EXISTENCE THEOREMS FOR ENTIRE SOLUTIONS OF STATIONARY KIRCHHOFF FRACTIONAL p–LAPLACIAN EQUATIONS

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Abstract. The paper deals with existence, multiplicity and asymptotic behavior of entire solutions for a series of stationary Kirchhoff fractional p–Laplacian equations. The existence presents several difficulties due to the intrinsic lack of compactness arising from different reasons and the suitable strategies adopted to overcome the technical hurdles depend on the specific problem under consideration. The results of the paper extend in several directions recent theorems. Furthermore, the main assumptions required in the paper weaken the hypotheses used in the recent literature on stationary Kirchhoff fractional problems. Some equations treated in the paper cover the so called *degenerate* case, that is the case in which the Kirchhoff function M is zero at zero. In other words, from a physical point of view, when the base tension of the string modeled by the equation is zero: a very realistic case. Last but not least no monotonicity assumption is required on M and also this aspect makes the models more believable in several physical applications.

Keywords: stationary Kirchhoff problems, nonlocal p–Laplacian operators, Hardy coefficients and critical exponents
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1. Introduction

In this paper we study a series of problems involving the stationary Kirchhoff fractional p–Laplacian operator $u \mapsto M([u]_{s,p}^p)(-\Delta)_p^s u$ for functions u, defined in the entire $\mathbb{R}^N$ and belonging to suitable fractional Sobolev spaces in which the Gagliardo semi–norm

$$[u]_{s,p}^p = \iint_{\mathbb{R}^{2N}} \frac{|u(x)-u(y)|^p}{|x-y|^{N+ps}} dxdy$$

is well defined. The operator $(-\Delta)_p^s$ is the fractional p–Laplacian, which for every function $\varphi \in C_0^\infty(\mathbb{R}^N)$ may be defined, up to normalization factors, as

$$(-\Delta)_p^s \varphi(x) = 2 \lim_{\varepsilon \to 0^+} \int_{\mathbb{R}^N \setminus B_\varepsilon(x)} \frac{|\varphi(x)-\varphi(y)|^{p-2}(\varphi(x)-\varphi(y))}{|x-y|^{N+ps}} dy$$

for all $x \in \mathbb{R}^N$, where $B_\varepsilon(x) = \{y \in \mathbb{R}^N : |x-y| < \varepsilon\}$.

The nonlocal coefficient M is the main Kirchhoff function related to the elliptic part of the problem and is assumed throughout the paper, without further mentioning, that

($\mathcal{M}$) $0 < s < 1 < p < \infty$, $sp < N$, $M : \mathbb{R}_0^+ \to \mathbb{R}_0^+$ *is a nonnegative continuous function and there exists* $\theta \in [1, N/(N-ps))$ *such that* $tM(t) \le \theta \mathscr{M}(t)$ *for any* $t \in \mathbb{R}_0^+$, *where* $\mathscr{M}(t) = \int_0^t M(\tau)d\tau$

holds.

A typical prototype for M, due to Kirchhoff in 1883, is given by $M(t) = a + b\, t^{\theta-1}$ for $t \ge 0$, with $a, b \ge 0$ and $a + b > 0$. When $M(t) > 0$ for all $t \ge 0$, Kirchhoff problems are said to be *non–degenerate* and this happens for example if $a > 0$ and $b \ge 0$ in the model case. Otherwise, if $M(0) = 0$ and $M(t) > 0$ for all $t > 0$, the Kirchhoff problems are called *degenerate* and this occurs in the model case when $a = 0$ and $b > 0$.

In the large literature on Kirchhoff problems, the transverse oscillations of a stretched string, with non–local flexural rigidity, depends continuously on the Sobolev deflection norm of u via $M([u]_{s,2}^2)$. Sometimes the Kirchhoff function M is assumed Lipschitz continuous, but not always monotone, as in [10], even if the model proposed by Kirchhoff is clearly monotone. Note that when the inertial effects of longitudinal modes can be neglected, the tension is spatially uniform along the string and can be directly computed from the elongation of the string according to the Hooke law and arriving to the form of M proposed by Kirchhoff and derived properly by Carrier. In any case, M measures the change of the tension on the string caused by the change of its length during the vibration. The presence of the nonlinear coefficient M is crucial to be considered when the changes in tension during the motion cannot be neglected. In the case of linear string vibrations, the tension is constant that is $M(t) \equiv M(0)$, but nonlinear vibrations are more realistic.

The existence theorems we prove should use new techniques in order to overcome the nonlocal nature of the problems as well as the lack of compactness, and the suitable strategies adopted depend of course on the problem under consideration.

The first equation we treat is

$$M([u]_{s,p}^p)(-\Delta)_p^s u - \gamma \frac{|u|^{p-2}u}{|x|^{ps}} = \lambda w(x)|u|^{q-2}u + K(x)|u|^{p_s^*-2}u \quad \text{in } \mathbb{R}^N, \tag{1.1}$$

where γ and λ are real parameters, the exponent q is such that $\theta p < q < p_s^*$, with $p_s^* = Np/(N-ps)$ the critical exponent in the sense of Sobolev, and the positive weights w and K satisfy

(w) $w > 0$ a.e. in $\mathbb{R}^N$ and $w \in L^{\wp}(\mathbb{R}^N)$, *with* $\wp = p_s^*/(p_s^* - q)$ *and* $\theta p < q < p_s^*$,

(K) $K \geq 0$ a.e. in $\mathbb{R}^N$ and $K \in L^\infty(\mathbb{R}^N)$.

When yielding with (1.1) we assume also (w) and (K), without further mentioning.

Problem (1.1) is fairly delicate due to the intrinsic lack of compactness, which arise from the Hardy term and the nonlinearity with critical exponent p_s^*. For this reason we assume that (1.1) is non–degenerate, that is

$$\inf_{t \in \mathbb{R}_0^+} M(t) = a > 0. \tag{1.2}$$

Stationary non–degenerate Kirchhoff problems have been extensively studied in the last decades, but usually under the request that M is increasing in $\mathbb{R}_0^+$, as in [17, 26, 29] and the reference therein. We replace the monotonicity assumption by $(\mathcal{M})$. In particular, as shown in [2, 31], the Kirchhoff function $M(t) = (1+t)^k + (1+t)^{-1}$, $k \in (0,1)$, verifies both $(\mathcal{M})$ and (1.2), but is not monotone. In fact, we have that $\inf_{t\in\mathbb{R}_0^+} M(t) = a > 0$, with $a = k^{-k/(k+1)}(1+k) < M(0) = 2$. Furthermore, $(\mathcal{M})$ is satisfied taking $\theta = k+1$, provided that k is so small that $k < sp/(N-sp)$.

The main solution space for (1.1) is the fractional Beppo–Levi space $D^{s,p}(\mathbb{R}^N)$, that is the completion of $C_0^\infty(\mathbb{R}^N)$ with respect to $[\,\cdot\,]_{s,p}$. As it is well known, $D^{s,p}(\mathbb{R}^N) = (D^{s,p}(\mathbb{R}^N), [\,\cdot\,]_{s,p})$ is a uniformly convex Banach space. By Theorems 1 and 2 of [23]

$$\|u\|_{p_s^*}^p \leq C_{N,p} \frac{s(1-s)}{(N-ps)^{p-1}} [u]_{s,p}^p, \qquad \int_{\mathbb{R}^N} \frac{|u(x)|^p}{|x|^{ps}} \leq C_{N,p} \frac{s(1-s)}{(N-ps)^p} [u]_{s,p}^p$$

for all $u \in D^{s,p}(\mathbb{R}^N)$, where $C_{N,p}$ is a positive constant depending only on N and p. Thus, the fractional Sobolev embedding $D^{s,p}(\mathbb{R}^N) \hookrightarrow L^{p_s^*}(\mathbb{R}^N)$ and the fractional Hardy embedding $D^{s,p}(\mathbb{R}^N) \hookrightarrow L^p(\mathbb{R}^N, |x|^{-ps})$ are continuous, but not compact. However, we are able to introduce the best fractional critical Sobolev and Hardy constant $S = S(N,p,s)$ and $H = H(N,p,s)$ given by

$$S = \inf_{\substack{u \in D^{s,p}(\mathbb{R}^N) \\ u \neq 0}} \frac{[u]_{s,p}^p}{\|u\|_{p_s^*}^p}, \quad H = \inf_{\substack{u \in D^{s,p}(\mathbb{R}^N) \\ u \neq 0}} \frac{[u]_{s,p}^p}{\|u\|_H^p}, \quad \|u\|_H^p = \int_{\mathbb{R}^N} \frac{|u(x)|^p}{|x|^{ps}} dx. \tag{1.3}$$

Of course the numbers S and H are strictly positive. We refer to Theorem 1.1 of [18] for the sharp Hardy constant H.

Define $\kappa = \kappa(q, M)$ by

$$\kappa = \frac{a(q-\theta p)}{\theta(q-p)}.$$

Clearly $\kappa \in (0, a]$, being $\theta \geq 1$ and $p \leq \theta p < q$ by assumption on θ and q. There are cases, besides the obvious one $M \equiv a$, in which $\kappa = a$, that is $\theta = 1$ in $(\mathcal{M})$, as shown in Section 2. Problem (1.1) has a variational nature and, under the above structural assumptions, (weak) solutions of (1.1) are exactly the critical points of the underlying functional $J_{\gamma,\lambda}$, which satisfies the geometry of the mountain pass lemma. The main existence result for problem (1.1) is given in terms of critical points $u_{\gamma,\lambda}$ found at special mountain pass levels. These solutions are simply called *mountain pass solutions.*

Theorem 1.1. *Suppose that* (1.1) *is non–degenerate, that is* (1.2) *holds. Then for every* $\gamma \in (-\infty, \kappa H)$ *problem* (1.1) *admits a non–trivial mountain pass solution* $u_{\gamma,\lambda}$ *for any* $\lambda > 0$ *and* $u_{\gamma,\lambda}$ *satisfies the asymptotic behavior*

$$\lim_{\lambda \to \infty} [u_{\gamma,\lambda}]_{s,p} = 0, \tag{1.4}$$

whenever $\|K\|_\infty = 0$. *While if* $\|K\|_\infty > 0$*, then there exists* $\lambda^* = \lambda^*(\gamma) > 0$ *such that for any* $\lambda \geq \lambda^*$ *problem* (1.1) *admits a non–trivial mountain pass solution* $u_{\gamma,\lambda}$ *which satisfies again* (1.4).

Theorem 1.1 extends in several directions Theorem 1.3 of [16]. A very natural appealing open problem is to prove existence of nontrivial solutions for (1.1), when $M(0) = 0$ and $M(t) > 0$ for all $t > 0$.

Next, we study problem (1.1) in the degenerate case, that is when

$$M(0) = \inf_{t\in\mathbb{R}_0^+} M(t) = 0.$$

But we require that $\gamma = 0$, that is that the Hardy term does not appear any longer. A very intriguing open question is to prove existence of solutions in the *degenerate case*, assuming only that

$$\inf_{t\in\mathbb{R}_0^+} M(t) = 0.$$

Up to now this case has been never treated, since it seems particularly delicate to handle, even if extremely interesting, not only from a mathematical point of view, but especially in applications.

More precisely, we next consider

$$M([u]_{s,p}^p)(-\Delta)_p^s u = \lambda w(x)|u|^{q-2}u + K(x)|u|^{p_s^*-2}u \quad \text{in } \mathbb{R}^N, \tag{1.5}$$

when (1.2) is replaced by $M(0) = 0$ *and*

($\mathcal{M}_1$) *For any* $\tau > 0$ *there exists* $m_\tau > 0$ *such that* $M(t) \geq m_\tau$ *for all* $t \geq \tau$.

($\mathcal{M}_2$) *There exists* $b > 0$ *such that* $M(t) \geq b\,t$ *for any* $t \in [0, 1]$.

The degenerate case is very appealing and it is covered in famous well known papers in Kirchhoff theory, as [10, 27]. In particular, in [10] the Kirchhoff function M is assumed Lipschitz continuous, but not monotone. Also in the degenerate case there are several functions M which are not increasing in $\mathbb{R}_0^+$, but satisfy ($\mathcal{M}$), ($\mathcal{M}_1$) and ($\mathcal{M}_2$), see Section 3 for details.

From a physical point of view the fact that $M(0) = 0$ means that the base tension of the string is zero, a very realistic model. The existence theorem for (1.5) is

Theorem 1.2. *Let* $M(0) = 0$ *and* $ps < N < 2ps$. *Suppose that* M *satisfy* ($\mathcal{M}_1$) *and* ($\mathcal{M}_2$). *Then problem* (1.5) *admits a non–trivial mountain pass solution* u_λ *for any* $\lambda > 0$ *and* u_λ *satisfies the asymptotic behavior*

$$\lim_{\lambda\to\infty} [u_\lambda]_{s,p} = 0, \tag{1.6}$$

whenever $\|K\|_\infty = 0$. *While if* $\|K\|_\infty > 0$*, then there exists* $\lambda^* > 0$ *such that for any* $\lambda \geq \lambda^*$ *problem* (1.5) *admits a non–trivial mountain pass solution* u_λ *which satisfies again* (1.6).

When $p = 2$, the restriction $2s < N < 4s$ was also required in [2, 24] for somehow related fractional degenerate Kirchhoff Dirichlet problems in bounded domains and it implies $N \in \{1, 2, 3\}$. Similarly, condition ($\mathcal{M}_2$) already appears in [2], where it is largely explained as it weakens analogous growth conditions on M required at 0 in the degenerate case in the literature on Kirchhoff problems. We refer to [2] for further details.

Theorem 1.2 is very general and extends in several directions the existence results obtained in [21], when $\mu = 0$ in [21].

