# A SHARP STABILITY RESULT
# FOR THE GAUSS MEAN VALUE FORMULA

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Abstract. We prove a quantitative stability result for the Gauss mean value formula. We also show by an example that the estimate proved here is sharp.

## 1. Introduction

Among the various rigidity properties satisfied by balls one of the best known example is provided by the Gauss mean value formula for harmonic functions. A simple proof of this fact was given by Kuran in [7]. Denoting by $D$ an open set in $\mathbb{R}^n$, $n \geq 2$, and by $\mathcal{H}(D)$ the family of the harmonic functions in $D$, his result reads as follows.

**Theorem 1.1.** *Let $D \subset \mathbb{R}^n$ be an open set with finite measure and let $x_0 \in D$ be such that*

$$u(x_0) = \fint_D u(x)\,dx \qquad \forall u \in \mathcal{H}(D) \cap L^1(D). \tag{1.1}$$

*Then $D$ is a ball centered at $x_0$.*

In view of this nice characterization of balls it is natural to raise the question of the stability of the mean value equality (1.1). To this end, given a set $D$ of finite measure and $x_0 \in D$ we define the rescaled *Gauss mean value gap of $D$ relative to $x_0$* as

$$G(D, x_0) := \sup_{u \in \mathcal{H}(D) \cap L^1(D), u \not\equiv 0} \frac{\left| u(x_0) - \fint_D u(x)\,dx \right|}{\|u\|_{\widetilde{L}^1(D)}}, \tag{1.2}$$

where we have set

$$\|u\|_{\widetilde{L}^1(D)} := \fint_D |u(x)|\,dx.$$

In this paper we show that the mean value gap $G(D, x_0)$ controls the $L^1$ distance of $D$ from a ball. Precisely, setting $r_{x_0}$ :=dist$(x_0, \partial D)$, we prove the following stability inequality.

**Theorem 1.2.** *There exists a constant $C(n)$ such that if $D \subset \mathbb{R}^n$ is an open set of finite measure and $x_0 \in D$, then*

$$\frac{|D \setminus B(x_0, r_{x_0})|}{|D|} \leq C(n)\, G(D, x_0). \tag{1.3}$$

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Note that this result immediately implies Kuran's Theorem 1.1. In fact, if $G(D, x_0) = 0$ from (1.3) we have that $|D \setminus B(x_0, r_{x_0})| = 0$. Thus, since $D$ contains $B(x_0, r_{x_0})$ and is open, $D = B(x_0, r_{x_0})$.

A common way to measure the distance of a measurable set $D \subset \mathbb{R}^n$, $|D| < \infty$, from a ball is provided by the so called *Fraenkel asymmetry* which is defined by setting

$$\alpha(D) := \min_{x \in \mathbb{R}^n} \frac{|D \triangle B(x, r_D)|}{|D|},$$

where $r_D$ is the radius of a ball with the same measure of $D$. Since $|D \triangle B(x_0, r_D)| = 2|D \setminus B(x_0, r_D)| \le |D \setminus B(x_0, r_{x_0})|$, the stability estimate (1.3) implies that

*if $D$ is an open set of finite measure and $x_0 \in D$, then*

$$\alpha(D) \le 2C(n)\, G(D, x_0). \tag{1.4}$$

Inequality (1.4) is reminiscent of other stability estimates, such as the quantitative isoperimetric inequality, see [6] and [5] for the anisotropic case, which states that there exists a constant $c(n)$ such that if $D \subset \mathbb{R}^n$ is a measurable set of finite measure, then

$$\alpha(D)^2 \le c(n) \left( \frac{P(D) - P(B(0, r_D))}{P(B(0, r_D))} \right), \tag{1.5}$$

where $P(\cdot)$ stands for the perimeter. Here, the right hand side of (1.5) represents the gap between the perimeter of $D$ and the (minimal) perimeter of the ball with the same volume.

