–(iv) hold for $\varphi(t) = ct$ and for a general choice of $\omega(r,t)$.*

Then $z \equiv 0$ on $\mathbb{R}^n_+$.

3. The trace of a vector field with locally maximal normal trace

We start noticing that the weak normal trace $[\xi\cdot\nu_S]$ of a vector field ξ at an oriented $\mathcal{H}^{n-1}$-rectifiable set S does not necessarily coincide to any pointwise, almost-everywhere, or measure-theoretic limit of the scalar product $\xi(y)\cdot\nu_S(x)$, as $y \in B_1^{-\nu_S}(x)$ converges to x. To see this one can consider the following example. Let $\mathbb{R}^2_0$ be oriented in such a way that $-e_2$ is the outward normal. For $i \ge 1$ and $j = 1,\dots,2^i-1$ set $x_{ij} = \frac{j}{2^i}$, $y_i = \frac{1}{2^i}$, $r_i = \frac{1}{2^{i+2}}$. Then, for such i and j we take $\varphi_i \in C_c^\infty(\mathbb{R})$ with compact support in $(0,r_i)$, so that in particular $\varphi_i(0) = \varphi_i(r_i) = 0$, and define $p_{ij} = (x_{ij}, y_i) \in \mathbb{R}^2_+$. Notice that by our choice of parameters, the balls $\{B_{ij} = B_{r_i}(p_{ij})\}_{i,j}$ are pairwise disjoint. Whenever $p \in B_{ij}$ we set

$$\xi(p) = \varphi_i(|p-p_{ij}|)(p-p_{ij})^\perp\,,$$

while $\xi(p) = 0$ otherwise (see Figure 1). One can suitably choose φ_i so that $\|\xi\|_{L^\infty(B_{ij})} = 1$ for all i, j. Moreover $\operatorname{div}\xi = 0$ on $\mathbb{R}^2_+$ and thus for any $f \in C^1_c(\mathbb{R}^2)$, by the Gauss-Green formula and owing to the definition of ξ, one has

$$\int_{\mathbb{R}^2_0} f\,[\xi\cdot\nu]\,d\mathcal{H}^1 = \int_{\mathbb{R}^2_+} \xi\cdot Df = \sum_{i,j}\int_{B_{ij}} \xi\cdot Df = \sum_{i,j}\int_{\partial B_{ij}} f\,\xi\cdot\nu_{ij} = 0\,,$$

so that $[\xi\cdot\nu] = 0$ on $\mathbb{R}^2_0$. At the same time, ξ twists in any neighborhood of any point $p_0 = (x_0, 0)$, $x_0 \in (0,1)$. In conclusion, the scalar product $\xi(p)\cdot\nu(p_0)$ does not converge to 0 in any pointwise or measure-theoretic sense.

On the other hand, one might guess that this lack of convergence is mainly due to the averaging nature of the value of the weak normal trace, and that it should not occur when the weak normal trace attains an extremal value. To this aim we will employ the rigidity property for vector fields proved in the previous section in the 2-dimensional case, and show the following result. Whenever $n = 2$ and the weak normal trace is maximal at some point $x_0 \in S$ such that

(a) the normal vector $\nu_S(x_0)$ is defined at x_0;
(b) x_0 is a Lebesgue point for the weak normal trace of z on S, with respect to the measure $\mathcal{H}^{n-1}\llcorner S$;
(c) $|\operatorname{div} z|(B_r(x_0)\setminus S) = o(r^{n-1})$ as $r \to 0^+$;

then the vector field ξ admits a full, classical trace at x_0. Notice that the set of requests (a), (b), (c) is not stringent as by standard results (see [2, Theorem 2.56]) $\mathcal{H}^{n-1}$-almost every point $x_0 \in S$ satisfies these properties. In the following, any $x_0 \in S$ that satisfies (a), (b) and (c) will be called a *qualified* point of S.