In the last part of the work, we extend the results given in [31] for the nonhomogeneous fractional p–Laplacian Schrödinger–Kirchhoff equations to the degenerate case $M(0) = 0$ and consider

$$M([u]_{s,p}^p)(-\Delta)_p^s u + V(x)|u|^{p-2}u = f(x, u) + g(x, u) + h(x) \quad \text{in } \mathbb{R}^N. \tag{1.7}$$

For (1.7) we assume that M satisfies only ($\mathcal{M}$) and ($\mathcal{M}_1$). The function h can be viewed as a perturbation term and *h is assumed throughout the paper to be in* $L^{\nu'}(\mathbb{R}^N)$*, where* ν' *is the conjugate exponent of some fixed* $\nu \in [p, p_s^*]$.

On the potential function V we suppose

(V_1) $V \in C(\mathbb{R}^N)$ *satisfies* $\inf_{x\in\mathbb{R}^N} V(x) \geq V_0 > 0$.

(V_2) *There exists* $R > 0$ *such that* $\lim_{|y|\to\infty} \operatorname{meas}(\{x \in B_R(y) : V(x) \leq c\}) = 0$ *for any* $c > 0$.

Condition (V_2) is weaker than the coercivity assumption $V(x) \to \infty$ as $|x| \to \infty$ usually required in Schrödinger problems. Assumption (V_2) was originally introduced by Bartsch and Wang in [6] to overcome the lack of compactness in problems defined in the entire space $\mathbb{R}^N$.

The natural space where finding solutions for (1.7) is W, that is the completion of $C_0^\infty(\mathbb{R}^N)$ with respect to the norm

$$\|u\|_W = \left([u]_{s,p}^p + \|u\|_{p,V}^p\right)^{1/p}, \qquad \|u\|_{p,V}^p = \int_{\mathbb{R}^N} V(x)|u(x)|^p dx,$$

with V satisfying (V_1). By standard arguments, it is clear that also $W = (W, \|\cdot\|_W)$ is a uniformly convex Banach space, see Lemma 10 in the Appendix of [31].

The nonlinearity $f : \mathbb{R}^N \times \mathbb{R} \to \mathbb{R}$ *is a Carathéodory function verifying condition*

$(\mathcal{F})$ *There exists* $q \in (\theta p, p_s^*)$ *such that either*
- (f_1) $f(x,t) = w(x)|t|^{q-2}t$ *for a.a* $x \in \mathbb{R}^N$ *and all* $t \in \mathbb{R}$, *where* w *satisfies* (w), *or*
- (f_2) f *verifies both assumptions*
  - (a) *there exists a positive function* w *of class* $L^\infty(\mathbb{R}^N)$ *such that* $w(x) = o(1)$ *as* $|x| \to \infty$ *and* $|f(x,t)| \le w(x)|t|^{q-1}$ *for a.a* $x \in \mathbb{R}^N$ *and all* $t \in \mathbb{R}$,
  - (b) $0 < qF(x,t) \le t\, f(x,t)$ *for a.a* $x \in \mathbb{R}^N$ *and all* $t \in \mathbb{R} \setminus \{0\}$, *where* $F(x,t) = \displaystyle\int_0^t f(x,\tau)d\tau$.

The nonlinearity $g : \mathbb{R}^N \times \mathbb{R} \to \mathbb{R}$ *is a Carathéodory function satisfying condition*

$(\mathcal{G})$ *There are exponents* r *and* μ *in* $(\theta p, p_s^*)$ *such that for all* $\varepsilon > 0$ *there exists* $C_\varepsilon > 0$ *and*

$$|g(x,t)| \le \theta p \varepsilon |t|^{\theta p - 1} + r C_\varepsilon |t|^{r-1}$$

for a.a. $x \in \mathbb{R}^N$ *and all* $t \in \mathbb{R}$, *and either*
- (i) $\theta p < \mu < q$ *and* $\mu G(x,t) \le t\, g(x,t)$ *for a.a* $x \in \mathbb{R}^N$ *and all* $t \in \mathbb{R}$, *where* $G(x,t) = \int_0^t g(x,\tau)d\tau$, *or*
- (ii) $q \le \mu < p_s^*$ *and* $0 \le \mu G(x,t) \le t\, g(x,t)$ *for a.a* $x \in \mathbb{R}^N$ *and all* $t \in \mathbb{R}$.

Simple examples of subcritical nonlinear terms which satisfy conditions $(\mathcal{F})$ and $(\mathcal{G})$ are given by $f(x,t) = w(x)|t|^{q-2}t$, with $w > 0$ a.e. in $\mathbb{R}^N$ and w either satisfying (w), or with w in $L^\infty(\mathbb{R}^N)$ and $w(x) = o(1)$ as $|x| \to \infty$, and by $g(x,t) = \varphi(x)\psi(t)$, with $\varphi \in L^\infty(\mathbb{R}^N)$, $\varphi > 0$ a.e in $\mathbb{R}^N$ and $\psi(t) = r|t|^{r-2}t + \mu|t|^{\mu-2}t$, provided that $\theta p < \min\{q, \mu\}$ and $\mu \le r < p_s^*$.

The condition $\inf\{G(x,t) : x \in \mathbb{R}^N, \ |t| = 1\} > 0$, assumed in [31], is no longer required in this paper thanks to the presence of the nontrivial nonlinearity f. We state below the main existence result for problem (1.7).

Theorem 1.3. *Let* $M(0) = 0$ *and suppose that* M, V, f *and* g *satisfy* $(\mathcal{M}_1)$, (V_1)*–*(V_2), $(\mathcal{F})$ *and* $(\mathcal{G})$, *respectively. Then there exists a number* $\delta > 0$ *such that for all perturbations* $h \in L^{\nu'}(\mathbb{R}^N)$, *with* $\|h\|_{\nu'} \le \delta$, *problem* (1.7) *admits a nontrivial mountain pass solution* u_0 *in* W. *If furthermore* h *is nontrivial, then* (1.7) *admits at least a second independent nontrivial solution* u_1 *in* W.

Finally, if $g \equiv 0$, *then the assertion above continues to hold assuming only* (V_1) *on the potential* V.

The last part of Theorem 1.3, that is when $g \equiv 0$ in (1.7), takes inspiration from the paper [19] and covers also the interesting case in which V is a positive constant. Furthermore, Theorem 1.3 extends in several directions previous accomplishments, as the existence results obtained in [7, 15, 22, 33, 34] and in a broad sense [8].

Finally, we study equation (1.7), still requiring only (V_1) on the potential V, but including the term g, provided that V, f, g and h are radial functions in x.

Theorem 1.4. *Let* $N \ge 2$ *and* $M(0) = 0$. *Suppose that* $(\mathcal{M}_1)$, (V_1), $(\mathcal{F})$ *and* $(\mathcal{G})$ *hold, and that* V, f, g *and* h *are radial functions in* x. *Then there exists a number* $\delta > 0$ *such that for all radial perturbations* $h \in L^{\nu'}(\mathbb{R}^N)$, *with* $\|h\|_{\nu'} \le \delta$, *problem* (1.7) *admits a nontrivial radial mountain pass solution* u_0 *in* W. *If furthermore* h *is nontrivial, then* (1.7) *admits at least a second independent nontrivial radial solution* u_1 *in* W.

Theorem 1.4 not only covers the so called degenerate and nonlocal case, but also it encloses general nonlinearities and perturbations. Therefore Theorem 1.4 extends in several directions the existence results obtained in [14, 25].

The paper is organized as follows. In Section 2 we prove the existence Theorem 1.1 for the Hardy problem (1.1) and the asymptotic behavior (1.4). Section 3 is devoted to the study of the degenerate problem (1.5) and to the proof of the existence Theorem 1.2 and of the validity of (1.6). Finally, Sections 4 and 5 deal with the proof of Theorems 1.3 and 1.4 for the degenerate Schrödinger–Kirchhoff equation (1.7), respectively.

2. The non-degenerate problem (1.1)

In this section we prove the existence result for problem (1.1) and we recall that throughout the section $(\mathcal{M})$, (w), (K) and (1.2) hold.

Let $L^q(\mathbb{R}^N, w)$ be the weighted Lebesgue space endowed with the norm

$$\|u\|_{q,w}^q = \int_{\mathbb{R}^N} w(x)|u|^q dx$$

By Proposition A.6 of [3] the Banach space $L^q(\mathbb{R}^N, w) = (L^q(\mathbb{R}^N, w), \|\cdot\|_{q,w})$ is uniformly convex. Furthermore, combining some ideas of Lemma 2.3 of [3], Lemma 2.2 of [4] and Lemma 2.6 of [32], see also Lemma 2.3 of [30], we have

Lemma 2.1. *The embedding $D^{s,p}(\mathbb{R}^N) \hookrightarrow L^q(\mathbb{R}^N, w)$ is continuous, with*

$$\|u\|_{q,w} \le C_w[u]_{s,p} \quad \text{for all } u \in D^{s,p}(\mathbb{R}^N), \tag{2.1}$$

and $C_w = S^{-1/p}\|w\|_{\wp}^{1/q} > 0$. Furthermore, the above embedding is also compact.

Proof. By (w), the Hölder inequality and (1.3), for all $u \in D^{s,p}(\mathbb{R}^N)$

$$\|u\|_{q,w} \le \left(\int_{\mathbb{R}^N} w(x)^{\wp} dx\right)^{1/\wp q} \cdot \left(\int_{\mathbb{R}^N} |u|^{p_s^*} dx\right)^{1/p_s^*} \le S^{-1/p}\|w\|_{\wp}^{1/q}[u]_{s,p},$$

that is (2.1) holds.

To prove the second part of the lemma, we need to show that if $u_n \rightharpoonup u$ in $D^{s,p}(\mathbb{R}^N)$, then $\|u_n - u\|_{q,w} \to 0$ as $n \to \infty$. Thanks to the Hölder inequality

$$\int_{\mathbb{R}^N \setminus B_R} w(x)|u_n - u|^q dx \le L\left(\int_{\mathbb{R}^N \setminus B_R} w(x)^{\wp} dx\right)^{1/\wp} = o(1) \tag{2.2}$$

as $R \to \infty$, being $w \in L^{\wp}(\mathbb{R}^N)$ and $\|u_n - u\|_{p_s^*}^q = L < \infty$ for all $n \in \mathbb{N}$ by (1.3). Moreover, for all $R > 0$ the embedding $D^{s,p}(\mathbb{R}^N) \hookrightarrow W^{s,p}(B_R)$ is continuous and so the embedding $D^{s,p}(\mathbb{R}^N) \hookrightarrow\hookrightarrow L^{\nu}(B_R)$ is compact for all $\nu \in [1, p_s^*)$, by Corollary 7.2 of [12]. In fact, using (1.3) and the Hölder inequality

$$\|u\|_{W^{s,p}(B_R)}^p \le C_R\|u\|_{p_s^*}^p + [u]_{s,p}^p \le (C_R/S + 1)[u]_{s,p}^p$$

for all $u \in D^{s,p}(\mathbb{R}^N)$, where $C_R = (\omega_N/N)^{ps/N}R^{ps}$ and ω_N is the measure of the unit sphere $S^{N-1} = \{x \in \mathbb{R}^N : |x| = 1\}$ of $\mathbb{R}^N$.

Fix $\varepsilon > 0$. There exists $R_\varepsilon > 0$ so large that $\int_{\mathbb{R}^N \setminus B_{R_\varepsilon}} w(x)|u_n - u|^q dx < \varepsilon$ by (2.2). Take a subsequence $(u_{n_k})_k \subset (u_n)_n$. Since $u_{n_k} \to u$ in $L^{\nu}(B_{R_\varepsilon})$ for all $\nu \in [1, p_s^*)$, then up to a further subsequence, still denoted by $(u_{n_k})_k$, we have that $u_{n_k} \to u$ a.e. in B_{R_ε}. Thus $w(x)|u_n - u|^q \to 0$ a.e. in B_{R_ε}. Furthermore, for each measurable subset $E \subset B_{R_\varepsilon}$, by the Hölder inequality we have

$$\int_E w(x)|u_{n_k} - u|^q dx \le L\left(\int_E w(x)^{\wp} dx\right)^{1/\wp}.$$

Hence, $(w(x)|u_{n_k} - u|^q)_k$ is equi–integrable and uniformly bounded in $L^1(B_{R_\varepsilon})$, since $w \in L^{\wp}(\mathbb{R}^N)$ by (w). Then, the Vitali convergence theorem implies

$$\lim_{k\to\infty} \int_{B_{R_\varepsilon}} w(x)|u_{n_k} - u|^q dx = 0$$

and so $u_n \to u$ in $L^q(B_{R_\varepsilon}, w)$, since the sequence $(u_{n_k})_k$ is arbitrary.

