A DIRICHLET PROBLEM WITH FREE GRADIENT DISCONTINUITY

Michele Carriero
Università del Salento, Dipartimento di Matematica "Ennio De Giorgi"
Strada provinciale per Arnesano, Lecce, 73100 Italy
(michele.carriero@unisalento.it)

Antonio Leaci
Università del Salento, Dipartimento di Matematica "Ennio De Giorgi"
Strada provinciale per Arnesano, Lecce, 73100 Italy
(antonio.leaci@unisalento.it)

Franco Tomarelli
Politecnico di Milano, Dipartimento di Matematica "Francesco Brioschi"
Piazza Leonardo da Vinci, 32, Milano, 20133 Italy
(franco.tomarelli@polimi.it)

Abstract. We prove the existence of a strong solution for Blake & Zisserman functional under Dirichlet boundary condition. The result is obtained by showing partial regularity of weak solutions up to the boundary through blow-up technique and a decay property for bi-harmonic functions in half-disk.

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1. Introduction

Minimization of Blake & Zisserman functional is a variational approach to contour detection in image analysis which deals with free discontinuity and second derivatives.

Inpainting is the process of filling in the missing or desired image information on domains where it is unavailable: this blank domains may correspond to scratches in a camera picture, occlusion by objects, blotches in an old movie film, aging of canvas and colors of a painting ([18], [19]).

In this paper we study Blake & Zisserman functional with Dirichlet boundary conditions. There are several motivations for this kind of boundary condition, among them we mention the possibility of facing some version of the inpainting problem. Moreover we emphasize the relevance of global minimizer under Dirichlet boundary conditions in the study of local minimizer under classic Neumann boundary conditions problem for Blake & Zisserman functional, since the last one is defined as a global minimizer under Dirichlet boundary condition ([14], [17]).

We refer to [5], [8], [9], [10], [14], [27], [28] for motivation and background analysis of variational approach to image segmentation and digital image processing.

Here we focus on the functional

$$\begin{aligned} F(K_0,K_1,v) \;=\; & \int_{\widetilde{\Omega}\setminus(K_0\cup K_1)} \left(|\,D^2 v\,|^2 + \mu|v-g|^q \right) d\mathbf{x} \\ & + \alpha\,\mathcal{H}^1\left(K_0\cap\widetilde{\Omega}\right) + \beta\,\mathcal{H}^1\left((K_1\setminus K_0)\cap\widetilde{\Omega}\right), \end{aligned}$$

with the aim of minimizing F among admissible triplets (K_0,K_1,v), say triplets fulfilling

$$\begin{cases} K_0\,,\ K_1 \text{ Borel subsets of } \mathbb{R}^2, \qquad K_0\cup K_1 \text{ closed}, \\ v\in C^2\left(\widetilde{\Omega}\setminus(K_0\cup K_1)\right),\ v \text{ approximately continuous in } \widetilde{\Omega}\setminus K_0, \\ v=w \text{ a.e. in } \widetilde{\Omega}\setminus\Omega\,, \end{cases}$$

where Ω, $\widetilde{\Omega}$ are open sets fulfilling $\Omega\subset\subset\widetilde{\Omega}\subset\subset\mathbb{R}^2$ and w is a given function in $\widetilde{\Omega}$.

If (K_0,K_1,u) is a minimizing triplet of F, then $K_0\cup K_1$ can be interpreted as an optimal segmentation of the monochromatic image of brightness intensity g, while the three elements of a minimizing triplet (K_0,K_1,u) play respectively the role of edges, creases and smoothly varying intensity in the region $\widetilde{\Omega}\setminus(K_0\cup K_1)$ for the segmented image. The second order functional F was introduced to overcome the over-segmentation of steep gradients (ramp effect) and other inconvenient which occur in lower order models as in case of Mumford & Shah functional ([5], [28], [29]). Here we mention our result in the simplified case of smooth boundary and smooth boundary datum in order to avoid technicalities in the assumptions.

Theorem 1.1. *Let α, β, μ, q, g, Ω, $\widetilde{\Omega}$ and w be s.t.*

$$0<\beta\le\alpha\le 2\beta,\ \ \mu>0,\ \ q>1,\ \ g\in L^q(\widetilde{\Omega})\cap L^{2q}_{loc}(\widetilde{\Omega}),\ \ w\in L^q(\widetilde{\Omega}), \tag{1.1}$$

$$\Omega\subset\subset\widetilde{\Omega}\subset\subset\mathbb{R}^2\,, \tag{1.2}$$

$$\Omega \textit{ is an open set with } C^2 \textit{ boundary } \partial\Omega\,,\ \widetilde{\Omega} \textit{ is an open set}, \tag{1.3}$$

$$w\in C^2(\widetilde{\Omega})\,, \tag{1.4}$$

$$D^2 w \in L^\infty(\widetilde{\Omega})\,. \tag{1.5}$$

Then there exists a triplet (C_0, C_1, u) which minimizes the functional F among admissible triplets (K_0, K_1, v), with $F(C_0, C_1, u) < +\infty$.
Moreover any minimizing triplet (K_0, K_1, v) fulfils:

$$K_0 \cap \widetilde{\Omega} \text{ and } K_1 \cap \widetilde{\Omega} \text{ are } (\mathcal{H}^1, 1) \text{ rectifiable sets,} \tag{1.6}$$

$$\mathcal{H}^1(K_0 \cap \widetilde{\Omega}) \;=\; \mathcal{H}^1(\overline{S_v})\,, \quad \mathcal{H}^1(K_1 \cap \widetilde{\Omega}) \;=\; \mathcal{H}^1(\overline{S_{\nabla v}} \setminus S_v)\,, \tag{1.7}$$

$$\begin{cases} v \in GSBV^2(\widetilde{\Omega}), \text{ hence} \\ v \text{ and } \nabla v \text{ have well defined two-sided traces, finite } \mathcal{H}^1 \text{ a.e. on } K_0 \cup K_1\,, \end{cases} \tag{1.8}$$

where S_v and $S_{\nabla v}$ respectively denote the singular sets of v and ∇v.

Actually more accurate results than Theorem 1.1 are needed for our purposes (see Remarks 2.4, 2.8): this will be achieved by Theorems 2.1, 2.2, 2.3 in Section 2, which allow discontinuous Dirichlet datum w and piece-wise C^2 boundary $\partial\Omega$. About the case of inpainting we emphasize that all these theorems can be proved exactly by the same technique whenever the integral of fidelity term $|v-g|^q$ is evaluated on $U \setminus (K_0 \cup K_1)$ instead of $\widetilde{\Omega} \setminus (K_0 \cup K_1)$, where $U \subset \Omega$ is a given, possibly empty Borel set: e.g. the image is noise-free in the region $\widetilde{\Omega} \setminus U$ and should be restored in U.
Theorems 2.1, 2.2, 2.3 are achieved by showing partial regularity of a suitably defined weak solution with penalized Dirichlet datum (Theorem 3.1). The novelty here consists in the regularization at the boundary for a free gradient discontinuity problem; the regularity is proven at points which have 2-dimensional energy density by: suitable joining along lunulae filling half-disk in order to take into account boundary effects (Lemma 4.1); blow-up at boundary points and at points near the boundary (Theorems 7.1, 7.2) proven by a new procedure, due to the presence of two parameters describing the lunulae; a decay estimate of the weak functionals evaluated at local minimizers (Theorems 7.3, 7.4).
Usually performing regularity analysis at boundary points requires a smooth extension with suitable estimates of the blown-up solution. The extension of bi-harmonic functions is quite different from extension of an harmonic function vanishing at the diameter, the last one is based on classical Schwarz reflection principle (see Remark 6.4) and doubles L^2 norm of the gradient in the whole disk: this doubling property was exploited in [6] to prove decay property for local minimizers of Mumford & Shah functional with Dirichlet boundary condition (see also [26]); unfortunately bi-harmonic extension lacks this doubling property (see Remark 6.5). We overcome this difficulty by a new tool, precisely an L^2 decay estimate of hessian for a bi-harmonic function in a half-disk vanishing together with its normal derivative on the diameter (Theorem 6.1): proving this decay requires a careful application of Duffin extension formula [22] and Almansi decomposition [1], since the bi-harmonic extension in the whole disk may increase a lot the L^2 norm of the hessian in the complementary half-disk.
The present paper focuses on the two dimensional case, nevertheless all the results proven here are valid in the n dimensional case except the compactness property (Theorem 5.4) and hessian decay of bi-harmonic functions in half-disk (Theorem 6.1 and Remark 7.5).
About minimization of the functional F under Neumann boundary condition we refer to [8], [9], [13]. About the description of the rich list of (differential, integral and geometric) extremality conditions for F we refer to [14].

The framework of the present paper will allow to prove existence of strong local minimizers for Blake & Zisserman functional and several extremality conditions fulfilled by strong local minimizers in the paper [17]. In general uniqueness of minimizers of the functionals F fails due to lack of convexity: we refer to [4] for explicit examples of multiplicity and property of generic uniqueness with respect to data α, β, g.
Moreover the results of the present paper are deeply exploited in [15], [16] and [17] to study fine properties of local minimizers of Blake & Zisserman functional under Neuman boundary condition, particularly about their singular set related to optimal segmentation.

2. Main result

The main result of this paper is the existence of strong minimizer for 2-D Blake & Zisserman functional F with Dirichlet boundary datum: Theorems 2.1, 2.2, 2.3. The boundary condition is prescribed by imposing a penalization whenever the competing function and its gradient do not coincide with the exterior traces of w and ∇w at $\partial\Omega$. Motivation and comments about the assumptions on Dirichlet datum are given in the Remarks at the end of this Section.
First we focus on the main part of F, say functional E:

$$E(K_0, K_1, v) = \int_{\widetilde{\Omega}\setminus(K_0\cup K_1\cup M)} \left|D^2 v\right|^2 \, d\mathbf{x} + \alpha\,\mathcal{H}^1\Big(K_0\cap\widetilde{\Omega}\Big) + \beta\,\mathcal{H}^1\Big((K_1\setminus K_0)\cap\widetilde{\Omega}\Big), \tag{2.1}$$

where $M\subset\partial\Omega$ is a prescribed finite set (possible corners in $\partial\Omega$), with the aim of minimizing it among admissible triplets (K_0, K_1, v), say triplets fulfilling

$$\begin{cases} K_0\,,\ K_1 \text{ Borel subsets of } \mathbb{R}^2, \qquad K_0\cup K_1 \text{ closed}, \\ v\in C^2\Big(\widetilde{\Omega}\setminus(K_0\cup K_1\cup M)\Big)\,,\ v \text{ approximately continuous in } \widetilde{\Omega}\setminus(K_0\cup M), \\ v = w \ \text{ a.e. in } \widetilde{\Omega}\setminus\Omega\,. \end{cases} \tag{2.2}$$

Theorem 2.1. *(***Strong solution of Dirichlet problem for the functional** *E)*
Let α, β, Ω, $\widetilde{\Omega}$, M, T_0, T_1 and w be s.t.

$$0<\beta\le\alpha\le 2\beta, \tag{2.3}$$

$$\Omega\subset\subset\widetilde{\Omega}\subset\subset\mathbb{R}^2\,, \tag{2.4}$$

$$\Omega \textit{ is an open set with Lipschitz boundary, } \widetilde{\Omega} \textit{ is an open set,} \tag{2.5}$$

$$\exists\ M \textit{ finite set s.t. each connected component of } (\partial\Omega\setminus M) \textit{ is uniformly } C^2, \tag{2.6}$$

$$(T_0\cup T_1)\cap\partial\Omega \ \textit{ is a finite set}\,, \tag{2.7}$$

$$T_0, T_1 \textit{ Borel sets}\,,\ T_0\cup T_1 \textit{ closed subset of } \mathbb{R}^2\,,\ \mathcal{H}^1\Big((T_0\cup T_1)\cap\widetilde{\Omega}\Big) < +\infty, \tag{2.8}$$

$$w\in C^2\Big(\widetilde{\Omega}\setminus(T_0\cup T_1)\Big)\,, \quad w \textit{ approximately continuous in } \widetilde{\Omega}\setminus T_0\,, \tag{2.9}$$

$$\text{(2.10)}\qquad \begin{cases} D^2 w \in L^2(\widetilde{\Omega} \setminus (T_0 \cup T_1)), \ D^2 w \in L^\infty\big(A \setminus (T_0 \cup T_1)\big) \\ \ \textit{with } A \textit{ open set s.t. } \partial\Omega \subset A \subset \widetilde{\Omega}, \\ \exists C > 0 \ : \ \|w\|_{L^\infty}, \|\nabla w\|_{L^\infty}, \|\nabla^2 w\|_{L^\infty} \le C \ \ \textit{in } A, \\ \mathrm{Lip}(\gamma') \le C \ \ \textit{with } \gamma \textit{ arc-length parametrization of } \partial\Omega, \\ \exists \bar{\varrho} > 0 \ : \ \mathcal{H}^1\big(\partial\Omega \cap B_\varrho(\mathbf{x})\big) < C\varrho \quad \forall \mathbf{x} \in \partial\Omega, \ \forall \varrho \le \bar{\varrho}, \end{cases}$$

$$\text{(2.11)}\qquad \begin{array}{l} \textit{there is no triplet } (\mathfrak{T}_0, \mathfrak{T}_1, \omega) \textit{ fulfilling:} \\ (2.8), (2.9), \ \ \omega = \mathrm{aplim}\, w \ \textit{in } \widetilde{\Omega} \setminus \mathfrak{T}_0, \ \textit{and } (\mathfrak{T}_0 \cup \mathfrak{T}_1) \underset{\neq}{\subset} (T_0 \cup T_1). \end{array}$$

Then there exists a triplet (C_0, C_1, u) *minimizing the functional* E *defined by (2.1) with finite energy, among admissible triplets* (K_0, K_1, v) *fulfilling (2.2).* $E(C_0, C_1, u) < \infty$. *Moreover any minimizing triplet* (K_0, K_1, v) *fulfills:*

$$\text{(2.12)}\qquad K_0 \cap \widetilde{\Omega} \textit{ and } K_1 \cap \widetilde{\Omega} \textit{ are } (\mathcal{H}^1, 1) \textit{ rectifiable sets,}$$

$$\text{(2.13)}\qquad \mathcal{H}^1(K_0 \cap \widetilde{\Omega}) \ = \ \mathcal{H}^1(\overline{S_v}), \quad \mathcal{H}^1(K_1 \cap \widetilde{\Omega}) \ = \ \mathcal{H}^1(\overline{S_{\nabla v}} \setminus S_v),$$

$$\text{(2.14)}\qquad \begin{cases} v \in GSBV^2(\widetilde{\Omega}), \ \textit{hence } v \textit{ and } \nabla v \\ \textit{have well defined two-sided traces, finite } \mathcal{H}^1 \textit{ a.e. on } K_0 \cup K_1, \end{cases}$$

$$\text{(2.15)}\qquad \begin{cases} v \textit{ minimizes the functional } \mathcal{E} \textit{ defined below by (3.1)} \\ \textit{among } v \textit{ s.t. } v = w \ \ \textit{a.e. in } \ \widetilde{\Omega} \setminus \Omega, \end{cases}$$

$$\text{(2.16)}\qquad \mathcal{E}(v) = \ E(K_0, K_1, \widetilde{v}).$$

Theorem 2.2. *(***Strong solution of Dirichlet problem for Blake & Zisserman functional***)*
Let α, β, μ, q, g, Ω, $\widetilde{\Omega}$, M, T_0, T_1 *and* w *be s.t. (2.3)-(2.11) and*

$$\text{(2.17)}\qquad \mu > 0, \ \ q > 1, \ \ g \in L^q(\widetilde{\Omega}) \cap L^{2q}_{loc}(\widetilde{\Omega}), \ \ w \in L^q(\widetilde{\Omega})$$

hold true.
Then there exists a triplet (C_0, C_1, u) *which minimizes Blake & Zisserman functional* F :

$$\text{(2.18)}\qquad F(K_0, K_1, v) \ = \ E(K_0, K_1, v) \ + \ \mu \int_{\widetilde{\Omega}} |\, v - g\,|^q \, d\mathbf{x},$$

among triplets (K_0, K_1, v) *fulfilling (2.2);* $F(C_0, C_1, u) < +\infty$*; (2.12), (2.13) and (2.14) hold true for any minimizing triplet* (K_0, K_1, v)*; any third element* v *of a minimizing triplet minimizes also the weak functional* $\mathcal{F}$ *defined by (3.7) below under constraint* $v = w$ *a.e. in* $\widetilde{\Omega} \setminus \Omega$*, and*

$$\text{(2.19)}\qquad \mathcal{F}(v) = \ F(K_0, K_1, \widetilde{v}).$$

Theorem 2.3. *Assume* α, β, μ, q, g, Ω, $\widetilde{\Omega}$, M, T_0, T_1, *and* w *fulfil (2.3)-(2.11), and*

$$\text{(2.20)}\qquad \alpha \ = \ \beta.$$

Then there exists a pair (K, v) *which minimizes the functional*

$$\text{(2.21)}\qquad \int_{\widetilde{\Omega} \setminus (K \cup M)} \big|\, D^2 v\, \big|^2 \, d\mathbf{x} \ + \ \alpha\, \mathcal{H}^1\left(K \cap \widetilde{\Omega}\right)$$

among pairs (K, v) *with closed* $K \subset \mathbb{R}^2$, $v \in C^2(\widetilde{\Omega} \setminus (K \cup M))$ *and* $v = w$ *a.e. in* $\widetilde{\Omega} \setminus \Omega$.

If in addition (2.17) holds true then there exists a pair (K, v) *which minimizes the functional*

$$\int_{\widetilde{\Omega}\setminus(K\cup M)} \left(\left| D^2 v \right|^2 + \mu \, |v-g|^q \right) d\mathbf{x} + \alpha \, \mathcal{H}^1 \left(K \cap \widetilde{\Omega} \right) \tag{2.22}$$

among pairs (K, v) *with closed* $K \subset \mathbb{R}^2$, $v \in C^2(\widetilde{\Omega} \setminus (K \cup M))$, *and* $v = w$ *a.e. in* $\widetilde{\Omega} \setminus \Omega$.
In both cases $K \cap \widetilde{\Omega}$ *is* $(\mathcal{H}^1, 1)$ *rectifiable for optimal* K.
Moreover:
if the pair (K, v) *minimizes (2.22) then* v *minimizes* $\mathcal{F}$ *among* $v \in GSBV^2(\widetilde{\Omega})$ *s.t.* $v = w$ *a.e. in* $\widetilde{\Omega} \setminus \Omega$;
if the pair (K, v) *minimizes (2.21) then* v *minimizes* $\mathcal{E}$ *among* $v \in GSBV^2(\widetilde{\Omega})$ *s.t.* $v = w$ *a.e. in* $\widetilde{\Omega} \setminus \Omega$.

Remark 2.4. *Theorems 2.1-2.3 can be exploited in the analysis of locally minimizing triplets of* E *and* F. *About locally minimizing triplets we refer to Definition 2.6 in* [17].

Remark 2.5. *Hypothesis (2.11) seems a technical assumption; actually it is a natural requirement: (2.11) simply says that the datum is expressed by an essential triplet (in the sense of Definition 2.7 in* [17], *see also* [9]*); in particular (2.11) is automatically fulfilled in case of a smooth datum.*

Remark 2.6. *The technical hypotheses* (2.7)-(2.10) *are natural assumptions: actually the whole set of hypotheses says that* w *represents a Dirichlet datum which is noise-free in the region* $\widetilde{\Omega} \setminus \Omega$, *as it is typical when facing inpainting problem if noise, blotches and all artifact to be removed are contained in* $\overline{\Omega}$.

Remark 2.7. *Finiteness property for* $(T_0 \cup T_1) \cap \partial\Omega$ *in hypothesis (2.7) can be weakened as follows (in Theorems 2.1, 2.2, 2.3):*

$$\mathcal{H}^1 \left((T_0 \cup T_1) \cap \partial\Omega \right) = 0 . \tag{2.7'}$$

Remark 2.8. *Assumption (2.7) is fulfilled when* $(T_0 \cup T_1) \cap \partial\Omega$ *is a single point; assumption (2.10) requires* $D^2 w$ *in* L^∞ *only around the boundary and far away from singular set* $(T_0 \cup T_1)$ *of Dirichlet datum. Hence Theorem 2.1 applies with the choices* $T_0 =$ *negative real axis,* $T_1 = \emptyset$, $\Omega = \mathbb{R}^2$ *and* $w = W$, *where in polar coordinates*

$$W(r,\theta) = \pm \sqrt{\frac{\alpha}{193\,\pi}}\; r^{3/2} \left(\sqrt{21} \left(\sin\frac{\theta}{2} - \frac{5}{3} \sin\left(\frac{3}{2}\theta\right) \right) \pm \left(\cos\frac{\theta}{2} - \frac{7}{3} \cos\left(\frac{3}{2}\theta\right) \right) \right)$$

is a candidate nontrivial local minimizer for $\mathcal{E}$ *in* $\mathbb{R}^2$ *(see* [17]*). Notice that* W *belongs to* $H^2(B_1(\mathbf{0}) \setminus T_0)$ *while* $D^2 W$ *is not bounded around the origin:* $D^2 W$ *has a singularity of order* $r^{-1/2}$.