Lemma 3.1. *Let $\eta, \{z_k\}_k$ be vector fields in $\mathcal{DM}^\infty$ with $\sup_k\|z_k\|_\infty < +\infty$. Let $\Sigma, \{S_k\}_k$ be oriented, closed $\mathcal{H}^{n-1}$–rectifiable sets with locally finite $\mathcal{H}^{n-1}$ measure, satisfying the following properties:*

(i) *$z_k \to \eta$ in L^∞–w^*;*
(ii) *$\mathcal{H}^{n-1}\llcorner S_k \rightharpoonup \mathcal{H}^{n-1}\llcorner\Sigma$;*
(iii) *$|\operatorname{div} z_k|\llcorner(\mathbb{R}^n\setminus S_k) \rightharpoonup 0$.*

Then, $\operatorname{div}\eta = 0$ *in* $\mathbb{R}^n\setminus\Sigma$.

Proof. Fix $\varphi \in C^1_c(\mathbb{R}^n\setminus\Sigma)$ and set $\mu_k = \operatorname{div} z_k$ and $\mu = \operatorname{div}\eta$. By the divergence theorem coupled with the formula (see [1, Proposition 3.2])

$$\mu_k\llcorner S_k = \Big([z_k\cdot\nu_{S_k}]^+ - [z_k\cdot\nu_{S_k}]^-\Big)\,\mathcal{H}^{n-1}\llcorner S_k\,,$$

we have

$$\begin{aligned}\int_{\mathbb{R}^n\setminus S_k}\varphi\,d\mu_k &= -\int_{S_k}\varphi\,d\mu_k - \int_{\mathbb{R}^n\setminus S_k}\nabla\varphi\cdot z_k\\ &= -\int_{S_k}\varphi\Big([z_k\cdot\nu_{S_k}]^+ - [z_k\cdot\nu_{S_k}]^-\Big)\,d\mathcal{H}^{n-1} - \int_{\mathbb{R}^n}\nabla\varphi\cdot z_k\,,\end{aligned}$$

hence by (ii) and (iii)

$$\left|\int_{\mathbb{R}^n}\nabla\varphi\cdot z_k\right| \le \int_{\mathbb{R}^n\setminus S_k}|\varphi|\,d\mu_k + 2\|z_k\|_\infty\int|\varphi|\,d\,\mathcal{H}^{n-1}\llcorner S_k \to 0 \qquad \text{as } k\to\infty\,.$$

This shows that

$$\int_{\mathbb{R}^n}\varphi\,d\mu = \lim_k\int_{\mathbb{R}^n}\varphi\,d\mu_k = \lim_k\int_{\mathbb{R}^n}\nabla\varphi\cdot z_k = 0\,,$$

which proves the thesis. □

Proposition 3.2. *Let $z \in \mathcal{DM}^\infty$ and let S be a closed, oriented $\mathcal{H}^{n-1}$–rectifiable set with locally finite $\mathcal{H}^{n-1}$ measure. Then, for $\mathcal{H}^{n-1}$–almost all $x_0 \in S$ and for any decreasing and infinitesimal sequence $\{r_k\}_k$, the sequence z_k of vector fields defined by $z_k(y) = z(x_0 + r_k y)$ converges up to subsequences to a vector field $\eta \in \mathcal{DM}^\infty$ in L^∞–w^*, such that setting $\Sigma = [\nu_S(x_0)]^\perp$, we have $\operatorname{div}\eta = 0$ on $\mathbb{R}^n \setminus \Sigma$ and $[\eta \cdot \nu_\Sigma] = [z \cdot \nu_S](x_0)$ on Σ.*