Consequently, $\int_{B_{R_\varepsilon}} w(x)|u_n - u|^q dx = o(1)$ as $n \to \infty$. In conclusion, as $n \to \infty$

$$\|u_n - u\|_{q,w}^q = \int_{\mathbb{R}^N \setminus B_{R_\varepsilon}} w(x)|u_n - u|^q dx + \int_{B_{R_\varepsilon}} w(x)|u_n - u|^q dx \le \varepsilon + o(1).$$

This completes the proof, being $\varepsilon > 0$ arbitrary. $\square$

We say that $u \in D^{s,p}(\mathbb{R}^N)$ is a (weak) solution of (1.1) if

$$M([u]_{s,p}^p)\langle u,\varphi\rangle_{s,p} - \gamma \int_{\mathbb{R}^N} \frac{|u(x)|^{p-2}u(x)\varphi(x)}{|x|^{ps}}dx$$
$$= \lambda \int_{\mathbb{R}^N} w(x)|u(x)|^{q-2}u(x)\varphi(x)dx + \int_{\mathbb{R}^N} K(x)|u(x)|^{p_s^*-2}u(x)\varphi(x)dx,$$

for all $\varphi \in D^{s,p}(\mathbb{R}^N)$, where

$$\langle u, \varphi\rangle_{s,p} = \iint_{\mathbb{R}^{2N}} \frac{|u(x)-u(y)|^{p-2}(u(x)-u(y))(\varphi(x)-\varphi(y))}{|x-y|^{N+ps}}dxdy.$$

Problem (1.1) has a variational structure and $J_{\gamma,\lambda} : D^{s,p}(\mathbb{R}^N) \to \mathbb{R}$, defined by

$$J_{\gamma,\lambda}(u) = \frac{1}{p}\mathscr{M}([u]_{s,p}^p) - \frac{\gamma}{p}\|u\|_H^p - \frac{\lambda}{q}\|u\|_{q,w}^q - \frac{1}{p_s^*}\|u\|_{p_s^*,K}^{p_s^*},$$

where $\|u\|_{p_s^*,K} = (\int_{\mathbb{R}^N} K(x)|u(x)|^{p_s^*}dx)^{1/p_s^*}$, is the underlying functional associated to (1.1). Obviously, $J_{\gamma,\lambda}$ is well defined and of class $C^1(D^{s,p}(\mathbb{R}^N))$.

Condition (1.2) gives that $M(t) > 0$ for any $t \in \mathbb{R}_0^+$ and $(\mathcal{M})$ yields that $t \mapsto t^{-\theta}\mathscr{M}(t)$ is nonincreasing in $\mathbb{R}^+$. Consequently, for all $t_0 > 0$

$$t_0^\theta \mathscr{M}(t) \le \mathscr{M}(t_0)t^\theta \quad \text{for all } t \ge t_0. \tag{2.3}$$

Now, as in [17], we prove that the functional $J_{\gamma,\lambda}$ has the geometric features required to apply the mountain pass theorem of Ambrosetti and Rabinowitz of [1].

Lemma 2.2. *For every $\gamma \in (-\infty, aH)$ and $\lambda > 0$ there exist $\alpha > 0$ and $\rho \in (0, 1]$ such that $J_{\gamma,\lambda}(u) \ge \alpha$ for all $u \in D^{s,p}(\mathbb{R}^N)$, with $[u]_{s,p} = \rho$, and a function $e \in C_0^\infty(\mathbb{R}^N)$, with $[e]_{s,p} > \rho$ and $J_{\gamma,\lambda}(e) < 0$. The function e depends only on γ^- when $K > 0$ a.e. in $\mathbb{R}^N$.*

Proof. Fix $\gamma \in (-\infty, aH)$ and $\lambda > 0$. By (K), (1.2), (1.3) and (2.1) there exists a positive constant S_K such that for all $u \in D^{s,p}(\mathbb{R}^N)$

$$J_{\gamma,\lambda}(u) \ge \frac{a}{p}[u]_{s,p}^p - \frac{\gamma}{p}\|u\|_H^p - \frac{\lambda}{q}\|u\|_{q,w}^q - \frac{1}{p_s^*}\|u\|_{p_s^*,K}^{p_s^*}$$
$$\ge \left(\frac{a}{p} - \frac{\gamma^+}{pH}\right)[u]_{s,p}^p - \frac{\lambda}{q}C_w^q[u]_{s,p}^q - S_K[u]_{s,p}^{p_s^*}.$$

Setting

$$\eta_{\gamma,\lambda}(t) = \left(\frac{a}{p} - \frac{\gamma^+}{pH}\right)t^p - \frac{\lambda}{q}C_w^q t^q - S_K t^{p_s^*} \quad \text{for all } t \in [0, 1],$$

we note that there exists $\rho \in (0, 1]$ such that $\max_{t\in[0,1]} \eta_{\gamma,\lambda}(t) = \eta_{\gamma,\lambda}(\rho) > 0$, since $\gamma^+ < aH$ and $p < q < p_s^*$ by $(\mathcal{M})$ and (w). Consequently, $J_{\gamma,\lambda}(u) \ge \alpha = \eta_{\gamma,\lambda}(\rho) > 0$ for all $u \in D^{s,p}(\mathbb{R}^N)$, with $[u]_{s,p} = \rho$.

Now take $v \in C_0^\infty(\mathbb{R}^N)$ such that $[v]_{s,p} = 1$. By (2.3) we have for $t \to \infty$

$$J_{\gamma,\lambda}(tv) \le \mathscr{M}(1)\frac{t^{\theta p}}{p}[v]_{s,p}^{\theta p} + \gamma^-\frac{t^p}{p}\|v\|_H^p - \lambda\frac{t^q}{q}\|v\|_{q,w}^q - \frac{t^{p_s^*}}{p_s^*}\|v\|_{p_s^*,K}^{p_s^*} \to -\infty,$$

since $p \le \theta p < q < p_s^*$ by $(\mathcal{M})$ and (w). Hence, taking $e = \tau_0 v$, with $\tau_0 > 0$ sufficiently large, we obtain at once that $[e]_{s,p} \ge 2$ and $J_{\gamma,\lambda}(e) < 0$. In particular, $[e]_{s,p} > \rho$ and e depends on γ^-. Furthermore, e can be taken independent of λ whenever $K > 0$ a.e. in $\mathbb{R}^N$, otherwise e could depend also on λ, as claimed. $\square$

From the proof of Lemma 2.2 it is apparent that if e is the function determined at some $\gamma \in (-\infty, aH)$ and $\lambda_0 > 0$, then e is such that $J_{\gamma,\lambda}(e) < 0$ for all $\lambda \ge \lambda_0$ and $[e]_{s,p} \ge 2 > \rho = \rho(\gamma, \lambda)$, being $\rho \in (0, 1]$.

We recall in passing that, if X is a real Banach space, a $C^1(X)$ functional J satisfies the Palais–Smale condition at level $c \in \mathbb{R}$ if any Palais–Smale sequence $(u_n)_n$ at level c, that is such that

$$J(u_n) \to c \quad \text{and} \quad J'(u_n) \to 0 \quad \text{in } X' \text{ as } n \to \infty \tag{2.4}$$

admits a convergent subsequence in X.

Fix $\gamma \in (-\infty, aH)$, $\lambda > 0$ and put

$$c_{\gamma,\lambda} = \inf_{g\in\Gamma} \max_{t\in[0,1]} J_{\gamma,\lambda}(g(t)),$$
$$\Gamma = \{g \in C([0,1], D^{s,p}(\mathbb{R}^N)) : g(0) = 0,\, g(1) = e\}.$$

Obviously $c_{\gamma,\lambda} > 0$ thanks to Lemma 2.2.

Before proving that $J_{\gamma,\lambda}$ satisfies the Palais–Smale condition at level $c_{\gamma,\lambda}$ in $D^{s,p}(\mathbb{R}^N)$, we introduce an asymptotic condition for the level $c_{\gamma,\lambda}$. This result is exactly Lemma 4.3 of [16] (see also Lemma 6 of [17] for a somehow related fractional non–degenerate Kirchhoff Dirichlet problem) and will be crucial not only to get (1.4), but above all to overcome the lack of compactness due to the presence of a Hardy term and a critical nonlinearity.

Lemma 2.3. *For all $\gamma \in (-\infty, aH)$ it results*

$$\lim_{\lambda\to\infty} c_{\gamma,\lambda} = 0.$$

Proof. Fix $\gamma \in (-\infty, aH)$ and $\lambda_0 > 0$. Let $e \in C_0^\infty(\mathbb{R}^N)$ be the function obtained by Lemma 2.2, depending on γ^- and possibly on λ_0. Hence the functional $J_{\gamma,\lambda}$ satisfies the mountain pass geometry at 0 and such e for all $\lambda \geq \lambda_0$. In particular, there exists $t_{\gamma,\lambda} > 0$ verifying $J_{\gamma,\lambda}(t_{\gamma,\lambda}e) = \max_{t\geq 0} J_{\gamma,\lambda}(te)$. Therefore, $\langle J'_{\gamma,\lambda}(t_{\gamma,\lambda}e), e\rangle = 0$ and then

$$\begin{aligned} t_{\gamma,\lambda}^{p-1}\big(M(t_{\gamma,\lambda}^p[e]_{s,p}^p)[e]_{s,p}^p - \gamma\|e\|_H^p\big) &= \lambda t_{\gamma,\lambda}^{q-1}\|e\|_{q,w}^q + t_{\gamma,\lambda}^{p_s^*-1}\|e\|_{p_s^*,K}^{p_s^*} \\ &\geq \lambda_0 t_{\gamma,\lambda}^{q-1}\|e\|_{q,w}^q \end{aligned} \tag{2.5}$$

by the fact that $\lambda \geq \lambda_0$. We claim that $\{t_{\gamma,\lambda}\}_{\lambda\geq\lambda_0}$ is bounded in $\mathbb{R}$. Indeed, using $(\mathcal{M})$ and (2.3) we get

$$\begin{aligned} t_{\gamma,\lambda}^p[e]_{s,p}^p M(t_{\gamma,\lambda}^p[e]_{s,p}^p) - \gamma t_{\gamma,\lambda}^p\|e\|_H^p &\leq \theta\mathscr{M}(t_{\gamma,\lambda}^p[e]_{s,p}^p) + \frac{\gamma^-}{H}t_{\gamma,\lambda}^p[e]_{s,p}^p \\ &\leq \left(\theta\mathscr{M}(1) + \frac{\gamma^-}{H}\right)t_{\gamma,\lambda}^{\theta p}[e]_{s,p}^{\theta p} \end{aligned} \tag{2.6}$$

for any $\lambda \in \Lambda$, with $\Lambda = \{\lambda \geq \lambda_0 : t_{\gamma,\lambda}[e]_{s,p} \geq 1\}$. Hence, from (2.5) and (2.6) it follows

$$\left(\theta\mathscr{M}(1) + \frac{\gamma^-}{H}\right)[e]_{s,p}^{\theta p} \geq \lambda_0 t_{\gamma,\lambda}^{q-\theta p}\|e\|_{q,w}^q \quad \text{for any } \lambda \in \Lambda,$$

which implies the boundedness of $\{t_{\gamma,\lambda}\}_{\lambda\in\Lambda}$, since $\theta p < q$ by (w), $\|e\|_{q,w} > 0$ and e depends on γ^- and λ_0 by Lemma 2.2 and its remark. It follows at once that indeed $\{t_{\gamma,\lambda}\}_{\lambda\geq\lambda_0}$ is bounded. This proves the claim.

Fix now a sequence $(\lambda_k)_k \subset [\lambda_0, \infty)$ such that $\lambda_k \to \infty$ as $k \to \infty$. Obviously $(t_{\gamma,\lambda_k})_k$ is bounded in $\mathbb{R}$. Hence, there exists a subsequence of $(\lambda_k)_k$, still relabeled $(\lambda_k)_k$, and a constant $\tau \geq 0$ such that $t_{\gamma,\lambda_k} \to \tau$ as $k \to \infty$. By the continuity of M, also $\big(M(t_{\gamma,\lambda_k}^p[e]_{s,p}^p)\big)_k$ is bounded, and so by (2.5) there exists L_{γ^-} such that

$$\lambda_k t_{\gamma,\lambda_k}^{q-1}\|e\|_{q,w}^q + t_{\gamma,\lambda_k}^{p_s^*-1}\|e\|_{p_s^*,K}^{p_s^*} \leq L_{\gamma^-} \quad \text{for any } k \in \mathbb{N}. \tag{2.7}$$

We assert that $\tau = 0$. In fact, if $\tau > 0$ we get

$$\lim_{k\to\infty}\left(\lambda_k t_{\gamma,\lambda_k}^{q-1}\|e\|_{q,w}^q + t_{\gamma,\lambda_k}^{p_s^*-1}\|e\|_{p_s^*,K}^{p_s^*}\right) = \infty,$$

which contradicts (2.7). Thus $\tau = 0$ and $t_{\gamma,\lambda} \to 0$ as $\lambda \to \infty$, since the sequence $(\lambda_k)_k$ is arbitrary.

Consider now the path $g(t) = te$, $t \in [0,1]$, belonging to Γ. By Lemma 2.2

$$0 < c_{\gamma,\lambda} \leq \max_{t\in[0,1]} J_{\gamma,\lambda}(g(t)) \leq J_{\gamma,\lambda}(t_{\gamma,\lambda}e) \leq \frac{1}{p}\mathscr{M}(t_{\gamma,\lambda}^p[e]_{s,p}^p) + \frac{\gamma^-}{p}\|e\|_H^p t_{\gamma,\lambda}^p.$$

Moreover, $\mathscr{M}(t_{\gamma,\lambda}^p[e]_{s,p}^p) \to 0$ as $\lambda \to \infty$ by the continuity of $\mathscr{M}$. This completes the proof of the lemma, since e depends only on γ^-. ∎

Now, following the key idea of the proof of Lemma 4.5 in [16], given when $M \equiv 1$ and $p = 2$, we prove the validity of the Palais–Smale condition for $J_{\gamma,\lambda}$ at level $c_{\gamma,\lambda}$ in $D^{s,p}(\mathbb{R}^N)$.

