ANZELLOTTI'S PAIRING THEORY AND THE GAUSS–GREEN THEOREM

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Abstract. In this paper we obtain a very general Gauss-Green formula for weakly differentiable functions and sets of finite perimeter. This result is obtained by revisiting the Anzellotti's pairing theory and by characterizing the measure pairing $(\boldsymbol{A}, Du)$ when $\boldsymbol{A}$ is a bounded divergence measure vector field and u is a bounded function of bounded variation.

1. Introduction

In the pioneering paper [6], Anzellotti established a pairing theory between weakly differentiable vector fields and BV functions. Among other applications that will be mentioned below, this theory can be used to extend the validity of the Gauss–Green formula to such vector fields and to non smooth domains.

As a means of comparison, there are mainly two kinds of generalizations of the Gauss–Green formula. On one hand, one may consider weakly differentiable vector fields but fairly regular (e.g. Lipschitz) domains, see e.g. [10]. On the other hand, starting from the fundamental results of De Giorgi and Federer (see e.g. [3, Theorem 3.36]), many other generalizations mainly concern regular vector fields but only weakly regular domains (i.e., sets of finite perimeter), see e.g. [11, 14–16, 28].

In this paper we will prove a Gauss–Green formula valid for weakly differentiable vector fields and weakly regular domains. This unifying result is obtained by revisiting the Anzellotti's pairing theory in the general case of divergence measure vector fields and BV functions. The core of the work is the characterization of the normal traces of these vector fields and the analysis of the singular part of the pairing measure. This will allow us to establish some nice formulas (coarea, chain rule, Leibnitz rule) for the pairing and, eventually, to prove our general Gauss–Green formula. We mention that, with our approach, no approximation step with smooth fields or smooth subdomains, in the spirit of [7] and [11], is needed. On top of that, our feeling is that the approximation with smooth fields may not work in our framework (see the discussion before Proposition 4.15).

Let us describe in more details the functional setting of the problem. Let $\mathcal{DM}^\infty$ denote the class of bounded divergence measure vector fields $\boldsymbol{A}\colon \mathbb{R}^N \to \mathbb{R}^N$, i.e. the vector fields with the properties $\boldsymbol{A} \in L^\infty$ and $\operatorname{div} \boldsymbol{A}$ is a finite Radon measure. If $\boldsymbol{A} \in \mathcal{DM}^\infty$ and u is a function of bounded variation with precise representative u^*, then the distribution $(\boldsymbol{A}, Du)$, defined by

$$(1) \qquad \langle (\boldsymbol{A}, Du)\,,\,\varphi\rangle := -\int_{\mathbb{R}^N} u^*\varphi\, d\operatorname{div}\boldsymbol{A} - \int_{\mathbb{R}^N} u\,\boldsymbol{A}\cdot\nabla\varphi\, dx, \qquad \varphi \in C_c^\infty(\mathbb{R}^N)$$

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is a Radon measure in $\mathbb{R}^N$, absolutely continuous with respect to $|Du|$. This fact has been proved by Anzellotti in [6] for several combinations of $\boldsymbol{A}$ and u (for instance div $\boldsymbol{A} \in L^1$ or u a BV continuous function), excluding the general case of $\boldsymbol{A} \in \mathcal{DM}^\infty$ and $u \in BV$. Indeed, at that time, it was not clear how the discontinuities of u interact with the discontinuities of the vector field $\boldsymbol{A}$. The pairing (1) has been defined in the general setting by Chen–Frid in the celebrated paper [11], where the authors also characterized the absolutely continuous part of the measure $(\boldsymbol{A}, Du)$ as $\boldsymbol{A} \cdot \nabla u$. Nevertheless, they have not characterized the singular part of the measure, and, as far as we know, this problem has remained unsolved, at least in this general setting.

On the other hand, the pairing in its full generality has been revealed a fundamental tool in several contexts. We cite, for example, [11–15, 17, 30] for applications in the theory of hyperbolic systems of conservation and balance laws, and [1] for the case of vector fields induced by functions of bounded deformation, with the aim of extending the Ambrosio–DiPerna–Lions theory of the transport equations (see also [2]).

The divergence measure vector fields play a crucial role also in the theory of capillarity and in the study of the Prescribed Mean Curvature problem (see e.g. [28] and the references therein), and in the context of continuum mechanics (see e.g. [22, 33, 34]).

Another field of application is related to the Dirichlet problem for equations involving the 1–Laplacian operator (see [4, 10, 25, 26, 32]). The interest in this setting comes out from an optimal design problem, in the theory of torsion and from the level set formulation of the Inverse Mean Curvature Flow. To deal with the 1–Laplacian $\Delta_1 u := \operatorname{div}\left(\frac{Du}{|Du|}\right)$, the main difficulty is to define the quotient $\frac{Du}{|Du|}$, being Du a Radon measure. This difficulty has been overcome in [4, 5] through the Anzellotti's theory of pairings. Namely, the role of this quotient is played by a vector field $\boldsymbol{A} \in \mathcal{DM}^\infty$ such that $\|\boldsymbol{A}\|_\infty \leq 1$ and $(\boldsymbol{A}, Du) = |Du|$.

Finally, in some lower semicontinuity problems for integral functionals defined in Sobolev spaces and in BV, the vector fields with measure–derivative occurred as natural dependence of the integrand with respect to the spatial variable (see [8, 19, 21]).

Let us now describe in more details the results proved in this paper.

Our first aim is to characterize the measure $(\boldsymbol{A}, Du)$ in the general case $\boldsymbol{A} \in \mathcal{DM}^\infty$ and $u \in BV$. As we have already recalled above, the absolutely continuous part of $(\boldsymbol{A}, Du)$ has been characterized in [11] as $\boldsymbol{A} \cdot \nabla u$, hence only the jump and the Cantor parts have to be studied.

The analysis of the jump part of the pairing requires, in particular, a detailed study of the normal traces of $u\boldsymbol{A}$ on a countably $\mathcal{H}^{N-1}$-rectifiable set Σ. Following the arguments in [1], in Proposition 3.1 below we will prove that, if $\boldsymbol{A} \in \mathcal{DM}^\infty$ and $u \in BV \cap L^\infty$, then $u\boldsymbol{A} \in \mathcal{DM}^\infty$ and the normal traces of $u\boldsymbol{A}$ on Σ are given by

$$\operatorname{Tr}^\pm(u\boldsymbol{A}, \Sigma) = u^\pm \operatorname{Tr}^\pm(\boldsymbol{A}, \Sigma), \qquad \mathcal{H}^{N-1} - \text{a.e. in } \Sigma.$$

This allows us to give a precise description of the jump part $(\boldsymbol{A}, Du)^j$ of the measure $(\boldsymbol{A}, Du)$ in terms of the trace of u and the normal trace of $\boldsymbol{A}$.

Under the additional assumption $|D^c u|(S_{\boldsymbol{A}}) = 0$, where $D^c u$ is the Cantor part of Du and $S_{\boldsymbol{A}}$ is the approximate discontinuity set of $\boldsymbol{A}$, we are able to give a representation formula for the Cantor part $(\boldsymbol{A}, Du)^c$ of the pairing measure. In Remark 3.4 we will discuss some cases of interest where this condition is satisfied.

In conclusion, in Section 3 we will prove that the measure $(\boldsymbol{A}, Du)$ admits the following decomposition:

(i) absolutely continuous part: $(\boldsymbol{A}, Du)^a = \boldsymbol{A} \cdot \nabla u \, \mathcal{L}^N$;

(ii) jump part: $(\boldsymbol{A}, Du)^j = \dfrac{\mathrm{Tr}^+(\boldsymbol{A}, J_u) + \mathrm{Tr}^-(\boldsymbol{A}, J_u)}{2}\,(u^+ - u^-)\,\mathcal{H}^{N-1} \llcorner J_u$;

(iii) Cantor part: if $|D^c u|(S_{\boldsymbol{A}}) = 0$, then $(\boldsymbol{A}, Du)^c = \widetilde{\boldsymbol{A}} \cdot D^c u$.

In Section 4, by using the above decomposition, we will be able to describe the Radon–Nikodým derivative of the measure $(\boldsymbol{A}, Du)$ with respect to $|Du|$. As a consequence, we will prove the coarea formula, the chain rule for the pairing $(\boldsymbol{A}, Dh(u))$, and the Leibniz formula for $(\boldsymbol{A}, D(uv))$ and $(v\boldsymbol{A}, Du)$.

Finally, in Section 5, exploiting the formulas proved in Section 4, we will prove our generalized Gauss-Green formula: if $\boldsymbol{A} \in \mathcal{DM}^\infty$, $u \in BV \cap L^\infty$, and $E \subset \mathbb{R}^N$ is a bounded set with finite perimeter, then

$$\int_{E^1} u^*\, d\operatorname{div}\boldsymbol{A} + \int_{E^1} (\boldsymbol{A}, Du) = -\int_{\partial^* E} u^+\ \mathrm{Tr}^+(\boldsymbol{A}, \partial^* E)\ d\mathcal{H}^{N-1}\,, \tag{2}$$

$$\int_{E^1 \cup \partial^* E} u^*\, d\operatorname{div}\boldsymbol{A} + \int_{E^1 \cup \partial^* E} (\boldsymbol{A}, Du) = -\int_{\partial^* E} u^-\ \mathrm{Tr}^-(\boldsymbol{A}, \partial^* E)\ d\mathcal{H}^{N-1}\,, \tag{3}$$

where E^1 is the measure theoretic interior of E, $\partial^* E$ is the reduced boundary of E and $\partial^* E$ is oriented with respect to the interior unit normal vector.

As we have already underlined in this introduction, a number of Gauss–Green formulas that can be found in the literature are a particular case of (2) and (3).

For example, the case $u \equiv 1$ has been considered in the classical De Giorgi–Federer formula with $\boldsymbol{A}$ a regular vector field (see e.g. [3, Theorem 3.36]), by Vol'pert [35, 36] for $\boldsymbol{A} \in BV(\Omega, \mathbb{R}^N)$ and finally by Chen–Torres–Ziemer [15] in the general case $\boldsymbol{A} \in \mathcal{DM}^\infty$.

The case of a non-constant u has been considered by Anzellotti [7] if $\operatorname{div}\boldsymbol{A} \in L^1$, by Comi–Payne [16] if u is a locally Lipschitz function, and by Leonardi–Saracco if $\boldsymbol{A} \in \mathcal{DM}^\infty \cap C^0$ (with some additional conditions on E).

2. Preliminaries

In the following Ω will always denote a nonempty open subset of $\mathbb{R}^N$.

Let $u \in L^1_{\mathrm{loc}}(\Omega)$. We say that u has an approximate limit at $x_0 \in \Omega$ if exists $z \in \mathbb{R}$ such that

$$\lim_{r \to 0^+} \frac{1}{\mathcal{L}^N\left(B_r(x_0)\right)} \int_{B_r(x_0)} |u(x) - z|\ dx = 0. \tag{4}$$

The set $S_u \subset \Omega$ of points where this property does not hold is called the approximate discontinuity set of u. For every $x_0 \in \Omega \setminus S_u$ the number z, uniquely determined by (4), is called the approximate limit of u at x_0 and denoted by $\widetilde{u}(x_0)$.

We say that $x_0 \in \Omega$ is an approximate jump point of u if there exist $a, b \in \mathbb{R}$ and a unit vector $\nu \in \mathbb{R}^n$ such that $a \neq b$ and

$$\begin{aligned} &\lim_{r \to 0^+} \frac{1}{\mathcal{L}^N(B_r^+(x_0))} \int_{B_r^+(x_0)} |u(y) - a|\, dy = 0, \\ &\lim_{r \to 0^+} \frac{1}{\mathcal{L}^N(B_r^-(x_0))} \int_{B_r^-(x_0)} |u(y) - b|\, dy = 0, \end{aligned} \tag{5}$$

where $B_r^\pm(x_0) := \{y \in B_r(x_0) :\ \pm(y - x_0) \cdot \nu > 0\}$. The triplet (a, b, ν), uniquely determined by (5) up to a permutation of (a, b) and a change of sign of ν, is denoted by $(u^+(x_0), u^-(x_0), \nu_u(x_0))$. The set of approximate jump points of u will be denoted by J_u.

The notions of approximate discontinuity set, approximate limit and approximate jump point can be obviously extended to the vectorial case (see [3, §3.6]).

In the following we shall always extend the functions $u^\pm$ to $\Omega \setminus (S_u \setminus J_u)$ by setting

$$u^\pm \equiv \widetilde{u} \text{ in } \Omega \setminus S_u.$$

In some occasions it will be useful to choose the orientation of ν in such a way that $u^- < u^+$ in J_u. These particular choices of u^- and u^+ will be called the approximate lower limit and the approximate upper limit of u respectively.

Here and in the following we will denote by $\rho \in C_c^\infty(\mathbb{R}^N)$ a symmetric convolution kernel with support in the unit ball, and by $\rho_\varepsilon(x) := \varepsilon^{-N}\rho(x/\varepsilon)$.

In the sequel we will use often the following result.

Proposition 2.1. *Let $u \in L^1_{\rm loc}(\Omega)$ and define*

$$u_\varepsilon(x) = \rho_\varepsilon * u(x) := \int_\Omega \rho_\varepsilon(x-y)\, u(y)\, dy.$$

(i) *If $x_0 \in \Omega \setminus S_u$, then $u_\varepsilon(x_0) \to \widetilde{u}(x_0)$ as $\varepsilon \to 0^+$.*
(ii) *If u is differentiable at $x_0 \in \mathbb{R}^N$, then $\nabla u_\varepsilon(x_0) \to \nabla u(x_0)$ as $\varepsilon \to 0^+$.*

2.1. **Functions of bounded variation and sets of finite perimeter.** We say that $u \in L^1(\Omega)$ is a function of bounded variation in Ω if the distributional derivative Du of u is a finite Radon measure in Ω. The vector space of all functions of bounded variation in Ω will be denoted by $BV(\Omega)$. Moreover, we will denote by $BV_{\rm loc}(\Omega)$ the set of functions $u \in L^1_{\rm loc}(\Omega)$ that belongs to $BV(A)$ for every open set $A \Subset \Omega$ (i.e., the closure $\overline{A}$ of A is a compact subset of Ω).

