Optimal estimates for the triple junction function and other surprising aspects of the area functional

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Dedicated to Francesco

Abstract

We consider the relaxed area functional for vector valued maps and its exact value on the triple junction function $u : B_1(O) \to \mathbb{R}^2$, a specific function which represents the first example of map whose graph area shows nonlocal effects. This is a map taking only three different values $\alpha, \beta, \gamma \in \mathbb{R}^2$ in three equal circular sectors of the unit radius ball $B_1(O)$. We prove a conjecture due to G. Bellettini and M. Paolini asserting that the recovery sequence provided in [5] (and the corresponding upper bound for the relaxed area functional of the map u) is optimal. At the same time, we show by means of a counterexample that such construction is not optimal if we consider different domains than $B_1(O)$, which still contain the same discontinuity set of u in $B_1(O)$. Such domains are obtained from $B_1(O)$ erasing part of interior of the sectors where u is constant.

Key words: area functional, relaxation, lower-semicontinuous envelope, minimal surfaces, integral currents, policonvexity.

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1 Introduction

The analysis of polyconvex energies arises in many branches of calculus of variations, and more specifically in problems coming from the mechanics of solids, like elasticity theory [2]. Particular attention has been given to energies with linear growth, and special issues concern the property of lower-semicontinuity on the class of admissible states (see [1] and references therein). A fundamental example of polyconvex function with linear growth is the area functional, the functional which measure the area of the graph of a given map. This is the simplest example of polyconvex energy related to a variable of a (physics, mechanics) system, and already shows many particular features and issues which are surprising and row against intuition.

The area functional is introduced as follows. Let $\Omega \subset \mathbb{R}^n$ be an open set. The graph of a smooth function $v : \Omega \to \mathbb{R}^N$ is defined as the subset G_v of $\Omega \times \mathbb{R}^N$ given by

$$G_v := \{(x, y) \in \Omega \times \mathbb{R}^N : y = v(x)\}. \tag{1.1}$$

The graph G_v is a surface of dimension n embedded in $\mathbb{R}^{n+N}$, and then its area can be computed, namely its n-dimensional Hausdorff measure. Considering the embedding $\Phi : x \mapsto (x, v(x))$, easy computation brings to the formula that, in the specific case $N = 1$ is given by

$$\mathcal{A}(v) := \int_\Omega (1 + |\nabla v|^2)^{\frac{1}{2}} dx, \tag{1.2}$$

whereas, if, for example, $n = N = 2$, reads as

$$\mathcal{A}(v) := \int_\Omega \big(1 + |\frac{\partial v_1}{\partial x_1}|^2 + |\frac{\partial v_1}{\partial x_2}|^2 + |\frac{\partial v_2}{\partial x_1}|^2 + |\frac{\partial v_2}{\partial x_2}|^2 + |J(v)|^2\big)^{\frac{1}{2}} dx. \tag{1.3}$$

Here $J(v)$ stands for the Jacobian determinant of v, i.e.,

$$J(v) := \frac{\partial v_1}{\partial x_1}\frac{\partial v_2}{\partial x_2} - \frac{\partial v_2}{\partial x_1}\frac{\partial v_1}{\partial x_2}. \tag{1.4}$$

It is easy to realize that such definition can be extended to all maps $v \in W^{1,\min\{n,N\}}(\Omega; \mathbb{R}^N)$. More in general, one can try to define the area of the graph of still less regular maps, proceeding by approximating them by regular functions (for the theory of polyconvexity in $W^{1,p}$ see [16]). To this respect, one is led to define the area functional for any map $v \in L^1(\Omega; \mathbb{R}^N)$, given by

$$\mathcal{A}(v) := \inf\{\liminf_{n\to+\infty} \mathcal{A}(v_n)\}, \tag{1.5}$$

where the infimum is computed on all sequences of functions $v_n \in C^1(\Omega; \mathbb{R}^N)$ such that $v_n \to v$ in $L^1(\Omega; \mathbb{R}^N)$. However, in general, it is not true that the relaxed functional (1.5) coincides with the original area functional (1.3) in $W^{1,1}(\Omega; \mathbb{R}^N)$, which is not lower-semicontinuous (see [1]). Moreover, it might happen that the value of the lower semicontinuous envelope $\mathcal{A}(v)$ be not finite for some function $v \in L^1(\Omega; \mathbb{R}^N) \setminus W^{1,\min\{n,N\}}(\Omega; \mathbb{R}^N)$. Therefore the first natural question arising from definition (1.5) is to determine the exact domain $D(\mathcal{A}) \subset L^1(\Omega; \mathbb{R}^N)$ of the functional $\mathcal{A}$. A second natural question is, of course, to determine the exact value of it, namely a general formula like (1.2) or (1.3). This very challenging problem has been completely solved in codimension 1, that is in the case the target space is $\mathbb{R}$ ($N = 1$) (see [8]). In this case, the lower semicontinuous envelope of the area functional $\mathcal{A} : C^1(\Omega; \mathbb{R}) \to \mathbb{R}$ is the functional

$$\mathcal{A}(v) = \begin{cases} \int_\Omega \sqrt{1 + |\nabla v|^2} dx + |D^s v|(\Omega) & \text{if } v \in BV(\Omega), \\ +\infty & \text{otherwise,} \end{cases} \tag{1.6}$$

where ∇v represents the absolutely continuous (with respect to the Lebesgue measure $\mathcal{L}^n$) part of the gradient Dv of v, and $D^s v$ its singular part. In other words, the area functional has as natural domain $BV(\Omega)$ the space of functions of bounded variations where it assumes the general integral form (1.6). In particular, thanks to the good properties of the integral form, it turns out that the area functional is subadditive if seen as function on sets. More precisely, let us consider on any open set $U \subset \Omega$ the area functional restricted to U, defined as

$$\mathcal{A}(v; U) := \int_U (1 + |\nabla v|^2)^{\frac{1}{2}} dx. \tag{1.7}$$

Then, for fixed $v \in BV(\Omega)$, we can look at $\mathcal{A}(v; \cdot)$ as a function on Borel sets. As a consequence of the expression (1.6) it turns out that $\mathcal{A}(v; \cdot)$ is subadditive, namely

$$\mathcal{A}(v; U_1 \cup U_2) \le \mathcal{A}(v; U_1) + \mathcal{A}(v; U_2) \quad \text{for all } U_1, U_2 \subset \Omega. \tag{1.8}$$

In higher dimension $N \ge 2$ all these good properties fail. First, it is only possible to prove that

$$\mathcal{A}(v; U) \ge \int_U \sqrt{1 + |\nabla v|^2} dx + |D^s v|(U), \tag{1.9}$$

and the inequality is strict in some cases. Furthermore an explicit example in [1] (which consider a slight modification of an example in [2]) shows that the subadditivity property does not hold true in general. In this example, first suggested by De Giorgi in [9], it is exhibited a simple function $u : \Omega \subset \mathbb{R}^2 \to \mathbb{R}^2$, called *triple junction function*. The function u takes only three values α, β, and γ, which are the vertices of an equilateral triangle of side $\sqrt{3}$ centered at the origin O of $\mathbb{R}^2$. The plane $\mathbb{R}^2$ is divided in three sectors D^A, D^B, and D^C, which have as boundaries three halflines with endpoint O and forming three equal angles of $2\pi/3$. The function u then is defined by setting

$$u = \alpha \text{ on } D^A, \quad u = \beta \text{ on } D^B, \quad u = \gamma \text{ on } D^C, \tag{1.10}$$

thus showing three jumps on the halflines meeting in the triple junction O. In [1] it is proved that for the function $u : \mathbb{R}^2 \to \{\alpha, \beta, \gamma\}$ the following happens:

(a) Let $R > 0$ be fixed and let $B_R(O)$ be the ball centered in O and with radius R. In any open subdomain $U \subset B_R(O)$ such that $O \notin U$ the relaxed area functional $\mathcal{A}(u; U)$ takes the form (1.9), and therefore its value is

$$\mathcal{A}(u; U) = \mathcal{L}^2(U) + |D^s u|(U).$$

Specifically, if $\rho \in (0, R)$ and $U = B_R(O) \setminus B_\rho(O)$, then

$$\mathcal{A}(u; U) = \pi(R^2 - \rho^2) + 3\sqrt{3}(R - \rho).$$

(b) The following two inequalities are provided

$$\mathcal{A}(u; B_R(O)) \leq \mathcal{L}^2(B_R(O)) + 4\sqrt{3}R, \tag{1.11}$$

$$\mathcal{A}(u; B_R(O)) > \mathcal{L}^2(B_R(O)) + 3\sqrt{3}R. \tag{1.12}$$

(c) If $s > R > \rho > 0$, then

$$\mathcal{A}(u; B_R(O)) > \mathcal{A}(u; B_\rho(O)) + \mathcal{A}(u; B_s(O) \setminus \overline{B_{\rho/2}(O)}). \tag{1.13}$$

The estimate (1.11), proved in [1], is not optimal. In [5] this bound has been improved. In order to give the precise value of the upper bound found in [5] we need some preliminary. Let us define the rectangle $\mathcal{R} := (0, R) \times (-\sqrt{3}/2, \sqrt{3}/2)$. Consider the function $\varphi : (-\sqrt{3}/2, \sqrt{3}/2) \to \mathbb{R}^+$ defined as

$$\varphi(-\frac{\sqrt{3}}{2}) = \varphi(\frac{\sqrt{3}}{2}) = 0, \quad \varphi(0) = \frac{1}{2}, \ \ \varphi \text{ is affine on } (-\frac{\sqrt{3}}{2}, 0) \text{ and } (0, \frac{\sqrt{3}}{2}). \tag{1.14}$$

We will deal with the following minimal problem: we want to minimize the area of the graph of continuous functions $v : \mathcal{R} \to \mathbb{R}$ belonging to the family

$$\mathcal{A}^1_\varphi(\mathcal{R}) := \{v \in W^{1,1}(\mathcal{R}) : v = 0 \text{ on } (0, R) \times \{-\sqrt{3}/2, \sqrt{3}/2\},\ v(0, \cdot) = \varphi(\cdot)\}. \tag{1.15}$$

If $\overline{v}$ is a minimizer for this minimum problem, the correspondent value of the area of the graph is denoted by m_R, namely

$$m_R := \mathcal{A}(\overline{v}; \mathcal{R}) = \inf\{\mathcal{A}(v; \mathcal{R}) : v \in \mathcal{A}^1_\varphi(\mathcal{R})\}. \tag{1.16}$$

Hence, in [5], the following inequality has been proved

$$\mathcal{A}(u; B_R(O)) \leq \mathcal{L}^2(B_R(O)) + 3m_R. \tag{1.17}$$

Furthermore Bellettini and Paolini [5] conjectured that such value is optimal, that is, for any sequence of maps $v_k \in C^1(B_R(O); \mathbb{R}^2)$ such that $v_k \to u$ strongly in $L^1(B_R(O); \mathbb{R}^2)$ it holds

$$\liminf_{k\to\infty} \mathcal{A}(v_k; B_R(O)) \geq \mathcal{L}^2(B_R(O)) + 3m_R. \tag{1.18}$$

In the present paper we propose a proof of this conjecture. Actually, without lose of generality, we work in the specific case $R = 1$ and denote $m_1 = m$. Therefore we prove

$$\mathcal{A}(u; B_1(O)) = \pi + 3m. \tag{1.19}$$

In order to show this result, we have to introduce some preliminary on currents and the concept of Cartesian maps. We thus exploit some well-known cornerstone Theorems of calculus with Cartesian currents, as their properties of closure and compactness. Then, the proof of (1.19) is articulated in three sections. In the first one, Section 3, we introduce the problem in the domain $\Omega = B_1(O)$, and start by taking a sequence $v_k \in C^1(\Omega; \mathbb{R}^2)$ approaching u, supposing it is optimal, namely

$$\mathcal{A}(v_k; \Omega) \to \mathcal{A}(u; \Omega).$$

Then we divide the domain in more sectors in order to detect the different behavior of the approaching sequence $\{v_k\}$. In particular we consider one small triangular sector containing the junction point O, and three other main sectors each containing one of the lines forming the jump set of u. We first look at the graphs of v_k in these sectors, treating them as integral currents in $\Omega \times \mathbb{R}^2$. Choosing suitable maps from $\mathbb{R}^4$ to $\mathbb{R}^3$, and considering the push forward by them, we then reduce to consider integral currents in $\mathbb{R}^3$, which have the advantage of being currents of codimension 1. This procedure of dimension reduction leads to four integral currents $\widehat{S}^1$, $\widehat{S}^2$, $\widehat{S}^3$, and $\mathcal{T}$, which satisfy the following key inequality[1]

$$|\widehat{S}^1| + |\widehat{S}^2| + |\widehat{S}^3| + |\mathcal{T}| + \mathcal{L}^2(\Omega) \leq \mathcal{A}(u; \Omega). \tag{1.20}$$

The currents $\widehat{S}^1$, $\widehat{S}^2$, $\widehat{S}^3$, and $\mathcal{T}$ show the following properties: they are supported in the prism $P := [0, 1) \times T$, where T is the closed triangle in $\mathbb{R}^2$ with vertices α, β, and γ, $\mathcal{T}$ is supported in $\{0\} \times T$, the sum $\widehat{S}^1 + \widehat{S}^2 + \widehat{S}^3 + \mathcal{T}$ is a closed current in $(-\infty, 1) \times \mathbb{R}^2$, and each $\widehat{S}^i$ shows a specific boundary $\partial \widehat{S}^i$ which, up to an error (see formula below), is supported on the edges of the prism: more specifically, there are integral 1-currents N^A, N^B, and N^C, such that,

$$\partial \widehat{S}^1 = -N^A + N^B - (Id \times \alpha)_\sharp [\![0, 1]\!] + (Id \times \gamma)_\sharp [\![0, 1]\!] + \mathcal{V}^1,$$

with $(Id \times \alpha)_\sharp [\![0, 1]\!]$ representing the graph of the constant map $f = \alpha$ on the segment $(0, 1)$, and $\mathcal{V}^1$ being a current supported on $\{0\} \times T$. Similar formulas hold for $\widehat{S}^2$ and $\widehat{S}^3$ (see Section 3 for details).

Afterward we are ready to state our main result, Theorem 3.7. This asserts that

$$|\widehat{S}^1| + |\widehat{S}^2| + |\widehat{S}^3| + |\mathcal{T}| \geq 3m, \tag{1.21}$$

which, together with (1.20), will provide the lower bound

$$\mathcal{A}(u; \Omega) \geq \mathcal{L}^2(\Omega) + 3m.$$

Combining this with the upper bound proved in [5], namely (1.17), we finally conclude (1.19).

Let us spend some words on the optimal construction obtained in [5]. For the recovery sequence therein the limit currents $\widehat{S}^1$, $\widehat{S}^2$, $\widehat{S}^3$ will coincide with the minimal surfaces providing the solution of problem (1.16). In particular the current $\mathcal{V}^1$ (and similarly $\mathcal{V}^2$ and $\mathcal{V}^3$) turns out to be the graph of φ appearing in (1.14). Moreover in this case the current $\mathcal{T}$ turn out to be null, as for the currents N^A, N^B, and N^C, which do not appear for the optimal recovery sequence. In some sense, the presence of $\mathcal{T}$ and N^A, N^B, N^C, do not provide better estimates for the area functional, and at optimality, they must vanish.

[1] To be precise, we prove this inequality with $\mathcal{L}^2(\Omega)$ replaced by $\mathcal{L}^2(\Omega) - \epsilon$, where $\epsilon > 0$ is a small parameter depending on the geometric construction; the inequality in (1.20) follows from the fact that we can render ϵ as small as we want optimizing the geometry of the construction, see Section 3 and Theorem 3.7.

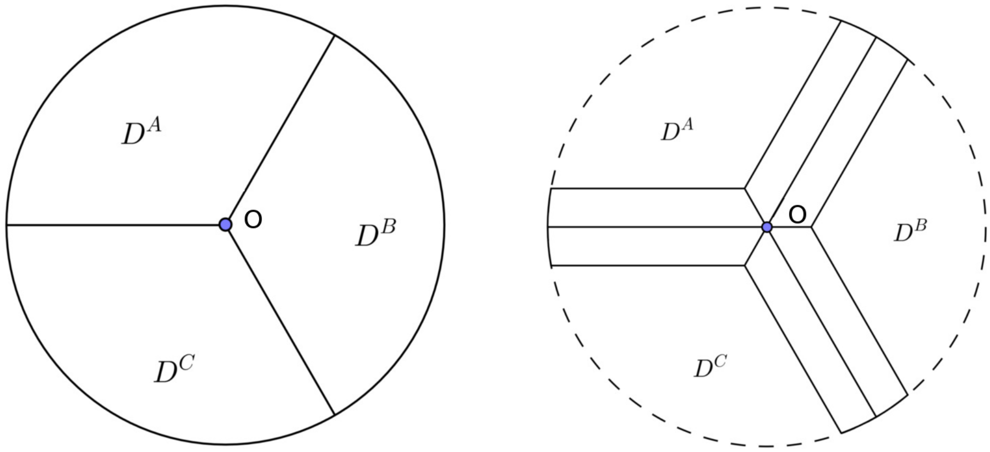

Figure 1: On the left it is represented the domain $B_1(O)$ and the sectors D^A, D^B, and D^C where the function u takes the values α, β, and γ respectively. The three segments meeting at O are the jump sets of u. The picture on the right represents the thin domain U_b, obtained from $B_1(O)$ by cutting part of the interior of the sectors where u is constant; the jump set is still drawn in black, together with three small segments connecting O to ∂U_b which represent the set where the vertical currents $\mathcal{G}_{v_k}$ concentrate.

In order to prove Theorem 3.7 we need to get rid of the currents N^A, N^B, and N^C, appearing in the boundaries of $\widehat{S}^i$. To this aim, we introduce a Steiner type symmetrization technique in Section 4. This is the heaviest part of the paper, and the more technical. The main idea relies into construct three symmetrization operators $\mathbb{S}_A$, $\mathbb{S}_B$, $\mathbb{S}_C$, each symmetrizing the currents $\widehat{S}^i$ and $\mathcal{T}$ with respect to one of the heights of the triangle T, and with the property of decreasing the masses of $\widehat{S}^i$, $\mathcal{T}$, and of their boundaries (see Lemma 4.23). Then, applying repeatedly these operators, we are able to reduce to integral currents S^1, S^2, S^3, and $\mathcal{T}$ which still satisfy (1.20), but have now well-properties at the boundaries; in particular the new currents N^A, N^B, and N^C, are null. This brings us to Section 5, where we finally prove Theorem 3.7. First we list some key features of the brand new currents S^1, S^2, S^3, and $\mathcal{T}$ (see properties (i) and (ii) at the beginning of Section 5). Observing that such properties are closed in the class of integral currents, we reduce our argument to a problem of minimal surfaces. This problem consists of minimizing the mass $|S^1| + |S^2| + |S^3| + |\mathcal{T}|$ among the class of integral currents satisfying properties (i) and (ii), which in particular contain a fixed boundary condition for such currents (see problem (5.12)). Some additional Lemmas bring us to deduce that the minimizers of this variational problem consists of three currents S^i (the currents $\mathcal{T}$ turns out to be zero) which can be identified with three Cartesian currents on the rectangle $\mathcal{R}^1 = (0,1) \times (-\sqrt{3}/2, \sqrt{3}/2)$. The boundaries of these Cartesian currents are shown to satisfy the same Dirichlet boundary datum as in (1.15). From this it easily turns out that the minimal mass of each S^i must be m, and (1.21) is achieved.

At this last step it is evident how we use the good feature of the class of Cartesian currents in codimension 1. In fact we strongly exploit the fact that every Cartesian currents is approximable by graphs of smooth functions, which is a property that is true only if the target space of these functions is $\mathbb{R}$ (i.e., one dimensional). We stress that at this point the dimension reduction exploited in Section 3 becomes crucial.

In the following Section 6 we face the problem of studying the optimality of the bound in (1.17) for different domains Ω still containing the triple junction. First let us emphasize that part of the conjecture in [5] also asserts that the same bound holds in the case that the lines meeting in O, boundaries of the regions D^A, D^B, and D^C, form angle not necessary equal to $2\pi/3$. We do not treat this case directly, but a sharp inspection of the proof we provide should show that it can be adapted to such a case, encouraging us to assert that also for this more general geometry the conjecture is true (however we do not detail this argument here and then are not in position to state a general result). On the one hand, as a consequence of the lack of subadditivity, it is not possible to express the area functional with an integral formula like (1.9). The example of the triple junction function and the corresponding features described in (a) above shows that it is evident that the nonlocal behavior of $\mathcal{A}(u;\cdot)$ strongly depends on the presence of the junction point. In absence of it the additivity comes back. Furthermore, the recovery sequence $\{v_k\} \in C^1(\Omega;\mathbb{R}^2)$ provided in [5] such that

$$\mathcal{L}^2(B_R(O)) + 3m_R = \liminf_{k\to\infty} \mathcal{A}(v_k; B_R(O)),$$

shows the following feature: if we look at the graphs of v_k as integral currents in $B_R(O)\times\mathbb{R}^2$, they concentrate in the singular set $J_u \times \mathbb{R}^2$, J_u being the union of the three radii with endpoint the triple junction O (i.e. the jump set of u). In other words, if $\mathcal{G}_{v_k} \in \mathcal{D}_2(B_R(O)\times\mathbb{R}^2)$ denotes the current carried by the graph of v_k, then

$$\mathcal{G}_{v_k} \rightharpoonup S,$$

with S a Cartesian current which writes as $S = \mathcal{G}_u + V$, where $V \in \mathcal{D}_2(B_R(O)\times\mathbb{R}^2)$ represents the vertical part originated by the concentration of $\mathcal{G}_{v_k}$, and supported on the set $J_u \times \mathbb{R}^2$. This phenomenon might lead to the following issue: if, let us say, $u \in SBV(\Omega;\mathbb{R}^2)$ and J_u represents the jump set of u, and if v_k are $C^1(\Omega;\mathbb{R}^2)$ functions providing

$$\mathcal{A}(u;U) = \liminf_{k\to\infty} \mathcal{A}(v_k;U), \tag{1.22}$$

then is it true that the graphs $\mathcal{G}_{v_k}$ tends to a Cartesian current $S = \mathcal{G}_u + V$ where the vertical part V is concentrated to the set $J_u \times \mathbb{R}^2$? If this question had a positive answer, we would be led to conjecture that $\mathcal{A}(u;\cdot)$ writes as

$$\mathcal{A}(u;U) = |\mathcal{G}_u| + \mathcal{A}_{nl}(u^+,u^-;J_u), \tag{1.23}$$

where $\mathcal{A}_{nl}$ is a nonlocal term whose value depends only on the jump set J_u and on the traces of u on it, namely u^+ and u^-. To my opinion this reasoning is misleading and the answer to the previous demand is, in general, negative. To justify this assertion, we provide an example in which the domain U_b of the triple junction function u is a subdomain of $B_1(O)$ obtained by biting part of the area where u is constant (namely the inner part of the sectors D^A, D^B, and D^C). This domain still contain the whole jump set J_u of u in $B_1(O)$, and in particular the junction point O, since it contains a neighborhood of it (see Figure 1, on the right). Contrarily to what one might aspect, the area functional computed on this domain is less then $\mathcal{L}^2(U_b) + 3m$, i.e.

$$\mathcal{A}(u;U_b) < \mathcal{L}^2(U_b) + 3m. \tag{1.24}$$

This example prove the following assertions:

- The recovery sequence provided in [5] is not optimal for the domain U_b, even if it contains the same discontinuity set of u in $B_1(O)$.
- A formula as (1.23) is false. Indeed, in the case $\Omega = B_1(O)$ it turns out from (1.19) that $\mathcal{A}_{nl}(u^+,u^-;J_u) = 3m$. However inequality (1.24) gives rise to a different value of $\mathcal{A}_{nl}(u^+,u^-;J_u)$, even if J_u and the traces $u^\pm$ does not change.

We do not conjecture that the sequence v_k of approximate functions we construct in Section 6 and such that

$$\liminf_{k\to\infty} \mathcal{A}(v_k;U_b) < \mathcal{L}^2(U_b) + 3m$$

are optimal. At the same time, we believe that for this specific domain the graphs $\mathcal{G}_{v_k}$ of an optimal sequence concentrate outside the set $J_u \times \mathbb{R}^2$. At least, in the specific example of Section 6, this graphs converge to a Cartesian current $\mathcal{G}_u + V$ where the vertical part V is supported on a set $K \times \mathbb{R}^2$, where K contains, besides of J_u, three additional segments connecting O to the boundary of U_b, lying on the bisectors of the halflines forming the triple junction (see Figure 1 on the right, where the set K is emphasized). Similar examples of this behavior have been provided in [7, Section 7], where the authors study the relaxed area functional in the presence of a function u with a prescribed discontinuity on a curve. Our construction of the approximating sequence v_k is similar to the one used in [7], where the jump set of u is somehow prolonged on a path reaching the boundary. In our case, this path is not fixed, but depends on k and at the limit as $k \to \infty$ becomes exactly the union of the three lines in 1 connecting O to the boundary. What is crucial here is that on this set we do not have uniform convergence of v_k to u.

Let us conclude this discussion emphasizing that the highly bad behavior of the area functional becomes evident in the presence of junction points as for the map u. It is possible that, when the jump set consists of a simple curve non self-intersecting, a formula as (1.23) holds true. There are important contributions in this direction in the very interesting papers [6, 7], where the authors study exactly this kind of singularities. More specifically they prove a formula like (1.23) (with inequality $\leq$ replacing the equality $=$) that in some cases can be shown to be optimal (that is equality holds). The nonlocal term $\mathcal{A}_{nl}(u^+, u^-; J_u)$ is related to a problem of minimal surfaces (see Theorem 1.1 in [6]).

Under the light of these last observations we realize that the problem of a full understanding of the relaxation of the area functional, and, more in general, of polyconvex energies in codimension greater than 1, is still a challenging issue we are far from.

2 Preliminaries

k-forms. Let α be a multi-index, i.e., an ordered (increasing) subset of $\{1, 2, \dots, n\}$. We denote by $|\alpha|$ the cardinality (or length) of α, and we denote by $\overline{\alpha}$ the complementary set of α, i.e., the multi-index given by the ordered set $\{1, 2, \dots, n\} \setminus \alpha$.

For all integers $n > 0$ and $k \geq 0$ with $k \leq n$, we denote by $\Lambda_k \mathbb{R}^n$ the space of k-vectors and by $\Lambda^k \mathbb{R}^n$ the space of k-covectors. Let $\Omega \subset \mathbb{R}^n$ be an open set. The symbol $\mathcal{D}^k(\Omega)$ stands for the topological vector space of smooth and compactly supported k-forms (that is the topological vector space of compactly supported and smooth maps on Ω with values in $\Lambda^k \mathbb{R}^n$). Any k-form $\omega \in \mathcal{D}^k(\Omega)$ can be written as sums of elementary forms, namely

$$\omega = \sum_{|\alpha|=k} \varphi_\alpha dx^\alpha,$$

where φ_α is a smooth compactly supported real function, and dx^α is the simple covector defined as $dx^\alpha = dx^{\alpha_1} \wedge \dots \wedge dx^{\alpha_k}$.

Assume $U \subset \mathbb{R}^n$ and $V \subset \mathbb{R}^N$ be open sets and $F : U \to V$ be a smooth map; then, for any $\omega \in \mathcal{D}^k(V)$ is defined a form $F^\sharp \omega \in \mathcal{D}^k(U)$ called pull-back of ω by F; if $\omega = \varphi_\alpha dy^\alpha$, $|\alpha| = k$, then

$$F^\sharp \omega = (\varphi_\alpha \circ F) dF_\alpha, \tag{2.1}$$

with

$$dF_\alpha = dF_{\alpha_1} \wedge dF_{\alpha_2} \wedge \dots \wedge dF_{\alpha_k},$$

where

$$dF_{\alpha_i} := \sum_k \frac{\partial F_{\alpha_i}}{\partial x_k} dx_k.$$

For a $N \times n$ matrix A with real entries and for multi-indices α and β with $|\alpha| = |\beta| = k \leq \min\{n, N\}$, $M_\alpha^\beta(A)$ denotes the determinant of the submatrix of A obtained by erasing the i-th columns and the j-th rows, for all $i \in \overline{\alpha}$ and $j \in \overline{\beta}$. We denote by $M(A)$ the n-vector in $\Lambda_n\mathbb{R}^{n+N}$ given by

$$M(A) := \sum_{k=0}^{n} \sum_{|\alpha|=|\beta|=k} \sigma(\alpha,\overline{\alpha}) M_\alpha^\beta(A) e_{\overline{\alpha}} \wedge \varepsilon_\beta,$$

where $\{e_i\}_{i\leq n}$ is the canonical basis of $\mathbb{R}^n$, $\{\varepsilon_i\}_{i\leq N}$ the canonical basis of $\mathbb{R}^N$, and $\sigma(\alpha,\overline{\alpha})$ is the sign of the permutation $(\alpha,\overline{\alpha})$ (see [11, pag. 230]). Accordingly, we set

$$|M(A)| := (1 + \sum_{k=1}^{\min\{n,N\}} \sum_{|\alpha|=|\beta|=k} |M_\alpha^\beta(A)|^2)^{1/2}.$$

Generalities on currents. The dual space of $\mathcal{D}^k(\Omega)$, denoted by $\mathcal{D}_k(\Omega)$, is the space of k-currents on Ω. We define a weak convergence in $\mathcal{D}_k(\Omega)$ setting $\mathcal{T}_j \rightharpoonup \mathcal{T}$ as currents if for all $\omega \in \mathcal{D}^k(\Omega)$ we have $\mathcal{T}_j(\omega) \to \mathcal{T}(\omega)$. For all currents $\mathcal{T} \in \mathcal{D}_k(\Omega)$ the mass of $\mathcal{T}$ in $U \subset \Omega$ is the number $|\mathcal{T}|_U \in [0,+\infty]$ defined by

$$|\mathcal{T}|_U := \sup_{\omega\in\mathcal{D}^k(U),\ |\omega|\leq 1} \mathcal{T}(\omega).$$

The boundary $\partial\mathcal{T} \in \mathcal{D}_{k-1}(\mathbb{R}^n)$ of a current $\mathcal{T} \in \mathcal{D}_k(\mathbb{R}^n)$ is defined as

$$\partial\mathcal{T}(\omega) = \mathcal{T}(d\omega) \quad \forall\Omega \in \mathcal{D}^{k-1}(\mathbb{R}^n). \tag{2.2}$$

A current $\mathcal{T}$ is said closed if it has null boundary, namely if $\partial\mathcal{T} = 0$ as current.

Given an oriented surface S of dimension $k \leq n$ embedded in $\mathbb{R}^n$, this defines a current in $\mathcal{D}_k(\mathbb{R}^n)$, obtained as integration of k-forms over it (the "volume form" is given by the orienting k-vector). We will often identify surfaces with currents and use the same notation for both. Given a k-rectifiable set K (a countable union of subsets of Lipschitz surfaces) and a summable real function θ on it (with respect to the k-dimensional Hausdorff measure) we can define a current $\mathcal{K}$ integrating k-forms over K as follows

$$\mathcal{K}(\omega) := \int_K \langle\omega(x), \tau\theta(x)\rangle d\mathcal{H}^k(x), \tag{2.3}$$

where $\langle\cdot,\cdot\rangle$ is the duality product between covectors and vectors. Here $\tau : S \to \Lambda_k(\mathbb{R}^n)$ and $\theta : S \to \mathbb{R}$ are such that $\tau(x) \in T_xS$ is a simple unit k-vector for $\mathcal{H}^k$-a.e. $x \in S$ and θ is a $\mathcal{H}^k$-integrable function. The current $\mathcal{K}$, denoted by $\mathcal{K} = \{K,\tau,\theta\}$ is said rectifiable. If $\mathcal{K}$ has rectifiable boundary and θ is an integer-valued function, then $\mathcal{K}$ is said rectifiable with integer multiplicity (or simply integer multiplicity current, i.m.c.). An integral current is an integer multiplicity current with finite mass and finite boundary mass. We use the notation

$$N(\mathcal{T}) := |\mathcal{T}| + |\partial\mathcal{T}|.$$

An integral current $\mathcal{T} \in \mathcal{D}_k(\mathbb{R}^n)$ is said indecomposable if there exists no integral current $\mathcal{R}$ such that $\mathcal{R} \neq 0 \neq \mathcal{T} - \mathcal{R}$ and

$$N(\mathcal{T}) = N(\mathcal{R}) + N(\mathcal{T} - \mathcal{R}).$$

The very specific case in which the integer mutiplicity current $\mathcal{K} \in \mathcal{D}_n(\mathbb{R}^n)$ is of the form $\mathcal{K} = \{K,\tau,\theta\}$ with $\theta = 1$ and $\tau = e_1 \wedge \cdots \wedge e_n$, then $\mathcal{K}$ turns out to be the standard integration over the set K and is denoted by

$$\mathcal{K} = [\![K]\!].$$

Moreover if K is a set of finite perimeter then the current $[\![K]\!]$ is integral.

The following theorem provides the decomposition of every integral current and the structure of integer multiplicity indecomposable 1-currents (see [10, Section 4.2.25]).