Another important property of balls, the Faber-Krahn inequality, states that they minimize the first Dirichlet eigenvalue of the Laplacian $\lambda(D)$ among all open sets $D$ with the same volume. The quantitative version of this inequality has been recently established in [2], where it is proved that there exists a constant $\kappa(n)$ such that if $D$ is an open set of finite measure then

$$\alpha(D)^2 \le \kappa(n) \left( \frac{\lambda(D) - \lambda(B(0, r_D))}{\lambda(B(0, r_D))} \right). \tag{1.6}$$

Note that in (1.5) and (1.6), as well as in many other stability estimates of this kind, the Fraenkel asymmetry always appears with a power 2. In our case we have

$$\alpha(D)^\gamma \le C(n, \gamma)\, G(D, x_0) \qquad \forall \gamma \ge 1 \tag{1.7}$$

with $C(n, \gamma) = 2^\gamma C(n)$. This immediately follows by (1.4), since $\alpha(D) \le 2$. Using our estimate from above of the mean value gap proved in Theorem 3.1, we will show in Section 4 that inequality (1.7) does not hold for $\gamma < 1$. Hence (1.3) is sharp.

Established that (1.3) gives an optimal estimate from below of the mean value gap, it is of interest to understand the behavior of the Gauss gap $G(D, x_0)$ when $D$ converges to a ball centered in $x_0$. Precisely, one would like to know which is the coarsest convergence on open sets such that, given a sequence $D_h$ converging to a ball $B(x_0, r)$ and points $x_h \in D_h$ with $x_h \to x_0$, one has that $G(D_h, x_h) \to 0$. We will show with an example that the $L^1$ convergence of the sets $D_h$ to the unit ball does not guarantee the convergence to zero of the Gauss gap $G(D_h, x_h)$, see Section 4. However a positive answer is given in the above mentioned Theorem 3.1.

We remark that our technique to prove the stability result Theorem 1.2 does not seem suitable to obtain a similar result for the Gauss gap related to the surface average.

An interesting stability result in this direction has been obtained in dimension $n = 2$ by Agostiniani and Magnanini in [1].

The plan of the paper is the following. In Section 2 we prove the stability result Theorem 1.2. In Section 3 we provide an estimate from above of the Gauss mean value gap. We will use this estimate in Section 4 to prove the sharpness of our stability estimates (1.3) and (1.4). In the same section we exhibit an example showing that the Gauss gap is not continuous with respect to the pointwise convergence of the characteristic functions of the domains.

## 2. Proof of Theorem 1.2

Let $D \subseteq \mathbb{R}^n$, $n \geq 2$, be an open set with finite Lebesgue measure. Fixed $x_0 \in D$, denote $r_{x_0} :=\text{dist}(x_0, \partial D)$. Then there exists $x_1 \in \partial B(x_0, r_{x_0}) \cap \partial D$.

As in Kuran [7], we define $h : D \to \mathbb{R}$,

$$h(x) := 1 + r_{x_0}^{n-2} \frac{|x - x_0|^2 - r_{x_0}^2}{|x - x_1|^n}.$$

**Lemma 2.1.** *$h \in \mathcal{H}(D) \cap L^1(D)$ and*

$$\|h\|_{L^1(D)} \leq c(n, |D|),$$

*where $c(n, |D|)$ denotes a constant only depending on $n$ and $|D|$.*

*Proof.* Let $P$ be the Poisson kernel for the ball $B(0, r_{x_0})$. Then

$$P(x, y) = c(r_{x_0}, n) \frac{r_{x_0}^2 - |x|^2}{|x - y|^n} \quad x \in B(0, r_{x_0}), y \in \partial B(0, r_{x_0}),$$

for some positive constant $c(r_{x_0}, n)$. It is well known that $x \mapsto P(x, y)$ is harmonic in $\mathbb{R}^n \setminus \{y\}$. By a translation argument and taking into account that $x_1 \notin D$, we conclude that $h \in \mathcal{H}(D)$.

