NONTRIVIAL SOLUTIONS OF QUASILINEAR ELLIPTIC EQUATIONS WITH NATURAL GROWTH CONDITIONS

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Dedicated to Gianni Gilardi

Abstract. We prove the existence of multiple solutions for a quasilinear elliptic equation containing a term with natural growth, under assumptions that are invariant by diffeomorphism. To this purpose we develop an adaptation of degree theory.

1. Introduction

Consider the quasilinear elliptic problem

$$
(1.1)\qquad \begin{cases} -\mathrm{div}\,[A(x,u)\nabla u] + B(x,u)|\nabla u|^2 = g(x,u) & \text{in } \Omega\,, \\ u = 0 & \text{on } \partial\Omega\,, \end{cases}
$$

where Ω is a bounded, connected and open subset of $\mathbb{R}^n$ and

$$A, B, g : \Omega \times \mathbb{R} \to \mathbb{R}$$

are Carathéodory functions such that:

(h_1) *for every $R > 0$ there exist $\beta_R > 0$ and $\nu_R > 0$ satisfying*

$$\nu_R \leq A(x,s) \leq \beta_R\,, \qquad |B(x,s)| \leq \beta_R\,, \qquad |g(x,s)| \leq \beta_R\,|s|\,,$$

for a.e. $x \in \Omega$ and every $s \in \mathbb{R}$ with $|s| \leq R$.

The existence of a weak solution $u \in W_0^{1,2}(\Omega) \cap L^\infty(\Omega)$ of (1.1), namely a solution of

$$
(1.2)\qquad \int_\Omega \left[A(x,u)\nabla u \cdot \nabla v + B(x,u)|\nabla u|^2\, v\right]\, dx = \int_\Omega g(x,u) v\, dx
$$

$$\text{for any } v \in W_0^{1,2}(\Omega) \cap L^\infty(\Omega)\,,$$

follows from the results of [6, 7], provided that a suitable *a priori* L^∞-estimate holds, possibly related to the existence of a pair of sub-/super-solutions (see e.g. [6, Théorème 2.1] and [7, Theorems 1 and 2]). Moreover, each weak solution $u \in W_0^{1,2}(\Omega) \cap L^\infty(\Omega)$ is locally Hölder continuous in Ω (see e.g. [10, Theorem VII.1.1]).

The existence of multiple solutions, in the semilinear case and under suitable regularity assumptions, has also been proved for instance in [2]. Here we are interested in the existence of multiple nontrivial solutions, as (h_1) implies that $g(x, 0) = 0$, under assumptions that do not imply an *a priori* $W^{1,\infty}$-estimate. Let us state our main result.

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Theorem 1.1. *Assume* (h_1) *and also that:*

(h_2) *there exist* $\underline{M} < 0 < \overline{M}$ *such that*

$$g(x, \underline{M}) \geq 0 \geq g(x, \overline{M}) \qquad \text{for a.e. } x \in \Omega\,;$$

(h_3) *the function* $g(x, \cdot)$ *is differentiable at* $s = 0$ *for a.e.* $x \in \Omega$.

Consider the eigenvalue problem

$$\begin{cases} -\text{div}\,[A(x,0)\nabla u] - D_s g(x,0)u = \lambda u & \text{in } \Omega\,, \\ u = 0 & \text{on } \partial\Omega\,, \end{cases} \tag{1.3}$$

and denote by (λ_k), $k \geq 1$, *the sequence of the eigenvalues repeated according to multiplicity.*

If there exists $k \geq 2$ *with* $\lambda_k < 0 < \lambda_{k+1}$ *and* k *even, then problem* (1.2) *admits at least three nontrivial solutions* u_1, u_2, u_3 *in* $W_0^{1,2}(\Omega) \cap L^\infty(\Omega) \cap C(\Omega)$ *with*

$$u_1 < 0 \text{ in } \Omega, \qquad u_2 > 0 \text{ in } \Omega, \qquad u_3 \text{ sign-changing}.$$

If A is constant and $B = 0$, the result is essentially contained in [3, 15], which in turn developed previous results of [4]. Actually, in those papers it is enough to assume that $\lambda_2 < 0$, because in that case (1.2) is the Euler-Lagrange equation of a suitable functional and variational methods, e.g. Morse theory, can be applied.

In our case there is no functional and also degree theory arguments cannot be applied in a standard way, because of the presence of the term $B(x,u)|\nabla u|^2$. Let us point out that our assumptions do not imply that the solutions u of (1.2) belong to $W^{1,\infty}(\Omega)$, so that the natural growth term $B(x,u)|\nabla u|^2$ plays a true role.

Let us also mention that our statement has an invariance property.

Remark 1.2. Let $\varphi : \mathbb{R} \to \mathbb{R}$ be an increasing C^2-diffeomorphism with $\varphi(0) = 0$. Then the following facts hold:

(*a*) the functions A, B and g satisfy the assumptions of Theorem 1.1 if and only A^φ, B^φ and g^φ, defined as

$$\begin{aligned} A^\varphi(x,s) &= (\varphi'(s))^2\, A(x,\varphi(s))\,, \\ B^\varphi(x,s) &= \varphi'(s)\varphi''(s)A(x,\varphi(s)) + (\varphi'(s))^3 B(x,\varphi(s))\,, \\ g^\varphi(x,s) &= \varphi'(s)g(x,\varphi(s))\,, \end{aligned}$$

do the same; in particular, we have

$$A^\varphi(x,0) = (\varphi'(0))^2\, A(x,0)\,, \qquad D_s g^\varphi(x,0) = (\varphi'(0))^2\, D_s g(x,0)\,;$$

(*b*) a function $u \in W_0^{1,2}(\Omega) \cap L^\infty(\Omega)$ is a solution of (1.2) if and only if $\varphi^{-1}(u)$ is a solution of (1.2) with A, B and g replaced by A^φ, B^φ and g^φ, respectively.

From the definition of B^φ we see that the term $B(x,u)|\nabla u|^2$ cannot be omitted in the equation, if we want to ensure this kind of invariance.

When (1.2) is the Euler-Lagrange equation associated to a functional, the question of invariance under suitable classes of diffeomorphisms has been already treated in [14], where it is shown that problems with degenerate coercivity can be reduced, in some cases, to coercive problems.

In the next sections we develop an adaptation of degree theory suited for our setting and then we prove Theorem 1.1 by a degree argument. Under assumptions that are not diffeomorphism-invariant, a degree theory for quasilinear elliptic equations with natural growth conditions has been already developed in [1]. Here we find it more convenient to reduce the equation (1.2) to a variational inequality possessing as obstacles a pair of sub-/super-solutions, according to an approach already considered for instance in [12].

2. Topological degree in reflexive Banach spaces

Let X be a reflexive real Banach space.

Definition 2.1. A map $F : D \to X'$, with $D \subseteq X$, is said to be *of class* $(S)_+$ if, for every sequence (u_k) in D weakly convergent to some u in X with

$$\limsup_k \langle F(u_k), u_k - u\rangle \leq 0\,,$$

it holds $\|u_k - u\| \to 0$.

More generally, if T is a metrizable topological space, a map $H : D \to X'$, with $D \subseteq X \times T$, is said to be *of class* $(S)_+$ if, for every sequence (u_k, t_k) in D with (u_k) weakly convergent to u in X, (t_k) convergent to t in T and

$$\limsup_k \langle H_{t_k}(u_k), u_k - u\rangle \leq 0\,,$$

it holds $\|u_k - u\| \to 0$ (we write $H_t(u)$ instead of $H(u,t)$).

Assume now that U is a bounded and open subset of X, $F : \overline{U} \to X'$ a bounded and continuous map of class $(S)_+$, K a closed and convex subset of X and $\varphi \in X'$. We aim to consider the variational inequality

$$\begin{cases} u \in K\,, \\ \langle F(u), v - u\rangle \geq \langle \varphi, v - u\rangle & \forall v \in K\,. \end{cases} \tag{2.1}$$

Remark 2.2. It is easily seen that the set

$$\left\{u \in \overline{U}:\ u \text{ is a solution of } (2.1)\right\}$$

is compact (possibly empty).

According to [8, 11, 13], if the variational inequality (2.1) has no solution $u \in \partial U$, one can define the topological degree

$$\deg((F, K), U, \varphi) \in \mathbb{Z}\,.$$

Let us recall some basic properties.

Proposition 2.3. *If* (2.1) *has no solution* $u \in \partial U$*, then*

$$\deg((F, K), U, \varphi) = \deg((F - \varphi, K), U, 0)\,.$$

Proposition 2.4. *If* (2.1) *has no solution* $u \in \partial U$*,* $u_0 \in X$ *and we set*

$$\begin{aligned} \widehat{U} \quad &= \{u - u_0:\ u \in U\}\,, \\ \widehat{K} \quad &= \{u - u_0:\ u \in K\}\,, \\ \widehat{F}(u) &= F(u_0 + u)\,, \end{aligned}$$

then

$$\deg((\widehat{F}, \widehat{K}), \widehat{U}, \varphi) = \deg((F, K), U, \varphi)\,.$$

Theorem 2.5. *If* (2.1) *has no solution* $u \in \overline{U}$, *then* $\deg((F,K),U,\varphi) = 0$.

Theorem 2.6. *If* (2.1) *has no solution* $u \in \partial U$ *and there exists* $u_0 \in K \cap U$ *such that*

$$\langle F(v), v - u_0\rangle \geq \langle \varphi, v - u_0\rangle \qquad \text{for any } v \in K \cap \partial U\,,$$

then $\deg((F,K),U,\varphi) = 1$.

Theorem 2.7. *If* U_0 *and* U_1 *are two disjoint open subsets of* U *and* (2.1) *has no solution* $u \in \overline{U} \setminus (U_0 \cup U_1)$, *then*

$$\deg((F,K),U,\varphi) = \deg((F,K),U_0,\varphi) + \deg((F,K),U_1,\varphi)\,.$$

Definition 2.8. Let K_k, K be closed and convex subsets of X. The sequence (K_k) is said to be *Mosco-convergent* to K if the following facts hold:

(a) if $k_j \to \infty$, $u_{k_j} \in K_{k_j}$ for any $j \in \mathbb{N}$ and (u_{k_j}) is weakly convergent to u in X, then $u \in K$;

(b) for every $u \in K$ there exist $\overline{k} \in \mathbb{N}$ and a sequence (u_k) strongly convergent to u in X with $u_k \in K_k$ for any $k \geq \overline{k}$.

Theorem 2.9. *Let* W *be a bounded and open subset of* $X \times [0,1]$, $H : \overline{W} \to X'$ *be a bounded and continuous map of class* $(S)_+$ *and* (K_t), $0 \leq t \leq 1$, *be a family of closed and convex subsets of* X *such that, for every sequence* (t_k) *convergent to* t *in* $[0,1]$, *the sequence* (K_{t_k}) *is Mosco-convergent to* K_t.