If $u \in BV(\Omega)$, then Du can be decomposed as the sum of the absolutely continuous and the singular part with respect to the Lebesgue measure, i.e.

$$Du = D^a u + D^s u, \qquad D^a u = \nabla u\, \mathcal{L}^N,$$

where ∇u is the approximate gradient of u, defined $\mathcal{L}^N$-a.e. in Ω. On the other hand, the singular part $D^s u$ can be further decomposed as the sum of its Cantor and jump part, i.e.

$$D^s u = D^c u + D^j u, \qquad D^c u := D^s u \llcorner (\Omega \setminus S_u), \quad D^j u := D^s u \llcorner J_u,$$

where the symbol $\mu \llcorner B$ denotes the restriction of the measure μ to the set B. We will denote by $D^d u := D^a u + D^c u$ the diffuse part of the measure Du.

In the following, we will denote by $\theta_u \colon \Omega \to S^{N-1}$ the Radon–Nikodým derivative of Du with respect to $|Du|$, i.e. the unique function $\theta_u \in L^1(\Omega, |Du|)^N$ such that the polar decomposition $Du = \theta_u\, |Du|$ holds. Since all parts of the derivative of u are mutually singular, we have

$$D^a u = \theta_u\, |D^a u|, \quad D^j u = \theta_u\, |D^j u|, \quad D^c u = \theta_u\, |D^c u|$$

as well. In particular $\theta_u(x) = \nabla u(x)/|\nabla u(x)|$ for $\mathcal{L}^N$-a.e. $x \in \Omega$ such that $\nabla u(x) \neq 0$ and $\theta_u(x) = \operatorname{sign}(u^+(x) - u^-(x))\, \nu_u(x)$ for $\mathcal{H}^{N-1}$-a.e. $x \in J_u$.

Let E be an $\mathcal{L}^N$-measurable subset of $\mathbb{R}^N$. For every open set $\Omega \subset \mathbb{R}^N$ the perimeter $P(E, \Omega)$ is defined by

$$P(E,\Omega) := \sup\left\{ \int_E \operatorname{div}\varphi\, dx : \ \varphi \in C_c^1(\Omega, \mathbb{R}^N),\ \|\varphi\|_\infty \leq 1 \right\}.$$

We say that E is of finite perimeter in Ω if $P(E, \Omega) < +\infty$.

Denoting by χ_E the characteristic function of E, if E is a set of finite perimeter in Ω, then $D\chi_E$ is a finite Radon measure in Ω and $P(E, \Omega) = |D\chi_E|(\Omega)$.

If $\Omega \subset \mathbb{R}^N$ is the largest open set such that E is locally of finite perimeter in Ω, we call reduced boundary $\partial^* E$ of E the set of all points $x \in \Omega$ in the support of $|D\chi_E|$ such that the limit

$$\widetilde{\nu}_E(x) := \lim_{\rho \to 0^+} \frac{D\chi_E(B_\rho(x))}{|D\chi_E|(B_\rho(x))}$$

exists in $\mathbb{R}^N$ and satisfies $|\widetilde{\nu}_E(x)| = 1$. The function $\widetilde{\nu} \colon \partial^* E \to S^{N-1}$ is called the measure theoretic unit interior normal to E.

A fundamental result of De Giorgi (see [3, Theorem 3.59]) states that $\partial^* E$ is countably $(N-1)$-rectifiable and $|D\chi_E| = \mathcal{H}^{N-1} \llcorner \partial^* E$.

Let E be an $\mathcal{L}^N$-measurable subset of $\mathbb{R}^N$. For every $t \in [0,1]$ we denote by E^t the set

$$E^t := \left\{ x \in \mathbb{R}^N : \lim_{\rho \to 0^+} \frac{\mathcal{L}^N(E \cap B_\rho(x))}{\mathcal{L}^N(B_\rho(x))} = t \right\}$$

of all points where E has density t. The sets E^0, E^1, $\partial^e E := \mathbb{R}^N \setminus (E^0 \cup E^1)$ are called respectively the measure theoretic exterior, the measure theoretic interior and the essential boundary of E. If E has finite perimeter in Ω, Federer's structure theorem states that $\partial^* E \cap \Omega \subset E^{1/2} \subset \partial^e E$ and $\mathcal{H}^{N-1}(\Omega \setminus (E^0 \cup \partial^e E \cup E^1)) = 0$ (see [3, Theorem 3.61]).

2.2. **Capacity.** Given an open set $A \subset \mathbb{R}^N$, the 1-*capacity* of A is defined by setting

$$C_1(A) := \inf \left\{ \int_{\mathbb{R}^N} |D\varphi| \, dx \ : \ \varphi \in W^{1,1}(\mathbb{R}^N), \quad \varphi \geq 1 \quad \mathcal{L}^N\text{-a.e. on } A \right\}.$$

Then, the 1-capacity of an arbitrary set $B \subset \mathbb{R}^N$ is given by

$$C_1(B) := \inf\{ C_1(A) \ : \ A \supseteq B, \ A \text{ open} \}.$$

It is well known that capacities and Hausdorff measure are closely related. In particular, we have that for every Borel set $B \subset \mathbb{R}^N$

$$C_1(B) = 0 \qquad \Longleftrightarrow \qquad \mathcal{H}^{N-1}(B) = 0.$$

Definition 2.2. *Let $B \subset \mathbb{R}^N$ be a Borel set with $C_1(B) < +\infty$. Given $\varepsilon > 0$, we call capacitary ε-quasi-potential (or simply capacitary quasi-potential) of B a function $\varphi_\varepsilon \in W^{1,1}(\mathbb{R}^N)$, such that $0 \leq \tilde{\varphi}_\varepsilon \leq 1$ $\mathcal{H}^{N-1}$-a.e. in $\mathbb{R}^N$, $\tilde{\varphi}_\varepsilon = 1$ $\mathcal{H}^{N-1}$-a.e. in B and*

$$\int_{\mathbb{R}^N} |D\varphi_\varepsilon| \, dx \leq C_1(B) + \varepsilon.$$

We recall that a function $u \colon \mathbb{R}^N \to \mathbb{R}$ is said C_1-quasi continuous if for every $\varepsilon > 0$ there exists an open set A, with $C_1(A) < \varepsilon$, such that the restriction $u \llcorner A^c$ is continuous on A^c; C_1-quasi lower semicontinuous and C_1-quasi upper semicontinuous functions are defined similarly.

It is well known that if u is a $W^{1,1}$-function, then its precise representative $\widetilde{u}$ is C_1-quasi continuous (see [23, Sections 9 and 10]). Moreover, to every BV-function u, it is possible to associate a C_1-quasi lower semicontinuous and a C_1-quasi upper semicontinuous representative, as stated by the following theorem (see [9], Theorem 2.5).

Theorem 2.3. *For every function $u \in BV(\Omega)$, the approximate upper limit u^+ and the approximate lower limit u^- are C_1-quasi upper semicontinuous and C_1-quasi lower semicontinuous, respectively.*

In particular, if B is a Borel subset of $\mathbb{R}^N$ with finite perimeter, then χ_B^- is C_1-quasi lower semicontinuous and χ_B^+ is C_1-quasi upper semicontinuous.

We recall the following lemma which is an approximation result due to Dal Maso (see [18], Lemma 1.5 and §6).

Lemma 2.4. *Let $u\colon \mathbb{R}^N \to [0,+\infty)$ be a C_1-quasi lower semicontinuous function. Then there exists an increasing sequence of nonnegative functions $\{u_h\} \subseteq W^{1,1}(\mathbb{R}^N)$ such that, for every $h \in \mathbb{N}$, u_h is approximately continuous $\mathcal{H}^{N-1}$-almost everywhere in $\mathbb{R}^N$ and $\widetilde{u}_h(x) \to u(x)$, when $h \to +\infty$, for $\mathcal{H}^{N-1}$-almost every $x \in \mathbb{R}^N$.*

2.3. **Divergence–measure fields.** We will denote by $\mathcal{DM}^\infty(\Omega)$ the space of all vector fields $\boldsymbol{A} \in L^\infty(\Omega, \mathbb{R}^N)$ whose divergence in the sense of distribution is a bounded Radon measure in Ω. Similarly, $\mathcal{DM}^\infty_{\rm loc}(\Omega)$ will denote the space of all vector fields $\boldsymbol{A} \in L^\infty_{\rm loc}(\Omega, \mathbb{R}^N)$ whose divergence in the sense of distribution is a Radon measure in Ω. We set $\mathcal{DM}^\infty = \mathcal{DM}^\infty(\mathbb{R}^N)$.

We recall that, if $\boldsymbol{A} \in \mathcal{DM}^\infty_{\rm loc}(\Omega)$, then $|\operatorname{div} \boldsymbol{A}| \ll \mathcal{H}^{N-1}$ (see [11, Proposition 3.1]). As a consequence, the set

$$\Theta_{\boldsymbol{A}} := \left\{ x \in \Omega :\ \limsup_{r \to 0+} \frac{|\operatorname{div} \boldsymbol{A}|(B_r(x))}{r^{N-1}} > 0 \right\},$$

is a Borel set, σ-finite with respect to $\mathcal{H}^{N-1}$, and the measure $\operatorname{div} \boldsymbol{A}$ can be decomposed as

$$\operatorname{div} \boldsymbol{A} = \operatorname{div}^a \boldsymbol{A} + \operatorname{div}^c \boldsymbol{A} + \operatorname{div}^j \boldsymbol{A},$$

where $\operatorname{div}^a \boldsymbol{A}$ is absolutely continuous with respect to $\mathcal{L}^N$, $\operatorname{div}^c \boldsymbol{A}(B) = 0$ for every set B with $\mathcal{H}^{N-1}(B) < +\infty$, and

$$\operatorname{div}^j \boldsymbol{A} = f\, \mathcal{H}^{N-1} \llcorner \Theta_{\boldsymbol{A}}$$

for some Borel function f (see [2, Proposition 2.5]).

2.4. **Normal traces.** The traces of the normal component of the vector field $\boldsymbol{A} \in \mathcal{DM}^\infty_{\rm loc}(\Omega)$ can be defined as distributions $\operatorname{Tr}^\pm(\boldsymbol{A}, \Sigma)$ on every countably $\mathcal{H}^{N-1}$–rectifiable set $\Sigma \subset \Omega$ in the sense of Anzellotti (see [1, 6, 11]).

More precisely, let us briefly recall the construction given in [1] (see Propositions 3.2, 3.4 and Remark 3.3). First of all, given a domain $\Omega' \Subset \Omega$ of class C^1, we define the trace of the normal component of $\boldsymbol{A}$ on $\partial\Omega'$ as a distribution as follows:

$$\left\langle \operatorname{Tr}(\boldsymbol{A}, \partial\Omega') \,,\, \varphi \right\rangle := \int_{\Omega'} \boldsymbol{A} \cdot \nabla\varphi \, dx + \int_{\Omega'} \varphi \, d \operatorname{div} \boldsymbol{A}, \qquad \forall \varphi \in C^\infty_c(\Omega). \tag{6}$$

It turns out that this distribution is induced by an L^∞ function on $\partial\Omega'$, still denoted by $\operatorname{Tr}(\boldsymbol{A}, \partial\Omega')$, and

$$\| \operatorname{Tr}(\boldsymbol{A}, \partial\Omega') \|_{L^\infty(\partial\Omega')} \leq \|\boldsymbol{A}\|_{L^\infty(\Omega')}.$$

Since Σ is countably $\mathcal{H}^{N-1}$–rectifiable, we can find countably many *oriented* C^1 hypersurfaces Σ_i, with classical normal ν_{Σ_i}, and pairwise disjoint Borel sets $N_i \subseteq \Sigma_i$ such that $\mathcal{H}^{N-1}(\Sigma \setminus \bigcup_i N_i) = 0$.

Moreover, it is not restrictive to assume that, for every i, there exist two open bounded sets Ω_i, Ω_i' with C^1 boundary and exterior normal vectors ν_{Ω_i} and $\nu_{\Omega_i'}$ respectively, such that $N_i \subseteq \partial\Omega_i \cap \partial\Omega_i'$ and

$$\nu_{\Sigma_i}(x) = \nu_{\Omega_i}(x) = -\nu_{\Omega_i'}(x) \qquad \forall x \in N_i.$$

At this point we choose, on Σ, the orientation given by $\nu_\Sigma(x) := \nu_{\Sigma_i}(x)$ $\mathcal{H}^{N-1}$-a.e. on N_i.

Using the localization property proved in [1, Proposition 3.2], we can define the normal traces of $\boldsymbol{A}$ on Σ by

$$\operatorname{Tr}^-(\boldsymbol{A},\Sigma) := \operatorname{Tr}(\boldsymbol{A},\partial\Omega_i), \quad \operatorname{Tr}^+(\boldsymbol{A},\Sigma) := -\operatorname{Tr}(\boldsymbol{A},\partial\Omega_i'), \qquad \mathcal{H}^{N-1}-\text{a.e. on } N_i.$$

These two normal traces belong to $L^\infty(\Sigma,\mathcal{H}^{N-1}\llcorner\Sigma)$ (see [1, Proposition 3.2]) and

$$\operatorname{div}\boldsymbol{A}\llcorner\Sigma = \left[\operatorname{Tr}^+(\boldsymbol{A},\Sigma) - \operatorname{Tr}^-(\boldsymbol{A},\Sigma)\right]\,\mathcal{H}^{N-1}\llcorner\Sigma\,. \tag{7}$$

2.5. **Anzellotti's pairing.** As in Anzellotti [6] (see also [11]), for every $\boldsymbol{A}\in\mathcal{DM}^\infty_{\rm loc}(\Omega)$ and $u\in BV_{\rm loc}(\Omega)\cap L^\infty_{\rm loc}(\Omega)$ we define the linear functional $(\boldsymbol{A},Du)\colon C^\infty_0(\Omega)\to\mathbb{R}$ by

$$\langle(\boldsymbol{A},Du)\,,\,\varphi\rangle := -\int_\Omega u^*\varphi\,d\operatorname{div}\boldsymbol{A} - \int_\Omega u\,\boldsymbol{A}\cdot\nabla\varphi\,dx. \tag{8}$$

The distribution $(\boldsymbol{A},Du)$ is a Radon measure in Ω, absolutely continuous with respect to $|Du|$ (see [6, Theorem 1.5] and [11, Theorem 3.2]), hence the equation

$$\operatorname{div}(u\boldsymbol{A}) = u^*\operatorname{div}\boldsymbol{A} + (\boldsymbol{A},Du) \tag{9}$$

holds in the sense of measures in Ω (We remark that, in [11], the measure $(\boldsymbol{A},Du)$ is denoted by $\overline{\boldsymbol{A}\cdot Du}$.) Furthermore, Chen and Frid in [11] proved that the absolutely continuous part of this measure with respect to the Lebesgue measure is given by $(\boldsymbol{A},Du)^a = \boldsymbol{A}\cdot\nabla u\,\mathcal{L}^N$.