Let us prove that $h \in L^1(D)$ and let us estimate its $L^1$-norm.
Using the inequalities

$$||x - x_0| - r_{x_0}| = ||x - x_0| - |x_1 - x_0|| \leq |x - x_1|,$$

$$|x - x_0| + r_{x_0} \leq |x - x_1| + 2r_{x_0},$$

and taking into account that

$$r_{x_0} \leq \left( \frac{|D|}{\omega_n} \right)^{\frac{1}{n}}, \tag{2.1}$$

where $\omega_n$ is the Lebesgue measure of the unit ball of $\mathbb{R}^n$, we get

$$\begin{aligned} \frac{||x - x_0|^2 - r_{x_0}^2|}{|x - x_1|^n} &\leq \frac{|x - x_1|(|x - x_1| + 2r_{x_0})}{|x - x_1|^n} \\ &\leq \frac{1}{|x - x_1|^{n-2}} + \left( \frac{|D|}{\omega_n} \right)^{\frac{1}{n}} \frac{2}{|x - x_1|^{n-1}}. \end{aligned} \tag{2.2}$$

Thus, recalling (2.1), $h$ is summable in $D \cap B(x_1, 1)$ and

$$\|h\|_{L^1(D \cap B(x_1, 1))} \leq c(|D|, n).$$

If $x \notin B(x_1, 1)$ then $|x - x_1| \geq 1$, so (2.2) implies

$$\frac{||x-x_0|^2 - r_{x_0}^2|}{|x-x_1|^n} \leq 1 + 2\left(\frac{|D|}{\omega_n}\right)^{\frac{1}{n}} \qquad \forall x \in D \setminus B(x_1, 1).$$

Therefore, using (2.1),

$$\int_{D\setminus B(x_1,1)} |h(x)|\, dx \leq \left(1 + \left(\frac{|D|}{\omega_n}\right)^{\frac{n-2}{n}} + 2\left(\frac{|D|}{\omega_n}\right)^{\frac{n-1}{n}}\right)|D| < \infty.$$

We conclude that the $L^1$-norm of $h$ in $D$ can be estimated by a constant only depending on $n$ and $|D|$. □

We are now ready to prove our main stability result.

*Proof of Theorem 1.2.* Since $G(D, x_0) = G(D - x_0, 0)$, we may assume without loss of generality that $x_0 = 0$. Moreover, since both $G(D, 0)$ and the left hand side of (1.3) are scaling invariant, we may also assume that $|D| = 1$.
By Lemma 2.1, $h \in \mathcal{H}(D) \cap L^1(D)$, moreover $h(0) = 0$. Therefore

$$G(D,0) \geq \frac{\left|h(0) - \fint_D h(x)\, dx\right|}{\|h\|_{\tilde{L}^1(D)}} = \frac{\left|\fint_D h(x)\, dx\right|}{\|h\|_{\tilde{L}^1(D)}} = \frac{\left|\int_D h(x)\, dx\right|}{\|h\|_{L^1(D)}}.$$

Using $B(0, r_0) \subseteq D$ and the Gauss mean value Theorem, we have

$$\int_D h(x)\, dx = \int_{D\setminus B(0,r_0)} h(x)\, dx + |B(0,r_0)| h(0) = \int_{D\setminus B(0,r_0)} h(x)\, dx.$$

So we have proved that

$$G(D,0) \geq \frac{\left|\int_{D\setminus B(0,r_0)} h(x)\, dx\right|}{\|h\|_{L^1(D)}}.$$

Taking into account that $h(x) \geq 1$ in $D \setminus B(0, r_0)$, we obtain

$$|D \setminus B(0, r_0)| \leq G(D, 0) \int_D |h(x)|\, dx.$$

We now observe that $r_0 \leq |B(0,1)|^{-1/n}$ and $|D| = 1$. Then Lemma 1.1 immediately implies

$$\int_D |h(x)|\, dx \leq C(n)$$

for some positive constant $C(n)$ depending only on the dimension. Hence, (1.3) follows. □

## 3. An estimate from above of the Gauss mean value gap

In this section we prove an estimate from above of the mean value gap $G(D, 0)$ for a particular class of smooth open sets close to a ball. To state it we denote $d_e$ the euclidean norm; i.e., $d_e(x) = |x|$.

Given a homogeneous norm $d$ in $\mathbb{R}^n$, such that $d \in C^\infty(\mathbb{R}^n \setminus \{0\})$, we denote by $D$ the $d$-unit ball; i.e,

$$D := \{x \in \mathbb{R}^n \,:\, d(x) < 1\}.$$

By definition of $D$,

$$\partial D = \{x \,:\, d(x) = 1\}.$$

We assume that

$$\frac{1}{R} d_e(x) \leq d(x) \leq R d_e(x) \qquad \forall x \in \mathbb{R}^n, \tag{3.1}$$

for some constant $R \geq 1$. In particular, (3.1) implies

$$\partial D \subseteq A_R, \tag{3.2}$$

where

$$A_R := \overline{B(0,R)} \setminus B(0, 1/R).$$

The main result in this section is the following one.