Then the following facts hold:

(a) *the set of pairs* $(u,t) \in \overline{W}$, *satisfying*

$$\begin{cases} u \in K_t\,, \\ \langle H_t(u), v - u\rangle \geq \langle \varphi, v - u\rangle \qquad \forall v \in K_t\,, \end{cases} \tag{2.2}$$

is compact (possibly empty);

(b) *if the problem* (2.2) *has no solution* $(u,t) \in \partial_{X\times[0,1]} W$ *and we set*

$$W_t = \{u \in X :\ (u,t) \in W\}\,,$$

then $\deg((H_t,K_t),W_t,\varphi)$ *is independent of* $t \in [0,1]$.

Proof. If (u_k,t_k) is a sequence in $\overline{W}$ constituted by solutions of (2.2), then up to a subsequence (u_k) is weakly convergent to some u in X and (t_k) is convergent to some t in $[0,1]$. Then $u \in K_t$ and there exists a sequence $(\hat{u}_k)$ strongly convergent to u in X with $\hat{u}_k \in K_{t_k}$. It follows

$$\begin{aligned} \langle H_{t_k}(u_k), u_k - u\rangle &= \langle H_{t_k}(u_k), \hat{u}_k - u\rangle + \langle H_{t_k}(u_k), u_k - \hat{u}_k\rangle \\ &\leq \langle H_{t_k}(u_k), \hat{u}_k - u\rangle + \langle \varphi, u_k - \hat{u}_k\rangle\,, \end{aligned}$$

whence

$$\limsup_k \langle H_{t_k}(u_k), u_k - u\rangle \leq 0\,.$$

Then $\|u_k - u\| \to 0$ and $(u,t) \in \overline{W}$. For every $v \in K_t$ there exists a sequence (v_k) strongly convergent to v in X with $v_k \in K_{t_k}$. From

$$\langle H_{t_k}(u_k), v_k - u_k\rangle \geq \langle \varphi, v_k - u_k\rangle$$

it follows

$$\langle H_t(u), v - u\rangle \geq \langle \varphi, v - u\rangle$$

so that the set introduced in assertion (a) is compact.

Assume now that the problem (2.2) has no solution $(u,t) \in \partial_{X\times[0,1]} W$. It is enough to prove that $\{t \mapsto \deg((H_t, K_t), W_t, \varphi)\}$ is locally constant.

Suppose first that $K_t \neq \emptyset$ for any $t \in [0,1]$. By Michael selection theorem (see e.g. [5, Theorem 1.11.1]) there exists a continuous map $\gamma : [0,1] \to X$ such that $\gamma(t) \in K_t$ for any $t \in [0,1]$.

If we set

$$\begin{aligned} \widehat{W} &= \{(u-\gamma(t), t) : \ (u,t) \in W\} \ , \\ \widehat{K}_t &= \{u - \gamma(t) : \ u \in K_t\} \ , \\ \widehat{H}_t(u) &= H_t(\gamma(t) + u) \, , \end{aligned}$$

then $\widehat{W}$, $\widehat{K}_t$ and $\widehat{H}$ satisfy the same assumptions and

$$\deg((\widehat{H}_t, \widehat{K}_t), \widehat{W}_t, \varphi) = \deg((H_t, K_t), W_t, \varphi)$$

by Proposition 2.4. Moreover $0 \in \widehat{K}_t$ for any $t \in [0,1]$. Therefore we may assume, without loss of generality, that $0 \in K_t$ for any $t \in [0,1]$.

Given $t \in [0,1]$, there exist a bounded and open subset U of X and $\delta > 0$ such that

$$U \times ([t-\delta, t+\delta] \cap [0,1]) \subseteq W$$

and such that (2.2) has no solution (u,τ) in

$$W \setminus (U \times [t-\delta, t+\delta])$$

with $t - \delta \leq \tau \leq t + \delta$. From Theorem 2.7 we infer that

$$\deg((H_\tau, K_\tau), W_\tau, \varphi) = \deg((H_\tau, K_\tau), U, \varphi) \qquad \text{for any } \tau \in [t-\delta, t+\delta] \, .$$

From [11, Theorem 4.53 and Proposition 4.61] we deduce that $\{\tau \mapsto \deg((H_\tau, K_\tau), U, \varphi)\}$ is constant on $[t-\delta, t+\delta]$.

In general, given $t \in [0,1]$, let us distinguish the cases $K_t \neq \emptyset$ and $K_t = \emptyset$.

If $K_t \neq \emptyset$, by the Mosco-convergence there exists $\delta > 0$ such that $K_\tau \neq \emptyset$ for any $\tau \in [t-\delta, t+\delta]$. By the previous step we infer that $\{\tau \mapsto \deg((H_\tau, K_\tau), W_\tau, \varphi)\}$ is constant on $[t-\delta, t+\delta]$.

If $K_t = \emptyset$, from Theorem 2.5 we infer that $\deg((H_t, K_t), W_t, \varphi) = 0$. Assume, for a contradiction, that there exists a sequence (t_k) convergent to t with $\deg((H_{t_k}, K_{t_k}), W_{t_k}, \varphi) \neq 0$ for any $k \in \mathbb{N}$. Again from Theorem 2.5 we infer that the problem (2.2) has a solution $(u_k, t_k) \in \overline{W}$, in particular $u_k \in K_{t_k}$, for any $k \in \mathbb{N}$. Up to a subsequence, (u_k) is weakly convegent to some u, whence $u \in K_t$ by the Mosco-convergence, and a contradiction follows. □

Now let Ω be a bounded and open subset of $\mathbb{R}^n$, let T be a metrizable topological space and let

$$a : \Omega \times (\mathbb{R} \times \mathbb{R}^n \times T) \to \mathbb{R}^n \, ,$$

$$b : \Omega \times (\mathbb{R} \times \mathbb{R}^n \times T) \to \mathbb{R}$$

be two Carathéodory functions. We will denote by $\| \ \|_p$ the usual norm in L^p and write $a_t(x,s,\xi)$, $b_t(x,s,\xi)$ instead of $a(x,(s,\xi,t))$, $b(x,(s,\xi,t))$.

In this section, we assume that a_t and b_t satisfy the *controllable growth conditions* in the sense of [10], uniformly with respect to t. In a simplified form enough for our purposes, this means that:

(UC) *there exist* $p \in]1, \infty[$, $\alpha^{(0)} \in L^1(\Omega)$, $\alpha^{(1)} \in L^{p'}(\Omega)$, $\beta > 0$ *and* $\nu > 0$ *such that*

$$|a_t(x,s,\xi)| \leq \alpha^{(1)}(x) + \beta\,|s|^{p-1} + \beta\,|\xi|^{p-1}\,,$$

$$|b_t(x,s,\xi)| \leq \alpha^{(1)}(x) + \beta\,|s|^{p-1} + \beta\,|\xi|^{p-1}\,,$$

$$a_t(x,s,\xi)\cdot\xi \geq \nu|\xi|^p - \alpha^{(0)}(x) - \beta\,|s|^p\,,$$

for a.e. $x \in \Omega$ *and every* $s \in \mathbb{R}$, $\xi \in \mathbb{R}^n$, $t \in T$*; such a* p *is clearly unique.*

It follows

$$\begin{cases} a_t(x,u,\nabla u) \in L^{p'}(\Omega) \\ b_t(x,u,\nabla u) \in L^{p'}(\Omega) \subseteq W^{-1,p'}(\Omega) \end{cases} \qquad \text{for any } t \in T \text{ and } u \in W_0^{1,p}(\Omega)$$

and the map $H : W_0^{1,p}(\Omega) \times T \to W^{-1,p'}(\Omega)$ defined by

$$H_t(u) = -\mathrm{div}\,[a_t(x,u,\nabla u)] + b_t(x,u,\nabla u)$$

is continuous and bounded on $B \times T$, whenever B is bounded in $W_0^{1,p}(\Omega)$.

Theorem 2.10. *Assume* (UC) *and also that:*

(UM) *we have*

$$\left[a_t(x,s,\xi) - a_t(x,s,\hat{\xi})\right]\cdot(\xi - \hat{\xi}) > 0$$

for a.e. $x \in \Omega$ *and every* $s \in \mathbb{R}$, $\xi, \hat{\xi} \in \mathbb{R}^n$, $t \in T$, *with* $\xi \neq \hat{\xi}$.

Then $H : W_0^{1,p}(\Omega) \times T \to W^{-1,p'}(\Omega)$ *is of class* $(S)_+$.

Proof. See e.g. [13, Theorem 1.2.1]. □

3. Quasilinear elliptic variational inequalities with natural growth conditions

Again, let Ω be a bounded and open subset of $\mathbb{R}^n$ and let now

$$a : \Omega \times (\mathbb{R} \times \mathbb{R}^n) \to \mathbb{R}^n\,,$$

$$b : \Omega \times (\mathbb{R} \times \mathbb{R}^n) \to \mathbb{R}$$

be two Carathéodory functions. In this paper we are interested in the case in which a and b satisfy the *natural growth conditions* in the sense of [10]. More precisely, we assume that:

(N) *there exist* $p \in]1, \infty[$ *and, for every* $R > 0$, $\alpha_R^{(0)} \in L^1(\Omega)$, $\alpha_R^{(1)} \in L^{p'}(\Omega)$, $\beta_R > 0$ *and* $\nu_R > 0$ *such that*

$$|a(x,s,\xi)| \leq \alpha_R^{(1)}(x) + \beta_R\,|\xi|^{p-1}\,,$$

$$|b(x,s,\xi)| \leq \alpha_R^{(0)}(x) + \beta_R\,|\xi|^p\,,$$

$$a(x,s,\xi)\cdot\xi \geq \nu_R\,|\xi|^p - \alpha_R^{(0)}(x)\,,$$

for a.e. $x \in \Omega$ *and every* $s \in \mathbb{R}$, $\xi \in \mathbb{R}^n$ *with* $|s| \leq R$*; such a* p *is clearly unique;*

(M) *we have*

$$\left[a(x,s,\xi) - a(x,s,\hat{\xi})\right] \cdot (\xi - \hat{\xi}) > 0$$

for a.e. $x \in \Omega$ *and every* $s \in \mathbb{R}$, $\xi, \hat{\xi} \in \mathbb{R}^n$ *with* $\xi \neq \hat{\xi}$.