3. Characterization of the Anzellotti's pairing

Proposition 3.1. *Let $\boldsymbol{A}\in\mathcal{DM}^\infty_{\rm loc}(\Omega)$, $u\in BV_{\rm loc}(\Omega)\cap L^\infty_{\rm loc}(\Omega)$ and let $\Sigma\subset\Omega$ be a countably $\mathcal{H}^{N-1}$–rectifiable set, oriented as in Section 2.4. Then $u\boldsymbol{A}\in\mathcal{DM}^\infty_{\rm loc}(\Omega)$ and the normal traces of $u\boldsymbol{A}$ on Σ are given by*

$$\operatorname{Tr}^\pm(u\boldsymbol{A},\Sigma) = \begin{cases} u^\pm\operatorname{Tr}^\pm(\boldsymbol{A},\Sigma), & \mathcal{H}^{N-1}-a.e.\ in\ J_u\cap\Sigma,\\ \widetilde{u}\ \operatorname{Tr}^\pm(\boldsymbol{A},\Sigma), & \mathcal{H}^{N-1}-a.e.\ in\ \Sigma\setminus J_u. \end{cases} \tag{10}$$

Proof. The fact that $u\boldsymbol{A}\in\mathcal{DM}^\infty_{\rm loc}(\Omega)$ has been proved in [11, Theorem 3.1].

We will use the same notations of Section 2.4. It is not restrictive to assume that J_u is oriented with ν_Σ on $J_u\cap\Sigma$.

Let us prove (10) for Tr^-. Let $x\in\Sigma$ satisfy:

(a) $x\in(\Omega\setminus S_u)\cup J_u$, $x\in N_i$ for some i, the set N_i has density 1 at x, and x is a Lebesgue point of $\operatorname{Tr}^-(\boldsymbol{A},\Sigma)$ with respect to $\mathcal{H}^{N-1}\llcorner\partial\Omega_i$;
(b) $|\operatorname{div}\boldsymbol{A}|\llcorner\Omega_i(B_\varepsilon(x)) = o(\varepsilon^{N-1})$ as $\varepsilon\to 0$;
(c) $|\operatorname{div}(u\boldsymbol{A})|\llcorner\Omega_i(B_\varepsilon(x)) = o(\varepsilon^{N-1})$.

We remark that $\mathcal{H}^{N-1}$-a.e. $x\in\Sigma$ satisfies these conditions. In particular, (a) is satisfied because $\mathcal{H}^{N-1}((\Omega\setminus S_u)\cup J_u) = 0$, whereas (b) and (c) follow from [3, Theorem 2.56 and (2.41)].

In order to simplify the notation, in the following we set $u^-(x) := \widetilde{u}(x)$ if $x\in\Omega\setminus S_u$.

Let us choose a function $\varphi\in C^\infty_c(\mathbb{R}^N)$, with support contained in $B_1(0)$, such that $0\le\varphi\le 1$. For every $\varepsilon>0$ let $\varphi_\varepsilon(y) := \varphi\left(\frac{y-x}{\varepsilon}\right)$.

By the very definition of normal trace, the following equality holds for every $\varepsilon > 0$ small enough:

$$
\begin{aligned}
&\frac{1}{\varepsilon^{N-1}} \int_{\partial\Omega_i} [\mathrm{Tr}(u\boldsymbol{A}, \partial\Omega_i) - u^-(x)\,\mathrm{Tr}(\boldsymbol{A}, \partial\Omega_i)]\, \varphi_\varepsilon(y)\, d\mathcal{H}^{N-1}(y) \\
&= \frac{1}{\varepsilon^{N-1}} \int_{\Omega_i} \nabla\varphi_\varepsilon(y) \cdot [u(y)\boldsymbol{A}(y) - u^-(x)\boldsymbol{A}(y)]\, dy \\
&\quad + \frac{1}{\varepsilon^{N-1}} \int_{\Omega_i} \varphi_\varepsilon(y)\, d[\mathrm{div}(u\boldsymbol{A}) - u^-(x)\,\mathrm{div}\,\boldsymbol{A}](y)\,.
\end{aligned}
\tag{11}
$$

Using the change of variable $z = (y - x)/\varepsilon$, as $\varepsilon \to 0$ the left hand side of this equality converges to

$$
[\mathrm{Tr}^-(u\boldsymbol{A}, \Sigma)(x) - u^-(x)\,\mathrm{Tr}^-(\boldsymbol{A}, \Sigma)] \int_{\Pi_x} \varphi(z)\, d\mathcal{H}^{N-1}(z)\,,
$$

where Π_x is the tangent plane to Σ_i at x. Clearly φ can be chosen in such a way that $\int_{\Pi_x} \varphi\, d\mathcal{H}^{N-1} > 0$.

In order to prove (10) for Tr^- it is then enough to show that the two integrals $I_1(\varepsilon)$ and $I_2(\varepsilon)$ at the right hand side of (11) converge to 0 as $\varepsilon \to 0$.

With the change of variables $z = (y-x)/\varepsilon$ and by the very definition of $\mathbf{v}$ we have that

$$
I_1(\varepsilon) = \int_{\Omega_i^\varepsilon} [u(x+\varepsilon z) - u^-(x)] \nabla\varphi(z) \cdot \boldsymbol{A}(x+\varepsilon z)\, dz,
$$

where

$$
\Omega_i^\varepsilon := \frac{\Omega_i - x}{\varepsilon}.
$$

As $\varepsilon \to 0$, these sets locally converge to the half space $P_x := \{z \in \mathbb{R}^N : \ \langle z\,,\, \nu(x)\rangle < 0\}$, hence

$$
\lim_{\varepsilon\to 0} \int_{\Omega_i^\varepsilon \cap B_1} |u(x+\varepsilon z) - u^-(x)|\, dz = \lim_{\varepsilon\to 0} \int_{P_x \cap B_1} |u(x+\varepsilon z) - u^-(x)|\, dz = 0
$$

(see [3, Remark 3.85]) so that

$$
|I_1(\varepsilon)| \leq \|\boldsymbol{A}\|_{L^\infty(B_\varepsilon(x))}\, \|\nabla\varphi\|_\infty \int_{\Omega_i^\varepsilon \cap B_1} |u(x+\varepsilon z) - u^-(x)|\, dz \to 0.
$$

From (b) we have that

$$
\lim_{\varepsilon\to 0} \frac{1}{\varepsilon^{N-1}} \left| \int_{\Omega_i} \varphi_\varepsilon(y)\, u^-(x)\, d\,\mathrm{div}\,\boldsymbol{A}(y) \right| \leq \limsup_{\varepsilon\to 0} |u^-(x)| \frac{|\,\mathrm{div}\,\boldsymbol{A}|(B_\varepsilon(x))}{\varepsilon^{N-1}} = 0.
$$

In a similar way, using (c), we get

$$
\lim_{\varepsilon\to 0} \frac{1}{\varepsilon^{N-1}} \left| \int_{\Omega_i} \varphi_\varepsilon\, d\,\mathrm{div}(u\boldsymbol{A}) \right| = 0,
$$

so that $I_2(\varepsilon)$ vanishes as $\varepsilon \to 0$.

The proof of (12) for Tr^+ is entirely similar. □

The following result has been proved in [24, Lemma 2.5].

Corollary 3.2. *Let $\boldsymbol{A} \in \mathcal{DM}^\infty_{\mathrm{loc}}(\Omega)$ and $u \in BV_{\mathrm{loc}}(\Omega) \cap L^\infty_{\mathrm{loc}}(\Omega)$. Then $u\boldsymbol{A} \in \mathcal{DM}^\infty_{\mathrm{loc}}(\Omega)$ and the normal traces of $u\boldsymbol{A}$ on J_u are given by*

$$\mathrm{Tr}^\pm(u\boldsymbol{A}, J_u) = u^\pm \,\mathrm{Tr}^\pm(\boldsymbol{A}, J_u), \qquad \mathcal{H}^{N-1} - a.e.\ in\ J_u. \tag{12}$$

In particular

$$\mathrm{div}(u\boldsymbol{A}) \llcorner J_u = \left[u^+ \,\mathrm{Tr}^+(\boldsymbol{A}, J_u) - u^- \,\mathrm{Tr}^-(\boldsymbol{A}, J_u)\right] \mathcal{H}^{N-1} \llcorner J_u. \tag{13}$$

Theorem 3.3. *Let $\boldsymbol{A} \in \mathcal{DM}^\infty_{\mathrm{loc}}(\Omega)$ and $u \in BV_{\mathrm{loc}}(\Omega) \cap L^\infty_{\mathrm{loc}}(\Omega)$. Then the measure $(\boldsymbol{A}, Du)$ admits the following decomposition:*

(i) *absolutely continuous part: $(\boldsymbol{A}, Du)^a = \boldsymbol{A} \cdot \nabla u \,\mathcal{L}^N$;*

(ii) *jump part: $(\boldsymbol{A}, Du)^j = \dfrac{\mathrm{Tr}^+(\boldsymbol{A}, J_u) + \mathrm{Tr}^-(\boldsymbol{A}, J_u)}{2} (u^+ - u^-) \,\mathcal{H}^{N-1} \llcorner J_u$;*

(iii) *diffuse part: if, in addition,*

$$|D^c u|(S_{\boldsymbol{A}}) = 0, \tag{14}$$

where $S_{\boldsymbol{A}}$ is the approximate discontinuity set of $\boldsymbol{A}$, then $(\boldsymbol{A}, Du)^d = \widetilde{\boldsymbol{A}} \cdot D^d u$.

Remark 3.4. Since $\mathcal{L}^N(S_{\boldsymbol{A}}) = 0$, assumption (14) is equivalent to $|D^d u|(S_{\boldsymbol{A}}) = 0$. In particular, it is satisfied, for example, if $S_{\boldsymbol{A}}$ is σ–finite with respect to $\mathcal{H}^{N-1}$ (see [3, Proposition 3.92(c)]). This is always the case if $\boldsymbol{A} \in BV_{\mathrm{loc}}(\Omega, \mathbb{R}^N) \cap L^\infty_{\mathrm{loc}}(\Omega, \mathbb{R}^N)$ and, notably, if $N = 1$. Another relevant situation for which (14) holds happens when $D^c u = 0$, i.e. if u is a special function of bounded variation, e.g. if u is the characteristic function of a set of finite perimeter.

Remark 3.5 (BV vector fields). If $\boldsymbol{A} \in BV_{\mathrm{loc}}(\Omega, \mathbb{R}^N) \cap L^\infty_{\mathrm{loc}}(\Omega, \mathbb{R}^N)$, then clearly $\boldsymbol{A} \in \mathcal{DM}^\infty_{\mathrm{loc}}(\Omega)$ and

$$\mathrm{Tr}^\pm(\boldsymbol{A}, J_u) = \boldsymbol{A}^\pm_{J_u} \cdot \nu_u\,, \qquad \mathcal{H}^{N-1}\text{-a.e. in } J_u,$$

where $\boldsymbol{A}^\pm_{J_u}$ are the traces of $\boldsymbol{A}$ on J_u (see [3, Theorem 3.77]). Hence, the jump part of $(\boldsymbol{A}, Du)$ can be written as

$$(\boldsymbol{A}, Du)^j = \frac{\boldsymbol{A}^+ + \boldsymbol{A}^-}{2} \cdot D^j u.$$

Proof. Let $u_\varepsilon := \rho_\varepsilon * u$. It has been proved in [11, Theorem 3.2] that

$$\langle (\boldsymbol{A}, Du)\,,\, \varphi \rangle = \lim_{\varepsilon \to 0} \langle (\boldsymbol{A}, Du_\varepsilon)\,,\, \varphi \rangle = \lim_{\varepsilon \to 0} \int_\Omega \varphi\, \boldsymbol{A} \cdot \nabla u_\varepsilon \, dx, \qquad \forall \varphi \in C^\infty_0(\Omega)$$

and that (i) holds. We remark that, if $K \Subset U \subset \overline{U} \Subset \Omega$ with U open, then

$$|(\boldsymbol{A}, Du)|(K) \le \|\boldsymbol{A}\|_{L^\infty(U)} \,|Du|(U),$$

hence, in particular

$$|(\boldsymbol{A}, Du)|(E) \le \|\boldsymbol{A}\|_{L^\infty(U)} \,|Du|(E) \qquad \text{for every Borel set } E \subset U.$$

It remains to prove (ii) and (iii). In order to simplify the notation, let us denote $\mu := (\boldsymbol{A}, Du)$.