**Theorem 3.1.** *Let $d$ be a homogeneous norm in $\mathbb{R}^n$, $d \in C^\infty(\mathbb{R}^n \setminus \{0\})$, satisfying* (3.1) *and let $D$ be the $d$-unit ball. Then, for all $\alpha \in (0,1]$,*

$$G(D,0) \leq c \, \|d - d_e\|_{C^{2,\alpha}(A_{2R})} \,,$$

*where $c$ is a positive constant depending only on $n$, $R$, $\alpha$, and $\|d\|_{C^{2,\alpha}(A_{2R})}$.*

As a straightforward consequence we have the following result.

**Corollary 3.2.** *Let $(d_h)$ be a sequence of homogeneous norms in $\mathbb{R}^n$, $d_h \in C^\infty(\mathbb{R}^n \setminus \{0\})$, such that*

$$d_h \to d_e \quad \text{in} \quad C^{2,\alpha}_{\text{loc}}(\mathbb{R}^n \setminus \{0\})$$

*for some $\alpha \in (0,1]$. Then*

$$\lim_{h \to \infty} G(D_h; 0) = 0,$$

*where $D_h = \{x \,:\, d_h(x) < 1\}$.*

Using Theorem 3.1 we will establish with an example the sharpness of (1.3) in any dimension $n \geq 2$, see Section 4.

In the proof of Theorem 3.1 we need the following estimate.

**Lemma 3.3.** *Let $d$ be a homogeneous norm in $\mathbb{R}^n$, satisfying* (3.1) *and such that $d \in C^1(\mathbb{R}^n \setminus \{0\})$.*
*Then*

$$\sup_{D \setminus \{0\}} \big| |\nabla d| - 1 \big\| \leq \|d - d_e\|_{C^1(A_R)}. \tag{3.3}$$

*Moreover,*

$$\big| |D| - \omega_n \big| \leq C(n,R) \|d - d_e\|_{C^0(A_R)} \tag{3.4}$$

*with $C(n,R)$ only depending on $n$ and $R$.*

*Proof.* Since $|\nabla d|$ is homogeneous of degree 0, then

$$\sup_{D \setminus \{0\}} \big| |\nabla d| - 1 \big| = \sup_{\partial D} \big| |\nabla d| - 1 \big| = \sup_{\partial D} \big| |\nabla d| - |\nabla d_e| \big| \leq \sup_{\partial D} |\nabla d - \nabla d_e| \,.$$

Taking into account (3.2), the inequality (3.3) follows.

Let us now prove (3.4).
Let us denote
$$\delta := \|d - d_e\|_{C^0(A_R)}.$$
Using (3.1) we have that
$$||D| - \omega_n| \le \omega_n(R^n + 1) \le C(n, R)\delta \qquad \text{if } \delta \ge 1$$
where $C(n, R) = \omega_n(R^n + 1)$. Thus, (3.4) is proved in the case $\delta \ge 1$.
Let us now consider the case $\delta < 1$. We claim that
$$B(0, 1 - \delta) \subseteq D \subseteq B(0, 1 + \delta), \qquad \text{if } \delta < 1. \tag{3.5}$$
If $x \in B(0, 1 - \delta) \cap B(0, 1/R)$ then (3.1) implies $x \in D$.
Let us consider $x \in B(0, 1 - \delta)$ (that trivially implies $|x| < R$) and $|x| \ge \frac{1}{R}$. Then
$$d(x) = d_e(x) + (d(x) - d_e(x)) < 1 - \delta + \delta = 1.$$
By definition of $D$, this implies $x \in D$.
So we have proved that
$$B(0, 1 - \delta) \subseteq D.$$
Let us prove that
$$D \subseteq B(0, 1 + \delta). \tag{3.6}$$
If $x \in D \cap B(0, 1/R)$ then trivially $|x| < \frac{1}{R} < 1+\delta$; i.e., $x \in B(0, 1+\delta)$. If $x \in D \setminus B(0, 1/R)$, then, by definition of $D$,
$$|x| = d_e(x) = d(x) + (d_e(x) - d(x)) < 1 + \delta.$$
So we have proved (3.6) and eventually (3.5).
To conclude, it suffices to notice that (3.5) implies
$$\omega_n(1 - \delta)^n \le |D| \le \omega_n(1 + \delta)^n, \qquad \text{if } \delta < 1,$$
which implies (3.4) also for the case $\delta < 1$. □

We are now ready to prove the main result of this section.