Lemma 2.4. *Let $\gamma \in (-\infty, \kappa H)$ be fixed. If $\|K\|_\infty = 0$, then $J_{\gamma,\lambda}$ satisfies the Palais–Smale condition at level $c_{\gamma,\lambda}$ for all $\lambda > 0$. While if $\|K\|_\infty > 0$, then there exists $\lambda^* = \lambda^*(\gamma) > 0$ such that $J_{\gamma,\lambda}$ satisfies the Palais–Smale condition at level $c_{\gamma,\lambda}$ for any $\lambda \geq \lambda^*$.*

Proof. Fix $\gamma < \kappa H$ and let $(u_n)_n \subset D^{s,p}(\mathbb{R}^N)$ be a Palais–Smale sequence of $J_{\gamma,\lambda}$ at level $c_{\gamma,\lambda}$ for all $\lambda > 0$. By $(\mathcal{M})$ and (1.3)

$$\begin{aligned} J_{\gamma,\lambda}(u_n) - \frac{1}{q}\langle J'_{\gamma,\lambda}(u_n), u_n\rangle &\geq \frac{1}{p}\mathscr{M}([u_n]^p_{s,p}) - \frac{1}{q}M([u_n]^p_{s,p})[u_n]^p_{s,p} \\ &\qquad - \gamma\left(\frac{1}{p} - \frac{1}{q}\right)\|u_n\|^p_H + \left(\frac{1}{q} - \frac{1}{p^*_s}\right)\|u_n\|^{p^*_s}_{p^*_s,K} \\ &\geq \left(\frac{1}{\theta p} - \frac{1}{q}\right)M([u_n]^p_{s,p})[u_n]^p_{s,p} - \frac{\gamma^+}{H}\left(\frac{1}{p} - \frac{1}{q}\right)[u_n]^p_{s,p} + \left(\frac{1}{q} - \frac{1}{p^*_s}\right)\|u_n\|^{p^*_s}_{p^*_s,K}. \end{aligned} \tag{2.8}$$

Then, thanks to (1.2), (2.4) and (2.8) there exists $\sigma_{\gamma,\lambda} > 0$ such that as $n \to \infty$

$$\begin{aligned} &c_{\gamma,\lambda} + \sigma_{\gamma,\lambda}[u_n]_{s,p} + o(1) \geq \mu_\gamma [u_n]^p_{s,p}, \\ &\mu_\gamma = a\left(\frac{1}{\theta p} - \frac{1}{q}\right) - \frac{\gamma^+}{H}\left(\frac{1}{p} - \frac{1}{q}\right) > 0 \end{aligned} \tag{2.9}$$

since $\gamma < \kappa H$. Therefore, $(u_n)_n$ is bounded in $D^{s,p}(\mathbb{R}^N)$.

By (1.3) and Lemma 2.1, there exists $u_{\gamma,\lambda} \in D^{s,p}(\mathbb{R}^N)$ such that, going if necessary to a subsequence

$$\begin{aligned} &u_n \rightharpoonup u_{\gamma,\lambda} \text{ in } D^{s,p}(\mathbb{R}^N), && [u_n]_{s,p} \to \alpha_{\gamma,\lambda}, \\ &u_n \rightharpoonup u_{\gamma,\lambda} \text{ in } L^{p^*_s}(\mathbb{R}^N), && \|u_n - u_{\gamma,\lambda}\|_{p^*_s,K} \to \ell_{\gamma,\lambda}, \\ &\frac{u_n}{|x|^s} \rightharpoonup \frac{u_{\gamma,\lambda}}{|x|^s} \text{ in } L^p(\mathbb{R}^N), && \|u_n - u_{\gamma,\lambda}\|_H \to \imath_{\gamma,\lambda}, \\ &u_n \to u_{\gamma,\lambda} \text{ in } L^q(\mathbb{R}^N, w), && u_n \to u_{\gamma,\lambda} \text{ a.e. in } \mathbb{R}^N. \end{aligned} \tag{2.10}$$

In particular, by (2.4) and (2.8)

$$c_{\gamma,\lambda} + o(1) \geq \mu_\gamma [u_n]^p_{s,p} + \left(\frac{1}{q} - \frac{1}{p^*_s}\right)\|u_n\|^{p^*_s}_{p^*_s,K}, \tag{2.11}$$

where μ_γ is given in (2.9).

First, we assert that

$$\lim_{\lambda\to\infty} \alpha_{\gamma,\lambda} = 0. \tag{2.12}$$

Otherwise, $\limsup_{\lambda\to\infty} \alpha_{\gamma,\lambda} = \alpha_\gamma > 0$. Hence there is a sequence $k \to \lambda_k \uparrow \infty$ such that $\alpha_{\gamma,\lambda_k} \to \alpha_\gamma$ as $k \to \infty$. Then, letting $k \to \infty$ we get from (2.11) and Lemma 2.3 that

$$0 \geq \mu_\gamma \alpha^p_\gamma > 0.$$

This contradiction proves the assertion (2.12). Moreover,

$$[u_{\gamma,\lambda}]_{s,p} \leq \lim_{n\to\infty} [u_n]_{s,p} = \alpha_{\gamma,\lambda},$$

since $u_n \rightharpoonup u_{\gamma,\lambda}$, and so (K), (1.3) and (2.12) implies that

$$\lim_{\lambda\to\infty} \|u_{\gamma,\lambda}\|_{p^*_s,K} = \lim_{\lambda\to\infty} \|u_{\gamma,\lambda}\|_H = \lim_{\lambda\to\infty} [u_{\gamma,\lambda}]_{s,p} = 0. \tag{2.13}$$

Thanks to (2.4), we have as $n \to \infty$

$$\begin{aligned} o(1) = M([u_n]^p_{s,p})\langle u_n, \varphi\rangle_{s,p} &- \gamma \int_{\mathbb{R}^N} \frac{|u_n(x)|^{p-2}u_n(x)\varphi(x)}{|x|^{ps}}dx \\ - \lambda \int_{\mathbb{R}^N} w(x)|u_n(x)|^{q-2}u_n(x)\varphi(x)dx &- \int_{\mathbb{R}^N} K(x)|u_n(x)|^{p^*_s-2}u_n(x)\varphi(x)dx, \end{aligned}$$

for any $\varphi \in D^{s,p}(\mathbb{R}^N)$. By (2.10), the sequence $(\mathcal{U}_n)_n$, defined in $\mathbb{R}^{2N} \setminus \mathrm{Diag}\,(\mathbb{R}^{2N})$ by

$$(x,y) \mapsto \mathcal{U}_n(x,y) = \frac{|u_n(x) - u_n(y)|^{p-2}(u_n(x) - u_n(y))}{|x-y|^{(N+ps)/p'}},$$

is bounded in $L^{p'}(\mathbb{R}^{2N})$ as well as $\mathcal{U}_n \to \mathcal{U}_{\gamma,\lambda}$ a.e. in $\mathbb{R}^{2N}$, where

$$\mathcal{U}_{\gamma,\lambda}(x,y) = \frac{|u_{\gamma,\lambda}(x) - u_{\gamma,\lambda}(y)|^{p-2}(u_{\gamma,\lambda}(x) - u_{\gamma,\lambda}(y))}{|x-y|^{(N+ps)/p'}}.$$

Thus, going if necessary to a further subsequence, we get $\mathcal{U}_n \rightharpoonup \mathcal{U}_{\gamma,\lambda}$ in $L^{p'}(\mathbb{R}^{2N})$, and so

$$\langle u_n, \varphi\rangle_{s,p} \to \langle u_{\gamma,\lambda}, \varphi\rangle_{s,p}$$

for any $\varphi \in D^{s,p}(\mathbb{R}^N)$, since $|\varphi(x) - \varphi(y)| \cdot |x-y|^{-(N+ps)/p} \in L^p(\mathbb{R}^{2N})$. Then, using (2.10) and the facts that $|u_n|^{q-2}u_n \rightharpoonup |u_{\gamma,\lambda}|^{q-2}u_{\gamma,\lambda}$ in $L^{q'}(\mathbb{R}^N, w)$ and $|u_n|^{p_s^*-2}u_n \rightharpoonup |u_{\gamma,\lambda}|^{p_s^*-2}u_{\gamma,\lambda}$ in $L^{p_s^{*\prime}}(\mathbb{R}^N, K)$, by Preposition A.8 of [3], we obtain

$$\begin{aligned} &M(\alpha^p_{\gamma,\lambda})\langle u_{\gamma,\lambda}, \varphi\rangle_{s,p} - \gamma \int_{\mathbb{R}^N} \frac{|u_{\gamma,\lambda}(x)|^{p-2}u_{\gamma,\lambda}(x)\varphi(x)}{|x|^{ps}}dx \\ &= \lambda \int_{\mathbb{R}^N} w(x)|u_{\gamma,\lambda}(x)|^{q-2}u_{\gamma,\lambda}(x)\varphi(x)dx + \int_{\mathbb{R}^N} K(x)|u_{\gamma,\lambda}(x)|^{p_s^*-2}u_{\gamma,\lambda}(x)\varphi(x)dx \end{aligned}$$

for any $\varphi \in D^{s,p}(\mathbb{R}^N)$. Hence, $u_{\gamma,\lambda}$ is a critical point of the $C^1(D^{s,p}(\mathbb{R}^N))$ functional

$$J_{\alpha_{\gamma,\lambda}}(u) = \frac{1}{p}M(\alpha^p_{\gamma,\lambda})[u]^p_{s,p} - \frac{\gamma}{p}\|u\|^p_H - \frac{\lambda}{q}\|u\|^q_{q,w} - \frac{1}{p_s^*}\|u\|^{p_s^*}_{p_s^*,K}. \tag{2.14}$$

By the Hölder inequality we have

$$|\langle u, v\rangle_{s,p}| \le [u]^{p-1}_{s,p}[v]_{s,p} \quad \text{for all } u, v \in D^{s,p}(\mathbb{R}^N)$$

and so, for any $u \in D^{s,p}(\mathbb{R}^N)$, the functional $\langle u, \cdot\rangle_{s,p}$ is linear and continuous on $D^{s,p}(\mathbb{R}^N)$. Consequently, (2.4), (2.10) and (2.14) give as $n \to \infty$

$$\begin{aligned} o(1) &= \langle J'_{\gamma,\lambda}(u_n) - J'_{\alpha_{\gamma,\lambda}}(u_{\gamma,\lambda}), u_n - u_{\gamma,\lambda}\rangle = M([u_n]^p_{s,p})[u_n]^p_{s,p} \\ &\quad + M(\alpha^p_{\gamma,\lambda})[u_{\gamma,\lambda}]^p_{s,p} - M([u_n]^p_{s,p})\langle u_n, u_{\gamma,\lambda}\rangle_{s,p} - M(\alpha^p_{\gamma,\lambda})\langle u_{\gamma,\lambda}, u_n\rangle_{s,p} \\ &\quad - \gamma \int_{\mathbb{R}^N} \frac{(|u_n|^{p-2}u_n - |u_{\gamma,\lambda}|^{p-2}u_{\gamma,\lambda})(u_n - u_{\gamma,\lambda})}{|x|^{ps}}dx \\ &\quad - \lambda \int_{\mathbb{R}^N} w(x)(|u_n|^{q-2}u_n - |u_{\gamma,\lambda}|^{q-2}u_{\gamma,\lambda})(u_n - u_{\gamma,\lambda})dx \\ &\quad - \int_{\mathbb{R}^N} K(x)(|u_n|^{p_s^*-2}u_n - |u_{\gamma,\lambda}|^{p_s^*-2}u_{\gamma,\lambda})(u_n - u_{\gamma,\lambda})dx \\ &= M(\alpha^p_{\gamma,\lambda})(\alpha^p_{\gamma,\lambda} - [u_{\gamma,\lambda}]^p_{s,p}) - \gamma\|u_n\|^p_H + \gamma\|u_{\gamma,\lambda}\|^p_H - \|u_n\|^{p_s^*}_{p_s^*,K} \\ &\quad + \|u_{\gamma,\lambda}\|^{p_s^*}_{p_s^*,K} + o(1) \\ &= M(\alpha^p_{\gamma,\lambda})([u_n - u_{\gamma,\lambda}]^p_{s,p}) - \gamma\|u_n - u_{\gamma,\lambda}\|^p_H - \|u_n - u_{\gamma,\lambda}\|^{p_s^*}_{p_s^*,K} + o(1). \end{aligned}$$

In fact, thanks to (2.10) it results

$$\lim_{n\to\infty} \int_{\mathbb{R}^N} w(x)(|u_n|^{q-2}u_n - |u_{\gamma,\lambda}|^{q-2}u_{\gamma,\lambda})(u_n - u_{\gamma,\lambda})dx = 0.$$

Furthermore, using again (2.10) and the celebrate Brézis & Lieb lemma of [5]

$$\begin{aligned} [u_n]^p_{s,p} &= [u_n - u_{\gamma,\lambda}]^p_{s,p} + [u_{\gamma,\lambda}]^p_{s,p} + o(1), \\ \|u_n\|^{p_s^*}_{p_s^*,K} &= \|u_n - u_{\gamma,\lambda}\|^{p_s^*}_{p_s^*,K} + \|u_{\gamma,\lambda}\|^{p_s^*}_{p_s^*,K} + o(1), \\ \|u_n\|^p_H &= \|u_n - u_{\gamma,\lambda}\|^p_H + \|u_{\gamma,\lambda}\|^p_H + o(1) \end{aligned}$$

as $n \to \infty$. Finally, we have used the fact that $[u_n]_{s,p} \to \alpha_{\gamma,\lambda}$ as $n \to \infty$.

Hence, we have obtained the main formula

$$\begin{aligned} M(\alpha^p_{\gamma,\lambda}) \lim_{n\to\infty}[u_n - u_{\gamma,\lambda}]^p_{s,p} &= \lim_{n\to\infty}\|u_n - u_{\gamma,\lambda}\|^{p_s^*}_{p_s^*,K} + \gamma \lim_{n\to\infty}\|u_n - u_{\gamma,\lambda}\|^p_H \\ &= \ell^{p_s^*}_{\gamma,\lambda} + \gamma\imath^p_{\gamma,\lambda}. \end{aligned} \tag{2.15}$$

Let us divide the proof in two parts.