Theorem 2.1. *For every integral current $\mathcal{T}$ there exists a sequence of indecomposable integral currents $\mathcal{T}_i$ such that*

$$\mathcal{T} = \sum_i \mathcal{T}_i \quad \text{and} \quad N(\mathcal{T}) = \sum_i N(\mathcal{T}_i).$$

Suppose $\mathcal{T}$ is an indecomposable integer multiplicity 1-current on $\mathbb{R}^n$. Then there exists a Lipschitz function $: \mathbb{R} \to \mathbb{R}^n$ with $\mathrm{Lip}(f) \le 1$ such that

$$f \llcorner (0, |\mathcal{T}|) \text{ is injective and } \mathcal{T} = f_\sharp [\![(0, |\mathcal{T}|)]\!].$$

Moreover $\partial \mathcal{T} = 0$ if and only if $f(0) = f(|\mathcal{T}|)$.

Assume $U \subset \mathbb{R}^n$ and $V \subset \mathbb{R}^N$ be open sets and $F : U \to V$ be a smooth map. The *push-forward* of a current $\mathcal{T} \in \mathcal{D}_k(U)$ by F is defined as

$$F_\sharp \mathcal{T}(\omega) := \mathcal{T}(\zeta F^\sharp \omega) \quad \text{for } \omega \in \mathcal{D}^k(V),$$

where $F^\sharp \omega$ is the standard pull-back of ω and ζ is any C^∞ function that is equal to 1 on $\mathrm{spt}\mathcal{T} \cap \mathrm{spt}F^\sharp \omega$. It turns out that $F_\sharp \mathcal{T} \in \mathcal{D}_k(V)$ does not depend on ζ and satisfies

$$\partial F_\sharp \mathcal{T} = F_\sharp \partial \mathcal{T}. \tag{2.4}$$

We will also employ the following crucial fact, which actually is valid for every dimension but, in our setting, will be used only in codimension 1.

Theorem 2.2. *Let $n \ge 1$ be an integer. Let $\mathcal{T} \in \mathcal{D}_{n-1}(\mathbb{R}^n)$ be an integral current such that $\partial \mathcal{T} = 0$. Then there exists an integral current $\mathcal{S} \in \mathcal{D}_n(\mathbb{R}^n)$ such that $\partial \mathcal{S} = \mathcal{T}$.*

This is a standard result; in particular the current $\mathcal{S}$ can be the so-called cone over $\mathcal{T}$, see [14, Section 7.4.4]. Besides, $\mathcal{S}$ can be given by the isoperimetric inequality Theorem, see [14, Theorem 7.9.1].

Cartesian currents and graphs. Let $\Omega \subset \mathbb{R}^n$ be an open set, and let $u : \Omega \to \mathbb{R}^N$ be a smooth map. The graph of u is the set

$$G_u := \{(x, y) \in \Omega \times \mathbb{R}^N : y = u(x)\}.$$

This is the support of the current $\mathcal{G}_u \in \mathcal{D}_n(\Omega \times \mathbb{R}^N)$ given by

$$\mathcal{G}_u := (Id \times u)_\sharp [\![\Omega]\!]. \tag{2.5}$$

This turns out to be an integer multiplicity current whose mass is obtained as the result of

$$|\mathcal{G}_u|_{\Omega \times \mathbb{R}^N} = \int_\Omega |M(Du)| dx. \tag{2.6}$$

Notice that this is exactly the area of the graph of u. In the specific case $n = N = 2$ this formula reads as (1.3), namely $\mathcal{A}(u; \Omega) = |\mathcal{G}_u|_{\Omega \times \mathbb{R}^2}$. In order that $\mathcal{G}_u$ be an integer multiplicity current much less regularity of u is needed. Indeed it suffices that u is approximately differentiable a.e. in Ω and that all the minors $M^\alpha_\beta(Du)$ (for all $|\alpha| = |\beta| = k$, for all $k \le \min\{n, N\}$) belong to $L^1(\Omega)$. We denote the class of functions $u \in L^1(\Omega; \mathbb{R}^N)$ satisfying these conditions by $\mathcal{A}^1(\Omega; \mathbb{R}^N)$, namely

$$\begin{aligned}\mathcal{A}^1(\Omega; \mathbb{R}^N) := \{u \in L^1(\Omega; \mathbb{R}^N) :\ & u \text{ is appr. diff. a.e. in } \Omega, \\ & \text{and } M^\alpha_\beta(Du) \in L^1(\Omega)\ \forall\, |\alpha| = |\beta| = k,\ k \le \min\{n, N\}\}\end{aligned}$$

The class of Cartesian maps is $\mathrm{Cart}(\Omega; \mathbb{R}^N)$ defined as

$$\mathrm{Cart}(\Omega; \mathbb{R}^N) := \{u \in \mathcal{A}^1(\Omega; \mathbb{R}^N) : |\mathcal{G}_u| < +\infty,\ \partial \mathcal{G}_u = 0 \text{ in } \Omega \times \mathbb{R}^N\}. \tag{2.7}$$

Let $\mathcal{T}$ be an i.m.c. in $\mathcal{D}_n(\Omega \times \mathbb{R}^N)$. For all multi-indices α and β with $|\alpha| + |\beta| = n$ we define

$$\mathcal{T}^{\alpha\beta}(\omega) := \mathcal{T}(\omega_{\alpha\beta} dx^\alpha \wedge dy^\beta),$$

the $\alpha\beta$-component of T. The $\mathcal{T}^{\bar{0}0}$ component can be identified with a Radon measure on Ω. If the component $T^{\bar{0}0}$ is a Radon measure with bounded variation it is well defined the norm

$$\|\mathcal{T}\|_1 := \sup\{\mathcal{T}(\varphi(x,y)|y|dy) : \varphi \in C^0_c(\Omega \times \mathbb{R}^N),\ |\varphi| \leq 1\}.$$

We define the class of graphs as

$$\begin{aligned} \mathrm{graph}(\Omega \times \mathbb{R}^N) := \{&\mathcal{T} \in \mathcal{D}_n(\Omega \times \mathbb{R}^N) \text{ s.t. } \mathcal{T} \text{ is an i.m.c. with } M(\mathcal{T}) < \infty, \\ &\|\mathcal{T}\|_1 < \infty, M(\partial\mathcal{T}) < \infty,\ \mathcal{T}^{\bar{0}0} \geq 0,\ \pi_\sharp \mathcal{T} = [\![\Omega]\!]\}, \end{aligned} \tag{2.8}$$

where $\pi : \Omega \times \mathbb{R}^N \to \Omega$ is the standard projection into the Ω. A proper subclass of the graphs is the class of Cartesian currents defined as follows:

$$\begin{aligned} \mathrm{cart}(\Omega \times \mathbb{R}^N) := \{&\mathcal{T} \in \mathcal{D}_n(\Omega \times \mathbb{R}^N) \text{ s.t. } \mathcal{T} \text{ is an i.m.c. with } M(\mathcal{T}) < \infty, \\ &\|\mathcal{T}\|_1 < \infty, \partial\mathcal{T} \llcorner (\Omega \times \mathbb{R}^N) = 0,\ \mathcal{T}^{\bar{0}0} \geq 0,\ \pi_\sharp \mathcal{T} = [\![\Omega]\!]\}. \end{aligned} \tag{2.9}$$

By the structure theorem for Cartesian currents (see [11, Section 4.2.3]) we can always decompose a Cartesian current $\mathcal{T}$ as a graph plus a vertical part, namely

$$\mathcal{T} = \mathcal{G}_u + \mathcal{S}, \tag{2.10}$$

where $\mathcal{S}$ is concentrated on a set $\Omega_0 \times \mathbb{R}^N$, $\mathcal{L}^n(\Omega_0) = 0$, and satisfies

$$\mathcal{S}(\omega_{\alpha\beta} dx^\alpha \wedge dy^\beta) = 0 \ \text{ if } \alpha \neq 0.$$

In codimension 1 every Cartesian current can be approximated by graphs of Cartesian maps: if $\mathcal{T} \in \mathrm{cart}(\Omega \times \mathbb{R})$ then there exists a sequence of smooth functions u_k such that $\mathcal{G}_{u_k} \rightharpoonup \mathcal{T}$. Moreover if $\mathcal{T} = \mathcal{G}_u + \mathcal{S}$ then $u_k \to u$ in $L^1(\Omega)$. This is a consequence of the approximability of BV-functions with real values (see [11, Section 4.2.4]).

Slicing. We will need some elementary application of the technique of slicing. Very often in the following of the paper this technique can be reduced to a generalized version of Fubini integration Theorem. For this reason we do not go into details and we refer to [14] (see also [10] and [11]) for a complete discussion.

Let $\mathcal{S}$ be an integral current in $\mathcal{D}_k(\mathbb{R}^3)$, $k \geq 1$ and let x be one of the three coordinate in $\mathbb{R}^3$. We denote by $\langle \mathcal{S}, t\rangle$ the slice of $\mathcal{S}$ on the plane $\{x = t\}$. This is an integral current of dimension $k-1$ with some important features related to $\mathcal{S}$. In particular (see [14, Lemma 7.6.3]), if $\mathcal{S}$ is supported on a rectifiable set (denoted by S), then $\langle \mathcal{S}, t\rangle$ is supported on $S \cap \{x = t\}$, and it holds

$$\int_{-\infty}^{\infty} |\langle \mathcal{S}, t\rangle| dt \leq |\mathcal{S}|. \tag{2.11}$$

Moreover it holds true, for $\mathcal{H}^1$-a.e. $t \in \mathbb{R}$,

$$\partial\langle \mathcal{S}, t\rangle = -\langle \partial\mathcal{S}, t\rangle. \tag{2.12}$$

2.1 Technical preliminaries

Lemma 2.3. *Let $D \subset \mathbb{R}^2$ be a bounded open set and $v_k \in C^1(D;\mathbb{R}^2)$ be such that $v_k \to v \equiv c$, a constant, in $L^1(D;\mathbb{R}^2)$. Assume that*

$$\|Dv_k\|_{L^1} + \|J(v_k)\|_{L^1} < C < +\infty \quad \textit{for all } k. \tag{2.13}$$

Then, up to a subsequence, $\mathcal{G}_{v_k} \rightharpoonup \mathcal{G}_v + S$ as currents, where S is the vertical part, and

$$|\mathcal{G}_v + S| = |\mathcal{G}_v| + |S| = \mathcal{L}^2(D) + |S|. \tag{2.14}$$

Moreover for all $\epsilon > 0$ sufficiently small there exists an open set $A_\epsilon \subset D$ with $|A_\epsilon| \le \epsilon$ such that, for a not relabeled subsequence,

$$\mathcal{G}_{v_k} \llcorner (A_\epsilon \times \mathbb{R}^2) \rightharpoonup \mathcal{G}_v \llcorner (A_\epsilon \times \mathbb{R}^2) + S. \tag{2.15a}$$

Let us write $\mathcal{G}_{v_k} \llcorner (D \times \mathbb{R}^2) = Z_\epsilon^k + \widetilde{Z}_\epsilon^k$ where, for any $\omega = \omega_{\alpha\beta} dx^\alpha \wedge dy^\beta \in \mathcal{D}^2(D \times \mathbb{R}^2)$,

$$Z_\epsilon^k(\omega_{\alpha\beta} dx^\alpha \wedge dy^\beta) = \int_{A_\epsilon \cap D} \omega_{\alpha\beta}(x, v_k(x)))M_\beta^\alpha(Dv_k)(x)dx,$$
$$\widetilde{Z}_\epsilon^k(\omega_{\alpha\beta} dx^\alpha \wedge dy^\beta) = \int_{A_\epsilon^c \cap D} \omega_{\alpha\beta}(x, v_k(x))M_\beta^\alpha(Dv_k)(x)dx,$$

and define $\widehat{\Pi} : D \times \mathbb{R}^2 \to \mathbb{R}^3$ the map $\widehat{\Pi} : (x_1, x_2, y_1, y_2) \mapsto (\sqrt{x_1^2 + x_2^2}, y_1, y_2)$. Then

$$\widehat{\Pi}_\sharp \widetilde{Z}_\epsilon^k \rightharpoonup 0. \tag{2.15b}$$

Proof. By the theory of Cartesian currents we know that the weak limit of the currents $\mathcal{G}_{v_k}$ is of the form $\mathcal{G}_v \llcorner D^+ + S$ where D^+ is a Borel subset of D such that $|D \setminus D^+| = 0$ (see Theorem 2 in [11, Section 4.2.3]). Expression (2.14) follows from the fact that $\mathcal{G}_v$ and S are singular with respect to each other, and furthermore $|\mathcal{G}_v| = \mathcal{L}^2(D)$, being $v \equiv c$ a constant.

Let us fix $\epsilon > 0$. By (2.13) and the biting Lemma [4] there exists a (not relabeled) subsequence and a Borel set $A_\epsilon \subset D$ with $|A_\epsilon| \le \epsilon$ such that Dv_k and $J(v_k)$ are equi-uniformly integrable in $L^1(D \setminus A_\epsilon; \mathbb{R}^2)$, and thus there exist the limits

$$Dv_k \rightharpoonup G \quad \text{weakly in } L^1(D \setminus A_\epsilon; \mathbb{R}^{2\times 2}),$$
$$J(v_k) \rightharpoonup d \quad \text{weakly in } L^1(D \setminus A_\epsilon).$$

From Theorem 5 in [11, Section 4.2.3] (see formula (17) for d and (18) for G with $|\beta| = 2$) we find out that $G = 0$ and $d = 0$, namely

$$Dv_k \rightharpoonup 0 \quad \text{weakly in } L^1(D \setminus A_\epsilon; \mathbb{R}^{2\times 2}),$$
$$J(v_k) \rightharpoonup 0 \quad \text{weakly in } L^1(D \setminus A_\epsilon). \tag{2.16}$$

Fix now any function $\varphi \in C_c^\infty(D \times \mathbb{R}^2)$, setting $\omega_{ij} = \varphi dx^i \wedge dy^j$, we infer

$$\mathcal{G}_{v_k} \llcorner (A_\epsilon \times \mathbb{R}^2)(\omega_{ij}) = \int_{A_\epsilon} \varphi(x, v_k(x)) D_{\bar{i}}(v_k)_j(x)dx \tag{2.17}$$
$$= \int_D \varphi(x, v_k(x)) D_{\bar{i}}(v_k)_j(x)dx - \int_{D \setminus A_\epsilon} \varphi(x, v_k(x)) D_{\bar{i}}(v_k)_j(x)dx, \tag{2.18}$$

tends to

$$\mathcal{G}_v(\omega_{ij}) + S(\omega_{ij}) - \int_{D\setminus A_\epsilon} \varphi(x, v(x)) D_{\bar{i}} v_j(x)dx = \mathcal{G}_v(\omega_{ij}) \llcorner (A_\epsilon \times \mathbb{R}^2) + S(\omega_{ij}).$$

To let the last term pass to the limit we have here used [11, Proposition 1, Section 1.2.4, pag. 54]. Arguing similarly for a form $\omega = \omega_{ij} = \varphi dx^i \wedge dx^j$ and for $\omega = \varphi dy^i \wedge dy^j$, thanks to the convergence of the Jacobian, we conclude (2.15a).

To prove (2.15b) we check that $\widehat{\Pi}_\sharp \widetilde{Z}_\epsilon^k(\omega) \to 0$ for all $\omega \in \mathcal{D}^2(\mathbb{R}^3)$. It suffices to consider the three cases $\omega = \varphi d\rho \wedge dy^i$, $i = 1, 2$ and $\omega = \varphi dy^1 \wedge dy^2$. Take $\omega = \varphi d\rho \wedge dy^i$, $i = 1, 2$,

$$\widehat{\Pi}^\sharp \omega = \varphi \circ \widehat{\Pi}(x)(\frac{x_1}{|x|} dx^1 \wedge dy^i + \frac{x_2}{|x|} dx^2 \wedge dy^i),$$

so that, thanks to (2.16),

$$\widehat{\Pi}_\sharp \mathcal{G}_{v_k} \llcorner ((D \setminus A_\epsilon) \times \mathbb{R}^2)(\omega) = \sum_{j=1,2} \int_{D \setminus A_\epsilon} \varphi \circ \widehat{\Pi}(x) \frac{x_j}{|x|} \frac{\partial (v_k)_i}{\partial x_{\bar{j}}} dx \to 0.$$

If we choose $\omega = \varphi dy^1 \wedge dy^2$ we have

$$\widehat{\Pi}^\sharp \omega = \varphi \circ \widehat{\Pi}(x) dy^1 \wedge dy^2,$$

and hence, from (2.16),

$$\widehat{\Pi}_\sharp \mathcal{G}_{v_k} \llcorner ((D \setminus A_\epsilon) \times \mathbb{R}^2)(\omega) = \int_{D \setminus A_\epsilon} \varphi \circ \widehat{\Pi}(x) J(v_k) dx \to 0.$$

so that (2.15b) follows. □

Let T be the triangle in $\mathbb{R}^2$ with vertices α, β, and γ. Let $\pi_T : \mathbb{R}^2 \to T$ be the orthogonal projection onto the convex set T.

Lemma 2.4. *Let $v \in C^1(\Omega; \mathbb{R}^2)$. Then $\mathcal{A}(\pi_T \circ v) \leq \mathcal{A}(v)$.*

Proof. We observe first that the map $\pi_T \circ v$ is Lipschitz, that is of class $W^{1,\infty}(\Omega; \mathbb{R}^2)$, and its Jacobian determinant satisfies, almost everywhere on Ω,

$$J(\pi_T \circ v) = J(\pi_T)(v) J(v) \leq J(v), \tag{2.19}$$

the inequality following from the fact that $J(\pi_T)$ is 1 on T and null elsewhere. Moreover since π_T is a contraction, it holds

$$\begin{aligned}
&|\frac{\partial (\pi_T \circ v)_1}{\partial x_1}|^2 + |\frac{\partial (\pi_T \circ v)_1}{\partial x_2}|^2 + |\frac{\partial (\pi_T \circ v)_2}{\partial x_1}|^2 + |\frac{\partial (\pi_T \circ v)_2}{\partial x_2}|^2 \\
&= |\frac{\partial (\pi_T \circ v)}{\partial x_1}|^2 + |\frac{\partial (\pi_T \circ v)}{\partial x_2}|^2 \leq |\frac{\partial v}{\partial x_1}|^2 + |\frac{\partial v}{\partial x_2}|^2 \\
&= |\frac{\partial v_1}{\partial x_1}|^2 + |\frac{\partial v_1}{\partial x_2}|^2 + |\frac{\partial v_2}{\partial x_1}|^2 + |\frac{\partial v_2}{\partial x_2}|^2.
\end{aligned} \tag{2.20}$$

Putting together (2.19) and (2.20) we conclude. □

Here we state a result which relies on standard techniques in the theory of minimal surfaces:

Lemma 2.5. *Let $\varphi : (-\sqrt{3}/2, \sqrt{3}/2) \to \mathbb{R}^+$ be the piecewise affine function defined in* (1.14). *Let l_j be an increasing sequence of positive numbers such that*

$$l_j \nearrow l > 0 \quad \text{as } j \to \infty, \tag{2.21}$$

and let $\mathcal{R}_j$ be the rectangle $(0, l_j) \times (-\sqrt{3}/2, \sqrt{3}/2)$. Let m_j be the area of the minimal surface satisfying problem (1.16) *in $\mathcal{R}_j$, namely $m_j = m_{l_j}$. Then*

$$m_j \to m_l \quad \text{as } j \to \infty. \tag{2.22}$$

Proof. On one hand it holds $m_j \leq m_l$ for all j. Indeed, let u_j be the minimizer of the minimum problem (1.16), i.e.

$$\mathcal{A}(u_j; \mathcal{R}_j) = \int_{-\sqrt{3}/2}^{\sqrt{3}/2} \int_0^{l_j} \sqrt{1 + |\nabla u_j|^2} dx_1 dx_2 = m_j, \tag{2.23}$$

and let u be the minimizer of the same problem in the domain $\mathcal{R}_l := (0, l) \times (-\sqrt{3}/2, \sqrt{3}/2)$. We easily see that $u \llcorner (0, l_j) \times (-\sqrt{3}/2, \sqrt{3}/2)$ is an immediate competitor for the problem (1.16) in $\mathcal{R}_j$, and therefore

$$m_j \leq \mathcal{A}(u_j; \mathcal{R}_j) \leq \mathcal{A}(u; \mathcal{R}_j) \leq \mathcal{A}(u; \mathcal{R}_l) = m_l. \tag{2.24}$$

We therefore deduce $\lim m_j \le m_l$. Let us prove the opposite inequality. For fixed j we define the function $\tilde{u}_j$ on the domain $\mathcal{R}_l = (0,l) \times (-\sqrt{3}/2, \sqrt{3}/2)$ as

$$\tilde{u}_j(x_1,x_2) = \begin{cases} u_j(x_1 - (l-l_j), x_2) & \text{if } x_1 \in ((l-l_j), l) \\ \varphi(x_2) & \text{otherwise.} \end{cases} \tag{2.25}$$

It is then checked that $\tilde{u}_j$ is an admissible competitor for the problem (1.16) in $\mathcal{R}_l$, and moreover

$$A(\tilde{u}_j; \mathcal{R}_l) = m_j + 2(l - l_j). \tag{2.26}$$

In conclusion we have found

$$m_j + 2(l-l_j) = A(\tilde{u}_j; \mathcal{R}_l) \le m_l, \tag{2.27}$$

and the thesis follows. $\square$

We will consider a suitable sequence $\{v_k\} \in C^1(\Omega; \mathbb{R}^2)$ approaching the triple junction function u and such that

$$\lim_{k\to\infty} \mathcal{A}(v_k; \Omega) = \mathcal{A}(u; \Omega). \tag{2.28}$$

Notice that if we focus our attention to sequences of Lipschitz functions, the value of the area functional does not change thanks to the approximability of functions of class $C^1(\Omega;\mathbb{R}^2)$ (see step 1 of the proof in [5, Section 2]).

3 The problem in $\Omega = B_1(O)$

We study the problem of the area functional in the domain $\Omega = B_1(O)$, the ball centered at the origin and with radius $R = 1$. In the sequel we will denote by $u : \Omega \to \{\alpha, \beta, \gamma\}$ the triple junction function defined in the introduction. Let $\{v_k\}$ be a sequence of functions in $C^1(\Omega;\mathbb{R}^2)$ with $v_k \to u$ in $L^1(\Omega;\mathbb{R}^2)$ such that (2.28) holds true for $\Omega = B_1(O)$. In particular we can assume that v_k converge to u pointwise a.e. in Ω. Thanks to Lemma 2.4, up to replacing v_k by $\pi_T \circ v_k$, it is not restrictive to assume that v_k takes values in T for all $k \in \mathbb{N}$. With this assumption we cannot ensure that v_k is of class C^1 everywhere, but we can still suppose that it is of class C^1 in the set $v_k^{-1}(\mathring{T})$, where $\mathring{T} = T \setminus \partial T$ is the interior of T. We will prove that

$$\lim_{k\to\infty} \mathcal{A}(v_k, \Omega) \ge \mathcal{L}^2(\Omega) + 3m = \pi + 3m, \tag{3.1}$$

with $m = m_1$ being the value introduced in (1.16).

Geometric setting. Let us denote by J_i, $i = 1,2,3$, the segments of length 1 which are the jump sets of the function u; specifically J_1 is the interface between the sets $\{u = \alpha\}$ and $\{u = \gamma\}$, J_2 is the interface between $\{u = \beta\}$ and $\{u = \alpha\}$, and J_3 is the interface between $\{u = \gamma\}$ and $\{u = \beta\}$.

We will now select three sequences of real numbers $\theta_j \in (-\pi/6, \pi/6)$, $\rho_j \in (0,1)$, and $\delta_j \in (0,1)$ with $\theta_j \to 0$, $\rho_j \to 0$, and $\delta_j \to 0$. We first set (identifying $\mathbb{R}^2$ with $\mathbb{C}$)

$$B_j := \rho_j e^{\theta_j i}, \quad A_j := e^{\frac{2\pi i}{3}} B_j, \quad C_j := e^{\frac{4\pi i}{3}} B_j.$$

The points B_j, A_j, and C_j are the vertices of equilateral triangles with edge $\sqrt{3}\rho_j$ centered at the origin. The numbers θ_j and $\rho_j > 0$ are then chosen in such a way that the sequence $\{v_k\}$ converges to u at the points B_j, A_j, and C_j, for all fixed $j = 1, 2, \dots$. Notice that such a choice is possible since v_k converges to u a.e. in Ω. Moreover thanks to the specific choice of θ_j, it is easy to see that $v_k(B_j) \to u(B_j) = \beta$, $v_k(A_j) \to u(A_j) = \alpha$, and $v_k(C_j) \to u(C_j) = \gamma$, for all $j = 1, 2, \dots$.

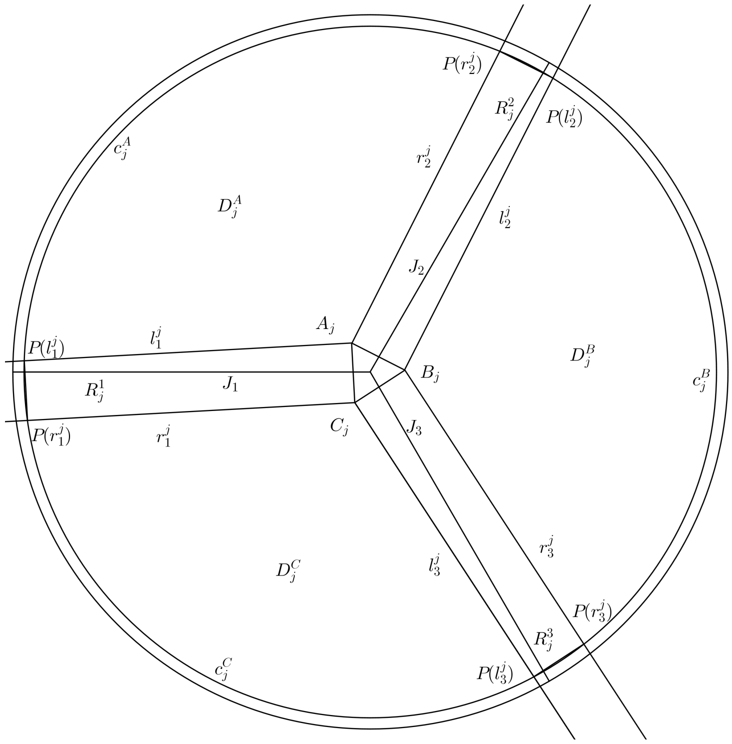

Figure 2: The domain $B_1(O)$ is decomposed in many sectors where we treat the graphs of v_n in different way.

Let l_1^j and r_1^j be two parallel halflines starting from the points A_j and C_j respectively, perpendicular to the edge $\overline{A_jC_j}$, and contained in the halfplane $\{x < 0\}$ (see figure 2). Similarly construct the halflines $l_2^j := e^{\frac{2\pi}{3}i}l_1^j$, $r_2^j := e^{\frac{2\pi}{3}i}r_1^j$, $l_3^j := e^{\frac{2\pi}{3}i}l_2^j$, and $r_3^j := e^{\frac{2\pi}{3}i}r_2^j$. Up to choosing θ_j small enough, we can assume that these halflines form a neighborhood of the three segments J_i, $i = 1, 2, 3$.

The halflines l_1^j and r_1^j meet $\partial B_{1-\delta_j}(O)$ at, say, $P(l_1^j)$ and $P(r_1^j)$. Similarly are defined the points $P(l_i^j)$ and $P(r_i^j)$ for $i = 2, 3$. Consider the rectangle R_j^1 of vertices $P(l_1^j)$, $P(r_1^j)$, C_j, and A_j, let $R_j^2 := e^{\frac{2\pi}{3}i}R_j^1$ and $R_j^3 := e^{\frac{2\pi}{3}i}R_j^2$. Let l_j be the lentgh of the segment $\overline{A_jP(l_1^j)}$. The region enclosed between the lines l_1^j, r_2^j, and the circle $\partial B_{l_j}(A_j)$ (consider the sector not containing O) is denoted by D_j^A. The arc obtained as intersection of the boundary of D_j^A and $\partial B_{l_j}(A_j)$ is denoted by c_j^A. It is here remarkable that if the number ρ_j is small enough with respect to δ_j, then it is easily seen that the sector D_j^A is well contained in $\Omega = B_1(O)$ (actually it suffices $\rho_j \leq 2\delta_j$). Similarly D_j^B, c_j^B are obtained by rotating D_j^A and c_j^A around O clockwise of an angle of $2\pi/3$. If the angle is $4\pi/3$ we get D_j^C and c_j^C respectively. We define the set

$$L_j := \cup_{i=1}^3 \partial R_j^i \cup c_j^A \cup c_j^B \cup c_j^C.$$

We will now suitably choose the sequence of real numbers $\delta_j > 0$. Let us first make some elementary deductions from (2.28). We observe that there exists a constant $C > 0$ such that

$$\sum_{i,h} \int_\Omega |\frac{\partial (v_k)_i}{\partial x_h}|dx + \int_\Omega |J(v_k)|dx \leq C \qquad \forall k \in \mathbb{N}. \tag{3.2}$$

In particular, by Fubini Theorem, it is not restrictive to assume (up to choosing suitably the numbers θ_j, ρ_j, and δ_j) that for all $j = 1, 2, \ldots$, there is a constant $C(j) > 0$ such that it holds

$$\liminf_{k\to\infty}\Big(\sum_{i,h}\int_{L_j}|\frac{\partial(v_k)_i}{\partial x_h}|d\mathcal{H}^1+\int_{L_j}|J(v_k)|d\mathcal{H}^1\Big)\le C(j)\qquad\forall j\in\mathbb{N}. \tag{3.3}$$

This is a consequence of the Fatou Lemma. Moreover we can also assume that the functions v_k pointwise converge to u $\mathcal{H}^1$-a.e. on L_j. Notice that in general the constant $C(j)$ depends on j.

Summarizing, we choose θ_j, ρ_j, and δ_j in such a way that:

(H1) The functions v_k converge to u at the points B_j, A_j, and C_j, and at the points $P(l_i^j)$, $P(r_i^j)$ for $i=1,2,3$ and for all fixed $j=1,2,\ldots$.

(H2) The functions v_k pointwise converge to u $\mathcal{H}^1$-a.e. on L_j, for all fixed $j=1,2,\ldots$.

(H3) The functions v_k admit a subsequence (depending on j) of uniform (with respect to k) bounded variation on L_j functions, (as a consequence of (3.3)) for all $j=1,2,\ldots$.

The set L_j consists of 3 arcs and 12 segments, six of the latters are the long sides of the rectangles R_j^i whose length is l_j, the other six are the short sides of these rectangles with length $\sqrt{3}\rho_j$. Denoting by $\{L_j^h\}_{h=1}^{15}$ these arcs and segments we parametrize each of them by a homomorphism $\phi_j^h:[0,1]\to L_j^h$. Notice that

$$l_j\nearrow 1,\quad \rho_j\searrow 0,\quad \delta_j\searrow 0. \tag{3.4}$$

We now fix the index $j\in\mathbb{N}$. We will pass to the limit as $j\to\infty$ only in the end of the proof of our main result (3.1) (see Theorem 3.7 below).

Exploiting hypotheses (H1)-(H3) it is not hard to see that we can extract a (non-relabeled) subsequence of $\{v_k\}$ such that

(H4) the functions $v_k\circ\phi_k^h$ converge in $L^1([0,1];\mathbb{R}^2)$, pointwise a.e. on $(0,1)$, and pointwise at the points $\{0,1\}$, for all $h=1,\ldots,15$, and converge weakly star in $BV([0,1];\mathbb{R}^2)$.

From hypothesis (H4) it follows that the image currents $(v_k\circ\phi_k^h)_\sharp[\![0,1]\!]$ admit limits in the weak topology [14] in the class of integral 1-currents (and will be identified with curves in $\mathbb{R}^2$ with specific endpoints). These will be crucial in the following discussion. Notice that with this notation, and still denoting by l_1^j the segment between A_j and $P(l_1^j)$ for instance, the current $(v_k)_\sharp[\![l_1^j]\!]$ coincides with $(v_k\circ\phi_k^h)_\sharp[\![0,1]\!]$, for some $h\in\{1,\ldots,15\}$.

Remark 3.1. The construction of the sets L_j depends on the parameters in (3.4). In what follows we will keep j fixed, and many objects we are going to define will depend on j. We will get rid of such dependence only at the end of this section, in the proof of Theorem 3.7 below.

The current S_k^1 originated from R_j^1. Let us now focus on the rectangle R_j^1 and let (x_1,x_2) be a system of Cartesian coordinates such that $R_j^1=(a,b)\times(-\sqrt{3}\rho_j/2,\sqrt{3}\rho_j/2)$. We can assume x_1 represents the distance between the point (x_1,x_2) and the segment $\overline{A_jC_j}$. In such a case we have $a=0$ and $b=l_j$, with l_j being the length of the part of l_1^j inside the ball $B_{1-\delta_j}(O)$, hence $R_j^1=(0,l_j)\times(-\sqrt{3}\rho_j/2,\sqrt{3}\rho_j/2)$. We define, following the idea in [6][2], the map $\Phi_k:R_j^1\to\mathbb{R}^3$ given by

$$\Phi_k(x_1,x_2):=\Big(x_1,v_k(x_1,x_2)\Big). \tag{3.5}$$

[2]This function, using the terminology introduced in [6,7], is a semicartesian parametrization, whose role of dimension reduction will be crucial in the following discussion.