*Proof of Theorem 3.1.* Let $d$ be a homogeneous norm in $\mathbb{R}^n$, $d \in C^\infty(\mathbb{R}^n \setminus \{0\})$, satisfying (3.1). Denote $D$ the corresponding $d$-unit ball. We provide here the proof for $n \ge 3$, since the case $n = 2$ goes exactly in the same way.
As in [3], define
$$P_d : \partial D \to \mathbb{R}, \qquad P_d = -\frac{\partial G}{\partial \nu},$$
where $G$ is the Green function of $D$ with pole at the origin.
Define a measure $\mu$ as follows:
$$d\mu(y) := n|\nabla d(y)| P_d\left(\frac{y}{d(y)}\right) dy \qquad y \in D \setminus \{0\}.$$
We now proceed by steps.

**Step 1.** In this first step we prove that
$$u(0) = \int_D u(y)\, d\mu(y) \qquad \forall u \in \mathcal{H}(D) \cap L^1(D). \tag{3.7}$$

Given $r > 0$, denote

$$D_r := \{v \in \mathbb{R}^n \, : \, d(x) < r\}.$$

By Theorem 1.2 in [3] (see also [4]), applied with $\alpha = n$, we have that

$$v(0) = \int_{D_r} v(y)\, d\mu_r(y), \qquad \forall v \in \mathcal{H}(D_r), v \geq 0, \tag{3.8}$$

where $\mu_r$ is the measure defined as

$$d\mu_r(y) := \frac{n}{r^n} |\nabla d(y)| P_d \left( \frac{y}{d(y)} \right) \, dy \qquad y \in D_r \setminus \{0\}.$$

In particular, choosing $v = 1$ we get $\mu_r(D_r) = 1$.
Consider $u \in \mathcal{H}(D) \cap L^1(D)$. Since $D = D_1$, then $\overline{D_r} \Subset D$ for every $r \in ]0, 1[$. The function

$$v := u - \min_{\overline{D_r}} u$$

is harmonic in $D_r$ and nonnegative. Thus, by (3.8),

$$v(0) = \int_{D_r} v(y)\, d\mu_r(y),$$

which implies

$$u(0) = \int_{D_r} u(y)\, d\mu_r(y) = \frac{1}{r^n} \int_{D_r} u(y)\, d\mu(y).$$

Letting $r$ go to 1 we get (3.7).

**Step 2.** In this second step we prove that

$$G(D, 0) \leq \sup_{x \in D \setminus \{0\}} \left| n\, |D|\, |\nabla d(x)| P_d \left( \frac{x}{d(x)} \right) - 1 \right| . \tag{3.9}$$

From the very definition of the Gauss mean value gap, see (1.2), we can write it as follows:

$$G(D, 0) := \sup_{u \in \mathcal{H}(D), \|u\|_{\widetilde{L}^1(D)} = 1} \left| u(0) - \fint_D u(x)\, dx \right| .$$

Using (3.7) we get

$$G(D, 0) = \sup_{u \in \mathcal{H}(D), \|u\|_{\widetilde{L}^1(D)} = 1} \left| \int_D u(x)\, d\mu(x) - \frac{1}{|D|} \int_D u(x)\, dx \right| .$$

We may write

$$\int_D u(x)\, d\mu(x) - \frac{1}{|D|} \int_D u(x)\, dx = \frac{1}{|D|} \int_D u(x) \left( n\, |D|\, |\nabla d(x)| P_d \left( \frac{x}{d(x)} \right) - 1 \right) dx.$$

Therefore, (3.9) follows at once using $\|u\|_{\widetilde{L}^1(D)} = 1$.