Proof of (ii). Since $(\boldsymbol{A}, Du) \ll |Du|$, it is clear that $(\boldsymbol{A}, Du)^j$ is supported in J_u. From (9) and (13) we have that

$$\begin{aligned}(\boldsymbol{A}, Du)^j &= (\boldsymbol{A}, Du) \llcorner J_u = \operatorname{div}(u\boldsymbol{A}) \llcorner J_u - u^* \operatorname{div} \boldsymbol{A} \llcorner J_u \\ &= \left[u^+ \operatorname{Tr}^+(\boldsymbol{A}, J_u) - u^- \operatorname{Tr}^-(\boldsymbol{A}, J_u)\right] \mathcal{H}^{N-1} \llcorner J_u \\ &\quad - \frac{u^+ + u^-}{2} \left[\operatorname{Tr}^+(\boldsymbol{A}, J_u) - \operatorname{Tr}^-(\boldsymbol{A}, J_u)\right] \mathcal{H}^{N-1} \llcorner J_u \\ &= \frac{\operatorname{Tr}^+(\boldsymbol{A}, J_u) + \operatorname{Tr}^-(\boldsymbol{A}, J_u)}{2} (u^+ - u^-) \mathcal{H}^{N-1} \llcorner J_u,\end{aligned}$$

and the proof is complete.

Proof of (iii). Let us consider the polar decomposition $Du = \theta_u |Du|$ of Du. By assumption (14), the approximate limit $\widetilde{\boldsymbol{A}}$ of $\boldsymbol{A}$ exists $|D^d u|$-a.e. in Ω. Hence, the equality in (iii) is equivalent to

$$\frac{d\mu}{d|D^d u|}(x) = \frac{d\mu^d}{d|D^d u|}(x) = \widetilde{\boldsymbol{A}}(x) \cdot \theta_u(x) \qquad \text{for } |D^d u|\text{-a.e. } x \in \Omega.$$

Let us choose $x \in \Omega$ such that

(a) x belongs to the support of $D^d u$, that is $|D^d u|(B_r(x)) > 0$ for every $r > 0$;

(b) there exists the limit $\lim_{r\to 0} \dfrac{\mu^d(B_r(x))}{|D^d u|(B_r(x))}$;

(c) $\lim_{r\to 0} \dfrac{|D^j u|(B_r(x))}{|Du|(B_r(x))} = 0$;

(d) $\lim_{r\to 0} \dfrac{1}{|D^d u|(B_r(x))} \displaystyle\int_{B_r(x)} \left|\boldsymbol{A}(y) \cdot \theta_u(y) - \widetilde{\boldsymbol{A}}(x) \cdot \theta_u(x)\right| d|D^d u|(y) = 0.$

We remark that these conditions are satisfied for $|D^d u|$-a.e. $x \in \Omega$.

Let $r > 0$ be such that

$$|D^j u| \left(\partial B_r(x)\right) = 0. \tag{15}$$

Observe that $\nabla u_\varepsilon = \rho_\varepsilon * Du = \rho_\varepsilon * D^d u + \rho_\varepsilon * D^j u$. Hence for every $\phi \in C_0(\mathbb{R}^N)$ with support in $B_r(x)$ it holds

$$\begin{aligned}&\left|\frac{1}{|D^d u|(B_r(x))} \int_{B_r(x)} \phi(y) \boldsymbol{A}(y) \cdot \rho_\varepsilon * Du(y)\, dy \right.\\ &\qquad \left. - \frac{1}{|D^d u|(B_r(x))} \int_{B_r(x)} \phi(y) \widetilde{\boldsymbol{A}}(x) \cdot \theta_u(x)\, d|D^d u|(y)\right| \\ &\le \left|\frac{1}{|D^d u|(B_r(x))} \int_{B_r(x)} \phi(y) \boldsymbol{A}(y) \cdot \rho_\varepsilon * D^d u(y)\, dy \right.\\ &\qquad \left. - \frac{1}{|D^d u|(B_r(x))} \int_{B_r(x)} \phi(y) \widetilde{\boldsymbol{A}}(x) \cdot \theta_u(x)\, d|D^d u|(y)\right| \\ &\qquad + \frac{1}{|D^d u|(B_r(x))} \|\phi\|_\infty \|\boldsymbol{A}\|_{L^\infty(B_r(x))} \int_{B_r(x)} \rho_\varepsilon * |D^j u|\, dy,\end{aligned} \tag{16}$$

where in the last inequality we use that $\left|\rho_\varepsilon * D^j u\right| \le \rho_\varepsilon * |D^j u|$.

We note that by (15)

$$\lim_{\varepsilon\to 0}\int_{B_r(x)} \rho_\varepsilon * |D^j u|\, dy = |D^j u|(B_r(x)).$$

Hence by taking the limit as $\varepsilon \to 0$ in (16) we obtain

$$\begin{aligned}
&\left|\frac{1}{|D^d u|(B_r(x))}\int_{B_r(x)} \phi(y)\, d\mu(y)\right.\\
&\quad\left. - \frac{1}{|D^d u|(B_r(x))}\int_{B_r(x)} \phi(y)\widetilde{\boldsymbol{A}}(x)\cdot\theta_u(x)\, d|D^d u|(y)\right|\\
&\le \frac{1}{|D^d u|(B_r(x))}\int_{B_r(x)} \phi(y)\left|\boldsymbol{A}(y)\cdot\theta_u(y) - \widetilde{\boldsymbol{A}}(x)\cdot\theta_u(x)\, d|D^d u|(y)\right|\\
&\quad + \frac{1}{|D^d u|(B_r(x))}\|\phi\|_\infty \|\boldsymbol{A}\|_{L^\infty(B_r(x))}\,|D^j u|(B_r(x)).
\end{aligned}$$

When $\phi(y) \to 1$ in $B_r(x)$, with $0 \le \phi \le 1$, we get

$$\begin{aligned}
&\left|\frac{\mu(B_r(x))}{|D^d u|(B_r(x))} - \widetilde{\boldsymbol{A}}(x)\cdot\theta_u(x)\right|\\
&\le \frac{1}{|D^d u|(B_r(x))}\int_{B_r(x)} \left|\boldsymbol{A}(y)\cdot\theta_u(y) - \widetilde{\boldsymbol{A}}(x)\cdot\theta_u(x)\right|\, d|D^d u|(y)\\
&\quad + \frac{1}{|D^d u|(B_r(x))}\|\boldsymbol{A}\|_{L^\infty(B_r(x))}|D^j u|(B_r(x)).
\end{aligned}$$

The conclusion is achieved now by taking $r \to 0$ and by using (c) and (d). □

Example 3.6. Let $\boldsymbol{A}\colon \mathbb{R}^2 \to \mathbb{R}^2$ be the vector field defined by $\boldsymbol{A}(x_1, x_2) = (1, 0)$ if $x_1 > 0$, $\boldsymbol{A}(x_1, x_2) = (-1, 0)$ if $x_1 < 0$. Clearly $\boldsymbol{A} \in \mathcal{DM}^\infty$ and $\operatorname{div}\boldsymbol{A} = 2\mathcal{H}^1 \llcorner S$, where $S := \{0\}\times\mathbb{R}$.

Let $E := (0, 1) \times (0, 1)$ and let $u := \chi_E \in BV(\mathbb{R}^2)$. Let us choose on $J_u = \partial E$ the orientation given by the interior unit normal ν to E, so that $u^+ = 1$ and $u^- = 0$ on ∂E.

Let us compute the normal traces $\alpha^\pm := \operatorname{Tr}^\pm(\boldsymbol{A}, J_u)$ of $\boldsymbol{A}$ on J_u, using the construction described in Section 2.4. Let $\partial E = J_u = S_1 \cup S_2 \cup S_3 \cup S_4$, where

$$S_1 = \{0\} \times [0, 1],\ S_2 = [0, 1] \times \{1\},\ S_3 = \{1\} \times [0, 1],\ S_4 = [0, 1] \times \{0\}.$$

Let us start with the computation of the normal traces on S_1. We can construct two open domains Ω and Ω' of class C^1, such that $\Omega \subset \{x_1 < 0\}$, $\Omega' \subset \{x_1 > 0\}$, and $S_1 \subset \partial\Omega \cap \partial\Omega'$. Indeed, with this choice we have

$$\nu = \nu_\Omega = (1, 0) = -\nu_{\Omega'} \qquad \text{on } S_1.$$

(Recall that ν_Ω is by definition the outward normal vector to Ω.) We thus have

$$\alpha^- := \operatorname{Tr}(\boldsymbol{A}, \partial\Omega) = -1, \quad \alpha^+ := -\operatorname{Tr}(\boldsymbol{A}, \partial\Omega') = 1, \qquad \text{on } S_1.$$

With similar constructions we get $\alpha^\pm = -1$ on S_3 and $\alpha^\pm = 0$ on $S_2 \cup S_4$, so that

$$\alpha^* := \frac{\alpha^+ + \alpha^-}{2} = \begin{cases} -1, & \text{on } S_3,\\ 0, & \text{on } S_1 \cup S_2 \cup S_4.\end{cases}$$

We can now check the validity of the relation

$$\operatorname{div}(u\boldsymbol{A}) = u^* \operatorname{div}\boldsymbol{A} + (\boldsymbol{A}, Du),$$

where $(\boldsymbol{A}, Du) = (u^+ - u^-)\alpha^* \mathcal{H}^1 \llcorner J_u$ (in this case the measure $(\boldsymbol{A}, Du)$ does not have a diffuse part). Indeed, we have

$$\operatorname{div}(u\boldsymbol{A}) = \mathcal{H}^1 \llcorner S_1 - \mathcal{H}^1 \llcorner S_3, \quad u^* \operatorname{div} \boldsymbol{A} = \mathcal{H}_1 \llcorner S_1, \quad (u^+ - u^-)\alpha^* \mathcal{H}^1 \llcorner J_u = -\mathcal{H}^1 \llcorner S_3.$$

By the way, observe that $u\boldsymbol{A} = u\boldsymbol{C}$, where $\boldsymbol{C}$ is the constant vector field $\boldsymbol{C} \equiv (1, 0)$ on $\mathbb{R}^2$. In this case the normal traces $\gamma^\pm$ of $\boldsymbol{C}$ on J_u are $\gamma^\pm = 1$ on S_1, $\gamma^\pm = -1$ on S_3, $\gamma^\pm = 0$ on $S_2 \cup S_4$, hence

$$u^* \operatorname{div} \boldsymbol{C} = 0, \quad (u^+ - u^-)\, \gamma^* \mathcal{H}^1 \llcorner J_u = \mathcal{H}^1 \llcorner S_1 - \mathcal{H}^1 \llcorner S_3 \,.$$

4. Some formulas

Since the measure $(\boldsymbol{A}, Du)$ is absolutely continuous with respect to $|Du|$, then

$$(\boldsymbol{A}, Du) = \theta(\boldsymbol{A}, Du, x)\, |Du|, \tag{17}$$

where $\theta(\boldsymbol{A}, Du, \cdot)$ denotes the Radon–Nikodým derivative of $(\boldsymbol{A}, Du)$ with respect to $|Du|$.

Let $Du = \theta_u |Du|$ be the polar decomposition of Du. From Theorem 3.3, if $|D^c u|(S_{\boldsymbol{A}}) = 0$ it holds

$$\theta(\boldsymbol{A}, Du, x) = \begin{cases} \left\langle \widetilde{\boldsymbol{A}}(x)\,,\, \theta_u(x) \right\rangle, & \text{for } |D^d u|\text{-a.e. } x \in \Omega, \\ \alpha^*(x) \operatorname{sign}(u^+(x) - u^-(x)), & \text{for } \mathcal{H}^{N-1}\text{-a.e. } x \in J_u, \end{cases} \tag{18}$$

where $\alpha^* := [\operatorname{Tr}^+(\boldsymbol{A}, J_u) + \operatorname{Tr}^-(\boldsymbol{A}, J_u)]/2$.

Remark 4.1. If $\operatorname{div} \boldsymbol{A} \in L^1(\Omega)$ and $u \in BV(\Omega) \cap L^\infty(\Omega)$, then $\operatorname{Tr}^+(\boldsymbol{A}, J_u) = \operatorname{Tr}^-(\boldsymbol{A}, J_u)$ $\mathcal{H}^{N-1}$-a.e. in J_u. Moreover Anzellotti has proved in [7, Theorem 3.6] that

$$\theta(\boldsymbol{A}, Du, x) = q_{\boldsymbol{A}}(x, \theta_u(x)) \qquad \text{for } |Du|\text{-a.e. } x \in \Omega,$$

where, for every $\zeta \in S^{N-1}$,

$$q_{\boldsymbol{A}}(x, \zeta) := \lim_{\rho \downarrow 0} \lim_{r \downarrow 0} \frac{1}{\mathcal{L}^N(C_{r,\rho}(x, \zeta))} \int_{C_{r,\rho}(x,\zeta)} \boldsymbol{A}(y) \cdot \zeta \, dy$$

with

$$C_{r,\rho}(x, \zeta) := \left\{ y \in \mathbb{R}^N : \ |(y - x) \cdot \zeta| < r, \ |(y - x) - [(y - x) \cdot \zeta]\zeta| < \rho \right\}$$

(the existence of the limit in the definition of $q_{\boldsymbol{A}}(x, \theta_u(x))$ for $|Du|$-a.e. $x \in \Omega$ is part of the statement). By using (18) in this framework, we can conclude that if $\operatorname{div} \boldsymbol{A} \in L^1(\Omega)$ and $|D^c u|(S_{\boldsymbol{A}}) = 0$, then we have

$$\left\langle \widetilde{\boldsymbol{A}}(x)\,,\, \theta_u(x) \right\rangle = q_{\boldsymbol{A}}(x, \theta_u(x)) \qquad \text{for } |D^d u|\text{-a.e. } x \in \Omega.$$

Finally, we remark that, when $\boldsymbol{A}$ is a $W^{1,1}(\Omega; \mathbb{R}^N)$ vector field, then $\operatorname{div} \boldsymbol{A} \in L^1(\Omega)$ and $|D^c u|(S_{\boldsymbol{A}}) = 0$.