Proof. Without loss of generality we can assume that x_0 is a qualified point, i.e., that (a), (b) and (c) hold. We now show that hypotheses (i)–(iii) of Lemma 3.1 are satisfied. Since $\|z_k\|_\infty = \|z\|_\infty$ for all k, thanks to Banach-Alaoglu Theorem (see also Theorem 3.28 of [5]) we can extract a not relabeled subsequence converging to $\eta \in \mathcal{DM}^\infty$, which gives (i). We set $S_k = r_k^{-1}(S - x_0)$, then thanks to (c) we have

$$|\operatorname{div} z_k|(B_R \setminus S_k) = r_k^{1-n}|\operatorname{div} z|(B_{Rr_k}(x_0) \setminus S) \to 0 \qquad \text{as } k \to \infty\,, \tag{12}$$

for all $R > 0$, which gives (iii). Owing to the localization property proved in [1, Proposition 3.2] we can replace S with the boundary of an open set Ω of class C^1, such that $x_0 \in \partial\Omega$ and $\nu_S(x_0) = \nu_\Omega$. Defining $\Omega_k = r_k^{-1}(\Omega - x_0)$, the proof of (ii) is reduced to showing that $\mathcal{H}^{n-1} \llcorner \partial\Omega_k$ weakly converge as measures to $\mathcal{H}^{n-1} \llcorner \partial H$, where H is the tangent half-space to Ω at x_0 (so that $\Sigma = \partial H$). This fact is a consequence of Theorem 1.4. Now we can apply Lemma 3.1 and obtain $\operatorname{div}\eta = 0$ on $\mathbb{R}^n \setminus \Sigma$. In order to prove the last part of the statement we have to show that for any $\varphi \in C_c^1(\mathbb{R}^n)$ one has

$$\int_H \varphi \, d\operatorname{div}\eta + \int_H \nabla\varphi \cdot \eta = \tau_0 \int_{\partial H} \varphi \, d\mathcal{H}^{n-1}\,,$$

where $\tau_0 = [z \cdot \nu_S(x_0)]$. Since we have proved that $\operatorname{div}\eta = 0$ on $\mathbb{R}^n \setminus \Sigma$, we only have to show that

$$\int_H \nabla\varphi \cdot \eta = \tau_0 \int_{\partial H} \varphi \, d\mathcal{H}^{n-1}\,. \tag{13}$$

Note that by the convergence of z_k to η, the L^1_{loc}-convergence of Ω_k to H, and the convergence of the measures $\mathcal{H}^{n-1} \llcorner \partial\Omega_k$ to $\mathcal{H}^{n-1} \llcorner \partial H$, we have

$$\int_H \nabla\varphi \cdot \eta - \tau_0 \int_{\partial H} \varphi \, d\mathcal{H}^{n-1} = \lim_k \int_{\Omega_k} \nabla\varphi \cdot z_k - \tau_0 \int_{\partial\Omega_k} \varphi \, d\mathcal{H}^{n-1}\,. \tag{14}$$

Therefore by (12) and (b) we infer that

$$\begin{aligned}
\left| \int_{\Omega_k} \nabla\varphi \cdot z_k - \tau_0 \int_{\partial\Omega_k} \varphi \, d\mathcal{H}^{n-1} \right| &\le \left| \int_{\partial\Omega_k} \varphi \left([z_k \cdot \nu_{\Omega_k}] - \tau_0 \right) d\mathcal{H}^{n-1} - \int_{\Omega_k} \varphi \, d\operatorname{div} z_k \right| \\
&\le \|\varphi\|_\infty \int_{\partial\Omega_k \cap \operatorname{spt}\varphi} |[z_k \cdot \nu_{\Omega_k}] - \tau_0| + \int_{\mathbb{R}^n \setminus \partial\Omega_k} |\varphi| \, |\operatorname{div} z_k| \\
&\to 0, \qquad \text{as } k \to \infty\,.
\end{aligned}$$

Combining this last fact with (14) implies (13) at once, and concludes the proof. □