**Step 3.** We now prove that

$$\sup_{x \in D \setminus \{0\}} \left| n\, |D|\, |\nabla d(x)| P_d \left( \frac{x}{d(x)} \right) - 1 \right| \leq c \, \|d - d_e\|_{C^{2,\alpha}(A_{2R})} , \tag{3.10}$$

where $c$ is a positive constant depending only on $n$, $R$ and $\|d\|_{C^{2,\alpha}(A_{2R})}$.
By Lemma 3.3 inequality (3.10) follows from the next one:

$$\sup_{x\in D\setminus\{0\}}\left|n\,\omega_n\,P_d\left(\frac{x}{d(x)}\right)-1\right|\le c\,\|d-d_e\|_{C^{2,\alpha}(A_{2R})}\,. \tag{3.11}$$

Let us recall that

$$P_d:\partial D\to\mathbb{R},\qquad P_d:=-\frac{\partial G}{\partial\nu},$$

where $G$ is the Green function of $D$ with pole at the origin; i.e.,

$$G(x,0)=\Gamma(x)-h(x).$$

Here $\Gamma$ denotes the fundamental solution of the Laplace operator, while $h$ solves

$$\begin{cases}\Delta u=0 & \text{in } D,\\ u=\Gamma & \text{on } \partial D.\end{cases}$$

For every $y\in\partial D$

$$\begin{aligned}n\omega_nP_d(y)-1&=-n\omega_n\langle\nabla\Gamma(y),\nu(y)\rangle-1+n\omega_n\langle\nabla h(y),\nu(y)\rangle\\&=:N_1(y)-1+N_2(y).\end{aligned}\tag{3.12}$$

Keeping in mind that

$$\Gamma(y)=\frac{1}{n\omega_n(n-2)}|y|^{2-n},$$

we have

$$N_1(y)=|y|^{1-n}\langle\frac{y}{|y|},\frac{\nabla d(y)}{|\nabla d(y)|}\rangle.$$

Writing

$$\nabla d(y)=\nabla(|y|+d(y)-|y|)=\frac{1}{|y|}\left(y+|y|\nabla(d-d_e)(y)\right)$$

and denoting

$$g(y):=|y|\nabla(d-d_e)(y),$$

we obtain

$$\frac{\nabla d(y)}{|\nabla d(y)|}=\frac{y+g(y)}{|y+g(y)|}.$$

Thus,

$$N_1(y)=|y|^{1-n}\langle\frac{y}{|y|},\frac{y+g(y)}{|y+g(y)|}\rangle.$$

We want to estimate $\sup_{\partial D}|N_1-1|$.
Let us first assume that $|g(y)|\ge\frac{1}{2R}$.
By the very definition of $g$ and using that $\frac{1}{R}\le|y|\le R$ (see (3.2)) we get

$$\|d-d_e\|_{C^1(A_R)}\ge\frac{|g(y)|}{|y|}\ge\frac{1}{2R^2}.$$

Therefore, by (3.2) and $R\ge1$,

$$\begin{aligned}|N_1(y)-1|&\le|N_1(y)|+1\le|y|^{1-n}+1\le R^{n-1}+1\le 2R^{n-1}\\&\le 4R^{n+1}\|d-d_e\|_{C^1(A_R)}.\end{aligned}\tag{3.13}$$

If instead $|g(y)| \le \frac{1}{2R}$, we consider the smooth function

$$F : D(R) \to \mathbb{R}, \qquad F(y,z) := |y|^{1-n}\langle \frac{y}{|y|}, \frac{y+z}{|y+z|}\rangle,$$

where

$$D(R) := \left\{ (y,z) \,:\, \frac{1}{R} \le |y| \le R,\ |z| \le \frac{1}{2R} \right\}.$$

Letting

$$C(n,R) := \sup_{D(R)} |\nabla_{x,z} F|,$$

and keeping in mind that if $y \in \partial D$ then $d(y) = 1$ and (3.2) holds,

$$\begin{aligned}
&|N_1(y) - 1| = |F(y, g(y)) - F(\frac{y}{|y|}, 0)| \\
&\le C(n,R) \left( \left| y - \frac{y}{|y|} \right| + |g(y)| \right) = C(n,R)\ (||y| - 1| + |g(y)|) \\
&\le C(n,R)\ (|d_e(y) - d(y)| + R|\nabla(d_e - d)(y)|) \\
&\le R\, C(n,R)\, \|d - d_e\|_{C^1(A_R)}.
\end{aligned} \tag{3.14}$$

Thus, collecting (3.13) and (3.14), we get

$$\sup_{\partial D} |N_1 - 1| \le c \|d - d_e\|_{C^1(A_R)}, \tag{3.15}$$

for some $c$ depending on $n$ and $R$.