Theorem 4.2 (Coarea formula)**.** *Let $\boldsymbol{A} \in \mathcal{DM}^\infty_{\mathrm{loc}}(\Omega)$, let $u \in BV_{\mathrm{loc}}(\Omega)$ and assume that $u^* \in L^1_{\mathrm{loc}}(\mathbb{R}^N, \operatorname{div} \boldsymbol{A})$. Then*

$$\langle (\boldsymbol{A}, Du)\,,\, \varphi \rangle = \int_{\mathbb{R}} \left\langle (\boldsymbol{A}, D\chi_{\{u>t\}})\,,\, \varphi \right\rangle dt, \qquad \forall \varphi \in C_c(\Omega) \tag{19}$$

and, for any Borel set $B \subset \Omega$,

$$(\boldsymbol{A}, Du)(B) = \int_{\mathbb{R}} (\boldsymbol{A}, D\chi_{\{u>t\}})(B)\, dt. \tag{20}$$

Furthermore, for $\mathcal{L}^1$-a.e. $t \in \mathbb{R}$,

$$\theta(\boldsymbol{A}, Du, x) = \theta(\boldsymbol{A}, D\chi_{\{u>t\}}, x) \qquad \text{for } |D\chi_{\{u>t\}}|\text{-a.e. } x \in \Omega. \tag{21}$$

Remark 4.3. Formulas (19) and (20) have been proved by Anzellotti (see [6, Proposition 2.7]) for $u \in BV(\Omega)$ and $\boldsymbol{A} \in L^\infty(\Omega, \mathbb{R}^N)$ with $\operatorname{div} \boldsymbol{A} \in L^N(\Omega)$. Moreover they have been proved in [27, Propositions 2.4 and 2.5] when $D^j u = 0$.

Proof. Let us first consider the case $u \in L^\infty(\Omega)$. By possibly replacing u with $u + \|u\|_\infty$, it is not restrictive to assume that $u \geq 0$

Let us fix a test function $\varphi \in C_c^\infty(\Omega)$. From the definition (8) of pairing, we have that

$$\begin{aligned} \int_{\mathbb{R}} \left\langle (\boldsymbol{A}, D\chi_{\{u>t\}}) \,,\, \varphi \right\rangle dt = & -\int_0^{+\infty} \left(\int_\Omega \chi^*_{\{u>t\}} \varphi \, d \operatorname{div} \boldsymbol{A} \right) dt \\ & - \int_0^{+\infty} \left(\int_\Omega \chi_{\{u>t\}} \boldsymbol{A} \cdot \nabla\varphi \, dx \right) dt =: -I_1 - I_2. \end{aligned} \tag{22}$$

The integral I_2 can be immediately computed as

$$I_2 = \int_\Omega u \, \boldsymbol{A} \cdot \nabla \varphi \, dx. \tag{23}$$

The first integral I_1 requires more care. From [19, Lemma 2.2] we have that, for $\mathcal{L}^1$-a.e. $t \in \mathbb{R}$, there exists a Borel set $N_t \subset \Omega$, with $\mathcal{H}^{N-1}(N_t) = 0$, such that

$$\forall x \in \Omega \setminus N_t: \qquad \chi^*_{\{u>t\}}(x) = \begin{cases} 1, & \text{if } u^-(x) > t, \\ 0, & \text{if } u^+(x) < t, \\ 1/2, & \text{if } u^-(x) \leq t \leq u^+(x)\,. \end{cases}$$

Since $|\operatorname{div} \boldsymbol{A}| \ll \mathcal{H}^{N-1}$, we deduce that, for $\mathcal{L}^1$-a.e. $t \in \mathbb{R}$,

$$\chi^*_{\{u>t\}}(x) = \frac{\chi_{\{u^->t\}}(x) + \chi_{\{u^+>t\}}(x)}{2}\,, \qquad \text{for } |\operatorname{div} \boldsymbol{A}|\text{-a.e. } x \in \Omega. \tag{24}$$

From (24), we can rewrite I_1 in the following way:

$$\begin{aligned} I_1 & = \int_0^{+\infty} \int_\Omega \left(\frac{\chi_{\{u^->t\}} + \chi_{\{u^+>t\}}}{2} \, \varphi \, d \operatorname{div} \boldsymbol{A} \right) dt \\ & = \int_\Omega \frac{u^- + u^+}{2} \, \varphi \, d \operatorname{div} \boldsymbol{A} = \int_\Omega u^* \, \varphi \, d \operatorname{div} \boldsymbol{A}\,. \end{aligned} \tag{25}$$

Hence, from (22), (23), (25) and the definition (8) of $(\boldsymbol{A}, Du)$, we conclude that (19) holds for every test function $\varphi \in C_c^\infty(\Omega)$. On the other hand, since both sides in (19) are measures in Ω, they coincide not only as distributions, but also as measures. Hence (19) and (20) follow.

Since, for $\mathcal{L}^1$-a.e. $t \in \mathbb{R}$, it holds

$$\frac{dDu}{d|Du|} = \frac{dD\chi_{\{u>t\}}}{d|D\chi_{\{u>t\}}|} \qquad |D\chi_{\{u>t\}}|\text{-a.e. in } \Omega,$$

we conclude that (21) follows.

Finally, the general case $u^* \in L^1_{\mathrm{loc}}(\mathbb{R}^N, \operatorname{div} \boldsymbol{A})$ follows using the previous step on the truncated functions $u_k := T_k(u)$, where, given $k > 0$, T_k is defined by

$$T_k(s) := \max\{\min\{s, k\}, -k\}\,, \qquad s \in \mathbb{R}. \tag{26}$$

Since T_k is a Lipschitz continuous function, we get that

$$u_k \in BV_{\rm loc}(\Omega) \cap L^\infty(\Omega), \quad u_k^\pm = T_k(u^\pm), \quad |Du_k| \leq |Du| \text{ in the sense of measures.}$$

Then $|u_k^\pm| \leq |u^\pm|$ and $|u_k^*| \leq |u^*|$, which implies that $u_k^* \in L^1_{\rm loc}(\Omega, \operatorname{div} \boldsymbol{A})$. □

Remark 4.4 (Representation of $\theta(\boldsymbol{A}, Du, x)$). Let $\boldsymbol{A} \in \mathcal{DM}^\infty_{\rm loc}(\Omega)$ and let $u \in BV_{\rm loc}(\Omega) \cap L^\infty_{\rm loc}(\Omega)$. If $E \Subset \Omega$ is a set of finite perimeter, then $|D\chi_E| = \mathcal{H}^{N-1} \llcorner \partial^* E$ hence, by Theorem 3.3, we have that

$$(\boldsymbol{A}, D\chi_E) = \frac{\operatorname{Tr}^+(\boldsymbol{A}, \partial^* E) + \operatorname{Tr}^-(\boldsymbol{A}, \partial^* E)}{2} |D\chi_E|,$$

that is

$$\theta(\boldsymbol{A}, D\chi_E, x) = \frac{\operatorname{Tr}^+(\boldsymbol{A}, \partial^* E) + \operatorname{Tr}^-(\boldsymbol{A}, \partial^* E)}{2} \qquad \text{for } \mathcal{H}^{N-1}\text{-a.e. } x \in \partial^* E.$$

Since, for $\mathcal{L}^1$-a.e. $t \in \mathbb{R}$, the set $E_{u,t} := \{u > t\}$ is of finite perimeter, then from (21) we deduce that, for these values of t,

$$\theta(\boldsymbol{A}, Du, x) = \frac{\operatorname{Tr}^+(\boldsymbol{A}, \partial^* E_{u,t}) + \operatorname{Tr}^-(\boldsymbol{A}, \partial^* E_{u,t})}{2} \qquad \text{for } \mathcal{H}^{N-1}\text{-a.e. } x \in \partial^* E_{u,t}.$$

Proposition 4.5 (Chain Rule). *Let $\boldsymbol{A} \in \mathcal{DM}^\infty_{\rm loc}(\Omega)$ and let $u \in BV_{\rm loc}(\Omega) \cap L^\infty_{\rm loc}(\Omega)$. Let $h : \mathbb{R} \to \mathbb{R}$ be a locally Lipschitz function. Then the following properties hold:*

(i) $(\boldsymbol{A}, Dh(u))^a = h'(\widetilde{u})\, \boldsymbol{A} \cdot \nabla u\, \mathcal{L}^N$ *and, if* $|D^c u|(S_{\boldsymbol{A}}) = 0$, *then* $(\boldsymbol{A}, Dh(u))^d = h'(\widetilde{u})\, (\boldsymbol{A}, Du)^d$;

(ii) $(\boldsymbol{A}, Dh(u))^j = \dfrac{h(u^+) - h(u^-)}{u^+ - u^-} (\boldsymbol{A}, Du)^j$;

(iii) *if h is non-decreasing, then*

$$\theta(\boldsymbol{A}, Dh(u), x) = \theta(\boldsymbol{A}, Du, x), \qquad \text{for } |Dh(u)|\text{-a.e. } x \in \Omega. \tag{27}$$

Remark 4.6. Formula (27) has been proved by Anzellotti (see [6, Proposition 2.8]) for $h \in C^1$, $u \in BV(\Omega)$ and $\boldsymbol{A} \in L^\infty(\Omega, \mathbb{R}^N)$ with $\operatorname{div} \boldsymbol{A} \in L^N(\Omega)$. Moreover it has been proved in [27, Proposition 2.7] when $D^j u = 0$.

Remark 4.7. The same characterization of $(\boldsymbol{A}, Dh(u))$ holds true if $h \colon I \to \mathbb{R}$ is a locally Lipschitz function in a interval I, provided that $u(\Omega) \subset I$ and $h \circ u \in BV_{\rm loc}(\Omega)$.

Proof. From the Chain Rule Formula (see [3, Theorem 3.99]), we have that

$$D^d h(u) = h'(\widetilde{u}) D^d u, \qquad D^j h(u) = (h(u^+) - h(u^-))\, \nu\, \mathcal{H}^{N-1} \llcorner J_u.$$

On the other hand, $(h(u))^\pm = h(u^\pm)$, hence (i) and (ii) follow from Theorem 3.3.

The proof of (iii) can be done as in [27, Proposition 2.7]. □

Aim of the next results is the characterization of the pairing $(v\boldsymbol{A}, Du)$. We first present a preliminary result in the case $u = v$ in Lemma 4.8. The general case will follow in Proposition 4.9. The same results, under the assumption $D^j u = D^j v = 0$, have been proved in [29, Proposition 2.3].

Lemma 4.8. *Let $\boldsymbol{A} \in \mathcal{DM}^\infty_{\rm loc}(\Omega)$ and $u \in BV_{\rm loc}(\Omega) \cap L^\infty_{\rm loc}(\Omega)$. Then*

$$(u\boldsymbol{A}, Du) = u^*(\boldsymbol{A}, Du) + \frac{(u^+ - u^-)^2}{4} \operatorname{div} \boldsymbol{A} \llcorner J_u, \tag{28}$$

that is

$$(u\boldsymbol{A}, Du)^d = u^*(\boldsymbol{A}, Du)^d, \tag{29}$$

$$(u\boldsymbol{A}, Du)^j = \frac{\alpha^+ u^+ + \alpha^- u^-}{2}\,(u^+ - u^-)\,\mathcal{H}^{N-1} \llcorner J_u, \tag{30}$$

where $\alpha^\pm := \mathrm{Tr}^\pm(\boldsymbol{A}, J_u)$*. In particular, if* $D^j u = 0$ *then* $(u\boldsymbol{A}, Du) = u^*(\boldsymbol{A}, Du)$*.*

Proof. Since the statement is local in nature, it is not restrictive to assume that $u \in L^\infty(\Omega)$.

Let us first assume that $u > 0$. Since $D(u^2) = 2u^* Du$, from Proposition 4.5(iii) we have that

$$\theta(\boldsymbol{A}, D(u^2), x) = \theta(\boldsymbol{A}, Du, x) \qquad \text{for } |Du|\text{-a.e. } x \in \Omega,$$

hence

$$(\boldsymbol{A}, D(u^2)) = \theta(\boldsymbol{A}, D(u^2), x)|D(u^2)| = \theta(\boldsymbol{A}, Du, x) 2u^*|Du| = 2u^*(\boldsymbol{A}, Du).$$

Starting from the relation

$$\operatorname{div}(u^2\boldsymbol{A}) = (u^2)^* \operatorname{div}\boldsymbol{A} + (\boldsymbol{A}, D(u^2)) = (u^2)^* \operatorname{div}\boldsymbol{A} + 2u^*(\boldsymbol{A}, Du)$$

we get

$$\begin{aligned} 2u^*(\boldsymbol{A}, Du) &= \operatorname{div}(u^2\boldsymbol{A}) - (u^2)^* \operatorname{div}\boldsymbol{A} = u^* \operatorname{div}(u\boldsymbol{A}) + (u\boldsymbol{A}, Du) - (u^2)^* \operatorname{div}\boldsymbol{A} \\ &= [(u^*)^2 - (u^2)^*] \operatorname{div}\boldsymbol{A} + u^*(\boldsymbol{A}, Du) + (u\boldsymbol{A}, Du), \end{aligned}$$

that is

$$(u\boldsymbol{A}, Du) = u^*(\boldsymbol{A}, Du) - [(u^*)^2 - (u^2)^*] \operatorname{div}\boldsymbol{A}.$$

Hence (28) follows after observing that $(u^*)^2 - (u^2)^* = 0$ in $\Omega \setminus S_u$ and $(u^*)^2 - (u^2)^* = -(u^+ - u^-)^2/4$ on J_u. The relations (29) and (30) now follow from Theorem 3.3(ii).