The image of Φ_k is a surface in $\mathbb{R}^3$ which is identified with an integral current $S_k^1 \in \mathcal{D}_2(\mathbb{R}^3)$. Thus

$$S_k^1 := (\Phi_k)_\sharp [\![R_j^1]\!]. \tag{3.6}$$

In a similar way we construct the maps $\Phi_k : R_j^i \to \mathbb{R}^3$ and the associated image currents S_k^i, for $i = 2, 3$. Let us introduce the projection $\Pi : \mathbb{R}^4 \to \mathbb{R}^3$ given by

$$\Pi(x_1, x_2, y_1, y_2) = (x_1, y_1, y_2). \tag{3.7}$$

If we denote by $\Psi_k : \mathbb{R}^2 \to \mathbb{R}^2 \times \mathbb{R}^2$ the function $\Psi_k := Id \times v_k : (x_1, x_2) \mapsto (x_1, x_2, v_k(x_1, x_2))$ we can write

$$\Phi_k = \Pi \circ \Psi_k. \tag{3.8}$$

The current $S_k^1 \in \mathcal{D}_2(\mathbb{R}^3)$ satisfies

$$S_k^1 = \Pi_\sharp (\Psi_k)_\sharp [\![R_j^1]\!] = \Pi_\sharp (\mathcal{G}_{v_k} \llcorner (R_j^1 \times \mathbb{R}^2)). \tag{3.9}$$

Now, if $\mathcal{T} \in \mathcal{D}_2(\mathbb{R}^4)$, for any 2-form $\omega \in \mathcal{D}^2(\mathbb{R}^3)$ the push-forward of $\mathcal{T}$ by Π is defined as

$$\Pi_\sharp \mathcal{T}(\omega) = \mathcal{T}(\Pi^\sharp \omega),$$

$\Pi^\sharp \omega$ being the pull-back of ω by Π. It is easily seen that $\Pi^\sharp \omega$ is ω itself (can be identified with it). As a consequence we see that $\Pi_\sharp : \mathcal{D}_2(\mathbb{R}^4) \to \mathcal{D}_2(\mathbb{R}^3)$ does not increase the mass, namely

$$|S_k^1| \le |\mathcal{G}_{v_k}|_{R_j^1 \times \mathbb{R}^2}. \tag{3.10}$$

By definition, the currents S_k^i have boundaries $\partial S_k^i = (\Phi_k)_\sharp [\![\partial R_j^i]\!]$, $i = 1, 2, 3$. Thanks to (H3) and the fact that $\Pi_\sharp$ does not increase the mass, it is easily checked that the masses of these boundaries are uniformly bounded with respect to k. Let us consider again the case $i = 1$ (we will argue similarly for $i = 2, 3$); the boundary can be split in four parts, each corresponding to one edge of R_j^1. Remembering that $R_j^1 = (0, l_j) \times (-\sqrt{3}\rho_j/2, \sqrt{3}\rho_j/2)$, set

$$T_k^1 = (\Phi_k)_\sharp [\![(0, l_j) \times \{\sqrt{3}\rho_j/2\}]\!], \tag{3.11a}$$
$$\overline{T}_k^1 = (\Phi_k)_\sharp [\![(0, l_j) \times \{-\sqrt{3}\rho_j/2\}]\!], \tag{3.11b}$$
$$V_k^1 = (\Phi_k)_\sharp [\![\{0\} \times (-\sqrt{3}\rho_j/2, \sqrt{3}\rho_j/2)]\!], \tag{3.11c}$$
$$\overline{V}_k^1 = (\Phi_k)_\sharp [\![\{l_j\} \times (-\sqrt{3}\rho_j/2, \sqrt{3}\rho_j/2)]\!], \tag{3.11d}$$

(see Figure 3). We have

$$\partial S_k^1 = T_k^1 - \overline{T}_k^1 + V_k^1 - \overline{V}_k^1. \tag{3.12}$$

We then use the compactness Theorem for integral currents (see [14]), and letting $k \to \infty$ we find an integral current $S^1 \in \mathcal{D}_2(\mathbb{R}^3)$ such that, up to a not relabeled subsequence,

$$S_k^1 \rightharpoonup S^1,$$

(we remark that S^1 depends on j; not to overburden notation we drop the label j here). By lower semicontinuity and (3.10), we get

$$|S^1| \le \liminf_{k \to \infty} |\mathcal{G}_{v_k}|_{R_j^1 \times \mathbb{R}^2}. \tag{3.13}$$

The current S_k^A on the sector D_j^A. On the sector D_j^A we consider polar coordinates (ρ, θ) centered at A_j. Consider the map $\widetilde{\Pi} : D_j^A \times \mathbb{R}^2 \to [0, l_j] \times \mathbb{R}^2$ given by

$$\widetilde{\Pi}(\rho, \theta, y_1, y_2) = (\rho, y_1, y_2).$$

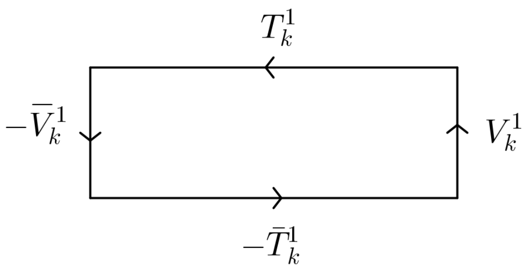

Figure 3: The rectangle R_j^1 is depicted with the standard orientation. The push forward of the integration on the edges by Φ_k gives rise to the currents denoted in the figure.

Let $\mathcal{T}$ be an integral current in $\mathcal{D}_2(D_j^A \times \mathbb{R}^2)$, and consider the push-forward by $\widetilde{\Pi}$, namely $\widetilde{\Pi}_\sharp \mathcal{T}$. Writing $\widetilde{\Pi}$ in euclidean coordinates it is an easy check that the map $\widetilde{\Pi}$ is a contraction and that $\widetilde{\Pi}_\sharp$ does not increase the mass of $\mathcal{T}$.

In the spirit of what we have made on the set R_j^1 let us now consider the following map $\widetilde{\Phi}_k : D_j^A \to \mathbb{R}^3$,

$$\widetilde{\Phi}_k : (\rho, \theta) \mapsto (\rho, v_k(\rho, \theta)). \tag{3.14}$$

By definition, it is checked that

$$(\widetilde{\Phi}_k)_\sharp [\![D_j^A]\!] = \widetilde{\Pi}_\sharp \mathcal{G}_{v_k} \llcorner (D_j^A \times \mathbb{R}^2).$$

We thus define

$$S_k^A := (\widetilde{\Phi}_k)_\sharp [\![D_j^A]\!]. \tag{3.15}$$

Let S^A be a weak limit for (a not-relabeled subsequence of) $\{S_k^A\}$, namely

$$\widetilde{\Pi}_\sharp \mathcal{G}_{v_k} \llcorner (D_j^A \times \mathbb{R}^2) = S_k^A \rightharpoonup S^A. \tag{3.16}$$

We emphasize the dependence of S^A on j. Fix $\epsilon > 0$ and let A_ϵ be as in Lemma 2.3 with $D = D_j^A$, so $|A_\epsilon| \leq \epsilon$. We now split the current $\mathcal{G}_{v_k} \llcorner (D_j^A \times \mathbb{R}^2) = Z_\epsilon^k + \widetilde{Z}_\epsilon^k$ where

$$Z_\epsilon^k(\omega_{\alpha\beta} dx^\alpha \wedge dy^\beta) = \int_{A_\epsilon \cap D_j^A} \omega_{\alpha\beta}(x, v_k(x)) M_\beta^\alpha (Dv_k)(x) dx, \tag{3.17}$$

$$\widetilde{Z}_\epsilon^k(\omega_{\alpha\beta} dx^\alpha \wedge dy^\beta) = \int_{A_\epsilon^c \cap D_j^A} \omega_{\alpha\beta}(x, v_k(x)) M_\beta^\alpha (Dv_k)(x) dx, \tag{3.18}$$

for all $\omega \in \mathcal{D}^2(D_j^A \times \mathbb{R}^2)$. By Lemma 2.3 we know that

$$\widetilde{\Pi}_\sharp \widetilde{Z}_\epsilon^k \rightharpoonup 0, \tag{3.19}$$

so that

$$\widetilde{\Pi}_\sharp Z_\epsilon^k = \widetilde{\Pi}_\sharp \mathcal{G}_{v_k} \llcorner (D_j^A \times \mathbb{R}^2) - \widetilde{\Pi}_\sharp \widetilde{Z}_\epsilon^k = S_k^A - \widetilde{\Pi}_\sharp \widetilde{Z}_\epsilon^k \rightharpoonup S^A. \tag{3.20}$$

By lowersemicontinuity we infer

$$
\begin{aligned}
|S^A| &\leq \liminf_{k\to\infty} |\widetilde{\Pi}_\sharp \mathcal{G}_{v_k}|_{(D_j^A\cap A_\epsilon)\times\mathbb{R}} \leq \liminf_{k\to\infty} |\mathcal{G}_{v_k}|_{(D_j^A\cap A_\epsilon)\times\mathbb{R}^2} \\
&= \liminf_{k\to\infty} \Big(|\mathcal{G}_{v_k}|_{D_j^A\times\mathbb{R}^2} - |\mathcal{G}_{v_k}|_{(D_j^A\cap A_\epsilon^c)\times\mathbb{R}^2} \Big) \\
&\leq \liminf_{k\to\infty} |\mathcal{G}_{v_k}|_{D_j^A\times\mathbb{R}^2} - \limsup_{k\to\infty} |\mathcal{G}_{v_k}|_{(D_j^A\cap A_\epsilon^c)\times\mathbb{R}^2} \\
&= \liminf_{k\to\infty} |\mathcal{G}_{v_k}|_{D_j^A\times\mathbb{R}^2} - \limsup_{k\to\infty} |\mathcal{G}_{v_k}|_{(D_j^A\setminus A_\epsilon)\times\mathbb{R}^2} \\
&\leq \liminf_{k\to\infty} |\mathcal{G}_{v_k}|_{D_j^A\times\mathbb{R}^2} - |D_j^A| + \epsilon. \qquad (3.21)
\end{aligned}
$$

The last inequality is due to the fact that $|\mathcal{G}_{v_k}|_{(D_j^A\setminus A_\epsilon)\times\mathbb{R}^2} \geq |D_j^A \setminus A_\epsilon| \geq |D_j^A| - \epsilon$. Thus by arbitrariness of $\epsilon > 0$ we conclude

$$
|S^A| + |D_j^A| \leq \liminf_{k\to\infty} |\mathcal{G}_{v_k}|_{D_j^A\times\mathbb{R}^2}. \qquad (3.22)
$$

Let us now restrict our attention to the boundary of $S_k^A = (\Phi_k)_\sharp [\![D_j^A]\!]$ in $\mathbb{R}^3$. For any fixed k, the current

$$
(\widetilde{\Phi}_k)_\sharp [\![l_1^j]\!]
$$

coincides with the current T_k^1 defined in (3.11a). As a consequence, if we set

$$
\widehat{S}_k^1 := S_k^1 + S_k^A, \qquad (3.23)
$$

we infer that the boundary of $\widehat{S}_k^1$ coincides with the current

$$
\partial \widehat{S}_k^1 = \overline{T}_k^2 + V_k^1 - \overline{T}_k^1 - \overline{V}_k^1 - C_k^A, \qquad (3.24)
$$

where V_k^1, $\overline{T}_k^1$, and $\overline{V}_k^1$ are defined in (3.11), and C_k^A and $\overline{T}_k^2$ are the currents

$$
C_k^A := (\widetilde{\Phi}_k)_\sharp [\![c_j^A]\!] \qquad \overline{T}_k^2 := (\widetilde{\Phi}_k)_\sharp [\![r_2^j]\!]. \qquad (3.25)
$$

A similar construction as above can be done for the sectors D_k^B and D_k^C. Thus we are led to define

$$
\begin{aligned}
\widehat{S}_k^2 &:= S_k^2 + (\widetilde{\Phi}_k)_\sharp [\![D_j^B]\!] = S_k^2 + S_k^B, \\
\widehat{S}_k^3 &:= S_k^3 + (\widetilde{\Phi}_k)_\sharp [\![D_j^C]\!] = S_k^3 + S_k^C, \qquad (3.26)
\end{aligned}
$$

whose boundaries are, respectively,

$$
\begin{aligned}
\partial \widehat{S}_k^2 &= (\widetilde{\Phi}_k)_\sharp [\![r_3^j]\!] + V_k^2 - \overline{T}_k^2 - \overline{V}_k^2 - C_k^B, \\
\partial \widehat{S}_k^3 &= (\widetilde{\Phi}_k)_\sharp [\![r_1^j]\!] + V_k^3 - \overline{T}_k^3 - \overline{V}_k^3 - C_k^C. \qquad (3.27)
\end{aligned}
$$

Since by mere observations we have $\overline{T}_k^3 = (\widetilde{\Phi}_k)_\sharp [\![r_3^j]\!]$, and $\overline{T}_k^1 = (\widetilde{\Phi}_k)_\sharp [\![r_1^j]\!]$ (compare (3.11)), we conclude

$$
\partial(\widehat{S}_k^1 + \widehat{S}_k^2 + \widehat{S}_k^3) = \sum_{i=1}^{3} (V_k^i - \overline{V}_k^i) - C_k^A - C_k^B - C_k^C.
$$

We now aim to pass to the limit as $k \to \infty$. Considering the limit of the terms in (3.24), as a consequence of hypothesis (H4), we can find five Lipschitz curves $(\varphi_1)_j$, $(h = 1, \ldots, 5)$ defined on the interval $I := [0, 1]$ such that the push-forward of the integrations on I by $(\varphi_1)_h$, are the limit currents of $\overline{T}_k^2$, V_k^1, $\overline{T}_k^1$, $\overline{V}_k^1$, and C_k^A. In particular (renaming such curves) we have $\varphi_A : I \to \mathbb{R}^3$ and $\varphi_C : I \to \mathbb{R}^3$ such that

$$
\begin{aligned}
\overline{T}_k^2 &\rightharpoonup (\varphi_A)_\sharp [\![I]\!], \\
\overline{T}_k^1 &\rightharpoonup -(\varphi_C)_\sharp [\![I]\!]. \qquad (3.28)
\end{aligned}
$$

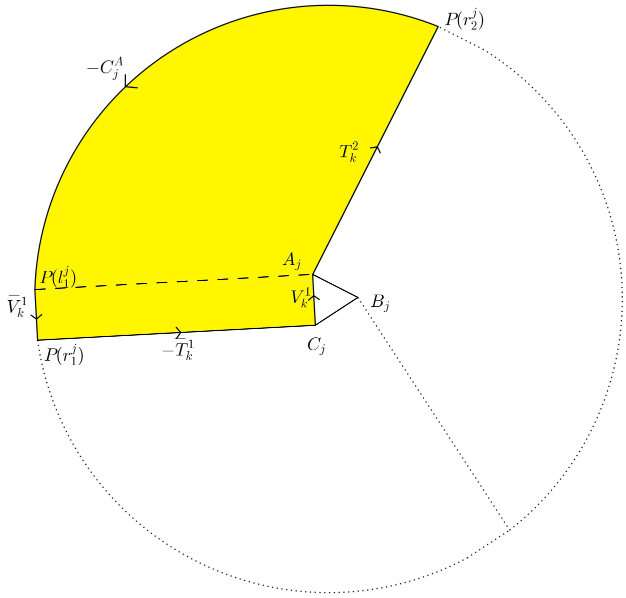

Figure 4: The area obtained by the union of R_j^1 and D_j^A is depicted and painted in yellow, with the standard orientation of its boundary. The push forward of the integration on R_j^1 by Φ_k and on D_j^A by $\widetilde{\Phi}_k$ has as sum the current $\widehat{S}_j^1$, whose boundary is the images by such maps of the edges of the area, as showed in the figure (see (3.24)). The two integrations over the traced segment $\overline{P(l_1^j)A_j}$ cancel out, since the orientation of this segment has opposite sign when seen as part of the boundary of R_j^1 and D_j^A.

Thanks to the fact that the maps v_k are converging pointwise on r_2^j and r_1^j to α and γ (respectively)[3], we again infer from the theory of Cartesian currents that the currents $(\varphi_A)_\sharp[\![I]\!]$ and $(\varphi_C)_\sharp[\![I]\!]$ are the graphs over the interval $[0,1]$ of the constants α and γ, respectively, (possibly) plus an additional vertical part.

As for the case $i=1$, we have that all currents on the right hand side of (3.27) admit limits as $k\to\infty$. Indeed we see that there exists a Lipschitz curve $\varphi_B : I \to \mathbb{R}^3$ such that

$$(\widetilde{\Phi}_k)_\sharp[\![r_3^j]\!] \rightharpoonup (\varphi_B)_\sharp[\![I]\!]. \tag{3.29}$$

Let us first state:

Proposition 3.2. *There exist three Lipschitz curves $\varphi_A : I \to \mathbb{R}^3$, $\varphi_B : I \to \mathbb{R}^3$, and $\varphi_C : I \to \mathbb{R}^3$ such that* (3.28) *and* (3.29) *hold true and for a.e. $s\in[0,1]$ we have*

$$\varphi_A(I)\cap(\{s\}\times\mathbb{R}^2) = \{(s,\alpha)\}, \tag{3.30}$$
$$\varphi_B(I)\cap(\{s\}\times\mathbb{R}^2) = \{(s,\beta)\}, \tag{3.31}$$
$$\varphi_C(I)\cap(\{s\}\times\mathbb{R}^2) = \{(s,\gamma)\}. \tag{3.32}$$

Proof. The current $\overline{T}_k^2 = (\widetilde{\Phi}_k)_\sharp[\![r_2^j]\!] \in \mathcal{D}_1([0,l_j]\times\mathbb{R}^2)$ is exactly the graph on $[0,l_j]$ of $((v_k)_1,(v_k)_2)$. Moreover such functions restricted on $[0,l_j]$ have equi-uniformly bounded variations, and are continuous, so that, in particular, their graphs are Cartesian currents on $[0,l_j]\times\mathbb{R}^2$. Up to re-parametrize this functions on $[0,l_j]$ we can apply the structure Theorem for Cartesian Currents (see [11, Section 4.2.3]) which asserts that the limit graph has the form $(Id\times u)_\sharp[\![[0,l_j]]\!] + N^A$, with u the limit of v_k in $L^1([0,l_j];\mathbb{R}^2)$, and N^A a vertical part

[3] More precisely, referring to hypothesis (H4), such convergence takes place a.e. on the interval $[0,1]$.

which is supported on a singular set $S \times \mathbb{R}^2$. Namely, we have,

$$(Id \times v_k)_\sharp[\![0, l_j]\!] \rightharpoonup (Id \times \alpha)_\sharp[\![0, l_j]\!] + N^A, \tag{3.33}$$

where we have denoted the constant map equal to α by the symbol α itself. Therefore there is a subset I^+ of full measure in $[0, l_j]$ such that N^A is concentrated in $([0,1] \setminus I^+) \times \mathbb{R} = S \times \mathbb{R}^2$, and on the complement $I^+ \times \mathbb{R}$ the limit current is the integration over the segment $[0, l_j] \times \alpha \subset \mathbb{R}^3$. The fact that the limit current can be parametrized by only one path $\varphi_A = ((\varphi_A)_1, (\varphi_A)_2)$ is a consequence of the fact that for all k the current $\overline{T}^2_k$ is the image of the integration over r_2^j by uniformly bounded BV functions[4]. The thesis then follows for φ_A, and a similar argument applies for φ_B and φ_C. □

Regarding the convergence of the other terms in (3.24) and (3.27) we have proved the following:

Proposition 3.3. *There exist integral currents $\widehat{S}^1, \widehat{S}^2, \widehat{S}^3 \in \mathcal{D}_2([0, l_j] \times \mathbb{R}^2)$, $\mathcal{V}^1, \mathcal{V}^2, \mathcal{V}^3 \in \mathcal{D}_1(\{0\} \times \mathbb{R}^2)$, and $\overline{\mathcal{V}}^1, \overline{\mathcal{V}}^2, \overline{\mathcal{V}}^3, C^A, C^B, C^C \in \mathcal{D}_1(\{l_j\} \times \mathbb{R}^2)$, such that*[5]

$$\widehat{S}^i_k \rightharpoonup -\widehat{S}^i, \quad i = 1, 2, 3, \tag{3.34}$$

$$V^i_k \rightharpoonup -\mathcal{V}^i, \quad i = 1, 2, 3, \tag{3.35}$$

$$\overline{V}^i_k \rightharpoonup \overline{\mathcal{V}}^i, \quad i = 1, 2, 3, \tag{3.36}$$

$$C^A_k \rightharpoonup C^A, \quad C^B_k \rightharpoonup C^B, \quad C^C_k \rightharpoonup C^C, \tag{3.37}$$

and

$$\partial\widehat{S}^1 = -(\varphi_A)_\sharp[\![I]\!] + \mathcal{V}^1 + (\varphi_C)_\sharp[\![I]\!] + \overline{\mathcal{V}}^1 + C^A, \tag{3.38}$$

$$\partial\widehat{S}^2 = -(\varphi_B)_\sharp[\![I]\!] + \mathcal{V}^2 + (\varphi_A)_\sharp[\![I]\!] + \overline{\mathcal{V}}^2 + C^B, \tag{3.39}$$

$$\partial\widehat{S}^3 = -(\varphi_C)_\sharp[\![I]\!] + \mathcal{V}^3 + (\varphi_B)_\sharp[\![I]\!] + \overline{\mathcal{V}}^3 + C^C. \tag{3.40}$$

Actually, we can say more about the currents φ_A, φ_B, and φ_C. From the proof of Proposition 3.2 we have found that there is a vertical current N^A (see (3.33)) such that

$$(\varphi_A)_\sharp[\![I]\!] = (Id \times \alpha)_\sharp[\![0, l_j]\!] + N^A, \tag{3.41a}$$

and similarly we will have

$$(\varphi_B)_\sharp[\![I]\!] = (Id \times \beta)_\sharp[\![0, l_j]\!] + N^B, \tag{3.41b}$$

$$(\varphi_C)_\sharp[\![I]\!] = (Id \times \gamma)_\sharp[\![0, l_j]\!] + N^C. \tag{3.41c}$$

The currents N^A, N^B, N^C will be concentrated on a set $S \times \mathbb{R}^2$, with $S = \{s_i\}_{i\in\mathbb{N}} \subset [0, l_j]$ at most countable. Indeed, using the decomposition theorem for 1-currents (Theorem 2.1) we conclude that N^A can be decomposed as a countable sum of closed loops $\overline{\alpha}_i : [0, t_i^A] \to \{s_i\} \times \mathbb{R}^2$ such that $\overline{\alpha}_i(0) = \overline{\alpha}_i(t_i^A) = (s_i, \alpha)$. In particular the cardinality of such possible set $\{s_i\}$ is at most countable. This is summarized in the following:

Proposition 3.4. *There is a countable set $S = \{s_i\}_{i\in\mathbb{N}} \subset [0, l_j]$ and a family of closed curves $\overline{\alpha}_i : [0, t_i^A] \to \{s_i\} \times T$, $\overline{\beta}_i : [0, t_i^B] \to \{s_i\} \times T$, $\overline{\gamma}_i : [0, t_i^C] \to \{s_i\} \times T$ (we recall that T is the closed triangle with vertices α, β, and γ) such that for all $i \in \mathbb{N}$*

$$\overline{\alpha}_i(0) = \overline{\alpha}_i(t_i^A) = (s_i, \alpha), \tag{3.42}$$

$$\overline{\beta}_i(0) = \overline{\beta}_i(t_i^B) = (s_i, \beta), \tag{3.43}$$

$$\overline{\gamma}_i(0) = \overline{\gamma}_i(t_i^C) = (s_i, \gamma), \tag{3.44}$$

[4]Equivalently, this is a consequence of the fact that the currents $\overline{T}^2_k$ can be parametrized by uniformly bounded BV maps defined on the same interval $[0, 1]$ (see hypothesis (H4)). We will later see that the set S is at most countable. This will follow from the fact that N^A is a vertical 1-current with no boundary, see Proposition 3.4.

[5]The choice of the sign in front of $\widehat{S}^i$ and $\mathcal{V}^i$ is just a definition which will turn out to be helpful in order to simplify some notation in the next Section.

and

$$(\varphi_A)_\sharp[\![I]\!] = (Id \times \alpha)_\sharp[\![[0,t_j]]\!] + \sum_i (\overline{\alpha}_i)_\sharp[\![[0,t_i^A]]\!], \tag{3.45}$$

$$(\varphi_B)_\sharp[\![I]\!] = (Id \times \beta)_\sharp[\![[0,t_j]]\!] + \sum_i (\overline{\beta}_i)_\sharp[\![[0,t_i^B]]\!], \tag{3.46}$$

$$(\varphi_C)_\sharp[\![I]\!] = (Id \times \gamma)_\sharp[\![[0,t_j]]\!] + \sum_i (\overline{\gamma}_i)_\sharp[\![[0,t_i^C]]\!]. \tag{3.47}$$

We remark that the sum of the lengths of the curves $\overline{\alpha}_i$, $\overline{\beta}_i$, and $\overline{\gamma}_i$, is finite, and therefore up to reparametrization we can choose t_i^A, t_i^B, t_i^B with finite sum.

Finally, from (3.23), by (3.13) and (3.22), we infer

$$|\widehat{S}^1| + |D_j^A| \le |S^1| + |S^A| + |D_j^A| \le \liminf_{k\to\infty} |\mathcal{G}_{v_k}|_{R_j^1\times\mathbb{R}^2} + \liminf_{k\to\infty} |\mathcal{G}_{v_k}|_{D_j^A\times\mathbb{R}^2}, \tag{3.48}$$

and similarly

$$|\widehat{S}^2| + |D_j^B| \le \liminf_{k\to\infty} |\mathcal{G}_{v_k}|_{R_k^2\times\mathbb{R}^2} + \liminf_{k\to\infty} |\mathcal{G}_{v_k}|_{D_k^B\times\mathbb{R}^2}, \tag{3.49}$$

$$|\widehat{S}^3| + |D_j^C| \le \liminf_{k\to\infty} |\mathcal{G}_{v_k}|_{R_k^3\times\mathbb{R}^2} + \liminf_{k\to\infty} |\mathcal{G}_{v_k}|_{D_k^C\times\mathbb{R}^2}. \tag{3.50}$$

Triangle current. Consider the triangle T_j with vertices A_j, B_j, and C_j, and let $[\![T_j]\!]$ be the current given by integration on T_j. Let $J : \mathbb{R}^2 \to \mathbb{R}^3$ be given by

$$J(y_1, y_2) = (0, y_1, y_2).$$

The map $J_k := J \circ v_k : T_j \to \mathbb{R}^3$, induces the current $(J_k)_\sharp[\![T_j]\!] \in \mathcal{D}_2(\{0\}\times\mathbb{R}^2)$ whose total mass is easily seen to be smaller than that of $\mathcal{G}_{v_k}$ in $T_j \times \mathbb{R}^2$. Indeed, the map J is a natural immersion and preserves the mass, whereas the mass of $(v_k)_\sharp[\![T_j]\!]$ is given by

$$\int_{T_j} |J(v_k)(x)|dx < \mathcal{A}(v_k, T_k) = |\mathcal{G}_{v_k}|_{T_j\times\mathbb{R}^2}. \tag{3.51}$$

Notice here that the inequality is strict since we are integrating only the Jacobian of v. As for the boundary of $(J_k)_\sharp[\![T_j]\!]$, this is given by the sum of the push-forward by J_k of the integration over the edges of T_j, i.e.,

$$(J_k)_\sharp[\![\overline{A_jC_j}]\!] + (J_k)_\sharp[\![\overline{C_jB_j}]\!] + (J_k)_\sharp[\![\overline{B_jA_j}]\!].$$

Going back to the definition of Φ_k and of V_k^1 (see (3.8) and (3.11c)), it is observed that $J_k \llcorner \overline{A_jC_j} \equiv \Phi_k$, and similarly for the other indices. In particular we have

$$\partial(J_k)_\sharp[\![T_j]\!] = -V_k^1 - V_k^2 - V_k^3. \tag{3.52}$$

Since by hypothesis (H3) the mass of this current is uniformly bounded with respect to k, we infer the existence of an integral current $\mathcal{T} \in \mathcal{D}_2(\{0\}\times\mathbb{R}^2)$ such that

$$-(J_k)_\sharp[\![T_j]\!] \rightharpoonup \mathcal{T}, \tag{3.53}$$

and thus

$$|\mathcal{T}| \le \liminf_{k\to\infty} |(J_k)_\sharp[\![T_j]\!]| \le \liminf_{k\to\infty} |\mathcal{G}_{v_k}|_{T_j\times\mathbb{R}^2}, \tag{3.54}$$

by (3.51). Let us finally study the boundary of $\mathcal{T}$. From (3.53) we infer $-\partial(J_k)_\sharp[\![T_j]\!] \rightharpoonup \partial\mathcal{T}$ and by (3.52), we find that

$$V_k^i \rightharpoonup -\mathcal{V}^i \quad \text{for } i = 1,2,3, \tag{3.55}$$

$$\partial\mathcal{T} = -\mathcal{V}^1 - \mathcal{V}^2 - \mathcal{V}^3, \tag{3.56}$$

where $\mathcal{V}^i$, $i = 1,2,3,$, are given in Proposition 3.3. By construction and again hypothesis (H4) there exist three Lipschitz paths $\psi_i : [0,1] \to \mathbb{R}^2$, $i = 1,2,3$, with

$$\mathcal{V}_i = (\psi_i)_\sharp[\![[0,1]]\!], \tag{3.57}$$

$$\psi_1(0) = \alpha, \quad \psi_1(1) = \gamma = \psi_3(0), \quad \psi_3(1) = \beta = \psi_2(0), \quad \psi_2(1) = \alpha. \tag{3.58}$$

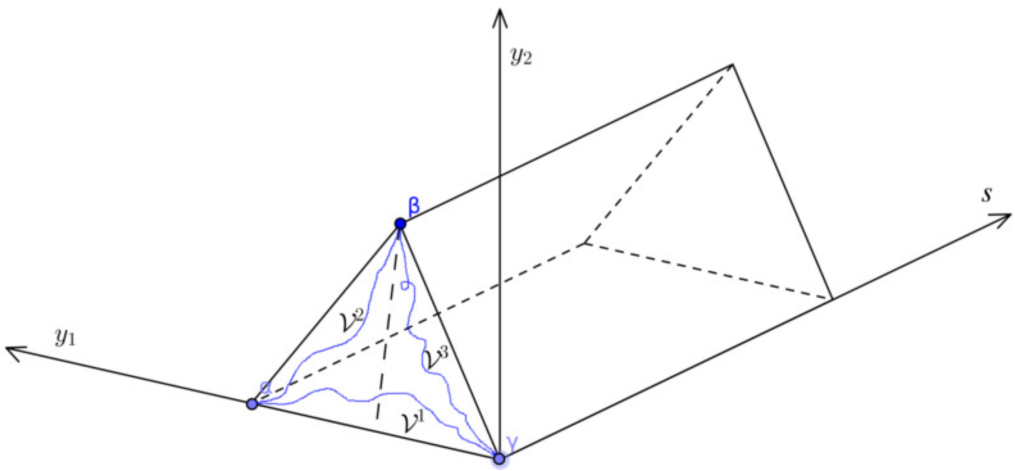

Figure 5: This is the prism $P = (0, l_j) \times T$. On the bottom face $\{0\} \times T$ the three currents $\mathcal{V}^i$, $i = 1, 2, 3$, can be seen in blue. The area enclosed by them is the support of the current $\mathcal{T}$.

Remark 3.5. Notice again the choice of signs in front of $\mathcal{T}$ and $\mathcal{V}^i$; this convention is convenient to simplify notation in the next section. Notice also that with this convention the currents $\mathcal{V}^1$, $\mathcal{V}^2$, $\mathcal{V}^3$ can be written as integration over paths connecting α to γ, β to α, and γ to β, respectively.

Total current. Consider now the currents $\widehat{S}^i_k$ for $i = 1, 2, 3$ and $(J_k)_\sharp[\![T_j]\!]$. With (3.24) and (3.52) at disposal, we readly infer that the current

$$U_k := -\widehat{S}^1_k - \widehat{S}^2_k - \widehat{S}^3_k - (J_k)_\sharp[\![T_j]\!], \tag{3.59}$$

has boundary

$$\partial U_k = \overline{V}^1_k + C^A_k + \overline{V}^2_k + C^B_k + \overline{V}^3_k + C^C_k. \tag{3.60}$$

Moreover, since the maps v_k take values in the triangle T, by definition of $\widehat{S}^i_k$ and $(J_k)_\sharp[\![T_j]\!]$ we find out that each current $\widehat{S}^i_k$ and $(J_k)_\sharp[\![T_j]\!]$ have support in closure of the prism $P := [0, l_j) \times T$, namely

$$\overline{P} := [0, l_j] \times T. \tag{3.61}$$

Moreover, by (3.60), ∂U_k is supported in $\{l_j\} \times T$, or, in other words, the currents U_k are closed as currents in $\mathcal{D}_2((-\infty, l_j) \times \mathbb{R}^2)$.

Passing to the limit in $k \to \infty$ and appealing to Propositions 3.2 and 3.3:

Proposition 3.6. *The current $U \in \mathcal{D}_2(\mathbb{R}^3)$ given by*

$$U = \widehat{S}^1 + \widehat{S}^2 + \widehat{S}^3 + \mathcal{T},$$

has boundary

$$\partial U = \overline{\mathcal{V}}^1 + C^A + \overline{\mathcal{V}}^2 + C^B + \overline{\mathcal{V}}^3 + C^C.$$

Moreover U is supported in $\overline{P}$ and ∂U is supported in $\{l_j\} \times T$. In particular U is a closed current in $\mathcal{D}_2((-\infty, l_j) \times \mathbb{R}^2)$.