Case $\|K\|_\infty = 0$. Clearly $\ell_{\gamma,\lambda} = 0$ in (2.15). Assume for contradiction that $\imath_{\gamma,\lambda} > 0$. Then, from (1.3) and (2.15)

$$M(\alpha^p_{\gamma,\lambda}) \lim_{n\to\infty}[u_n - u_{\gamma,\lambda}]^p_{s,p} = \gamma \lim_{n\to\infty}\|u_n - u_{\gamma,\lambda}\|^p_H < aH \lim_{n\to\infty}\|u_n - u_{\gamma,\lambda}\|^p_H \\ \leq M(\alpha^p_{\gamma,\lambda}) \lim_{n\to\infty}[u_n - u_{\gamma,\lambda}]^p_{s,p},$$

which is impossible. Hence, $\imath_{\gamma,\lambda} = 0$ for all $\lambda > 0$. Thus, using also (2.15) and the fact that $\ell_{\gamma,\lambda} = 0$, we get

$$\lim_{n\to\infty}[u_n - u_{\gamma,\lambda}]^p_{s,p} = \lim_{n\to\infty}\|u_n - u_{\gamma,\lambda}\|^p_H = 0$$

by (1.2). In conclusion, $u_n \to u_{\gamma,\lambda}$ in $D^{s,p}(\mathbb{R}^N)$ as $n \to \infty$ for all $\lambda > 0$ as required.

Case $\|K\|_\infty > 0$. By (2.11) and the Brézis & Lieb lemma, we get as $n \to \infty$

$$c_{\gamma,\lambda} + o(1) \geq \left(\frac{1}{q} - \frac{1}{p^*_s}\right)\|u_n\|^{p^*_s}_{p^*_s,K} = \left(\frac{1}{q} - \frac{1}{p^*_s}\right)\left[\ell^{p^*_s}_{\gamma,\lambda} + \|u_{\gamma,\lambda}\|^{p^*_s}_{p^*_s,K}\right] + o(1).$$

Then, Lemma 2.3 and (2.13) imply that

$$\lim_{\lambda\to\infty} \ell_{\gamma,\lambda} = 0. \tag{2.16}$$

Since $\gamma < aH$ there exists $c \in [0, 1)$ such that $\gamma^+ = c\,a\,H$. Thus, (2.15) can be rewritten as

$$(1-c)M(\alpha^p_{\gamma,\lambda}) \lim_{n\to\infty}[u_n - u_{\gamma,\lambda}]^p_{s,p} + cM(\alpha^p_{\gamma,\lambda}) \lim_{n\to\infty}[u_n - u_{\gamma,\lambda}]^p_{s,p} = \ell^{p^*_s}_{\gamma,\lambda} + \gamma\,\imath^p_{\gamma,\lambda}.$$

Now, for all $\lambda > 0$ we have $\ell^{p^*_s}_{\gamma,\lambda} + \gamma^+\imath^p_{\gamma,\lambda} \geq (1-c)S\|K\|^{-p/p^*_s}_\infty a\,\ell^p_{\gamma,\lambda} + c\,a\,H\imath^p_{\gamma,\lambda}$ by (K), (1.3) and (1.2), being $c \geq 0$. Therefore, since $\gamma^+ = c\,a\,H$,

$$\ell^{p^*_s}_{\gamma,\lambda} \geq (1-c)S\,\|K\|^{-p/p^*_s}_\infty a\,\ell^p_{\gamma,\lambda}. \tag{2.17}$$

Consequently, (2.16) and (2.17) imply at once that there exists $\lambda^* = \lambda^*(\gamma) > 0$ such that $\ell_{\gamma,\lambda} = 0$ for all $\lambda \geq \lambda^*$. In other words,

$$\lim_{n\to\infty}\|u_n - u_{\gamma,\lambda}\|_{p^*_s,K} = 0$$

for all $\lambda \geq \lambda^*$. From now on we can proceed as in the first case, and prove that $\imath_{\gamma,\lambda} = 0$ for all $\lambda \geq \lambda^*$. Thus, using also (2.15), we get $u_n \to u_{\gamma,\lambda}$ in $D^{s,p}(\mathbb{R}^N)$ as $n \to \infty$ for all $\lambda \geq \lambda^*$ as required, and the proof is complete. □

As already noted in the Introduction, besides the obvious case $M \equiv a$, in which $\kappa = a$, there are several non monotone Kirchhoff functions for which $\kappa = a$, that is $\theta = 1$. For instance, $M(t) = 2$ in $[0, 1]$, $M(t) = 3-t$ in $[1, 2]$, $M(t) = 1$ in $[2, 3]$, $M(t) = t-2$ in $[3, 2\sqrt{3}]$ and finally $M(t) = 2\sqrt{3}-2$ in $[2\sqrt{3}, \infty)$ is a non monotone function satisfying $(\mathcal{M})$, with $\theta = 1$.

Proof of Theorem 1.1 Fix $\gamma \in (-\infty, \kappa H)$. Thanks to Lemmas 2.2 and 2.4 the functional $J_{\gamma,\lambda}$ satisfies all the assumptions of the mountain pass theorem for any $\lambda > 0$ when $\|K\|_\infty = 0$ and for any $\lambda \geq \lambda^*$, with $\lambda^* = \lambda^*(\gamma) > 0$ if $\|K\|_\infty > 0$. This guarantees the existence of a critical point $u_{\gamma,\lambda} \in D^{s,p}(\mathbb{R}^N)$ for $J_{\gamma,\lambda}$ at level $c_{\gamma,\lambda}$. Since $J_{\gamma,\lambda}(u_{\gamma,\lambda}) = c_{\gamma,\lambda} > 0 = J_{\gamma,\lambda}(0)$ we have that $u_{\gamma,\lambda} \neq 0$. Moreover the asymptotic behavior (1.4) holds thanks to (2.13). □

3. The degenerate problem (1.5)

In this section we study the degenerate problem (1.5) and in passing we recall that $(\mathcal{M})$ is assumed throughout the paper.

If $\mathscr{M}(t_0) > 0$ for some $t_0 > 0$, then $(\mathcal{M})$ yields that

$$t_0^\theta \mathscr{M}(t) \geq \mathscr{M}(t_0)t^\theta \quad \text{for all } t \in [0, t_0] \tag{3.1}$$

and we have the following

Lemma 3.1. *Suppose that $\mathscr{M}$ is not identically zero in $\mathbb{R}^+$ and that $(\mathcal{M})$ holds, with $\theta = 1$. Then $M(0) = M_0 > 0$.*

Proof. Suppose that $\mathscr{M}(1) > 0$, for simplicity. By (3.1) we know that

$$\mathscr{M}(1)t \leq \mathscr{M}(t) = \int_0^t M(\tau)d\tau$$

for all $t \in [0, 1]$. Now, let $(t_n)_n$ be a sequence such that $t_n \in (0, 1)$ for all $n \in \mathbb{N}$ and $t_n \downarrow 0$ as $n \to \infty$. By the mean value theorem, we obtain the existence of a sequence $(\tau_n)_n$, with $\tau_n \in (0, t_n)$, such that

$$\mathscr{M}(1)t_n \leq M(\tau_n)t_n \quad \text{for all } n \in \mathbb{N} \tag{3.2}$$

and so $\mathscr{M}(1) \leq M(\tau_n)$ for all $n \in \mathbb{N}$, being $t_n > 0$. Observe that $M(\tau_n) \to M(0)$ as $n \to \infty$, since $\tau_n \to 0$ as $n \to \infty$ and M is continuous. Hence, letting $n \to \infty$ in (3.2) we get $M(0) \geq \mathscr{M}(1)$. This completes the proof, since $\mathscr{M}(1) > 0$ by assumption. □

As note in [9], condition $(\mathcal{M}_1)$ gives that $M(t) > 0$ for any $t > 0$. In particular, under $(\mathcal{M})$ and $(\mathcal{M}_1)$, *Lemma* 3.1 *implies that* $\theta > 1$ *when* $M(0) = 0$. The vice versa is false. Indeed, when $a > 0$ and $b > 0$, the canonical Kirchhoff function $M(t) = a + b\,t$ has the property that $M(0) = a > 0$, but M satisfies $(\mathcal{M})$ for any $\theta \geq 2$.

We say that $u \in D^{s,p}(\mathbb{R}^N)$ is a (weak) *solution* of (1.5) if

$$\begin{aligned} M([u]_{s,p}^p)\langle u, \varphi\rangle_{s,p} = \lambda &\int_{\mathbb{R}^N} w(x)|u(x)|^{q-2}u(x)\varphi(x)dx \\ &+ \int_{\mathbb{R}^N} K(x)|u(x)|^{p_s^*-2}u(x)\varphi(x)dx, \end{aligned}$$

for all $\varphi \in D^{s,p}(\mathbb{R}^N)$. The underlying functional associated to problem (1.5) is $J_\lambda : D^{s,p}(\mathbb{R}^N) \to \mathbb{R}$, given by

$$J_\lambda(u) = \frac{1}{p}\mathscr{M}([u]_{s,p}^p) - \frac{\lambda}{q}\|u\|_{q,w}^q - \frac{1}{p_s^*}\|u\|_{p_s^*,K}^{p_s^*}.$$

Obviously, also J_λ is well defined and of class $C^1(D^{s,p}(\mathbb{R}^N))$. For (1.5) we somehow follow the strategies developed in Section 2, but for convenience of the reader we give details.

Lemma 3.2. *Suppose that $\mathscr{M}$ is not identically zero. For any $\lambda > 0$ there exist $\alpha > 0$ and $\rho > 0$ such that $J_\lambda(u) \geq \alpha$ for all $u \in D^{s,p}(\mathbb{R}^N)$, with $[u]_{s,p} = \rho$, and a function $e \in C_0^\infty(\mathbb{R}^N)$, with $[e]_{s,p} > \rho$ and $J_\lambda(e) < 0$. The function e is independent of λ when $K > 0$ a.e. in $\mathbb{R}^N$.*

Proof. By assumption there exists $t_0 > 0$ such that $\mathscr{M}(t_0^p) > 0$.

Fix $\lambda > 0$ and take $u \in D^{s,p}(\mathbb{R}^N)$, with $[u]_{s,p} \leq t_0$. By $(\mathcal{M})$, (K), (1.3), (2.1) and (3.1) there exists a positive constant S_K such that

$$\begin{aligned} J_\lambda(u) &\geq m[u]_{s,p}^{\theta p} - \frac{\lambda}{q}\|u\|_{q,w}^q - \frac{1}{p_s^*}\|u\|_{p_s^*,K}^{p_s^*} \\ &\geq m[u]_{s,p}^{\theta p} - \frac{\lambda}{q}C_w^q[u]_{s,p}^q - S_K[u]_{s,p}^{p_s^*}, \end{aligned}$$

where $m = \mathscr{M}(t_0^p)t_0^{-\theta p}/p > 0$, as shown above. Setting

$$\eta_\lambda(t) = m\,t^{\theta p} - \frac{\lambda}{q}C_w^q t^q - S_K t^{p_s^*} \quad \text{for all } t \in [0, t_0],$$

we note that there exists $\rho \in (0, t_0]$ such that $\max_{t\in[0,t_0]}\eta_\lambda(t) = \eta_\lambda(\rho) > 0$, since $\theta p < q < p_s^*$ by (w). Consequently, $J_\lambda(u) \geq \alpha = \eta_\lambda(\rho) > 0$ for all $u \in D^{s,p}(\mathbb{R}^N)$, with $[u]_{s,p} = \rho$.

Now, take $v \in C_0^\infty(\mathbb{R}^N)$ such that $[v]_{s,p} = 1$. By (2.3) we have as $t \to \infty$

$$J_\lambda(tv) \leq m\,t^{\theta p} - \lambda\frac{t^q}{q}\|v\|_{q,w}^q - \frac{t^{p_s^*}}{p_s^*}\|v\|_{p_s^*,K}^{p_s^*} \to -\infty,$$

since $\theta p < q < p_s^*$ by (w). Hence, taking $e = \tau_0 v$, with $\tau_0 > 0$ sufficiently large, we obtain at once that $[e]_{s,p} \geq 2t_0$ and $J_\lambda(e) < 0$. In particular, $[e]_{s,p} > \rho$ and e does not depend on λ whenever K >0 a.e. in $\mathbb{R}^N$. □

Again from the proof of Lemma 3.2 it is apparent that if e is the function determined at some $\lambda_0 > 0$, then e is such that $J_\lambda(e) < 0$ for all $\lambda \geq \lambda_0$ and $[e]_{s,p} \geq 2t_0 > \rho = \rho(\lambda)$, being $\rho \in (0, t_0]$.

Fix $\lambda > 0$ and set

$$c_\lambda = \inf_{g\in\Gamma} \max_{t\in[0,1]} J_\lambda(g(t)),$$
$$\Gamma = \{g \in C([0,1], D^{s,p}(\mathbb{R}^N)) : g(0) = 0,\, g(1) = e\}.$$

Obviously $c_\lambda > 0$ thanks to Lemma 3.2.