The general case of $u \in L^\infty(\Omega)$ can be obtained from the previous case, considering the function $v := u + c$, which is positive if $c > \|u\|_\infty$. Namely, (28) easily follows observing that

$$(v\boldsymbol{A}, Dv) = (u\boldsymbol{A}, Du) + c(\boldsymbol{A}, Du), \quad v^* = u^* + c, \quad J_v = J_u, \quad v^+ - v^+ = u^+ - u^-. \qquad \square$$

Proposition 4.9. *Let* $\boldsymbol{A} \in \mathcal{DM}^\infty_{\rm loc}(\Omega)$ *and* $u, v \in BV_{\rm loc}(\Omega) \cap L^\infty_{\rm loc}(\Omega)$*. Then*

$$(v\boldsymbol{A}, Du) = v^*(\boldsymbol{A}, Du) + \frac{(u^+ - u^-)(v^+ - v^-)}{4} \operatorname{div}\boldsymbol{A} \llcorner (J_u \cap J_v), \tag{31}$$

that is

$$(v\boldsymbol{A}, Du)^d = v^*(\boldsymbol{A}, Du)^d, \tag{32}$$

$$(v\boldsymbol{A}, Du)^j = \frac{\alpha^+ v^+ + \alpha^- v^-}{2}\,(u^+ - u^-)\,\mathcal{H}^{N-1} \llcorner J_u, \tag{33}$$

where $\alpha^\pm := \mathrm{Tr}^\pm(\boldsymbol{A}, J_u)$*.*

Proof. From Lemma 4.8 we have that

$$\begin{aligned} ((u+v)\boldsymbol{A}, D(u+v)) &= (u+v)^*(\boldsymbol{A}, D(u+v)) \\ &\quad + \frac{(u^+ + v^+ - u^- - v^-)^2}{4} \operatorname{div}\boldsymbol{A} \llcorner (J_u \cup J_v). \end{aligned} \tag{34}$$

Let us compute the two sides of this equality. We have that

$$\begin{aligned} LHS &= (u\boldsymbol{A}, Du) + (v\boldsymbol{A}, Dv) + (v\boldsymbol{A}, Du) + (u\boldsymbol{A}, Dv) \\ &= u^*(\boldsymbol{A}, Du) + \frac{(u^+ - u^-)^2}{4} \operatorname{div} \boldsymbol{A} \llcorner J_u + v^*(\boldsymbol{A}, Dv) + \frac{(v^+ - v^-)^2}{4} \operatorname{div} \boldsymbol{A} \llcorner J_v \\ &\quad + (v\boldsymbol{A}, Du) + (u\boldsymbol{A}, Dv) \end{aligned}$$

On the other hand, the right–hand side of (34) is computed as

$$\begin{aligned} RHS = {} & u^*(\boldsymbol{A}, Du) + u^*(\boldsymbol{A}, Dv) + v^*(\boldsymbol{A}, Du) + v^*(\boldsymbol{A}, Dv) \\ & + \frac{(u^+ - u^-)^2}{4} \operatorname{div} \boldsymbol{A} \llcorner J_u + \frac{(v^+ - v^-)^2}{4} \operatorname{div} \boldsymbol{A} \llcorner J_v \\ & + \frac{(u^+ - u^-)(v^+ - v^-)}{2} \operatorname{div} \boldsymbol{A} \llcorner (J_u \cap J_v). \end{aligned}$$

Hence, after some simplifications (34) gives

$$\begin{aligned} (v\boldsymbol{A}, Du) + (u\boldsymbol{A}, Dv) = {} & u^*(\boldsymbol{A}, Dv) + v^*(\boldsymbol{A}, Du) \\ & + \frac{(u^+ - u^-)(v^+ - v^-)}{2} \operatorname{div} \boldsymbol{A} \llcorner (J_u \cap J_v). \end{aligned} \tag{35}$$

Since

$$\operatorname{div}(uv\boldsymbol{A}) = u^* \operatorname{div}(v\boldsymbol{A}) + (v\boldsymbol{A}, Du), \qquad \operatorname{div}(uv\boldsymbol{A}) = v^* \operatorname{div}(u\boldsymbol{A}) + (u\boldsymbol{A}, Dv),$$

it holds

$$(v\boldsymbol{A}, Du) - (u\boldsymbol{A}, Dv) = v^*(\boldsymbol{A}, Du) - u^*(\boldsymbol{A}, Dv). \tag{36}$$

Summing together (35) and (36) we get (31). The relations (32) and (33) now follow from Theorem 3.3(ii). □

Remark 4.10. Observe that, in general,

$$(v\boldsymbol{A}, Du) \neq v^*(\boldsymbol{A}, Du),$$

because the jump part of the two measures can differ on points of $J_u \cap J_v$ (see also the case of $u = v = \chi_E$ in [16, Remark 3.4]).

Proposition 4.11 (Leibniz rule)**.** *Let $\boldsymbol{A} \in \mathcal{DM}^\infty_{\rm loc}(\Omega)$ and $u, v \in BV_{\rm loc}(\Omega) \cap L^\infty_{\rm loc}(\Omega)$. Then*

$$(\boldsymbol{A}, D(uv)) = v^*(\boldsymbol{A}, Du) + u^*(\boldsymbol{A}, Dv). \tag{37}$$

More precisely, the measure $(\boldsymbol{A}, D(uv))$ admits the following decomposition:

(i) *absolutely continuous part: $(\boldsymbol{A}, D(uv))^a = \boldsymbol{A} \cdot \nabla(uv)\, \mathcal{L}^N$, with $\nabla(uv) = u\nabla v + v\nabla u$;*

(ii) *jump part:*

$$(\boldsymbol{A}, D(uv))^j = \frac{\alpha^+ + \alpha^-}{2} (u^+v^+ - u^-v^-) \mathcal{H}^{N-1} \llcorner (J_u \cup J_v).$$

where $\alpha^\pm := \mathrm{Tr}^\pm(\boldsymbol{A}, J_u \cup J_v)$;

(iii) *diffuse part: if, in addition, $|D^c(uv)|(S_{\boldsymbol{A}}) = 0$, then $(\boldsymbol{A}, D(uv))^d = \widetilde{\boldsymbol{A}} \cdot D^d(uv)$, with $D^d(uv) = \widetilde{u} D^d v + \widetilde{v} D^d u$.*

Proof. We have that

$$\begin{aligned}
(\boldsymbol{A}, D(uv)) &= \operatorname{div}(uv\boldsymbol{A}) - (uv)^* \operatorname{div}\boldsymbol{A} = \frac{1}{2}\operatorname{div}(uv\boldsymbol{A}) + \frac{1}{2}\operatorname{div}(uv\boldsymbol{A}) - (uv)^*\operatorname{div}\boldsymbol{A} \\
&= \frac{1}{2}\left[u^*\operatorname{div}(v\boldsymbol{A}) + (v\boldsymbol{A}, Du)\right] + \frac{1}{2}\left[v^*\operatorname{div}(u\boldsymbol{A}) + (u\boldsymbol{A}, Dv)\right] - (uv)^*\operatorname{div}\boldsymbol{A} \\
&= \frac{1}{2}u^*\left[\operatorname{div}(v\boldsymbol{A}) - v^*\operatorname{div}\boldsymbol{A}\right] + \frac{1}{2}v^*\left[\operatorname{div}(u\boldsymbol{A}) - u^*\operatorname{div}\boldsymbol{A}\right] \\
&\quad + \frac{1}{2}(v\boldsymbol{A}, Du) + \frac{1}{2}(u\boldsymbol{A}, Dv) + \left[u^*v^* - (uv)^*\right]\operatorname{div}\boldsymbol{A} \\
&= \frac{1}{2}u^*(\boldsymbol{A}, Dv) + \frac{1}{2}v^*(\boldsymbol{A}, Du) + \frac{1}{2}(v\boldsymbol{A}, Du) + \frac{1}{2}(u\boldsymbol{A}, Dv) \\
&\quad + \left[u^*v^* - (uv)^*\right]\operatorname{div}\boldsymbol{A}\,.
\end{aligned}$$

A direct computation shows that

$$u^*v^* - (uv)^* = -\frac{(u^+ - u^-)(v^+ - v^-)}{4} \qquad \mathcal{H}^{N-1}\text{-a.e. in } J_u \cup J_v,$$

whereas $u^*v^* - (uv)^* = 0$ in $\Omega \setminus (S_u \cup S_v)$.

Hence, using (31) on $(u\boldsymbol{A}, Dv)$ and $(v\boldsymbol{A}, Du)$, we finally get (37). □

Using the results proved so far, Theorem 3.3 can be slightly extended to the case of unbounded BV functions as follows.

Theorem 4.12. *Let $\boldsymbol{A} \in \mathcal{DM}^\infty_{\mathrm{loc}}(\Omega)$, $u \in BV_{\mathrm{loc}}(\Omega)$ and assume that $u^* \in L^1_{\mathrm{loc}}(\Omega, \operatorname{div}\boldsymbol{A})$. Then the pairing $(\boldsymbol{A}, Du)$, defined as a distribution by (8), is a Radon measure in Ω and admits the decomposition given in Theorem 3.3.*

Proof. The fact that $(\boldsymbol{A}, Du)$ is a Radon measure in Ω, with $|(\boldsymbol{A}, Du)| \ll |Du|$, has been proved in [20, Corollary 2.3].

Properties (i), (ii) and (iii) in Theorem 3.3 will follow with a truncation argument similar to that used in the proof of Proposition 2.7 in [6].

More precisely, let us define the truncated functions $u_k := T_k(u)$ where T_k is defined in (26).

Since $|u_k^*| \leq |u^*|$, by the Dominated Convergence Theorem we can pass to the limit in the relation

$$\langle (\boldsymbol{A}, Du_k)\,,\, \varphi\rangle = -\int_\Omega u_k^*\varphi\, d\operatorname{div}\boldsymbol{A} - \int_\Omega u_k\boldsymbol{A}\cdot\nabla\varphi\, dx$$

obtaining that

$$\langle (\boldsymbol{A}, Du_k)\,,\, \varphi\rangle \to \langle (\boldsymbol{A}, Du)\,,\, \varphi\rangle \qquad \forall \varphi \in C_c^\infty(\Omega).$$

Since $|D^c u_k| \leq |D^c u|$, from Theorem 3.3 it holds

$$(\boldsymbol{A}, Du_k)^d = \widetilde{\boldsymbol{A}}\cdot D^d u_k, \qquad \text{if } |D^c u|(S_{\boldsymbol{A}}) = 0, \tag{38}$$

$$(\boldsymbol{A}, Du_k)^j = \operatorname{Tr}^*(\boldsymbol{A}, J_{u_k})(u_k^+ - u_k^-)\mathcal{H}^{N-1} \llcorner J_{u_k} = \operatorname{Tr}^*(\boldsymbol{A}, J_u)(u_k^+ - u_k^-)\mathcal{H}^{N-1} \llcorner J_u. \tag{39}$$

From the Chain Rule Formula (see [3, Example 3.100]) we have that

$$D^d u_k \llcorner \{|\widetilde{u}| < k\} = D^d u \llcorner \{|\widetilde{u}| < k\}.$$

Since, for every $x \in \Omega \setminus S_u$ there exists $k > 0$ such that $x \in \{|\widetilde{u}| < k\}$, from (38) we conclude that (i) and (ii) in Theorem 3.3 hold.

Concerning the jump part, observe that if $x \in J_u$ and $k > \max\{|u^+(x)|, |u^-(x)|\}$, then $x \in J_{u_k}$ and $u_k^\pm(x) = T_k(u^\pm(x)) = u^\pm(x)$. Hence from (39) we can conclude that also property (iii) in Theorem 3.3 holds. □

Remark 4.13. We extract the following fact from the proof of Theorem 4.12. Let $\boldsymbol{A} \in \mathcal{DM}^\infty_{\rm loc}(\Omega)$, $u \in BV_{\rm loc}(\Omega) \cap L^1_{\rm loc}(\Omega, \operatorname{div} \boldsymbol{A})$, and let $u_k := T_k(u)$ be the truncated functions of u, where T_k is defined in (26). If we define

$$\Omega_k := \{x \in J_u : \ |u^\pm(x)| < k\} \cup \{x \in \Omega \setminus S_u : \ |\widetilde{u}(x)| < k\},$$

then it holds

$$(\boldsymbol{A}, Du_k) \llcorner \Omega_k = (\boldsymbol{A}, Du) \llcorner \Omega_k \qquad \forall k > 0.$$

Remark 4.14. Let $\boldsymbol{A} \in \mathcal{DM}^\infty_{\rm loc}(\Omega)$ and $u \in BV_{\rm loc}(\Omega)$. Then $u^* \in L^1_{\rm loc}(\Omega, \operatorname{div} \boldsymbol{A})$ if at least one of the following conditions holds:

(a) $u \in L^\infty_{\rm loc}$;
(b) $\operatorname{div} \boldsymbol{A} \geq 0$ or $\operatorname{div} \boldsymbol{A} \leq 0$.

The first case is trivial. For case (b) the proof follows from [31, Remark 8.3].

We conclude this section with an approximation result in the spirit of [11, Theorem 1.2]. This kind of approximation has been used for example in [6] and [11] as an essential tool in order to pass from smooth vector fields to less regular fields. Unfortunately, in our general setting, properties (iv) and (v) below can be proved only under the additional assumption $|D^c u|(S_{\boldsymbol{A}}| = 0$, so we cannot use this approximation to obtain the Gauss–Green formula in Section 5. Nevertheless, we think that Proposition 4.17 may be useful in order to get semicontinuity results for functionals depending linearly in ∇u.

Proposition 4.15 (Approximation by C^∞ functions)**.** *Let $\boldsymbol{A} \in \mathcal{DM}^\infty(\Omega)$. Then there exists a sequence $(\boldsymbol{A}_k)_k$ in $C^\infty(\Omega, \mathbb{R}^N) \cap L^\infty(\Omega, \mathbb{R}^N)$ satisfying the following properties.*

(i) $\boldsymbol{A}_k \to \boldsymbol{A}$ *in* $L^1(\Omega, \mathbb{R}^N)$ *and* $\int_\Omega |\operatorname{div} \boldsymbol{A}_k| \, dx \to |\operatorname{div} \boldsymbol{A}|(\Omega)$.
(ii) $\operatorname{div} \boldsymbol{A}_k \stackrel{*}{\rightharpoonup} \operatorname{div} \boldsymbol{A}$ *in the weak* sense of measures in* Ω.
(iii) *For every countably $\mathcal{H}^{N-1}$-rectifiable set $\Sigma \subset \Omega$ it holds*

$$\left\langle \operatorname{Tr}^\pm(\boldsymbol{A}_k, \Sigma)\,,\, \varphi \right\rangle \to \langle \operatorname{Tr}^*(\boldsymbol{A}, \Sigma)\,,\, \varphi\rangle \qquad \forall \varphi \in C_c(\Omega),$$

where $\operatorname{Tr}^*(\boldsymbol{A}, \Sigma) := [\operatorname{Tr}^+(\boldsymbol{A}, \Sigma) + \operatorname{Tr}^-(\boldsymbol{A}, \Sigma)]/2$.