Key inequality. We can write

$$\begin{aligned}\mathcal{A}(v_k, \Omega) = |\mathcal{G}_{v_k}|_{\Omega \times \mathbb{R}^2} = &|\mathcal{G}_{v_k}|_{D^A_j \times \mathbb{R}^2} + |\mathcal{G}_{v_k}|_{D^B_j \times \mathbb{R}^2} + |\mathcal{G}_{v_k}|_{D^C_j \times \mathbb{R}^2} \\ &+ \sum_{i=1}^{3} |\mathcal{G}_{v_k}|_{R^i_j \times \mathbb{R}^2} + |\mathcal{G}_{v_k}|_{T_j \times \mathbb{R}^2} + |\mathcal{G}_{v_k}|_{E_j \times \mathbb{R}^2},\end{aligned}$$

where $E_j := B_1(O) \setminus (\cup_{i=1}^3 R_j^i \cup T_j \cup D_j^A \cup D_j^B \cup D_j^C)$, so that, passing to the liminf and taking into account (3.48)-(3.50) and (3.54), we conclude

$$\begin{aligned}\mathcal{A}(u, \Omega) &\geq |\widehat{S}^1| + |\widehat{S}^2| + |\widehat{S}^3| + |\mathcal{T}| + |D_j^A| + |D_j^B| + |D_j^C| + |T_j| \\ &= |\widehat{S}^1| + |\widehat{S}^2| + |\widehat{S}^3| + |\mathcal{T}| + \pi - |E_j|. \end{aligned} \tag{3.62}$$

We now state our main result:

Theorem 3.7. *We have*

$$|\widehat{S}^1| + |\widehat{S}^2| + |\widehat{S}^3| + |\mathcal{T}| \geq 3m_{l_j}. \tag{3.63}$$

From this result we can easily address inequality (3.1). Indeed using (3.4) we infer $E_j \to 0$ as $j \to \infty$. Inequality (3.1) is then achived from (3.62) if we show that $m_{l_j} \to m_1 = m$. But this is the content of Lemma 2.5.

4 A symmetrization technique

In this section we construct three symmetrization operators for currents, denoted by $\mathbb{S}_A$, $\mathbb{S}_B$, and $\mathbb{S}_C$. These are substantially based on a Steiner-type symmetrization technique, and their role is crucial in order to prove Theorem 3.7. Indeed these operators satisfy the feature of non-increasing the total mass of their arguments (see Lemma 4.23). Hence, after suitably applying $\mathbb{S}_A$, $\mathbb{S}_B$, and $\mathbb{S}_C$ to the currents $\widehat{S}^1$, $\widehat{S}^2$, $\widehat{S}^3$, and $\mathcal{T}$, we will obtain the currents $\overline{S}^1$, $\overline{S}^2$, $\overline{S}^3$, and $\overline{\mathcal{T}}$ which will satisfy

$$|\overline{S}^1| + |\overline{S}^2| + |\overline{S}^3| + |\overline{\mathcal{T}}| \leq |\widehat{S}^1| + |\widehat{S}^2| + |\widehat{S}^3| + |\mathcal{T}|. \tag{4.1}$$

On the other hand, the symmetrization operators have the advantage of decreasing the mass of the currents N^A, N^B, and N^C in (3.41) (Lemma 4.23). In particular, after a suitable combination of applications of $\mathbb{S}_A$, $\mathbb{S}_B$, and $\mathbb{S}_C$, they vanish. We then arrive to the currents $\overline{S}^1$, $\overline{S}^2$, $\overline{S}^3$, and $\overline{\mathcal{T}}$ whose corresponding N^A, N^B, and N^C, are null, and this will be a key ingredient in order to prove that

$$3m_{l_j} \leq |\overline{S}^1| + |\overline{S}^2| + |\overline{S}^3| + |\overline{\mathcal{T}}|, \tag{4.2}$$

(this last inequality will be addressed in Section 5). The last two inequalities together prove Theorem 3.7.

In order to introduce the symmetrization operators we first start by setting some notation. Let us denote by h_A, h_B, and h_C the heights of the triangle T passing through $A = \alpha$, $B = \beta$, and $C = \gamma$ respectively. We will denote the lines (axes) obtained prolonging them by $\widehat{h}_A$, $\widehat{h}_B$, and $\widehat{h}_C$, respectively. We will now construct an operator $\mathbb{S}_B$ which symmetrizes the currents $\overline{S}^1$, $\overline{S}^2$, $\overline{S}^3$, and $\overline{\mathcal{T}}$ with respect to the axis $\widehat{h}_B$ (and similarly there will be operators relative to C and A).

Suppose for simplicity that the coordinates of $\alpha \in \mathbb{R}^2$ and $\gamma \in \mathbb{R}^2$ have the same ordinate (i.e. we choose a coordinate system in $\mathbb{R}^2$ such that $\alpha_2 = \gamma_2$). Moreover the coordinates of $\mathbb{R}^3$ are denoted by (x, y_1, y_2). Let P_B be the foot of the height h_B, namely the intersection between $\widehat{h}_B$ and the segment $\overline{AC}$. Let l_B^- be the halfline starting from P_B and obtained by prolonging the height h_B below the segment $\overline{AC}$. Let $l_B^+ = \widehat{h}_B \setminus l_B^-$. Let $\mathcal{R}^1 := [\![(0, l_j) \times (\alpha_1, \gamma_1) \times \{\alpha_2\}]\!]$ be the current of integration over the rectangle in $\mathbb{R}^3$ with vertices $(0, \alpha)$, $(0, \gamma)$, (l_j, γ), (l_j, α). Consider the current $\mathcal{B}^1 \in \mathcal{D}_3(\mathbb{R}^3)$ obtained as integration over the set

$$B^1 := \mathcal{R}^1 \times l_B^-, \tag{4.3}$$

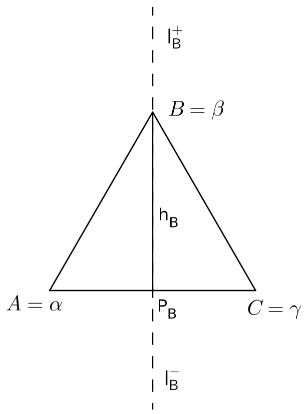

Figure 6: The triangle T with the notation introduced in Section 4.

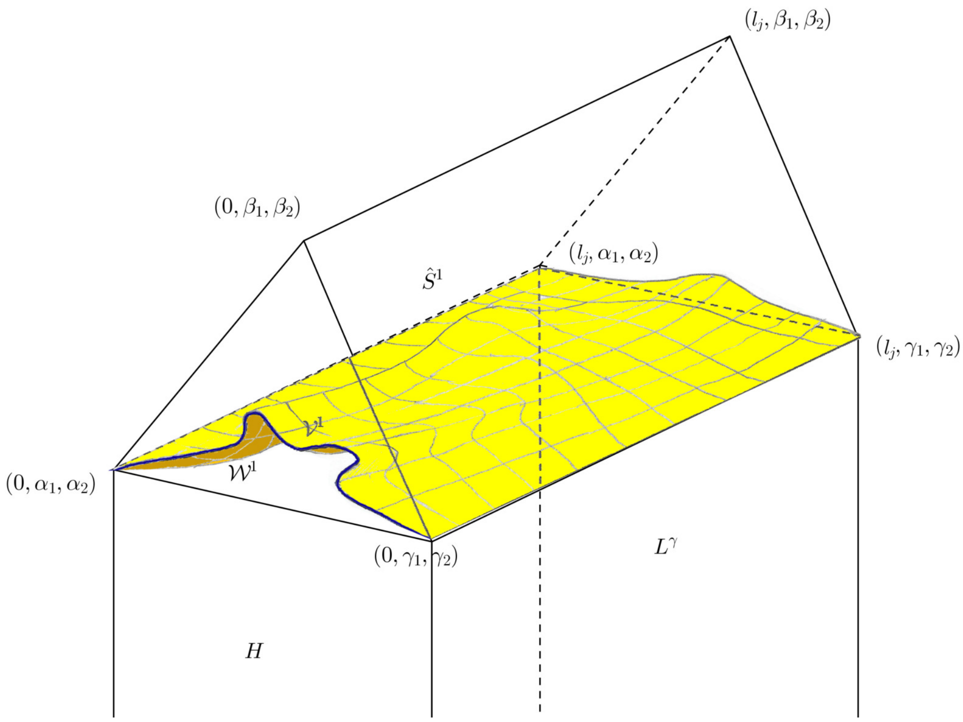

Figure 7: In this picture is depicted in yellow the surface $\widehat{S}^1$. The curve in blue is instead $\mathcal{V}^1$, whereas the region contained between $\mathcal{V}^1$ and the segment with vertices $(0,\alpha_1,\alpha_2)$ and $(0,\gamma_1,\gamma_2)$ is $\mathcal{W}^1$. The rectangle with vertices $(0,\alpha_1,\alpha_2)$, $(0,\gamma_1,\gamma_2)$, (l_j,γ_1,γ_2), and (l_j,α_1,α_2) is $\mathcal{R}^1$, while the parallelepiped (with infinite height) below it is B^1. There are labeled the two faces of it, H and L^γ. The face opposite to H is $\overline{H}$, and the one opposite to L^γ is L^α. For simplicity we have depicted the simpler case in which the currents Y^C and Y^A are null (as a consequence N^A and N^B are null).

i.e. $\mathcal{B}^1 := [\![B^1]\!]$. It is seen that

$$\partial\mathcal{B}^1 = L^\alpha - L^\gamma - H + \overline{H} + \mathcal{R}^1, \tag{4.4}$$

where

$$\begin{aligned} L^\alpha &= [\![(0,l_j)\times\{\alpha_1\}\times(-\infty,\alpha_2)]\!],\\ L^\gamma &= [\![(0,l_j)\times\{\gamma_1\}\times(-\infty,\alpha_2)]\!],\\ H &= [\![\{0\}\times(\gamma_1,\alpha_1)\times(-\infty,\alpha_2)]\!],\\ \overline{H} &= [\![\{l_j\}\times(\gamma_1,\alpha_1)\times(-\infty,\alpha_2)]\!]. \end{aligned} \tag{4.5}$$

Moreover (see (3.11c) and (3.35)), $\mathcal{V}^1 + [\![\{0\}\times(\gamma_1,\alpha_1)\times\{\alpha_2\}]\!]$ is a closed current in $\mathcal{D}_1(\{0\}\times\mathbb{R}^2)$ (by convention $\mathcal{V}^1$ has the orientation in such a way it connects α to γ), so that there is a current $\mathcal{W}_1 \in \mathcal{D}_2(\{0\}\times\mathbb{R}^2)$ with

$$\partial\mathcal{W}_1 = -\mathcal{V}^1 - [\![\{0\}\times(\gamma_1,\alpha_1)\times\{\alpha_2\}]\!].$$

By Proposition 3.3 and Proposition 3.4 the boundary of the current $\widehat{S}^1$ is

$$\begin{aligned} \partial\widehat{S}^1 = &- (Id\times\alpha)_\sharp[\![[0,l_j]]\!] - \sum_i(\overline{\alpha}_i)_\sharp[\![[0,t_i^A]]\!] + \mathcal{V}^1\\ &+ (Id\times\gamma)_\sharp[\![[0,l_j]]\!] + \sum_i(\overline{\gamma}_i)_\sharp[\![[0,t_i^C]]\!] + \overline{\mathcal{V}}^1 + C^A, \end{aligned} \tag{4.6}$$

Since (see (3.25) and (3.37)) $\overline{\mathcal{V}}^1 + C^A$ is supported on $\{l_j\}\times\mathbb{R}^2$ we have

$$\begin{aligned} \partial\widehat{S}^1 \llcorner ((-\infty,l_j)\times\mathbb{R}^2) = &- (Id\times\alpha)_\sharp[\![[0,l_j]]\!] - \sum_i(\overline{\alpha}_i)_\sharp[\![[0,t_i^A]]\!] + \mathcal{V}^1\\ &+ (Id\times\gamma)_\sharp[\![[0,l_j]]\!] + \sum_i(\overline{\gamma}_i)_\sharp[\![[0,t_i^C]]\!]. \end{aligned} \tag{4.7}$$

Recall that the arcs $\overline{\alpha}_i$ have image in $\{s_i\}\times T$, and are closed. We infer the existence (see Theorem 2.2) of integral currents $Y_i^A \in \mathcal{D}_2(\{s_i\}\times\overline{T})$ such that

$$\partial Y_i^A = (\overline{\alpha}_i)_\sharp[\![[0,t_i^A]]\!], \quad i\in\mathbb{N}.$$

There exist sets with finite perimeter $(A_i)_h^+$ and $(A_i)_h^-$ in $\{s_i\}\times\mathbb{R}^2$ such that

$$Y_i^A = \sum_h [\![(A_i)_h^+]\!] - \sum_h [\![(A_i)_h^-]\!]. \tag{4.8}$$

Assume also that this decomposition is made of undecomposable components, as in Theorem 2.1. Accordingly, we set

$$(Y_i^A)^+ := \sum_h [\![(A_i)_h^+]\!], \qquad (Y_i^A)^- = \sum_h [\![(A_i)_h^-]\!]. \tag{4.9}$$

Similarly

$$Y_i^B = (Y_i^B)^+ - (Y_i^B)^- = \sum_h [\![(B_i)_h^+]\!] - \sum_h [\![(B_i)_h^-]\!], \tag{4.10}$$

$$Y_i^C = (Y_i^C)^+ - (Y_i^C)^- = \sum_h [\![(C_i)_h^+]\!] - \sum_h [\![(C_i)_h^-]\!]. \tag{4.11}$$

Notice that since these components are undecomposable we have, essentially, $(A_i)_h^+\cap(A_i)_k^- = \varnothing$ for all h,k, and similarly for B and C (by essentially we mean that the intersection has null $\mathcal{H}^2$-measure). Denote

$$Y^A := \sum_i Y_i^A, \quad Y^B := \sum_i Y_i^B, \quad Y^C := \sum_i Y_i^C.$$

We define $G_1 \in \mathcal{D}_2((-\infty, l_j) \times \mathbb{R}^2)$ as

$$G_1 := \widehat{S}^1 + Y^A - Y^C + L^\alpha - L^\gamma + \mathcal{W}_1 - H. \tag{4.12}$$

By (4.5) and (4.7) we observe that G_1 is closed (notice that in the last formula we consider G_1 as a current in $(-\infty, l_j) \times \mathbb{R}^2$ instead of $\mathbb{R}^3$, so that we do not need to add $\overline{H}$ to make it boundaryless.). Then there exists an integral current $\mathcal{G}_1 \in \mathcal{D}_3((-\infty, l_j) \times \mathbb{R}^2)$ with $\partial\mathcal{G}_1 = G_1$ (see, again, Theorem 2.2). The current $\mathcal{G}_1$ turns out to be a sort of "subgraph" of the surface $\widehat{S}^1$ (see figure 7). The idea now is to symmetrize G_1. To this aim we symmetrize $\mathcal{G}_1$ by Steiner symmetrization and we will define the symmetrized of G_1 as the boundary of the obtained set. More precisely, let us explain this procedure in details. From Theorem 2.1 there are measurable sets with locally finite perimeter $U_h^1 \subset (-\infty, l_j) \times \mathbb{R}^2$ such that

$$\mathcal{G}_1 = \sum_h \theta_h [\![U_h^1]\!], \quad \theta_h \in \{-1, 1\}, \tag{4.13}$$

and for every bounded open set $A \subset (-\infty, l_j) \times \mathbb{R}^2$ it holds

$$|G_1|_A = \sum_h |\partial U_h^1|_A.$$

Up to translating the sets U_i^1 in the y_1 direction $((x, y_1, y_2) \mapsto (x, y_1 + t, y_2))$ we can assume they are all mutually disjoint and with multiplicity $+1$ or -1 . Then it is well defined the set $\mathbb{S}_B(U^1)$ obtained by Steiner symmetrization of the set $U^1 := \cup_h U_h^1$ with respect to the plane containing $[0, l_j] \times h_B$ (see [15, Section 14.1] for the definition of Steiner symmetrization and its properties). The new set $\mathbb{S}_B(U^1)$ defines a current

$$\widehat{\mathbb{S}}_B(\mathcal{G}_1) := [\![\mathbb{S}_B(U^1)]\!], \tag{4.14}$$

whose boundary satisfies

$$|\partial\widehat{\mathbb{S}}_B(\mathcal{G}_1)|_A \leq \sum_j |\partial U_j^1|_A = |G_1|_A, \tag{4.15}$$

for every bounded open set $A \subset (-\infty, l_j) \times \mathbb{R}^2$ (see [15, Section 14.1]). It is important here to observe that it might happen that the set $\mathbb{S}_B(U^1)$ is not contained in the solid

$$\overline{Q} := \overline{B^1} \cup \overline{P},$$

(where B^1 is defined in (4.3) and with P being the prism $P = [0, l_j) \times T$), as it is for the original current $\mathcal{G}_1$. This is due to the fact that before symmetrizing it, the current $\mathcal{G}_1$ might have high multiplicity in $\overline{Q}$, while the symmetrization enforces it to have multiplicity 1. In the case $\mathbb{S}_B(U^1)$ exceeds $\overline{Q}$ we need to restrict it to $\overline{Q}$, and hence we set

$$\mathbb{S}_B(\mathcal{G}_1) := [\![\mathbb{S}_B(U^1) \cap \overline{Q}]\!].$$

It is easy to see that, since $\overline{Q}$ is a convex set, inequality (4.15) still holds true, namely

$$|\partial\mathbb{S}_B(\mathcal{G}_1)|_A \leq |G_1|_A, \tag{4.16}$$

for every bounded open set $A \subset (-\infty, l_j) \times \mathbb{R}^2$.

Definition 4.1. The symmetrization with respect to the h_B axis of G_1 is

$$\mathbb{S}_B(G_1) := \partial\mathbb{S}_B(\mathcal{G}_1). \tag{4.17}$$

Remark 4.2. Let us emphasize that the symmetrization of the current $\mathcal{G}_1$, obtained as integration over the symmetrized set $\mathbb{S}_B(U^1)$ is well defined and does not depend on the specific decomposition in (4.13). Indeed it is not difficult to see that $\mathbb{S}_B(U^1)$ can be obtained also without the decomposition theorem for currents, in the following way. Consider the plane $\mathbb{R} \times \widehat{h}_B$, containing the height h_B and the edge $[0, l_j] \times \{\beta\}$ of the prism P. Let (s,t) be two orthogonal coordinates on this plane, and let $r_{s,t}$ be the line passing through the point (s,t) and orthogonal to $\mathbb{R} \times \widehat{h}_B$. By slicing it is possible to consider the 1-current $\langle \mathcal{G}_1, (s,t) \rangle$, which represents the restriction of $\mathcal{G}_1$ to the line $r_{s,t}$. This is uniquely determined for a.e. $(s,t) \in \mathbb{R}^2$. Hence we can consider the mass $m_{s,t} := |\langle \mathcal{G}_1, (s,t) \rangle|$ and define the set $\mathbb{S}_B(U^1)$ symmetric with respect to $\mathbb{R} \times \widehat{h}_B$ in such a way that, if $r_{s,t}$ is endowed with a coordinate x such that $x = 0$ at $r_{s,t} \cap (\mathbb{R} \times \widehat{h}_B)$, then

$$\mathbb{S}_B(U^1) \cap r_{s,t} := \Big(-\frac{m_{s,t}}{2}, \frac{m_{s,t}}{2}\Big).$$

Remark 4.3. Let us also observe that the presence of the current $\mathcal{B}^1$ in the definition of $\mathcal{G}_1$ (see (4.3) and (4.4)) is not crucial but it is convenient for exposition. Nevertheless the symmetrization of $\mathcal{G}_1$ is trivial below the plane containing $\mathcal{R}^1$, since it transforms $\mathcal{B}^1$ into itself. The fact that the symmetrization of $\mathcal{G}_1$ might have support exceeding the solid $\overline{Q}$ can only take place in the upper halfspace $\mathbb{R}^2 \times l_B^+$.

In order to define the symmetrization of the current $\widehat{S}^1$ (introduced in Definition 4.13 below), we first need to symmetrize the currents Y^A and Y^C (whose symmetrizations are given in Definitions 4.7 and 4.10). To this aim, let us analyze what happens to the vertical parts of the current G_1 after the symmetrization. We consider two cases.

Case $s_i > 0$**.** Let $s_i \in (0, l_j)$ be as in Proposition 3.4, and consider the corresponding decomposition in (4.8). The currents Y_i^A and Y_i^C satisfy

$$\partial Y_i^A = \sum_h [\![\partial A_h^+]\!] - \sum_h [\![\partial A_h^-]\!], \qquad \partial Y_i^C = \sum_h [\![\partial C_h^+]\!] - \sum_h [\![\partial C_h^-]\!]. \tag{4.18}$$

Let $\widehat{S}^1 \llcorner s_i = \widehat{S}^1 \llcorner (\{s_i\} \times \mathbb{R}^2)$ be the part of the current $\widehat{S}^1$ with support in the plane $\{s_i\} \times \mathbb{R}^2$. Notice that $G_1 \llcorner (\{s_i\} \times \mathbb{R}^2) := Y_i^A - Y_i^C + \widehat{S}^1 \llcorner s_i$; this is the part of G_1 in $\{s_i\} \times \mathbb{R}^2$. More precisely, if, as in (4.13), $\mathcal{G}_1 = \sum_h \theta_h [\![U_h^1]\!]$, then

$$G_1 \llcorner (\{s_i\} \times \mathbb{R}^2) := \sum_h \sigma_h \theta_h [\![\partial U_h^1 \cap (\{s_i\} \times \mathbb{R}^2)]\!], \tag{4.19}$$

where ∂U_h^1 is the reduced boundary of U_h^1 and σ_h is 1 or -1 according to whether ∂U_h^1 has external normal vector equal to $(1,0,0)$ or $(-1,0,0)$, respectively (the orientation of ∂U_h^1 is given by the volume form inherited by its normal unit vector). Let I^+ and I^- be the sets of indices for which $\sigma_h = \pm 1$ respectively. Equivalently (4.19) writes as

$$G_1 \llcorner (\{s_i\} \times \mathbb{R}^2) := \sum_{h \in I^+} \theta_h [\![\partial U_h^1 \cap (\{s_i\} \times \mathbb{R}^2)]\!] - \sum_{h \in I^-} \theta_h [\![\partial U_h^1 \cap (\{s_i\} \times \mathbb{R}^2)]\!]. \tag{4.20}$$

Accordingly set

$$G_1 \llcorner (\{s_i\} \times \mathbb{R}^2)^{\pm} := \sum_{h \in I^{\pm}} \theta_h [\![\partial U_h^1 \cap (\{s_i\} \times \mathbb{R}^2)]\!], \tag{4.21}$$

so that

$$G_1 \llcorner (\{s_i\} \times \mathbb{R}^2) = G_1 \llcorner (\{s_i\} \times \mathbb{R}^2)^+ - G_1 \llcorner (\{s_i\} \times \mathbb{R}^2)^-.$$

Now we want to study the boundary of the current $\mathbb{S}_B(\mathcal{G}_1)$ which is concentrated on the plane $\{s_i\} \times \mathbb{R}^2$, i.e. the restriction of $\mathbb{S}_B(G_1)$ to such plane. Let $(\dot{U}_h^1)_i := (U_h^1 \setminus \partial U_h^1) \cap (\{s_i\} \times \mathbb{R}^2)$.

Definition 4.4. Let $\widehat{E}_i^0$ be the Steiner symmetrization with respect to $\widehat{h}_B$ of the set $\cup_h (\dot{U}_h^1)_i$ (seen as a subset of $\{s_i\} \times \mathbb{R}^2$). Let $\widehat{E}_i^+$ be the Steiner symmetrization of the set

$$(\cup_h (\dot{U}_h^1)_i) \bigcup (\cup_{h \in I^+} (\partial U_h^1) \cap (\{s_i\} \times \mathbb{R}^2)),$$

(again considered union of disjoint sets, up to translation) and let $\widehat{E}_i^-$ be the Steiner symmetrization of the set

$$(\cup_h(\dot{U}_h^1)_i)\bigcup(\cup_{h\in I^-}(\partial U_h^1)\cap(\{s_i\}\times\mathbb{R}^2)).$$

Again it might happen that $\widehat{E}_i^0$, $\widehat{E}_i^+$, $\widehat{E}_i^-$ intersect $\mathbb{R}^3\setminus\overline{Q}$, so that we set

$$E_i^0:=\widehat{E}_i^0\cap\overline{Q},\qquad E_i^+:=\widehat{E}_i^+\cap\overline{Q},\qquad E_i^-:=\widehat{E}_i^-\cap\overline{Q}. \tag{4.22}$$

Observe that $E_i^0=\widehat{E}_i^0\cap\overline{Q}=\widehat{E}_i^0\cap T_i$ where $T_i=\{s_i\}\times T$. This holds true since the original currents $G_1\llcorner(\{s_i\}\times\mathbb{R}^2)$ have support in the triangle $\{s_i\}\times T$.

Lemma 4.5. *It holds*

$$\mathbb{S}_B(G_1)\llcorner(\{s_i\}\times\mathbb{R}^2)=[\![E_i^+\setminus E_i^-]\!]-[\![E_i^-\setminus E_i^+]\!]. \tag{4.23}$$

In particular

$$|\mathbb{S}_B(G_1)\llcorner(\{s_i\}\times\mathbb{R}^2)|=|E_i^+\Delta E_i^-|.$$

Proof. We can always split

$$\mathcal{G}_1=(\mathcal{G}_i^1)^++(\mathcal{G}_i^1)^-,$$

where $(\mathcal{G}_i^1)^+:=\sum_h\theta_h[\![U_h^1\cap(\{x<s_i\})]\!]$ and $(\mathcal{G}_i^1)^-:=\sum_h\theta_j[\![U_h^1\cap(\{x>s_i\})]\!]$. It is then easy to see that their boundaries (seen as sets, with a little abuse of notation) are

$$\partial(\mathcal{G}_i^1)^+=(\cup_h\dot{U}_h^1)_i\bigcup(\cup_{h\in I^+}\partial U_h^1\cap(\{s_i\}\times\mathbb{R}^2)),$$

and similarly

$$\partial(\mathcal{G}_i^1)^-=(\cup_h\dot{U}_h^1)_i\bigcup(\cup_{h\in I^-}\partial U_h^1\cap(\{s_i\}\times\mathbb{R}^2)).$$

In particular the symmetrizations of $(\mathcal{G}_i^1)^+$ and $(\mathcal{G}_i^1)^-$, namely $\mathbb{S}_B(\mathcal{G}_i^1)^+$ and $\mathbb{S}_B(\mathcal{G}_i^1)^-$, have boundaries on $\{s_i\}\times\mathbb{R}^2$ given by $\widehat{E}_i^+$ and $\widehat{E}_i^-$ respectively. To study the symmetrization of $\mathcal{G}_i^1$ we consider the sum of $\mathbb{S}_B(\mathcal{G}_i^1)^+$ and $\mathbb{S}_B(\mathcal{G}_i^1)^-$. Since their orientations are opposite, the thesis follows just by considering the restrictions to $\overline{Q}$. $\square$

The positive and negative part of the current $\mathbb{S}_B(G_1)\llcorner(\{s_i\}\times\mathbb{R}^2)$ are

$$\begin{aligned}\mathbb{S}_B(G_1)^+\llcorner(\{s_i\}\times\mathbb{R}^2)&=[\![E_i^+\setminus E_i^-]\!],\\ \mathbb{S}_B(G_1)^-\llcorner(\{s_i\}\times\mathbb{R}^2)&=[\![E_i^-\setminus E_i^+]\!]\end{aligned} \tag{4.24}$$

shortly denoted by

$$\mathbb{S}_B(G_i^1)^+=\mathbb{S}_B(G_1)^+\llcorner(\{s_i\}\times\mathbb{R}^2),\qquad \mathbb{S}_B(G_i^1)^-=\mathbb{S}_B(G_1)^-\llcorner(\{s_i\}\times\mathbb{R}^2), \tag{4.25}$$

and

$$\mathbb{S}_B(G_1)\llcorner(\{s_i\}\times\mathbb{R}^2)=\mathbb{S}_B(G_i^1)=\mathbb{S}_B(G_i^1)^+-\mathbb{S}_B(G_i^1)^-. \tag{4.26}$$

At this stage it is convenient to define $Y_i=Y_i^A-Y_i^C=(Y_i^A)^+-(Y_i^C)^+-(Y_i^A)^-+(Y_i^C)^-$ (the second equality due to (4.9)); it turns out that

$$Y_i=\sum_h[\![(D_i)_h^+]\!]-\sum_h[\![(D_i)_h^-]\!],$$

for suitable sets $(D_i)_h^+$ and $(D_i)_h^-$ in $\{s_i\}\times\mathbb{R}^2$ (notice that by hypothesis of undecomposibility it turns out that $\cup_h(D_i)_h^+$ and $\cup_h(D_i)_h^-$ are essentially disjoint). Hence we decompose Y^i in a positive and negative part, namely $Y_i=Y_i^+-Y_i^-$, where

$$Y_i^\pm:=\sum_h[\![(D_i)_h^\pm]\!]. \tag{4.27}$$

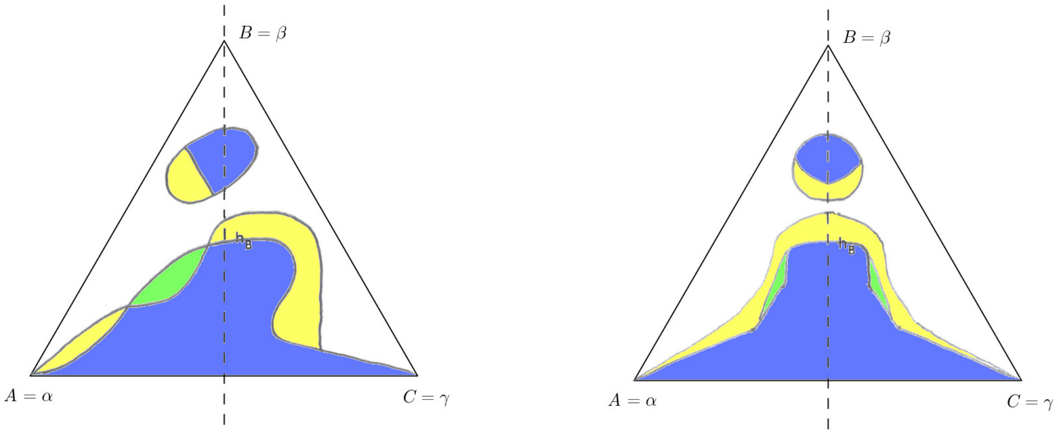

Figure 8: In the picture on the left we have drawn the current $G_1 \llcorner (\{s_i\} \times \mathbb{R}^2)$. The set colored in blue is $\cup_h (\dot{U}^1_h)_i$, the one in yellow is $(\cup_j (\dot{U}^1_h)_i) \bigcup (\cup_{h \in I^+} (\partial U^1_h) \cap (\{s_i\} \times \mathbb{R}^2))$ whereas the one in green is $(\cup_j (\dot{U}^1_h)_i) \bigcup (\cup_{h \in I^-} (\partial U^1_h) \cap (\{s_i\} \times \mathbb{R}^2))$. The picture on the right is the symmetrized set. In particular, the blue zone is the set E^0_i, the one in blue green is the set E^-_i, while the one in blue and yellow is E^+_i. In this case we see that E^+_i contains E^-_i (we have to consider that the yellow area overlaps the green one, i.e. the green area is part of the yellow one in this example).