Let us now estimate $N_2$.
For every $y \in \partial D$ we have

$$\begin{aligned}
|N_2(y)| &= n\omega_n |\langle \nabla h\,(y)\,, \nu\,(y) \rangle| = n\omega_n |\langle \nabla (h\,(y) - \Gamma(1)), \nu\,(y)\rangle| \\
&\le n\omega_n |\nabla (h\,(y) - \Gamma(1))|.
\end{aligned}$$

We observe that

$$\begin{cases} \Delta(h - \Gamma(1)) = 0 & \text{in } D \\ h(y) - \Gamma(1) = \psi(y)(\Gamma(y) - \Gamma(1)) & \text{if } y \in \partial D \end{cases}$$

where $\psi \in C^\infty(\mathbb{R}^n; [0,1])$,

$$\psi = 1 \quad \text{in } A_R, \qquad \psi = 0 \quad \text{in } \mathbb{R}^n \setminus A_{2R}.$$

On the other hand, if $y \in \partial D$,

$$\begin{aligned}
\psi(y)(\Gamma(y) - \Gamma(1)) &= \frac{\psi(y)}{n(n-2)\omega_n} \left( |y|^{2-n} - 1 \right) \\
&= \frac{\psi(y)}{n(n-2)\omega_n} \left( d_e(y)^{2-n} - d(y)^{2-n} \right) \\
&=: \Phi(y)(d(y) - d_e(y)).
\end{aligned}$$

Here $\Phi : \mathbb{R}^n \to \mathbb{R}$,

$$\Phi(x) := \frac{\psi(x)}{n(n-2)\omega_n} \cdot \frac{d_e(x)^{2-n} - d(x)^{2-n}}{d(x) - d_e(x)}.$$

Hence $\Phi \in C^\infty(\mathbb{R}^n)$ and

$$\|\Phi\|_{C^{2,\alpha}(B(0,R))} \leq c\big(1 + \|d\|_{C^{2,\alpha}(A_{2R})}\big) \tag{3.16}$$

and $c$ depends only on $n$, $R$ and $\alpha$.
Thus, by the Schauder estimates and (3.16) we have

$$\begin{aligned}\sup_{\partial D} |\nabla(h - \Gamma(1))| \leq \sup_{D} |\nabla(h - \Gamma(1))| &\leq \|h - \Gamma(1)\|_{C^{2,\alpha}(D)} \\ &\leq c\|\Phi(d - d_e)\|_{C^{2,\alpha}(D)} \leq c\|\Phi(d - d_e)\|_{C^{2,\alpha}(B(0,R))} \\ &\leq c\|d - d_e\|_{C^{2,\alpha}(A_{2R})},\end{aligned}$$

for some $c$ depending on $n$, $R$, $\alpha$ and $\|d\|_{C^{2,\alpha}(A_{2R})}$.
So we have proved that

$$\sup_{\partial D} |N_2| \leq c(n, R, \alpha, \|d\|_{C^{2,\alpha}(A_{2R})})\|d - d_e\|_{C^{2,\alpha}(A_{2R})}. \tag{3.17}$$

The estimate (3.11) follows by (3.12), (3.15) and (3.17).

□

## 4. On the sharpness of the stability estimate

The following example establishes the sharpness of our stability result, see (1.3).