3. Weak Dirichlet problem for Blake & Zisserman functional

We denote by $B_\varrho(\mathbf{x})$ the open ball $\{\mathbf{y} \in \mathbb{R}^2;\ |\mathbf{y} - \mathbf{x}| < \varrho\}$, and set $B_\varrho = B_\varrho(\mathbf{0})$, $B_\varrho^+ = B_\varrho \cap \{(x,y) :\ y > 0\}$, $B_\varrho^- = B_\varrho \cap \{(x,y) :\ y < 0\}$. We denote by χ_U the characteristic function of U for any $U \subset \mathbb{R}^2$. If x, y are real numbers we denote by $[x]$ the integer part of x and set $x \vee y = \max\{x, y\}$, $x \wedge y = \min\{x, y\}$. For any pair of vectors $\mathbf{a}$, $\mathbf{b}$ we denote the tensor product by $\mathbf{a} \otimes \mathbf{b}$ and set $\mathbf{a} \odot \mathbf{b} = (1/2)(\mathbf{a} \otimes \mathbf{b} + \mathbf{b} \otimes \mathbf{a})$.
For any pair of 2×2 matrices A, B we set $A : B = \sum_{i,j=1}^2 A_{ij} B_{ij}$. For any real $s > 1$ we

denote by s' the conjugate exponent $s' = s/(s-1)$.
For any Borel function $v : \Omega \to \mathbb{R}$ and $\mathbf{x} \in \Omega$, $z \in \overline{\mathbb{R}} := \mathbb{R} \cup \{-\infty, +\infty\}$, we set $z = \operatorname{ap}\lim_{\mathbf{y}\to\mathbf{x}} v(\mathbf{y})$ (*approximate limit* of v at $\mathbf{x}$) if, for every $g \in C^0(\overline{\mathbb{R}})$,

$$g(z) = \lim_{\varrho\to 0} \fint_{B_\varrho(\mathbf{0})} g(v(\mathbf{x}+\boldsymbol{\xi}))\, d\boldsymbol{\xi}\,;$$

the function $\widetilde{v}(\mathbf{x}) = \operatorname{ap}\lim_{\mathbf{y}\to\mathbf{x}} v(\mathbf{y})$ is called *representative* of v; the *singular set* of v is $S_v = \{\mathbf{x} \in \Omega : \nexists z \; s.t. \operatorname{ap}\lim_{\mathbf{y}\to\mathbf{x}} v(\mathbf{y}) = z\}$.
A Borel function $v : \Omega \to \mathbb{R}$ is *approximately continuous at* $\mathbf{x} \in \Omega$ iff $v(\mathbf{x}) = \operatorname{ap}\lim_{\mathbf{y}\to\mathbf{x}} v(\mathbf{y})$.
By referring to [2], [9], [14]: Dv denotes the distributional gradient of v, $\nabla v(\mathbf{x})$ denotes the approximate gradient of v, $SBV(\Omega)$ denotes the De Giorgi class of functions $v \in BV(\Omega)$ such that

$$\int_\Omega |Dv| = \int_\Omega |\nabla v|\, d\mathbf{x} + \int_{S_v} |v^+ - v^-|\, d\mathcal{H}^1.$$

$$SBV_{loc}(\Omega) := \{v \in SBV(\Omega') : \; \forall \Omega' \subset\subset \Omega\},$$

$$GSBV(\Omega) := \left\{v : \Omega \to \mathbb{R} \text{ Borel function}; -k \vee v \wedge k \in SBV_{loc}(\Omega) \; \forall k \in \mathbb{N}\right\}.$$

$$GSBV^2(\Omega) := \left\{v \in GSBV(\Omega), \; \nabla v \in \big(GSBV(\Omega)\big)^2\right\}.$$

We will exploit the weak formulation $\mathcal{E}$ of the functional E introduced in [8]:

$$\mathcal{E}(v) = \int_{\widetilde{\Omega}} |\nabla^2 v|^2\, d\mathbf{x} + \alpha\, \mathcal{H}^1(S_v) + \beta\, \mathcal{H}^1(S_{\nabla v} \setminus S_v)\,. \tag{3.1}$$

Theorem 3.1. (**Dirichlet problem for weak form of Blake & Zisserman functionals $\mathcal{F}$ and $\mathcal{E}$**)
Assume (2.3), (2.4), (2.5) and

$$w \in C^2\left(\widetilde{\Omega} \setminus \overline{(S_w \cup S_{\nabla w})}\right), \quad w \text{ approximately continuous in } \widetilde{\Omega} \setminus S_w\,, \tag{3.2}$$

$$\mathcal{E}(w) < +\infty \tag{3.3}$$

$$\mathcal{H}^1\left(\overline{(S_w \cup S_{\nabla w})} \setminus (S_w \cup S_{\nabla w})\right) = 0\,, \tag{3.4}$$

$$\mathcal{H}^1\left(\overline{(S_w \cup S_{\nabla w})} \cap \partial\Omega\right) = 0 \qquad (\text{ or } \overline{(S_w \cup S_{\nabla w})} \cap \partial\Omega \text{ finite})\,. \tag{3.5}$$

Set

$$X(\widetilde{\Omega}) \stackrel{\text{def}}{=} \left\{ v \in GSBV^2(\widetilde{\Omega}) \; s.t. \; v = w \; a.e. \; in \; \widetilde{\Omega} \setminus \Omega \right\}. \tag{3.6}$$

Then there exists u minimizing the the functional $\mathcal{E}$ in $X(\widetilde{\Omega})$ with finite energy.
Moreover, if $\mu > 0$, $g \in L^q(\widetilde{\Omega})$ and $w \in L^q(\widetilde{\Omega})$, then there exists u which minimizes the functional $\mathcal{F}$ in $X(\widetilde{\Omega})$ with finite energy:

$$\mathcal{F}(v) = \int_{\widetilde{\Omega}} |\nabla^2 v|^2\, d\mathbf{x} + \alpha\, \mathcal{H}^1(S_v) + \beta\, \mathcal{H}^1(S_{\nabla v} \setminus S_v) + \mu \int_{\widetilde{\Omega}} |v - g|^q\, d\mathbf{x} \tag{3.7}$$

Proof. Assume $\mu > 0$, $g \in L^q(\widetilde{\Omega})$, $w \in L^q(\widetilde{\Omega})$. Obviously $\mathcal{F}(v) \geq 0 \;\; \forall v \in X(\widetilde{\Omega})$.
Assumptions (3.2), (3.3), (3.4), (3.5) entail $w \in X(\widetilde{\Omega})$ and $\mathcal{F}(w) < +\infty$.
Let $v_h \in X$ be a minimizing sequence for $\mathcal{F}$. By Theorem 8 in [8] there is v_∞ in $X(\widetilde{\Omega})$ and a subsequence s.t., without relabeling, $v_h \to v_\infty$ a.e. in $\widetilde{\Omega}$.
$v_h = w$ in $\widetilde{\Omega} \setminus \Omega$ entails $v_\infty = w$ in $\widetilde{\Omega} \setminus \Omega$. By Theorem 10 in [8]:

$$\mathcal{F}(v_\infty) \;\leq\; \liminf_h \, \mathcal{F}(v_h)\,,$$

hence $\mathcal{F}(v_\infty) = \inf_{v \in X(\widetilde{\Omega})} \mathcal{F}(v)$. The same argument applies to $\mathcal{E}$. □

Remark 3.2. *Assumptions of Theorems 2.1, 2.2 and 2.3 (about Dirichlet datum for strong formulation) on triplets (T_0, T_1, w) entail the assumptions (about Dirichlet datum for weak formulation) on w of Theorem 3.1.*

Remark 3.3. *So far we know that both $\mathcal{E}$ and $\mathcal{F}$ achieve finite minimum under Dirichlet boundary condition provided the structural assumptions ((2.3)-(2.11) and (2.17)) hold true. We want to show that also E, F have the same property.*

Definition 3.4. About the functionals defined by (2.1), (2.18), (3.1), (3.7) we will often use the short notation $E, F, \mathcal{E}, \mathcal{F}$; nevertheless, whenever required by clearness of exposition about interchange of various ingredients (functions, parameters, Dirichlet datum, domains A) we will use several different (self-explaining) notation:

$$\mathcal{F}(v),\; \mathcal{F}_g(v),\; \mathcal{F}_{g\,w}(v),\; \mathcal{F}(v, A),\; \mathcal{F}_{g\,w}(v, \mu, \alpha, \beta, A); \qquad F(K_0, K_1, v),\; F_g(K_0, K_1, v),$$

$$\mathcal{E}(v),\; \mathcal{E}(v, A),\; \mathcal{E}(v, \alpha, \beta, A); \qquad E(K_0, K_1, v)\,.$$

Lemma 3.5. *(***Scaling***) Let $v \in GSBV^2(B_r(\mathbf{x}_0))$ where $\mathbf{x}_0 = (x_0, y_0)$.
For $\lambda > 0$ and for every $\mathbf{x} \in B_1$ set*

$$v_r(\mathbf{x}) \;=\; \frac{v(\mathbf{x}_0 + r\mathbf{x})}{\lambda^{1/2}\, r^{3/2}}\,, \qquad g_r(\mathbf{x}) \;=\; \frac{g(\mathbf{x}_0 + r\mathbf{x})}{\lambda^{1/2}\, r^{3/2}}\,. \tag{3.8}$$

Then $v_r \in GSBV^2(B_1)$ and

$$\mathcal{F}_g(v, \mu, \alpha, \beta, B_r(\mathbf{x}_0)) = \lambda r \mathcal{F}_{g_r}(v_r, \mu\lambda^{\frac{q}{2}-1} r^{1+\frac{3}{2}q}, \frac{\alpha}{\lambda}, \frac{\beta}{\lambda}, B_1) \tag{3.9}$$

and, by setting $K_{0r} = (K_0 - \mathbf{x}_0)/r$, $K_{1r} = (K_1 - \mathbf{x}_0)/r$,

$$F_g(K_0, K_1, v, \mu, \alpha, \beta, B_r(\mathbf{x}_0)) = \lambda r F_{g_r}(K_{0r}, K_{1r}, v_r, \mu\lambda^{\frac{q}{2}-1} r^{1+\frac{3}{2}q}, \frac{\alpha}{\lambda}, \frac{\beta}{\lambda}, B_1). \tag{3.10}$$

Proof. The thesis follows by change of variables. □

Definition 3.6. For any $\mathbf{x} \in \overline{\Omega}$ and r s.t. $0 < r < dist(\mathbf{x}, \partial\widetilde{\Omega})$, we say that u is an Ω *local minimizer of* $\mathcal{F}_{g\,w}(\cdot, \mu, \alpha, \beta, B_r(\mathbf{x}))$ if

$$\begin{cases} u \in GSBV^2(B_r(\mathbf{x})):\; u = w \;\text{ a.e. in }\; B_r(\mathbf{x})\backslash\Omega,\; \mathcal{F}_{g\,w}(u, \mu, \alpha, \beta, A) < +\infty, \\[2ex] \mathcal{F}_{g\,w}(u, \mu, \alpha, \beta, A) \leq \mathcal{F}_{g\,w}(u + \eta, \mu, \alpha, \beta, A) \\[2ex] \forall A \subset\subset B_r(\mathbf{x}), \forall \eta \in GSBV^2(B_r(\mathbf{x})):\; \text{spt}\,\eta \subset A,\; \eta = 0 \;\text{ a.e. in } B_r(\mathbf{x})\backslash\Omega. \end{cases} \tag{3.11}$$

Definition 3.7. For any $\mathbf{x} \in \overline{\Omega}$ and r s.t. $0 < r < dist(\mathbf{x}, \partial\widetilde{\Omega})$, we say that u is an Ω *local minimizer of* $\mathcal{E}_w(\cdot, \alpha, \beta, B_r(\mathbf{x}))$ if

$$(3.12)\qquad \begin{cases} u \in GSBV^2(B_r(\mathbf{x})) : \ u = w \ \text{ a.e. in } \ B_r(\mathbf{x}) \setminus \Omega, \ \mathcal{E}_w(u, \alpha, \beta, A) < +\infty, \\ \mathcal{E}_w(u, \alpha, \beta, A) \le \mathcal{E}_w(u + \eta, \alpha, \beta, A) \\ \forall A \subset\subset B_r(\mathbf{x}), \ \forall \eta \in GSBV^2(B_r(\mathbf{x})) : \ \operatorname{spt} \eta \subset A, \ \eta = 0 \ \text{ a.e. in } B_r(\mathbf{x}) \setminus \Omega. \end{cases}$$

Remark 3.8. *Definitions 3.6 and 3.7 will be used also with open sets different from* Ω *provided suitable data* g, w *are defined in the contest.*
Definitions 3.6 and 3.7 are in fact equivalent to the ones given in [9]*, though they are slightly different, since sublevels of energy* $\mathcal{F}$ *(or* $\mathcal{E}$*) are linear subspaces of* $GSBV^2$ *(due to Corollary 4.5 in* [3]*).*

Remark 3.9. *Due to Lemma 3.5, if* u *is a* $B_\varrho(\mathbf{x})$ *local minimizer of* $\mathcal{F}_{g\,w}(\cdot, \mu, \alpha, \beta, B_\varrho(\mathbf{x}))$ *then for any* $\mathbf{a}, \lambda, c$

$$\mathbf{y} \to u(\mathbf{y}) = \lambda^{-1/2} \varrho^{-3/2} \big(u(\mathbf{x} + \varrho\, \mathbf{y}) - \varrho\, \mathbf{a} \cdot \mathbf{y} - c \big)$$

is a $B_1(\mathbf{0})$ *local minimizer of* $\mathcal{F}_{\gamma\,\omega}(\cdot, \mu \lambda^{\frac{q}{2}-1} r^{1+\frac{3}{2}q}, \alpha/\lambda, \beta/\lambda, B_1(\mathbf{0}))$ *where*

$$\begin{aligned} \gamma &= \lambda^{-1/2} \varrho^{-3/2} \big(g(\mathbf{x} + \varrho\, \mathbf{y}) - \varrho\, \mathbf{a} \cdot \mathbf{y} - c \big)\,, \\ \omega &= \lambda^{-1/2} \varrho^{-3/2} \big(w(\mathbf{x} + \varrho\, \mathbf{y}) - \varrho\, \mathbf{a} \cdot \mathbf{y} - c \big)\,. \end{aligned}$$

The same property holds true for $\mathcal{E}_w$.

Now we prove a uniform density upper bound (Theorem 3.10 below) for the weak functional $\mathcal{F}$ at the points $\mathbf{x} \in \overline{\Omega}$; this result is analogous to the estimate which holds true at interior points in case of Neuman boundary conditions (Theorem 2.12 and Remark 2.13 in [14]) and is a particular case of a global estimate in case of Dirichlet boundary conditions which holds true uniformly up to the boundary.
We define a suitable constant in order to handle boundary conditions:

$$(3.13)\qquad \begin{cases} L = \big(\mathrm{C}(\partial\Omega)\big)^2 + \big(\mathrm{Lip}(\nabla w)\big)^2, \text{ where } \mathrm{C}(\partial\Omega) \text{ is an uniform estimate for} \\ \text{second derivatives of piecewise arc-length parametrization of } \partial\Omega \text{ and} \\ \mathrm{Lip}(\nabla w) \text{ is the Lipschitz constant of } \nabla w \text{ in the neighborhood } A \text{ of } \partial\Omega. \end{cases}$$

Theorem 3.10. (Density upper bound for the functional $\mathcal{F}$ at the boundary) *Let* u *be a minimizer in* $X(\widetilde{\Omega})$ *for the functional* $\mathcal{F}$ *with* (2.3)–(2.11), (2.17)*. Then there exist* $C \in (0, +\infty)$ *and* $\bar{\varrho} = \bar{\varrho}(\alpha, \beta, L, \|w\|_{L^q}, \|g\|_{L^q}) > 0$*, where* L *is defined by* (3.13)*, such that*

$$(3.14)\qquad \mathcal{H}^1\big(\partial\Omega \cap B_\varrho(\mathbf{x})\big) < C\varrho \quad \forall \mathbf{x} \in \overline{\Omega}\,, \ \forall \varrho \le \bar{\varrho}\,,$$

and

$$(3.15)\qquad \mathcal{F}(u, \overline{B_\varrho}(\mathbf{x})) \le c_0 \varrho \quad \forall \varrho \text{ s.t. } 0 < \varrho \le (\bar{\varrho} \wedge 1) \text{ and } \forall\, \mathbf{x} \in \partial\Omega \text{ s.t. } B_\varrho(\mathbf{x}) \subset \widetilde{\Omega},$$

where $c_0 = L\pi + 2^{q-1} \pi^{\frac{1}{2}} \mu \big(\|w\|^q_{L^{2q}(B_\varrho(\mathbf{x}))} + \|g\|^q_{L^{2q}(B_\varrho(\mathbf{x}))} \big) + (2\pi + C)\, \alpha.$
If $q = 2$ *and* $g\,, w \in L^\infty(\widetilde{\Omega})$*, then we can choose*

$$c_0 = L\pi + 2\pi\mu \big(\|w\|^2_{L^\infty} + \|g\|^2_{L^\infty} \big) + (2\pi + C)\, \alpha\,.$$

Proof. Estimate (3.14) follows by (3.13) and the finiteness of L, which is due to (2.6), (2.9), (2.10). By minimality of u for $\mathcal{F}$ we get

$$\mathcal{F}(u) \le \mathcal{F}(v)\,,$$

where

$$v = u\,\chi_{\widetilde{\Omega}\setminus(B_\varrho(\mathbf{x})\cap\Omega)}.$$

By taking into account $\beta \le \alpha$ and $\mathcal{F}(u, \widetilde{\Omega}\setminus\overline{B_\varrho}(\mathbf{x})) = \mathcal{F}(v, \widetilde{\Omega}\setminus\overline{B_\varrho}(\mathbf{x}))$, we obtain

$$\begin{aligned}
\mathcal{F}(u, \overline{B_\varrho}(\mathbf{x})) \;&\le\; \mathcal{F}(v, \overline{B_\varrho}(\mathbf{x})) \le \int_{B_\varrho(\mathbf{x})\setminus\overline{\Omega}} \big(|\nabla^2 w|^2 + \mu|w-g|^q\big)\,d\mathbf{y} + \mu\int_{B_\varrho(\mathbf{x})\cap\Omega} |g|^q\,d\mathbf{y}\\
&\qquad + \alpha\mathcal{H}^1\big(\partial B_\varrho(\mathbf{x})\cap\Omega\big) + \alpha\mathcal{H}^1\big(\partial\Omega\cap B_\varrho(\mathbf{x})\big)\\
&\le\; L\,\pi\,\varrho^2 + 2^{q-1}\mu\int_{B_\varrho(\mathbf{x})\setminus\overline{\Omega}} (|w|^q + |g|^q)\,d\mathbf{y}\\
&\qquad + \mu\int_{B_\varrho(\mathbf{x})\cap\Omega} |g|^q\,d\mathbf{y} + 2\pi\alpha\varrho + \alpha\mathcal{H}^1\big(\partial\Omega\cap B_\varrho(\mathbf{x})\big)\\
&\le\; L\,\pi\,\varrho^2 + 2^{q-1}\mu\big(\|w\|^q_{L^{2q}(B_\varrho(\mathbf{x}))} + \|g\|^q_{L^{2q}(B_\varrho(\mathbf{x}))}\big)(\pi\varrho^2)^{\frac12} + 2\pi\alpha\varrho + C\alpha\varrho\,,
\end{aligned}$$

hence we achieve the proof. □

4. Joining and matching between lunulae

In this Section we prove some technical tools aimed to the proof of partial regularity at boundary points.