It turns out

$$|Y_i| = |Y^A_i - Y^C_i| = |Y^+_i| + |Y^-_i|. \tag{4.28}$$

Consider now the currents $\mathcal{F}^+_i, \mathcal{F}^-_i \in \mathcal{D}_2(\{s_i\} \times \mathbb{R}^2)$ given by

$$\mathcal{F}^{\pm}_i := G_1 \llcorner (\{s_i\} \times \mathbb{R}^2)^{\pm} - Y^{\pm}_i, \tag{4.29}$$

where we employed the notation (4.21). Note that

$$\mathcal{F}^+_i - \mathcal{F}^-_i = G_1 \llcorner (\{s_i\} \times \mathbb{R}^2) - Y_i.$$

Decomposing $\mathcal{F}^{\pm}_i$ in undecomposable components, we find two families of sets $(Z^+_i)_h$ and $(Z^-_i)_h$ such that

$$\mathcal{F}^+_i = \sum_h \theta_h [\![(Z^+_i)_h]\!], \qquad \mathcal{F}^-_i = \sum_h \theta_h [\![(Z^-_i)_h]\!] \quad \theta_h \in \{-1, 1\}. \tag{4.30}$$

We are then led to define:

Definition 4.6. The set $\widehat{F}^+_i$ is the Steiner symmetrization with respect to $\widehat{h}_B$ of the set

$$(\cup_h \dot{U}^1_h)_i \bigcup \cup_h (Z^+_i)_h$$

(again considered as union of disjoint sets in $\{s_i\} \times \mathbb{R}^2$, up to translation) and $\widehat{F}^-_i$ is the Steiner symmetrization of the set

$$(\cup_h \dot{U}^1_h)_i \bigcup \cup_h (Z^-_i)_h.$$

We consider their restrictions to $\overline{Q}$, and, since also in this case $\mathcal{F}_i$ have supports in T_i, such restrictions coincide with

$$F^+_i := \widehat{F}^+_i \cap T_i, \qquad F^-_i := \widehat{F}^-_i \cap T_i.$$

The symmetrizations of the currents Y_i^A and Y_i^C are then defined as follows:

Definition 4.7. Let Π_α be the halfplane in $\mathbb{R}^2$ bounded by the axis $\widehat{h}_B$ and containing α. Let Π_γ be the complementary halfplane. We define

$$\mathbb{S}_B(Y_i^A) := [\![E_i^+ \cap \Pi_\alpha]\!] - [\![E_i^- \cap \Pi_\alpha]\!] - [\![F_i^+ \cap \Pi_\alpha]\!] + [\![F_i^- \cap \Pi_\alpha]\!], \tag{4.31}$$

$$-\mathbb{S}_B(Y_i^C) := [\![E_i^+ \cap \Pi_\gamma]\!] - [\![E_i^- \cap \Pi_\gamma]\!] - [\![F_i^+ \cap \Pi_\gamma]\!] + [\![F_i^- \cap \Pi_\gamma]\!]. \tag{4.32}$$

It is also convenient to define $\mathbb{S}_B(Y_i) = \mathbb{S}_B(Y_i^A) - \mathbb{S}_B(Y_i^C) = \mathbb{S}_B(Y_i^+) - \mathbb{S}_B(Y_i^-)$ with

$$\mathbb{S}_B(Y_i^\pm) := [\![E_i^\pm]\!] - [\![F_i^\pm]\!].$$

First, by definition, it turns out that the currents $\mathbb{S}_B(Y_i^A)$ and $\mathbb{S}_B(Y_i^C)$ are supported on disjoint sets. Therefore

$$|\mathbb{S}_B(Y_i^A) - \mathbb{S}_B(Y_i^C)| = |\mathbb{S}_B(Y_i^A)| + |\mathbb{S}_B(Y_i^C)|.$$

Moreover, we have the following

Lemma 4.8. *The currents* $\mathbb{S}_B(Y_i^A)$ *and* $\mathbb{S}_B(Y_i^C)$ *satisfy*

$$|\mathbb{S}_B(Y_i^A)| + |\mathbb{S}_B(Y_i^C)| = |\mathbb{S}_B(Y_i^A) - \mathbb{S}_B(Y_i^C)| \le |Y_i^A - Y_i^C| \le |Y_i^A| + |Y_i^C|. \tag{4.33}$$

Proof. By writing

$$[\![E_i^\pm]\!] - [\![F_i^\pm]\!] = [\![E_i^\pm \setminus F_i^\pm]\!] - [\![F_i^\pm \setminus E_i^\pm]\!],$$

we infer

$$\mathbb{S}_B(Y_i^A) - \mathbb{S}_B(Y_i^C) = [\![E_i^+ \setminus F_i^+]\!] - [\![F_i^+ \setminus E_i^+]\!] - [\![E_i^- \setminus F_i^-]\!] + [\![F_i^- \setminus E_i^-]\!].$$

Now, since $|Y_i^+| + |Y_i^-| = |Y_i^A - Y_i^C|$ (by hypothesis on the decomposition), the thesis will be proved if we show that

$$|[\![E_i^+ \setminus F_i^+]\!] - [\![F_i^+ \setminus E_i^+]\!]| \le |Y_i^+|, \tag{4.34}$$

$$|[\![E_i^- \setminus F_i^-]\!] - [\![F_i^- \setminus E_i^-]\!]| \le |Y_i^-|. \tag{4.35}$$

To see the first inequality (the second is similar) we argue by slicing, considering sections of the currents $G_i^+ = G_1 \llcorner (\{s_i\} \times \mathbb{R}^2)^+$ and $(\mathcal{F}_i)^+$ at $\{y_2 = t\}$. First observe that the mass

$$|[\![E_i^+ \setminus F_i^+]\!] - [\![F_i^+ \setminus E_i^+]\!]| = |E_i^+ \Delta F_i^+| = \int_{-\infty}^{+\infty} |(E_i^+ \Delta F_i^+) \cap \{y_2 = t\}| dt,$$

and then that $|(E_i^+ \Delta F_i^+) \cap \{y_2 = t\}| \le |(\widehat{E}_i^+ \Delta \widehat{F}_i^+) \cap \{y_2 = t\}|$. Moreover, at fixed t it follows, by Definitions 4.4 and 4.6, that $|(\widehat{E}_i^+ \Delta \widehat{F}_i^+) \cap \{y_2 = t\}| = ||\langle G_i^+, t\rangle| - |\langle \mathcal{F}_i^+, t\rangle||$ (here we use that the decompositions in (4.13) and (4.30) are made of undecomposable components; see also Remark 4.2), and hence $|(E_i^+ \Delta F_i^+) \cap \{y_2 = t\}| \le ||\langle G_i^+, t\rangle| - |\langle \mathcal{F}_i^+, t\rangle|| \le |\langle G_i^+ - \mathcal{F}_i^+, t\rangle| = |\langle Y_i^+, t\rangle|$. Therefore we conclude

$$|[\![E_i^+ \setminus F_i^+]\!] - [\![F_i^+ \setminus E_i^+]\!]| \le \int_{-\infty}^{+\infty} |\langle Y_i^+, t\rangle| dt \le |Y_i^+|. \tag{4.36}$$

□

Case $s_i = 0$. In this case we define the sets $\widehat{E}_0^0$, $\widehat{E}_0^+$, $\widehat{E}_0^-$, E_0^0, E_0^+, E_0^- as in Definition 4.4. First let us observe that the component $G_1 \llcorner (\{0\} \times \mathbb{R}^2)^+$ is null together with the sets $(\dot{U}_j^1)_0$ (see (4.21)). As a consequence the sets $\widehat{E}_0^0$, $\widehat{E}_0^+$, E_0^0, and E_0^+, are all empty. In this case we need a different definition for the symmetrization of $Y_0^A = Y^A \llcorner (\{0\} \times \mathbb{R}^2)$ and $Y_0^C = Y^C \llcorner (\{0\} \times \mathbb{R}^2)$. As before, we define

$$Y_0 = Y_0^A - Y_0^C.$$

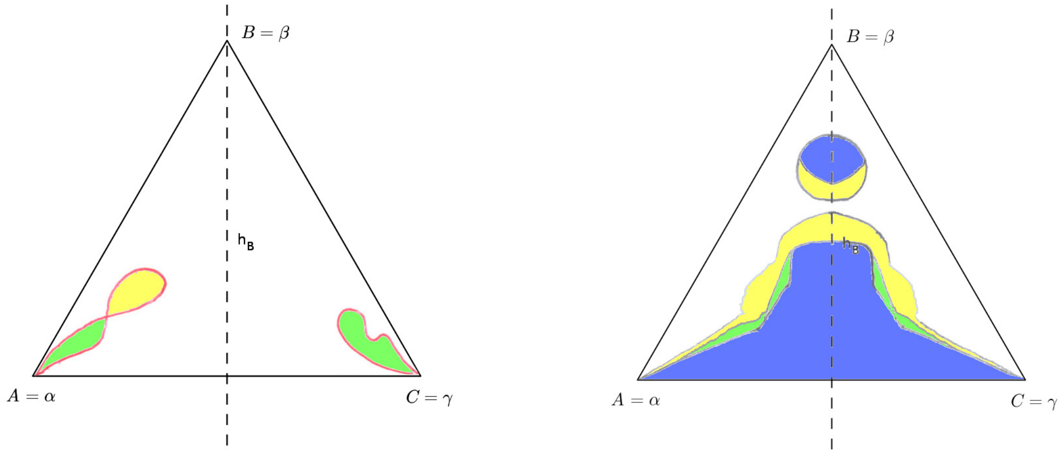

Figure 9: The figure on the left represents the two currents Y_i^A and Y_i^C. On the right, referring also to Figure 8, the two sets F_i^+ and F_i^- are depicted; the first one is the union of all the colored area (yellow, that overlaps the green one, and blue) whereas F_i^- is the union of the blue and green areas. These areas are a bit bigger than the corresponding in Figure 8, their difference will give rise to the symmetrized currents $\mathbb{S}_B(Y^A)$ and $\mathbb{S}_B(Y^C)$.

Then, in place of (4.29), we define

$$-\mathcal{F}_0^- := -G_1 \llcorner (\{0\} \times \mathbb{R}^2)^- - Y_0. \tag{4.37}$$

Decomposing $\mathcal{F}_0^-$ in undecomposable components we find

$$\mathcal{F}_0^- = \sum_h \theta_h [\![(Z_0)_h]\!] \quad \theta_h \in \{-1, 1\},$$

and therefore we arrive at:

Definition 4.9. The set $\widehat{F}_0^-$ is the Steiner symmetrization with respect to $\widehat{h}_B$ of the set $\cup_h (Z_0)_h$ (again considered as union of disjoint sets in $\{0\} \times \mathbb{R}^2$). We consider its restriction to $\overline{Q}$,

$$F_0^- := \widehat{F}_0^- \cap \overline{Q}.$$

We can now introduce the symmetrizations of the currents Y_0^A and Y_0^C:

Definition 4.10. We define

$$\mathbb{S}_B(Y_0^A) := -[\![E_0^- \cap \Pi_\alpha]\!] + [\![F_0^- \cap \Pi_\alpha]\!], \tag{4.38}$$

$$-\mathbb{S}_B(Y_0^C) := -[\![E_0^- \cap \Pi_\gamma]\!] + [\![F_0^- \cap \Pi_\gamma]\!]. \tag{4.39}$$

We also set $\mathbb{S}_B(Y_0) = \mathbb{S}_B(Y_0^A) - \mathbb{S}_B(Y_0^C)$ so that

$$\mathbb{S}_B(Y_0) := -[\![E_0^-]\!] + [\![F_0^-]\!].$$

Lemma 4.11. *The currents $\mathbb{S}_B(Y_0^A)$ and $\mathbb{S}_B(Y_0^C)$ satisfy*

$$|\mathbb{S}_B(Y_0^A)| + |\mathbb{S}_B(Y_0^C)| = |\mathbb{S}_B(Y_0^A) - \mathbb{S}_B(Y_0^C)| \le |Y_0^A - Y_0^C| \le |Y_0^A| + |Y_0^C|. \tag{4.40}$$

Proof. As in Lemma 4.8 we infer

$$\mathbb{S}_B(Y_0^A) - \mathbb{S}_B(Y_0^C) = -[\![E_0^- \setminus F_0^-]\!] + [\![F_0^- \setminus E_0^-]\!].$$

Then we will prove that

$$|[\![E_0^- \setminus F_0^-]\!] - [\![F_0^- \setminus E_0^-]\!]| \le |Y_0|, \tag{4.41}$$

$Y_0 = Y_0^A - Y_0^C$. Also in this case we proceed by slicing considering sections of the currents

$$(G_0^1)^- = G_1 \llcorner (\{0\} \times \mathbb{R}^2)^- \quad \text{and} \quad \mathcal{F}_0^- \quad \text{at} \quad \{y_2 = t\}.$$

We then conclude as in the proof of Lemma 4.8 by observing that

$$|[\![E_0^- \setminus F_0^-]\!] - [\![F_0^- \setminus E_0^-]\!]| = |E_0^- \Delta F_0^-| = \int_{-\infty}^{+\infty} |(E_0^- \Delta F_0) \cap \{y_2 = t\}| dt,$$

and using the inequality $|(E_0^- \Delta F_0^-) \cap \{y_2 = t\}| \leq ||\langle (G_0^1)^-, t\rangle| - |\langle \mathcal{F}_0^-, t\rangle|| \leq |\langle (G_0^1)^- - \mathcal{F}_0^-, t\rangle| = |\langle Y_0, t\rangle|$. □

We now define the symmetrization of the current $\mathcal{W}_1$. We recall that this is the current in $\mathcal{D}_2(\{0\} \times \mathbb{R}^2)$ such that $\partial \mathcal{W}_1 = -\mathcal{V}^1 - [\![\{0\} \times (\alpha_1, \gamma_1) \times \{\alpha_2\}]\!]$. There exist sets $W_h \subset \{0\} \times \mathbb{R}^2$ with

$$\mathcal{W}_1 - H = \sum_h \theta_h [\![W_h]\!] \quad \theta_h \in \{-1, 1\}.$$

Definition 4.12. The Steiner symmetrization of $\cup_h W_h$ with respect to the axis $\widehat{h}_B$ is the set $\widehat{\mathbb{S}}_B(W_1)$, and its intersection with $\overline{Q}_0 := \overline{Q} \cap (\{0\} \times \mathbb{R}^2)$ is denoted by $\mathbb{S}_B(W_1)$. We define the current

$$\mathbb{S}_B(\mathcal{W}_1) := -[\![\mathbb{S}_B(W_1)]\!] + H,$$

where, by convention, the set $\mathbb{S}_B(W_1)$ is oriented by the unit vector $(1, 0, 0)$. We define

$$\mathbb{S}_B(\mathcal{V}^1) := -[\![\{0\} \times (\alpha_1, \gamma_1) \times \{\alpha_2\}]\!] - \partial \mathbb{S}_B(\mathcal{W}_1). \tag{4.42}$$

It turns out that

$$|\mathbb{S}_B(\mathcal{W}_1)| \leq |\mathcal{W}_1|. \tag{4.43}$$

Notice that such inequality will be an equality if the symmetrization of $\cup_h W_h$ is already enclosed in $\overline{Q}_0$. Moreover it is observed that $\mathbb{S}_B(\mathcal{V}_1)$ coincides with the restriction of $\partial \mathbb{S}_B(\mathcal{W}_1)$ to the halfplane $\{0\} \times \mathbb{R} \times l_B^+$. Hence by the property of the Steiner symmetrization it follows that

$$|\mathbb{S}_B(\mathcal{V}^1)| \leq |\mathcal{V}^1|, \tag{4.44}$$

(it is straightforward that, in addition, such inequality is strict if the symmetrization of $\cup_h W_h$ exceeds the set $\overline{Q}_0$, since the left-hand side becomes even smaller after the intersection). Finally, observe that $\mathbb{S}_B(\mathcal{V}^1)$ will have support in $T_0 = T \cap (\{0\} \times \mathbb{R}^2)$.

We are now in position to define the symmetrization of the current $\widehat{S}^1$.

Definition 4.13. The symmetrization of the current $\widehat{S}^1$ with respect to the axis $\widehat{h}_B$ is the current $\mathbb{S}_B(\widehat{S}^1) \in \mathcal{D}_2((-\infty, l_j) \times \mathbb{R}^2)$ defined as

$$\mathbb{S}_B(\widehat{S}^1) := \mathbb{S}_B(G_1) - \mathbb{S}_B(Y^A) + \mathbb{S}_B(Y^C) - L^\alpha + L^\gamma + H - \mathbb{S}_B(\mathcal{W}_1). \tag{4.45}$$

Since $\mathbb{S}_B(G_1)$ is closed it turns out that

$$\partial \mathbb{S}_B(\widehat{S}^1) = -\partial \mathbb{S}_B(Y^A) + \partial \mathbb{S}_B(Y^C) - (Id \times \alpha)_\sharp [\![0, l_j]\!] + (Id \times \gamma)_\sharp [\![0, l_j]\!] + \mathbb{S}_B(\mathcal{V}^1) \tag{4.46}$$

We now prove the crucial result:

Theorem 4.14. *It holds*

$$|\mathbb{S}_B(\widehat{S}^1)| \leq |\widehat{S}^1|.$$

Proof. We first decompose $\mathbb{S}_B(\widehat{S}^1)$ as

$$\mathbb{S}_B(\widehat{S}^1) = \mathbb{S}_B(\mathcal{L}^1) + \sum_i \mathbb{S}_B(\widehat{S}^1) \llcorner (\{s_i\} \times \mathbb{R}^2),$$

where $\{s_i\} \subset [0, l_j)$ is the countable set such that for all s_i the current $\mathbb{S}_B(\widehat{S}^1) \llcorner (\{s_i\} \times \mathbb{R}^2)$ is nonnegligible. The complementary current is $\mathbb{S}_B(\mathcal{L}^1)$. Roughly speaking, $\mathbb{S}_B(\mathcal{L}^1)$ is the lateral part of the current $\mathbb{S}_B(\widehat{S}^1)$. It is easy to see, by the definition of the Steiner symmetrization, that the symmetrization $\mathbb{S}_B(U)$ of a set U does not increase the mass of both the lateral part of the set, and its complementary part $(\partial U \cap \{s_i\} \times \mathbb{R}^2)$. Moreover intersecting $\mathbb{S}_B(U)$ with the solid $\overline{Q}$ gives rise to a still smaller lateral part. In particular we infer

$$|\mathbb{S}_B(\mathcal{L}^1)| \leq |\mathcal{L}^1|, \tag{4.47}$$

so that to conclude the proof we have to show that for all sections $\{s_i\} \times \mathbb{R}^2$ it turns out

$$|\mathbb{S}_B(\widehat{S}^1)| \llcorner (\{s_i\} \times \mathbb{R}^2) \leq |\widehat{S}^1| \llcorner (\{s_i\} \times \mathbb{R}^2). \tag{4.48}$$

We distinguish the cases $s_i = 0$ and $s_i \neq 0$. In the previous one we have

$$\mathbb{S}_B(\widehat{S}^1) \llcorner (\{0\} \times \mathbb{R}^2) = -\mathbb{S}_B(G_0^1)^- - \mathbb{S}_B(Y_0) - \mathbb{S}_B(\mathcal{W}^1) + H, \tag{4.49}$$

(indeed in this case $\mathbb{S}_B(G_0^1)^+ = 0$) whereas in the latter

$$\mathbb{S}_B(\widehat{S}^1) \llcorner (\{s_i\} \times \mathbb{R}^2) = \mathbb{S}_B(G_i^1)^+ - \mathbb{S}_B(G_i^1)^- - \mathbb{S}_B(Y_0^+) + \mathbb{S}_B(Y_0^-), \tag{4.50}$$

(recall notation (4.25)). Let us treat the first case. We establish (4.48) arguing by slicing, as in the proof of Lemma 4.8, namely restricting to every section of this currents at $\{0\} \times \mathbb{R} \times \{t\}$, $t \geq \alpha_2$ (since the currents involved are integration over sets, this argument can be reduced to Fubini Theorem). Recall that $\widehat{S}^1 \llcorner (\{0\} \times \mathbb{R}^2) = -(G_0^1)^- - Y_0 - \mathcal{W}_1 + H$, so that, by (4.43), for $t \geq \alpha_2$,

$$\begin{aligned} |\langle \widehat{S}^1 \llcorner (\{0\} \times \mathbb{R}^2), t\rangle| &\geq ||\langle (G_0^1)^- + Y_0), t\rangle| - |\langle \mathcal{W}_1, t\rangle|| \\ &= ||\widehat{F}_0^- \cap \{y_2 = t\}| - |\widehat{\mathbb{S}}_B(W_1) \cap \{y_2 = t\}||. \end{aligned} \tag{4.51}$$

Here we have used $\widehat{\mathbb{S}}_B(W_1)$, the set obtained by Steiner symmetrization of $\cup_h W_h$ without intersection with $\overline{Q}_0$ (see Definition 4.12). Notice that, by definition of Steiner symmetrization with respect to h_B, and taking into account that the edge of T_0 has length $\sqrt{3}$, it turns out that

$$\begin{aligned} &|F_0^- \cap \{y_2 = t\}| = \tau(|\widehat{F}_0^- \cap \{y_2 = t\}|), \\ &|\mathbb{S}_B(W_1) \cap \{y_2 = t\}| = \tau(|\widehat{\mathbb{S}}_B(W_1) \cap \{y_2 = t\}|), \end{aligned} \tag{4.52}$$

where $\tau(x) = \min\{|x|, l(t)\}$, with $l(t) = 3/2 - t$ be the width of the triangle T at height t. Since τ is Lipschitz continuous with constant 1, from (4.51) it follows that

$$|\langle \widehat{S}^1 \llcorner (\{0\} \times \mathbb{R}^2), t\rangle| \geq ||F_0^- \cap \{y_2 = t\}| - |\mathbb{S}_B(W_1) \cap \{y_2 = t\}||. \tag{4.53}$$

On the other hand, recalling that $-\mathbb{S}_B(G_0^1)^- = -[\![E_0^-]\!]$ and $-\mathbb{S}_B(Y_0) = -[\![F_0^-]\!] + [\![E_0^-]\!]$, from (4.49) we infer

$$\langle \mathbb{S}_B(\widehat{S}^1) \llcorner (\{0\} \times \mathbb{R}^2), t\rangle = -\langle [\![F_0^-]\!], t\rangle - \langle \mathbb{S}_B(\mathcal{W}_1), t\rangle, \tag{4.54}$$

and, since for every $t \geq \alpha_2$ it holds $(F_0^- \cap \{y_2 = t\}) \subset (\mathbb{S}_B(\mathcal{W}_1) \cap \{y_2 = t\})$ (or viceversa), we conclude

$$|\langle \mathbb{S}_B(\widehat{S}^1) \llcorner (\{0\} \times \mathbb{R}^2), t\rangle| = ||F_0^- \cap \{y_2 = t\}| - |\mathbb{S}_B(W_1) \cap \{y_2 = t\}||.$$

Combining this with (4.53) and integrating over $t \geq \alpha_2$ we get (4.48) for $s_i = 0$.

Let us now treat the case $s_i \neq 0$. Starting from (4.50) and taking into account that $\mathbb{S}_B(G_i^1) = \mathbb{S}_B(G_i^1)^+ - \mathbb{S}_B(G_i^1)^- = [\![E_i^+]\!] - [\![E_i^-]\!]$ and $\mathbb{S}_B(Y_i) = \mathbb{S}_B(Y_i^+) - \mathbb{S}_B(Y_i^-) = [\![E_i^+]\!] - [\![F_i^+]\!] - [\![E_i^-]\!] + [\![F_i^-]\!]$ we obtain

$$|\langle \mathbb{S}_B(\widehat{S}^1) \llcorner (\{s_i\} \times \mathbb{R}^2), t\rangle| = |\langle [\![F_i^+]\!], t\rangle - \langle [\![F_i^-]\!], t\rangle| = |(F_i^+ \Delta F_i^-) \cap \{y_2 = t\}|.$$

This is less or equal to

$$|\langle \mathcal{F}_i^+ - \mathcal{F}_i^-, t\rangle| = |\langle \widehat{S}^1 \llcorner (\{s_i\} \times \mathbb{R}^2), t\rangle|.$$

by (4.12) and (4.29). Integrating over $t \geq \alpha_2$ we conclude (4.48).

□

We are going to define the symmetrizations of the currents $\widehat{S}^2$ and $\widehat{S}^3$. We proceed as for $\widehat{S}^1$, and we replace G_1 defined in (4.12) by $\widetilde{G}_1$ given by

$$\widetilde{G}_1 := -\widehat{S}^2 - \widehat{S}^3 - Y^A + Y^C + L^\alpha - L^\gamma + \widetilde{\mathcal{W}}_1 - H. \tag{4.55}$$

that is closed in $\mathcal{D}_2((-\infty, l_j) \times \mathbb{R}^2)$ as well. Here $\widetilde{\mathcal{W}}_1$ is a current in $\mathcal{D}_2(\{0\} \times \mathbb{R}^2)$ such that

$$\partial \widetilde{\mathcal{W}}_1 = \mathcal{V}^2 + \mathcal{V}^3 - [\![\{0\} \times (\gamma_1, \alpha_1) \times \{\alpha_2\}]\!].$$

Defining $\widetilde{\mathcal{G}}_1 \in \mathcal{D}_3((-\infty, l_j) \times \mathbb{R}^2)$ with $\partial \widetilde{\mathcal{G}}_1 = \widetilde{G}_1$, we are again led to write, as for (4.13),

$$\widetilde{\mathcal{G}}^1 = \sum_h \theta_h [\![\widetilde{U}_h^1]\!], \quad \theta_h \in \{-1, 1\}, \tag{4.56}$$

for some Borel sets $\widetilde{U}_h^1 \subset [0, l_j) \times \mathbb{R}^2$ with local finite perimeters such that

$$|\widetilde{G}_1|_A = \sum_h |\partial \widetilde{U}_h^1|_A,$$

for any bounded open set $A \subset (-\infty, l_j) \times \mathbb{R}^2$. The symmetrization of $\widetilde{\mathcal{G}}^1$, namely $\mathbb{S}_B(\widetilde{\mathcal{G}}^1)$, is then defined as for $\mathbb{S}_B(\widetilde{\mathcal{G}}^1)$, the Steiner symmetrization of the union of the (disjoint) sets $\widetilde{U}_h^1$, and then restricting it to $\overline{Q}$. Therefore:

Definition 4.15. The symmetrization with respect to the $\widehat{h}_B$ axis of $\widetilde{G}_1$ is

$$\mathbb{S}_B(\widetilde{G}_1) := \partial \mathbb{S}_B(\widetilde{\mathcal{G}}_1). \tag{4.57}$$

As for $\mathcal{W}_1$, we first symmetrize $\widetilde{\mathcal{W}}_1$. We find sets $\widetilde{W}_h \subset \{0\} \times \mathbb{R}^2$ such that

$$\widetilde{\mathcal{W}}_1 - H = \sum_h \theta_h [\![\widetilde{W}_h]\!].$$

Definition 4.16. The Steiner symmetrization of $\cup_h \widetilde{W}_h$ with respect to the axis $\widehat{h}_B$ and restricted to $\overline{Q}_0$ is denoted by $\mathbb{S}_B(\widetilde{W}_1)$ (again $\{\widetilde{W}_h\}_h$ are considered mutually disjoint). We define the current

$$\mathbb{S}_B(\widetilde{\mathcal{W}}_1) := -[\![\mathbb{S}_B(\widetilde{W}_1)]\!] + H.$$

Moreover we set

$$J = \mathbb{S}_B(\mathcal{V}^2 + \mathcal{V}^3) := [\![\{0\} \times (\gamma_1, \alpha_1) \times \{\alpha_2\}]\!] + \partial \mathbb{S}_B(\widetilde{\mathcal{W}}_1).$$

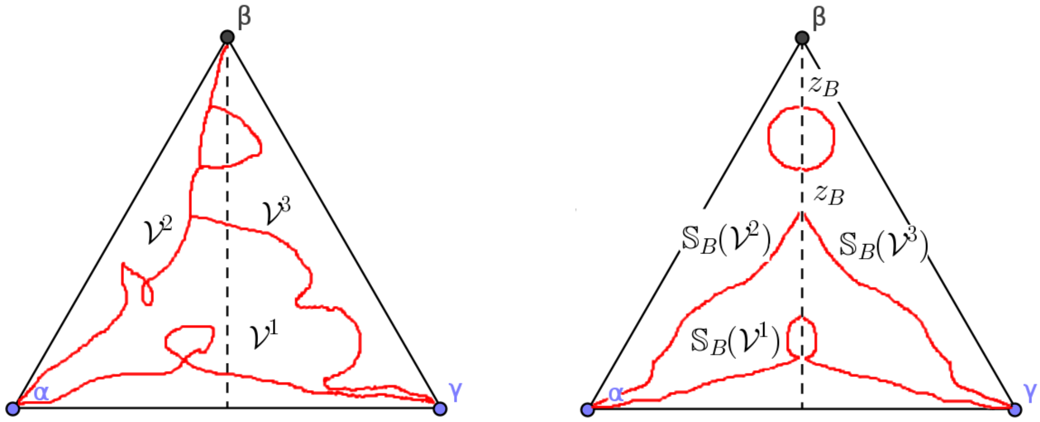

Figure 10: In this picture is depicted the bottom face $\{0\} \times T$ of the prism $P = (0, l_j) \times T$. On the left are drawn in red the three currents $\mathcal{V}^i$, $i = 1, 2, 3$, before applying the operator $\mathbb{S}_B$. The picture of the right represents the same currents after the symmetrization; on the two segments denoted by $z_B \subset h_B$ the two currents $\mathbb{S}_B(\mathcal{V}^2)$ and $\mathbb{S}_B(\mathcal{V}^3)$ overlap, and thus cancel each other. The set z_B is the support of the current r_B (see Definition 4.17).

It turns out that

$$|\mathbb{S}_B(\widetilde{\mathcal{W}}_1)| \leq |\widetilde{\mathcal{W}}_1|, \tag{4.58}$$

with strict inequality if the symmetrization of $\cup_h \widetilde{W}_h$ exceeds $\overline{Q}_0$ (also in this case it is easily observed that $\mathbb{S}_B(\widetilde{\mathcal{W}}_1)$ has support in T_0). In order to define $\mathbb{S}_B(\mathcal{V}^2)$ and $\mathbb{S}_B(\mathcal{V}^3)$ we still need some preliminary. The current J is supported on a 1-set that is symmetric with respect to h_B and has boundary $\delta_\alpha - \delta_\gamma$. In particular the restriction of J to the halfplane Π^α, namely J_α, has boundary $\delta_\alpha + \sum_h \delta_{P_h} - \sum_h \delta_{Q_h}$ with $\{P_h\}_h$ and $\{Q_h\}_h$ a sequence of points on h_B (and similarly J_γ has boundary $-\sum_h \delta_{P_h} + \sum_h \delta_{Q_h} - \delta_\gamma$). Let r_B be the (unique) 1-current supported on h_B with boundary $-\sum_h \delta_{P_h} + \sum_h \delta_{Q_h} - \delta_\beta$, and let us denote by z_B its support (see Figure 10). Therefore

Definition 4.17. The currents $\mathbb{S}_B(\mathcal{V}^2)$ and $\mathbb{S}_B(\mathcal{V}^3)$ are defined as

$$\mathbb{S}_B(\mathcal{V}^2) := J_\alpha + r_B, \qquad \mathbb{S}_B(\mathcal{V}^3) := J_\gamma - r_B. \tag{4.59}$$

Notice that

$$\partial \mathbb{S}_B(\widetilde{\mathcal{W}}_1) = \mathbb{S}_B(\mathcal{V}^2) + \mathbb{S}_B(\mathcal{V}^3) - [\![\{0\} \times (\alpha_1, \gamma_1) \times \{\alpha_2\}]\!]. \tag{4.60}$$

It can be proved that

$$|\mathbb{S}_B(\mathcal{V}_2)| + |\mathbb{S}_B(\mathcal{V}_3)| \leq |\mathcal{V}_3| + |\mathcal{V}_2|. \tag{4.61}$$

This will be addressed in Lemma 4.23 below.

Set $K = \widehat{S}^2 + \widehat{S}^3$. Let us recall that

$$\partial K \llcorner ((-\infty, l_j) \times \mathbb{R}^2) = (Id \times \alpha)_\sharp [\![0, l_j]\!] + \partial Y^A - (Id \times \gamma)_\sharp [\![0, l_j]\!] - \partial Y^C + \mathcal{V}^2 + \mathcal{V}^3. \tag{4.62}$$

In the spirit of Definition 4.13 we are led to:

Definition 4.18. The symmetrization of the current $K = \widehat{S}^2 + \widehat{S}^3$ with respect to the axis $\widehat{h}_B$ is defined as

$$\mathbb{S}_B(K) = \mathbb{S}_B(\widehat{S}^2 + \widehat{S}^3) := -\mathbb{S}_B(\widetilde{G}_1) + \mathbb{S}_B(Y^A) - \mathbb{S}_B(Y^C) + L^\alpha - L^\gamma + \mathbb{S}_B(\widetilde{\mathcal{W}}_1) - H. \tag{4.63}$$

The current $\mathbb{S}_B(K)$ is symmetric with respect to h_B and contained in $\overline{P}$, in particular it is an integral current $\mathbb{S}_B(K) = \{K_S, \tau, \theta\}$ where the rectifiable set K_S is symmetric with respect to $\widehat{h}_B$. Therefore let $K_S = K_S^\alpha \cup K_S^\gamma$ with $K_S^\alpha = K_S \cap \Xi_\alpha$, $K_S^\gamma = K_S \cap \Xi_\gamma$, where $\Xi_\alpha = \mathbb{R} \times \Pi_\alpha$ ($\Xi_\gamma = \mathbb{R} \times \Pi_\gamma$) is the halfspace bounded by $\mathbb{R} \times \widehat{h}_B$ and containing α (γ respectively). Notice that, by symmetry, the component of the current $\mathbb{S}_B(K)$ on the plane $\mathbb{R} \times \widehat{h}_B$ is null. The currents $\mathbb{S}_B(K_\alpha)$ and $\mathbb{S}_B(K_\gamma)$ are then defined as

$$\mathbb{S}_B(K_\alpha) := \mathbb{S}_B(K) \llcorner K_S^\alpha, \qquad \mathbb{S}_B(K_\gamma) := \mathbb{S}_B(K) \llcorner K_S^\gamma. \tag{4.64}$$

By (4.63) it is easily seen that

$$\begin{aligned}\partial \mathbb{S}_B(K) \llcorner ((-\infty, l_j) \times \mathbb{R}^2) =& (Id \times \alpha)_\sharp [\![0, l_j]\!] + \partial \mathbb{S}_B(Y^A) - (Id \times \gamma)_\sharp [\![0, l_j]\!] \\ & - \mathbb{S}_B(\partial Y^C) + \mathbb{S}_B(\mathcal{V}^2) + \mathbb{S}_B(\mathcal{V}^3),\end{aligned}$$

and therefore

$$\partial \mathbb{S}_B(K^\alpha) \llcorner ((-\infty, l_j) \times \mathbb{R}^2) = (Id \times \alpha)_\sharp [\![0, l_j]\!] - \partial \mathbb{S}_B(Y^A) - (\psi_\beta)_\sharp [\![0, t]\!] + \mathbb{S}_B(\mathcal{V}^2), \tag{4.65}$$

where ψ_β is a parametrization of the set $\mathbb{S}_B(K) \cap ([0, l_j) \times h_B)$, $t > 0$ (more precisely, ψ_β might be a countable sum of disjoint curves; this is not an issue, and for simplicity of exposition, in what follows we will still denote by ψ_β the sum of these currents[6]). Let K_0 be the 2-current in $\mathcal{D}_2((0, l_j) \times h_B)$ with boundary

$$\partial K_0 = (\psi_\beta)_\sharp [\![0, t]\!] - (Id \times \beta)_\sharp [\![0, l_j]\!].$$

(That is, the integration over the stripe enclosed between the set $\psi_\beta((0, t))$ and the line $(0, l_j) \times \{\beta\}$). Notice that by definition K_0 will be the integration over a set and hence is an integral current with multiplicity 1. This will be important to prove Theorem 4.20 below.