Lemma 3.3. *If $\mathscr{M}$ is not identically zero, then*

$$\lim_{\lambda\to\infty} c_\lambda = 0.$$

Proof. We can proceed exactly as in the proof of Lemmas 3.2 and 2.3, formally taking $\gamma = 0$, $\lambda_0 > 0$, replacing $\mathscr{M}(1)$ by $\mathscr{M}(t_0^p)/t_0^{\theta p}$ in (2.6) and defining

$$\Lambda = \{\lambda > \lambda_0 \,:\, t_\lambda[e]_{s,p} \geq t_0\}.$$

We leave the further details to the reader. □

Let $(u_n)_n \subset D^{s,p}(\mathbb{R}^N)$ be a Palais–Smale sequence for J_λ at level $c_\lambda \in \mathbb{R}$. Then

$$J_\lambda(u_n) \to c_\lambda \quad \text{and} \quad J'_\lambda(u_n) \to 0 \quad \text{as } n \to \infty. \tag{3.3}$$

Next, we prove the validity of the Palais–Smale condition for J_λ at level c_λ in $D^{s,p}(\mathbb{R}^N)$, following somehow the main idea of the proof of Lemma 3.4 in [2], given for $p = 2$, see also [9]. It is exactly at this point that we need also $(\mathcal{M}_1)$ and $(\mathcal{M}_2)$. Just to clarify the two simple examples,

$$M_1(t) = \begin{cases} 4t & \text{se } t \in [0,1], \\ 6-2t & \text{se } t \in [1,2], \\ t & \text{se } t \in [2,\infty[, \end{cases} \qquad M_2(t) = \sqrt{t} + \arctan t,$$

are not monotone, but satisfy $(\mathcal{M})$, with $\theta = 2$, as well as $(\mathcal{M}_1)$ and $(\mathcal{M}_2)$.

Lemma 3.4. *Let $(\mathcal{M}_1)$–$(\mathcal{M}_2)$ hold and suppose that $ps < N < 2ps$. If $\|K\|_\infty = 0$, then the functional J_λ satisfies the Palais–Smale condition at level c_λ for all $\lambda > 0$. While if $\|K\|_\infty > 0$, then there exists $\lambda^* > 0$ such that J_λ satisfies the Palais–Smale condition at level c_λ for any $\lambda \geq \lambda^*$.*

Proof. Fix $\lambda > 0$. Let $(u_n)_n \subset D^{s,p}(\mathbb{R}^N)$ be a Palais–Smale sequence of J_λ at level c_λ. Due to the degenerate nature of (1.5), we have to consider two situations. Either $\inf_{n\in\mathbb{N}}[u_n]_{s,p} = d_\lambda > 0$ or $\inf_{n\in\mathbb{N}}[u_n]_{s,p} = 0$. Hence, let us divide the proof in two parts.

Case $\inf_{n\in\mathbb{N}}[u_n]_{s,p} = d_\lambda > 0$. We claim that $(u_n)_n$ is bounded in $D^{s,p}(\mathbb{R}^N)$. By $(\mathcal{M}_1)$, with $\tau_\lambda = d_\lambda^p$, there exists $m_\lambda = m_{\tau_\lambda} > 0$ such that

$$M([u_n]^p_{s,p}) \geq m_\lambda \quad \text{for any } n \in \mathbb{N}. \tag{3.4}$$

Using $(\mathcal{M})$, we get

$$\begin{aligned} J_\lambda(u_n) - &\frac{1}{q}\langle J'_\lambda(u_n), u_n\rangle \\ &\geq \frac{1}{p}\mathscr{M}([u_n]^p_{s,p}) - \frac{1}{q}M([u_n]^p_{s,p})[u_n]^p_{s,p} + \left(\frac{1}{q} - \frac{1}{p_s^*}\right)\|u_n\|^{p_s^*}_{p_s^*,K} \\ &\geq \left(\frac{1}{\theta p} - \frac{1}{q}\right)M([u_n]^p_{s,p})[u_n]^p_{s,p} + \left(\frac{1}{q} - \frac{1}{p_s^*}\right)\|u_n\|^{p_s^*}_{p_s^*,K}. \end{aligned} \tag{3.5}$$

Then, thanks to (3.3) and (3.4), there exists $\sigma_\lambda > 0$ such that as $n \to \infty$

$$c_\lambda + \sigma_\lambda[u_n]_{s,p} + o(1) \geq m_\lambda\left(\frac{1}{\theta p} - \frac{1}{q}\right)[u_n]^p_{s,p}.$$

Therefore, $(u_n)_n$ is bounded in $D^{s,p}(\mathbb{R}^N)$, since $1 < p < \theta p < q$ by $(\mathcal{M})$, (w) and Lemma 3.1.

By (1.3) and Lemma 2.1, there exists $u_\lambda \in D^{s,p}(\mathbb{R}^N)$ such that, going if necessary to a subsequence

$$u_n \rightharpoonup u_\lambda \text{ in } D^{s,p}(\mathbb{R}^N), \qquad [u_n]_{s,p} \to \alpha_\lambda,$$

$$u_n \rightharpoonup u_\lambda \text{ in } L^{p_s^*}(\mathbb{R}^N), \qquad \|u_n - u_\lambda\|_{p_s^*,K} \to \ell_\lambda, \tag{3.6}$$
$$u_n \to u_\lambda \text{ in } L^q(\mathbb{R}^N, w), \qquad u_n \to u_\lambda \text{ a.e. in } \mathbb{R}^N.$$

In particular, by (3.5) as $n \to \infty$

$$c_\lambda + o(1) \geq \left(\frac{1}{\theta p} - \frac{1}{q}\right) M([u_n]_{s,p}^p)[u_n]_{s,p}^p. \tag{3.7}$$

Moreover $\alpha_\lambda > 0$, since $d_\lambda > 0$. Therefore $M([u_n]_{s,p}^p) \to M(\alpha_\lambda^p) > 0$ as $n \to \infty$, by the continuity of M and the fact that 0 is the unique zero of M thanks to $(\mathcal{M}_1)$.

First, we assert that

$$\lim_{\lambda \to \infty} \alpha_\lambda = 0. \tag{3.8}$$

Otherwise, $\limsup_{\lambda \to \infty} \alpha_\lambda = \alpha > 0$. Hence there is a sequence $k \to \lambda_k \uparrow \infty$ such that $\alpha_{\lambda_k} \to \alpha$ as $k \to \infty$. Then, letting $k \to \infty$ we get from (3.7) and Lemma 3.3 that

$$0 \geq \left(\frac{1}{\theta p} - \frac{1}{q}\right) M(\alpha^p)\alpha^p > 0$$

by $(\mathcal{M}_1)$. This contradiction proves the assertion (3.8). Moreover,

$$[u_\lambda]_{s,p} \leq \lim_{n \to \infty} [u_n]_{s,p} = \alpha_\lambda,$$

since $u_n \rightharpoonup u_\lambda$ in $D^{s,p}(\mathbb{R}^N)$, and so (K), (1.3) and (3.8) implies that

$$\lim_{\lambda \to \infty} \|u_\lambda\|_{p_s^*,K} = \lim_{\lambda \to \infty} [u_\lambda]_{s,p} = 0. \tag{3.9}$$

Thanks to (3.3), as in Lemma 2.4, we obtain

$$M(\alpha_\lambda^p)\langle u_\lambda, \varphi\rangle_{s,p} = \lambda \int_{\mathbb{R}^N} w(x)|u_\lambda(x)|^{q-2}u_\lambda(x)\varphi(x)dx + \int_{\mathbb{R}^N} K(x)|u_\lambda(x)|^{p_s^*-2}u_\lambda(x)\varphi(x)dx$$

for any $\varphi \in D^{s,p}(\mathbb{R}^N)$. Hence, u_λ is a critical point of the $C^1(D^{s,p}(\mathbb{R}^N))$ functional

$$J_{\alpha_\lambda}(u) = \frac{1}{p} M(\alpha_\lambda^p)[u]_{s,p}^p - \frac{\lambda}{q}\|u\|_{q,w}^q - \frac{1}{p_s^*}\|u\|_{p_s^*,K}^{p_s^*}. \tag{3.10}$$

Consequently, (3.3), (3.6) and (3.10) give as $n \to \infty$

$$\begin{aligned} o(1) = \langle J'_\lambda(u_n) - J'_{\alpha_\lambda}(u_\lambda), u_n - u_\lambda\rangle &= M([u_n]_{s,p}^p)[u_n]_{s,p}^p + M(\alpha_\lambda^p)[u_\lambda]_{s,p}^p \\ &- M([u_n]_{s,p}^p)\langle u_n, u_\lambda\rangle_{s,p} - M(\alpha_\lambda^p)\langle u_\lambda, u_n\rangle_{s,p} \\ &- \lambda \int_{\mathbb{R}^N} w(x)(|u_n|^{q-2}u_n - |u_\lambda|^{q-2}u_\lambda)(u_n - u_\lambda)dx \\ &- \int_{\mathbb{R}^N} K(x)(|u_n|^{p_s^*-2}u_n - |u_\lambda|^{p_s^*-2}u_\lambda)(u_n - u_\lambda)dx \\ &= M(\alpha_\lambda^p)(\alpha_\lambda^p - [u_\lambda]_{s,p}^p) - \|u_n\|_{p_s^*,K}^{p_s^*} + \|u_\lambda\|_{p_s^*,K}^{p_s^*} + o(1) \\ &= M(\alpha_\lambda^p)([u_n - u_\lambda]_{s,p}^p) - \|u_n - u_\lambda\|_{p_s^*,K}^{p_s^*} + o(1). \end{aligned} \tag{3.11}$$

Thus, we have obtained the main formula

$$M(\alpha_\lambda^p) \lim_{n \to \infty} [u_n - u_\lambda]_{s,p}^p = \lim_{n \to \infty} \|u_n - u_\lambda\|_{p_s^*,K}^{p_s^*}. \tag{3.12}$$

Suppose that $\|K\|_\infty = 0$. Then $\ell_\lambda = 0$ in (3.12) and so $u_n \to u_\lambda$ in $D^{s,p}(\mathbb{R}^N)$ as $n \to \infty$ for all $\lambda > 0$, being $M(\alpha_\lambda^p) > 0$.

Otherwise, if $\|K\|_\infty > 0$, by (3.3), (3.5), (3.6) and the Brézis & Lieb lemma, we get as $n \to \infty$

$$\begin{aligned} c_\lambda + o(1) = J_\lambda(u_n) - \frac{1}{q}\langle J'_\lambda(u_n), u_n\rangle &\geq \left(\frac{1}{q} - \frac{1}{p_s^*}\right)\|u_n\|_{p_s^*,K}^{p_s^*} \\ &= \left(\frac{1}{q} - \frac{1}{p_s^*}\right)\left[\ell_\lambda^{p_s^*} + \|u_\lambda\|_{p_s^*,K}^{p_s^*}\right] + o(1). \end{aligned}$$

Then, Lemma 3.3 and (3.9) imply that
$$\lim_{\lambda\to\infty} \ell_\lambda = 0. \tag{3.13}$$
Therefore, by (K) and (3.12), we have as $n \to \infty$
$$\|u_n - u_\lambda\|_{p_s^*,K}^{p_s^*} \geq S\|K\|_\infty^{-p/p_s^*} M(\alpha_\lambda^p)\|u_n - u_\lambda\|_{p_s^*,K}^p + o(1),$$
where S is the best fractional Sobolev constant given in (1.3). Consequently, using (3.6), for all $\lambda \in \mathbb{R}^+$
$$\ell_\lambda^{p_s^*} \geq S\|K\|_\infty^{-p/p_s^*} M(\alpha_\lambda^p)\ell_\lambda^p. \tag{3.14}$$
We claim that there exists $\lambda^* > 0$ such that $\ell_\lambda = 0$ for all $\lambda \geq \lambda^*$. Otherwise there exists a sequence $k \to \lambda_k \uparrow \infty$ such that $\ell_{\lambda_k} = \ell_k > 0$. Observe that (3.11) implies in particular
$$M(\alpha_\lambda^p)\big(\alpha_\lambda^p - [u_\lambda]_{s,p}^p\big) = \ell_\lambda^{p_s^*},$$
for any $\lambda > 0$. Then, denoting $\alpha_{\lambda_k} = \alpha_k$ and $u_{\lambda_k} = u_k$, by (3.14) we get
$$(\ell_k^{p_s^*})^{ps/N} = M(\alpha_k^p)^{ps/N}\big(\alpha_k^p - [u_k]_{s,p}^p\big)^{ps/N} \geq S\|K\|_\infty^{-p/p_s^*} M(\alpha_k^p).$$
Thanks to the above inequality, $(\mathcal{M}_2)$ and (3.8), we obtain for k sufficiently large
$$\alpha_k^{p^2 s/N} \geq \big(\alpha_k^p - [u_k]_{s,p}^p\big)^{ps/N} \geq S\|K\|_\infty^{-p/p_s^*} M(\alpha_k^p)^{1-ps/N} \geq cS\|K\|_\infty^{-p/p_s^*}\alpha_k^{p(1-ps/N)},$$
where $c = b^{1-ps/N}$. Hence, since $\alpha_k > 0$ for all $k \in \mathbb{N}$, it follows that for all k sufficiently large
$$\alpha_k^{p(2ps/N-1)} \geq cS\|K\|_\infty^{-p/p_s^*}.$$
This is impossible by (3.8), being $2ps > N$ by assumption. The claim is so proved.

Therefore, for all $\lambda \geq \lambda^*$
$$\lim_{n\to\infty} \|u_n - u_\lambda\|_{p_s^*,K} = 0.$$
Consequently, by (3.12), $u_n \to u_\lambda$ in $D^{s,p}(\mathbb{R}^N)$ as $n \to \infty$ for all $\lambda \geq \lambda^*$, as required. This completes the proof of the first case.