If, in addition, $u \in BV_{\rm loc}(\Omega) \cap L^\infty_{\rm loc}(\Omega)$ and $|D^c u|(S_{\boldsymbol{A}}) = 0$, then

(iv) $(\boldsymbol{A}_k, Du) \stackrel{*}{\rightharpoonup} (\boldsymbol{A}, Du)$ *locally in the weak* sense of measures in* Ω;
(v) $\theta(\boldsymbol{A}_k, Du, x) \to \theta(\boldsymbol{A}, Du, x)$ *for* $|Du|$*-a.e.* $x \in \Omega$.

Remark 4.16. It is not difficult to show that a similar approximation result holds also for $\boldsymbol{A} \in \mathcal{DM}^\infty_{\rm loc}(\Omega)$ with a sequence $(\boldsymbol{A}_k)$ in $C^\infty(\Omega, \mathbb{R}^N)$.

Proof. (i) This part is proved in [11, Theorem 1.2]. We just recall, for later use, that for every k the vector field $\boldsymbol{A}_k$ is of the form

$$\boldsymbol{A}_k = \sum_{i=1}^{\infty} \rho_{\varepsilon_i} * (\boldsymbol{A}\varphi_i), \tag{40}$$

where (φ_i) is a partition of unity subordinate to a locally finite covering of Ω depending on k and, for every i, $\varepsilon_i \in (0, 1/k)$ is chosen in such a way that

$$\int_\Omega |\rho_{\varepsilon_i} * (\boldsymbol{A} \cdot \nabla\varphi_i) - \boldsymbol{A} \cdot \nabla\varphi_i| \, dx \leq \frac{1}{k\, 2^i} \tag{41}$$

(see [11], formula (1.8)).

(ii) From (i) we have that

$$\int_\Omega \boldsymbol{A}_k \cdot \nabla\varphi\, dx \to \int_\Omega \boldsymbol{A} \cdot \nabla\varphi\, dx \qquad \forall \varphi \in C^1_c(\Omega),$$

hence (ii) follows from $\sup_k \int_\Omega |\operatorname{div} \boldsymbol{A}_k|\, dx < +\infty$ and the density of $C^0_c(\Omega)$ in $C^1_c(\Omega)$ in the norm of $L^\infty(\Omega)$.

(iii) Before proving (iii), we need to prove the following claim: if $E \Subset \Omega$ is a set of finite perimeter, then

$$\lim_{k\to+\infty} \int_\Omega \chi_E\, \varphi \operatorname{div} \boldsymbol{A}_k\, dx = \int_\Omega \chi^*_E\, \varphi\, d \operatorname{div} \boldsymbol{A}, \qquad \forall \varphi \in C^\infty_c(\Omega). \tag{42}$$

Namely, from the definition (40) of $\boldsymbol{A}_k$ and the identity $\sum_i \nabla\varphi_i = 0$ we have that

$$\operatorname{div} \boldsymbol{A}_k = \sum_i \rho_{\varepsilon_i} * (\varphi_i \operatorname{div} \boldsymbol{A}) + \sum_i \left[\rho_{\varepsilon_i} * (\boldsymbol{A}\cdot\nabla\varphi_i) - \boldsymbol{A}\cdot\nabla\varphi_i\right].$$

From (41) we have that

$$\left|\sum_i \int_\Omega \chi_E \varphi \left[\rho_{\varepsilon_i} * (\boldsymbol{A}\cdot\nabla\varphi_i) - \boldsymbol{A}\cdot\nabla\varphi_i\right]\, dx\right| < \frac{1}{k}\, \|\varphi\|_\infty\,,$$

hence, to prove (42), it is enough to show that

$$\lim_{k\to+\infty} \sum_i \int_\Omega \chi_E \varphi\, \rho_{\varepsilon_i} * (\varphi_i \operatorname{div} \boldsymbol{A}) = \int_\Omega \chi^*_E\, \varphi\, d \operatorname{div} \boldsymbol{A}. \tag{43}$$

On the other hand,

$$\sum_i \int_\Omega \chi_E \varphi\, \rho_{\varepsilon_i} * (\varphi_i \operatorname{div} \boldsymbol{A}) = \sum_i \int_\Omega \rho_{\varepsilon_i} * (\chi_E \varphi)\, \varphi_i\, d \operatorname{div} \boldsymbol{A}\,,$$

hence (43) follows observing that, $\mathcal{H}^{N-1}$–a.e. in Ω,

$$\chi^*_E \varphi - \sum_i \varphi_i \rho_{\varepsilon_i} * (\chi_E \varphi) = \sum_i \varphi_i \left[\chi^*_E \varphi - \rho_{\varepsilon_i} * (\chi_E \varphi)\right] \to 0.$$

Let us now prove (iii). Let $\Omega' \Subset \Omega$ be a set of class C^1. By the definition (6), by (i), (ii) and (42), for every $\varphi \in C^\infty_c(\Omega)$ we have that

$$\begin{aligned}
\left\langle \operatorname{Tr}(\boldsymbol{A}_k, \partial\Omega')\,,\, \varphi\right\rangle &= \int_{\Omega'} \boldsymbol{A}_k \cdot \nabla\varphi\, dx + \int_{\Omega'} \varphi \operatorname{div} \boldsymbol{A}_k\, dx \\
&= \int_\Omega \chi_{\Omega'}\, \boldsymbol{A}_k \cdot \nabla\varphi\, dx + \int_\Omega \chi_{\Omega'}\, \varphi \operatorname{div} \boldsymbol{A}_k\, dx \\
&\to \int_\Omega \chi_{\Omega'}\, \boldsymbol{A} \cdot \nabla\varphi\, dx + \int_\Omega \chi^*_{\Omega'}\, \varphi\, d \operatorname{div} \boldsymbol{A} \\
&= \int_{\Omega'} \boldsymbol{A} \cdot \nabla\varphi\, dx + \int_{\Omega'} \varphi\, d \operatorname{div} \boldsymbol{A} + \frac{1}{2} \int_{\partial\Omega'} \varphi\, d \operatorname{div} \boldsymbol{A}.
\end{aligned}$$

Hence, using the notations of Section 2.4, by (7) on the set $N_i \subset \partial\Omega_i \cap \partial\Omega'_i$ it holds

$$\operatorname{Tr}^-(\boldsymbol{A}_k, \Sigma) = \operatorname{Tr}(\boldsymbol{A}_k, \partial\Omega_i) \to \operatorname{Tr}^-(\boldsymbol{A}, \Sigma) + \frac{1}{2}\left[\operatorname{Tr}^+(\boldsymbol{A}, \Sigma) - \operatorname{Tr}^-(\boldsymbol{A}, \Sigma)\right] = \operatorname{Tr}^*(\boldsymbol{A}, \Sigma),$$

where the convergence is in the weak* sense of L^∞. A similar computation holds for $\operatorname{Tr}^+(\boldsymbol{A}_k, \Sigma)$.

(iv) From the very definition (40) of $\boldsymbol{A}_k$, we have that

$$\boldsymbol{A}_k(x) \to \widetilde{\boldsymbol{A}}(x) \qquad \text{for } \mathcal{H}^{N-1}\text{-a.e. } x \in \Omega. \tag{44}$$

From Theorem 3.3, (44) and (iii) we have that

$$\boldsymbol{A}_k(x) \cdot D^d u \to \widetilde{\boldsymbol{A}}(x) \cdot D^d u, \qquad |D^d u|\text{-a.e. in } \Omega,$$
$$\operatorname{Tr}(\boldsymbol{A}_k, J_u)(u^+ - u^-) \to \operatorname{Tr}^*(\boldsymbol{A}, J_u)(u^+ - u^-), \qquad \mathcal{H}^{N-1}\text{-a.e. in } J_u,$$

hence

$$(\boldsymbol{A}_k, Du)^d \to (\boldsymbol{A}, Du)^d, \qquad (\boldsymbol{A}_k, Du)^j \to (\boldsymbol{A}, Du)^j .$$

(v) Using the definition (17) of θ, we have that, for every $\varphi \in C_c(\Omega)$,

$$\int_\Omega \theta(\boldsymbol{A}_k, Du, x)\varphi(x)\, d|Du| = \langle (\boldsymbol{A}_k, Du)\, ,\, \varphi\rangle$$
$$\to \langle (\boldsymbol{A}, Du)\, ,\, \varphi\rangle = \int_\Omega \theta(\boldsymbol{A}, Du, x)\varphi(x)\, d|Du|,$$

hence (v) follows. □

Proposition 4.17. *Let $(\boldsymbol{A}_k)$ be a sequence in $\mathcal{DM}^\infty(\Omega)$ such that $\boldsymbol{A}_k \to \boldsymbol{A} \in \mathcal{DM}^\infty(\Omega)$ in $L^1_{\rm loc}(\Omega;\mathbb{R}^N)$ and the sequence $\mu_k := \operatorname{div} \boldsymbol{A}_k$ locally weakly* converges to $\mu := \operatorname{div} \boldsymbol{A}$. Let $u \in BV(\Omega) \cap L^\infty(\Omega)$ be compactly supported in Ω. Then the following hold:*

(a) *If the measures μ_h are positive and $u \geq 0$, then*

$$\int_\Omega u^- \, d\mu \leq \liminf_{h\to\infty} \int_\Omega u^- \, d\mu_h, \tag{45}$$

$$\int_\Omega u^+ \, d\mu \geq \limsup_{h\to\infty} \int_\Omega u^+ \, d\mu_h, \tag{46}$$

where u^- (resp. u^+) is the approximate lower (resp. upper) limit of u.

(b) *Assume that $|\mu_h| \overset{*}{\rightharpoonup} |\mu|$ locally weakly*. If $|\mu|(J_u) = 0$, then*

$$\int_\Omega u^* \, d\mu = \lim_{h\to+\infty} \int_\Omega u^* \, d\mu_h, \qquad \int_\Omega u^\pm \, d\mu = \lim_{h\to+\infty} \int_\Omega u^\pm \, d\mu_h. \tag{47}$$

Proof. (a) Let us first consider the case $u \in W^{1,1}(\Omega) \cap L^\infty(\Omega)$. Since u has compact support in Ω, it follows that

$$\int_\Omega \widetilde{u}\, d\mu = -\int_\Omega \nabla u \cdot \boldsymbol{A}\, dx = \lim_{k\to\infty} -\int_\Omega \nabla u \cdot \boldsymbol{A}_k\, dx = \lim_{k\to\infty} \int_\Omega \widetilde{u}\, d\mu_k. \tag{48}$$

Let us now consider the general case $u \in BV(\Omega)$. From Theorem 2.3, the approximate upper limit u^+ and the approximate lower limit u^- are C_1-quasi upper semicontinuous and C_1-quasi lower semicontinuous, respectively. In order to prove (45), we remark that by Lemma 2.4 there exists an increasing sequence of nonnegative functions $(u_h) \subseteq W^{1,1}(\Omega)$ such that, for every $h \in \mathbb{N}$, u_h is approximately continuous $\mathcal{H}^{N-1}$-almost everywhere in Ω and $\widetilde{u}_h(x) \to u^-(x)$, when $h \to +\infty$, for $\mathcal{H}^{N-1}$-almost every $x \in \Omega$.

Therefore for $\mathcal{H}^{N-1}$-almost every $x \in \Omega$

$$u^-(x) = \sup_{h\in\mathbb{N}} \widetilde{u}_h(x)$$

and for every $\phi \in C^0_c(\Omega)$, with $0 \leq \phi \leq 1$, we have

$$\int_\Omega \phi u^- \, d\mu = \sup_{h\in\mathbb{N}} \int_\Omega \phi \widetilde{u}_h \, d\mu.$$

Moreover, since $u \in L^\infty(\Omega)$, we can assume that, for every $h \in \mathbb{N}$, $u_h \in L^\infty(\Omega)$, then $\phi\, u_h \in W^{1,1}(\Omega) \cap L^\infty(\Omega)$, with compact support, and $\mu(S_{\phi u_h}) = 0$. Hence, by (48),

$$\int_\Omega \phi\, \widetilde{u}_h\, d\mu = \lim_{k\to\infty} \int_\Omega \phi\, \widetilde{u}_h\, d\mu_k \le \liminf_{k\to\infty} \int_\Omega \phi\, u^-\, d\mu_k.$$

The conclusion follows taking the supremum among all the functions $\phi \in C^0_c(\Omega)$, with $0 \le \phi \le 1$, and among the $h \in \mathbb{N}$.

The proof of (46) is similar, since by Lemma 2.4 there exists a decreasing sequence of nonnegative functions $(v_h) \subseteq W^{1,1}(\Omega)$ such that, for every $h \in \mathbb{N}$, v_h is approximately continuous $\mathcal{H}^{N-1}$-almost everywhere in Ω and $\widetilde{v}_h(x) \to u^+(x)$, when $h \to +\infty$, for $\mathcal{H}^{N-1}$-almost every $x \in \Omega$. Therefore for $\mathcal{H}^{N-1}$-almost every $x \in \Omega$

$$u^+(x) = \inf_{h\in\mathbb{N}} \widetilde{v}_h(x)$$

and we have

$$\int_\Omega u^+\, d\mu = \inf_{h\in\mathbb{N}} \int_\Omega \widetilde{v}_h\, d\mu.$$

Moreover, since $u \in L^\infty(\Omega)$, we have that $v_h \in L^\infty(\Omega)$ for any h sufficiently large, and since the support of u is compact and $u \in L^\infty(\Omega)$ there exists a relatively compact neighborhood U of the support of u which contains the support of v_h for any h sufficiently large. Therefore $v_h \in W^{1,1}(\Omega) \cap L^\infty(\Omega)$ and it has compact support for h sufficiently large, and $\mu(S_{v_h}) = 0$. Hence we get

$$\int_\Omega \widetilde{v}_h\, d\mu = \lim_{k\to\infty} \int_\Omega \widetilde{v}_h\, d\mu_k \ge \limsup_{k\to\infty} \int_\Omega v^+\, d\mu_k.$$

The conclusion follows taking the infimum among the $h \in \mathbb{N}$.