Lemma 4.1. *(***Joining between lunulae***)*
Assume (2.3), (2.4), (2.17), z, u in $GSBV^2(\widetilde{\Omega})$, $\mathbf{x}_0 = (x_0, y_0) \in \partial\Omega$, $0 < d < \sigma < s < t < 1$ and $\sigma - d < t - s$ s.t. $\overline{B_t(\mathbf{x}_0)} \subset \widetilde{\Omega}$, $\partial\Omega\cap B_t(\mathbf{x}_0) \in C^2$ and $\mathbf{d}$ denotes the inner normal to $\partial\Omega$ at $\mathbf{x}_0$. Set

$$B_t^d = B_t(\mathbf{x}_0)\cap\{(\mathbf{x}-\mathbf{x}_0)\cdot\mathbf{d} > d\}\,, \qquad M_{t,s}^{d,\sigma} = \overline{B_t^d}\setminus B_s^\sigma\,. \tag{4.1}$$

Then for every $\theta \in (0,1)$ there are $c = c(\theta) > 0$ and a cut-off function Ψ in $C^2(B_t(\mathbf{x}_0))\cap C_0^2(B_t^d(\mathbf{x}_0))$ s.t. $\Psi \equiv 1$ in a neighborhood of $B_s^\sigma = B_s(\mathbf{x}_0)\cap\{(\mathbf{x}-\mathbf{x}_0)\cdot\mathbf{d} > \sigma\}$ and, by setting

$$U = \Psi u + (1-\Psi)z$$

we have

$$\begin{aligned}
\mathcal{F}(U, \overline{B_t^d}) \;&\le\; (1+\theta)\,\Big(\mathcal{F}(u, \overline{B_t^d}) + \mathcal{F}(z, \overline{B_t^d}\setminus B_s^\sigma)\Big) +\\
&+ \frac{c}{(\sigma-d)^2}\left(\int_{B_t^d\setminus B_s^\sigma} |\nabla(u-z)|^2 d\mathbf{x} + \frac{c}{\theta\, d^2\,(\sigma-d)^2}\int_{B_t^d\setminus B_s^\sigma} |u-z|^2 d\mathbf{x}\right)
\end{aligned}$$

and

$$\begin{aligned}
\mathcal{E}(U, \overline{B_t^d}) \;&\le\; (1+\theta)\,\Big(\mathcal{E}(u, \overline{B_t^d}) + \mathcal{E}(z, \overline{B_t^d}\setminus B_s^\sigma)\Big) +\\
&+ \frac{c}{(\sigma-d)^2}\left(\int_{B_t^d\setminus B_s^\sigma} |\nabla(u-z)|^2 d\mathbf{x} + \frac{c}{\theta\, d^2\,(\sigma-d)^2}\int_{B_t^d\setminus B_s^\sigma} |u-z|^2 d\mathbf{x}\right)
\end{aligned}$$

Proof. Proving the estimates for the terms containing $|\nabla^2\cdot|^2$ will be enough.
Without loss of generality we assume $\mathbf{x}_0=\mathbf{0}$ and $\mathbf{d}=\mathbf{e}_2$ so that

$$\{(\mathbf{x}-\mathbf{x}_0)\cdot\mathbf{d}>d\} \;=\; \{y>d\}$$

We fix $\theta\in(0,1)$ and $N=N(\theta)$ where $N=1+[C/\theta]$ and C is a suitable constant. Let

$$s_j \;=\; s+j\,\frac{t-s}{N}\,,\qquad j=0,\dots,N,$$

$$d_0\;=\;\sigma\,,\quad d_j\;=\;d_0-j\,\frac{\sigma-d}{N}\,,\qquad j=0,\dots,N,$$

and ψ_j $(j=0,\dots,N-1)$ a list of C^2 cut-off functions between B_{s_j} and $B_{s_{j+1}}$ (say $0\le\psi_j\le 1$, $\psi_j\equiv 1$ in a neighborhood of B_{s_j}, ψ_j vanishes outside $\overline{B_{s_{j+1}}}$) with

$$|D\psi_j|\;\le\;\frac{2\,N}{\sigma-d}\,,\qquad \left|D^2\psi_j\right|\;\le\;\frac{8\,N^2}{d\,(\sigma-d)^2}\,,\qquad \text{in } C_j\stackrel{\text{def}}{=}B_{s_{j+1}}\setminus B_{s_j}$$

and a list of 1-dimensional cut-off functions η_j, $j=1,\dots,N$, between $\{y>d_j\}$ and $\{y>d_{j+1}\}$ (say $0\le\eta_j\le 1$, $\eta_j\equiv 1$ in a neighborhood of $\{y>d_j\}$, η_j vanishes outside $\{y>d_{j+1}\}$) with

$$|D\eta_j|\;\le\;\frac{2\,N}{\sigma-d}\,,\qquad \left|D^2\eta_j\right|\;\le\;\frac{2\,N^2}{(\sigma-d)^2}\,,\qquad \text{in } E_j\stackrel{\text{def}}{=}\{d_{j+1}<y<d_j\}$$

Then define

$$U_j\;=\;\eta_j\,\psi_j\,u\;+\;(1-\eta_j\psi_j)\,z\,.$$

For any w,

$$\nabla^2(\eta_j\,\psi_j\,w)\;=\;\eta_j\,\nabla^2(\psi_j\,w)\;+\;\begin{pmatrix} 0 & (\eta_j)_y\,(\psi_j w)_x\\ (\eta_j)_y\,(\psi_j w)_x & (\eta_j)_{yy}\,\psi_j\,w\end{pmatrix}$$

We introduce the handles $M_j\stackrel{\text{def}}{=}(C_j\cap\{y>d_{j+1}\})\cup(E_j\cap\{|\mathbf{x}|<s_{j+1}\})$ for $j=1,\dots,N-1$ and the lunula $M_0=E_0\cap\{|\mathbf{x}|<s\}$: we notice that the sets M_j, $j=0,\dots,N-1$, are pair-wise disjoint ($j\ne k\;\Rightarrow\; M_j\cap M_k=\emptyset$) and their union covers the whole lunula $\{|\mathbf{x}|<t\}\cap\{y>d\}$ up to a set of measure 0.
Since ψ_j is a radial function we obtain, for every j,

$$\begin{aligned}\int_{B_t^d}|\nabla^2U_j|^2\,d\mathbf{x}\;\le&\int_{B_{s_j}^d}|\nabla^2u|^2\,d\mathbf{x}+\int_{B_t^d\setminus B_{s_j}^{d_{j+1}}}|\nabla^2z|^2\,d\mathbf{x}\\ &+\int_{M_j}\Big|\,\eta_j\left(\psi_j\,\nabla^2u\,+\,(1-\psi_j)\,\nabla^2z\,+\,2\,D\psi_j\odot\nabla(u-z)\;+\;D^2\psi_j\,(u-z)\right)\\ &\quad+(\mathbf{e}_1\otimes\mathbf{e}_2+\mathbf{e}_2\otimes\mathbf{e}_1)\,(\eta_j)_y\,(\psi_ju)_x\,+\,\mathbf{e}_2\otimes\mathbf{e}_2\,(\eta_j)_{yy}\,\psi_j\,u\\ &\quad+(\mathbf{e}_1\otimes\mathbf{e}_2+\mathbf{e}_2\otimes\mathbf{e}_1)\,(\eta_j)_y\,((1-\psi_j)z)_x\,+\,\mathbf{e}_2\otimes\mathbf{e}_2\,(\eta_j)_{yy}\,(1-\psi_j)\,z\Big|^2\,d\mathbf{x}\end{aligned}$$

$$\begin{aligned}\le&\int_{B_t^d}|\nabla^2u|^2\,d\mathbf{x}+\int_{B_t^d\setminus B_s^\sigma}|\nabla^2z|^2\,d\mathbf{x}\\ &+C\int_{M_j}\Big(|\nabla^2u|^2+|\nabla^2z|^2+|D\psi_j|^2|\nabla(u-z)|^2+|D^2\psi_j|^2|u-z|^2\\ &\quad+|D\eta_j|^2|D\psi_j|^2|u-z|^2\;+\;|D\eta_j|^2|\nabla(u-z)|^2\;+\;|D^2\eta_j|^2|u-z|^2\Big)\,d\mathbf{x}\,.\end{aligned}$$

By taking into account that $\mathrm{spt}(\eta_j\psi_j) \subset \bigcup_{k=0}^{j+1} \overline{M_k}$ we add the last inequalities with respect to j from 0 to $N-1$:

$$\min_j \int_{B_t^d} |\nabla^2 U_j|^2 \, d\mathbf{x} \;\le\; \int_{B_t^d} |\nabla^2 u|^2 \, d\mathbf{x} + \int_{B_t^d \setminus B_s^\sigma} |\nabla^2 z|^2 \, d\mathbf{x}$$
$$+\frac{C}{N} \int_{B_t^d \setminus B_s^\sigma} \left(|\nabla^2 u|^2 + |\nabla^2 z|^2 + \left(\frac{2N}{\sigma - d}\right)^2 |\nabla(u-z)|^2 + \left(\frac{12N^2}{d(\sigma-d)^2}\right)^2 |u-z|^2 \right) d\mathbf{x}$$

We select the index j achieving such minimum and set $U = U_j$. Hence

$$\int_{B_t^d} |\nabla^2 U|^2 \, d\mathbf{x} \;\le\; \int_{B_t^d} |\nabla^2 u|^2 \, d\mathbf{x} + \int_{B_t^d \setminus B_s^\sigma} |\nabla^2 z|^2 \, d\mathbf{x}$$
$$+\theta \int_{B_t^d \setminus B_s^\sigma} \left(|\nabla^2 u|^2 + |\nabla^2 z|^2 + \left(\frac{2(C+1)}{\theta(\sigma - d)}\right)^2 |\nabla(u-z)|^2 + \left(\frac{12(C+1)^2}{\theta d(\sigma-d)^2}\right)^2 |u-z|^2 \right) d\mathbf{x}$$

and the thesis follows by inequalities with $c = 12\,(C+1)^2/\,\theta\,,\ \Psi = \eta_j\psi_j$, since the terms not containing ∇^2 fulfill the inequality in the thesis with $\theta = 0$:

$$\mathcal{H}^1\left(S_U \cap \overline{B_t^d}\right) = \mathcal{H}^1\left(S_u \cap \overline{B_{s_j}^{d_j}}\right) + \mathcal{H}^1\left(S_z \cap (\overline{B_t^d} \setminus B_{s_{j+1}}^{d_{j+1}})\right)$$
$$+\mathcal{H}^1\left(S_u \cap (\overline{B_{s_{j+1}}^{d_{j+1}}} \setminus B_{s_j}^{d_j})\right) + \mathcal{H}^1\left(S_z \cap (\overline{B_{s_{j+1}}^{d_{j+1}}} \setminus B_{s_j}^{d_j})\right) ,$$

$$\mathcal{H}^1\left((S_{\nabla U} \setminus S_U) \cap \overline{B_t^d}\right)$$
$$= \mathcal{H}^1\left((S_{\nabla u} \setminus S_u) \cap \overline{B_{s_j}^{d_j}}\right) + \mathcal{H}^1\left((S_{\nabla z} \setminus S_z) \cap (\overline{B_t^d} \setminus B_{s_{j+1}}^{d_{j+1}})\right)$$
$$+\mathcal{H}^1\left((S_{\nabla u} \setminus S_v) \cap (\overline{B_{s_{j+1}}^{d_{j+1}}} \setminus B_{s_j}^{d_j})\right) + \mathcal{H}^1\left((S_{\nabla z} \setminus S_z) \cap (\overline{B_{s_{j+1}}^{d_{j+1}}} \setminus B_{s_j}^{d_j})\right),$$

$$\int_{B_t^d} |U-g|^q \, d\mathbf{x} \;\le\; \int_{B_{s_j}^{d_j}} |u-g|^q \, d\mathbf{x} + \int_{B_t^d \setminus B_{s_{j+1}}^{d_{j+1}}} |z-g|^q \, d\mathbf{x}$$
$$+ \int_{\overline{B_{s_{j+1}}^{d_{j+1}}} \setminus B_{s_j}^{d_j}} (\psi_j |u-g|^q + (1-\psi_j)|z-g|^q) \, d\mathbf{x} .$$

□

Lemma 4.2. (**Matching with lunulae**) *Let* $\mathbf{x}_0 = (x_0, y_0)$, $z, v \in GSBV^2(\widetilde{\Omega})$, $\overline{B}_t(\mathbf{x}_0) \subset \widetilde{\Omega}$ *and*

$$\mathcal{H}^1\big(S_z \cap \partial B_t^d\big) = \mathcal{H}^1\big(S_{\nabla z} \cap \partial B_t^d\big) = \mathcal{H}^1\big(S_v \cap \partial B_t^d\big) = \mathcal{H}^1\big(S_{\nabla v} \cap \partial B_t^d\big) = 0$$

where $B_t^d = B_t(\mathbf{x}_0) \cap \{y > y_0 + d\}$. *Then, by setting*

$$u = \begin{cases} z & in\ B_t^d \\ v & in\ \widetilde{\Omega} \setminus B_t^d \end{cases}$$

we have

$$\begin{aligned}\mathcal{F}_g(u,\mu,\alpha,\beta,\widetilde{\Omega}) \leq\ & \mathcal{F}_g(z,\mu,\alpha,\beta,B^d_t) + \mathcal{F}_g(v,\mu,\alpha,\beta,\widetilde{\Omega}\setminus\overline{B^d_t}) \\ & +\alpha\,\mathcal{H}^1\big(\{\widetilde{z}\neq\widetilde{v}\}\cap\partial B^d_t\big) + \beta\,\mathcal{H}^1\Big(\big(\{\widetilde{\nabla z}\neq\widetilde{\nabla v}\}\setminus\{\widetilde{z}\neq\widetilde{v}\}\big)\cap\partial B^d_t\Big),\end{aligned}$$

$$\begin{aligned}\mathcal{E}(u,\alpha,\beta,\widetilde{\Omega}) \leq\ & \mathcal{E}(z,\alpha,\beta,B^d_t) + \mathcal{E}(v,\alpha,\beta,\widetilde{\Omega}\setminus\overline{B^d_t}) \\ & +\alpha\,\mathcal{H}^1\big(\{\widetilde{z}\neq\widetilde{v}\}\cap\partial B^d_t\big) + \beta\,\mathcal{H}^1\Big(\big(\{\widetilde{\nabla z}\neq\widetilde{\nabla v}\}\setminus\{\widetilde{z}\neq\widetilde{v}\}\big)\cap\partial B^d_t\Big).\end{aligned}$$

Proof. The thesis follows by the definitions. □

Lemma 4.3. *Let $v\in GSBV^2(\widetilde{\Omega})$ s.t.*

$$\mathcal{F}_g(v,\mu,\alpha,\beta,T)\ <\ +\infty \qquad \textit{for every compact set } T\subset\widetilde{\Omega}\,.$$

Then

$$\lim_{\varrho\to 0}\ \varrho^{-1}\,\mathcal{F}_g(\,v,\mu,\alpha,\beta,B_\varrho(\mathbf{x})\,)\ =\ 0 \qquad \textit{for } \mathcal{H}^1 \textit{ a.e. } \mathbf{x}\in\widetilde{\Omega}\setminus(S_v\cup S_{\nabla v})\ .$$

Proof. Apply the same argument of Lemma 2.6 in [21]. □

5. Truncation, Poincaré inequalities and compactness properties in $GSBV$ and $GSBV^2$

We recall a Poincaré-Wirtinger type inequality in the class $GSBV$ which was proven in [9] allowing surgical truncations of non integrable functions of several variables and we refine its statement with the aim of taming blow-up at boundary points in case of functions vanishing in a sector of positive measure. About this inequality we emphasize that $v\in GSBV^2(\Omega)$ does not even entail that either v or ∇v belongs to $L^1_{loc}(\Omega)$.
Let B be an open ball in $\mathbb{R}^2$. For every measurable function $v:B\to\mathbb{R}$ we define the least median of v in B as

$$\mathrm{med}(v,B)=\inf\{\ t\in\mathbb{R};\ |\{v<t\}\cap B|\geq\frac{1}{2}|B|\}.$$

We remark that $\mathrm{med}(\cdot,B)$ is a non linear operator and in general it has no relationship with the averaged integral $\int_B\cdot\,dy\,/\,|B|$.
Obviously we have $\mathrm{med}(v\chi_{B\setminus E}+\mathrm{med}(v,B)\chi_E,B)=\mathrm{med}(v,B)$ for every $E\subset B$. For every $v\in GSBV(B)$ and $a\in\mathbb{R}$ with $(2\gamma_2\mathcal{H}^1(S_v))^2\leq a\leq\frac{1}{2}|B|$, we set

$$\tau'(v,a,B)=\inf\{t\in\mathbb{R};\ |\{v<t\}|\ \geq\ a\}\,,$$
$$\tau''(v,a,B)=\inf\{t\in\mathbb{R};\ |\{v\geq t\}|\ \leq\ a\}\,,$$

here γ_2 is the isoperimetric constant relative to the balls of $\mathbb{R}^2$, i.e.

$$\min\{|E\cap B|^{\frac{1}{2}},|B\setminus E|^{\frac{1}{2}}\}\leq\gamma_2 P(E,B) \qquad \text{for every Borel set } E\,,$$

and $P(E,B)$ denotes the perimeter of E in B : $P(E,B)=\int_B|D\chi_E|$.
For $\eta\geq 0$ we define the truncation operator

$$T(v,a,\eta)=(\tau'(v,a,B)-\eta)\vee v\wedge(\tau''(v,a,B)+\eta). \tag{5.1}$$

We easily get $T(T(v,a,\eta),a,\eta)=T(v,a,\eta)$, $|\nabla T(v,a,\eta)|\leq|\nabla v|$ a.e. on B. Moreover $\mathrm{med}(T(v,a,\eta),B)=\mathrm{med}(v,B)$ and $T(\lambda v,a,\lambda\eta)=\lambda T(v,a,\eta)$ for every $\lambda>0$, and

$$|\{v\neq T(v,a,\eta)\}|\leq 2a. \tag{5.2}$$

In case v is vector-valued the operators med and T are defined componentwise.

For any given function in $GSBV$, we define an affine polynomial correction such that both median and gradient median vanish.

Let $B_r(\mathbf{x}) \subset \Omega$ and $v \in GSBV(B_r(\mathbf{x}))$; for every $\mathbf{y} \in \mathbb{R}^2$ we set

$$(M_{\mathbf{x},r}\, v)(\mathbf{y}) = \mathrm{med}(\nabla v, B_r(\mathbf{x})) \cdot (\mathbf{y} - \mathbf{x}) \tag{5.3}$$

$$(\mathcal{P}_{\mathbf{x},r}\, v)(\mathbf{y}) = (M_{\mathbf{x},r}\, v)(\mathbf{y}) + \mathrm{med}(v - M_{\mathbf{x},r}\, v, B_r(\mathbf{x})). \tag{5.4}$$

Since $\mathrm{med}(v - c, B_r(\mathbf{x})) = \mathrm{med}(v, B_r(\mathbf{x})) - c$ for every $c \in \mathbb{R}$ and $\nabla(\mathcal{P}_{\mathbf{x},r}\, v) = \nabla(M_{\mathbf{x},r}\, v) = \mathrm{med}(\nabla v, B_r(\mathbf{x}))$ then we have $\mathcal{P}_{\mathbf{x},r}\,(v - \mathcal{P}_{\mathbf{x},r}\, v) = 0$, say

$$\mathrm{med}(v - \mathcal{P}_{\mathbf{x},r}\, v, B_r(\mathbf{x})) = 0, \qquad \mathrm{med}(\nabla(v - \mathcal{P}_{\mathbf{x},r}\, v), B_r(\mathbf{x})) = \mathbf{0}.$$

We notice that there are v such that $\mathrm{med}(v, B_r(\mathbf{x})) \neq \mathrm{med}(\mathcal{P}_{\mathbf{x},r}\, v, B_r(\mathbf{x}))$, take e.g. $v(x,y) = (x^2 - x)H(-x) - \frac{x}{2}H(x)$, where H is the Heaviside function.