Definition 4.19. We set

$$\mathbb{S}_B(\widehat{S}^2) = \mathbb{S}_B(K_\alpha) + K_0, \qquad \mathbb{S}_B(\widehat{S}^3) = \mathbb{S}_B(K_\gamma) - K_0.$$

Eventually, we define

$$\mathbb{S}_A(Y^B) := 0.$$

In such a way it holds

$$\begin{aligned}\partial \mathbb{S}_B(\widehat{S}^2) \llcorner ((-\infty, l_j) \times \mathbb{R}^2) &= (Id \times \alpha)_\sharp [\![0, l_j]\!] + \partial \mathbb{S}_B(Y^A) - (Id \times \beta)_\sharp [\![0, l_j]\!] + \mathbb{S}_B(\mathcal{V}^2), \\ \partial \mathbb{S}_B(\widehat{S}^3) \llcorner ((-\infty, l_j) \times \mathbb{R}^2) &= (Id \times \beta)_\sharp [\![0, l_j]\!] - (Id \times \gamma)_\sharp [\![0, l_j]\!] - \partial \mathbb{S}_B(Y^C) + \mathbb{S}_B(\mathcal{V}^3).\end{aligned} \tag{4.66}$$

Theorem 4.20. *It holds*

$$|\mathbb{S}_B(\widehat{S}^2)| + |\mathbb{S}_B(\widehat{S}^3)| \le |\widehat{S}^2| + |\widehat{S}^3|. \tag{4.67}$$

Proof. In the case that $(\psi_\beta)_\sharp [\![0, t]\!] = (Id \times \beta)_\sharp [\![0, l_j]\!]$, namely $K_0 = 0$, the thesis easily follows arguing as for Theorem 4.14. Then we have to treat the case $K_0 \neq 0$. In this case, following the lines of the proof of Theorem 4.14, we first infer

$$|\mathbb{S}_B(\widehat{S}^2 + \widehat{S}^3)| \le |\widehat{S}^2 + \widehat{S}^3|. \tag{4.68}$$

Let us identify K_0 with its support set; by construction $\mathbb{S}_B(\widetilde{\mathcal{G}}_1)$ is null in $K_0 \times \mathbb{R}^2$, and since $\mathbb{S}_B(\widetilde{\mathcal{G}}_1) \llcorner (K_0 \times \mathbb{R}^2)$ corresponds to the symmetrization of $\widetilde{\mathcal{G}}_1$ on $K_0 \times \mathbb{R}$, it follows that $\widetilde{\mathcal{G}}_1$ is

[6]If there is a unique parametrization, let us emphasize that such curve might be non-injective and might cross two times, with opposite directions, a segment; this might happen if the set $\mathbb{S}_B(K) \cap ([0, l_j) \times h_B)$ is not connected.

null in the set $K_0 \times \mathbb{R}$ (recall that here $\mathbb{R}$ denotes the line orthogonal to the plane containing $[0, l_j) \times h_B$). In particular it follows that the two currents $\widehat{S}^2$ and $\widehat{S}^3$ have null sum in this set, that is

$$\widehat{S}^2 \llcorner (K_0 \times \mathbb{R}) = -\widehat{S}^3 \llcorner (K_0 \times \mathbb{R}). \tag{4.69}$$

Now, by (4.68), writing

$$|\mathbb{S}_B(\widehat{S}^2)| + |\mathbb{S}_B(\widehat{S}^3)| = |\mathbb{S}_B(\widehat{S}^2 + \widehat{S}^3)| + 2|K_0| \leq |\widehat{S}^2 + \widehat{S}^3| + 2|K_0|, \tag{4.70}$$

it remains to prove that

$$|K_0| \leq |\widehat{S}^2|_{K_0 \times \mathbb{R}} = |\widehat{S}^3|_{K_0 \times \mathbb{R}}. \tag{4.71}$$

Indeed thanks to (4.69), from (4.71) we infer

$$|\widehat{S}^2 + \widehat{S}^3| + |\widehat{S}^2|_{K_0 \times \mathbb{R}}| + |\widehat{S}^3|_{K_0 \times \mathbb{R}} = |\widehat{S}^2|_{K_0^c \times \mathbb{R}} + |\widehat{S}^3|_{K_0^c \times \mathbb{R}} + |\widehat{S}^2|_{K_0 \times \mathbb{R}} + |\widehat{S}^3|_{K_0 \times \mathbb{R}} \leq |\widehat{S}^2| + |\widehat{S}^3|,$$

that with (4.70) addresses the result. The claim (4.71) follows by an argument of slicing: for $s \in (0, l_j)$ let us denote by $\sigma(s)$ the length of the intersection between K_0 and the plane $\{s\} \times \mathbb{R}^2$, namely

$$\sigma(s) = \mathcal{H}^1(K_0 \cap (\{s\} \times \mathbb{R}^2)), \tag{4.72}$$

then it holds,

$$|\widehat{S}^2 \llcorner (K_0 \times \mathbb{R})| \geq \int_0^{l_j} |\langle \widehat{S}^2 \llcorner (K_0 \times \mathbb{R}), s \rangle| ds \geq \int_0^{l_j} \sigma(s) ds = |K_0|.$$

(Observe that the projection of the support of $\langle \widehat{S}^2 \llcorner (K_0 \times \mathbb{R}), s \rangle$ onto $\widehat{h}_B$ coincides with $K_0 \cap (\{s\} \times \mathbb{R}^2)$ for $\mathcal{H}^1$-a.e. $s \in (0, l_j)$)[7]. The last equality follows since K_0 has multiplicity 1.

□

Finally we define the symmetrization of the current $\mathcal{T}$. Recalling Definitions 4.12 and 4.16 of $\mathbb{S}_B(\mathcal{W}_1)$ and $\mathbb{S}_B(\widetilde{\mathcal{W}}_1)$, we set:

Definition 4.21. The symmetrization of the current $\mathcal{T} \in \mathcal{D}_2(\{0\} \times \mathbb{R}^2)$ is the current $\mathbb{S}_B(\mathcal{T}) \in \mathcal{D}_2(\{0\} \times \mathbb{R}^2)$ defined as

$$\mathbb{S}_B(\mathcal{T}) := \mathbb{S}_B(\mathcal{W}) - \mathbb{S}_B(\widetilde{\mathcal{W}}). \tag{4.73}$$

From (4.42) and (4.60) it follows

$$\partial \mathbb{S}_B(\mathcal{T}) = -\mathbb{S}_B(\mathcal{V}^1) - \mathbb{S}_B(\mathcal{V}^2) - \mathbb{S}_B(\mathcal{V}^3). \tag{4.74}$$

Moreover we can prove

Proposition 4.22. *It holds*

$$|\mathbb{S}_B(\mathcal{T})| \leq |\mathcal{T}|. \tag{4.75}$$

Proof. We employ a simple argument of slicing, as in the proof of Theorem 4.14. Let $t \in \mathbb{R}$, and consider the section line $\{0\} \times \mathbb{R} \times \{y_2 = t\}$. First we observe that $\mathcal{T} = \mathcal{W}_1 - \widetilde{\mathcal{W}}_1$, so that for all $t \in \mathbb{R}$ we have

$$|\langle \mathcal{T}, t \rangle| \geq ||\langle \mathcal{W}_1, t \rangle| - |\langle \widetilde{\mathcal{W}}_1, t \rangle||. \tag{4.76}$$

[7]This is a consequence of the Constancy Lemma and (4.62). Indeed, roughly speaking, for a.e. $s \in (0, l_j)$ the slice $\langle \widehat{S}^2, s \rangle$ is a curve connecting β to α, and hence its projection onto h_B is surjective.

By Definition 4.12 and 4.16 we have

$$
\begin{aligned}
&||\mathbb{S}_B(W_1)\cap\{y_2=t\}|-|\mathbb{S}_B(\widetilde{W}_1)\cap\{y_2=t\}||=|(\mathbb{S}_B(W_1)\Delta\mathbb{S}_B(\widetilde{W}_1))\cap\{y_2=t\}|\\
&=|\langle\mathbb{S}_B(\mathcal{W}_1)-\mathbb{S}_B(\widetilde{\mathcal{W}}_1),t\rangle|=|\langle\mathbb{S}_B(\mathcal{T}),t\rangle|,
\end{aligned}
$$

where we have used that $(\mathbb{S}_B(W_1)\cap\{y_2=t\})\subset(\mathbb{S}_B(\widetilde{W}_1)\cap\{y_2=t\})$ (or viceversa). This, together with (4.76), and integrated over $\mathbb{R}$ gives (4.75) since

$$
\begin{aligned}
&||\mathbb{S}_B(W_1)\cap\{y_2=t\}|-|\mathbb{S}_B(\widetilde{W}_1)\cap\{y_2=t\}||\\
&=|\tau(\widehat{\mathbb{S}}_B(W_1)\cap\{y_2=t\})-\tau(\widehat{\mathbb{S}}_B(\widetilde{W}_1)\cap\{y_2=t\})|\leq||\langle\mathcal{W}_1,t\rangle|-|\langle\widehat{\mathcal{W}}_1,t\rangle||,
\end{aligned}
$$

(recall that $\widehat{\mathbb{S}}_B(W_1)$ and $\widehat{\mathbb{S}}_B(\widetilde{W}_1)$ are the symmetrizations of the sets $\{W_h\}_h$ and $\{\widetilde{W}_h\}_h$ before intersecting with $\overline{Q}$; then we employ the same argument in (4.52)). □

We finally observe that the current

$$\mathbb{S}_B(\mathcal{T})+\mathbb{S}_B(\widehat{S}^1)+\mathbb{S}_B(\widehat{S}^2)+\mathbb{S}_1(\widehat{S}^3),$$

is closed in $\mathcal{D}_2((-\infty,l_j)\times\mathbb{R}^2)$. This follows from (4.46), (4.66), and (4.74).

Let us collect some crucial observations about the symmetrization operator, summarized in the following lemma.

Lemma 4.23. *The symmetrization operator $\mathbb{S}_B$ enjoys the following features:*

(a) It holds

$$|\mathbb{S}_B(Y^A)|+|\mathbb{S}_B(Y^C)|\leq|Y^A|+|Y^C|,$$

whereas

$$|\mathbb{S}_B(Y^B)|=0.$$

Moreover, by definition, $|\mathbb{S}_B(Y^A)|=|\mathbb{S}_B(Y^C)|$.

(b) The currents $\mathbb{S}_B(\widehat{S}^i)$, $i=1,2,3$, satisfy

$$
\begin{aligned}
\partial\mathbb{S}_B(\widehat{S}^1)\llcorner((-\infty,l_j)\times\mathbb{R}^2)=&-(Id\times\alpha)_\sharp[\![0,l_j]\!]-\mathbb{S}_B(\partial Y^A)\\
&+(Id\times\gamma)_\sharp[\![0,l_j]\!]+\mathbb{S}_B(\partial Y^C)+\mathbb{S}_B(\mathcal{V}^1),\\
\partial\mathbb{S}_B(\widehat{S}^2)\llcorner((-\infty,l_j)\times\mathbb{R}^2)=&(Id\times\alpha)_\sharp[\![0,l_j]\!]+\mathbb{S}_B(\partial Y^A)\\
&-(Id\times\beta)_\sharp[\![0,l_j]\!]+\mathbb{S}_B(\mathcal{V}^2),\\
\partial\mathbb{S}_B(\widehat{S}^3)\llcorner((-\infty,l_j)\times\mathbb{R}^2)=&(Id\times\beta)_\sharp[\![0,l_j]\!]-\mathbb{S}_B(\partial Y^C)\\
&-(Id\times\gamma)_\sharp[\![0,l_j]\!]+\mathbb{S}_B(\mathcal{V}^3).
\end{aligned}
$$

(c) The current $\mathbb{S}_B(\mathcal{T})$ satisfies

$$\partial\mathbb{S}_B(\mathcal{T})=-\mathbb{S}_B(\mathcal{V}^1)-\mathbb{S}_B(\mathcal{V}^2)-\mathbb{S}_B(\mathcal{V}^3),\tag{4.77}$$

and

$$|\mathbb{S}_B(\mathcal{V}^1)|\leq|\mathcal{V}^1|,\qquad|\mathbb{S}_B(\mathcal{V}^2)|+|\mathbb{S}_B(\mathcal{V}^3)|\leq|\mathcal{V}^2|+|\mathcal{V}^3|.\tag{4.78}$$

(d) We have

$$|\mathbb{S}_B(\mathcal{T})|+|\mathbb{S}_B(\widehat{S}^1)|+|\mathbb{S}_B(\widehat{S}^2)|+|\mathbb{S}_B(\widehat{S}^3)|\leq|\mathcal{T}|+|\widehat{S}^1|+|\widehat{S}^2|+|\widehat{S}^3|.\tag{4.79}$$

Proof. Statement (a) is given by Lemma 4.8 and by definition of $\mathbb{S}_B(Y^B)$. Item (b) follows from (4.46) and (4.66). The first equation in (c) is (4.74). Let us demonstrate the second equation in (4.78) (the first inequality is (4.44)). The argument is very similar to the one employed in the proof of Theorem 4.20; let us sketch it. Recalling Definition 4.17, we want to prove that

$$|J|+2|z_B|\le|\mathcal{V}^2|+|\mathcal{V}^3|=|\mathcal{V}^2|_{\mathbb{R}\times z_B}+|\mathcal{V}^2|_{\mathbb{R}\times(h_B\setminus z_B)}+|\mathcal{V}^3|_{\mathbb{R}\times z_B}+|\mathcal{V}^3|_{\mathbb{R}\times(h_B\setminus z_B)},\quad(4.80)$$

where z_B is the support set of the current r_B (see Figure 10). Now, by Steiner symmetrization it is easily seen that $|J|\le|\mathcal{V}^2|_{\mathbb{R}\times(h_B\setminus z_B)}+|\mathcal{V}^3|_{\mathbb{R}\times(h_B\setminus z_B)}$, whereas the proof that $|z_B|\le|\mathcal{V}^2|_{\mathbb{R}\times z_B}$ follows by noticing that z_B is exactly the projection on h_B of the support of $\mathcal{V}^2\llcorner(\mathbb{R}\times z_B)$, that is onto on z_B since $\mathcal{V}^2$ is an arc connecting α to β. Finally, we achieve (d) just gathering together (4.75) with Theorems 4.14 and 4.20. □

The operators $\mathbb{S}_A$ and $\mathbb{S}_C$ are the symmetrizations with respect to the plane $\mathbb{R}\times\widehat{h}_A$ and $\mathbb{R}\times\widehat{h}_C$, respectively, constructed like $\mathbb{S}_B$ switching the role of A, B, and C, accordingly.

5 Proof of Theorem 3.7

We are now ready to prove Theorem 3.7. Our strategy will be to apply repeatedly the symmetrization operators to the currents $\widehat{S}^i$ and $\mathcal{T}$. We proceed as follows: we define $\mathbb{S}:=\mathbb{S}_A\circ\mathbb{S}_B\circ\mathbb{S}_C$ and set

$$\begin{aligned}&(\widehat{S}^i)_k:=\mathbb{S}^k(\widehat{S}^i),\quad i=1,2,3,\\&\mathcal{T}_k:=\mathbb{S}^k(\mathcal{T}),\end{aligned}$$

for every $h\in\mathbb{N}$. We will prove the following:

Proposition 5.1. *There exists integral currents $\overline{S}^i,\overline{\mathcal{T}}\in\mathcal{D}_2((-\infty,l_j)\times\mathbb{R}^2)$ such that*

$$\begin{aligned}&(\widehat{S}^i)_k\rightharpoonup\overline{S}^i\qquad\textit{for } i=1,2,3,\\&\mathcal{T}_k\rightharpoonup\overline{\mathcal{T}},\end{aligned}\tag{5.1}$$

and

$$|\overline{S}^1|+|\overline{S}^2|+|\overline{S}^3|+|\overline{\mathcal{T}}|\le|(\widehat{S}^1)_k|+|(\widehat{S}^2)_k|+|(\widehat{S}^3)_k|+|\mathcal{T}_k|,\tag{5.2}$$

for all $k\in\mathbb{N}$. Moreover $\overline{S}^1+\overline{S}^2+\overline{S}^3+\overline{\mathcal{T}}$ is a closed current in $\mathcal{D}_2((-\infty,l_j)\times\mathbb{R}^2)$, and

$$\begin{aligned}\partial\overline{S}^1\llcorner((0,l_j)\times\mathbb{R}^2)=&-(Id\times\alpha)_\sharp[\![0,l_j]\!]+(Id\times\gamma)_\sharp[\![0,l_j]\!],\\\partial\overline{S}^2\llcorner((0,l_j)\times\mathbb{R}^2)=&(Id\times\alpha)_\sharp[\![0,l_j]\!]-(Id\times\beta)_\sharp[\![0,l_j]\!],\\\partial\overline{S}^3\llcorner((0,l_j)\times\mathbb{R}^2)=&(Id\times\beta)_\sharp[\![0,l_j]\!]-(Id\times\gamma)_\sharp[\![0,l_j]\!].\end{aligned}\tag{5.3}$$

Remark 5.2. Notice that after one application of $\mathbb{S}$ nothing ensures us that the currents N^A, N^B, and N^C vanish. This is because every application of a symmetrization operator reduces their mass but not necessarily nullify it. For these reason we will need to apply $\mathbb{S}$ infinite times.

Proof. The weak convergences (5.1) entail

$$|\overline{S}^1|+|\overline{S}^2|+|\overline{S}^3|+|\overline{\mathcal{T}}|\le\liminf_{k\to\infty}|(\widehat{S}^1)_k|+|(\widehat{S}^2)_k|+|(\widehat{S}^3)_k|+|\mathcal{T}_k|,\tag{5.4}$$

and Lemma 4.23 (d) implies that the sequence on the right-hand side is nonincreasing, so that for all $h\in\mathbb{N}$ we have

$$\liminf_{k\to\infty}\Big(|(\widehat{S}^1)_k|+|(\widehat{S}^2)_k|+|(\widehat{S}^3)_k|+|\mathcal{T}_k|\Big)\le|(\widehat{S}^1)_h|+|(\widehat{S}^2)_h|+|(\widehat{S}^3)_h|+|\mathcal{T}_h|,$$

and inequality (5.2) follows. Let us prove (5.1). We first focus on the currents Y^A, Y^B, and Y^C. Owing to Lemma 4.23 (a) it is easy to prove that after an application of $\mathbb{S}$ we have

$$|\mathbb{S}(Y^A)| + |\mathbb{S}(Y^B)| + |\mathbb{S}(Y^C)| \le \frac{1}{4}(|Y^A| + |Y^B| + |Y^C|).$$

Thus by induction we get

$$|\mathbb{S}^k(Y^A)| + |\mathbb{S}^k(Y^B)| + |\mathbb{S}^k(Y^C)| \le \frac{1}{4^k}(|Y^A| + |Y^B| + |Y^C|).$$

In particular

$$\mathbb{S}^k(Y_A) \to 0, \quad \mathbb{S}^k(Y_B) \to 0, \quad \mathbb{S}^k(Y_C) \to 0. \tag{5.5}$$

Let us set

$$\begin{aligned} P_k^1 &:= (\widehat{S}^1)_k + \mathbb{S}^k(Y^A) - \mathbb{S}^k(Y^C), \\ P_k^2 &:= (\widehat{S}^2)_k + \mathbb{S}^k(Y^B) - \mathbb{S}^k(Y^A), \\ P_k^3 &:= (\widehat{S}^3)_k + \mathbb{S}^k(Y^C) - \mathbb{S}^k(Y^B); \end{aligned} \tag{5.6}$$

from Lemma 4.23 (b) we infer that

$$\begin{aligned} \partial P_k^1 \llcorner ((-\infty, l_j) \times \mathbb{R}^2) &= -(Id \times \alpha)_\sharp [\![0, l_j]\!] + (Id \times \gamma)_\sharp [\![0, l_j]\!] + \mathbb{S}^k(\mathcal{V}^1), \\ \partial P_k^2 \llcorner ((-\infty, l_j) \times \mathbb{R}^2) &= -(Id \times \beta)_\sharp [\![0, l_j]\!] + (Id \times \alpha)_\sharp [\![0, l_j]\!] + \mathbb{S}^k(\mathcal{V}^2), \\ \partial P_k^3 \llcorner ((-\infty, l_j) \times \mathbb{R}^2) &= -(Id \times \gamma)_\sharp [\![0, l_j]\!] + (Id \times \beta)_\sharp [\![0, l_j]\!] + \mathbb{S}^k(\mathcal{V}^3). \end{aligned} \tag{5.7}$$

Since $\mathbb{S}^k(\mathcal{V}^i)$ have uniformly bounded masses by Lemma 4.23 (c), thanks to (4.79) as well, we find limit integral currents $\overline{S}^i, \overline{\mathcal{T}} \in \mathcal{D}_2((-\infty, l_j) \times \mathbb{R}^2)$ such that, up to subsequences,

$$\begin{aligned} &(P^i)_k \rightharpoonup \overline{S}^i \qquad \text{for } i = 1, 2, 3, \\ &\mathcal{T}_k \rightharpoonup \overline{\mathcal{T}}. \end{aligned}$$

Thanks to (5.5) and (5.6) we infer (5.1). The fact that $\overline{S}^1 + \overline{S}^2 + \overline{S}^3 + \overline{\mathcal{T}}$ is a closed current in $\mathcal{D}_2((-\infty, l_j) \times \mathbb{R}^2)$ follows from the fact that $(\widehat{S}^1)_k + (\widehat{S}^2)_k + (\widehat{S}^3)_k + \mathcal{T}_k$ is closed for all k and tends to $\overline{S}^1 + \overline{S}^2 + \overline{S}^3 + \overline{\mathcal{T}}$. Finally (5.3) follows from (5.7) passing to the limit. $\square$

The currents $\overline{S}^i, \overline{\mathcal{T}} \in \mathcal{D}_2((-\infty, l_j) \times \mathbb{R}^2)$ satisfy the following properties:

(i) The integral current $\overline{\mathcal{T}}$ is supported in $\{0\} \times T$ and has boundary

$$\partial \overline{\mathcal{T}} = -\overline{\mathcal{V}}^1 - \overline{\mathcal{V}}^2 - \overline{\mathcal{V}}^3.$$

There exist three Lipschitz functions $\psi_i : [0, 1] \to T$, $i = 1, 2, 3$, such that

$$\begin{aligned} &\mathcal{V}_i = (\psi_i)_\sharp [\![(0, 1)]\!] \quad i = 1, 2, 3, \\ &\psi_1(0) = \alpha, \quad \psi_1(1) = \gamma = \psi_3(0), \quad \psi_3(1) = \beta = \psi_2(0), \quad \psi_2(1) = \alpha. \end{aligned}$$

Moreover there is a constant $C > 0$ such that

$$\sum_{i=1}^{3} |\mathcal{V}_i| \le C. \tag{5.8}$$

(ii) The three currents $\overline{S}^i$ $i = 1, 2, 3$ are integral and satisfy

$$\partial \overline{S}^1 = -(Id \times \alpha)_\sharp [\![0, l_j]\!] + (Id \times \gamma)_\sharp [\![0, l_j]\!] + \overline{\mathcal{V}}^1, \tag{5.9}$$

$$\partial \overline{S}^2 = (Id \times \alpha)_\sharp [\![0, l_j]\!] - (Id \times \beta)_\sharp [\![0, l_j]\!] + \overline{\mathcal{V}}^2, \tag{5.10}$$

$$\partial \overline{S}^3 = (Id \times \beta)_\sharp [\![0, l_j]\!] - (Id \times \gamma)_\sharp [\![0, l_j]\!] + \overline{\mathcal{V}}^3. \tag{5.11}$$

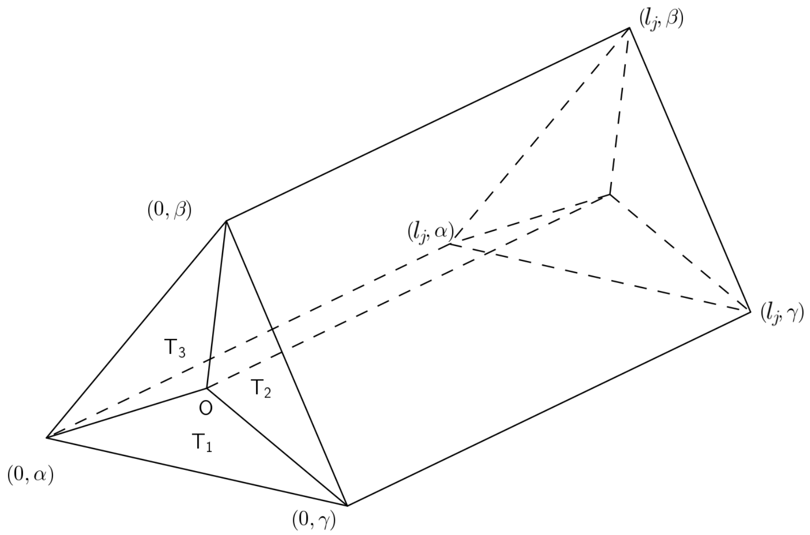

Figure 11: This figure represents the geometric setting introduced before Proposition 5.3. The prism P is the union of the three prisms P_i with base T_i, $i = 1, 2, 3$. The lateral surface of the prism P is made of three rectangles $\mathcal{R}_i$, $i = 1, 2, 3$. For instance, the rectangle $\mathcal{R}_1$ is the one with vertices $(0, \alpha)$, $(0, \gamma)$, (l_j, γ), and (l_j, α).

We can write down an additional condition, which however is a consequence of (i) and (ii):

(ii') The current $U := \overline{S}^1 + \overline{S}^2 + \overline{S}^3 + \overline{\mathcal{T}}$ is a closed current in $\mathcal{D}_2((-\infty, l_j) \times \mathbb{R}^2)$.

We then are led to the following minimum problem

$$\min \big\{ |S^1| + |S^2| + |S^3| + |\mathcal{T}| : S^i \ (i = 1, 2, 3), \text{ and } \mathcal{T} \text{ satisfy (i) and (ii)} \big\}. \tag{5.12}$$

The existence of a minimizer follows from the Compactness Theorem for integral currents. Let $(S^1, S^2, S^3, \mathcal{T})$ be a minimizer. Of course, thanks to (5.2) for $k = 0$ and the definition of $(S^1, S^2, S^3, \mathcal{T})$ we have

$$|\widehat{S}^1| + |\widehat{S}^2| + |\widehat{S}^3| + |\mathcal{T}| \geq |S^1| + |S^2| + |S^3| + |\mathcal{T}|.$$

Therefore if we prove that $(S^1, S^2, S^3, \mathcal{T})$ satisfies (3.63) then the proof of Theorem 3.7 is complete. To this aim we will first prove three preliminary results. We begin with some geometric definitions. The triangle T with vertices α, β, and γ, can be seen as the union of the three triangles T_i, $i = 1, 2, 3$, where T_1 has vertices α, γ and O, T_2 has vertices α, β, and O, while T_3 has vertices β, γ, and O. The prism $P = (0, l_j) \times T$ can be seen as the union of the three prisms P_1, P_2, and P_3, given by $P_i = (0, l_j) \times T_i$, $i = 1, 2, 3$. Let us recall that $\mathcal{R}_1$, $\mathcal{R}_2$, and $\mathcal{R}_3$ are the rectangles with edges $\overline{\alpha\gamma} \times (0, l_j)$, $\overline{\beta\alpha} \times (0, l_j)$, and $\overline{\gamma\beta} \times (0, l_j)$, respectively (see Figure 11).

Proposition 5.3. *There is a minimizer $(S^1, S^2, S^3, \mathcal{T})$ of the minimum problem (5.12) such that the currents S^i, $i = 1, 2, 3$, are the graphs of Cartesian maps on $\mathcal{D}_2(\mathcal{R}^i \times \mathbb{R})$, $i = 1, 2, 3$. Namely, there are functions $u_i \in BV(\mathcal{R}^i; \mathbb{R})$, $i = 1, 2, 3$ such that*

$$S^i = (Id \times u_i)_\sharp [\![\mathcal{R}_i]\!], \qquad |S^i \llcorner (\mathcal{R}^i \times \mathbb{R})| = \mathcal{A}(u_i; \mathcal{R}_i), \tag{5.13}$$

for $i = 1, 2, 3$.

Proof. We will use the fact that, by the minimality assumption of $(S^1, S^2, S^3, \mathcal{T})$, if we apply a symmetrization operator to these currents, their total mass cannot strictly decrease. We proceed in three steps.

Step 1. We consider lines in $\mathbb{R}^3$ which are orthogonal to the s-axis and to the height h_B (i.e. parallel to the y_1-axis). These lines are $r_{s,t} := \{s\} \times \mathbb{R} \times \{t\}$ with $s \in [0, l_j]$ and $t \in (\alpha_2, \beta_2)$ (recall we choose the system (y_1, y_2) is such a way that $\alpha_2 = \gamma_2$). Let us identify the current S^i with its support set. We claim that

$$\mathcal{H}^2(\{(s,t) \in (0, l_j) \times (\alpha_2, \beta_2) : \sharp\{S^i \cap r_{s,t}\} > 1 \text{ for some } i = 2, 3\}) = 0. \tag{5.14}$$

First notice that for both $i = 2, 3$ it holds

$$\sharp\{S^i \cap r_{s,t}\} \geq 1 \text{ for } \mathcal{H}^2 - \text{a.e. } (s,t). \tag{5.15}$$

To see (5.14) we argue by contradiction, and denoting by A the set in (5.14), suppose $\mathcal{H}^2(A) > 0$. Let $\widetilde{\mathcal{G}}_1$ be the current in $\mathcal{D}_3((0, l_j) \times \mathbb{R}^2)$ with boundary $S^2 + S^3 + \mathcal{R}^1$ (we are neglecting the boundaries in $\{0\} \times T$ if we look at $S^2 + S^3 + \mathcal{R}^1$ as currents in $(0, l_j) \times \mathbb{R}^2$). We identify $\widetilde{\mathcal{G}}_1$ with its support set (which coincides with the area enclosed between the surfaces S^2, S^3, and $\mathcal{R}^1$). Define

$$C_K = \{(s,t) \in (0, l_j) \times (\alpha_2, \beta_2) : \mathcal{H}^1(r_{s,t} \cap \widetilde{\mathcal{G}}_1) > 0\}$$

and $C_K^c := ((0, l_j) \times (\alpha_2, \beta_2)) \setminus C_K$[8]. By definition of $\mathbb{S}_B(S^2)$ it is seen that the operator $\mathbb{S}_B$ transforms $S^2 \cap (C_K \times \mathbb{R})$ into $\mathbb{S}_B(\partial\widetilde{\mathcal{G}}_1) \cap K_\alpha$ (see Definition 4.18) and sends $S^2 \cap (C_K^c \times \mathbb{R})$ into K_0 (see Definition 4.19); similarly for S^3. If $\mathcal{H}^2(A) > 0$ then either $\mathcal{H}^2(A \cap C_K^c) > 0$ or $\mathcal{H}^2(A \cap C_K) > 0$. Let us treat the two cases separately:

(1) (Case $\mathcal{H}^2(A \cap C_K^c) > 0$) suppose that (5.14) takes place in C_K^c and for the index $i = 2$, namely
$$\mathcal{H}^2(\{(s,t) \in (0, l_j) \times (\alpha_2, \beta_2) \cap C_K^c : \sharp\{S^2 \cap r_{s,t}\} > 1\}) > 0.$$
Since both the sets $S^2 \cap (C_K^c \times \mathbb{R})$ and $S^3 \cap (C_K^c \times \mathbb{R})$ are transformed into K_0 by $\mathbb{S}_B$, we can write
$$\begin{aligned}
&|S^2| + |S^3| = \\
&\quad = |S^2 \cap (C_K \times \mathbb{R})| + |S^2 \cap (C_K^c \times \mathbb{R})| + |S^3 \cap (C_K \times \mathbb{R})| + |S^3 \cap (C_K^c \times \mathbb{R})| \\
&\quad \geq |\mathbb{S}_B(S^2) \cap (C_K \times \mathbb{R})| + |\mathbb{S}_B(S^3) \cap (C_K \times \mathbb{R})| \\
&\qquad + \int_{C_K^c} \sharp\{r_{s,t} \cap S^2\} d\mathcal{H}^2 + \int_{C_K^c} \sharp\{r_{s,t} \cap S^3\} d\mathcal{H}^2 \\
&\quad > |\mathbb{S}_B(S^2) \cap (C_K \times \mathbb{R})| + |\mathbb{S}_B(S^3) \cap (C_K \times \mathbb{R})| + \int_{C_K^c} 2 d\mathcal{H}^2 \\
&\quad = |\mathbb{S}_B(S^2) \cap (C_K \times \mathbb{R})| + |\mathbb{S}_B(S^3) \cap (C_K \times \mathbb{R})| + 2|K_0 \cap (C_K^c \times \mathbb{R})| \\
&\quad = |\mathbb{S}_B(S^2)| + |\mathbb{S}_B(S^3)|.
\end{aligned}$$
The fact that such inequality is strict contradicts the assumption that $(S^1, S^2, S^3, \mathcal{T})$ is a minimizer.