(b) In order to prove (47) firstly we assume that $\mu_k \ge 0$. We observe that $\widetilde{v}_h - \widetilde{u}_h \to 0$ $\mathcal{H}^{N-1}$-a.e. on $\Omega \setminus S_u$ and, since $\mu(S_u) = 0$,

$$\lim_{h\to+\infty} \int_\Omega (\widetilde{v}_h - \widetilde{u}_h)\, d\mu = 0.$$

We have

$$\begin{aligned}\int_\Omega \widetilde{u}_h\, d\mu &= \lim_{k\to\infty} \int_\Omega \widetilde{u}_h\, d\mu_k \le \liminf_{k\to\infty} \int_\Omega u^-\, d\mu_k \le \limsup_{k\to\infty} \int_\Omega u^+\, d\mu_k \\ &\le \lim_{k\to\infty} \int_\Omega \widetilde{v}_h\, d\mu_k = \int_\Omega \widetilde{v}_h\, d\mu.\end{aligned}$$

By taking $h \to +\infty$, we obtain that

$$\int_\Omega u^-\, d\mu = \lim_{k\to\infty} \int_\Omega u^-\, d\mu_k = \lim_{k\to\infty} \int_\Omega u^+\, d\mu_k = \int_\Omega u^+\, d\mu.$$

By the definition of u^* we get

$$\lim_{k\to\infty} \int_\Omega u^-\, d\mu_k = \lim_{k\to\infty} \int_\Omega u^+\, d\mu_k = \lim_{k\to\infty} \int_\Omega u^*\, d\mu_k.$$

The general case can we obtained by writing the measure μ as the difference between its positive and its negative part. This concludes the proof. □

Remark 4.18. We would like to underline two consequences of Proposition 4.17.

(a) By (47), for every $u \in BV_{\rm loc}(\Omega) \cap L^\infty_{\rm loc}(\Omega)$, if $|\operatorname{div} \boldsymbol{A}|(J_u) = 0$, then

$$\langle (\boldsymbol{A}_k, Du)\,,\, \phi\rangle \to \langle (\boldsymbol{A}, Du)\,,\, \phi\rangle \qquad \forall \phi \in C^0_c(\Omega)\,.$$

(b) If $-\operatorname{div}\boldsymbol{A}_k \geq 0$, then

$$-\int_\Omega u^- \operatorname{div}\boldsymbol{A} \leq \liminf_{k\to\infty}\left(-\int_\Omega u^- \operatorname{div}\boldsymbol{A}_k\right).$$

5. The Gauss–Green formula

In this section we will prove a generalized Gauss–Green formula for vector fields $\boldsymbol{A} \in \mathcal{DM}^\infty_{\rm loc}(\mathbb{R}^N)$ on a set $E \subset \mathbb{R}^N$ of finite perimeter.

Using the conventions of Section 2.4, we will assume that the generalized normal vector on $\partial^* E$ coincides $\mathcal{H}^{N-1}$-a.e. on $\partial^* E$ with the measure–theoretic *interior* unit normal vector $\widetilde{\nu}_E$ to E. Hence, if $\alpha^\pm := \operatorname{Tr}^\pm(\boldsymbol{A}, \partial^* E)$ are the normal traces of $\boldsymbol{A}$ on $\partial^* E$ according to our definition in Section 2.4, then, using the notation of [16], $\alpha^+ \equiv (\mathcal{A}_i\cdot\widetilde{\nu}_E)$ and $\alpha^- \equiv (\mathcal{A}_e\cdot\widetilde{\nu}_E)$ correspond respectively to the interior and the exterior normal traces on $\partial^* E$.

Since $|D\chi_E| = \mathcal{H}^{N-1} \llcorner \partial^* E$, from Proposition 4.9 we deduce that α^+ and α^- are respectively the Radon–Nikodým derivatives with respect to $|D\chi_E|$ of the measures

$$\sigma_i := 2\,(\chi_E\boldsymbol{A}, D\chi_E), \qquad \sigma_e := 2\,(\chi_{\mathbb{R}^N\setminus E}\boldsymbol{A}, D\chi_E),$$

that are both absolutely continuous with respect to $|D\chi_E|$, hence

$$\sigma_i = \alpha^+\,\mathcal{H}^{N-1}\llcorner\partial^* E, \qquad \sigma_e = \alpha^-\,\mathcal{H}^{N-1}\llcorner\partial^* E$$

(see also [16, Theorem 3.2]).

For example, if E is an open bounded set of class C^1 and $\boldsymbol{A}$ is a piecewise continuous vector field that can be extended continuously by vector fields $\boldsymbol{A}_i$ and $\boldsymbol{A}_e$ in $\overline{E}$ and $\mathbb{R}^N\setminus E$ respectively, then

$$\alpha^+ = -\operatorname{Tr}(\boldsymbol{A}, \partial E) = -\boldsymbol{A}_i\cdot\nu_E = \boldsymbol{A}_i\cdot\widetilde{\nu}_E, \qquad \alpha^- = \boldsymbol{A}_e\cdot\widetilde{\nu}_E.$$

If $u \in BV_{\rm loc}(\mathbb{R}^N)$, in the following formulas we will understand

$$u^\pm(x) := \widetilde{u}(x) \qquad \forall x\in\mathbb{R}^N\setminus S_u.$$

Theorem 5.1. *Let $\boldsymbol{A} \in \mathcal{DM}^\infty_{\rm loc}(\mathbb{R}^N)$, $u \in BV_{\rm loc}(\mathbb{R}^N)$ and assume that $u^* \in L^1_{\rm loc}(\mathbb{R}^N, \operatorname{div}\boldsymbol{A})$. Let $E\subset\mathbb{R}^N$ be a bounded set with finite perimeter. Then the following Gauss–Green formulas hold:*

$$\int_{E^1} u^*\,d\operatorname{div}\boldsymbol{A} + \int_{E^1}(\boldsymbol{A}, Du) = -\int_{\partial^* E}\alpha^+u^+\ d\mathcal{H}^{N-1}\,, \tag{49}$$

$$\int_{E^1\cup\partial^* E} u^*\,d\operatorname{div}\boldsymbol{A} + \int_{E^1\cup\partial^* E}(\boldsymbol{A}, Du) = -\int_{\partial^* E}\alpha^-u^-\ d\mathcal{H}^{N-1}\,, \tag{50}$$

where E^1 is the measure theoretic interior of E and $\alpha^\pm := \operatorname{Tr}^\pm(\boldsymbol{A}, \partial^* E)$ are the normal traces of $\boldsymbol{A}$ when $\partial^* E$ is oriented with respect to the interior unit normal vector.

Remark 5.2. This result extends Theorem 5.3 of [15] where $u = \phi \in C^\infty_c$ (see also [16, Theorem 4.1] where $u = \phi \in \operatorname{Lip}_{\rm loc}$). Leonardi and Saracco (see Theorem 2.2 in [28]) established a similar formula by considering the collection $X(\Omega)$ of vector fields $\boldsymbol{A} \in L^\infty(\Omega;\mathbb{R}^N)\cap C^0(\Omega;\mathbb{R}^N)$ such that $\operatorname{div}\boldsymbol{A}\in L^\infty(\Omega)$ and by assuming that the set E with finite perimeter satisfies an additional weak regularity condition.

Proof. Since E is bounded, without loss of generality we can assume that $\boldsymbol{A}\in\mathcal{DM}^\infty(\mathbb{R}^N)$ and $u\in BV(\mathbb{R}^N)$. We divide the proof in two steps.

Step 1. Firstly, we consider the case $u \in L^\infty(\mathbb{R}^N)$. Since E is a bounded set with finite perimeter, we have that $\chi_E \in BV(\mathbb{R}^N)$ and the reduced boundary $\partial^* E$ is a $\mathcal{H}^{N-1}$-rectifiable set. Moreover, the vector field $\chi_E u\boldsymbol{A}$ is compactly supported, so that

$$\operatorname{div}(\chi_E u\boldsymbol{A})(\mathbb{R}^N) = 0$$

(see [16, Lemma 3.1]). Hence by choosing in (9) χ_E instead of u and $u\boldsymbol{A}$ instead of $\boldsymbol{A}$, we get

$$\int_{\mathbb{R}^N} \chi_E^* \, d \operatorname{div}(u\boldsymbol{A}) = -(u\boldsymbol{A}, D\chi_E)(\mathbb{R}^N). \tag{51}$$

We recall that

$$\chi_E^* = \chi_{E^1} + \frac{1}{2}\chi_{\partial^* E},$$

and, by Proposition 3.1 and the definition of normal traces it holds

$$\operatorname{div}(u\boldsymbol{A}) \llcorner \partial^* E = (u^+\alpha^+ - u^-\alpha^-)\mathcal{H}^{N-1} \llcorner \partial^* E\,.$$

Hence

$$\int_{\mathbb{R}^N} \chi_E^* \, d \operatorname{div}(u\boldsymbol{A}) = \int_{E^1} d \operatorname{div}(u\boldsymbol{A}) + \frac{1}{2}\int_{\partial^* E} [u^+\alpha^+ - u^-\alpha^-] \; d\mathcal{H}^{N-1}\,. \tag{52}$$

On the other hand $D\chi_E = \widetilde{\nu}_E \, \mathcal{H}^{N-1} \llcorner \partial^* E$ so that, by Proposition 4.9,

$$(u\boldsymbol{A}, D\chi_E) = (u\alpha)^* \, \mathcal{H}^{N-1} \llcorner \partial^* E,$$

that in turn gives

$$(u\boldsymbol{A}, D\chi_E)(\mathbb{R}^N) = \int_{\partial^* E} \frac{1}{2}[u^+\alpha^+ + u^-\alpha^-] \; d\mathcal{H}^{N-1}\,. \tag{53}$$

Finally, substituting (52) and (53) in (51) and simplifying, we obtain (49).

On the other hand,

$$\begin{aligned}\int_{E^1 \cup \partial^* E} d \operatorname{div}(u\boldsymbol{A}) &= \int_{E^1} d \operatorname{div}(u\boldsymbol{A}) + \int_{\partial^* E} [u^+\alpha^+ - u^-\alpha^-] \; d\mathcal{H}^{N-1} \\ &= -\int_{\partial^* E} u^+\alpha^+ \; d\mathcal{H}^{N-1} + \int_{\partial^* E} [u^+\alpha^+ - u^-\alpha^-] \; d\mathcal{H}^{N-1}\,,\end{aligned}$$

hence (50) follows. This concludes the proof of Step 1.

Step 2. Let us consider now $u \in BV(\mathbb{R}^N)$ such that $u^* \in L^1_{\text{loc}}(\mathbb{R}^N, \operatorname{div}\boldsymbol{A})$. As in the proof of Theorem 4.12, let $u_k := T_k(u)$ be the truncated functions of u, where T_k is the truncation operator defined in (26).

By Step 1, since $T_k(u) \in L^\infty(\mathbb{R}^N)$ we obtain

$$\int_{E^1} T_k(u)^* \, d \operatorname{div}\boldsymbol{A} + \int_{E^1} (\boldsymbol{A}, DT_k(u)) = -\int_{\partial^* E} \alpha^+ T_k(u^+) \; d\mathcal{H}^{N-1}\,, \tag{54}$$

for every $k > 0$. We have that

$$T_k(u)^* = \frac{T_k(u)^+ + T_k(u)^-}{2} = \frac{T_k(u^+) + T_k(u^-)}{2} \to \frac{u^+ + u^-}{2} = u^*, \qquad \mathcal{H}^{N-1}\text{-a.e.},$$

hence $T_k(u)^*(x) \to u^*(x)$ for $|\operatorname{div}\boldsymbol{A}|$-a.e. $x \in \mathbb{R}^N$. Since $|T_k(u)^*| \le |u^*| \in L^1_{\text{loc}}(\mathbb{R}^N, \operatorname{div}\boldsymbol{A})$, from the Dominated Convergence Theorem we have that

$$\lim_{k \to +\infty} \int_{E^1} T_k(u)^* \, d \operatorname{div}\boldsymbol{A} = \int_{E^1} u^* \, d \operatorname{div}\boldsymbol{A}. \tag{55}$$

With a similar argument we also get that

$$\lim_{k\to+\infty}\int_{\partial^* E}\alpha^+ T_k(u)^+\,d\mathcal{H}^{N-1}=\int_{\partial^* E}\alpha^+ u^+\,d\mathcal{H}^{N-1}. \tag{56}$$

On the other hand, by the definition (9) of pairing, for every $\varphi\in C_c^\infty(\mathbb{R}^N$ it holds

$$\langle(\boldsymbol{A},DT_k(u)),\varphi\rangle=-\int_{\mathbb{R}^N}T_k(u)^*\varphi\,d\operatorname{div}\boldsymbol{A}-\int_{\mathbb{R}^N}T_k(u)\boldsymbol{A}\cdot\nabla\varphi\,dx\,.$$

We can use the Dominated Convergence Theorem in both integrals at the right–hand side (for the first one we can reason as in (55)), obtaining

$$\lim_{k\to\infty}\int_{E^1}(\boldsymbol{A},DT_k(u))=\int_{E^1}(\boldsymbol{A},Du)\,. \tag{57}$$

Finally, from (54), (55), (56) and (57) we get (49). Formula (50) can be obtained in a similar way. □

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