Theorem 5.1. *(***Poincaré-Wirtinger inequality for** $GSBV$ **functions in a ball***)*
Let $B \subset \mathbb{R}^2$ be an open ball, $v \in GSBV(B)$ and $a \in \mathbb{R}$ with

$$\big(2\gamma_2 \mathcal{H}^1(S_v)\big)^2 \le a \le \frac{1}{2}|B|, \tag{5.5}$$

let $\eta \ge 0$ and $T(v, a, \eta)$ as in (5.1). Then

$$\int_B |D\, T(v, a, \eta)| \le 2|B|^{\frac{1}{2}} \left(\int_B |\nabla T(v, a, \eta)|^2\, dy \right)^{\frac{1}{2}} + 2\eta \mathcal{H}^1(S_v). \tag{5.6}$$

We have also, for every $s \ge 2$,

$$\int_B |T(v, a, \eta) - \mathrm{med}(v, B)|^s dy \le 2^{s-1} (\gamma_2 s)^s \left(\int_B |\nabla T(v, a, 0)|^2\, dy \right)^{\frac{s}{2}} |B| + (2\eta)^s a\,. \tag{5.7}$$

Proof. See [9], Theorem 4.1. □

Theorem 5.2. *(***Classical Poincaré inequality in BV***)*
For any $\mathbf{x} \in \mathbb{R}^2$, $r > 0$, and $0 < \vartheta < 1$ there is K_ϑ such that

$$\|v\|_{L^2(B_r(\mathbf{x}))} \;\le\; K_\vartheta \int_{B_r(\mathbf{x})} |Dv| \qquad \forall v \in BV(B_r(\mathbf{x}))\ s.t. \tag{5.8}$$

$$|\{\mathbf{y} \in B_r(\mathbf{x}) \,:\, v(\mathbf{y}) = 0\}\,| \,/\, |B_r(\mathbf{x})| \;\ge\; \vartheta\,. \tag{5.9}$$

Proof. See [23], Theorem 5.6.1(iii). □

Theorem 5.3. *(***Poincaré-Wirtinger inequality for** $GSBV$ **functions vanishing in a sector***)*
Let $B \subset \mathbb{R}^2$ be an open ball, $v \in GSBV(B)$ s.t. (5.9) holds true and $a \in \mathbb{R}$ with

$$\big(2\gamma_2 \mathcal{H}^1(S_v)\big)^2 \le a \le \frac{1}{2}|B|, \tag{5.10}$$

let $\eta \ge 0$ and $T(v, a, \eta)$ as in (5.1). Then

$$\int_B |D\, T(v, a, \eta)| \le 2|B|^{\frac{1}{2}} \left(\int_B |\nabla T(v, a, \eta)|^2\, dy \right)^{\frac{1}{2}} + 2\eta \mathcal{H}^1(S_v). \tag{5.11}$$

We have also, for every $s \geq 2$,

$$\int_B |T(v,a,\eta)|^s dy \leq 2^{s-1}\,(K_\vartheta s)^s \left(\int_B |\nabla T(v,a,0)|^2\,dy\right)^{\frac{s}{2}} |B| + (2\eta)^s a\,. \tag{5.12}$$

Proof. Similar to the proof of Theorem 4.1 in [9] except for the use of Theorem 5.2 instead of Poincaré-Wirtinger inequality (4.12) in [9], since we do not need to force vanishing of least median of v. □

Theorem 5.4. ***(Compactness and lower semicontinuity for*** $GSBV^2$ ***functions vanishing in a set of full measure)***
Assume $B_r(\mathbf{x}) \subset \mathbb{R}^2$, $u_h \in GSBV^2(B_r(\mathbf{x}))$, $0 < \vartheta < 1$

$$|\{\mathbf{y} \in B_r(\mathbf{x})\,:\; u_h(\mathbf{y}) = 0\}|\;/\;|B_r(\mathbf{x})|\;\geq\;\vartheta\,, \tag{5.13}$$

$$\sup_h \int_{B_r(\mathbf{x})} |\nabla^2 u_h|^2\,d\mathbf{y}\;<\;+\infty \tag{5.14}$$

and

$$\lim_h\; L_h\;=\;0\,, \qquad \text{where } L_h = \mathcal{H}^1(S_{u_h} \cup S_{\nabla u_h})\,. \tag{5.15}$$

Then there are a positive constant c (dependent on the left-hand side of (5.14)), $u_\infty \in H^2(B_r(\mathbf{x}))$ and a sequence $z_h \in GSBV^2(B_r(\mathbf{x}))$ (whose explicit construction is given by (5.23)-(5.28)) s.t., up to a finite number of indices,

$$|\,\{z_h \neq u_h\}\,| \leq\; c\,{L_h}^2 \tag{5.16}$$

$$P(\,\{z_h \neq u_h\},\, B_r(\mathbf{x}))\;\leq\; c\,L_h \tag{5.17}$$

and there is a subsequence z_{h_k} such that

$$\lim_k\; z_{h_k}\;=\;u_\infty \qquad \text{strongly in } L^p(B_r(\mathbf{x}))\,,\; \forall p \geq 1\,, \tag{5.18}$$

$$\lim_k\; \nabla\, z_{h_k}\;=\;Du_\infty \qquad \text{strongly in } L^p(B_r(\mathbf{x}))\,,\; \forall p \geq 1\,, \tag{5.19}$$

$$\int_{B_r(\mathbf{x})} |D^2 u_\infty|^2\,d\mathbf{y}\;\leq\;\liminf_k \int_{B_r(\mathbf{x})} |\nabla^2 z_{h_k}|^2\,d\mathbf{y}\;\leq\;\liminf_k \int_{B_r(\mathbf{x})} |\nabla^2 u_{h_k}|^2\,d\mathbf{y}\,, \tag{5.20}$$

$$\lim_k\; u_{h_k}\;=\;u_\infty \qquad \text{a.e. in } B_r(\mathbf{x})\,, \tag{5.21}$$

$$\lim_k\; \nabla\, u_{h_k}\;=\;Du_\infty \qquad \text{a.e. in } B_r(\mathbf{x})\,. \tag{5.22}$$

Proof. Identical to the proof of Theorem 4.3 in [9], except for the fact that we can avoid forcing least median of u_h and ∇u_h to vanish since we can use Theorem 5.3 for functions vanishing in a sector instead of *GSBV* Poincaré-Wirtinger inequality given by Theorem 4.1 in [9].
For reader convenience we recall the explicit construction of the sequence z_h :
by setting $a_h = 4{\gamma_2}^2{L_h}^2$ we have $a_h \leq |B_r|/2$ for large h. Hence there is c dependent on the left-hand side of (5.14) and there are $\eta_h^k \in (0,1)\,, h \in \mathbb{N},\; k = 1,2$, s.t.

$$\left|\,\{\,T(\nabla_k\, u_h, a_h, \eta_h^k)\;\neq\;\nabla_k\, u_h\,\}\,\right|\;\leq\; c\,{L_h}^2 \tag{5.23}$$

$$P\left(\,\{\,T(\nabla_k\, u_h, a_h, \eta_h^k)\;\neq\;\nabla_k\, u_h\,\},\, B_r\,\right)\;\leq\; c\left(L_h + \mathcal{H}^1(S_{\nabla_k u_h})\right) \tag{5.24}$$

Referring to definition (5.1) of truncating operator T, we set

$$E_h = \bigcup_{k=1,2} \{ \mathbf{y} \in B_r : T(\nabla_k u_h, a_h, \eta_h^k) \neq \nabla_k u_h \} \tag{5.25}$$

$$\xi_h = u_h \, \chi_{B_r \setminus E_h} \tag{5.26}$$

$$b_h = 4 K_\vartheta{}^2 \left(\mathcal{H}^1(S_{\xi_h} \cup S_{\nabla \xi_h}) \right)^2 \leq \frac{1}{2} |B_r| \tag{5.27}$$

$$z_h = T(\xi_h, b_h, \eta_h) \tag{5.28}$$

□

6. Hessian decay for bi-harmonic functions in half-disk

In this Section we prove that any function which is bi-harmonic in a half-disk and vanishes together with its normal derivative on the diameter has a suitable decay of hessian L^2-norm.

Theorem 6.1. *(**L^2-hessian decay for bi-harmonic functions in half-disk which vanish together with normal derivative along diameter**)*
Set $B_1^+ = B_1(\mathbf{0}) \cap \{(x,y) \in \mathbb{R}^2 : y > 0\} \subset \mathbb{R}^2$, $\Gamma = B_1(\mathbf{0}) \cap \{(x,y) \in \mathbb{R}^2 : y = 0\}$.
Assume $z \in H^2(B_1^+)$, $\Delta^2 z = 0$ *on* B_1^+, $z = \partial z/\partial y = 0$ *on* Γ.
Then

$$\|D^2 z\|^2_{L^2(B_\varrho^+)} \leq \varrho^2 \, \|D^2 z\|^2_{L^2(B_1^+)} \qquad \forall \varrho \leq 1 . \tag{6.1}$$

Moreover there exists an unique extension Z *of* z *in whole* B_1 *such that* $\Delta^2 Z \equiv 0$ *and both* z, Z *have the following expansion in polar coordinates, which is strongly convergent in* $L^2(B_1)$ *and strongly convergent in* $H^2(B_1^+)$:

$$Z(x,y) = \sum_{k=0}^{\infty} \left(a_k \cos(k\vartheta) + b_k \sin(k\vartheta) + (\alpha_k \cos(k\vartheta) + \beta_k \sin(k\vartheta))\, r^2 \right) r^k . \tag{6.2}$$

Proof. Since $z \in H^2(B_1^+)$, is bi-harmonic in B_1^+ and $z = \partial z/\partial y = 0$ on $B_1(\mathbf{0}) \cap \{y = 0\}$, then z solves a boundary value problem in B_1^+ for the bilaplacian operator with homogeneous Dirichlet boundary conditions on the diameter; hence regularity properties at a flat portion of the boundary (see [25], Chap.7) entail $z \in C^1(B_1^+(\mathbf{0}) \cup (B_1(\mathbf{0}) \cap \{y = 0\}))$. So the classical Duffin formula [22] holds true:
z has a bi-harmonic extension Z in B_1 defined by

$$\begin{cases} Z(x,y) = z(x,y) & \forall\, (x,y) \in B_1^+ , \\ Z(x,-y) = -z(x,y) + 2y z_y(x,y) - y^2 \Delta z(x,y) & \forall\, (x,-y) \in B_1^- . \end{cases}$$

Function Z belongs to $L^2(B_1)$ by construction. Z is bi-harmonic in B_1, hence by Almansi decomposition [1], there exist two harmonic functions ψ, φ in $L^2(B_1)$ such that $Z = \psi + (x^2 + y^2)\varphi$: in polar coordinates,

$$\varphi(r,\vartheta) = \frac{1}{4\,r} \int_0^r \Delta z(\varrho, \vartheta)\, d\varrho , \qquad \psi = z - r^2 \varphi . \tag{6.3}$$

Hence Z can be represented, with suitable coefficients, by the expansion (6.2) which is strongly convergent in $L^2(B_1)$ and hence in $H^2(B_\varrho)$ for all $\varrho < 1$.
Notice that only suitable combinations of terms in expansion (6.2), say

$$(6.4)\qquad \begin{cases} v_k \;=\; r^{k+1}\left(\sin((k-1)\vartheta)) - \frac{k-1}{k+1}\sin((k+1)\vartheta)\right)\,, & k = 2,3,4,\dots \\ \omega_k \;=\; r^{k+1}\Big(\cos((k-1)\vartheta)) - \cos((k+1)\vartheta)\Big)\,, & k=-1 \text{ and } 1,2,3,\dots \end{cases}$$

fulfill also the conditions on diameter, nevertheless we disregard this complicate relationship on coefficients (though it is implicitly understood) which is useless in the sequel since system (6.4) is strongly entangled and far from providing any orthogonal basis either in $H^2(B_r^+)$ or in $L^2(B_r^+)$.
By denoting f_k the k-th term of the expansion (6.2), we compute the second derivatives of f_0 and f_1:

$$\begin{aligned} D^2_{xx}f_0 &= 2\alpha_0\,, \qquad D^2_{xy}f_0 \;=\; 0 \qquad D^2_{yy}f_0 \;=\; 2\alpha_0\,, \\ D^2_{xx}f_1 &= 2r\left(3\alpha_1\cos(\vartheta) + \beta_1\sin(\vartheta)\right)\,, \quad D^2_{xy}f_1 \;=\; 2r\left(\beta_1\cos(\vartheta) + \alpha_1\sin(\vartheta)\right)\,, \\ D^2_{yy}f_1 &= 2r\left(\alpha_1\cos(\vartheta) + 3\beta_1\sin(\vartheta)\right)\,, \end{aligned}$$

then we compute the second derivatives of f_k, with $k \geq 2$:

$$\begin{aligned} D^2_{xx}f_k \;&=\; r^{k-2}\Big(k\left(a_k\,(k-1) + \alpha_k\,(k+1)\,r^2\right)\cos((k-2)\,\vartheta) + 2\alpha_k\,(k+1)\,r^2\cos(k\vartheta) \\ &\qquad + k\left(b_k\,(k-1) + \beta_k\,(k+1)\,r^2\right)\sin((k-2)\,\vartheta) + 2\beta_k\,(k+1)\,r^2\sin(k\vartheta)\Big)\,, \\ D^2_{xy}f_k \;&=\; kr^{k-2}\Big(\left(b_k\,(k-1) + \beta_k\,(k+1)\,r^2\right)\cos((k-2)\,\vartheta) \\ &\qquad - \left(a_k\,(k-1) + \alpha_k\,(k+1)\,r^2\right)\sin((k-2)\,\vartheta)\Big)\,, \\ D^2_{yy}f_k \;&=\; r^{k-2}\Big(-\left(k\left(a_k\,(k-1) + \alpha_k\,(k+1)\,r^2\right)\cos((k-2)\,\vartheta)\right) + 2\alpha_k\,(k+1)\,r^2\cos(k\vartheta) \\ &\qquad - k\left(b_k\,(k-1) + \beta_k\,(k+1)\,r^2\right)\sin((k-2)\,\vartheta) + 2\beta_k\,(k+1)\,r^2\sin(k\vartheta)\Big)\,. \end{aligned}$$

Hence for suitable coefficients $c_k = c_k^{i,j}, d_k = d_k^{i,j}, \gamma_k = \gamma_k^{i,j}, \delta_k = \delta_k^{i,j}$, any second derivative of z has the following strongly $L^2(B_\varrho^+)$ convergent expansion, for every $\varrho < 1$ and $i, j = 1, 2$:

$$(6.5)\qquad D^2_{ij}z = \sum_{k=0}^{\infty}\left(c_k\cos(k\vartheta) \;+\; d_k\sin(k\vartheta) \;+ \left(\gamma_k\cos(k\vartheta) + \delta_k\sin(k\vartheta)\right)r^2\,\right)\,r^k\,.$$

Due to strong convergence, we can select partial sums in (6.5) as follows, by splitting terms with different arguments in trigonometric functions,

$$(6.6)\qquad \begin{aligned} D^2_{ij}z \;=\;& c_0 + \gamma_0\,r^2 \\ &+ \left(c_1\cos(\vartheta) \;+\; d_1\sin(\vartheta)\right)\,r \;+ \left(\gamma_1\cos(\vartheta) + \delta_1\sin(\vartheta)\right)\,r^3 \\ &+ \left(c_2\cos(2\vartheta) \;+\; d_2\sin(2\vartheta)\right)\,r^2 + \left(\gamma_2\cos(2\vartheta) + \delta_2\sin(2\vartheta)\right)r^4 \\ &+ \left(c_3\cos(3\vartheta) \;+\; d_3\sin(3\vartheta)\right)\,r^3 \;+ \left(\gamma_3\cos(3\vartheta) + \delta_3\sin(3\vartheta)\right)r^5 \\ &+ \left(c_4\cos(4\vartheta) \;+\; d_4\sin(4\vartheta)\right)\,r^4 \;+ \left(\gamma_4\cos(4\vartheta) + \delta_4\sin(4\vartheta)\right)r^6 \\ &+ \;\dots \end{aligned}$$

The system $\{\cos(2k\vartheta),\ \sin(2k\vartheta)\}_{k\in\mathbb{N}}$ is an orthogonal complete system in $L^2(0,\pi)$. Then odd lines in (6.6) are mutually orthogonal also in $L^2(B_r^+)$ and we can expand all the trigonometric functions with odd multiple of ϑ with respect to this system, in such a way that even lines will be absorbed by odd ones. This is carefully performed by suppressing even lines (the ones where $(2k+1)\vartheta$ appears) in (6.6) one at each step and taking into account the $L^2(B_r^+)$ orthogonal splitting $L^2(B_r^+) = V \oplus V^\perp$, where V is the space

$$V \stackrel{\text{def}}{=} \text{span}\,\Big\{ 1,\, r\,, r^2\,, \{r^{2k+1}\,,\; k=1,2,\dots\} \Big\}.$$

At first the $L^2(0,\pi)$ convergent expansions

$$\cos(\vartheta) = \sum_{n=1}^{\infty} \xi_n^1 \sin(2n\vartheta)\,, \qquad \sin(\vartheta) = \phi_0^1 + \sum_{n=1}^{\infty} \phi_n^1 \cos(2n\vartheta)$$

allow to cancel second line (related to ϑ) in (6.6) by allocating the two terms with trigonometric functions evaluated at $2n\vartheta$ on $(2n+1)$-th line and, taking into account r powers and convergence properties, writing

$$D_{ij}^2 z = S_1 + \Sigma_1\,, \quad \text{with} \quad S_1 \in V, \quad \text{and} \quad \Sigma_1 \quad \text{with empty first and second line of (6.6)}\;:$$

$$S_1 = c_0 + \gamma_0\, r^2 + d_1\, \phi_0^1\, r + \delta_1\, \phi_0^1\, r^3\,.$$

Then the $L^2(0,\pi)$ convergent expansions

$$\cos(3\vartheta) = \sum_{n=1}^{\infty} \xi_n^3 \sin(2n\vartheta)\,, \qquad \sin(3\vartheta) = \phi_0^3 + \sum_{n=1}^{\infty} \phi_n^3 \cos(2n\vartheta)$$

allow to cancel fourth line (related to 3ϑ) in (6.6) by allocating all terms with trigonometric functions evaluated at $2n\vartheta$ on $(2n+1)$-th line and, taking into account r powers and convergence properties, writing

$$D_{ij}^2 z = S_2 + \Sigma_2 \quad \text{with} \quad S_2 \in V, \quad \text{and} \quad \Sigma_2 \quad \text{with empty second and fourth lines of (6.6)}\;:$$

$$S_2 = S_1 + d_3\, \phi_0^3\, r^3 + \delta_3\, \phi_0^3\, r^5\,.$$

By iteration (after expanding $\sin((2k+1)\vartheta)$, $\cos((2k+1)\vartheta)$, $k=0,\dots,n-1$, and getting $D_{ij}^2 z = S_{n-1} + \Sigma_{n-1}$), we expand $\sin((2n+1)\vartheta)$, $\cos((2n+1)\vartheta)$, and get

$$D_{ij}^2 z = S_n + \Sigma_n \quad \text{with} \quad S_n \in V, \quad \text{and} \quad \Sigma_n \quad \text{with empty first } n \text{ odd lines}\,,$$

where S_n is a finite sum in the space V:

$$S_n = S_{n-1} + d_{2n-1}\, \phi_0^{2n-1}\, r^{2n-1} + \delta_{2n-1}\, \phi_0^{2n-1}\, r^{2n+1}\,.$$

Though Σ_n might not belong to $V^\perp$, by exploiting $L^2(B_r^+)$ convergence in (6.6) we denote by Ξ_n the modified odd lines from the third odd line (say the fifth one of (6.6)) to the n-th odd line, explicitly (referring to the lines position in (6.6)):

$$\begin{aligned}
\Xi_1 &= \text{expansion of the second line}\,,\\
\Sigma_1 &= \Xi_1 + \text{ all the lines after the second}\,,\\
\Xi_2 &= \Xi_1 + \text{third line} + \text{expansion of the fourth line}\,,\\
\Sigma_2 &= \Xi_2 + \text{ all the lines after the fourth}\,,\\
\Xi_n &= \Xi_{n-1} + (2n-1)\text{-th line} + \text{expansion of the } (2n)\text{-th line}\,,\\
\Sigma_n &= \Xi_n + \text{ all the lines after the } (2n)\text{-th line}\,.
\end{aligned}$$

By denoting ε_n the sum of all lines in (6.6) after the $(2n)$-th line, we get

$$D_{ij}^2 z = S_n + \Xi_n + \varepsilon_n \quad S_n \in V\,, \quad \Xi_n \in V^\perp, \quad \varepsilon_n \to 0 \text{ strongly in } L^2(B_r^+)\,. \tag{6.7}$$

(6.8) $\qquad S_n \to S$ strongly in $L^2(B_r^+)$, $\Xi_n \to \Xi$ strongly in $L^2(B_r^+)$.