Then we can define a map

$$F : W_0^{1,p}(\Omega) \cap L^\infty(\Omega) \to W^{-1,p'}(\Omega) + L^1(\Omega)$$

by

$$F(u) = -\mathrm{div}\,[a(x,u,\nabla u)] + b(x,u,\nabla u)\,.$$

Remark 3.1. Assume that $\hat{a}$ and $\hat{b}$ also satisfy (N) and (M). An easy density argument shows that, if

$$\int_\Omega \left[a(x,u,\nabla u)\cdot\nabla v + b(x,u,\nabla u)\,v\right]dx = \int_\Omega \left[\hat{a}(x,u,\nabla u)\cdot\nabla v + \hat{b}(x,u,\nabla u)\,v\right]dx$$
$$\text{for any } u \in W_0^{1,p}(\Omega)\cap L^\infty(\Omega) \text{ and any } v \in C_c^\infty(\Omega)\,,$$

then

$$\int_\Omega \left[a(x,u,\nabla u)\cdot\nabla v + b(x,u,\nabla u)\,v\right]dx = \int_\Omega \left[\hat{a}(x,u,\nabla u)\cdot\nabla v + \hat{b}(x,u,\nabla u)\,v\right]dx$$
$$\text{for any } u \in W_0^{1,p}(\Omega)\cap L^\infty(\Omega) \text{ and any } v \in W_0^{1,p}(\Omega)\cap L^\infty(\Omega)\,.$$

Consider also a p-quasi upper semicontinuous function $\underline{u} : \Omega \to \overline{\mathbb{R}}$ and a p-quasi lower semicontinuous function $\overline{u} : \Omega \to \overline{\mathbb{R}}$, and set

$$K = \left\{u \in W_0^{1,p}(\Omega)\cap L^\infty(\Omega) : \ \underline{u} \leq \tilde{u} \leq \overline{u} \ \ p\text{-q.e. in } \Omega\right\}\,,$$

where $\tilde{u}$ is any p-quasi continuous representative of u (see e.g. [9]).

We aim to consider the solutions of the variational inequality

$$(VI) \qquad \begin{cases} u \in K\,, \\ \displaystyle\int_\Omega \left[a(x,u,\nabla u)\cdot\nabla(v-u) + b(x,u,\nabla u)\,(v-u)\right]dx \geq 0 \\ \qquad\qquad\qquad\qquad\qquad\qquad \text{for every } v \in K\,. \end{cases}$$

We denote by $Z^{tot}(F,K)$ the set of solutions u of (VI). We will simply write Z^{tot}, if no confusion can arise.

For every $u \in K$, we also set

$$T_uK = \Big\{v \in W_0^{1,p}(\Omega)\cap L^\infty(\Omega) : \ \tilde{v} \geq 0 \ p\text{-q.e. in } \{\tilde{u} = \underline{u}\} \text{ and}$$
$$\tilde{v} \leq 0 \ p\text{-q.e. in } \{\tilde{u} = \overline{u}\}\Big\}\,.$$

Proposition 3.2. *A function* $u \in K$ *satisfies* (VI) *if and only if*

$$\int_\Omega \left[a(x,u,\nabla u)\cdot\nabla v + b(x,u,\nabla u)\,v\right]dx \geq 0 \qquad \text{for every } v \in T_uK\,.$$

Proof. Assume that u is a solution of (VI) and let $v \in T_uK$ with $v \leq 0$ a.e. in Ω. Since $\max\{k(\underline{u}-\tilde{u}), \tilde{v}\}$ is a nonincreasing sequence of nonpositive p-quasi upper semicontinuous functions converging to $\tilde{v}$ p-q.e. in Ω, by [9, Lemma 1.6] there exists a sequence (v_k) in $W^{1,p}_0(\Omega)$ converging to v in $W^{1,p}_0(\Omega)$ with $\tilde{v}_k \geq \max\{k(\underline{u} - \tilde{u}), \tilde{v}\}$ p-q.e. in Ω. Without loss of generality, we may assume that $\tilde{v}_k \leq 0$ p-q.e. in Ω.

Then it follows that $u + \frac{1}{k}\, v_k \in K$, whence

$$\int_\Omega \big[a(x,u,\nabla u)\cdot\nabla v_k + b(x,u,\nabla u)\, v_k\big]\, dx \geq 0\,.$$

Going to the limit as $k \to \infty$, we get

$$\int_\Omega \big[a(x,u,\nabla u)\cdot\nabla v + b(x,u,\nabla u)\, v\big]\, dx \geq 0\,.$$

If $v \in T_uK$ with $v \geq 0$ a.e. in Ω, the argument is similar. Since every $v \in T_uK$ can be written as $v = v^+ - v^-$ with $v^+, -v^- \in T_uK$, it follows

$$\int_\Omega \big[a(x,u,\nabla u)\cdot\nabla v + b(x,u,\nabla u)\, v\big]\, dx \geq 0 \qquad \text{for every } v \in T_uK\,.$$

Since $K \subseteq u + T_uK$, the converse is obvious. □

We are also interested in the invariance of the problem with respect to suitable transformations.

Let us denote by Φ the set of increasing C^2-diffeomorphisms $\varphi : \mathbb{R} \to \mathbb{R}$ such that $\varphi(0) = 0$ and by Θ the set of C^1-functions $\vartheta : \mathbb{R} \to]0, +\infty[$.

For any $\varphi \in \Phi$ and $\vartheta \in \Theta$, we define

$$F^\varphi\,,\ F_\vartheta : W^{1,p}_0(\Omega) \cap L^\infty(\Omega) \to W^{-1,p'}(\Omega) + L^1(\Omega)$$

by

$$F^\varphi(u) = F(\varphi(u))\,,\ F_\vartheta(u) = \vartheta(u)\, F(u)\,.$$

If we define the Carathéodory functions

$$a^\varphi, a_\vartheta : \Omega \times (\mathbb{R}\times\mathbb{R}^n) \to \mathbb{R}^n\,, \qquad b^\varphi, b_\vartheta : \Omega \times (\mathbb{R}\times\mathbb{R}^n) \to \mathbb{R}$$

by

$$\begin{aligned}
a^\varphi(x,s,\xi) &= \varphi'(s)\, a(x,\varphi(s),\varphi'(s)\xi)\,,\\
b^\varphi(x,s,\xi) &= \varphi''(s)\, a(x,\varphi(s),\varphi'(s)\xi)\cdot\xi + \varphi'(s)\, b(x,\varphi(s),\varphi'(s)\xi)\,,\\
a_\vartheta(x,s,\xi) &= \vartheta(s)\, a(x,s,\xi)\,,\\
b_\vartheta(x,s,\xi) &= \vartheta'(s)\, a(x,s,\xi)\cdot\xi + \vartheta(s)\, b(x,s,\xi)\,,
\end{aligned}$$

it easily follows that

$$\begin{aligned}
F^\varphi(u) &= -\mathrm{div}\,[a^\varphi(x,u,\nabla u)] + b^\varphi(x,u,\nabla u)\,,\\
F_\vartheta(u) &= -\mathrm{div}\,[a_\vartheta(x,u,\nabla u)] + b_\vartheta(x,u,\nabla u)\,.
\end{aligned}$$

We also set $u^\varphi = \varphi^{-1}(u)$ for any $u \in K$ and $E^\varphi = \{u^\varphi :\ u \in E\}$, whenever $E \subseteq K$.

It is easily seen that

$$(a^\varphi)^\psi = a^{\varphi\circ\psi}\,, \qquad (a_\vartheta)_\varrho = a_{\vartheta\varrho}\,, \qquad (u^\varphi)^\psi = u^{\varphi\circ\psi}\,,$$

for every $\varphi, \psi \in \Phi$ and $\vartheta, \varrho \in \Theta$.

We also say that (a, b) is of *Euler-Lagrange type*, if there exists a function

$$L : \Omega \times \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$$

such that

(a) $\{x \mapsto L(x, s, \xi)\}$ is measurable for every $(s, \xi) \in \mathbb{R} \times \mathbb{R}^n$;
(b) $\{(s, \xi) \mapsto L(x, s, \xi)\}$ is of class C^1 for a.e. $x \in \Omega$;
(c) we have

$$a(x, s, \xi) = \nabla_\xi L(x, s, \xi)\,, \qquad b(x, s, \xi) = D_s L(x, s, \xi)\,,$$

for a.e. $x \in \Omega$ and every $(s, \xi) \in \mathbb{R} \times \mathbb{R}^n$.

Taking into account Proposition 3.2, the next two results are easy to prove.

Proposition 3.3. *For every $\varphi \in \Phi$, the following facts hold:*

(a) *the functions a^φ, b^φ satisfy (N) and (M) with the same p;*
(b) *we have*

$$K^\varphi = \left\{u \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega) :\ \underline{u}^\varphi \leq \tilde{u} \leq \overline{u}^\varphi \ \ p\text{-q.e. in } \Omega\right\}$$

and $\underline{u}^\varphi : \Omega \to \mathbb{R}$ is p-quasi upper semicontinuous, while $\overline{u}^\varphi : \Omega \to \mathbb{R}$ is p-quasi lower semicontinuous;
(c) *u is a solution of (VI) if and only if u^φ is a solution of the corresponding variational inequality with a, b and K replaced by a^φ, b^φ and K^φ, respectively;*
(d) *the pair (a, b) is of Euler-Lagrange type if and only if the pair (a^φ, b^φ) does the same; moreover, if L is associated with (a, b), then*

$$L^\varphi(x, s, \xi) = L(x, \varphi(s), \varphi'(s)\xi)$$

is associated with (a^φ, b^φ).

Proposition 3.4. *For every $\vartheta \in \Theta$, the following facts hold:*

(a) *the functions a_ϑ, b_ϑ satisfy (N) and (M) with the same p;*
(b) *u is a solution of (VI) if and only if the same u is a solution of the corresponding variational inequality with a and b replaced by a_ϑ and b_ϑ, respectively.*

Remark 3.5. Let $\varphi \in \Phi$ with φ' nonconstant, let $w \in L^2(\Omega)$ and let $a(x, s, \xi) = \varphi'(s)\xi$, $b(x, s, \xi) = -w(x)$.

Then (a, b) is not of Euler-Lagrange type. However, if we take $\vartheta(s) = \varphi'(s)$, then $(a_\vartheta, b_\vartheta)$, which is given by

$$a_\vartheta\,(x, s, \xi) = [\varphi'(s)]^2\xi\,, \qquad b_\vartheta\,(x, s, \xi) = \varphi'(s)\varphi''(s)|\xi|^2 - w(x)\varphi'(s)\,,$$

is of Euler-Lagrange type with

$$L(x, s, \xi) = \frac{1}{2}\,[\varphi'(s)]^2|\xi|^2 - w(x)\varphi(s)\,.$$

Therefore the property of being of Euler-Lagrange type is not invariant under the transformation induced by ϑ, which plays in fact the role of "integrating factor". By the way, if then we take $\psi = \varphi^{-1}$, we get

$$(a_\vartheta)^\psi(x, s, \xi) = \xi\,, \qquad (b_\vartheta)^\psi(x, s, \xi) = -w(x)\,,$$

which are simply related to

$$L(x, s, \xi) = \frac{1}{2}\,|\xi|^2 - w(x)s\,.$$

Proposition 3.6. *For every $R > 0$ there exist $\vartheta_1, \vartheta_2 \in \Theta$ and $c > 0$, depending only on R, β_R and ν_R, such that*

$$b_{\vartheta_1}(x,s,\xi) \leq c\,\alpha_R^{(0)}(x)\,,$$
$$b_{\vartheta_2}(x,s,\xi) \geq -c\,\alpha_R^{(0)}(x)\,,$$

for a.e. $x \in \Omega$ and every $s \in \mathbb{R}$, $\xi \in \mathbb{R}^n$ with $|s| \leq R$.