**Example 4.1.** Consider the ellipsoids

$$D_\epsilon := \big\{x \in \mathbb{R}^n \,:\, (\epsilon x_1)^2 + x_2^2 + x_3^2 + \cdots + x_n^2 < 1\big\} \qquad \epsilon \in ]\frac{1}{2}, 1[.$$

Of course $1 = \operatorname{dist}(0, \partial D_\epsilon)$, so $B(0,1) \subseteq D_\epsilon$ and $\partial B(0,1)$ is tangent to $\partial D_\epsilon$.
We claim that there exists a constant $C' > 1$, independent of $\epsilon$, such that

$$\frac{1}{C(n)} \frac{|D_\epsilon \setminus B(0,1)|}{|D_\epsilon|} \leq G(D_\epsilon; 0) \leq C' \frac{|D_\epsilon \setminus B(0,1)|}{|D_\epsilon|} \qquad \forall \epsilon \in ]\frac{1}{2}, 1[, \tag{4.1}$$

where the left inequality comes from Theorem 1.2. Since

$$\frac{|D_\epsilon \setminus B(0,1)|}{|D_\epsilon|} = 1 - \frac{|B(0,1)|}{|D_\epsilon|} = 1 - \epsilon \to_{\epsilon \to 1^-} 0. \tag{4.2}$$

we conclude that the exponent 1 in the left hand side of (1.3) is sharp.

We need only to prove the right inequality in (4.1).
Define the homogeneous norms in $\mathbb{R}^n$

$$d_\epsilon(x) := \sqrt{(\epsilon x_1)^2 + x_2^2 + x_3^2 + \cdots + x_n^2} \qquad \epsilon \in ]\frac{1}{2}, 1[.$$

Of course $d_\epsilon \in C^\infty(\mathbb{R}^n \setminus \{0\})$ and the euclidean norm $d_e$ is $d_1$. It is trivial that

$$\frac{1}{2} d_e(x) \leq d_\epsilon(x) \leq d_e(x) \qquad \forall \epsilon \in ]\frac{1}{2}, 1[, \quad \forall x \in \mathbb{R}^n.$$

Therefore, by Theorem 3.1, used with $R = 2$, there exists $c$ independent of $\epsilon$,

$$G(D_\epsilon, 0) \leq c \, \|d_\epsilon - d_e\|_{C^3(B(0,4) \setminus B(0,1/4))} \qquad \forall \epsilon \in ]\frac{1}{2}, 1[. \tag{4.3}$$

An easy computation and (4.2) give

$$\|d_\epsilon - d_e\|_{C^3(B(0,4)\setminus B(0,1/4))} \le C(1-\epsilon) = C\frac{|D_\epsilon \setminus B(0,1)|}{|D_\epsilon|}, \tag{4.4}$$

with $C$ independent of $\epsilon$. By (4.3) and (4.4) we get the right inequality in (4.1).

We conclude with an example showing that the Gauss mean value is not continuous with respect to the pointwise convergence of the characteristic functions of the domains.

**Example 4.2.** Let us consider the unit ball $B$ in $\mathbb{R}^n$, $n \ge 2$, centered at the origin. Consider the open sets

$$D_k := B \setminus C_k,$$

where

$$C_k := \left\{ \lambda\xi \in \mathbb{R}^n \,:\, |\xi| = 1,\ \frac{1}{2} \le \lambda \le 1,\ \frac{1}{2k} \le |\xi - e_1| \le \frac{1}{k} \right\}.$$

As usual $e_1$ stands for the first vector of the canonical basis of $\mathbb{R}^n$.
Of course, $0 \in D_k$ and $\frac{1}{2} = \text{dist}(0, \partial D_k)$. By Theorem 1.2

$$G(D_k, 0) \ge \frac{1}{C(n)} \frac{|D_k \setminus B(0,1/2)|}{|D_k|} \ge \frac{1}{2C(n)}.$$

We claim that

$$\chi_{D_k} \to \chi_B \quad \text{everywhere in } B.$$

To prove this it is enough to show that

$$\text{if } x \in B \text{ then there exists } \bar{k} \text{ such that } x \in D_k \text{ for every } k \ge \bar{k}.$$

If $x \in B$ then $x = \lambda\xi$, with $0 \le \lambda < 1$, and some $\xi \in \mathbb{R}^n$, $|\xi| = 1$.
If $\xi = e_1$ then $x \notin C_k$, therefore $x \in D_k$ for every $k$.
If $0 < |\xi - e_1| =: \rho$ then $x \notin C_k$, for every $k > \frac{1}{\rho}$. Hence, the claim follows.

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