(2) (Case $\mathcal{H}^2(A \cap C_K) > 0$) now we take into account that $\mathbb{S}_B$ transforms $S^2 \cap (C_K \times \mathbb{R})$ into $\mathbb{S}_B(\partial\widetilde{\mathcal{G}}_1) \cap K_\alpha$ and $S^3 \cap (C_K \times \mathbb{R})$ into $\mathbb{S}_B(\partial\widetilde{\mathcal{G}}_1) \cap K_\gamma$. In $C_K \times \mathbb{R}$ it happens that $\partial\widetilde{\mathcal{G}}_1 \cap r_{s,t} \geq 2$. Suppose first that the subset $B \subset C_K$ defined as
$$B := \{(s,t) \in C_K : \sharp\{\partial\widetilde{\mathcal{G}}_1 \cap r_{s,t}\} > 2\}$$

[8] In other words C_k is the projection of $\widetilde{\mathcal{G}}_1$ onto the rectangle $(0, l_j) \times h_B$.

satisfies $\mathcal{H}^2(B) > 0$. In this case, as a property of Steiner symmetrization, it is known that $|\partial\mathbb{S}_B(\widetilde{\mathcal{G}}_1) \cap (B \times \mathbb{R})| < |\partial\widetilde{\mathcal{G}}_1 \cap (B \times \mathbb{R})|$, and thus we easily arrive to $|S^2| + |S^3| > |\mathbb{S}_B(S^2)| + |\mathbb{S}_B(S^3)|$, again a contradiction. Suppose then that

$$\sharp\{\partial\widetilde{\mathcal{G}}_1 \cap r_{s,t}\} = 2, \qquad \text{for } \mathcal{H}^2 - \text{a.e. } (s,t) \in C_K. \tag{5.16}$$

On the other hand we have, by hypothesis, $\mathcal{H}^2(A \cap C_K) > 0$, therefore we again can assume that the set

$$B_2 := \{(s,t) \in \big((0,l_j) \times (\alpha_2, \beta_2)\big) \cap C_K : \sharp\{S^2 \cap r_{s,t}\} = 2\} \tag{5.17}$$

has positive $\mathcal{H}^2$ measure (similarly we might assume this happens for S^3). At the same time, by (5.15), it must occur that

$$\sharp\{S^3 \cap r_{s,t}\} \geq 1 \text{ for } \mathcal{H}^2 - \text{a.e.}(s,t) \in C_K. \tag{5.18}$$

Since $\partial\widetilde{\mathcal{G}}_1 \subset S^2 \cup S^3$ we have two cases:

(a) $\mathcal{H}^2((S^2 \cup S^3) \cap (C_K \times \mathbb{R})) > \mathcal{H}^2(\partial\widetilde{\mathcal{G}}_1 \cap (C_K \times \mathbb{R}))$ and hence we have $\mathcal{H}^2((S^2 \cup S^3) \cap (C_K \times \mathbb{R})) > \mathcal{H}^2(\mathbb{S}_B(\partial\widetilde{\mathcal{G}}_1) \cap (C_K \times \mathbb{R})) = \mathcal{H}^2((\mathbb{S}_B(S^2) \cup \mathbb{S}_B(S^3)) \cap (C_K \times \mathbb{R}))$, again contradicting the minimality;

(b) $\mathcal{H}^2((S^2 \cup S^3) \cap (C_K \times \mathbb{R})) = \mathcal{H}^2(\partial\widetilde{\mathcal{G}}_1 \cap (C_K \times \mathbb{R}))$, we find that essentially $((S^2 \cup S^1) \cap (C_K \times \mathbb{R})) = \partial\widetilde{\mathcal{G}}_1 \cap (C_K \times \mathbb{R})$. Recall that, by (5.16), $\mathcal{H}^2$-a.e. $(s,t) \in C_K$ it holds $\partial\widetilde{\mathcal{G}}_1 \cap r_{s,t} = 2$; this together with (5.17) implies that, up to a negligible set, $\partial\widetilde{\mathcal{G}}_1 = S^2$ in $B_2 \times \mathbb{R}$ (see Figure 12). Thus, thanks to (5.18) we infer

$$\begin{aligned} |S^2 \cap (B_2 \times \mathbb{R})| + |S^3 \cap (B_2 \times \mathbb{R})| &> |S^2 \cap (B_2 \times \mathbb{R})| = |\partial\widetilde{\mathcal{G}}_1 \cap (B_2 \times \mathbb{R})| \\ &\geq |\mathbb{S}_B(\partial\widetilde{\mathcal{G}}_1) \cap (B_2 \times \mathbb{R})| = |\mathbb{S}_B(S^2) \cap (B_2 \times \mathbb{R})| + |\mathbb{S}_B(S^3) \cap (B_2 \times \mathbb{R})|, \end{aligned}$$

from which we again arrive at $\mathcal{H}^2((S^2 \cup S^3) \cap (C_K \times \mathbb{R})) > \mathcal{H}^2(\mathbb{S}_B(\partial\widetilde{\mathcal{G}}_1) \cap (C_K \times \mathbb{R})) = \mathcal{H}^2((\mathbb{S}_B(S^2) \cup \mathbb{S}_B(S^3)) \cap (C_K \times \mathbb{R}))$, concluding the proof of (5.14).

Step 2. From (5.14) it follows that for $\mathcal{H}^2$-a.e. $(s,t) \in (0,l_j) \times (\alpha_2, \beta_2)$ it holds $\sharp\{S^2 \cap r_{s,t}\} = 1$ where $r_{s,t}$ are lines parallel to $\overline{\alpha\gamma}$, i.e. $r_{s,t} \parallel \overline{\alpha\gamma}$. Arguing as in Step 1 we infer that the same is true if we consider lines $r_{s,t}$ parallel to the edge $\overline{\beta\gamma}$. Thus we have

$$\sharp\{S^2 \cap r_{s,t}\} = 1 \qquad r_{s,t} \parallel \overline{\alpha\gamma} \text{ and } r_{s,t} \parallel \overline{\beta\gamma}. \tag{5.19}$$

Consider now lines $r_{s,t}$ parallel to the height h_C, so that we can assume $(s,t) \in \mathcal{R}_2 = (0,l_j) \times (\beta_1, \alpha_1)$. We claim

$$\sharp\{S^2 \cap r_{s,t} : r_{s,t} \parallel h_C\} = 1 \qquad \text{for } \mathcal{H}^2 - \text{a.e } (s,t) \in \mathcal{R}_2. \tag{5.20}$$

Denote by E the set of all $(s,t) \in \mathcal{R}_2$ such that $\sharp\{S^2 \cap r_{s,t}\} > 1$, namely

$$E := \{(s,t) \in \mathcal{R}_2 : \sharp\{S^2 \cap r_{s,t}\} > 1\},$$

and assume by contradiction that E has positive $\mathcal{H}^2$-measure. Define $E_\theta := \{(s,t) \in E : s = \theta\}$. As a consequence the set

$$\Theta := \{\theta \in (0,l_j) : \mathcal{H}^1(E_\theta) > 0\}$$

has positive $\mathcal{H}^1$-measure. We are going to show that for $\mathcal{H}^1$-a.e. $\theta \in \Theta$ either the set

$$\{t \in (\beta_1, \alpha_1) : \sharp\{S^2 \cap r_{\theta,t}\} > 1 \text{ for } r_{\theta,t} \parallel \overline{\alpha\gamma}\} \text{ or } \{t \in (\beta_1, \alpha_1) : \sharp\{S^2 \cap r_{\theta,t}\} > 1 \text{ for } r_{\theta,t} \parallel \overline{\beta\gamma}\}$$

has positive measure. This will contradict (5.19) and hence prove (5.20).

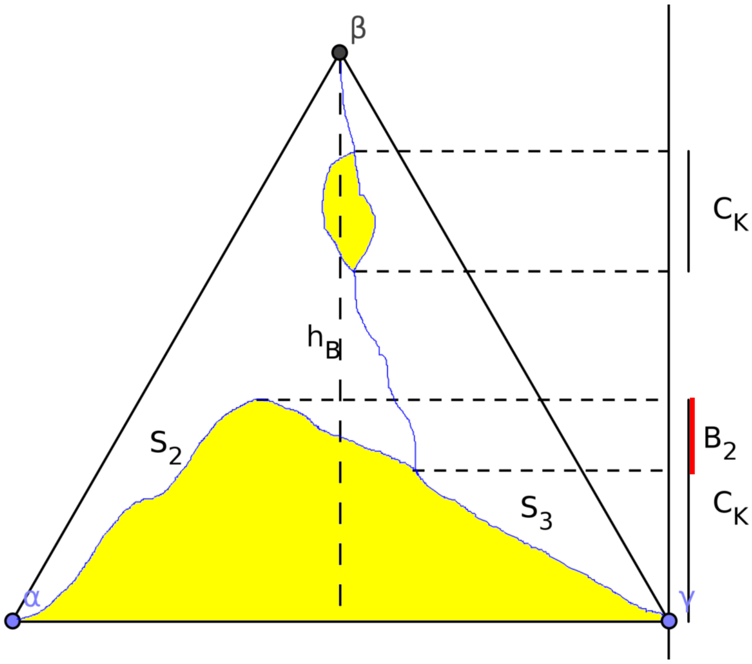

Figure 12: This figure is a section of the prism P at fixed $\theta \in (0, l_j)$. The area colored in yellow is the set $\widetilde{\mathcal{G}}_1$; on the right are represented the set C_K (black) and its subset B_2 in (5.17) (red). The dotted lines parallel to $\overline{\alpha\gamma}$ are $r_{\theta,t}$.

Let $\theta \in \Theta$ be fixed, we can find $t \in (\beta_1, \alpha_1)$ such that $\sharp\{S^2 \cap r_{s,t} : r_{s,t} \parallel h_C\} > 1$. For almost every $s \in (0, l_j)$ the section $S^2 \cap (\{s\} \times \mathbb{R}^2)$ is given by the union of Lipschitz curves $\{\gamma_i\}_{i \geq 0}$ such that γ_0 connects α and β, and $\{\gamma_i\}_{i>0}$ are closed (by the decomposition theorem for integral 1-currents, Theorem 2.1), therefore we can assume this for our choice of θ. Moreover each γ_i is injective. Since E_θ has positive measure, we can find t such that either (1) $r_{s,t}$ intersects γ_0 in two points, say P and Q, or (2) $r_{s,t}$ intersects γ_0 at one point and another curve γ_1. Let us treat the two cases separately:

(1) in this case, up to change the choice of t, we can assume that the tangent vectors to γ_0 at P and Q are defined and are not vertical, i.e. parallel to h_C (namely, the curve γ_0 crosses the lines $r_{\theta,t}$ at P and Q and is not tangent to that, see Figure 13). Since the curve γ_0 connects α to β, it is easy to see, as a consequence of the Theorem of the Jordan curve, that there must be another point, say R, in the intersection of $r_{\theta,t}$ and γ_0. Up to rename the points, suppose R stays between P and Q on the line $r_{\theta,t}$. Suppose first that R is also between P and Q on the curve γ_0 (see picture 13 left). In this case P is connected to α and Q to β, so that if P is below (above) R and Q above (below) it, we see that the line passing through R and parallel to $\overline{\alpha\gamma}$ ($\overline{\beta\gamma}$, respectively) will intersect γ_0 in three points. Instead, suppose that R is not between P and Q on the curve γ_0. Let Q be the middle point, and suppose that P is connected to β and P is below R (see picture 13 right; the other cases are similar). In such a case the line passing through R and parallel to $\overline{\beta\gamma}$ intersects γ_0 at least three times, one on the arc connecting β to P, one at R, and one in the sub-curve of γ_0 connecting P to Q.

(2) This case is simpler. Indeed all the lines parallel to $\overline{\beta\gamma}$ (and also $\overline{\alpha\gamma}$) passing through γ_1 also intersect γ_0. Moreover also almost every lines intersecting γ_1 must intersect it in at least two points (again thanks to the Theorem of the Jordan curve).

In both cases (1) and (2) we can find a set of lines parallel to $\overline{\alpha\gamma}$ or $\overline{\beta\gamma}$ intersecting S^2 in more than one point, which (suitably parametrized by coordinates in the corresponding rectangle) have a $\mathcal{H}^1$-nonnegligible measure. Since this happens for $\mathcal{H}^1$-a.e. $s \in \Theta$, and Θ

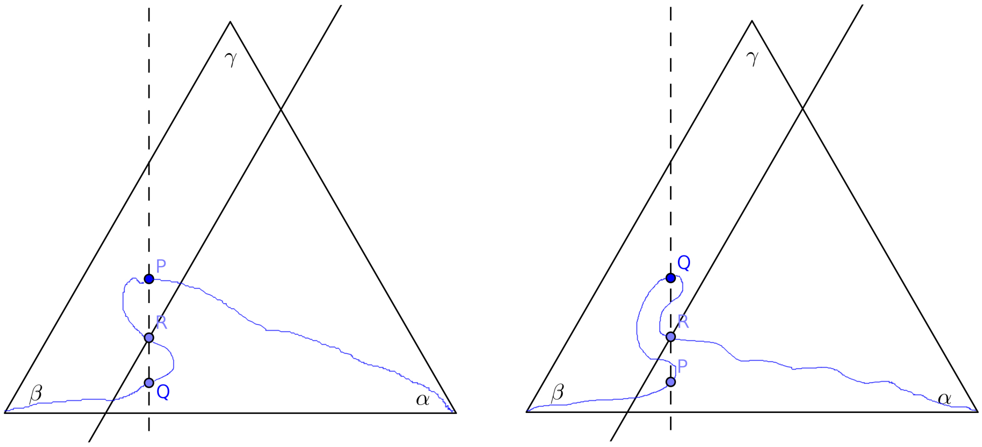

Figure 13: In this figure the case (1) of step 2 of the proof of Proposition 5.3 is depicted in two possible configurations.

has nonzero measure, by Fubini Theorem we contradict (5.19) and therefore, by absurd, get (5.20).

Step 3. Notice that assertion (5.20) holds true for all S^i, $i = 1, 2, 3$. Fix i, say $i = 1$.

$$\sharp\{S^1 \cap r_{s,t} : r_{s,t} = (s,t) \times \mathbb{R}\} = 1 \qquad \text{for } \mathcal{H}^2 - \text{a.e } (s,t) \in \mathcal{R}_1. \tag{5.21}$$

Note also that $S^1 \in \mathcal{D}_2(\mathcal{R}_1 \times \mathbb{R})$ is closed (its boundary is supported in $\partial\mathcal{R}_1 \times \mathbb{R}$). Recall that $\mathcal{R}_1 = (0, l_j) \times (\alpha_1, \gamma_1) \times \{\alpha_2\} \simeq (0, l_j) \times (\alpha_1, \gamma_1)$, again with an appropriate choice of the coordinate system (x, y_1, y_2). We assume $\alpha_2 = \gamma_2 = 0$. There exists an integral current $\mathcal{G}_1 \in \mathcal{D}_3(\mathcal{R}_1 \times \mathbb{R})$ with $\partial\mathcal{G}_1 = S^1$. Moreover there are sets U_i such that

$$\mathcal{G}_1 = \sum_h \theta_h [\![U_h]\!], \tag{5.22}$$

and it holds $S^1 = \sum_h [\![\partial U_h]\!]$. As a consequence the set $S^1 \Delta (\cup_h \partial U_h)$ has $\mathcal{H}^2$-null measure. We will prove that in this decomposition there is a unique set U_h (with boundary the whole S^1).

By (5.21) for $\mathcal{H}^2$-a.e. $(s,t) \in \mathcal{R}_1$ there is a unique point in the intersection of the vertical line $r_{s,t} = (s,t) \times \mathbb{R}$ with S^1. If this point is $Y_{s,t} := (s, t, y_2)$ we denote by $u_1(s,t) = y_2$ its last coordinate. We see that u_1 defines a map in $L^\infty(\mathcal{R}_1)$ (the measurability of u_1 easily follows from the fact that S^1 is an integral current, and thus it is the union of subsets of Lipschitz surfaces). We denote by $\pi : (s, t, y_2) \to (s,t) \in \mathcal{R}_1$ the projection of $\mathcal{R}_1 \times \mathbb{R}$ onto $\mathcal{R}_1$. From the fact that $S^1 = \cup_h \partial U_h$ (up to negligible sets) it follows that

$$\cup_h \pi(U_h) = \mathcal{R}_1. \tag{5.23}$$

By slicing it is easily seen that $U_h \cap r_{s,t}$ has boundary the unique point $Y_{s,t}$, for $\mathcal{H}^2$-a.e. $(s,t) \in \mathcal{R}_1$; hence $U_h \cap r_{s,t}$ is a halfline, either $(s,t) \times (-\infty, u_1(s,t))$ or $(s,t) \times (u_1(s,t), +\infty)$. Denote by

$$U_h^+ := \big\{(s,t,z) : z \in (u_1(s,t), +\infty),\ U_h \cap r_{s,t} = \{(s,t) \times (u_1(s,t), +\infty)\}\big\},$$

and

$$U_h^- := \big\{(s,t,z) : z \in (-\infty, u_1(s,t)),\ U_h \cap r_{s,t} = \{(s,t) \times (-\infty, u_1(s,t))\}\big\}.$$

(where equalities are intended up to negligible sets). But now it easily follows that $\partial U_h^+ = (S^1 \cap \partial U_h) \cup V_h^+$, where V_h^+ is the set

$$V_h^+ := \{(s,t,r) : (s,t) \in \partial\pi(U_h^+),\ z > u_1(s,t)\},$$

and similarly $\partial U_h^- = (S^1 \cap \partial U_h) \cup V_h^-$, with

$$V_h^- := \{(s,t,z) : (s,t) \in \partial\pi(U_h^-),\ z < u_1(s,t)\}.$$

Therefore, in order that $\partial U_h \subset S^1$ it must hold that both V_h^+ and V_h^- have null $\mathcal{H}^2$-measure in $(0,l_j) \times (\alpha_1,\gamma_1) \times \mathbb{R}$ (S^2 has support in the prism $\overline{P}$ and hence compact support while $V_h^\pm$ are unbounded). This implies that $\partial\pi(U_h^+) \cup \partial\pi(U_h^-)$ must be a subset of $\partial\mathcal{R}_1$. In particular $\partial\pi(U_h) \subset \partial\mathcal{R}_1$. This is possible only if $\pi(U_h) = \mathcal{R}_1$, and thus $\partial\pi(U_h)$ coincides with $\mathcal{R}_1$[9]. In particular we have proved that for some h we have $\pi(U_h) = \mathcal{R}_1$ and, since for every other index $i \neq h$ the set $\pi(U_i) \cap \pi(U_h)$ has null $\mathcal{H}^2$-measure (by (5.21)), we conclude that there is only one index h for which U_h has positive measure (namely, the decomposition of $\mathcal{G}_1$ in (5.22) consists of only one set, call it U). Finally, since the same argument applies to $\partial\pi(U_h^+)$ and $\partial\pi(U_h^-)$, we also have obtained that the relative sets U_h^+ and U_h^- cannot have both nonzero measure. Hence, say $U = U_h^-$ (up to change orientation of S^1).

The subgraph of u_1 is defined as the set

$$SG_1 := \{(s,t,z) \in \mathcal{R}_1 \times \mathbb{R} : z \leq u_1(s,t)\}.$$

Let $\widehat{\mathcal{G}}_1$ be the current defined as the integration on the subgraph of u, namely

$$\widehat{\mathcal{G}}_1 = [\![SG_1]\!]. \tag{5.24}$$

By definition, it turns out that $SG_1 = U_h^- = U$, and thus $\widehat{\mathcal{G}}_1$ coincides with $\mathcal{G}_1$ defined in (5.22). Therefore $\partial\widehat{\mathcal{G}}_1 = \partial\mathcal{G}_1 = S^1$. Now we invoke [11, Theorem 2, Section 4.2.4], that, combined with [11, Proposition 3, Section 4.2.4], implies that S is a Cartesian current in $\mathrm{Cart}(\mathcal{R}_1 \times \mathbb{R})$, $u_1 \in BV(\mathcal{R}_1;\mathbb{R})$, and

$$|S^1|_{\mathcal{R}_1\times\mathbb{R}} = \mathcal{A}(u_1,\mathcal{R}_1). \tag{5.25}$$

The assertion for $i = 2,3$ follows similarly. □

Lemma 5.4. *There is a minimizer $(S^1,S^2,S^3,\mathcal{T})$ satisfying the hypotheses of Proposition 5.3 such that $\mathcal{T} = 0$.*

Proof. Let $(S^1,S^2,S^3,\mathcal{T})$ be as in Lemma 5.3. Up to applying $\mathbb{S}_B$ again we can assume that S^1, S^2, S^3 are symmetric with respect to h_B. Moreover $\mathcal{V}^1 \llcorner \Pi_\alpha$ (see Definition 4.7 for Π_α) is the graph of a nondecreasing function u_1 defined on $[\alpha_1, 0]$ (here $0 = \beta_1$ is the ascissa corresponding to the segment h_B). Let P be the intersection between the curve $\mathcal{V}^1$ and h_B. Consider the segment $\overline{P\beta}$. The arc $\mathcal{V}^1 \llcorner \Pi_\alpha \cup \overline{P\beta}$ connects α to β.

The curve $\mathcal{V}^1 \llcorner \Pi_\alpha$ has $\mathcal{H}^1$-a.e. tangent vector $\vec{v}$ that forms an angle $\theta \in [0,\pi/2]$ with the segment $\overline{\alpha\gamma}$. As a consequence the angle between $\vec{v}$ and h_C is $\theta + \pi/6 \in [\pi/6, 2\pi/3]$ (see figure 14 left). This means that the curve $\mathcal{V}^1 \llcorner \Pi_\alpha \cup \overline{P\beta}$ can be seen as the graph of a function v_2 defined on the segment $\overline{\beta\alpha}$. Furthermore we know that $\mathcal{V}^2$ is the graph of a function u_2 on $\overline{\beta\alpha}$.

Notice that the current $\mathcal{T} \llcorner \Pi_\alpha$ is the integral over the area enclosed between the two graphs of u_2 and v_2. Let us denote by $\mathcal{T}_\alpha := \mathcal{T} \llcorner \Pi_\alpha$ such current. We hence redefine $\mathcal{V}^2$ as $\widehat{\mathcal{V}}^2 := \mathcal{V}^1 \llcorner \Pi_\alpha \cup \overline{P\beta}$, namely the graph of v_2. Moreover S^2 is redefined as $\widehat{S}^2 := S^2 + \mathcal{T}_\alpha$. A similar construction is made on the halfplane Π_γ and S^3 is defined in a symmetric way. It results that $\mathcal{T}$, seen as the current with boundary the new $\widehat{\mathcal{V}}^1 := \mathcal{V}^1$, $\widehat{\mathcal{V}}^2$, and $\widehat{\mathcal{V}}^3$, becomes null. Hence we infer

$$|\widehat{S}^2| + |\widehat{S}^3| \leq |S^2| + |\mathcal{T}_\alpha| + |S^3| + |\mathcal{T}_\gamma| = |S^2| + |S^3| + |\mathcal{T}|. \tag{5.26}$$

The thesis is achieved since we got a minimizer with the desired properties. □

[9] This is a consequence of the Constancy Lemma; if $\mathcal{R}_1 \setminus \pi(U_h)$ and $\pi(U_h)$ have both $\mathcal{H}^2$-positive measure, and considering sections $\mathcal{R}_s^1$ of $\mathcal{R}_1$ at s fixed, we find that for a positive $\mathcal{H}^1$-measure subset of $(0,l_j)$ the section $\mathcal{R}_s^1$ contains an inner point X_s that belongs to the mutual boundary of $\mathcal{R}_1 \setminus \pi(U_h)$ and $\pi(U_h)$; this would imply that such mutual boundary has positive $\mathcal{H}^1$-measure inside $\mathcal{R}_1$.

Consider now the baricenter O of the triangle T. Let us denote by $\Lambda_1 := \overline{\alpha O} \cup \overline{O\gamma}$, $\Lambda_3 := \overline{O\gamma} \cup \overline{O\beta}$, $\Lambda_2 := \overline{O\beta} \cup \overline{\alpha O}$.

Lemma 5.5. *There is a minimizer $(S^1, S^2, S^3, \mathcal{T})$ as in Lemma 5.4 with $\mathcal{V}^i = \Lambda_i$, for $i = 1, 2, 3$.*

Proof. Step 1. Apply $\mathbb{S}_B$ and assume $\mathcal{V}_2$ is symmetric to $\mathcal{V}_3$ with respect to h_B. By the previous lemma we have $\mathcal{V}_1 = -\mathcal{V}_2 - \mathcal{V}_3$. Let P be the (unique) intersection of $\mathcal{V}^1$ with h_B; by symmetry the segment $\overline{PB} \subset h_B$ is the common part of $\mathcal{V}_2$ and $\mathcal{V}_3$. When we apply $\mathbb{S}_A$ the point P is sent to $P' := \pi_{h_A}(P)$ the orthogonal projection of P onto h_A (see figure 14 right). Denote by $\mathcal{V}$ the union of the support of the currents $\mathcal{V}^i$. This is composed by three arcs $\mathcal{V}_A$, $\mathcal{V}_B$, $\mathcal{V}_C$ connecting P to α, β, and γ respectively. By definition of $\mathbb{S}_A$ this transforms $\mathcal{V}_A$ into $\mathbb{S}_A(\mathcal{V}_A) = \overline{\alpha P'}$. In particular $|\mathcal{V}_A| > |\mathbb{S}_A(\mathcal{V}_A)|$ if P is not on h_A (i.e. if P does not coincide with O). On the other hand it is easy to see that $|\mathcal{V}_B| + |\mathcal{V}_C| \geq |\mathbb{S}_A(\mathcal{V}_B)| + |\mathbb{S}_A(\mathcal{V}_C)|$, and therefore we arrive at

$$\sum_{i=1}^{3} |\mathcal{V}^i| > \sum_{i=1}^{3} |\mathbb{S}_A(\mathcal{V}^i)|, \tag{5.27}$$

if $P \neq O$.

Step 2. We now consider the following minimum problem:

$$\min\{\sum_i |\mathcal{V}^i| : (S^1, S^2, S^3, \mathcal{T}) \text{ is as in Lemma 5.4}\}. \tag{5.28}$$

From the features of the minimizers of problem (5.12) it is easily seen that such family is compact in the set of integral currents. Moreover, thanks to (5.8), also the corresponding currents $\{\mathcal{V}^i\}_{i=1,2,3}$ form a compact family, and hence we infer the existence of a solution of (5.28). We claim that the (not relabeled) minimizer $(S^1, S^2, S^3, \mathcal{T})$ satifies the thesis. Indeed, if not, we have two cases: $P \neq O$, and thus after applying some symmetrization operator as described in Step 1 we got a best minimizer, a contradiction. The second case is $P = O$ but some among $\mathcal{V}_B$, $\mathcal{V}_A$, $\mathcal{V}_C$ does not coincide with $\overline{O\beta}$, $\overline{O\alpha}$, or $\overline{O\gamma}$, respectively. Say $\mathcal{V}_A \neq \overline{O\alpha}$; now again $\mathbb{S}_A$ transforms $\mathcal{V}_A$ into $\mathbb{S}_A(\mathcal{V}_A) = \overline{\alpha O}$ and in particular $|\mathcal{V}_A| > |\mathbb{S}_A(\mathcal{V}_A)|$, again a contradiction.

□

We are finally ready to prove Theorem 3.7.

Proof of Theorem 3.7. We consider a minimizer $(S^1, S^2, S^3, \mathcal{T})$ as in Lemma 5.5. Let $u_1 : \mathcal{R}_1 \to \mathbb{R}$ be the map in Proposition 5.3. The graph of u_1, namely S^1, has boundary

$$\partial S^1 = -(Id \times \alpha)_\sharp [\![0, l_j]\!] + (Id \times \gamma)_\sharp [\![0, l_j]\!] + \mathcal{V}^1. \tag{5.29}$$

in $\mathcal{D}_1((-\infty, l_j) \times \mathbb{R}^2)$. Moreover, up to choose coodinates of $\mathbb{R}^2$ in such a way that $\alpha_2 = \gamma_2 = 0$ we see that the currents $(Id \times \alpha)_\sharp [\![0, l_j]\!]$ and $(Id \times \gamma)_\sharp [\![0, l_j]\!]$ are exactly the graph over $(0, l_j) \times \{\alpha_1\}$ and $(0, l_j) \times \{\gamma_1\}$ (respectively) of the function $u_1 = 0$. We also know that the current $\mathcal{V}^1$ is exactly the integration over the graph of the function φ in (1.14) on $\{0\} \times (\gamma_1, \alpha_1)$. Extending φ on $(0, l_j) \times \{\alpha_1\}$ and $(0, l_j) \times \{\gamma_1\}$ by setting $\varphi = 0$ we see that φ is then a Lipschitz function on $\partial \mathcal{R}_1 \cap R$ (set $R = (-\infty, l_j) \times \mathbb{R}$), and then it can be extended to a Lipschitz function (still denoted by φ) defined on $R \setminus \mathcal{R}_1$ (let us also take it with compact support on R, for simplicity). Consider the graph of φ over $R \setminus \mathcal{R}_1$, namely $(Id \times \varphi)_\sharp [\![R \setminus \mathcal{R}_1]\!]$; it is then easily observed that the current

$$\overline{S} := \begin{cases} S & \text{on } \mathcal{R}_1 \times \mathbb{R} \\ (Id \times \varphi)_\sharp [\![R \setminus \mathcal{R}_1]\!] & \text{on } (R \setminus \mathcal{R}_1) \times \mathbb{R}, \end{cases} \tag{5.30}$$

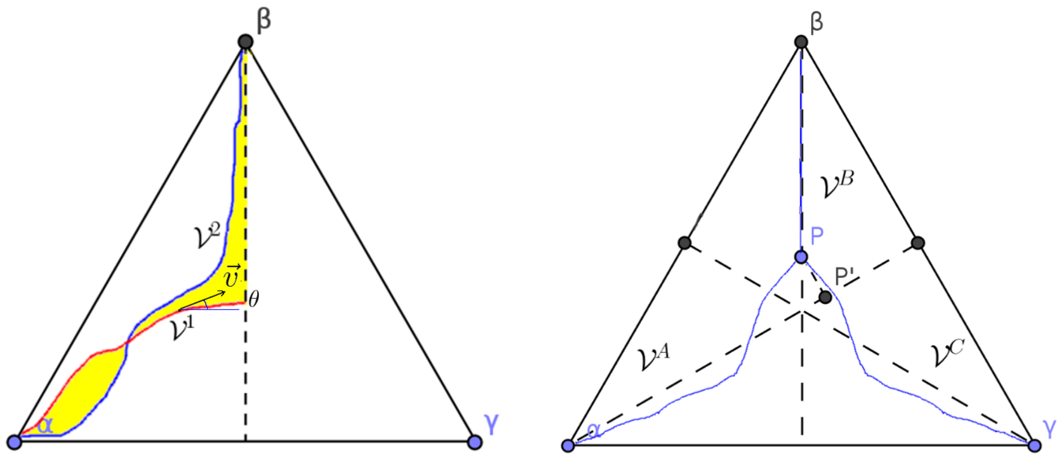

Figure 14: In the picture on the left is an example of the proof of Lemma 5.4. In yellow it is depicted the area enclosed between $\mathcal{V}^1$ and $\mathcal{V}^2$, support of the current $\mathcal{T}\llcorner\Pi_\alpha$. The tangent vector $\vec{v}$ to $\mathcal{V}^1$ forms an angle $\theta\in[0,\pi/2]$ with the line $\overline{\alpha\gamma}$. The picture on the right describes the proof of Lemma 5.5; the operator $\mathbb{S}_A$ projects P in $P'=\pi_{h_A}(P)$.

defines a Cartesian current in $\mathcal{D}_2(R\times\mathbb{R})$. We are then led to considering the following minimum problem:

$$\min\{|\widehat{S}|_{\overline{\mathcal{R}}_1\times\mathbb{R}}:\widehat{S}\in\mathrm{cart}^1(R\times\mathbb{R})\text{ and }\widehat{S}\llcorner((R\setminus\mathcal{R}_1)\times\mathbb{R})=\overline{S}\llcorner((R\setminus\mathcal{R}_1)\times\mathbb{R})\}. \tag{5.31}$$

By [12, Theorem 8, Section 6.1.2] (see also [13, Theorem 15.9]), it is well-known that this minimization problem admits a solution $\widehat{S}$, and moreover $\widehat{S}$ satisfies the following property: there exists $\widehat{u}\in BV(\mathcal{R}_1)$ such that $|S|_{\overline{\mathcal{R}}_1\times\mathbb{R}}=\mathcal{A}(\widehat{u};\overline{\mathcal{R}}_1)$, and

$$\widehat{u}\in\ \mathrm{argmin}\ \{\int_{\mathcal{R}_1}\sqrt{1+|Du|^2}dx+\int_{\partial\mathcal{R}_1\cap R}|u-\varphi|d\mathcal{H}^1:u\in BV(R)\}. \tag{5.32}$$

Finally, thanks to [5, Remark 2.1], it is observed that the minimum of the value in the last expression is exactly m_{l_j}, so that we infer $|S|_{\overline{\mathcal{R}}_1\times\mathbb{R}}=\mathcal{A}(\widehat{u};\overline{\mathcal{R}}_1)=m_{l_j}$ (the value of m_{l_j} is defined in (1.16)). From (5.31), since S^1 is a competitor, we conclude

$$|S^1|\geq m_{l_j}. \tag{5.33}$$

The same being true for S^2 and S^3, we have addressed Theorem 3.7. □

Remark 5.6. The equivalence of problems (5.31) and (5.32) only holds when the codimension of the Cartesian current is 1 (that is when we consider real valued BV-functions graphs). This is a consequence of the fact that, for $N=1$, it holds true $\mathrm{cart}^1(\Omega;\mathbb{R}^N)=\mathrm{Cart}^1(\Omega;\mathbb{R}^N)$ (see Proposition 3 in [11, Section 4.2.4]).