Proof. If we set $\vartheta(s) = \exp(\gamma s)$ with $\gamma\,\nu_R \geq \beta_R$, we have

$$\begin{aligned} b_\vartheta(x,s,\xi) &= [\gamma\, a(x,s,\xi)\cdot\xi + b(x,s,\xi)]\exp(\gamma s) \\ &\geq \left[\gamma\,\nu_R\,|\xi|^p - \gamma\,\alpha_R^{(0)}(x) - \alpha_R^{(0)}(x) - \beta_R\,|\xi|^p\right]\exp(\gamma s) \\ &\geq -(\gamma+1)\exp(\gamma R)\,\alpha_R^{(0)}(x)\,, \end{aligned}$$

whence the existence of ϑ_2. The existence of ϑ_1 can be proved in a similar way. □

4. Quasilinear elliptic variational inequalities with natural growth conditions depending on a parameter

Again, let Ω be a bounded and open subset of $\mathbb{R}^n$ and let now T be a metrizable topological space and

$$a : \Omega \times (\mathbb{R}\times\mathbb{R}^n \times T) \to \mathbb{R}^n\,,$$
$$b : \Omega \times (\mathbb{R}\times\mathbb{R}^n \times T) \to \mathbb{R}$$

be two Carathéodory functions satisfying (N) and (M) uniformly with respect to $t \in T$.

More precisely, we assume that a_t and b_t satisfy (UM) and:

(UN) *there exist $p \in]1,\infty[$ and, for every $R > 0$, $\alpha_R^{(0)} \in L^1(\Omega)$, $\alpha_R^{(1)} \in L^{p'}(\Omega)$, $\beta_R > 0$ and $\nu_R > 0$ such that*

$$|a_t(x,s,\xi)| \leq \alpha_R^{(1)}(x) + \beta_R\,|\xi|^{p-1}\,,$$
$$|b_t(x,s,\xi)| \leq \alpha_R^{(0)}(x) + \beta_R\,|\xi|^p\,,$$
$$a_t(x,s,\xi)\cdot\xi \geq \nu_R\,|\xi|^p - \alpha_R^{(0)}(x)\,,$$

for a.e. $x \in \Omega$ and every $s \in \mathbb{R}$, $\xi \in \mathbb{R}^n$ and $t \in T$ with $|s| \leq R$; again, such a p is clearly unique.

Then we can define a map

$$H : \left[W_0^{1,p}(\Omega) \cap L^\infty(\Omega)\right] \times T \to \left[W^{-1,p'}(\Omega) + L^1(\Omega)\right]$$

by

$$H_t(u) = -\mathrm{div}\,[a_t(x,u,\nabla u)] + b_t(x,u,\nabla u)\,.$$

Consider also, for each $t \in T$, a p-quasi upper semicontinuous function $\underline{u}_t : \Omega \to \overline{\mathbb{R}}$ and a p-quasi lower semicontinuous function $\overline{u}_t : \Omega \to \overline{\mathbb{R}}$, set

$$K_t = \left\{u \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega) :\ \underline{u}_t \leq \tilde{u} \leq \overline{u}_t \ \ p\text{-q.e. in } \Omega\right\}$$

and assume the following form of continuity related to the Mosco-convergence:

(MC) *for every sequence (t_k) convergent to t in T, the following facts hold:*

- *if (u_k) is a sequence weakly convergent to u in $W_0^{1,p}(\Omega)$, with $u_k \in K_{t_k}$ for any $k \in \mathbb{N}$ and (u_k) bounded in $L^\infty(\Omega)$, then $u \in K_t$;*
- *for every $u \in K_t$ there exist $\overline{k} \in \mathbb{N}$ and a sequence (u_k) in $W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$ which is bounded in $L^\infty(\Omega)$ and strongly convergent to u in $W_0^{1,p}(\Omega)$, with $u_k \in K_{t_k}$ for any $k \geq \overline{k}$.*

Then consider the parametric variational inequality

$$(PVI) \qquad \begin{cases} (u,t) \in [W_0^{1,p}(\Omega) \cap L^\infty(\Omega)] \times T\,, \\ u \in K_t\,, \\ \displaystyle\int_\Omega \left[a_t(x,u,\nabla u) \cdot \nabla(v-u) + b_t(x,u,\nabla u)\,(v-u)\right] dx \geq 0 \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{for every } v \in K_t\,. \end{cases}$$

Theorem 4.1. *Let (u_k, t_k) be a sequence of solutions of (PVI) with (u_k) bounded in $L^\infty(\Omega)$ and (t_k) convergent to some t in T with $K_t \neq \emptyset$.*

Then (u_k) admits a subsequence strongly convergent in $W_0^{1,p}(\Omega)$ to some u and (u,t) is a solution of (PVI).

Proof. Let $w \in K_t$ and let (w_k) be a sequence strongly convergent to w in $W_0^{1,p}(\Omega)$, with $w_k \in K_{t_k}$ for any $k \in \mathbb{N}$ and (w_k) bounded in $L^\infty(\Omega)$. If we set

$$\begin{gathered} \widehat{T} = \mathbb{N} \cup \{\infty\}\,, \\ \hat{a}_k(x,s,\xi) = a_{t_k}(x, w_k(x)+s, \nabla w_k(x)+\xi)\,, \\ \hat{a}_\infty(x,s,\xi) = a_t(x, w(x)+s, \nabla w(x)+\xi)\,, \\ \hat{b}_k(x,s,\xi) = b_{t_k}(x, w_k(x)+s, \nabla w_k(x)+\xi)\,, \\ \hat{b}_\infty(x,s,\xi) = b_t(x, w(x)+s, \nabla w(x)+\xi)\,, \\ \hat{\underline{u}}_k = \underline{u}_{t_k} - w_k\,, \qquad \hat{\underline{u}}_\infty = \underline{u}_t - w\,, \\ \hat{\overline{u}}_k = \overline{u}_{t_k} - w_k\,, \qquad \hat{\overline{u}}_\infty = \overline{u}_t - w\,, \\ \hat{u}_k = u_k - w_k\,, \end{gathered}$$

and define $\widehat{K}_k$, $\widehat{K}_\infty$ accordingly, it is easily seen that all the assumptions are still satisfied and now $0 \in \widehat{K}_k$. Therefore, we may assume without loss of generality that $0 \in K_{t_k}$ for any $k \in \mathbb{N}$.

Let $R > 0$ be such that $\|u_k\|_\infty \leq R$ for any $k \in \mathbb{N}$. We claim that (u_k^+) is bounded in $W_0^{1,p}(\Omega)$. Actually, by Propositions 3.4 and 3.6, we may assume, without loss of generality, that

$$b_t(x,s,\xi) \geq -c\,\alpha_R^{(0)}(x) \qquad \text{whenever } |s| \leq R\,.$$

Then the choice $v = -u_k^-$ in (PVI) yields

$$\begin{aligned} 0 \geq \int_\Omega &\left[a_{t_k}(x,u_k^+,\nabla u_k^+) \cdot \nabla u_k^+ + b_{t_k}(x,u_k^+,\nabla u_k^+)\,u_k^+\right] dx \\ &\geq \nu_R \int_\Omega |\nabla u_k^+|^p\,dx - \int_\Omega \alpha_R^{(0)}\,dx - c\int_\Omega \alpha_R^{(0)}\,u_k^+\,dx\,, \end{aligned}$$

which implies that (u_k^+) is bounded in $W_0^{1,p}(\Omega)$.

In a similar way one finds that (u_k^-) is bounded in $W_0^{1,p}(\Omega)$, so that (u_k) is weakly convergent, up to a subsequence, to some u in $W_0^{1,p}(\Omega)$ and $u \in K_t$. Let (z_k) be a sequence strongly convergent to u in $W_0^{1,p}(\Omega)$, with $z_k \in K_{t_k}$ for any $k \in \mathbb{N}$ and (z_k) bounded in $L^\infty(\Omega)$.

Let $\psi : \mathbb{R} \to [0,1]$ be a continuous function such that $\psi(s) = 1$ for $|s| \leq R$ and $\psi(s) = 0$ for $|s| \geq R+1$. Then, let

$$\begin{aligned} \breve{a}_t(x,s,\xi) &= \psi(s)\, a_t(x,s,\xi) + (1-\psi(s))\, |\xi|^{p-2}\xi \,, \\ \breve{b}_t(x,s,\xi) &= \psi(s)\, b_t(x,s,\xi) \,. \end{aligned}$$

It is easily seen that each (u_k, t_k) is also a solution of (PVI) with a_t and b_t replaced by $\breve{a}_t$ and $\breve{b}_t$. Moreover, $\breve{a}_t$ and $\breve{b}_t$ satisfy both the assumptions (UN), (UM) and the assumptions of [1, Theorem 4.2]. In particular, there exist $\alpha^{(0)} \in L^1(\Omega)$, $\beta > 0$ and $\nu > 0$ such that

$$\breve{a}_t(x,s,\xi)\cdot\xi \geq \nu\,|\xi|^p - \alpha^{(0)}(x)\,, \qquad |\breve{b}_t(x,s,\xi)| \leq \alpha^{(0)}(x) + \beta\,|\xi|^p$$

for a.e. $x \in \Omega$ and every $s \in \mathbb{R}$, $\xi \in \mathbb{R}^n$ and $t \in T$.

Now, if $p < n$, the proof of [1, Theorem 4.2] can be repeated in a simplified form, as (u_k) is bounded in $L^\infty(\Omega)$. We have only to observe that, if $\varphi : \mathbb{R} \to \mathbb{R}$ is the solution of

$$\begin{cases} \varphi'(s) = 1 + \dfrac{\beta}{\nu}\,|\varphi(s)|\,, \\ \varphi(0) = 0\,, \end{cases}$$

then there exists $\tau > 0$ such that

$$0 \leq \tau s \varphi(s) \leq s^2 \qquad \text{whenever } |s| \leq R + \sup_k\, \|z_k\|_\infty\,,$$

so that

$$u_k - \tau\,\varphi(u_k - z_k) \in K_{t_k} \qquad \text{for any } k \in \mathbb{N}\,.$$

It follows

$$\int_\Omega \left[\breve{a}_{t_k}(x,u_k,\nabla u_k)\cdot\nabla(\varphi(u_k - z_k)) + \breve{b}_{t_k}(x,u_k,\nabla u_k)\,\varphi(u_k - z_k)\right] dx \leq 0$$

and now the proof of [1, Theorem 4.2] can be repeated with minor modifications, showing that (u_k) is strongly convergent to u in $W_0^{1,p}(\Omega)$. If $p \geq n$, the argument is similar and simpler.

It is easily seen that (u,t) is a solution of (PVI). □

Remark 4.2. In the previous theorem the assumption $K_t \neq \emptyset$ is crucial to ensure that (u_k) is bounded in $W_0^{1,p}(\Omega)$.

Consider $\Omega =]0,1[$, $a_t(x,s,\xi) = \xi$, $b_t(x,s,\xi) = 0$, $\underline{u}_t = z - t$ and $\overline{u}_t = z + t$, where $z \in C_c(]0,1[) \setminus W_0^{1,2}(]0,1[)$.