Hence

$$D^2_{ij}z = \sum_{k=0}^{\infty}\left(\left(\sum_{h=0}^{\infty}(A_{h,k}+B_{h,k}r^2)r^h\right)\cos(2k\vartheta)+\left(\sum_{h=0}^{\infty}(C_{h,k}+D_{h,k}r^2)r^h\right)\sin(2k\vartheta)\right)$$

where the expansion is strongly $L^2(B_\varrho^+)$ convergent, $\forall \varrho \in (0,1)$.
Since $\int_0^\varrho r\,dr = \varrho^2/2$ and

$$\|1=\cos 0\|^2_{L^2(0,\pi)} = \pi, \quad \|\cos(2n\vartheta)\|^2_{L^2(0,\pi)} = \|\sin(2n\vartheta)\|^2_{L^2(0,\pi)} = \pi/2\,, \quad n=1,2,\ldots,$$

by setting $\lambda_2 = \lambda_2^{(i,j)} = \frac{\pi}{2}\left(\sum_{h=0}^{\infty}(A^{i,j}_{h,0})^2 + \sum_{h=0}^{\infty}(B^{i,j}_{h,0})^2\right)$, $\Lambda_2 = \sum_{ij}\lambda_2^{(i,j)}$,
via Plancherel identity in $L^2(B_r^+)$, we get the existence of $\lambda_l = \lambda_l^{(i,j)} \geq 0$, $l=3,4,\ldots$, s.t.

$$\begin{aligned}\|D^2_{ij}z\|^2_{L^2(B_\varrho^+)} &= \int_0^\varrho \left\{\pi\left|\sum_{h=0}^{\infty}(A_{h,0}+B_{h,0}\,r^2)\,r^h\right|^2\right. \\ &\quad \left. + \frac{\pi}{2}\sum_{k=1}^{\infty}\left(\left|\sum_{h=0}^{\infty}(A_{h,k}+B_{h,k}\,r^2)\,r^h\right|^2 + \left|\sum_{h=0}^{\infty}(C_{h,k}+D_{h,k}\,r^2)\,r^h\right|^2\right)\right\} r\,dr \\ &= \lambda_2\,\varrho^2 + \sum_{l=3}^{\infty}\lambda_l\,\varrho^l\,.\end{aligned}$$

The last power series with positive coefficients is convergent (so the inner sums do converge) and is estimated (uniformly in $\varrho < 1$) by $\|D^2_{ij}z\|^2_{L^2(B_1^+)} < +\infty$; then it is absolutely convergent even for $\varrho = 1$, and the sum is estimated in the same way. Then $D^2z \in L^2(B_1^+)$ and has the same expansion since coefficients $A_{h,k}, B_{h,k}, C_{h,k}, D_{h,k}$ are independent of $\varrho \in (0,1]$. Moreover $(S_n + \Xi_n)$ converges strongly in $L^2(B_1^+)$ to $D^2_{ij}z$, together with every reordering of its.
By summarizing the following expansion is strongly $L^2(B_\varrho^+)$ convergent, $\forall \varrho \in (0,1]$:

$$D^2_{ij}z = \sum_{k=0}^{\infty}\left(\left(\sum_{h=0}^{\infty}(A_{h,k}+B_{h,k}r^2)r^h\right)\cos(2k\vartheta)+\left(\sum_{h=0}^{\infty}(C_{h,k}+D_{h,k}r^2)r^h\right)\sin(2k\vartheta)\right)\,.$$

So, if $\|D^2z\|_{L^2(B_1^+)} \neq 0$, then

$$\frac{\|D^2z\|^2_{L^2(B_\varrho^+)}}{\|D^2z\|^2_{L^2(B_1^+)}} = \left(\Lambda_2\,\varrho^2 + \sum_{l=3}^{\infty}\Lambda_l\,\varrho^l\right)\Big/\left(\Lambda_2 + \sum_{l=3}^{\infty}\Lambda_l\right) \leq \varrho^2\,.$$

□

Remark 6.2. *Since* $\lambda_2^{(1,2)} = 0$, *by the proof of Theorem 6.1 we get a faster decay of mixed derivative:*

$$\|D^2_{xy}z\|^2_{L^2(B_\varrho^+)} \leq \varrho^3\,\|D^2_{xy}z\|^2_{L^2(B_1^+)}\,.$$

Remark 6.3. *No nontrivial harmonic function fulfils assumptions of Theorem 6.1. Precisely any* $z \in H^2(B_1^+)$ *s.t.* $\Delta^2 z = 0$ *on* B_1^+ *and* $z = \partial z/\partial y = 0$ *on* $B_1(\mathbf{0}) \cap \{y=0\}$ *satisfies also* $\Delta z = 0$ *in* B_1^+ *if and only if* $z \equiv 0$.
Nevertheless there are (simple) examples with $\Delta z \neq 0 = \Delta^2 z$ *on* B_1^+, $z = \partial z/\partial y = 0$

on $B_1(\mathbf{0}) \cap \{y = 0\}$ *with non trivial harmonic part in Almansi decomposition: e.g.* $z(x, y) = y^2 = (y^2 - x^2)/2 + (x^2 + y^2)/2$.

Remark 6.4. *Theorem 6.1 cannot be deduced by Schwarz reflection principle for harmonic functions vanishing on the diameter, since the Almansi decomposition on the half-disk* B_1^+ *(*[14], [1]*) may not respect the vanishing value on the diameter:*
e.g. $\varrho^3\big(\cos\vartheta - \cos(3\vartheta)\big) = \varrho^2\varphi + \psi$ *where* $\varphi = x$, $\psi = 3xy^2 - x^3$ *are both harmonic but do not vanish on the diameter* $\{y = 0\}$ *(see* [1] *and Theorem 3.2 of* [17]*).*

Remark 6.5. *While Schwarz reflection for harmonic functions vanishing on the diameter is bounded by 1 as a linear operator from* $H^1(B_1^+)$ *to* $H^1(B_1^-)$*, Duffin extension map for bi-harmonic functions vanishing on the diameter together with normal derivative provides a poor control of* $H^2(B_1^-)$ *in term of* $H^2(B_1^+)$ *as shown by the following example: referring to* (6.4)*, if we choose* $z = \omega_2 - v_3 + \omega_4 - v_5$ *then* $\|D^2 z\|_{L^2(B_1^-)} \approx 12.5761\ \|D^2 z\|_{L^2(B_1^+)}$.
This depends on the fact that bi-harmonic extension of z *may be either even in* y *(e.g.* $z = y^2$*) or odd in* y *(e.g.* $z = r^3(3\sin\vartheta - \sin(3\vartheta)) = 4y^3$*) or a mixing of the two (e.g.* $z = \omega_2 - v_3$*).*

Remark 6.6. *Several bi-harmonic functions like* $\varrho^3(\cos\vartheta - \cos(3\vartheta))$ *and, quite surprisingly, also combinations of multi-valued functions like* $\varrho^{3/2}\big(\cos(\vartheta/2) - \cos(3\vartheta/2)\big)$ *or like* $\varrho^{5/3}\big(\cos(\vartheta/3) - \cos(5\vartheta/3)\big)$ *actually turn out to be (*$H^2(B_1^+)$ *strongly convergent) infinite sums of the kind given by (6.2) above with vanishing value and normal derivative on* T*: hence they have single-valued analytic (and bi-harmonic) extension to the whole disk* B_ϱ *and fulfil decay property* (6.1).
In general if we set, for any $t\,,\,\tau \in \mathbb{N}$, $\varphi_t(\vartheta) = (\sin(t\vartheta) - \sin((t-2)\vartheta)\,t/(t-2))$ *and* $\psi_\tau(\vartheta) = (\cos(\tau\vartheta) - \cos((\tau-2)\vartheta))$*, then both* $r^t\varphi_t(\vartheta)$*,* $r^\tau\psi_\tau(\vartheta)$*, though built with polydromic functions, do have (unique) bi-harmonic extension to the whole disk* $B_\varrho(\mathbf{0})$ *: in fact* $\partial^h_\vartheta\,\varphi_t(\vartheta)\ |_{\vartheta=0} = \partial^h_\vartheta\,\varphi_t(\vartheta)\ |_{\vartheta=2\pi}\ \ \forall h$ *(due to* 2π *periodicity of* sin *and* cos*), so that their gluing at* 2π *is not only continuous but also analytic. The same argument holds true for* ψ_τ.

7. Blow-up and Decay at boundary points

In this Section we locally analyze the boundary around any point belonging to the set $\partial\Omega \setminus (T_0 \cup T_1 \cup M)$. At first (in Theorems 7.1, 7.2) we perform a blow-up of the functionals $\mathcal{F}$ and $\mathcal{E}$ around the origin under the additional assumption that $\mathbf{0}$ belongs to $\partial\Omega$. Then we exploit this results (by translating and scaling) to estimate the decay of these functionals when evaluated on local minimizers around boundary points (see Theorems 7.3, 7.4).

Theorem 7.1. ***(Blow-up of the functional*** $\mathcal{F}$ ***at boundary points)***
Assume (2.4)-(2.8) and:
$\mathbf{0} \in \partial\Omega \setminus (T_0 \cup T_1 \cup M)$, $B_r(\mathbf{0}) \subset \widetilde{\Omega}$, $\psi_h \in C^2(-r, r)$ *with* $\psi_h \to 0$ *in* $W^{2,\infty}(-r, +r)$, $\omega_h \in C^2(B_r)$ *with* $\omega_h \to \omega_\infty \equiv 0$ *in* $W^{2,\infty}(B_r(\mathbf{0}))$

$$(7.1)\qquad \begin{cases} \psi_h \in C^2(-r,r)\,, \quad \psi_h(0) = 0, \quad \psi_h'(0) = 0 \quad Lip\,(\psi_h') \le 1\,, \\[2ex] B^{\psi_h +} \stackrel{\text{def}}{=} B_r(\mathbf{0}) \cap \{y > \psi_h(x)\}\,, \quad B^{\psi_h -} \stackrel{\text{def}}{=} B_r(\mathbf{0}) \cap \{y < \psi_h(x)\}\,, \\[2ex] B^\tau_\varrho = \{\mathbf{x} = (x,y) : |\mathbf{x}| < \varrho,\ y > \tau)\} \ \text{for } 0 < \tau < \varrho < r\,. \end{cases}$$

$\gamma_h \in L^q(\widetilde{\Omega}) \cap L^{2q}_{loc}(\widetilde{\Omega})$, *let* α_h, β_h, μ_h, *three sequences of positive numbers with* $\beta_h \le \alpha_h$, *and let* $v_\infty \in H^2(B_r(\mathbf{0}))$ *s.t.* $v_\infty \equiv 0$ *in* $B_r^-(\mathbf{0})$. *Assume* $v_h \in GSBV^2(\widetilde{\Omega}) \cap L^q(\widetilde{\Omega})$, $v_h = \omega_h$ *a.e. in* $B^{\psi_h -}$ *and*

(i) v_h *are* Ω *local minimizers of* $\mathcal{F}_{\gamma_h\,\omega_h}(\,\cdot\,, \mu_h, \alpha_h, \beta_h, B_r(\mathbf{0}))$,

(ii) $\lim_h \mathcal{H}^1\left((S_{v_h} \cup S_{\nabla v_h}) \cap B_r(\mathbf{0})\right) \;=\; 0$,

(iii) $\exists\ \lim_h \mathcal{F}_{\gamma_h\,\omega_h}(\,v_h, \mu_h, \alpha_h, \beta_h, \overline{B^\tau_\varrho}\,) \stackrel{\text{def}}{=} \delta(\varrho,\tau) \;\le\; 1$
for a.e. $\varrho, \tau \in (0,r)$ *with* $\tau < \varrho$, *and set* $\delta(\varrho,\tau) \;=\; 0$ *if* $\varrho < \tau$.

(iv) $\lim_h v_h \;=\; v_\infty$ *a.e. in* $B_r(\mathbf{0})$,

(v) $\lim_h \mu_h \;=\; 0$, $\lim_h \mu_h \|\gamma_h\|^q_{L^q(B_r(\mathbf{0}))} \;=\; 0$.

Then, for every $\varrho \in (0,r), \tau \in (0,\varrho)$, v_∞ *minimizes the functional*

$$\int_{B^\tau_\varrho(\mathbf{0})} \left|\, D^2 v \,\right|^2 \, d\mathbf{x} \tag{7.2}$$

over $\{v \in H^2(B_r(\mathbf{0})) :\ v = v_\infty$ *in* $B_r(\mathbf{0}) \backslash B^\tau_\varrho$; *in particular* $v = 0$ *in* $\overline{B_r^-}(\mathbf{0})\}$.
Moreover

$$\delta(\varrho,\tau) \;=\; \int_{B^\tau_\varrho(\mathbf{0})} \left|\, D^2 v_\infty \right|^2 \, d\mathbf{x} \qquad \textit{for almost all } \varrho, \tau :\ 0 < \tau < \varrho < r\,. \tag{7.3}$$

In particular $\Delta^2 v_\infty \;=\; 0$ *in* $B_r^+(\mathbf{0})$, $v_\infty \;=\; 0 \;=\; \partial v_\infty / \partial y$ *in* $B_r(\mathbf{0}) \cap \{y \;=\; 0\}$, *and* $v_\infty \in C^1(B_r(\mathbf{0}))$.

Proof. By convergence assumptions on ψ_h, for any $\kappa \in (0, r/2)$ we can assume $|\psi_h| < \kappa < r/2$ for large h, that is the hypograph of ψ_h contains a fixed sector (of the disk B_r) where $v_h = \omega_h$, hence $u_h = v_h - \omega_h$ fulfil assumption (5.13) uniformly in h with $\vartheta \ge 1/3$, while Dirichlet datum ω_h is not imposed on the portion of the disk where $y > \kappa$.
By (iv) (v) we get, up to subsequences,

$$\lim_h \ {\mu_h}^{1/q}\, |v_h - \gamma_h| \;=\; 0 \qquad \text{a.e in } B_r\,.$$

By (ii) (iii) sequence $u_h = v_h - \omega_h$ fulfils all the assumptions of Theorem 5.4:

$$\sup_{\varrho,\tau} \sup_h \int_{B^\tau_\varrho} |\nabla^2 v_h|^2 \, d\mathbf{x} \;\le\; 1\,. \tag{7.4}$$

Then we can build a sequence z_h as in (5.23)-(5.28), choose subsequences (without relabeling) z_h, u_h, $v_h = u_h + \omega_h$ and $u_\infty \in H^2$ s.t. (5.16)-(5.20) hold true. Since $\omega_h \to 0$ and $v_h \to v_\infty$ a.e., we get $u_h \to u_\infty = v_\infty$, a.e. By (5.20), $\omega_h \to 0$ in $W^{2,\infty}$ and by (iii) we obtain, for a.e. ϱ, τ , $0 < \tau < \varrho < r$,

$$\int_{B^\tau_\varrho} \left|\, D^2 v_\infty \right|^2 \, d\mathbf{y} \;\le\; \liminf_h \int_{B^\tau_\varrho} \left(\left|\, \nabla^2 u_h \,\right|^2 \,+\, \mu_h |u_h - \gamma_h|^q \right) d\mathbf{y}$$

$$\le \liminf_h \int_{B^\tau_\varrho} \left(\left|\, \nabla^2 v_h \,\right|^2 \,+\, \mu_h |v_h - \gamma_h|^q \right) d\mathbf{y} \le \lim_h \ \mathcal{F}_{\gamma_h\,\omega_h}(v_h, \overline{B^\tau_\varrho}) \;=\; \delta(\varrho,\tau)\,.$$

To achieve the proof we have to show that for a.e. ϱ, τ, $0 < \tau < \varrho < r$, for every $\kappa \in (0, r/2)$, and every $u \in H^2(B_r)$ with $u = v_\infty$ in $B_r^+(\mathbf{0}) \setminus B^{2\kappa}_\varrho$ (hence $u = 0$ in $B_r^-(\mathbf{0})$):

$$\int_{B^\tau_\varrho} \left|\, D^2 u \,\right|^2 \, d\mathbf{y} \;\ge\; \delta(\varrho,\tau)\,. \tag{7.5}$$

In fact (7.5) implies $\Delta v_\infty = 0$, B_ϱ^+ and $v_\infty = \partial v_\infty/\partial y = 0$ on $B_1 \cap \{y = 0\}$; hence $v_\infty \in C^1(B_r^+ \cup (B_r \cap \{y=0\}))$ ([25]). We prove the inequality (7.5) for fixed $\kappa \in (0, r/2)$: the convergence property of ψ_h allows to repeat the proof for any such κ by selecting large enough h.

The map δ is monotone non decreasing in ϱ and monotone non incresing in τ, hence:
for any frozen τ, the map δ is continuous up to a countable set of values for ϱ,
for any frozen ϱ, the map δ is continuous up to a countable set of values for τ.
For any selection of ϱ, τ s.t. δ is *separately continuous* at ϱ, τ, we get by monotonicity that actually δ is a (two-variables) continuous map at ϱ, τ. This continuity property holds true for a.e. $\varrho, \tau \in (0, r)$.
Assume by contradiction that there exist $u \in H^2(B_r)$, $\varepsilon > 0$, s, σ, s.t. $2\kappa < \sigma < s < r$, δ is continuous at $\varrho = s, \tau = \sigma$, $u = v_\infty$ in $B_r^+ \setminus B_s^{2\kappa}$ (hence $u = 0$ in $B_r{}^-$) and

$$\int_{B_s^\sigma} \left| D^2 u \right|^2 d\mathbf{y} \ \le\ \delta(s,\sigma) - \varepsilon\,. \tag{7.6}$$

From now on we fix η, κ s.t.

$$s < \eta < r\,, \qquad 0 < \kappa < \frac{1}{2}\sigma < \frac{1}{2}\sqrt{r^2 - \eta^2} \tag{7.7}$$

Referring to (5.25), (5.27), we set

$$L_h = \mathcal{H}^1\left((S_{v_h} \cup S_{\nabla v_h}) \cap B_r(\mathbf{0})\right) \tag{7.8}$$

$$\Xi_h \ =\ \{\mathbf{y} \in B_r \ :\ z_h \neq \xi_h\}\,, \tag{7.9}$$

$$A_h \ =\ \{z_h \neq u_h\}\,. \tag{7.10}$$

In particular $A_h = E_h \cup \Xi_h$.
In order to get a contradiction we will paste together u_h and z_h along the boundary of a suitably chosen lunula (the energy addition will tend to 0 as $h \to \infty$ due to (iii), (5.17) and Matching Lemma 4.2), then we will join such new function in a smaller lunula with u (which has less squared hessian energy). Before some preliminary estimates are needed.
We emphasize that

$$\mathcal{H}^1(S_{v_h} \cap \partial B_\varrho) = \mathcal{H}^1(S_{\nabla v_h} \cap \partial B_\varrho) \ =\ 0 \qquad \text{for } a.e\ \varrho \in (0,r) \tag{7.11}$$

and by (5.17)

$$P(A_h, B_r) \ \le\ c\, L_h\,. \tag{7.12}$$

By integrating first in polar coordinates, then in cartesian coordinates and taking into account the isoperimetric inequality we get

$$\alpha_h \int_0^r \mathcal{H}^1(A_h \cap \partial B_\varrho)\, d\varrho \ =\ \alpha_h |A_h \cap B_r| \ \le\ \alpha_h(\gamma_2 P(A_h, B_r))^2 \ \le\ c^2\gamma_2^2 \alpha_h {L_h}^2 \tag{7.13}$$

$$\begin{aligned} &\alpha_h \int_0^{\sqrt{r^2-t^2}} \mathcal{H}^1(A_h \cap \{(x,y) :\ |x| \le t\})\, dy \\ &= \alpha_h \left| A_h \cap \{|x| \le t,\ 0 \le y \le \sqrt{r^2 - t^2}\} \right| \\ &\le \alpha_h |A_h \cap B_r| \ \le\ \alpha_h(\gamma_2 P(A_h, B_r))^2 \ \le\ c^2 \gamma_2^2 \alpha_h {L_h}^2 \end{aligned} \tag{7.14}$$

since the sequence $\alpha_h L_h$ is bounded by (iii), then (ii) entails

$$\lim_h \ \alpha_h \, {L_h}^2 \ = \ 0.$$

Hence, by (7.13), we have, up to subsequence and without relabeling,

$$\exists \ \lim_h \ \alpha_h \, \mathcal{H}^1 \left(A_h \cap \partial {B_t}^+ \right) \ = \ 0 \ \text{ for a.e. } t \in (0,r)\,. \tag{7.15}$$