Case $\inf_{n\in\mathbb{N}}[u_n]_{s,p} = 0$. If 0 is an accumulation point of $([u_n]_{s,p})_n$, then there is a subsequence strongly convergent to $u_\lambda = 0$ in $D^{s,p}(\mathbb{R}^N)$ and so $c_\lambda = J_\lambda(u_\lambda) = 0$, which contradicts $c_\lambda > 0$. Hence, 0 is an isolated point for the real sequence $([u_n]_{s,p})_n$. Then there is a subsequence $([u_{n_k}]_{s,p})_k$ such that $\inf_{k\in\mathbb{N}}[u_{n_k}]_{s,p} = d_\lambda > 0$, and we can proceed as before. This completes the proof of the second case and of the lemma. □

Proof of Theorem 1.2 Thanks to Lemmas 3.2 and 3.4 the functional J_λ satisfies all the assumptions of the mountain pass theorem for any $\lambda > 0$ when $\|K\|_\infty = 0$ and for any $\lambda \geq \lambda^*$, with $\lambda^* > 0$, if $\|K\|_\infty > 0$. This guarantees the existence of a critical point $u_\lambda \in D^{s,p}(\mathbb{R}^N)$ for J_λ at level c_λ. Since $J_\lambda(u_\lambda) = c_\lambda > 0 = J_\lambda(0)$ we have that $u_\lambda \neq 0$. Moreover the asymptotic behavior (1.6) holds thanks to (3.9). □

4. The Schrödinger–Kirchhoff equation (1.7)

This section is dedicated to the study of the Schrödinger–Kirchhoff equation (1.7). First, by Theorem 6.7 and Corollary 7.2 of [12] we have the following embedding result for the uniformly convex Banach space W defined in the Introduction.

Lemma 4.1. *Let (V_1) hold. If $\nu \in [p, p_s^*]$, then the embeddings*
$$W \hookrightarrow W^{s,p}(\mathbb{R}^N) \hookrightarrow L^\nu(\mathbb{R}^N)$$
are continuous. In particular, there exists a constant $C_\nu > 0$ such that
$$\|u\|_\nu \leq C_\nu\|u\|_W \quad \textit{for all } u \in W.$$
If $\nu \in [1, p_s^*)$, then the embedding $W \hookrightarrow\hookrightarrow L^\nu(B_R)$ is compact for any $R > 0$.

We say that $u \in W$ is a (weak) *solution* of problem (1.7) if

$$\begin{aligned} M([u]_{s,p}^p)\langle u, \varphi\rangle_{s,p} &+ \int_{\mathbb{R}^N} V(x)|u(x)|^{p-2}u(x)\varphi(x)dx \\ &= \int_{\mathbb{R}^N} f(x,u(x))\varphi(x)dx + \int_{\mathbb{R}^N} g(x,u(x))\varphi(x)dx + \int_{\mathbb{R}^N} h(x)\varphi(x)dx \end{aligned} \tag{4.1}$$

for any $\varphi \in W$. Obviously, also problem (1.7) has a variational structure and the underlying functional associated is I, defined in W by

$$I(u) = J(u) - H(u), \quad J(u) = \frac{1}{p}\big(\mathscr{M}([u]_{s,p}^p) + \|u\|_{p,V}^p\big),$$
$$H(u) = \int_{\mathbb{R}^N} F(x,u)dx + \int_{\mathbb{R}^N} G(x,u)dx + \int_{\mathbb{R}^N} h(x)udx.$$

By Lemma 2 of [31] we know that $J : W \to \mathbb{R}$ is of class $C^1(W)$ and

$$\langle J'(u), \varphi\rangle = M([u]_{s,p}^p)\langle u, \varphi\rangle_{s,p} + \int_{\mathbb{R}^N} V(x)|u(x)|^{p-2}u(x)\varphi(x)dx$$

for all $u, \varphi \in W$.

The next results takes somehow inspiration from Lemma 3.1 of [19].

Lemma 4.2. *Assume* (V_1) *and* $(\mathcal{F})$. *Then the functional* $\Phi(u) = \int_{\mathbb{R}^N} F(x,u)dx$ *is of class* $C^1(W)$ *and for any fixed* $u \in W$

$$\langle \Phi'(u), \varphi\rangle = \int_{\mathbb{R}^N} f(x,u(x))\varphi(x)dx \quad \textit{for all } \varphi \in W. \tag{4.2}$$

Furthermore, $\Phi' : W \to W'$ *is weak–to–strong sequentially continuous.*

Proof. First, Φ is well defined on W. Indeed, using the Hölder inequality and Lemma 4.1, we get

$$0 \le \int_{\mathbb{R}^N} F(x,u)dx \le \begin{cases} \dfrac{1}{q}\|w\|_{\wp}\|u\|_{p_s^*}^q \le \|w\|_{\wp}\dfrac{C_{p_s^*}^q}{q}\|u\|_W^q & \text{under } (f_1) \\ \dfrac{1}{q}\|w\|_\infty\|u\|_q^q \le \|w\|_\infty\dfrac{C_q^q}{q}\|u\|_W^q & \text{under } (f_2)-(a) \end{cases}.$$

Hence, denoting for simplicity by K_w either $\|w\|_{\wp}C_{p_s^*}^q/q$ or $\|w\|_\infty C_q^q/q$, we get at once

$$0 \le \int_{\mathbb{R}^N} F(x,u)dx \le K_w\|u\|_W^q. \tag{4.3}$$

Obviously, the functional Φ is Gâteaux differentiable in W and (4.2) holds for all $u, \varphi \in W$. Thus, we only need to prove that $\Phi'(u_n) \to \Phi'(u)$ in W' if $u_n \rightharpoonup u$ in W. Let $(u_n)_n \subset W$ and $u \in W$ such that $u_n \rightharpoonup u$ in W. Then there exists a constant C such that for all $n \in \mathbb{N}$

$$\|u_n\|_W \le C \quad \text{and} \quad \|u\|_W \le C.$$

First, suppose that (f_1) holds in $(\mathcal{F})$. Using Lemma 2.1 we know that $u_n \to u$ in $L^q(\mathbb{R}^M, w)$ and so $|u_n|^{q-2}u_n \to |u|^{q-2}u$ in $L^{q'}(\mathbb{R}^N, w)$, thanks to Proposition A.8 of [3]. Fixed $v \in W$, with $\|v\|_W = 1$, by the Hölder inequality and Lemma 2.1 we get as $n \to \infty$

$$\begin{aligned} \int_{\mathbb{R}^N}|f(x,u_n) - f(x,u)| \cdot |v|dx &\le \left(\int_{\mathbb{R}^N} w(x)\big||u_n|^{q-2}u_n - |u|^{q-2}u\big|^{q'}dx\right)^{1/q'}\|v\|_{q,w} \\ &\le C_w\left(\int_{\mathbb{R}^N} w(x)\big||u_n|^{q-2}u_n - |u|^{q-2}u\big|^{q'}dx\right)^{1/q'} = o(1). \end{aligned}$$

Consequently, $\Phi'(u_n) \to \Phi'(u)$ in W' as $n \to \infty$, as required.

Now suppose that (f_2)–(a) holds in $(\mathcal{F})$. Fixed $\varepsilon > 0$, there exists $R_\varepsilon > 0$ such that for $|x| \ge R_\varepsilon$

$$|f(x,u(x))| \le \varepsilon|u(x)|^{q-1} \quad \text{and} \quad |f(x,u_n(x))| \le \varepsilon|u_n(x)|^{q-1} \text{ for all } n \in \mathbb{N}. \tag{4.4}$$

Consider a subsequence $(u_{n_k})_k \subset (u_n)_n$. By Lemma 4.1 we get that $u_{n_k} \to u$ in $L^q(B_{R_\varepsilon})$. Hence, up to a further subsequence, still denoted by $(u_{n_k})_k$ for simplicity, there exists a function $\varphi \in L^q(B_{R_\varepsilon})$ such that

$$u_{n_k} \to u \text{ a.e. in } B_{R_\varepsilon} \quad \text{and} \quad |u_{n_k}| \le \varphi \text{ a.e. in } B_{R_\varepsilon} \text{ and for all } k \in \mathbb{N}.$$

Consequently, using (f_2)–(a) we have that $f(x, u_{n_k}(x)) \to f(x, u(x))$ for a.a. x in B_{R_ε} and

$$|f(x, u_{n_k})| \leq \|w\|_\infty \varphi^{q-1} \in L^{q'}(B_{R_\varepsilon}) \text{ for all } k \in \mathbb{N}.$$

The Lebesgue dominated convergence theorem implies that $f(x, u_{n_k}) \to f(x, u)$ in $L^{q'}(B_{R_\varepsilon})$ and so $f(x, u_n) \to f(x, u)$ in $L^{q'}(B_{R_\varepsilon})$ as $n \to \infty$, since the sequence $(u_{n_k})_k$ is arbitrary.

Fix $v \in W$, with $\|v\|_W = 1$. Then

$$\begin{aligned}\int_{B_{R_\varepsilon}} |f(x, u_n) - f(x, u)| \cdot |v| dx &\leq \left(\int_{B_{R_\varepsilon}} |f(x, u_n) - f(x, u)|^{q'} dx \right)^{1/q'} \|v\|_q \\ &\leq C_q \left(\int_{B_{R_\varepsilon}} |f(x, u_n) - f(x, u)|^{q'} dx \right)^{1/q'} = o(1)\end{aligned}$$

as $n \to \infty$. Now, by the Hölder inequality, Lemma 4.1, (f_2)–(a) and (4.4)

$$\begin{aligned}\int_{\mathbb{R}^N \setminus B_{R_\varepsilon}} |f(x, u_n) - f(x, u)| \cdot |v| dx &\leq \int_{\mathbb{R}^N \setminus B_{R_\varepsilon}} |f(x, u_n)| \cdot |v| dx + \int_{\mathbb{R}^N \setminus B_{R_\varepsilon}} |f(x, u)| \cdot |v| dx \\ &\leq \varepsilon \int_{\mathbb{R}^N \setminus B_{R_\varepsilon}} |u_n|^{q-1} \cdot |v| dx + \varepsilon \int_{\mathbb{R}^N \setminus B_{R_\varepsilon}} |u|^{q-1} \cdot |v| dx \\ &\leq \varepsilon \|v\|_q \big(\|u_n\|_q^{q-1} + \|u\|_q^{q-1} \big) \\ &\leq \varepsilon C_q^q \big(\|u_n\|_W^{q-1} + \|u\|_W^{q-1} \big) \leq \varepsilon \kappa,\end{aligned}$$

where $\kappa = 2C_q^q C^{q-1}$. Hence, as $n \to \infty$

$$|\langle \Phi'(u_n) - \Phi'(u), v \rangle| \leq \int_{\mathbb{R}^N} |f(x, u_n) - f(x, u)| \cdot |v| dx \leq \varepsilon \kappa + o(1).$$

Then, being $\varepsilon > 0$ arbitrary, we get at once that

$$\|\Phi'(u_n) - \Phi'(u)\|_{W'} = \sup_{v \in W, \|v\|_W = 1} |\langle \Phi'(u_n) - \Phi'(u), v \rangle| \to 0 \quad \text{as } n \to \infty.$$

This complete the proof. □

Clearly, the proof of Lemma 4.2 does not use at all (f_2)–(b) in $(\mathcal{F})$. Combining Lemma 4.2 with Lemma 3 of [31], we have that $H \in C^1(W)$ and

$$\langle H'(u), \varphi \rangle = \int_{\mathbb{R}^N} f(x, u(x)) \varphi(x) dx + \int_{\mathbb{R}^N} g(x, u(x)) \varphi(x) dx + \int_{\mathbb{R}^N} h(x) \varphi(x) dx$$

for all u, $\varphi \in W$. Consequently, the underlying functional I associated to problem (1.7) is well defined and of class $C^1(W)$.

Before proving Theorem 1.3. let us note that $(\mathcal{G})$ yields

$$|G(x, t)| \leq \varepsilon |t|^{\theta p} + C_\varepsilon |t|^r \quad \text{for a.a } x \in \mathbb{R}^N \text{ and all } t \in \mathbb{R}. \tag{4.5}$$

Lemma 4.3. *Suppose that $(\mathcal{M}_1)$, (V_1), $(\mathcal{F})$ and $(\mathcal{G})$ hold. Then there exist numbers $\alpha > 0$, $\rho > 0$ and $\delta > 0$ such that $I(u) \geq \alpha$ for all $u \in W$, with $\|u\|_W = \rho$, and for all $h \in L^{\nu'}(\mathbb{R}^N)$, with $\|h\|_{\nu'} \leq \delta$, and there exists a radial function $e \in C_0^\infty(\mathbb{R}^N)$, with $\|e\|_W > \rho$ and $I(e) < 0$.*

Proof. By the Hölder inequality, Lemma 4.1, (3.1) with $t_0 = 1$, (4.3) and (4.5), for $u \in W$, with $\|u\|_W \leq 1$, and putting $\beta = \min\{\mathscr{M}(1), 1\}/p$, we have

$$\begin{aligned}I(u) &\geq \frac{1}{p} \big(\mathscr{M}(1)[u]_{s,p}^{\theta p} + \|u\|_{p,V}^p \big) - K_w \|u\|_W^q - \varepsilon \|u\|_{\theta p}^{\theta p} - C_\varepsilon \|u\|_r^r - \|h\|_{\nu'} \|u\|_\nu \\ &\geq \beta \|u\|_W^{\theta p} - \varepsilon C_{\theta p}^{\theta p} \|u\|_W^{\theta p} - K_w \|u\|_W^q - C_\varepsilon C_r^r \|u\|_W^r - C_\nu \|h\|_{\nu'} \|u\|_W \\ &= \Big(\frac{\beta}{2} \|u\|_W^{\theta p - 1} - K_w \|u\|_W^{q-1} - C_\varepsilon C_r^r \|u\|_W^{r-1} - C_\nu \|h\|_{\nu'} \Big) \|u\|_W,\end{aligned}$$

choosing $\varepsilon = \beta / 2C_{\theta p}^{\theta p} > 0$. Define

$$\eta(t) = \frac{\beta}{2} t^{\theta p - 1} - K_w t^{q-1} - C_\varepsilon C_r^r t^{r-1} \qquad \text{for all } t \in [0, 1].$$

There exists $\rho \in (0,1]$ such that $\max_{t\in[0,1]} \eta(t) = \eta(\rho) > 0$, since $1 < \theta p < \min\{q,\, r\}$ by $(\mathcal{M})$, $(\mathcal{F})$ and $(\mathcal{G})$. Taking $\delta = \eta(\rho)/2C_\nu$, we obtain that $I(u) \ge \alpha = \rho\eta(\rho)/2 > 0$ for all $u \in W$, with $\|u\|_W = \rho$ and for all $h \in L^{\nu'}(\mathbb{R}^N)$, with $\|h\|_{\nu'} \le \delta$.