Theorem 3.3. *Let $\xi \in \mathcal{DM}^\infty(\mathbb{R}^2)$. Let S be a closed, oriented $\mathcal{H}^1$-rectifiable set with locally bounded $\mathcal{H}^1$ measure. Suppose that $[\xi \cdot \nu_S](x_0) = \|\xi\|_\infty$ at a qualified point $x_0 \in S$. Then one has*

$$\operatorname*{ap-lim}_{x \to x_0^-} \xi(x) = \|\xi\|_\infty \nu_S(x_0)\,.$$

Proof. Without loss of generality, up to a translation we can suppose $x_0 = 0$ and up to a rotation that $\nu_S(x_0) = -e_2$. Moreover, up to rescaling ξ we can suppose $\|\xi\|_\infty = 1$. Let

$$B_1^+ = B_1 \cap \mathbb{R}^2_+\,.$$

We then want to show that the set

$$N_\alpha := \{x \in B_1^+ \,:\, |\xi(x) + e_2| \geq \alpha\} \tag{15}$$

has density zero at 0 for all $\alpha > 0$. Argue by contradiction and suppose there exist $\alpha, \beta > 0$ and a sequence of radii $\{r_k\}_k$ decreasing to zero, such that

$$\frac{|N_\alpha \cap B_{r_k}|}{\pi r_k^2} \geq \beta \qquad \text{for all } k\,. \tag{16}$$

Define $z_0(x) := \xi(x) + e_2$ and the sequence $z_k(y) = z_0(r_k y)$ for $k \in \mathbb{N}$. Since the second component of z_k is $z_{k,2}(x) = \xi_2(r_k x) + 1$ one easily sees that

$$z_{k,2}(x) \geq \frac{|z_k(x)|^2}{2}\,, \tag{17}$$

for almost every $x \in \mathbb{R}^2$. By the definition of N_α and by (17), the contradiction hypothesis (16) reads equivalently as

$$\left|\left\{x \in r_k^{-1} B_1^+ \,:\, z_{k,2}(x) > \frac{\alpha^2}{2}\right\} \cap B_1\right| \geq \pi\beta\,. \tag{18}$$

On top of that, $z_0 \in \mathcal{DM}^\infty$ with $\|z_0\|_\infty \leq 2$. By Proposition 3.2 the sequence z_k defined above converges in L^∞-w^* (up to subsequences, we do not relabel) to a vector field η such that $\mathrm{div}(\eta) = 0$ on $\mathbb{R}^2_+$ and $[\eta \cdot (-e_2)] = [z_0 \cdot (-e_2)](0)$ on $\mathbb{R}^2_0$. We aim to show that η satisfies the hypotheses (i)-(iii) of Theorem 2.1. Were this the case, one would conclude $\eta \equiv 0$ in $\mathbb{R}^2_+$ and this would yield a contradiction with (18). Indeed, taking $\chi_{B_1^+}$ as a test function we get

$$\pi\frac{\alpha^2\beta}{2} \leq \int_{B_1^+} z_{k,2} \xrightarrow[k]{} \int_{B_1^+} \eta_2\,. \tag{19}$$

On one hand, we know that hypothesis (i) of Theorem 2.1 is satisfied as $\mathrm{div}(\eta) = 0$ on $\mathbb{R}^2_+$. On the other hand, as $[\eta \cdot \nu] = -[z_0 \cdot e_2](0)$ on $\mathbb{R}^2_0$, we get that

$$[\eta \cdot \nu] = -[\xi \cdot e_2](0) - e_2 \cdot e_2 = \|\xi\|_\infty - 1 = 0\,,$$

so that hypothesis (ii) of Theorem 2.1 holds as well. We are left to show that hypothesis (iii) of Theorem 2.1 is satisfied. Since z_k is equibounded, by (17) we infer as well that $|z_k|^2$ converges (up to subsequences, we do not relabel) to some function ζ in L^∞-w^*. Clearly, one has from (17) and the weak-$*$ convergence of z_k and of $|z_k|^2$ that $\zeta \leq 2\eta_2$ almost everywhere. We want to prove that the same holds with $|\eta|^2$ in place of ζ so to retrieve hypothesis (ii) of Theorem 2.1 with the choice $\varphi(t) = t^2/2$. Take a probability measure $f\,dx$ with $f \in L^1(\mathbb{R}^2)$. Then, by Jensen's inequality

$$\int |z_k|^2 f\,dx \geq \left|\int z_k f\,dx\right|^2 = \left(\int z_{k,1} f\,dx\right)^2 + \left(\int z_{k,2} f\,dx\right)^2.$$