6 An example in a thin domain

In this section we consider the problem of the area functional in a thin domain U_b, a tubular neighborhood of the jump set of u, instead of the whole ball $B_1(0)$. The domain U_b is depicted in Figure 1 in the introduction. We then construct a sequence of Lipschitz functions $\{v_k\}$ which converges in $L^1(U_b)$ to the triple junction function u and whose area of the graphs satisfies

$$\liminf_{k\to\infty}\mathcal{A}(v_k;U_b)<\mathcal{L}^2(U_b)+3m.$$

In particular we infer that the construction made in [5] for (done for Ω the disk) do not provide a recovery sequence for the area functional, which in fact satisfies

$$\mathcal{A}(u;U_b) < \mathcal{L}^2(U_b) + 3m.$$

This dependence on the domain has been pointed out, for a different function, in [7]. As explained in the introduction, the inequality above is due to a certain interaction between the jump set of u and the boundary of the domain. This interactions has been observed indeed already in [7, Section 7] (see also the example for the vortex map in [1] upon which the examples in [7] are inspired). The main issue is the absence of uniform convergence of v_k outside the jump set.

An auxiliary construction. We start by defining an auxiliary function. Consider two fixed real numbers $h_1, h_2 > 0$. Let $x_0 < x_1 < x_2 < x_3 < x_3 < x_4 < x_5$, and $\delta, \epsilon > 0$ be real numbers with $\epsilon < h_2$, and set $d_i := x_i - x_{i-1}$ for $i = 1, \dots, 5$. In a plane with Cartesian coordinates x and y consider the rectangles A_i, of vertices (x_{i-1}, δ), (x_i, δ), $(x_i, -\delta)$, $(x_{i-1}, -\delta)$, for $i = 1, \dots, 5$, and set $R := \cup_{i=1}^5 A_i$. The set R is a rectangle with basis of width $d = \sum_{i=1}^5 d_i$ and height 2δ. We define the following partition of A_4 and A_5: write $A_i = A_i^0 \cup A_i^+ \cup A_i^-$, $i = 4, 5$, where

$$\begin{aligned}
A_4^0 &:= A_4 \cap \{(x,y) : |y| \leq \frac{\delta\epsilon}{\epsilon(1 - \frac{x-x_3}{d_4}) + h_2 \frac{x-x_3}{d_4}}\},\\
A_4^+ &:= A_4 \cap \{(x,y) : \frac{\delta\epsilon}{\epsilon(1 - \frac{x-x_3}{d_4}) + h_2 \frac{x-x_3}{d_4}} < y \leq \delta\},\\
A_4^- &:= A_4 \cap \{(x,y) : -\delta \leq y < -\frac{\delta\epsilon}{\epsilon(1 - \frac{x-x_3}{d_4}) + h_2 \frac{x-x_3}{d_4}}\},\\
A_5^0 &:= A_5 \cap \{(x,y) : |y| \leq \frac{\delta\epsilon}{h_2}(1 - \frac{x-x_4}{d_5})\},\\
A_5^+ &:= A_5 \cap \{(x,y) : \frac{\delta\epsilon}{h_2}(1 - \frac{x-x_4}{d_5}) < y \leq \delta\},\\
A_5^- &:= A_5 \cap \{(x,y) : -\delta \leq y < -\frac{\delta\epsilon}{h_2}(1 - \frac{x-x_4}{d_5})\},
\end{aligned}$$

We will now define a continuous map $v = (v_1, v_2) : R \to \mathbb{R}^2$. The first component v_1 of v is defined as follows:

$$\begin{aligned}
v_1(x,y) &= 0 && \text{on } A_1 \cup A_2,\\
v_1(x,y) &= h_1 \frac{x - x_2}{d_3} && \text{on } A_3,\\
v_1(x,y) &= h_1 && \text{on } A_4^0 \cup A_5^0,\\
v_1(x,y) &= \frac{h_1}{h_2 - \epsilon}\Big(h_2 - \frac{y}{\delta}\big(\epsilon(1 - \frac{x-x_3}{d_4}) + h_2 \frac{x-x_3}{d_4}\big)\Big) && \text{on } A_4^+,\\
v_1(x,y) &= \frac{h_1}{h_2 - \epsilon}\Big(h_2 + \frac{y}{\delta}\big(\epsilon(1 - \frac{x-x_3}{d_4}) + h_2 \frac{x-x_3}{d_4}\big)\Big) && \text{on } A_4^-,\\
v_1(x,y) &= (h_2 - \frac{h_2}{\delta} y)\frac{h_1}{h_2 - \epsilon(1 - \frac{x-x_4}{d_5})} && \text{on } A_5^+,\\
v_1(x,y) &= (h_2 + \frac{h_2}{\delta} y)\frac{h_1}{h_2 - \epsilon(1 - \frac{x-x_4}{d_5})} && \text{on } A_5^-.
\end{aligned} \tag{6.1}$$

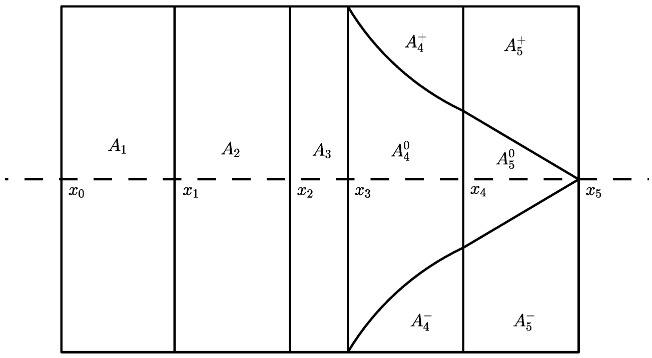

Figure 15: The rectangle R.

The component v_2 is instead defined as:

$$
\begin{aligned}
v_2(x,y) &= y\frac{h_2}{\delta} && \text{on } A_1 \cup A_5,\\
v_2(x,y) &= y\frac{h_2}{\delta}(1-\frac{x-x_1}{d_2}) + y\frac{\epsilon}{\delta}\frac{x-x_1}{d_2} && \text{on } A_2,\\
v_2(x,y) &= y\frac{\epsilon}{\delta} && \text{on } A_3,\\
v_2(x,y) &= y\frac{\epsilon}{\delta}(1-\frac{x-x_3}{d_4}) + y\frac{h_2}{\delta}\frac{x-x_3}{d_4} && \text{on } A_4.
\end{aligned} \tag{6.2}
$$

It is easily checked that the function v is Lipschitz continuous on R and has partial derivatives given by

$$
\begin{aligned}
&\frac{\partial v_1}{\partial x}(x,y) = \frac{\partial v_1}{\partial y}(x,y) = 0 && \text{on } A_1 \cup A_2 \cup A_4^0 \cup A_5^0,\\
&\frac{\partial v_1}{\partial x}(x,y) = \frac{h_1}{d_3}, \qquad \frac{\partial v_1}{\partial y}(x,y) = 0 && \text{on } A_3,\\
&\frac{\partial v_1}{\partial x}(x,y) = \frac{h_1}{h_2-\epsilon}(\frac{y\epsilon}{\delta d_4} - \frac{yh_2}{\delta d_4}) && \text{on } A_4^+,\\
&\frac{\partial v_1}{\partial x}(x,y) = -\frac{h_1}{h_2-\epsilon}(\frac{y\epsilon}{\delta d_4} - \frac{yh_2}{\delta d_4}) && \text{on } A_4^-,\\
&\frac{\partial v_1}{\partial y}(x,y) = -\frac{h_1}{\delta(h_2-\epsilon)}\big(\epsilon(1-\frac{x-x_3}{d_4}) + h_2\frac{x-x_3}{d_4}\big) && \text{on } A_4^+,\\
&\frac{\partial v_1}{\partial y}(x,y) = \frac{h_1}{\delta(h_2-\epsilon)}\big(\epsilon(1-\frac{x-x_3}{d_4}) + h_2\frac{x-x_3}{d_4}\big) && \text{on } A_4^-,\\
&\frac{\partial v_1}{\partial x}(x,y) = -\frac{(h_2-\frac{h_2}{\delta}y)\frac{\epsilon h_1}{d_5}}{\big|(\frac{x-x_4}{d_5}-1)\epsilon + h_2\big|^2} && \text{on } A_5^+,\\
&\frac{\partial v_1}{\partial x}(x,y) = -\frac{(h_2+\frac{h_2}{\delta}y)\frac{\epsilon h_1}{d_5}}{\big|(\frac{x-x_4}{d_5}-1)\epsilon + h_2\big|^2} && \text{on } A_5^-,\\
&\frac{\partial v_1}{\partial y}(x,y) = -\frac{h_2 h_1}{\delta}\frac{1}{(\frac{x-x_4}{d_5}-1)\epsilon + h_2} && \text{on } A_5^+,\\
&\frac{\partial v_1}{\partial y}(x,y) = \frac{h_2 h_1}{\delta}\frac{1}{(\frac{x-x_4}{d_5}-1)\epsilon + h_2} && \text{on } A_5^-,
\end{aligned}
$$

and

$$
\begin{aligned}
&\frac{\partial v_2}{\partial x}(x,y) = 0 && \text{on } A_1 \cup A_3 \cup A_5,\\
&\frac{\partial v_2}{\partial y}(x,y) = \frac{h_2}{\delta} && \text{on } A_1 \cup A_5,\\
&\frac{\partial v_2}{\partial y}(x,y) = \frac{\epsilon}{\delta} && \text{on } A_3,\\
&\frac{\partial v_2}{\partial x}(x,y) = y\frac{\epsilon}{\delta d_2} - y\frac{h_2}{\delta d_2}, \qquad \frac{\partial v_2}{\partial y}(x,y) = \frac{h_2}{\delta}(1-\frac{x-x_1}{d_2}) + \frac{\epsilon}{\delta}\frac{x-x_1}{d_2} && \text{on } A_2,\\
&\frac{\partial v_2}{\partial x}(x,y) = y\frac{h_2}{\delta d_4} - y\frac{\epsilon}{\delta d_4}, \qquad \frac{\partial v_2}{\partial y}(x,y) = \frac{\epsilon}{\delta}(1-\frac{x-x_3}{d_4}) + \frac{h_2}{\delta}\frac{x-x_3}{d_4} && \text{on } A_4.
\end{aligned}
$$

Moreover we can easily compute the Jacobian $J(v)$ of v which turns out to be nonzero only on sets A_3, A_5^+, and A_5^- where it holds

$$
\begin{aligned}
J(v)(x,y) &= \frac{\epsilon h_1}{\delta d_3} && \text{on } A_3,\\
J(v)(x,y) &= -\frac{(h_2 - \frac{h_2}{\delta}y)\frac{\epsilon h_1 h_2}{\delta d_5}}{\left|(\frac{x-x_4}{d_5}-1)\epsilon + h_2\right|^2} && \text{on } A_5^+,\\
J(v)(x,y) &= -\frac{(h_2 + \frac{h_2}{\delta}y)\frac{\epsilon h_1 h_2}{\delta d_5}}{\left|(\frac{x-x_4}{d_5}-1)\epsilon + h_2\right|^2} && \text{on } A_5^-.
\end{aligned}
$$

We now want to give an estimate of the area of the graph of v over R, considering a small value of ϵ, say $\epsilon < h_2/2$. Using the inequality

$$
\mathcal{A}(v,R) \le \int_R 1 + |\frac{\partial v_1}{\partial x}| + |\frac{\partial v_1}{\partial y}| + |\frac{\partial v_2}{\partial x}| + |\frac{\partial v_2}{\partial y}| + |J(v)|\, dxdy, \tag{6.3}
$$

we infer $\mathcal{A}(v,R) \le 2\delta d + \sum_{i=1}^5 I_i$, where

$$
I_i := \int_{A_i} |\frac{\partial v_1}{\partial x}| + |\frac{\partial v_1}{\partial y}| + |\frac{\partial v_2}{\partial x}| + |\frac{\partial v_2}{\partial y}| + |J(v)|\, dxdy,
$$

$i = 1,\dots,5$, $d = \sum_i d_i = x_5 - x_0$. Tedious computations lead to

$$
\begin{aligned}
I_1 &= 2h_2 d_1,\\
I_2 &= d_2 h_2 + d_2\epsilon + \delta(h_2 - \epsilon),\\
I_3 &= 2\delta h_1 + 2\epsilon d_3 + 2\epsilon h_1,\\
I_4 &= \delta h_1 + \delta h_1 \epsilon^2 + \delta(h_2-\epsilon) + d_4(h_2+\epsilon) + h_1 d_4,
\end{aligned}
$$

whereas, splitting $I_5 = I_5^1 + I_5^2$, with $I_5^2 = \int_{A_5} |J(v)| dxdy$, we can estimate

$$
\begin{aligned}
I_5^1 &\le \frac{\delta\epsilon h_1 h_2}{|h_2-\epsilon|^2} + \frac{2h_1 h_2 d_5}{|h_2-\epsilon|} + 2h_2 d_5,\\
I_5^2 &\le \frac{\epsilon h_1 h_2^2}{|h_2-\epsilon|^2}.
\end{aligned} \tag{6.4}
$$

To bound these terms we have used that $|y| \le \delta$ and $(\frac{x-x_4}{d_5}-1)\epsilon + h_2 \ge h_2 - \epsilon$ in A_5 and we integrated on the whole A_5. From (6.4) we see that there exists a constant $C > 0$ depending only on h_1 and h_2 (recall $\epsilon < h_2/2$) such that

$$
\mathcal{A}(v,R) \le C(\delta + d) + C\epsilon(d + \delta + \delta\epsilon). \tag{6.5}
$$

Geometry and construction of v. We consider the points in $\mathbb{R}^2$,

$$\alpha = (-1/2, \sqrt{3}/2), \quad \beta = (1,0), \quad \gamma = (-1/2, -\sqrt{3}/2),$$

and fix a positive real number $\eta < 1$. Notice that identifying the Cartesian plane with the complex one, we can also write $\alpha = e^{\frac{2\pi}{3}i}$, $\beta = 1$, $\gamma = e^{\frac{4\pi}{3}i}$. Let us introduce the following six halflines

$$\begin{aligned}
l_1 &= \{x < -\frac{\eta}{2},\ y = \eta\frac{\sqrt{3}}{2}\},\\
r_1 &= \{x < -\frac{\eta}{2},\ y = -\eta\frac{\sqrt{3}}{2}\},\\
l_2 &= \{x > \eta,\ y = \sqrt{3}x - \sqrt{3}\eta\},\\
r_2 &= \{x > -\frac{\eta}{2},\ y = \sqrt{3}x + \sqrt{3}\eta\},\\
l_3 &= \{x > -\frac{\eta}{2},\ y = -\sqrt{3}x - \sqrt{3}\eta\},\\
r_3 &= \{x > \eta,\ y = -\sqrt{3}x + \sqrt{3}\eta\},
\end{aligned}$$

which have endpoints in one of the points $A = \eta\alpha$, $B = \eta\beta$, $C = \eta\gamma$. Now we define three subsets of $B_1(O)$, the ball centered at the origin $O = (0,0)$ with radius 1. The set $\widetilde{\Omega}_1$ is defined as the subset of the plane which is enclosed by the two halflines l_1 and r_1, the segments $\overline{OA}$ and $\overline{OC}$, and which contains the halfaxis $\{x < 0, y = 0\}$. Then we set $\Omega_1 := \widetilde{\Omega}_1 \cap B_1(O)$. The sets Ω_2 is constructed similarly using the halflines l_2 and r_2, or in other words, is obtained clockwise rotating the set Ω_1 of an angle of $\frac{2\pi}{3}$ around O. Namely $\Omega_2 = e^{-\frac{2\pi}{3}i}\Omega_1$. Similarly, $\Omega_3 = e^{-\frac{2\pi}{3}i}\Omega_2$. Finally we set $\Omega := \cup_{i=1}^3 \Omega_i$.

Let $\xi > 0$ be a small parameter, $\xi < \eta$. Consider the triangle T^ξ with vertices $A^\xi = \xi\alpha$, $B^\xi = \xi\beta$, and $C^\xi = \xi\gamma$, and set $\Omega_i^\xi := \Omega_i \setminus T^\xi$, $i = 1,2,3$. Consider also the halflines $l_1^\xi = (\xi/\eta)l_1$, $r_1^\xi = (\xi/\eta)r_1$, which are parallel to l_1 and r_1, but have as endpoints A^ξ and B^ξ respectively. Similarly are constructed the halflines l_2^ξ, r_2^ξ, l_3^ξ, r_3^ξ, as shown in figure 16.

Let us focus now on the set Ω_1^ξ. This can be divided into three sectors

$$U_1^+ = \Omega_1 \cap \{y > \sqrt{3}\xi/2\}, \quad U_1^- = \Omega_1 \cap \{y < -\sqrt{3}\xi/2\},$$
$$U_1^0 = \Omega_1 \cap \{-\sqrt{3}\xi/2 < y < \sqrt{3}\xi/2\}.$$

Consider $x_0 < x_1 < x_2 < x_3 < x_4 < x_5 = -\xi/2$, and let $d = x_5 - x_0$. In the rectangle $R_1 := (x_0, x_5) \times (-\xi\sqrt{3}/2, \xi\sqrt{3}/2)$ we define a function v as follows

$$v \text{ is defined in (6.1) and (6.2) with } h_1 = 1/2,\ h_2 = \sqrt{3}/2, \quad \text{on } R_1. \tag{6.6}$$

In the remaining part of U_1^0 the function v is settled

$$v(x,y) = (0, y/\xi) \quad \text{on } U_1^0 \setminus R_1.$$

We then extend v on the sets U_1^+ in the following way: define

$$\begin{aligned}
V_1^+ &= U_1^+ \cap \{y - \xi < -\sqrt{3}(x - x_1)\},\\
V_2^+ &= U_1^+ \cap \{-\sqrt{3}(x - x_1) < y - \xi < -\sqrt{3}(x - x_2)\},\\
V_3^+ &= U_1^+ \cap \{-\sqrt{3}(x - x_2) < y - \xi < -\sqrt{3}(x - x_3)\},\\
V_4^+ &= U_1^+ \cap \{-\sqrt{3}(x - x_3) < y - \xi < -\sqrt{3}(x - x_4)\},\\
V_5^+ &= U_1^+ \cap \{-\sqrt{3}(x - x_4) < y - \xi < -\sqrt{3}(x - x_5)\},
\end{aligned}$$

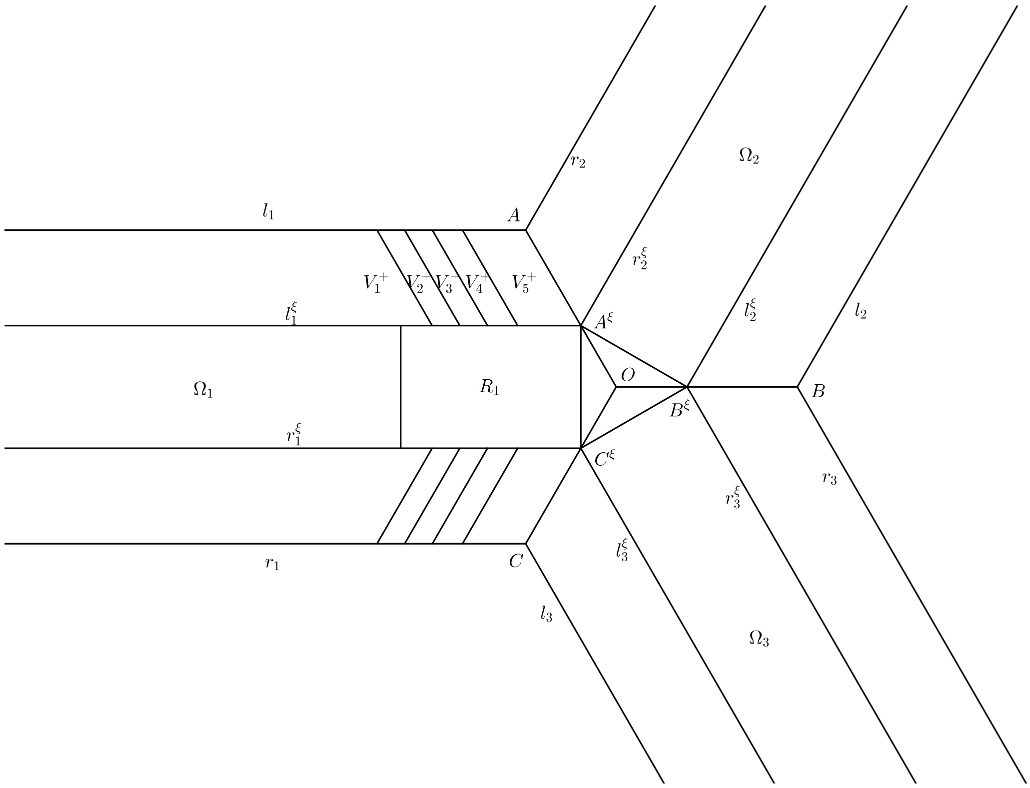

Figure 16: The thin domain $\Omega = U_b$.

and

$$
\begin{aligned}
v(x,y) &:= \big(0, \frac{\sqrt{3}}{2}\big) && \text{on } V_1^+ \cup V_5^+, \\
v(x,y) &:= \big(0, \frac{\sqrt{3}}{2}(1 - \frac{t(x) - x_1}{x_2 - x_1}) + \epsilon \frac{t(x) - x_1}{x_2 - x_1}\big) && \text{on } V_2^+, \\
v(x,y) &:= \big(\frac{1}{2}\frac{t(x) - x_2}{x_3 - x_2}, \epsilon\big) && \text{on } V_3^+, \\
v(x,y) &:= \big(\frac{1}{2}(1 - \frac{t(x) - x_3}{x_4 - x_3}), \epsilon(1 - \frac{t(x) - x_3}{x_4 - x_3}) + \frac{\sqrt{3}}{2}\frac{t(x) - x_3}{x_4 - x_3}\big) && \text{on } V_4^+,
\end{aligned}
$$

where, for brevity, we have set $t(x) = x + (y - \xi)\frac{1}{\sqrt{3}}$. In other words, the variable v_1 is constantly 0 on V_1^+, V_2^+, V_5^+, constantly $\frac{1}{2}$ on the common boundary of V_3^+ and V_4^+, and affine on V_3^+ and V_4^+. As for the variable v_2, it equals $\frac{\sqrt{3}}{2}$ on V_1^+ and V_5^+, equals ϵ on V_3^+, and is affine on V_2^+ and V_4^+. Moreover if z is the new variable $z := -\sqrt{3}x - y$, so that the line $\overline{OA}$ corresponds to the set where $z = 0$, we see that v on U_+^1 depends only on z, and it holds

$$
\begin{aligned}
&\frac{\partial v}{\partial z}(x,y) = \big(0,0\big) && \text{on } V_1^+,\\
&\frac{\partial v}{\partial z}(x,y) = \big(0, \frac{\sqrt{3}-2\epsilon}{2\sqrt{3}(x_2-x_1)}\big) && \text{on } V_2^+,\\
&\frac{\partial v}{\partial z}(x,y) = \big(-\frac{1}{2\sqrt{3}(x_3-x_2)},0\big) && \text{on } V_3^+,\\
&\frac{\partial v}{\partial z}(x,y) = \big(\frac{1}{2\sqrt{3}(x_4-x_3)},-\frac{\sqrt{3}-2\epsilon}{2\sqrt{3}(x_4-x_3)}\big) && \text{on } V_4^+,\\
&\frac{\partial v}{\partial z}(x,y) = \big(0,0\big) && \text{on } V_5^+.
\end{aligned}
$$

In U_1^- the function v is defined in such a way that v_1 is even with respect to the variable y, and v_2 is odd with respect to y.

We also write, in complex coordinates, $v = v_1 + iv_2$, and we set

$$\widetilde{v} := v - 1/2.$$

For convenience we still denote $\widetilde{v}$ by v. Notice that the function v is equal to $e^{\frac{2\pi}{3}i}$ on V_1^+ and V_5^+, and is equal to $e^{\frac{4\pi}{3}i}$ on V_1^- and V_5^-.

We now define v on Ω_2^ξ and Ω_3^ξ. In the complex coordinate $\omega \in \mathbb{C}$, this is defined as follows

$$
\begin{aligned}
&v(\omega) = e^{-\frac{2\pi}{3}i} v(e^{\frac{2\pi}{3}i}\omega) && \text{on } \Omega_2^\xi,\\
&v(\omega) = e^{-\frac{4\pi}{3}i} v(e^{\frac{4\pi}{3}i}\omega) && \text{on } \Omega_3^\xi.
\end{aligned}
\tag{6.7}
$$

It is easily checked that the function v is continuous on $\cup_{i=1}^3 \Omega_i^\xi$ and on the common boundaries of Ω_i^ξ, $i = 1,2,3$.

It remains to define v in the triangle T^ξ. Let T_1^ξ, T_2^ξ, and T_3^ξ be the midpoints of the edges of T^ξ, namely

$$T_1^\xi = -\frac{\xi}{2}, \qquad T_2^\xi = \xi e^{\frac{\pi}{6}i}, \qquad T_3^\xi = \xi e^{\frac{5\pi}{6}i}.$$

Notice that $v = 0$ at T_i^ξ. We set $v = 0$ on the triangle with vertices T_i^ξ, $i = 1,2,3$. Finally since $v = e^{\frac{2\pi}{3}i}$ at A^ξ, we set v to be linear in the triangle with vertices A^ξ, T_1^ξ, T_2^ξ. Similarly v is defined in the remaining triangles. It is straightforward to check that with such definition v is Lipschitz continuous.

We now want to compute the area of the graph associated to the map v on Ω. By symmetry, the area associated to the domains Ω_i, $i = 1,2,3$, are equal. Let us first estimate the area in U_1^0. In the rectangle R_1 we can use formula (6.5) with $d = x_5 - x_0$, $h_1 = \frac{1}{2}$, $h_2 = \sqrt{3}/2$, so that we find an absolute constant $C > 0$ such that

$$|\mathcal{G}_v|_{R_1\times\mathbb{R}^2} \le C(\xi + d) + C(d+\xi)\epsilon. \tag{6.8}$$

In $U_1^0 \setminus R_1$ the only nonzero component of the gradient of v is $\frac{\partial v_2}{\partial y} = \frac{1}{\xi}$, and thus

$$|\mathcal{G}_v|_{(U_1^0\setminus R_1)\times\mathbb{R}^2} \le \sqrt{3}\xi(1-d-\frac{\xi}{2}) + \sqrt{3}(1-d-\frac{\xi}{2}). \tag{6.9}$$

Let us now estimate the contribution on U_1^+. Using the inequality (6.3) and the values of the derivatives computed above, we easily get

$$|\mathcal{G}_v|_{U_1^+\times\mathbb{R}^2} \le \mathcal{L}^2(U_1^+) + \frac{(\eta-\xi)}{\sqrt{3}}\big(\sqrt{3}-2\epsilon+1\big). \tag{6.10}$$

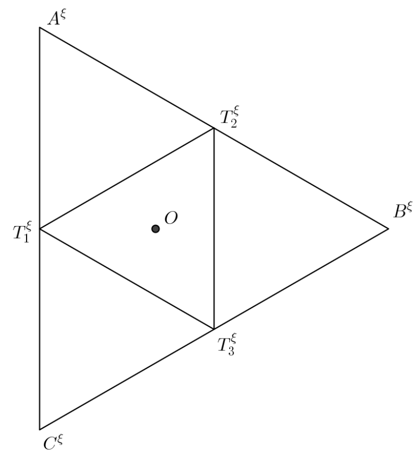

Figure 17: The triangle T^ξ.

The same estimate holds true in U_1^-. Finally the contribution in the triangle T^ξ is easily computed. Indeed all the derivatives are zero in the triangle with vertices T_i^ξ, $i = 1, 2, 3$, and using the linearity of v in the triangle with vertices A^ξ, T_1^ξ, T_2^ξ we get

$$|\mathcal{G}_v|_{T^\xi \times \mathbb{R}^2} = \mathcal{L}^2(T^\xi) + \frac{3\sqrt{3}}{4}\xi. \tag{6.11}$$

Summing all the bounds obtained so far, we infer that there is a constant C with

$$|\mathcal{G}_v| \leq \mathcal{L}^2(\Omega) + C(\xi + d + \epsilon + d\epsilon + \epsilon^2) + C\eta + 3\sqrt{3}. \tag{6.12}$$

The example. Let us introduce a parameter $k \in \mathbb{N}$ and let us choose a sequence ξ_k, d_k, ϵ_k of positive real numbers converging to 0. Let $v_k : \Omega \to \mathbb{R}^2$ be the Lipschitz function corresponding to these values. The functions v_k are almost everywhere converging to the function $u : \Omega \to \{\alpha, \beta, \gamma\}$ given by (1.10) restricted to the thin domain Ω. Moreover, since v_k are uniformly bounded in L^∞, they are converging to u in $L^1(\Omega; \mathbb{R}^2)$. Inequality (6.12) provides

$$\mathcal{A}(u, \Omega) \leq |\mathcal{G}_{v_k}|_\Omega \leq \mathcal{L}^2(\Omega) + C(\xi_k + d_k + \epsilon_k + d_k\epsilon_k + \epsilon_k^2) + C\eta + 3\sqrt{3}. \tag{6.13}$$

Passing to the limit in $k \to \infty$ we get

$$\mathcal{A}(u, \Omega) \leq \mathcal{L}^2(\Omega) + C\eta + 3\sqrt{3}. \tag{6.14}$$

Exploiting now the fact that $m > \sqrt{3}$, we can choose η small enough so that

$$\mathcal{A}(u, \Omega) \leq \mathcal{L}^2(\Omega) + C\eta + 3\sqrt{3} < \mathcal{L}^2(\Omega) + 3m.$$

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References

[1] E. Acerbi, G. Dal Maso, *New lower semicontinuity results for polyconvex integrals*. Calc. Var. Partial Differential Equations **2** (1994), 329-372.

[2] J. M. Ball, *Convexity conditions and existence theorems in nonlinear elasticity*, Arch. Rational Mech. Anal. **63** (1977), 337-403.

[3] J. M. Ball, F. Murat, *$W^{1,p}$-quasiconvexity and variational problems for multiple integrals*, J. Funct. Anal. **58** (1984), 225-253, and **66** (1986), 439-439.

[4] J. M. Ball, F. Murat, *Remarks on Chacon's Biting Lemma*, Proc. Amer. Math. Soc., **107**(3), (1989), 655-663.

[5] G. Bellettini, M. Paolini, *On the area of the graph of a singular map from the plane to the plane taking three values.* Advances in Calculus of Variations, **3**(4), (2010), 371-386.

[6] G. Bellettini, M. Paolini, L. Tealdi, *On the area of the graph of a piecewise smooth map from the plane to the plane with a curve discontinuity*, ESAIM: Control Optim. Calc. Var, **22**, 29-63, (2016).

[7] G. Bellettini, M. Paolini, L. Tealdi, *Semicartesian surfaces and the relaxed area of maps from the plane to the plane with a line discontinuity*, Ann. Mat. Pura Appl., **195**(6), 2131-2170, (2016).

[8] G. Dal Maso, *Integral representation on $BV(\Omega)$ of Γ-limits of variational integrals*, Manuscripta Math., **30**, (1979), 387-416.

[9] E. De Giorgi, *On the relaxation of functionals defined on cartesian manifolds*, in Developments in Partial Differential Equations and Applications in Mathematical Physics (Ferrara 1992), Plenum Press, New York (1992).

[10] H. Federer, *Geometric measure theory*, Springer-Verlag Berlin Heidelberg (1996).

[11] M. Giaquinta, G. Modica, J. Soucek, *Cartesian currens in the calculus of variations*, Volume I, Springer-Verlag Berlin Heidelberg (1998).

[12] M. Giaquinta, G. Modica, J. Soucek, *Cartesian currens in the calculus of variations*, Volume II, Springer-Verlag Berlin Heidelberg (1998).

[13] E. Giusti, *Minimal Surfaces and Functions of Bounded Variation*, Birkäuser, Boston, (1984).

[14] S. G. Krantz, H. R. Parks, *Geometric integration theory*, Springer Birkäuser, (2008).

[15] F. Maggi, *Sets of Finite Perimeter and Geometric Variational Problems*, Cambridge University Press, (2012).

[16] C. B. Jr. Morrey, *Multiple Integrals in the Calculus of Variations*, Springer, Berlin Heidelberg New York, (1966).