If $t_k \to 0$ with $t_k > 0$ and (u_k, t_k) are the solutions of (PVI), then the sequence (u_k) is unbounded in $W_0^{1,2}(\Omega)$.

5. Topological degree for quasilinear elliptic variational inequalities with natural growth conditions

Consider again the setting of Section 3. Throughout this section, we also assume that:

(B) *the functions* $\underline{u}$ *and* $\overline{u}$ *are bounded.*

It follows that Z^{tot} is automatically bounded in $L^\infty(\Omega)$.

Theorem 5.1. *The set* Z^{tot} *is (strongly) compact in* $W_0^{1,p}(\Omega)$ *(possibly empty).*

Proof. If $K = \emptyset$, we have $Z^{tot} = \emptyset$. Otherwise the assertion follows from Theorem 4.1. ☐

Definition 5.2. We denote by $\mathcal{Z}(F, K)$ the family of the subsets Z of Z^{tot} which are both open and closed in Z^{tot} with respect to the $W_0^{1,p}(\Omega)$-topology. We will simply write $\mathcal{Z}$, if no confusion can arise.

Fix a continuous function $\psi : \mathbb{R} \to [0, 1]$ such that

$$\psi(s) = 1 \quad \text{for } s \le 1\,, \qquad \psi(s) = 0 \quad \text{for } s \ge 2\,,$$

then set, for any $t \in [0, 1]$ and $s \in \mathbb{R}$,

$$\Psi_t(s) = \psi(t|s|)\, s\,.$$

If we consider $T = [0, 1]$ and define

$$\begin{aligned} a_t(x, s, \xi) &= \psi(t|s|)\, a(x, s, \xi) + [1 - \psi(t|s|)]\, |\xi|^{p-2}\xi\,, \\ b_t(x, s, \xi) &= \Psi_t\left(b(x, s, \xi)\right)\,, \end{aligned}$$

it is easily seen that a_t, b_t satisfy (UN) and (UM). Moreover, for every $\underline{t} \in]0, 1[$, they satisfy (UC), if t is restricted to $[\underline{t}, 1]$. In particular, we can define a continuous map $H : W_0^{1,p}(\Omega)\times]0, 1] \to W^{-1,p'}(\Omega)$ by

$$H_t(u) = -\mathrm{div}\left[a_t(x, u, \nabla u)\right] + b_t(x, u, \nabla u)$$

and, by Theorem 2.10, this map is of class $(S)_+$.

Proposition 5.3. *For every* $Z \in \mathcal{Z}$, *the following facts hold:*

(a) *there exist a bounded and open subset* U *of* $W_0^{1,p}(\Omega)$ *and* $\overline{t} \in]0, 1]$ *such that*

$$Z = Z^{tot} \cap U = Z^{tot} \cap \overline{U}\,,$$

while the variational inequality (PVI) *has no solution* $(u, t) \in \partial U \times [0, \overline{t}]$; *in particular, the degree* $\deg((H_t, K), U, 0)$ *is defined whenever* $t \in]0, \overline{t}]$;

(b) *if* $\psi_0, \psi_1 : \mathbb{R} \to [0, 1]$ *have the same properties of* ψ *and* U_0, $\overline{t}_0$ *and* U_1, $\overline{t}_1$ *are as in* (a), *then*

$$\deg((H_t, K), U_0, 0) = \deg((H_\tau, K), U_1, 0)$$

for every $t \in]0, \overline{t}_0]$ *and* $\tau \in]0, \overline{t}_1]$;

(c) *if* $\hat{a}$ *and* $\hat{b}$ *also satisfy* (N), (M) *and*

$$\begin{aligned} &\int_\Omega \big[a(x, u, \nabla u) \cdot \nabla v + b(x, u, \nabla u)\, v\big]\, dx \\ &\qquad = \int_\Omega \big[\hat{a}(x, u, \nabla u) \cdot \nabla v + \hat{b}(x, u, \nabla u)\, v\big]\, dx \end{aligned}$$

for any $u \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$ *and any* $v \in C_c^\infty(\Omega)$, *then we have*

$$\deg((H_t, K), U, 0) = \deg((\widehat{H}_\tau, K), \widehat{U}, 0)$$

for every $t \in]0, \bar{t}]$ *and* $\tau \in]0, \hat{t}]$, *provided that* U, $\bar{t}$, $\widehat{U}$, $\hat{t}$ *are as in* (a) *with respect to* a, b *and* $\hat{a}, \hat{b}$, *respectively.*

Proof. By definition of $\mathcal{Z}$, there exists a bounded and open subset U of $W_0^{1,p}(\Omega)$ such that

$$Z = Z^{tot} \cap U = Z^{tot} \cap \overline{U}\,.$$

If (u_k, t_k) is a sequence of solutions of (PVI) with $u_k \in \partial U$ and $t_k \to 0$, then (u_k) is bounded in $L^\infty(\Omega)$ by (B). By Theorem 4.1, up to a subsequence (u_k) is convergent to some u in $W_0^{1,p}(\Omega)$ and u is a solution of (VI). Then $u \in \partial U$ and a contradiction follows. Therefore, there exists $\bar{t} \in]0, 1]$ such that (PVI) has no solution $(u, t) \in \partial U \times [0, \bar{t}]$ and assertion (a) is proved.

To prove (b), define

$$\psi_\mu = (1 - \mu)\psi_0 + \mu\psi_1$$

and $H_{t,\mu}$ accordingly. Arguing as before, we find $\underline{t} > 0$ with $\underline{t} \leq \bar{t}_j$, $j = 0, 1$, such that

$$\begin{cases} u \in K\,, \\ \langle H_{t,\mu}(u), v - u\rangle \geq 0 \qquad \forall v \in K\,, \end{cases}$$

has no solution with $u \in (\overline{U_0} \setminus U_1) \cup (\overline{U_1} \setminus U_0)$, $t \in [0, \underline{t}]$ and $\mu \in [0, 1]$.

From Theorem 2.7 we infer that

$$\deg((H_{\underline{t},0}, K), U_0, 0) = \deg((H_{\underline{t},0}, K), U_0 \cap U_1, 0)\,,$$
$$\deg((H_{\underline{t},1}, K), U_1, 0) = \deg((H_{\underline{t},1}, K), U_0 \cap U_1, 0)\,.$$

On the other hand, we have

$$\deg((H_{\underline{t},0}, K), U_0 \cap U_1, 0) = \deg((H_{\underline{t},1}, K), U_0 \cap U_1, 0)\,,$$
$$\deg((H_{\underline{t},0}, K), U_0, 0) = \deg((H_{t,0}, K), U_0, 0) \qquad \forall t \in]0, \bar{t}_0]\,,$$
$$\deg((H_{\underline{t},1}, K), U_1, 0) = \deg((H_{\tau,1}, K), U_1, 0) \qquad \forall \tau \in]0, \bar{t}_1]\,,$$

by Theorem 2.9. Then assertion (b) also follows.

The proof of (c) is quite similar. □

Definition 5.4. For every $Z \in \mathcal{Z}(F, K)$, we set

$$\mathrm{ind}((F, K), Z) = \deg((H_t, K), U, 0)\,,$$

where ψ, U, $\bar{t}$ are as in (a) and $0 < t \leq \bar{t}$. We will simply write $\mathrm{ind}(Z)$, if no confusion can arise.

Proposition 5.5. *Assume that* a *and* b *satisfy, instead of* (N), *the more specific controllable growth condition:*

(C) *there exist* $p \in]1, \infty[$, $\alpha^{(0)} \in L^1(\Omega)$, $\alpha^{(1)} \in L^{p'}(\Omega)$, $\beta > 0$ *and* $\nu > 0$ *such that*

$$|a(x, s, \xi)| \leq \alpha^{(1)}(x) + \beta\,|s|^{p-1} + \beta\,|\xi|^{p-1}\,,$$

$$|b(x, s, \xi)| \leq \alpha^{(1)}(x) + \beta\,|s|^{p-1} + \beta\,|\xi|^{p-1}\,,$$

$$a(x, s, \xi) \cdot \xi \geq \nu|\xi|^p - \alpha^{(0)}(x) - \beta\,|s|^p\,,$$

for a.e. $x \in \Omega$ *and every* $s \in \mathbb{R}$, $\xi \in \mathbb{R}^n$.

Then the map $F : W_0^{1,p}(\Omega) \to W^{-1,p'}(\Omega)$ is continuous, bounded on bounded subsets and of class $(S)_+$. Moreover, if U is a bounded and open subset of $W_0^{1,p}(\Omega)$ such that (VI) has no solution $u \in \partial U$ and

$$Z = \{u \in U : \ u \text{ is a solution of } (VI)\} ,$$

then $Z \in \mathcal{Z}$ and

$$\mathrm{ind}(Z) = \deg((F, K), U, 0) .$$

Proof. It is easily seen that this time a_t and b_t satisfy (UC) and (UM), for t belonging to all $[0, 1]$.

Then the assertions follow from Theorems 2.10 and 2.9. □

Theorem 5.6. *Let $Z \in \mathcal{Z}$ with $\mathrm{ind}(Z) \neq 0$. Then $Z \neq \emptyset$.*

Proof. Let U and $\bar{t}$ be as in (a) of Proposition 5.3. If $Z = \emptyset$, from Theorem 4.1 we infer that there exists $t \in]0, \bar{t}]$ such that (PVI) has no solution (u, t) with $u \in \overline{U}$. From Theorem 2.5 we deduce that

$$\mathrm{ind}(Z) = \deg((H_t, K), U, 0) = 0$$

and a contradiction follows. □

Along the same line, the additivity property can be proved taking advantage of Theorems 2.7 and 4.1.

Theorem 5.7. *Let $Z_0, Z_1 \in \mathcal{Z}$ with $Z_0 \cap Z_1 = \emptyset$. Then $Z_0 \cup Z_1 \in \mathcal{Z}$ and*

$$\mathrm{ind}(Z_0 \cup Z_1) = \mathrm{ind}(Z_0) + \mathrm{ind}(Z_1) .$$

Theorem 5.8. *Let*

$$a : \Omega \times (\mathbb{R} \times \mathbb{R}^n \times [0, 1]) \to \mathbb{R}^n ,$$

$$b : \Omega \times (\mathbb{R} \times \mathbb{R}^n \times [0, 1]) \to \mathbb{R}$$

be two Carathéodory functions satisfying (UN) and (UM) with respect to $T = [0, 1]$ and set

$$H_t(u) = -\mathrm{div}\,[a_t(x, u, \nabla u)] + b_t(x, u, \nabla u) .$$

Let also, for each $t \in [0, 1]$, $\underline{u}_t : \Omega \to \overline{\mathbb{R}}$ be a p-quasi upper semicontinuous function and $\overline{u}_t : \Omega \to \overline{\mathbb{R}}$ a p-quasi lower semicontinuous function, define K_t as in section 4 and assume that:

- *the functions $\underline{u}_t, \overline{u}_t$ are bounded uniformly with respect to $t \in [0, 1]$, we have $K_t \neq \emptyset$ for any $t \in [0, 1]$ and assumption (MC) is satisfied.*

Then the following facts hold:

(a) *the set*

$$\widehat{Z}^{tot} := \left\{(u, t) \in [W_0^{1,p}(\Omega) \cap L^\infty(\Omega)] \times [0, 1] : \ (u, t) \text{ is a solution of } (PVI)\right\}$$

is (strongly) compact in $W_0^{1,p}(\Omega) \times [0, 1]$ (possibly empty);

(b) *if $\widehat{Z}$ is open and closed in $\widehat{Z}^{tot}$ with respect to the topology of $W_0^{1,p}(\Omega) \times [0, 1]$ and*

$$\widehat{Z}_t = \left\{u \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega) : \ (u, t) \in \widehat{Z}\right\} ,$$

then $\widehat{Z}_t \in \mathcal{Z}(H_t, K_t)$ for any $t \in [0, 1]$ and $\mathrm{ind}((H_t, K_t), \widehat{Z}_t)$ is independent of $t \in [0, 1]$.