By assumption (7.7), the interval $(2\kappa, \sqrt{r^2-\eta^2})$ is not empty and contains σ.
For any choice of $t \in (s,r)$ as above (fulfilling (7.15)) and for a.e. $d \in \left(2\kappa, \sqrt{r^2-\eta^2}\right)$ (thanks to (7.14)) we have

$$\lim_h \ \alpha_h \, \mathcal{H}^1 \left(A_h \cap \{|x| \le t, \ y = d\} \right) \ = \ 0\,; \tag{7.16}$$

by summarizing, for any t fulfilling (7.15) for a.e. $d \in \left(2\kappa, \sqrt{r^2-t^2}\right)$ both (7.15), (7.16) hold true, so that, up to subsequence and without relabeling,

$$\exists \quad \lim_h \ \alpha_h \, \mathcal{H}^1 \left(A_h \cap \partial B_t^d \right) \ = \ 0 \ \text{ for a.e. } t, d\,, \tag{7.17}$$

and by (7.11), (7.17), (ii) and $\beta_h \le \alpha_h$ we get,

$$\begin{gathered} \lim_h \ \left(\alpha_h \, \mathcal{H}^1 \left(S_{z_h} \cap \partial B_t^d \right) + \beta_h \, \mathcal{H}^1 \left((S_{\nabla z_h} \setminus S_{z_h}) \cap \partial B_t^d \right) \right) \ = \ 0 \\ \text{for a.e } t \in (s,\eta) \text{ and a.e } d \in \left(2\kappa, \sqrt{r^2-t^2}\right). \end{gathered} \tag{7.18}$$

Notice that the interval $(2\kappa, \sqrt{r^2-t^2})$ is not empty since it contains σ due to (7.7). By continuity of δ at $\varrho = s, \tau = \sigma$ and by (7.17), (7.18) and (iii) we can choose $t \in (s,\eta)$ close to s as needed, $d \in (2\kappa, \sigma)$ close to σ as needed (and let them fixed in the following) and $\widetilde{h} \in \mathbb{N}$ s.t. $\sigma - d < t - s$ and, setting $M_{t,s}^{d,\sigma} = \overline{B_t^d} \setminus B_s^\sigma$, the following list of inequalities hold true:

$$\delta(t,d) - \delta(s,\sigma) \ < \ \varepsilon/6\,, \tag{7.19}$$

$$\alpha_h \, \mathcal{H}^1 \left(A_h \cap \partial \, M_{t,s}^{d,\sigma} \right) \ \le \ \varepsilon/6 \qquad h > \widetilde{h}\,, \tag{7.20}$$

$$\alpha_h \, \mathcal{H}^1 \left(S_{z_h} \cap \partial \, M_{t,s}^{d,\sigma} \right) + \beta_h \, \mathcal{H}^1 \left((S_{\nabla z_h} \setminus S_{z_h}) \cap \partial \, M_{t,s}^{d,\sigma} \right) \ \le \ \varepsilon/6 \ \ h > \widetilde{h}\,, \tag{7.21}$$

$$\int_{B_t^d \setminus B_s^\sigma} |D^2 u|^2 \, d\mathbf{x} \ < \ \varepsilon/6\,, \tag{7.22}$$

$$\mathcal{F}\left(v_h, \overline{B_t^d} \setminus B_s^\sigma \right) \le \ 2\,\varepsilon/6 \qquad h > \widetilde{h}\,; \tag{7.23}$$

In fact (7.19) expresses the continuity of δ at (s,σ); feasibility of choices (7.20), (7.21) follows by (7.17), (7.18); inequality (7.22) follows by the absolute continuity of $\int_A |D^2 u|^2 d\mathbf{x}$ with respect to the Lebesgue measure of A, eventually (7.23) follows by

$$\lim_h \ \mathcal{F}\left(v_h, \overline{B_t^d} \setminus B_s^\sigma \right) \ = \ \delta(t,d) - \delta(s,\sigma) \ + \lim_h \ \mathcal{F}\left(v_h, \partial B_s^\sigma \right)$$

which is estimated by $2\,\varepsilon/6$ thanks to (7.19), (7.21).
We fix the matching:

$$\zeta_h \ = \ u_h \, \chi_{B_r \setminus \overline{B_t^d}} \ + \ z_h \, \chi_{\overline{B_t^d}}\ , \tag{7.24}$$

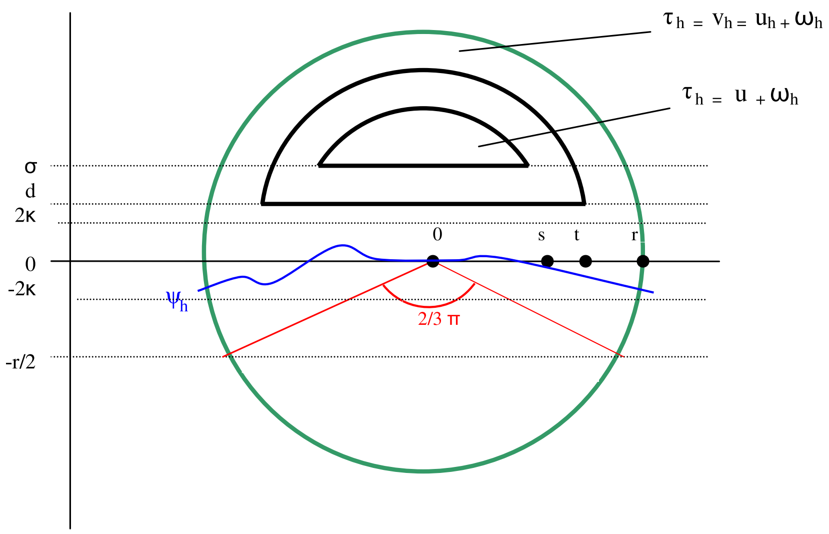

FIGURE 1. Joining along handle $M^{d,\sigma}_{t,s} = \overline{B^d_t} \setminus B^\sigma_s$

hence (7.17) and Lemma 4.2 entail, for a.e. $\varrho \in (0, r)$, $\tau \in (d, \sigma)$,

$$\lim_h \mathcal{F}_{\gamma_h\,\omega_h}(\zeta_h, \mu_h, \alpha_h, \beta_h, B^\tau_\varrho) = \lim_h \mathcal{F}_{\gamma_h\,\omega_h}(v_h, \mu_h, \alpha_h, \beta_h, B^\tau_\varrho) = \delta(\varrho, \tau) \le 1\,. \tag{7.25}$$

Eventually we perform the joining of $u + \omega_h$ and $\zeta_h + \omega_h$ between lunulae B^d_t and B^σ_s: by referring to Lemma 4.1, we choose $\Psi \equiv 1$ in a neighborhood of B^σ_s and set

$$\tau_h \;=\; \Psi\,(u + \omega_h) \,+\, (1 - \Psi)\,(\zeta_h + \omega_h)\,, \tag{7.26}$$

so that

$$\tau_h \;=\; u_h + \omega_h = v_h \quad \text{in } B_r \setminus \overline{B^d_t}\,, \qquad \tau_h \;=\; u + \omega_h \quad \text{in } \overline{B^\sigma_s}\,, \tag{7.27}$$

hence

$$\mathcal{F}(\tau_h, B_r \setminus \overline{B^d_t}) \;=\; \mathcal{F}(v_h, B_r \setminus \overline{B^d_t})\,. \tag{7.28}$$

Then by Lemma 4.1 we obtain, for any $\theta > 0$,

$$\begin{aligned} \mathcal{F}(\tau_h, \overline{B^d_t}) \;&\le\; (1+\theta)\,\Big(\mathcal{F}(u + \omega_h, \overline{B^d_t}) \,+\, \mathcal{F}(\zeta_h + \omega_h, \overline{B^d_t} \setminus B^\sigma_s)\Big) \,+ \\ &+ \frac{c}{(\sigma - d)^2}\left(\int_{B^d_t \setminus B^\sigma_s} |\nabla(u - \zeta_h)|^2 d\mathbf{x} \,+\, \frac{c}{\theta\, d^2\,(\sigma - d)^2}\int_{B^d_t \setminus B^\sigma_s} |u - \zeta_h|^2 d\mathbf{x}\right) \end{aligned} \tag{7.29}$$

By compactness Theorem 5.4, with our choice for d, σ fulfilling $2\kappa < d < \sigma$:

$$\lim_h \int_{B^d_t \setminus B^\sigma_s} |\nabla(v_\infty - \zeta_h)|^2 d\mathbf{x} \,=\, \lim_h \int_{B^d_t \setminus B^\sigma_s} |v_\infty - \zeta_h|^2 d\mathbf{x} \,=\, 0\,, \tag{7.30}$$

hence, thanks to $u = v_\infty$ in $B^+_r \setminus B^{2\kappa}_s$, possibly by extracting subsequences without relabeling and letting $h \to +\infty$ in (7.29) we obtain

$$\lim_h \mathcal{F}(\tau_h, \overline{B^d_t}) \le (1+\theta)\left(\lim_h \mathcal{F}(u + \omega_h, \overline{B^d_t}) + \lim_h \mathcal{F}(\zeta_h + \omega_h, \overline{B^d_t} \setminus B^\sigma_s)\right). \tag{7.31}$$

By convergence $\omega_h \to 0$ in $W^{2,\infty}(B_r)$ there is $h_0 \geq \widetilde{h}$ s.t.

$$\left| \int_{B_t^d \setminus B_s^\sigma} |D^2(u+\omega_h)|^2 \, d\mathbf{x} - \int_{B_t^d \setminus B_s^\sigma} |D^2 u|^2 d\mathbf{x} \right| < \frac{\varepsilon}{6} \qquad \text{for } h > h_0 \, . \tag{7.32}$$

By (5.26), $\omega_h \to 0$ in $W^{2,\infty}$, (7.23), (7.27) we get

$$\begin{aligned} &\lim_h \mathcal{F}\left(\zeta_h + \omega_h, \overline{B_t^d} \setminus B_s^\sigma\right) \\ &\quad = \lim_h \mathcal{F}\left(z_h + \omega_h, \overline{B_t^d} \setminus B_s^\sigma\right) \leq \lim_h \mathcal{F}\left(v_h, \overline{B_t^d} \setminus B_s^\sigma\right) \leq 2\,\frac{\varepsilon}{6} \end{aligned} \tag{7.33}$$

By letting $\vartheta \to 0$ in (7.31), taking into account (7.20)-(7.24),(7.27),(7.32),(7.33), we get

$$\mathcal{F}\left(\tau_h, \overline{B_t^d} \setminus B_s^\sigma\right) \leq \; 4\,\varepsilon/6 \qquad h > h_0 \, ; \tag{7.34}$$

By exploiting (7.6),(7.20), (7.21), (7.23), (7.34), $\tau_h = u + \omega_h$ in B_s^σ and eventually δ monotonicity with respect to inclusion of sets, we get the contradiction:

$$\begin{aligned} \delta(t,d) \; &= \lim_h \mathcal{F}\left(v_h \, , \, \overline{B_t^d}\right) \overset{\text{minimality of } v_h}{\leq} \lim_h \mathcal{F}\left(\tau_h \, , \, \overline{B_t^d}\right) \\ &= \lim_h \mathcal{F}\left(\tau_h \, , \, \overline{B_s^\sigma}\right) + \lim_h \mathcal{F}\left(\tau_h \, , \, \overline{B_t^d} \setminus B_s^\sigma\right) - \lim_h \mathcal{F}\left(\tau_h \, , \, \partial B_s^\sigma\right) \\ &\overset{(7.34)}{\leq} \lim_h \mathcal{F}\left(\tau_h \, , \, \overline{B_s^\sigma}\right) + 4\,\frac{\varepsilon}{6} \\ &= \lim_h \mathcal{F}\left(u + \omega_h \, , \, \overline{B_s^\sigma}\right) + 4\,\frac{\varepsilon}{6} = \lim_h \int_{B_s^\sigma} |D^2(u+\omega_h)|^2 \, d\mathbf{x} + 4\,\frac{\varepsilon}{6} \\ &\overset{(7.32)}{\leq} \int_{B_s^\sigma} |D^2 u|^2 \, d\mathbf{x} + 5\,\frac{\varepsilon}{6} \leq \delta(s,\sigma) - \varepsilon + 5\,\frac{\varepsilon}{6} \leq \delta(t,d) - \frac{\varepsilon}{6} \, . \end{aligned} \tag{7.35}$$

□

Theorem 7.2. ***(Blow-up of the functional $\mathcal{E}$ at boundary points)***
Assume (2.4)-(2.11), $\mathbf{0} \in \partial\Omega \setminus (T_0 \cup T_1 \cup M)$ and $B_r(\mathbf{0}) \subset \widetilde{\Omega}$, let $\alpha_h, \beta_h,$ two sequences of positive numbers with $\beta_h \leq \alpha_h$, $\psi_h \in C^2(-r,r)$ with $\psi_h \to 0$ in $W^{2,\infty}(-r,+r)$, $\omega_h \in C^2(B_r(\mathbf{0}))$ with $\omega_h \to \omega_\infty \equiv 0$ in $W^{2,\infty}$ and let $v_\infty \in H^2(B_r(\mathbf{0}))$ s.t. $v_\infty \equiv 0$ in $B_r^-(\mathbf{0})$. Assume (7.1), $v_h \in GSBV^2(\widetilde{\Omega})$, $v_h = \omega_h$ a.e. in $B^{\psi_h -}$ and

(i) *v_h are Ω local minimizers of $\mathcal{E}_{\omega_h}(\,\cdot\,, \alpha_h, \beta_h, B_r(\mathbf{0}))$,*

(ii) $\lim_h \mathcal{H}^1\left((S_{v_h} \cup S_{\nabla v_h}) \cap B_r(\mathbf{0})\right) = 0$,

(iii) $\exists \; \lim_h \mathcal{E}_{\omega_h}(\, v_h, \alpha_h, \beta_h, \overline{B_\varrho^\tau}(\mathbf{0})) \overset{\text{def}}{=} \delta(\varrho,\tau) \leq 1$
for a.e. $\varrho, \tau \in (0,r)$ with $\tau < \varrho$, and set $\delta(\varrho,\tau) = 0$ if $\varrho < \tau$.

(iv) $\lim_h v_h = v_\infty$ *a.e. in $B_r(\mathbf{0})$.*

Then, for every $\varrho, \tau \, : 0 < \tau < \varrho < r$, v_∞ minimizes the functional

$$\int_{B_r^+(\mathbf{0})} \left| \, D^2 u \, \right|^2 \, d\mathbf{x} \tag{7.36}$$

over $\{u \in H^2(B_\varrho(\mathbf{0})):\ u = 0 \ \text{in}\ B_r^-(\mathbf{0});\ u = v_\infty \ \text{in}\ B_r(\mathbf{0})\setminus B_\varrho^\tau(\mathbf{0})\}$.
Moreover

$$(7.37)\qquad \delta(\varrho,\tau) \;=\; \int_{B_\varrho^\tau(\mathbf{0})} \left| D^2 v_\infty \right|^2 d\mathbf{x} \qquad \text{for almost all } \tau,\varrho:\ 0<\tau<\varrho<r\,.$$

In particular $\Delta^2 v_\infty = 0$ *in* $B_r^+(\mathbf{0})$, $v_\infty = 0 = \partial v_\infty/\partial y$ *in* $B_r(\mathbf{0})\cap\{y=0\}$.

Proof. Repeat the proof of the previous Theorem with $\mu_h = 0$. □

Due to (2.5), (2.6) for any sequence of points $\mathbf{x}_h \in \partial\Omega \setminus (T_0 \cup T_1 \cup M)$ possibly after suitable rotations of coordinates around each $\mathbf{x}_h = (x_h, y_h)$, we can find ϱ_h and φ_h s.t., by setting

$$(7.38)\qquad \Omega_{\varphi_h+} = \Omega \cap B_{\varrho_h}(\mathbf{x}_h)\,, \qquad \Omega_{\varphi_h-} = B_{\varrho_h}(\mathbf{x}_h)\setminus\overline{\Omega}\,,$$

we have

$$(7.39)\qquad \begin{cases} w \in C^2(B_{\varrho_h}(\mathbf{x}_h))\,, \qquad \Omega_{\varphi_h+} \;=\; B_{\varrho_h}(\mathbf{x}_h)\cap\{y>\varphi_h(x)\}\,, \\ \varphi_h \in C^2(x_h-\varrho_h, x_h+\varrho_h),\ \varphi_h(x_h) = y_h,\ \varphi_h{}'(x_h) = 0, \mathrm{Lip}(\varphi_h{}') \le C. \end{cases}$$

Referring to (7.38), (7.39) we re-scale and translate the sets Ω_{φ_h} to the the origin and choose the graphs ψ_h to be used in the application of the blow-up Theorem (with $r = 1$) as follows:

$$(7.40)\qquad \psi_h(x) = \varrho_h{}^{-1}\left(\varphi_h(x_h+\varrho_h x) - y_h\right)$$

$$(7.41)\qquad B^{\psi_h+} = B_1(\mathbf{0})\cap\{(x,y):\ y_h+\varrho_h y > \varphi_h(x_h+\varrho_h x)\} = B_1(\mathbf{0})\cap\{y>\psi_h(x)\}$$

$$(7.42)\qquad B^{\psi_h-} = B_1(\mathbf{0})\cap\{(x,y):\ y_h+\varrho_h y < \varphi_h(x_h+\varrho_h x)\} = B_1(\mathbf{0})\cap\{y<\psi_h(x)\}\,;$$

we get

$$(7.43)\qquad \begin{cases} B^{\psi_h\pm} = \left(\Omega_{\varphi_h\pm} - \mathbf{x}_h\right)/\varrho_h \\ \psi_h(0) = 0\,,\ \psi_h{}'(0) = 0 \\ \psi_h{}'(x) = \varphi_h{}'(x_h+\varrho_h x) = \varrho_h x\, \varphi_h{}''(x_h) + o(\varrho_h) \\ \psi_h{}''(x) = \varrho_h \varphi_h{}''(x_h+\varrho_h x) \\ \mathrm{Lip}(\psi_h) = \mathrm{Lip}(\varphi_h)\,, \quad \mathrm{Lip}(\psi_h{}') = \varrho_h\, \mathrm{Lip}(\varphi_h{}')\,. \end{cases}$$

Theorem 7.3. *(**Decay of the functional $\mathcal{F}$ at boundary points**) Assume (2.3)-(2.10) and (2.17). Then, by referring to* (3.14) *and to* (3.15) *about the meaning of* $\bar\varrho$ *and* c_0,

$$(7.44)\qquad \forall k>2,\ \forall\eta,\sigma\in(0,1),\quad \exists\varepsilon_1>0,\ \exists\vartheta_1>0 \qquad \text{such that}$$

for all $\varepsilon \in (0,\varepsilon_1]$, *for any* $\mathbf{x} \in \partial\Omega\setminus(T_0\cup T_1\cup M)$, *for any* u *which is an* $\overline{\Omega_{\varphi+}}$ *local minimizer of* $\mathcal{F}_{gw}(\cdot,\mu,\alpha,\beta,\overline{\Omega_{\varphi+}})$, *for any* ϱ *s.t.* $B_\varrho(\mathbf{x}) \subset \widetilde{\Omega}$ *(we can assume without restriction (7.39)),* $0<\varrho\le\left(\varepsilon^k\wedge\bar\varrho\wedge(c_0\vee 1)^{-1}\right)$, $\int_{B_\varrho(\mathbf{x})}|g|^{2q}\le\varepsilon^k$ *and*

$$(7.45)\qquad \alpha\,\mathcal{H}^1\left(S_u\cap\overline{\Omega_{\varphi+}}\right) \;+\; \beta\,\mathcal{H}^1\left((S_{\nabla u}\setminus S_u)\cap\overline{\Omega_{\varphi+}}\right) \;<\; \varepsilon\,\varrho\,,$$

we have

$$\text{(7.46)}\qquad \begin{aligned}&\mathcal{F}_{g\,w}(u,B_{\eta\varrho}(\mathbf{x}))\\ &\quad\leq\ \eta^{2-\sigma}\ \max\left\{\mathcal{F}_{g\,w}(u,B_{\varrho}(\mathbf{x}))\ ,\ \varrho^2\,\vartheta_1\ \left(\left(\mathrm{Lip}(\varphi')\right)^2+\left(\mathrm{Lip}(Dw)\right)^2\right)\right\}.\end{aligned}$$