As $k \to \infty$, by the weak-$*$ convergence we get

$$\int \zeta f\,dx \geq \left(\int \eta_1 f\,dx\right)^2 + \left(\int \eta_2 f\,dx\right)^2.$$

Thus, by letting $f\,dx$ toward the Dirac measure centered at x, (ii) follows at once for almost every point x (more precisely, x must be a Lebesgue point for the functions ζ, η_1, η_2). A direct application of Theorem 2.1 yields the desired contradiction. □

Remark 3.4. An application of Theorem 3.3 can be found in the framework of capillarity. In [19] we considered the prescribed mean curvature equation

$$\operatorname{div}(Tu) = \operatorname{div}\left(\frac{\nabla u}{\sqrt{1+|\nabla u|^2}}\right) = H \qquad \text{in } \Omega\,, \tag{PMC}$$

where Ω is a *weakly regular* set, i.e., it is an open bounded set satisfying

$$P(\Omega) = \mathcal{H}^{n-1}(\partial\Omega) \tag{20}$$

together with the Poincaré trace inequality

$$\min\{P(E;\partial\Omega), P(\Omega\setminus E;\partial\Omega)\} \le k\, P(E;\Omega)\,.$$

A necessary and sufficient condition for the existence of solutions u is the validity of the inequality

$$\left|\int_A H\,dx\right| < P(A)\,,$$

for all proper subsets A of Ω. Moreover (see [19, Theorem 4.1]) we also proved that the following conditions are equivalent:

- **(E)** *(Extremality)* The pair (Ω, H) satisfies $\left|\int_\Omega H\,dx\right| = P(\Omega)$.
- **(U)** *(Uniqueness)* The solution of (PMC) is unique up to vertical translations.
- **(M)** *(Maximality)* Ω is maximal, i.e. no solution of (PMC) can exist in any domain strictly containing Ω.
- **(V)** *(weak Verticality)* There exists a solution u of (PMC) which is weakly vertical at $\partial\Omega$, i.e.

$$[Tu\cdot\nu] = 1\,, \qquad \mathcal{H}^{n-1}\text{-a.e. on } \partial\Omega\,,$$

  where $[Tu\cdot\nu]$ is the weak normal trace of Tu on $\partial\Omega$.
- **(V')** *(integral Verticality)* There exists a solution u of (PMC) and a sequence $\{\Omega_i\}_i$ of smooth subdomains, such that $\Omega_i \subset\subset \Omega$, $|\Omega\setminus\Omega_i| \to 0$, $P(\Omega_i) \to P(\Omega)$, and

$$\lim_{i\to\infty} \int_{\partial\Omega_i} Tu(x)\cdot\nu\, d\mathcal{H}^{n-1} = P(\Omega),$$

  as $i \to \infty$.

In the physically relevant case $n = 2$ (that is, when the graph of u is a 2-dimensional surface in the cylinder $\Omega\times\mathbb{R}$), by letting $S = \partial^*\Omega$, which coincides with $\partial\Omega$ up to $\mathcal{H}^1$ negligible sets due to (20), and by noticing that $|Tu| < 1$ inside Ω, we can apply Theorem 3.3 to Tu (extended to zero outside Ω) and obtain that (V) is equivalent to the following, classical condition

- **($\tilde{\mathbf{V}}$)** *(essential Verticality)* There exists a solution u of (PMC) which is essentially vertical at $\partial\Omega$, i.e. $Tu = \nu_\Omega$ $\mathcal{H}^1$-almost everywhere on $\partial\Omega$ in the classical trace sense.

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