Proof. First of all, the set $\widehat{Z}^{tot}$ is compact by Theorem 4.1. To prove assertion (b), let W be a bounded and open subset of $W_0^{1,p}(\Omega) \times [0,1]$ such that

$$\widehat{Z} = \widehat{Z}^{tot} \cap W = \widehat{Z}^{tot} \cap \overline{W}\,.$$

In particular, we have $\widehat{Z}_t \in \mathcal{Z}(H_t, K_t)$ for any $t \in [0,1]$.

Let $\widehat{T} = [0,1] \times [0,1]$, let

$$\begin{aligned} a_{t,\tau}(x,s,\xi) &= \psi(\tau|s|)\, a_t(x,s,\xi) + [1-\psi(\tau|s|)]\, |\xi|^{p-2}\xi\,,\\ b_{t,\tau}(x,s,\xi) &= \Psi_\tau(b_t(x,s,\xi))\,, \end{aligned}$$

for $(t,\tau) \in \widehat{T}$ and define

$$\begin{gathered} H_{t,\tau}(u) = -\mathrm{div}[a_{t,\tau}(x,u,\nabla u)] + b_{t,\tau}(x,u,\nabla u)\,,\\ K_{t,\tau} = K_t\,. \end{gathered}$$

It is easily seen that $a_{t,\tau}$ and $b_{t,\tau}$ satisfy (UN) and (UM) with respect to $\widehat{T}$, so that we can consider the problem

$$\begin{cases} (u,(t,\tau)) \in [W_0^{1,p}(\Omega) \cap L^\infty(\Omega)] \times \widehat{T}\,,\\ u \in K_{t,\tau}\,,\\ \displaystyle\int_\Omega \big[a_{t,\tau}(x,u,\nabla u)\cdot \nabla(v-u) + b_{t,\tau}(x,u,\nabla u)\,(v-u)\big]\,dx \ge 0\\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{for every } v \in K_{t,\tau}\,. \end{cases} \tag{5.1}$$

Since $[0,1]$ is compact, by Theorem 4.1 there exists $\overline{\tau} \in]0,1]$ such that (5.1) has no solution $(u,(t,\tau))$ with $(u,t) \in \partial W$ and $0 \le \tau \le \overline{\tau}$.

By Definition 5.4 we infer that

$$\mathrm{ind}((H_t,K_t),\widehat{Z}_t) = \deg((H_{t,\overline{\tau}},K_t),W_t,0) \qquad \text{for any } t \in [0,1]$$

and the assertion follows from Theorem 2.9. □

Remark 5.9. By Theorem 5.8 and Proposition 5.5, $\mathrm{ind}((F,K),Z)$ can be calculated also by other approximation techniques, with respect to the one used in Definition 5.4.

Theorem 5.10. *If* $K \neq \emptyset$*, then* $\mathrm{ind}(Z^{tot}) = 1$.

Proof. Define, for $0 \le t \le 1$,

$$\begin{aligned} a_t(x,s,\xi) &= t\,a(x,s,\xi) + (1-t)\,|\xi|^{p-2}\xi\,,\\ b_t(x,s,\xi) &= t\,b(x,s,\xi)\,. \end{aligned}$$

It is easily seen that a_t and b_t satisfy assumptions (UN) and (UM), so that Theorem 5.8 can be applied. If we take $\widehat{Z} = \widehat{Z}^{tot}$, we get

$$\mathrm{ind}(Z^{tot}) = \mathrm{ind}((H_1,K),\widehat{Z}_1) = \mathrm{ind}((H_0,K),\widehat{Z}_0)\,.$$

Let $u_0 \in K$ and let

$$U = \left\{u \in W_0^{1,p}(\Omega):\ \|\nabla u\|_2 < r\right\}\,,$$

with r large enough to guarantee that $u_0 \in U$, $\widehat{Z}_0 \subseteq U$ and

$$\int_\Omega |\nabla v|^{p-2}\nabla v \cdot \nabla(v-u_0)\,dx \ge 0 \qquad \text{for every } v \in \partial U\,.$$

From Proposition 5.5 and Theorem 2.6 we infer that

$$\mathrm{ind}((H_0, K), \widehat{Z}_0) = \deg((H_0, K), U, 0) = 1$$

and the assertion follows. □

Proposition 5.11. *Let $Z \in \mathcal{Z}(F, K)$ and let $\varphi \in \Phi$ and $\vartheta \in \Theta$. Then $Z^\varphi \in \mathcal{Z}(F^\varphi, K^\varphi)$, $Z \in \mathcal{Z}(F_\vartheta, K)$ and*

$$\mathrm{ind}((F, K), Z) = \mathrm{ind}((F^\varphi, K^\varphi), Z^\varphi) = \mathrm{ind}((F_\vartheta, K), Z)\,.$$

Proof. If we set

$$\varphi_t(s) = (1-t)s + t\varphi(s)\,, \qquad \vartheta_t(s) = (1-t) + t\vartheta(s)\,,$$

the assertion follows from Theorem 5.8. □

6. Proof of Theorem 1.1

We aim to apply the results of the previous sections to

$$a(x, s, \xi) = A(x, s)\xi\,, \qquad b(x, s, \xi) = B(x, s)|\xi|^2 - g(x, s)\,.$$

By hypothesis (h_1), assumptions (N) and (M) are satisfied with $p = 2$. Moreover, if $\underline{M}$ and $\overline{M}$ are as in hypothesis (h_2), then $\underline{u} = \underline{M}$ and $\overline{u} = \overline{M}$ satisfy assumption (B).

Denote by (λ_k), $k \geq 1$, the sequence of the eigenvalues of (1.3), repeated according to multiplicity, and set, for a matter of convenience, $\lambda_0 = -\infty$.

Finally, define F, K, Z^{tot}, $\mathcal{Z}$ and $\mathrm{ind}(Z)$ as before, observe that $K \neq \emptyset$ and set

$$Z_+ = \left\{u \in Z^{tot} \setminus \{0\} :\ u \geq 0 \text{ a.e. in } \Omega\right\}\,,$$
$$Z_- = \left\{u \in Z^{tot} \setminus \{0\} :\ u \leq 0 \text{ a.e. in } \Omega\right\}\,.$$

It is easily seen that

$$a^\varphi(x, s, \xi) = A^\varphi(x, s)\xi\,, \qquad b^\varphi(x, s, \xi) = B^\varphi(x, s)|\xi|^2 - g^\varphi(x, s)\,.$$

Let us also set

$$A_\vartheta(x, s) = \vartheta(s)\,A(x, s)\,, \qquad B_\vartheta(x, s) = \vartheta'(s)\,A(x, s) + \vartheta(s)\,B(x, s)\,,$$
$$g_\vartheta(x, s) = \vartheta(s)\,g(x, s)\,,$$

so that

$$a_\vartheta(x, s, \xi) = A_\vartheta(x, s)\xi\,, \qquad b_\vartheta(x, s, \xi) = B_\vartheta(x, s)|\xi|^2 - g_\vartheta(x, s)\,.$$

Proposition 6.1. *For every $R > 0$ there exist $\vartheta_1, \vartheta_2 \in \Theta$, depending only on β_R and ν_R, such that*

$$B_{\vartheta_1}(x, s) \leq 0 \leq B_{\vartheta_2}(x, s) \qquad \text{for a.e. } x \in \Omega \text{ and every } s \in \mathbb{R} \text{ with } |s| \leq R\,.$$

Proof. It is a simple variant of Proposition 3.6. □

Proposition 6.2. *Let*

$$\hat{g} : \Omega \times \mathbb{R} \to \mathbb{R}$$

be a Carathéodory function such that for every $R > 0$ there exists $\beta_R > 0$ satisfying

$$|\hat{g}(x, s)| \leq \beta_R \qquad \text{for a.e. } x \in \Omega \text{ and every } s \in \mathbb{R} \text{ with } |s| \leq R$$

and such that

$$\hat{g}(x, \underline{M}) \geq 0 \geq \hat{g}(x, \overline{M}) \qquad \text{for a.e. } x \in \Omega\,.$$

If u is a solution of the variational inequality (VI) with

$$a(x,s,\xi) = A(x,s)\xi\,, \qquad b(x,s,\xi) = B(x,s)|\xi|^2 - \hat{g}(x,s)\,,$$

then u satisfies the equation

$$\int_\Omega \left[A(x,u)\nabla u\cdot\nabla v + B(x,u)\,|\nabla u|^2\, v\right] dx = \int_\Omega \hat{g}(x,u)\, v\, dx \qquad \textit{for any } v \in W_0^{1,2}(\Omega)\cap L^\infty(\Omega)\,.$$

Proof. Let $v \in W_0^{1,2}(\Omega)\cap L^\infty(\Omega)$ with $v \geq 0$ a.e. in Ω, let $t > 0$ and let

$$u_t = \min\{u + tv, \overline{M}\}\,.$$

Since $u_t \in K$, it follows

$$\begin{aligned}\frac{1}{t}\int_\Omega A(x,u)\nabla u\cdot\nabla(u_t - u)\,dx &\geq -\int_\Omega B(x,u)\,|\nabla u|^2\,\frac{u_t - u}{t}\,dx + \int_\Omega \hat{g}(x,u)\,\frac{u_t-u}{t}\,dx \\ &= -\int_{\{u<\overline{M}\}} B(x,u)\,|\nabla u|^2\,\frac{u_t - u}{t}\,dx + \int_{\{u<\overline{M}\}} \hat{g}(x,u)\,\frac{u_t-u}{t}\,dx\,.\end{aligned}$$