Proof. Assume the Theorem is false. Then there are $k>2$, η, $\sigma\in(0,1)$; three sequences $\varrho_h,\varepsilon_h,\vartheta_h$ s.t. $h\geq 3$ and $0<\varrho_h\leq\left(\bar\varrho\wedge(c_0\vee 1)^{-1}\right)$, $\varepsilon_h>0$, $\vartheta_h>0$, $\varepsilon_h\downarrow 0$, $\lim_h\vartheta_h=+\infty$; a sequence $\mathbf{x}_h\in\partial\Omega\setminus(T_0\cup T_1\cup M)$; a sequence $w_h\in C^2(B_{\varrho_h}(\mathbf{x_h}))$ s.t. $|w_h|\leq C$, $Lip(Dw_h)\leq C$; a sequence $\varphi_h\in C^2\big((x_h-\varrho_h,x_h+\varrho_h)\big)$ with $\varphi_h(x_h)=y_h$, $\varphi_h'(x_h)=0$, $\mathrm{Lip}({\varphi_h}')\leq C$ and

$$\Omega_{\varphi_h+}=\Omega\cap B_{\varrho_h}(\mathbf{x_h})\cap\{y>\varphi_h(x)\}\,;$$

a sequence $u_h\in X(\widetilde\Omega)$ of Ω local minimizers of $\mathcal{F}_{g\,w_h}(\cdot,\mu,\alpha,\beta,B_{\varrho_h}(\mathbf{x}_h))$ among v s.t. $v=w_h$ on Ω_{φ_h-};

$$\text{(7.47)}\qquad \varrho_h\leq{\varepsilon_h}^k\,,\qquad \int_{B_{\varrho_h}(\mathbf{x}_h)}|g|^{2q}\leq{\varepsilon_h}^k\,;$$

$$\text{(7.48)}\qquad \alpha\,\mathcal{H}^1\left(S_{u_h}\cap\overline{\Omega_{\varphi_h+}}\right)\ +\ \beta\,\mathcal{H}^1\left((S_{\nabla u_h}\setminus S_{u_h})\cap\overline{\Omega_{\varphi_h+}}\right)\ <\ \varepsilon_h\,\varrho_h$$

and

$$\text{(7.49)}\qquad \begin{aligned}&\mathcal{F}_{g\,w_h}(u_h,\mu,\alpha,\beta,B_{\eta\,\varrho_h}(\mathbf{x}_h))\\ &\ >\ \eta^{2-\sigma}\max\left\{\mathcal{F}_{g\,w_h}(u_h,\mu,\alpha,\beta,B_{\varrho_h}(\mathbf{x}_h)),\varrho_h^2\,\vartheta_h\ \left(\left(\mathrm{Lip}({\varphi_h}')\right)^2+\left(\mathrm{Lip}(Dw_h)\right)^2\right)\right\}.\end{aligned}$$

By translating $\mathbf{x}_h$ to $\mathbf{0}$, re-scaling, and applying a common affine linear correction to data and local minimizers, we set, for $\mathbf{y}\in B_1(\mathbf{0})$:

$$\text{(7.50)}\qquad \omega_h(\mathbf{y})\ =\ \left(\lambda_h\,{\varrho_h}^3\right)^{-1/2}\left(w_h(\mathbf{x}_h+\varrho_h\mathbf{y})-\varrho_h Dw_h(\mathbf{x}_h)\cdot\mathbf{y}-w_h(\mathbf{x}_h)\right)$$

$$\text{(7.51)}\qquad \gamma_h(\mathbf{y})\ =\ \left(\lambda_h\,{\varrho_h}^3\right)^{-1/2}\left(g(\mathbf{x}_h+\varrho_h\mathbf{y})-\varrho_h Dw_h(\mathbf{x}_h)\cdot\mathbf{y}-w_h(\mathbf{x}_h)\right)$$

$$\text{(7.52)}\qquad v_h(\mathbf{y})\ =\ \left(\lambda_h\,{\varrho_h}^3\right)^{-1/2}\left(u_h(\mathbf{x}_h+\varrho_h\mathbf{y})-\varrho_h Dw_h(\mathbf{x}_h)\cdot\mathbf{y}-w_h(\mathbf{x}_h)\right)$$

where

$$\text{(7.53)}\qquad \lambda_h\ =\ \left({\varrho_h}^{-1}\,\mathcal{F}_{g\,w_h}(u_h,\mu,\alpha,\beta,B_{\varrho_h}(\mathbf{x}_h))\right)\bigvee\varepsilon_h\,.$$

Notice that, due to density upper bound in Theorem 3.10 and $\varrho_h\leq\bar\varrho\wedge(c_0\vee 1)^{-1}$,

$$\lambda_h\leq c_0\vee 1<+\infty\qquad\text{and}\qquad \lambda_h\,\varrho_h\leq 1\quad\forall h$$

and functions v_h and ω_h coincide on $B^{\psi_h-}=B_1(\mathbf{0})\cap\{y<\psi_h(x)\}$.
Moreover, by uniform C^2 property of w_h, and applying Lagrange Theorem to each component of $D\omega_h$

$$\forall\mathbf{y}\in B_1,\ i,j=1,2\ \exists\ \widetilde t(i)\in(0,1)\ \text{s.t. by setting}\ \widetilde{\mathbf{y}}(i)=\widetilde t(i)\mathbf{y}\,,\ \text{we get}$$

$$D_j\,\omega_h(\mathbf{y})\ =\ D_j\,\omega_h(\mathbf{0})+\sum_i D_{ij}\omega_h(\,\widetilde{\mathbf{y}}(i))\mathbf{y}_i$$

say, by denoting $\widetilde{D^2\omega_h}(\widetilde{\mathbf{y}})$ the hessian of ω_h with i-th row evaluated at $\widetilde{\mathbf{y}}(i)$,

$$D\omega_h(\mathbf{y})\ =\ D\omega_h(\mathbf{0})\ +\ \widetilde{D^2\omega_h}(\widetilde{\mathbf{y}})\cdot\mathbf{y}$$

then

$$
(7.54)\qquad
\begin{cases}
\omega_h(\mathbf{0}) = 0\,, \qquad D\omega_h(\mathbf{0}) = \mathbf{0}\,,\\[2mm]
D\omega_h(\mathbf{y}) = \widetilde{D^2\omega_h}(\widetilde{\mathbf{y}})\cdot \mathbf{y}\\[2mm]
D\omega_h(\mathbf{y}) = (\,\lambda_h\,\varrho_h\,)^{-1/2}\Big(\,Dw_h(\mathbf{x}_h+\varrho_h\mathbf{y}) - Dw_h(\mathbf{x}_h)\,\Big)\\[2mm]
D^2\omega_h(\mathbf{y}) = (\,\varrho_h/\lambda_h\,)^{1/2}\;D^2 w_h(\mathbf{x}_h+\varrho_h\mathbf{y})\\[2mm]
|D^2\omega_h(\mathbf{y})| \,\le\, (\varrho_h/\lambda_h)^{1/2}\;\mathrm{Lip}(Dw_h)\\[2mm]
|D\omega_h(\mathbf{y})| \le |\widetilde{D^2\omega_h}(\widetilde{\mathbf{y}})|\,|\mathbf{y}| \le (\varrho_h/\lambda_h)^{1/2}\,\mathrm{Lip}(Dw_h)\\[2mm]
\mathrm{Lip}(D\omega_h) = (\varrho_h/\lambda_h)^{1/2}\,\mathrm{Lip}(Dw_h)
\end{cases}
$$

hence

$$
(7.55)\qquad |D\omega_h(\mathbf{y})| \le C\,\varepsilon_h^{(k-1)/2}\,, \qquad |D^2\omega_h(\mathbf{y})| \,\le\, C\;\varepsilon_h^{(k-1)/2}\,.
$$

Due to (2.6) φ_h are uniformly C^2, hence (7.43) entails $\psi_h \to 0$ in $W^{2,\infty}(-1,1)$.
Estimates (7.55) entail strong $W^{2,\infty}(B_1)$ convergence of ω_h to $\omega_\infty \equiv 0$.
Due to Remark 3.9 functions v_h are Ω local minimizers of $\mathcal{F}_{\gamma_h,\omega_h}(\cdot,\mu_h,\alpha_h,\beta_h,B_1(\mathbf{0}))$ among v with $v=\omega_h$ in B^{ψ_h-} where

$$
(7.56)\qquad \alpha_h \;=\; \frac{\alpha}{\lambda_h}\,, \qquad \beta_h \;=\; \frac{\beta}{\lambda_h}\,, \qquad \mu_h \;=\; \mu\;\lambda_h^{\,\frac{q}{2}-1}\;\varrho_h^{\,1+\frac{3}{2}q}\,.
$$

By scaling Lemma 3.5, (7.49), last identity in (7.43), (7.54), and $\lambda_h\,\varrho_h \le 1$ we have

$$
(7.57)\qquad
\begin{cases}
\mathcal{F}_{g\,w_h}(u_h,\mu,\alpha,\beta,B_{\varrho_h}(\mathbf{x}_h)) \;=\; \lambda_h\,\varrho_h\,\mathcal{F}_{\gamma_h\,\omega_h}(v_h,\mu_h,\alpha_h,\beta_h,B_1(\mathbf{0}))\\
\mathcal{F}_{g\,w_h}(u_h,\mu,\alpha,\beta,B_{\eta\varrho_h}(\mathbf{x}_h)) \;=\; \lambda_h\,\varrho_h\,\mathcal{F}_{\gamma_h\,\omega_h}(v_h,\mu_h,\alpha_h,\beta_h,B_\eta(\mathbf{0}))
\end{cases}
$$

and

$$
(7.58)\qquad
\begin{aligned}
\mathcal{F}_{\gamma_h\,\omega_h}(v_h,\mu_h,\alpha_h,\beta_h,B_\eta(\mathbf{0})) =\;& \mathcal{F}_{g\,w_h}(u_h,\mu,\alpha,\beta,B_{\eta\varrho_h}(\mathbf{x}_h))/(\lambda_h\,\varrho_h)\\
>\;& \eta^{2-\sigma}\frac{\varrho_h}{\lambda_h}\vartheta_h\,\Big(\Big(\frac{\mathrm{Lip}(\psi_h{}')}{\varrho_h}\Big)^2 + \Big(\frac{\mathrm{Lip}(D\omega_h)}{\sqrt{\varrho_h/\lambda_h}}\Big)^2\Big)\\
=\;& \eta^{2-\sigma}\frac{\vartheta_h}{\lambda_h}\Big(\frac{(\mathrm{Lip}(\psi_h{}'))^2}{\varrho_h} + \lambda_h\,(\mathrm{Lip}(D\omega_h))^2\Big)\\
\ge\;& \eta^{2-\sigma}\vartheta_h\,\Big(\,(\mathrm{Lip}(\psi_h{}'))^2 + (\mathrm{Lip}(D\omega_h))^2\Big)
\end{aligned}
$$

so that by (7.53), (7.57)

$$
(7.59)\qquad \mathcal{F}_{\gamma_h\,\omega_h}(v_h,\mu_h,\alpha_h,\beta_h,B_1(\mathbf{0})) \;\le\; 1\,,
$$

$$
(7.60)\qquad \alpha_h\,\mathcal{H}^1\left(S_{v_h}\cap\overline{B^{\psi_h+}}\right) \;+\; \beta_h\,\mathcal{H}^1\left((S_{\nabla v_h}\setminus S_{v_h})\cap\overline{B^{\psi_h+}}\right) \;<\; \varepsilon_h
$$

and (7.49), (7.57), (7.58) entail

$$
(7.61)\qquad \mathcal{F}_{\gamma_h\,\omega_h}(v_h,\mu_h,\alpha_h,\beta_h,B_\eta(\mathbf{0})) \;>\; \eta^{2-\sigma}\,\mathcal{F}_{\gamma_h\,\omega_h}(v_h,\mu_h,\alpha_h,\beta_h,B_1(\mathbf{0}))\,.
$$

By (7.59) and Theorem 5.4, up to subsequences and without relabeling,

$$(7.62) \qquad \exists\, v_\infty \in H^2(B_1) : \qquad \lim_h v_h = v_\infty \qquad \text{a.e. in } B_1$$

say hypothesis (iv) of Theorem 7.1, which we want to apply to handle v_h and v_∞. We proceed by checking the other assumptions of the Theorem 7.1.
Since u_h is an Ω local minimizer of $\mathcal{F}_{g\,\omega_h}$ then v_h is a $\varrho_h{}^{-1}(\Omega - \mathbf{x}_h)$ local minimizer of $\mathcal{F}_{\gamma_h\,\omega_h}$, that is (i) holds true; (7.60) entails (ii); we must verify (iii), (v) and the structural assumptions.
Now we prove (iii): choose a dense (in (0,1)) sequence of radii $\varrho_j = \tau_j$. Thanks to (7.59), for any pair $j, l \in \mathbb{N}$ such that $0 < \tau_l < \varrho_j < 1$ we can extract a subsequence of v_h and then diagonalize (without relabeling) in such a way that

$$\exists \text{ finite } \delta(\varrho_j, \tau_l) \stackrel{\text{def}}{=} \lim_h \mathcal{F}_{\gamma_h\,\omega_h}(v_h, \mu_h, \alpha_h, \beta_h, B^{\tau_l}_{\varrho_j}) \ \ \forall j, l\,;\ \delta(\varrho_j, \tau_l) \in (0,1)\,.$$

Since $\varrho_j, \tau_l \ \rightarrow\ \delta(\varrho_j, \tau_l)$ is monotone non decreasing with respect to inclusion of lunulae, there is a (unique) monotone non decreasing with respect to inclusion and one-side continuous with respect to exterior approximation extension defined for all lunulae in the two parameters family, defined as follows:

$$\delta(\varrho, \tau) \ = \ \inf_{j,l} \{ \delta(\varrho_j, \tau_l) \ :\ \varrho_j > \varrho,\ \tau_l < \tau \}\ .$$

obviously:
$\forall \tau,\ \varrho \to \delta(\varrho, \tau)$ is right-continuous everywhere and continuous up to a countable set,
$\forall \varrho,\ \tau \to \delta(\varrho, \tau)$ is left-continuous everywhere and continuous up to a countable set.
Since 2-variables separate continuity together with monotonicity entail continuity, we get

$$\forall \tau,\ \varrho \to \delta(\varrho, \tau) \text{ is continuous for a.e. } \varrho\,, \quad \forall \varrho,\ \tau \to \delta(\varrho, \tau) \text{ is continuous for a.e. } \tau$$

say, δ is continuous with respect to τ, ϱ almost everywhere in $0 < \tau < \varrho < r$.

Hence by monotonicity of $\mathcal{F}_{\gamma_h,\omega_h}(v_h, \mu_h, \alpha_h, \beta_h, \cdot)$ with respect to inclusion of sets and the same monotonicity property of δ, together with the coincidence in a dense set of $\delta(\varrho, \tau)$ with the limit of $\mathcal{F}_{\gamma_h,\omega_h}(v_h, \mu_h, \alpha_h, \beta_h, B^\tau_\varrho)$ we get the existence of such limit almost everywhere and its coincidence with δ almost everywhere.
Estimate (7.59) entails $\mathcal{F}_{\gamma_h\,\omega_h}(v_h, \mu_h, \alpha_h, \beta_h, B^\tau_\varrho) \leq\ 1$, hence $\delta(\varrho, \tau) \leq 1$, for all $\varrho\,, \tau$, $0 \leq \tau \leq \varrho \leq r$. Hence also the estimate in (iii) of Theorem 7.1 holds true.
Now we show that (v) holds true: by (7.53) $\lambda_h \geq \varepsilon_h$, then by density upper bound in Theorem 3.10, and (7.47) we get

$$(7.63) \qquad 0 < \mu_h \leq \frac{\varrho_h}{\lambda_h} \leq \frac{\varrho_h}{\varepsilon_h} \leq {\varepsilon_h}^{k-1} \qquad \text{for large } h,$$

hence $\lim_h \mu_h = 0$; moreover, by (7.50), (7.51) and (7.56), changing variables, using Hölder inequality and (7.47) we find for large h:

$$
\begin{aligned}
&\mu_h \int_{B_1} |\gamma_h - \omega_h|^q \, d\mathbf{x} \\
&= \frac{\mu_h}{\varrho_h{}^2} \frac{1}{(\lambda_h \varrho_h{}^3)^{q/2}} \int_{B_{\varrho_h}(\mathbf{x}_h)} \big| g_h(\mathbf{x}_h) - w_h(\mathbf{x}_h) \pm \big(\varrho_h D w_h(\mathbf{x}_h)\cdot \mathbf{y} - w_h(\mathbf{x}_h)\big)\big|^q d\mathbf{y} \\
&\le \frac{\mu_h}{\varrho_h{}^2} \frac{1}{(\lambda_h \varrho_h{}^3)^{q/2}} \int_{B_{\varrho_h}(\mathbf{x}_h)} |g_h(\mathbf{x}_h) - w_h(\mathbf{x}_h)|^q \, d\mathbf{y} \\
&\le 2^{q-1} \mu \varrho_h{}^{-1} \lambda_h{}^{-1} \left(\int_{B_{\varrho_h}(\mathbf{x}_h)} |g_h|^q d\mathbf{y} + \int_{B_{\varrho_h}(\mathbf{x}_h)} |w_h|^q d\mathbf{y} \right) \\
&\le 2^{q-1} \mu \varrho_h{}^{-1} \varepsilon_h{}^{-1} \left(\int_{B_{\varrho_h}(\mathbf{x}_h)} |g_h|^{2q} d\mathbf{y} \right)^{1/2} \sqrt{\pi}\, \varrho_h \; + \; 2^{q-1} \mu \varrho_h{}^{-1} \varepsilon_h{}^{-1} C^q \pi \varrho_h^2 \\
&\le \; 2^{q-1} \mu \left(\sqrt{\pi}\, \varepsilon_h{}^{k/2-1} + \pi\, C^q\, \varepsilon_h{}^{k-1} \right) \; \le \; 2^q \mu \sqrt{\pi}\, \varepsilon_h^{k/2-1}
\end{aligned}
\tag{7.64}
$$

We know $\mu_h \int_{B_1} |\omega_h|^q \to 0$ as $h \to \infty$ by the first statements in (7.54), (7.55). Hence (7.64) entails $\lim_h \; \mu_h \int_{B_1} |\gamma_h|^q \, d\mathbf{x} = 0$.
By Theorem 7.1, v_∞ is bi-harmonic in $B_1^+(\mathbf{0})$, $v_\infty = 0$ in $B_1^-(\mathbf{0})$ by (7.62) and

$$
\int_{B_\varrho(\mathbf{0})} |D^2 v_\infty|^2 \, d\mathbf{x} = \int_{B_\varrho^+(\mathbf{0})} |D^2 v_\infty|^2 \, d\mathbf{x} \qquad \varrho \in (0,1)\,, \tag{7.65}
$$

hence, since $v_\infty \in H^2(B_1)$, (7.65) holds true also for $\varrho = 1$.
Since $v_\infty = 0$ in $B_1^-(\mathbf{0})$, traces continuity in H^2 entails $v_\infty = \partial v_\infty / \partial y = 0$ in $B_1(\mathbf{0}) \cap \{y = 0\}$.
By (6.1) of Theorem 6.1 and (7.65) we get

$$
\begin{aligned}
\int_{B_\eta(\mathbf{0})} |D^2 v_\infty|^2 \, d\mathbf{x} \; &= \int_{B_\eta^+(\mathbf{0})} |D^2 v_\infty|^2 \, d\mathbf{x} \\
&\le \eta^2 \int_{B_1^+(\mathbf{0})} |D^2 v_\infty|^2 \, d\mathbf{x} = \eta^2 \int_{B_1(\mathbf{0})} |D^2 v_\infty|^2 \, d\mathbf{x}\,.
\end{aligned}
\tag{7.66}
$$