On the other hand, we have

$$\begin{aligned}\frac{1}{t}\int_\Omega A(x,u)\nabla u\cdot\nabla(u_t - u)\,dx &= \int_{\{u+tv<\overline{M}\}} A(x,u)\nabla u\cdot\nabla v\,dx - \frac{1}{t}\int_{\{u+tv\geq\overline{M}\}} A(x,u)|\nabla u|^2\,dx \\ &\leq \int_{\{u+tv<\overline{M}\}} A(x,u)\nabla u\cdot\nabla v\,dx\,,\end{aligned}$$

whence

$$\int_{\{u+tv<\overline{M}\}} A(x,u)\nabla u\cdot\nabla v\,dx \geq -\int_{\{u<\overline{M}\}} B(x,u)\,|\nabla u|^2\,\frac{u_t - u}{t}\,dx + \int_{\{u<\overline{M}\}} \hat{g}(x,u)\,\frac{u_t-u}{t}\,dx\,.$$

Since $0 \leq u_t - u \leq tv$, we can go to the limit as $t \to 0^+$, obtaining

$$\begin{aligned}\int_\Omega A(x,u)\nabla u \cdot \nabla v\,dx &= \int_{\{u<\overline{M}\}} A(x,u)\nabla u \cdot \nabla v\,dx \\ &\geq -\int_{\{u<\overline{M}\}} B(x,u)\,|\nabla u|^2\,v\,dx + \int_{\{u<\overline{M}\}} \hat{g}(x,u)\,v\,dx \\ &= -\int_\Omega B(x,u)\,|\nabla u|^2\,v\,dx + \int_\Omega \hat{g}(x,u)\,v\,dx \\ &\qquad\qquad - \int_{\{u=\overline{M}\}} \hat{g}(x,\overline{M})\,v\,dx \\ &\geq -\int_\Omega B(x,u)\,|\nabla u|^2\,v\,dx + \int_\Omega \hat{g}(x,u)\,v\,dx\,.\end{aligned}$$

Arguing on $u_t = \max\{u - tv, \underline{M}\}$, one can prove in a similar way that

$$\int_\Omega A(x,u)\nabla u \cdot \nabla v\,dx \leq -\int_\Omega B(x,u)\,|\nabla u|^2\,v\,dx + \int_\Omega \hat{g}(x,u)\,v\,dx\,,$$

whence

$$\int_\Omega \left[A(x,u)\nabla u \cdot \nabla v + B(x,u)\,|\nabla u|^2\,v\right]dx = \int_\Omega \hat{g}(x,u)\,v\,dx$$
$$\text{for any } v \in W_0^{1,2}(\Omega) \cap L^\infty(\Omega) \text{ with } v \geq 0 \text{ a.e. in } \Omega$$

and the assertion follows. □

Proposition 6.3. *Let Ω be connected and assume that $u \in W_0^{1,2}(\Omega) \cap L^\infty(\Omega)$ satisfies $u \geq 0$ a.e. in Ω, $u > 0$ on a set of positive measure and*

$$\int_\Omega \left[A(x,u)\nabla u \cdot \nabla v + B(x,u)\,|\nabla u|^2\,v\right]dx$$
$$\geq \int_\Omega g(x,u)\,v\,dx \qquad \text{for any } v \in W_0^{1,2}(\Omega) \cap L^\infty(\Omega) \text{ with } v \geq 0 \text{ a.e. in } \Omega\,.$$

Then we have

$$\operatorname*{ess\,inf}_C u > 0 \text{ for every compact subset } C \text{ of } \Omega\,.$$

Proof. By Propositions 3.4 and 6.1, we may assume without loss of generality that

$$B(x,s) \leq 0 \qquad \text{for a.e. } x \in \Omega \text{ and every } s \in \mathbb{R} \text{ with } |s| \leq \|u\|_\infty\,.$$

Then for every $v \in W_0^{1,2}(\Omega) \cap L^\infty(\Omega)$ with $v \geq 0$ a.e. in Ω, we have

$$\int_\Omega A(x,u)\nabla u \cdot \nabla v\,dx \geq \int_\Omega g(x,u)\,v\,dx = \int_\Omega \gamma(x,u)\,uv\,dx$$

with $\gamma(x,u) \in L^\infty(\Omega)$.

From [11, Theorem 8.15 and Remark 8.16] the assertion follows. □

Lemma 6.4. *Assume that Ω is connected and that $\lambda_1 < 0$. Moreover, according to (h_1), let $\beta > 0$ be such that*

$$|g(x,s)| \leq \beta|s| \qquad \text{for a.e. } x \in \Omega \text{ and every } s \in \mathbb{R} \text{ with } \underline{M} \leq s \leq \overline{M}\,.$$

Let also $\psi : \mathbb{R} \to [0,1]$ *be a continuous function, with* $\psi(0) > 0$ *and* $\psi(s) = 0$ *outside* $]\underline{M}, \overline{M}[$, *and consider the problem*

$$\begin{cases} (u,t) \in K \times [0,1]\,, \\ \displaystyle\int_\Omega \left[A(x,u)\nabla u \cdot \nabla(v-u) + B(x,u)\,|\nabla u|^2\,(v-u)\right] dx \geq \int_\Omega g_t(x,u)(v-u) \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{for every } v \in K\,, \end{cases} \tag{6.1}$$

where

$$g_t(x,s) = g(x,s) + t(\psi(s) + \beta\, s^-)\,.$$

Denote by $\widehat{Z}^{tot}$ *the set of solutions* (u,t) *of* (6.1) *and let*

$$\widehat{Z} = \left\{(u,t) \in \widehat{Z}^{tot} :\ u \geq 0 \text{ a.e. in } \Omega \text{ and } u > 0 \text{ on a set of positive measure}\right\}\,.$$

Then there exist $0 < r_1 < r_2$ *such that*

$$\begin{aligned} \widehat{Z} = \left\{(u,t) \in \widehat{Z}^{tot} :\ \int_\Omega (u^-)^2\,dx < r_1^2 < r_2^2 < \int_\Omega (u^+)^2\,dx\right\} \\ = \left\{(u,t) \in \widehat{Z}^{tot} :\ \int_\Omega (u^-)^2\,dx \leq r_1^2 < r_2^2 \leq \int_\Omega (u^+)^2\,dx\right\}\,. \end{aligned}$$

Proof. If we set

$$a_t(x,s,\xi) = A(x,s)\xi\,, \qquad b_t(x,s,\xi) = B(x,s)|\xi|^2 - g_t(x,s)\,,$$

it is easily seen that assumptions (UN) and (UM) are satisfied. From Theorem 5.8 we infer that $\widehat{Z}^{tot}$ is compact in $W_0^{1,p}(\Omega) \times [0,1]$.

First of all, we claim that there exists $r_2 > 0$ such that

$$\widehat{Z} \subseteq \left\{(u,t) \in \widehat{Z}^{tot} :\ r_2^2 < \int_\Omega (u^+)^2\,dx\right\}\,.$$

By Propositions 3.4 and 6.1, we may assume without loss of generality that

$$B(x,s) \geq 0 \qquad \text{for a.e. } x \in \Omega \text{ and every } s \in \mathbb{R} \text{ with } \underline{M} \leq s \leq \overline{M}\,.$$

Assume, for a contradiction, that (u_k, t_k) is a sequence in $\widehat{Z}$ with $\|u_k\|_2 \to 0$. Then we may suppose, without loss of generality, that (u_k) is convergent to 0 in $W_0^{1,p}(\Omega)$ and a.e. in Ω and that (t_k) is convergent to some $t \in [0,1]$. Let $u_k = \tau_k z_k$ with $\tau_k = \|\nabla u_k\|_2$ and, up to a subsequence, (z_k) weakly convergent to some z in $W_0^{1,2}(\Omega)$.

If $v \in K \setminus \{0\}$ with $v \geq 0$ a.e. in Ω, we have

$$\begin{aligned} &\int_\Omega \left[A(x,u_k)\nabla z_k \cdot \nabla(v-u_k) + \tau_k\, B(x,u_k)\,|\nabla z_k|^2\,(v-u_k)\right] dx \\ &\qquad \geq \int_\Omega \frac{g(x,\tau_k z_k)}{\tau_k}\,(v-u_k)\,dx + \frac{t_k}{\tau_k}\int_\Omega \psi(u_k)(v-u_k)\,dx\,, \end{aligned}$$

which implies that (t_k/τ_k) is bounded hence convergent, up to a subsequence, to some $\sigma \geq 0$.

Then we also get

$$\int_\Omega A(x,0)\nabla z \cdot \nabla v\,dx \geq \int_\Omega D_s g(x,0) z v\,dx + \sigma\,\psi(0)\int_\Omega v\,dx \qquad \text{for any } v \in K\,,$$

whence

$$\int_\Omega [A(x,0)\nabla z \cdot \nabla v - D_s g(x,0) z v]\, dx = \sigma\, \psi(0) \int_\Omega v\, dx$$

$$\text{for any } v \in W_0^{1,2}(\Omega) \cap L^\infty(\Omega)\,.$$

If we choose $v = \varphi_1$, where φ_1 is a positive eigenfunction of (1.3) associated with $\lambda_1 < 0$, we get

$$\lambda_1 \int_\Omega z\varphi_1\, dx = \sigma\psi(0) \int_\Omega \varphi_1\, dx\,,$$

whence $z = 0$.

Finally, the choice $v = 0$ in (6.1) yields

$$\int_\Omega A(x,u_k)|\nabla z_k|^2\, dx \le \int_\Omega \left[A(x,u_k)|\nabla z_k|^2 + \tau_k\, B(x,u_k)\, |\nabla z_k|^2\, z_k\right] dx$$

$$\le \int_\Omega \frac{g(x,\tau_k z_k)}{\tau_k}\, z_k\, dx + \frac{t_k}{\tau_k} \int_\Omega \psi(u_k) z_k\, dx\,.$$

We infer that $\|\nabla z_k\|_2 \to 0$ and a contradiction follows.

With this choice of r_2, we also have

$$\widehat{Z} \subseteq \left\{(u,t) \in \widehat{Z}^{tot} :\ \int_\Omega (u^-)^2\, dx < r_1^2 < r_2^2 < \int_\Omega (u^+)^2\, dx\right\}$$

$$\subseteq \left\{(u,t) \in \widehat{Z}^{tot} :\ \int_\Omega (u^-)^2\, dx \le r_1^2 < r_2^2 \le \int_\Omega (u^+)^2\, dx\right\}$$

for every $r_1 \in]0, r_2[$. Now we claim that there exists $r_1 \in]0, r_2[$ such that

$$\left\{(u,t) \in \widehat{Z}^{tot} :\ \int_\Omega (u^-)^2\, dx \le r_1^2 < r_2^2 \le \int_\Omega (u^+)^2\, dx\right\} \subseteq \widehat{Z}\,.$$

By Propositions 3.4 and 6.1, now we may assume without loss of generality that

$$B(x,s) \le 0 \qquad \text{for a.e. } x \in \Omega \text{ and every } s \in \mathbb{R} \text{ with } \underline{M} \le s \le \overline{M}\,.$$

Assume, for a contradiction, that (u_k, t_k) is a sequence in the set at the left hand side with $(u_k, t_k) \notin \widehat{Z}$ and $\|u_k^-\|_2 \to 0$. Then, up to a subsequence, (u_k) is convergent to some $u \in \widehat{Z}$ in $W_0^{1,p}(\Omega)$ and a.e. in Ω, while (t_k) is convergent to some $t \in [0,1]$. By Propositions 6.2 and 6.3 we have $u > 0$ a.e. in Ω. If we write $u_k^- = \tau_k z_k$ with $\tau_k = \|\nabla u_k^-\|_2$, we have that (z_k) is weakly convergent, up to a subsequence, to some z in $W_0^{1,2}(\Omega)$ and, on the other hand, (z_k) is convergent to 0 a.e. in Ω, as $u > 0$. The choice $v = u_k^+$ in (6.1) yields

$$\int_\Omega \left[A(x,u_k)\nabla u_k \cdot \nabla u_k^- + B(x,u_k)\, |\nabla u_k|^2\, u_k^-\right] dx$$

$$\ge \int_\Omega g(x,u_k)\, u_k^-\, dx + t_k \int_\Omega \left[\psi(u_k) + \beta u_k^-\right] u_k^-\, dx\,,$$

whence

$$-\int_\Omega A(x,u_k)|\nabla z_k|^2\, dx \ge \int_\Omega \frac{g(x,-\tau_k z_k)}{\tau_k}\, z_k\, dx\,.$$

We infer that $\|\nabla z_k\|_2 \to 0$ and a contradiction follows. □

Proposition 6.5. *If Ω is connected and $\lambda_1 < 0$, then we have $Z_+, Z_- \in \mathcal{Z}$ and*

$$\mathrm{ind}\,(Z_+) = \mathrm{ind}\,(Z_-) = 1\,.$$

Proof. Let g_t, $\widehat{Z}^{tot}$, $\widehat{Z}$, r_1 and r_2 be as in Lemma 6.4. We aim to apply again Theorem 5.8 with

$$a_t(x,s,\xi) = A(x,s)\xi\,, \qquad b_t(x,s,\xi) = B(x,s)|\xi|^2 - g_t(x,s)\,.$$

By Lemma 6.4 the set $\widehat{Z}$ is open and closed in $\widehat{Z}^{tot}$. From Theorem 5.8 we infer that

$$Z_+ = \widehat{Z}_0 \in \mathcal{Z}(H_0,K) = \mathcal{Z}$$

and that

$$\mathrm{ind}\,(Z_+) = \mathrm{ind}\left((H_0,K),\widehat{Z}_0\right) = \mathrm{ind}\left((H_1,K),\widehat{Z}_1\right)\,.$$

Now we claim that $\widehat{Z}_1 = Z^{tot}(H_1,K)$. By Propositions 3.4 and 6.1 we may assume without loss of generality that

$$B(x,s) \le 0 \qquad \text{for a.e. } x\in\Omega \text{ and every } s\in\mathbb{R} \text{ with } \underline{M} \le s \le \overline{M}\,.$$

If we take $v \in K\setminus\{0\}$ with $v \ge 0$ a.e. in Ω in (6.1), we see that $0 \notin Z^{tot}(H_1,K)$. Moreover, if $u \in Z^{tot}(H_1,K)$, the choice $v = u^+$ yields $(v-u) = u^-$, hence

$$\begin{aligned}
-\int_\Omega A(x,u)|\nabla u^-|^2\,dx \ge \int_\Omega \left[A(x,u)\nabla u\cdot\nabla u^- + B(x,u)\,|\nabla u|^2\,u^-\right]dx \\
\ge \int_\Omega g_1(x,u)u^-\,dx \ge 0\,.
\end{aligned}$$

Therefore $u^- = 0$, whence $u \in \widehat{Z}_1$ and the claim is proved.

From Theorem 5.10 we infer that

$$\mathrm{ind}\left((H_1,K),\widehat{Z}_1\right) = \mathrm{ind}\left((H_1,K),Z^{tot}(H_1,K)\right) = 1$$

and the assertion concerning Z_+ follows.

The assertion concerning Z_- can be proved in a similar way. ☐

Proposition 6.6. *If there exists $k \ge 0$ with $\lambda_k < 0 < \lambda_{k+1}$, then $\{0\} \in \mathcal{Z}$ and*

$$\mathrm{ind}\,(\{0\}) = (-1)^k\,.$$

Proof. If we set

$$\begin{aligned}
A_t(x,s) &= A(x,ts)\,, \qquad B_t(x,s) = t\,B(x,ts)\,,\\
g_t(x,s) &= \begin{cases} \dfrac{g(x,ts)}{t} & \text{if } 0 < t \le 1\,,\\ D_sg(x,0)s & \text{if } t = 0\,,\end{cases}\\
a_t(x,s,\xi) &= A_t(x,s)\xi\,, \qquad b_t(x,s,\xi) = B_t(x,s)|\xi|^2 - g_t(x,s)\,,
\end{aligned}$$

it is easily seen that assumptions (UN) and (UM) are satisfied. We aim to apply Theorem 5.8.

We claim that there exists $r > 0$ such that, if $(u,t) \in \widehat{Z}^{tot}$ and $\|\nabla u\|_2 \le r$, then $u = 0$. Assume, for a contradiction, that (u_k,t_k) is a sequence in $\widehat{Z}^{tot}$ with $u_k \ne 0$ and $\|\nabla u_k\|_2 \to 0$. Let $u_k = \tau_k z_k$ with $\tau_k = \|\nabla u_k\|_2$ and, up to a subsequence, (z_k) weakly convergent to some z in $W^{1,2}_0(\Omega)$ and (t_k) convergent to some t in $[0,1]$.

Given $v \in K$, if $t_k > 0$ we have

$$\begin{aligned}(6.2)\quad \int_\Omega \big[A(x,t_k u_k)\nabla u_k \cdot \nabla(v-u_k) + t_k\, B(x,t_k u_k)\,|\nabla u_k|^2\,(v-u_k)\big]\,dx \\ \geq \int_\Omega \frac{g(x,t_k u_k)}{t_k}\,(v-u_k)\,dx\,,\end{aligned}$$

whence

$$\begin{aligned}\int_\Omega \big[A(x,t_k u_k)\nabla z_k \cdot \nabla(v-u_k) + \tau_k t_k\, B(x,t_k u_k)\,|\nabla z_k|^2\,(v-u_k)\big]\,dx \\ \geq \int_\Omega \frac{g(x,\tau_k t_k z_k)}{\tau_k t_k}\,(v-u_k)\,dx\,.\end{aligned}$$

Going to the limit as $k \to \infty$, we get

$$\int_\Omega A(x,0)\nabla z\cdot\nabla v\,dx \geq \int_\Omega D_s g(x,0) z v\,dx \qquad \text{for any } v \in K\,.$$

If $t_k = 0$, we have

$$\int_\Omega A(x,0)\nabla z_k\cdot\nabla(v-u_k)\,dx \geq \int_\Omega D_s g(x,0) z_k(v-u_k)\,dx$$

and the same conclusion easily follows.

Then we infer that

$$\int_\Omega A(x,0)\nabla z\cdot\nabla v\,dx = \int_\Omega D_s g(x,0) z v\,dx \qquad \text{for any } v \in W_0^{1,2}(\Omega)\cap L^\infty(\Omega)\,,$$

whence $z = 0$, as 0 is not in the sequence (λ_k).

By Propositions 3.4 and 6.1, we may assume without loss of generality that

$$B(x,s) \geq 0 \qquad \text{for a.e. } x \in \Omega \text{ and every } s \in \mathbb{R} \text{ with } \underline{M} \leq s \leq \overline{M}\,.$$

If $t_k > 0$, the choice $v = -u_k^-$ in (6.2) yields

$$\begin{aligned}\int_\Omega A(x,t_k u_k)|\nabla z_k^+|^2\,dx \leq \int_\Omega \big[A(x,t_k u_k)|\nabla z_k^+|^2 + \tau_k t_k\, B(x,t_k u_k)\,|\nabla z_k|^2\, z_k^+\big]\,dx \\ \leq \int_\Omega \frac{g(x,\tau_k t_k z_k)}{\tau_k t_k}\, z_k^+\,dx\,,\end{aligned}$$

which implies that $\|\nabla z_k^+\|_2 \to 0$. If $t_k = 0$ the argument is analogous and simpler. In a similar way one can show that $\|\nabla z_k^-\|_2 \to 0$ and a contradiction follows. Therefore, there exists $r > 0$ with the required property.

In particular, we can apply Theorem 5.8 with

$$\widehat{Z} = \{0\}\times[0,1]\,,$$

obtaining

$$\{0\} = \widehat{Z}_1 \in \mathcal{Z}(H_1,K) = \mathcal{Z}$$

and

$$\operatorname{ind}(\{0\}) = \operatorname{ind}\Big((H_1,K),\widehat{Z}_1\Big) = \operatorname{ind}\Big((H_0,K),\widehat{Z}_0\Big) = \operatorname{ind}((H_0,K),\{0\})\,.$$

On the other hand, if we set

$$U = \left\{u \in W_0^{1,2}(\Omega):\ \|\nabla u\|_2 < r\right\}\,,$$

from Proposition 5.5 we infer that

$$\mathrm{ind}((H_0, K), \{0\}) = \deg((H_0, K), U, 0)\,.$$

Now, if we set

$$K_t = \begin{cases} \frac{1}{t}\,K & \text{if } 0 < t \leq 1\,, \\ W_0^{1,2}(\Omega) & \text{if } t = 0\,, \end{cases}$$

from [11, Theorem 4.53 and Proposition 4.61] we deduce that

$$\deg((H_0, K), U, 0) = \deg((H_0, W_0^{1,2}(\Omega)), U, 0)\,.$$

Finally, from [13, Theorem 2.5.2] it follows that

$$\deg((H_0, W_0^{1,2}(\Omega)), U, 0) = (-1)^k\,.$$

□

Proof of Theorem 1.1.
From Proposition 6.5 we know that

$$\mathrm{ind}\,(Z_+) = \mathrm{ind}\,(Z_-) = 1\,.$$

By Theorem 5.6 and Propositions 6.2 and 6.3 we infer that there exist at least two solutions $u_1 \in Z_-$ and $u_2 \in Z_+$ of (1.2) with

$$\operatorname*{ess\,sup}_C\, u_1 < 0 < \operatorname*{ess\,inf}_C\, u_2 \quad \text{for every compact subset } C \text{ of } \Omega\,.$$

Assume, for a contradiction, that

$$Z^{tot} = Z_- \cup \{0\} \cup Z_+$$

with $\mathrm{ind}\,(\{0\}) = 1$ by Proposition 6.6. From Theorems 5.10 and 5.7 we infer that

$$1 = \mathrm{ind}\left(Z^{tot}\right) = 3$$

and a contradiction follows. Therefore there exists

$$u_3 \in Z^{tot} \setminus (Z_- \cup \{0\} \cup Z_+)\,.$$

By Proposition 6.2 u_3 is a sign-changing solution of (1.2). According to [10, Theorem VII.1.1], each u_j is locally Hölder continuous in Ω. □

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