Therefore, by exploiting (iii), (7.3) of Blow-up Theorem 7.1 and (7.66)

$$
\limsup_h \; \mathcal{F}_{\gamma_h, \omega_h}(v_h, \mu_h, \alpha_h, \beta_h, B_\eta(\mathbf{0})) \; \le \; \eta^2 \int_{B_1(\mathbf{0})} |D^2 v_\infty|^2 \, d\mathbf{x} \tag{7.67}
$$

whereas, by (7.61),

$$
\begin{aligned}
&\lim_h \mathcal{F}_{\gamma_h \omega_h}(v_h, \mu_h, \alpha_h, \beta_h, B_\eta(\mathbf{0})) \\
&\quad \ge \eta^{2-\sigma} \lim_h \mathcal{F}_{\gamma_h \omega_h}(v_h, \mu_h, \alpha_h, \beta_h, B_1(\mathbf{0})) = \eta^{2-\sigma} \int_{B_1(\mathbf{0})} |D^2 v_\infty|^2 \, d\mathbf{x}
\end{aligned}
\tag{7.68}
$$

contradicting the assumption on η and σ. □

Theorem 7.4. ***(Decay of the functional $\mathcal{E}$ at boundary points)*** *Assume (2.3)-(2.11). Then,*

$$\forall\, k > 2,\ \forall \eta, \sigma \in (0,1)\,, \qquad \exists \widetilde{\varepsilon} > 0,\ \exists \widetilde{\vartheta} > 0 \quad \text{such that} \tag{7.69}$$

for any $\varepsilon \in (0, \widetilde{\varepsilon}]$, any $\mathbf{x} \in \partial\Omega \setminus (T_0 \cup T_1 \cup M)$, any u an $\overline{\Omega_{\varphi+}}$ local minimizer of $\mathcal{E}(\cdot, \alpha, \beta, \overline{\Omega_{\varphi+}})$, any ϱ s.t. $B_\varrho(\mathbf{x}) \subset \widetilde{\Omega}$ and (7.39), $0 < \varrho \le \left(\varepsilon^k \wedge \bar{\varrho} \wedge (c_0 \vee 1)^{-1}\right)$ and

$$\alpha\, \mathcal{H}^1\left(S_u \cap \overline{\Omega_{\varphi+}}\right) \ + \ \beta\, \mathcal{H}^1\left((S_{\nabla u} \setminus S_u) \cap \overline{\Omega_{\varphi+}}\right) \ < \ \varepsilon\, \varrho\,, \tag{7.70}$$

we have

$$\mathcal{E}(u, B_{\eta\varrho}(\mathbf{x})) \ \le \ \eta^{2-\sigma} \max \left\{ \mathcal{E}(u, B_\varrho(\mathbf{x}))\,,\ \varrho^2\, \widetilde{\vartheta}\, \left(\left(\mathrm{Lip}(\varphi')\right)^2 + \left(\mathrm{Lip}(Dw)\right)^2 \right) \right\}\,. \tag{7.71}$$

Proof. Straightforward consequence of Theorem 7.3 □

Remark 7.5. ***(Hessian decay and decay of $\mathcal{F}$, $\mathcal{E}$ at points close to the boundary)*** *By exploiting L^2 hessian decay at the boundary (Theorem 6.1) and L^2-hessian decay at interior points (Theorem 5.2 in* [9]*) we can show an L^2-hessian decay at points close to the boundary. Explicitly, by setting*

$$B_\varrho^\tau = B_\varrho(\mathbf{0}) \cap \left\{(x,y) \in \mathbb{R}^2 :\ y > \tau\right\}\,, \quad \Gamma_\tau = B_1(\mathbf{0}) \cap \left\{(x,y) \in \mathbb{R}^2 :\ y = \tau\right\}\,, \tag{7.72}$$

if $-1 < \tau < 0$, $z \in H^2(B_1^\tau)$, $\Delta^2 z = 0$ on B_1^τ and $z = \partial z/\partial y = 0$ on Γ_τ, then there is c_3 such that

$$\|D^2 z\|^2_{L^2(B_\varrho^\tau)} \le \ c_3\, \varrho^2\, \|D^2 z\|^2_{L^2(B_1^\tau)} \qquad \forall \varrho \in (0,1). \tag{7.73}$$

Moreover a decay property of functional $\mathcal{F}$ at point close to the boundary analogous to Theorem 7.3 can be proven:

$$\begin{cases} \exists \widetilde{\varrho}\ :\ \forall k > 2,\ \forall \eta, \sigma \in (0,1),\ \text{with } \eta^\sigma < \frac{1}{c_3}, \quad \exists \varepsilon_2 > 0,\ \exists \vartheta_2 > 0 \qquad \text{s.t.} \\ \forall \varepsilon \in (0, \varepsilon_2],\ \forall \varrho \le \widetilde{\varrho},\ \forall\, \mathbf{x} \in \Omega \setminus (T_0 \cup T_1) :\ \mathrm{dist}(\mathbf{x}, \partial\Omega) < \frac{\varrho}{2}, \end{cases} \tag{7.74}$$

for any $u \in GSBV^2(\Omega)$ which is an $\overline{\Omega \cap B_\varrho(\mathbf{x})}$ local minimizer of $\mathcal{F}(\cdot, \overline{\Omega \cap B_\varrho(\mathbf{x})})$, and

$$\alpha\, \mathcal{H}^1\left(S_u \cap \overline{\Omega \cap B_\varrho(\mathbf{x})}\right) \ + \ \beta\, \mathcal{H}^1\left((S_{\nabla u} \setminus S_u) \cap \overline{\Omega \cap B_\varrho(\mathbf{x})}\right) \ < \ \varepsilon\, \varrho\,, \tag{7.75}$$

we have

$$\mathcal{F}(u, B_{\eta\varrho}(\mathbf{x})) \ \le \ \eta^{2-\sigma} \max \left\{ \mathcal{F}(u, B_\varrho(\mathbf{x}))\ ,\ \varrho^2\, \vartheta_2\, L \right\}\,. \tag{7.76}$$

8. Proof of main results

In this Section we prove the main results.

Proof of Theorem 2.2 - Assume that $v \in GSBV^2(\widetilde{\Omega}) \cap L^q(\widetilde{\Omega})$ is a minimizer of $\mathcal{F}$ among $v \in GSBV^2(\widetilde{\Omega}) \cap L^q(\widetilde{\Omega})$ s.t. $v = w$ a.e. in $\widetilde{\Omega} \setminus \Omega$. The existence of such v is proven by Theorem 3.1 and Remark 3.2.

First of all we notice that if $B \subset \left(\widetilde{\Omega} \setminus M\right)$ is an open ball and $\mathcal{H}^1\left(B \cap (S_v \cup S_{\nabla v})\right) = 0$ then v is smooth in B since $v \in H^2(B)$ so that by standard regularity theory we get $\widetilde{v} \in C^2(B \cap \overline{\Omega})$ (see [25]).

So we deduce $\widetilde{v} \in C^2\left(\widetilde{\Omega} \setminus \overline{(S_v \cup S_{\nabla v})}\right)$.
Now we evaluate $\mathcal{H}^1\left(\widetilde{\Omega} \cap \left(\overline{S_v \cup S_{\nabla v}} \setminus (S_v \cup S_{\nabla v})\right)\right)$. Set

$$\Omega_0 \;=\; \left\{ \mathbf{x} \in \widetilde{\Omega} \;:\; \lim_{\varrho \to 0} \, \varrho^{-1}\, \mathcal{F}(v, B_\varrho(\mathbf{x})) \;=\; 0 \right\} \tag{8.1}$$

We are going to prove that Ω_0 is open.
Notice that $\Omega_0 \cap \left(\widetilde{\Omega} \setminus \overline{\Omega}\right)$ is trivially open by assumptions (2.8), (2.9), (2.11) so that we have only to analyze points $\mathbf{x}$ in $\overline{\Omega} \cap \Omega_0$, and show that they are all in the interior part of Ω_0.
The interior points $\mathbf{x} \in \Omega$ can be handled by applying Theorems 5.1, 5.4 in [9], to get

$$\mathcal{H}^1\left(\Omega \cap \left(\overline{S_v \cup S_{\nabla v}} \setminus (S_v \cup S_{\nabla v})\right)\right) \;=\; 0\,, \tag{8.2}$$

so we have obtained the information that $\Omega_0 \cap \Omega$ is open, and thanks to (2.6) (2.7) we are left to show that all points $\mathbf{x} \in \Omega_0 \cap (\partial\Omega \setminus (T_0 \cup T_1 \cup M))$ are interior points of Ω_0.
From now on we fix

$$\mathbf{x} \in \Omega_0 \cap (\partial\Omega \setminus (T_0 \cup T_1 \cup M))\,. \tag{8.3}$$

Let c_0 be the constant in the density upper bound Theorem 3.10. In order to apply Decay property of local minimizers (Theorem 7.3) fix $k > 2$, $\eta \in (0,1)$, $\sigma \in (0,1)$, and related constants $\varepsilon_1 = \varepsilon_1(\eta, \sigma, \alpha, \beta, \ldots)$, $\vartheta_1 = \vartheta_1(\eta, \sigma, \alpha, \beta, \ldots)$ whose existence is warranted by Theorem 7.3 in the present paper and let ε_0 the constant whose existence is warranted by Theorem 5.4 of [9]. Choose $\eta' \in (0, \eta)$ s.t.

$$(\eta')^{1-\sigma} c_0 \;<\; \varepsilon_0 \wedge \varepsilon_1\,. \tag{8.4}$$

and denote by ε', ϑ' the related constants $\varepsilon' = \varepsilon'(\eta', \sigma, \alpha, \beta, \ldots)$, $\vartheta' = \vartheta'(\eta', \sigma, \alpha, \beta, \ldots)$ whose existence is warranted by Theorem 7.3.
Set $\varepsilon = (\varepsilon_0 \wedge \varepsilon_1 \wedge \varepsilon')/2$. Choose r s.t.

$$0 < r^2 \vartheta_1 < \vartheta_1\,, \quad 0 < r < \left(\varepsilon^k \wedge \bar{\varrho} \wedge (c_0 \vee 1)^{-1}\right), \quad \int_{B_r(\mathbf{x})} |g|^{2q}\, d\mathbf{y} \le \varepsilon^k, \tag{8.5}$$

$$r^2 (\vartheta_1 \vee \vartheta')\, L \;<\; (c_0 \wedge \varepsilon)\, r\,, \qquad B_r(\mathbf{x}) \cap (T_0 \cup T_1 \cup M) = \emptyset\,, \tag{8.6}$$

$$\mathcal{F}(v, B_r(\mathbf{x})) \;\le\; \varepsilon\, \eta'\, r\,. \tag{8.7}$$

We claim that $B_{(1-\eta)r}(\mathbf{x}) \;\subset\; \Omega_0$. In fact, if $\mathbf{y} \in B_{(1-\eta)r}(\mathbf{x})$ there are 3 cases: if $\mathbf{y} \in \widetilde{\Omega} \setminus \overline{\Omega}$ then v coincides with w which is $(C^2 \cap W^{2,\infty})\,(B_{\eta r}(\mathbf{y}))$ hence the functional has a nice decay; if $B_{\eta r}(\mathbf{y}) \subset \Omega$ then $\mathcal{F}(v, B_{\eta r}(\mathbf{y})) \le \mathcal{F}(v, B_r(\mathbf{x})) \le \varepsilon_0 \eta r$, hence we can repeat the argument in Section 6 of [9]; if

$$\mathbf{y} \in \overline{\Omega} \cap B_{(1-\eta)r}(\mathbf{x})\,, \quad \text{and} \quad \operatorname{dist}(\mathbf{y}, \partial\Omega) \le \eta r \tag{8.8}$$

additional analysis is required as follows.
In case (8.8) we have

$$\mathcal{F}(v, B_{\eta' r}(\mathbf{y})) \;\le\; \mathcal{F}(v, B_r(\mathbf{x})) \;\le\; \varepsilon\, \eta'\, r\,. \tag{8.9}$$

By (8.4)-(8.7), by choosing $\varrho = \eta' r < r$ in Theorem 7.3 and Remark 7.5, and by density upper bound estimate (Theorem 3.10), referring to (3.13), we deduce

$$\begin{aligned} \mathcal{F}(v, B_{\eta'\varrho}(\mathbf{y})) &\leq (\eta')^{2-\sigma} \left(\mathcal{F}(v, B_{\varrho}(\mathbf{y})) \vee (\varrho^2 \vartheta' L) \right) \\ &\leq (\eta')^{2-\sigma} \left((c_0\, \varrho) \vee ((c_0 \wedge \varepsilon)\, \varrho) \right) \leq \varepsilon_0\, \eta'\, \varrho , \end{aligned} \tag{8.10}$$

and

$$\alpha\, \mathcal{H}^1(S_v \cap B_{\eta'\varrho}(\mathbf{y})) + \beta\, \mathcal{H}^1\left((S_{\nabla v} \setminus S_v) \cap B_{\eta'\varrho}(\mathbf{y})\right) < \varepsilon_0\, \eta'\, \varrho \tag{8.11}$$

so that, by setting $\varrho' = \eta'\varrho$, the choice $\varrho = \eta$ in Theorem 7.3 and Remark 7.5 entails

$$\begin{aligned} \mathcal{F}(v, B_{\eta\,\varrho'}(\mathbf{y})) &\leq \eta^{2-\sigma} \left(\mathcal{F}(v, B_{\varrho'}(\mathbf{y})) \vee \left((\varrho')^2 \vartheta_0 L \right) \right) \\ &\leq \eta^{2-\sigma} \left(\mathcal{F}(v, B_{\varrho'}(\mathbf{y})) \vee (\varepsilon_0 \varrho') \right) . \end{aligned} \tag{8.12}$$

Inequalities (8.10),(8.12) together with $\varrho' = \eta'\varrho$ entail

$$\mathcal{F}(v, B_{\eta\,\varrho'}(\mathbf{y})) \leq \eta^{2-\sigma}\, \varepsilon_0\, \varrho' . \tag{8.13}$$

In the same way we get: for any $h \in \mathbb{N}$

$$\mathcal{F}(v, B_{\eta^h\,\varrho'}(\mathbf{y})) \leq \eta^{h(2-\sigma)}\, \varepsilon_0\, \varrho' \tag{8.14}$$

entails

$$\begin{cases} \mathcal{F}(v, B_{\eta^{h+1}\,\varrho'}(\mathbf{y})) \leq \eta^{(h+1)(2-\sigma)}\, \varepsilon_0\, \varrho' , \\ \alpha\, \mathcal{H}^1(S_v \cap B_{\eta^{h+1}\varrho'}(\mathbf{y})) + \beta\, \mathcal{H}^1\left((S_{\nabla v} \setminus S_v) \cap B_{\eta^{h+1}\varrho'}(\mathbf{y}) \right) < \varepsilon_0\, \eta^{h+1} \varrho' . \end{cases} \tag{8.15}$$

Since (8.14) holds true for $h = 1$ due to (8.13), by induction we know that (8.15) holds true for any $h \in \mathbb{N}$. Then $\forall h = 1, 2, ...$

$$\begin{aligned} \mathcal{F}(v, B_{\eta^{h+1}\varrho'}(\mathbf{y})) &\leq \eta^{(h+1)(2-\sigma)}\, \varepsilon_0\, \varrho' = \eta^{h(2-\sigma)}\, \eta^{2-\sigma}\, \varepsilon_0\, \varrho' \leq \\ &\leq \eta^{h(2-\sigma)}\, \varepsilon_0\, \eta\, \varrho' = \eta^{h(1-\sigma)}\, \varepsilon_0\, (\eta^{h+1}\, \varrho') \end{aligned} \tag{8.16}$$

For every t s.t. $0 < t < \eta^2 \varrho'$ there is $j \geq 3$ s.t. $\eta^j\, \varrho' \leq t \leq \eta^{j-1}\, \varrho'$, so that, by(8.16),

$$\begin{aligned} t^{-1}\, \mathcal{F}(v, B_t(\mathbf{y})) &\leq t^{-1}\, \mathcal{F}(v, B_{\eta^{j-1}\varrho'}(\mathbf{y})) \leq t^{-1}\, \eta^{(j-2)(1-\sigma)}\, \varepsilon_0\, \eta^{j-1}\, \varrho' \\ &= (t^{-1}\, \eta^j\, \varrho')\, \eta^{(j-2)(1-\sigma)-1}\, \varepsilon_0 \leq \eta^{(j-2)(1-\sigma)-1}\, \varepsilon_0 \end{aligned} \tag{8.17}$$

and passing to the limit as $t \to 0_+$ (say $j \to +\infty$) we get $\mathbf{y} \in \Omega_0$.
By summarizing we have shown that Ω_0 is an open set.
Since $S_v \cup S_{\nabla v}$ is countably $(\mathcal{H}^1, 1)$ rectifiable, by Theorem 3.2.19 in [24] we get $\mathcal{H}^1\left((S_v \cup S_{\nabla v}) \cap \Omega_0\right) = 0$, $\nabla v = Dv$ in Ω_0, $\nabla^2 v = D^2 v$ in Ω_0, so that $\widetilde{v} \in C^2(\Omega_0)$ and $(S_v \cup S_{\nabla v}) \cap \Omega_0 = \emptyset$. Since Ω_0 is open then $\Omega_0 \cap \overline{(S_v \cup S_{\nabla v})} = \emptyset$. By Lemma 4.3 we have

$$\mathcal{H}^1\left(\widetilde{\Omega} \cap \left(\overline{(S_v \cup S_{\nabla v})} \setminus (S_v \cup S_{\nabla v}) \right) \right) = 0 . \tag{8.18}$$

By setting

$$K_0 = \overline{S_v} \setminus (S_{\nabla v} \setminus S_v) , \qquad K_1 = \overline{S_{\nabla v}} \setminus S_v , \tag{8.19}$$

thanks to (2.6), (2.7), (2.8), (2.11), (8.18), (8.19) we obtain

$$(8.20) \qquad K_0 \cup K_1 \text{ is closed, } \mathcal{H}^1(K_0 \cap \widetilde{\Omega}) \;=\; \mathcal{H}^1(\overline{S_v})\,,\; \mathcal{H}^1(K_1 \cap \widetilde{\Omega}) \;=\; \mathcal{H}^1(\overline{S_{\nabla v}} \setminus S_v)\,,$$

hence $K_0 \cap \widetilde{\Omega}$ and $K_1 \cap \widetilde{\Omega}$ are $(\mathcal{H}^1, 1)$ rectifiable, moreover

$$(8.21) \qquad F(K_0, K_1, \widetilde{v}) = \mathcal{F}(v) = \min\{\mathcal{F}(z) :\, z \in GSBV^2(\widetilde{\Omega}) \cap L^q(\widetilde{\Omega}),\, z = w \;\widetilde{\Omega}\backslash\Omega\}.$$

Then (by Lemma 3.2 and Remark 3.3 of [9]) we conclude that $(K_0, K_1, \widetilde{v})$ is a minimizing triplet for F in the class of admissible triplets.
By (8.19), (8.20), (8.21) and trace properties of $GSBV^2$ functions, we can say that properties (2.12), (2.13), (2.14), (2.15) hold true for the minimizing triplet $(K_0, K_1, \widetilde{v})$ obtained by partial regularity of a weak minimizer for $\mathcal{F}$.
Eventually, by Lemma 3.2 of [9], we get for any other minimizing triplet $(\mathfrak{K}_0, \mathfrak{K}_1, u)$ of F

$$(8.22) \qquad S_u \subset \mathfrak{K}_0\,, \qquad (S_{\nabla u} \setminus S_u) \subset (\mathfrak{K}_1 \setminus \mathfrak{K}_0)$$

$$(8.23) \qquad \mathcal{F}(u) \;\le\; F(\mathfrak{K}_0, \mathfrak{K}_1, u) \;=\; F(K_0, K_1, v)\,;$$

assume by contradiction that inequality in (8.23) is strict, then (by Theorem 3.1) there is $z \in GSBV^2(\widetilde{\Omega}) \cap L^q(\widetilde{\Omega})$ s.t.

$$(8.24) \qquad \mathcal{F}(z) \;=\; \min \mathcal{F} \;\le\; \mathcal{F}(u)$$

and, by repeating on z the regularization procedure previously performed on v, we find a minimizing triplet $(\mathfrak{Z}_0, \mathfrak{Z}_1, \widetilde{z})$ for F fulfilling

$$(8.25) \qquad F(K_0, K_1, \widetilde{v}) \;=\; F(\mathfrak{Z}_0, \mathfrak{Z}_1, \widetilde{z}) \;=\; \mathcal{F}(z)\,.$$

Relationships (8.24) and (8.25) together contradict (8.23) with strict inequality; so in (8.23) we must have equality, hence properties (2.12), (2.13), (2.14), (2.15) hold true also for $(\mathfrak{K}_0, \mathfrak{K}_1, u)$. $\square$

Proof of Theorem 2.1 - The thesis follows immediately by Theorem 2.2 by dropping the term $\mu \int_{\widetilde{\Omega}} |v - g|^q d\mathbf{x}$ and exploiting Theorem 7.4 instead of Theorem 7.3. $\square$

Proof of Theorem 2.3 - The thesis follows immediately by Theorem 2.2 with the choice $K = \overline{S_v \cup S_{\nabla v}}$ where v minimizes $\mathcal{F}$ and taking into account the assumption $\alpha = \beta$. $\square$

Proof of Theorem 1.1 - It is a Corollary of Theorem 2.2 since (2.10), (2.11) follows from $\widetilde{w} = w$, $T_0 = T_1 = \emptyset$, $M = \emptyset$ e $F(\emptyset, \emptyset, w) < +\infty$. $\square$

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