Fix a radial function v of class $C_0^\infty(\mathbb{R}^N)$, with $\|v\|_W = 1$. For a.a $x \in \mathbb{R}^N$, by $(\mathcal{F})$ we obtain that the function $t \mapsto t^{-q}F(x, tv(x))$ is nondecreasing in $\mathbb{R}^+$ and so

$$F(x, tv(x)) \ge t^q F(x, v(x)) \quad \text{ for all } t \ge 1.$$

Clearly $\int_{\mathbb{R}^N} F(x, v)dx > 0$ by $(\mathcal{F})$ and the fact that $v \in C_0^\infty(\mathbb{R}^N)$ and $\|v\|_W = 1$. Hence, as $t \to \infty$

$$\int_{\mathbb{R}^N} F(x, tv)dx \ge t^q \int_{\mathbb{R}^N} F(x, v)dx \to \infty. \tag{4.6}$$

Similarly, for a.a $x \in \mathbb{R}^N$ the function $t \mapsto t^{-\mu}G(x, tv(x))$ is nondecreasing in $\mathbb{R}^+$ by $(\mathcal{G})$ in both cases (i) and (ii). Then

$$\int_{\mathbb{R}^N} G(x, tv)dx \ge t^\mu \int_{\mathbb{R}^N} G(x, v)dx \quad \text{ for all } t \ge 1. \tag{4.7}$$

Moreover, (1.3) implies that $[v]_{s,p} > 0$. Consequently, by (2.3) with $t_0 = 1$, (4.6) and (4.7), putting again $\beta = \max\{\mathscr{M}(1), 1\}/p$, we have for all $t \ge 1$

$$\begin{aligned} I(tv) &\le \frac{1}{p}\Big(\mathscr{M}(1)t^{\theta p}[v]_{s,p}^{\theta p} + t^p\|v\|_{p,V}^p\Big) - \int_{\mathbb{R}^N} F(x, tv)dx - \int_{\mathbb{R}^N} G(x, tv)dx - t\int_{\mathbb{R}^N} h(x)vdx \\ &\le \beta\, t^{\theta p} - t\int_{\mathbb{R}^N} h(x)vdx - \begin{cases} \left(t^q \displaystyle\int_{\mathbb{R}^N} F(x, v)dx + t^\mu \int_{\mathbb{R}^N} G(x, v)dx\right) & \text{under } (\mathcal{G})-(i) \\ t^q \displaystyle\int_{\mathbb{R}^N} F(x, v)dx & \text{under } (\mathcal{G})-(ii) \end{cases} \\ &\to -\infty, \end{aligned}$$

since $1 < \theta p < q$ by $(\mathcal{M})$, $(\mathcal{F})$ and Lemma 3.1 since $M(0) = 0$. Choosing $e = \tau_0 v$, with $\tau_0 > 0$ large enough, we get at once that e is radial, $\|e\|_W > \rho$ and $I(e) < 0$. This completes the proof. $\square$

Let us recall a standard definition. Let X be a real Banach space and $\mathcal{I}$ be a functional of class $C^1(X)$. We say that $\mathcal{I}$ *satisfies the Palais–Smale condition*, if any Palais–Smale sequence $(u_n)_n \subset X$, that is with the properties that $(\mathcal{I}(u_n))_n$ is bounded and $\mathcal{I}'(u_n) \to 0$ in X' as $n \to \infty$, admits a subsequence which (strongly) converges in X.

It is exactly for proving that I possesses this property in W that we essentially use (V_2). This is also evident from the next crucial lemma.

Lemma 4.4 (Theorem 2.1 of [31])**.** *Let (V_1)–(V_2) hold and let $\nu \in [p, p_s^*)$ be a fixed exponent. For any bounded sequence $(v_n)_n$ in W there exists $v \in W$ such that, up to a subsequence,*

$$v_n \to v \quad \textit{strongly in } L^\nu(\mathbb{R}^N) \quad \textit{as } n \to \infty.$$

Lemma 4.5. *Let $(\mathcal{M}_1)$, (V_1)–(V_2), $(\mathcal{F})$ and $(\mathcal{G})$ hold. Then I satisfies the Palais–Smale condition.*

Proof. Let $(u_n)_n$ be a Palais–Smale sequence of I in W. Then there exists $C > 0$ such that

$$|\langle I'(u_n), u_n\rangle| \le C\|u_n\|_W \quad \text{and} \quad |I(u_n)| \le C \quad \text{for all } n \in \mathbb{N}. \tag{4.8}$$

Suppose that $\theta p < \mu < q$. Thus, by $(\mathcal{M})$, $(\mathcal{F})$, $(\mathcal{G})$–(i) and (4.8) we get

$$\begin{aligned} C + C\|u_n\|_W &\ge I(u_n) - \frac{1}{\mu}\langle I'(u_n), u_n\rangle \\ &\ge \left(\frac{1}{\theta p} - \frac{1}{\mu}\right)M([u_n]_{s,p}^p)[u_n]_{s,p}^p + \left(\frac{1}{p} - \frac{1}{\mu}\right)\|u_n\|_{p,V}^p - \int_{\mathbb{R}^N} F(x, u_n)dx \\ &\quad + \frac{1}{\mu}\int_{\mathbb{R}^N} f(x, u_n)u_n dx - \int_{\mathbb{R}^N} G(x, u_n)dx + \frac{1}{\mu}\int_{\mathbb{R}^N} g(x, u_n)u_n dx \\ &\quad - \frac{\mu - 1}{\mu}\int_{\mathbb{R}^N} h(x)u_n dx \\ &\ge \left(\frac{1}{\theta p} - \frac{1}{\mu}\right)M([u_n]_{s,p}^p)[u_n]_{s,p}^p + \left(\frac{1}{p} - \frac{1}{\mu}\right)\|u_n\|_{p,V}^p - \int_{\mathbb{R}^N} F(x, u_n)dx \end{aligned}$$

$$
\begin{aligned}
&+\frac{1}{q}\int_{\mathbb{R}^N} f(x,u_n)u_n dx - \|h\|_{\nu'}\|u_n\|_\nu \\
&\geq \left(\frac{1}{\theta p}-\frac{1}{\mu}\right)M([u_n]_{s,p}^p)[u_n]_{s,p}^p + \left(\frac{1}{p}-\frac{1}{\mu}\right)\|u_n\|_{p,V}^p - \|h\|_{\nu'}\|u_n\|_\nu.
\end{aligned}
$$

Similarly, if $q \leq \mu < p_s^*$, replacing now $(\mathcal{G})$–(i) by $(\mathcal{G})$–(ii), we have

$$
\begin{aligned}
C + C\|u_n\|_W &\geq I(u_n) - \frac{1}{q}\langle I'(u_n), u_n\rangle \\
&\geq \left(\frac{1}{\theta p}-\frac{1}{q}\right)M([u_n]_{s,p}^p)[u_n]_{s,p}^p + \left(\frac{1}{p}-\frac{1}{q}\right)\|u_n\|_{p,V}^p - \int_{\mathbb{R}^N} G(x,u_n)dx \\
&\quad + \frac{1}{\mu}\int_{\mathbb{R}^N} g(x,u_n)u_n dx - \|h\|_{\nu'}\|u_n\|_\nu \\
&\geq \left(\frac{1}{\theta p}-\frac{1}{q}\right)M([u_n]_{s,p}^p)[u_n]_{s,p}^p + \left(\frac{1}{p}-\frac{1}{q}\right)\|u_n\|_{p,V}^p - \|h\|_{\nu'}\|u_n\|_\nu.
\end{aligned}
$$

Therefore, we have proved that for all n

$$
\begin{aligned}
C + C\|u_n\|_W \geq \min&\left\{\frac{1}{\theta p}-\frac{1}{q}, \frac{1}{\theta p}-\frac{1}{\mu}\right\}M([u_n]_{s,p}^p)[u_n]_{s,p}^p \\
&+ \min\left\{\frac{1}{p}-\frac{1}{q}, \frac{1}{p}-\frac{1}{\mu}\right\}\|u_n\|_{p,V}^p - \|h\|_{\nu'}\|u_n\|_\nu.
\end{aligned}
\tag{4.9}
$$

Due to the degenerate nature of (1.7), we have to consider two situations. Either $\inf_{n\in\mathbb{N}}[u_n]_{s,p} = d > 0$ or $\inf_{n\in\mathbb{N}}[u_n]_{s,p} = 0$. Hence, let us divide the proof in two cases.

Case $\inf_{n\in\mathbb{N}}[u_n]_{s,p} = d > 0$. By $(\mathcal{M}_1)$, with $\tau = d^p$, there exists $m = m_\tau > 0$ such that

$$
M([u_n]_{s,p}^p) \geq m \quad \text{for any } n \in \mathbb{N}. \tag{4.10}
$$

Thus, (4.9) yields at once that for all n

$$
C + C\|u_n\|_W \geq \min\left\{m\left(\frac{1}{\theta p}-\frac{1}{q}\right), m\left(\frac{1}{\theta p}-\frac{1}{\mu}\right), \frac{1}{p}-\frac{1}{q}, \frac{1}{p}-\frac{1}{\mu}\right\}\|u_n\|_W^p - \gamma\|u_n\|_W,
$$

where $\gamma = C_\nu\|h\|_{\nu'}$ by Lemma 4.1. Consequently, $(u_n)_n$ is bounded in W, since $1 < p < \theta p < \min\{q, \mu\}$ by $(\mathcal{M})$, $(\mathcal{F})$, $(\mathcal{G})$ and Lemma 3.1, being $M(0) = 0$.

Hence, there exists a function $u \in W$ such that, going if necessary to a subsequence,

$$
u_n \rightharpoonup u \quad \text{in } W, \tag{4.11}
$$

being W a uniformly convex Banach space. Let us prove that $(u_n)_n$ converges strongly to u in W.

Since M is continuous in $\mathbb{R}_0^+$, then $\big(M([u_n]_{s,p}^p) - M([u]_{s,p}^p)\big)_n$ is bounded in $\mathbb{R}$. Hence, (4.11) gives that

$$
\lim_{n\to\infty}\big[M([u_n]_{s,p}^p) - M([u]_{s,p}^p)\big]\langle u, u_n - u\rangle_{s,p} = 0. \tag{4.12}
$$

Moreover, Lemma 4.2 implies that

$$
\lim_{n\to\infty}\int_{\mathbb{R}^N}(f(x,u_n) - f(x,u))(u_n - u)dx = 0. \tag{4.13}
$$

Thanks also to (V_2), going if necessary to a further subsequence, still denoted by $(u_n)_n$, we have

$$
u_n \to u \qquad \text{in } L^{\theta p}(\mathbb{R}^N) \text{ and in } L^r(\mathbb{R}^N) \tag{4.14}
$$

by Lemma (4.4), since $p < \theta p < r < p_s^*$ by $(\mathcal{M})$, $(\mathcal{G})$ and Lemma 3.1, being $M(0) = 0$. Furthermore, using $(\mathcal{G})$, with $\varepsilon = 1/\theta p$,

$$
|g(x,t)| \leq |t|^{\theta p - 1} + K_r|t|^{r-1} \quad \text{for a.a. } x \in \mathbb{R}^N \text{ and all } t \in \mathbb{R}, \tag{4.15}
$$

where $K_r = rC_{1/\theta p}$. Then, the Hölder inequality, (4.14) and (4.15) give as $n \to \infty$

$$
\left|\int_{\mathbb{R}^N}(g(x,u_n) - g(x,u))(u_n - u)dx\right|
$$

$$\le \int_{\mathbb{R}^N} \big(|u_n|^{\theta p-1} + |u|^{\theta p-1} + K_r|u_n|^{r-1} + K_r|u|^{r-1}\big)\big|u_n - u\big|dx$$
$$\le \big(\|u_n\|_{\theta p}^{\theta p-1} + \|u\|_{\theta p}^{\theta p-1}\big)\|u_n - u\|_{\theta p} + K_r\big(\|u_n\|_r^{r-1} + \|u\|_r^{r-1}\big)\|u_n - u\|_r = o(1).$$

Clearly $\langle I'(u_n) - I'(u), u_n - u\rangle \to 0$ as $n \to \infty$ by (4.11) and the fact that $I'(u_n) \to 0$ in W'. Therefore, combining with (4.11)–(4.13), we obtain as $n \to \infty$

$$\begin{aligned} o(1) = \langle I'(u_n) - I'(u), u_n - u\rangle &= M([u_n]_{s,p}^p)\big(\langle u_n, u_n - u\rangle_{s,p} - \langle u, u_n - u\rangle_{s,p}\big) \\ &\quad + [M([u_n]_{s,p}^p) - M([u]_{s,p}^p)]\langle u, u_n - u\rangle_{s,p} \\ &\quad + \int_{\mathbb{R}^N} V(x)(|u_n|^{p-2}u_n - |u|^{p-2}u)(u_n - u)dx + o(1) \\ &= M([u_n]_{s,p}^p)\big(\langle u_n, u_n - u\rangle_{s,p} - \langle u, u_n - u\rangle_{s,p}\big) \\ &\quad + \int_{\mathbb{R}^N} V(x)(|u_n|^{p-2}u_n - |u|^{p-2}u)(u_n - u)dx + o(1), \end{aligned}$$

that is

$$\lim_{n\to\infty}\Bigg(M([u_n]_{s,p}^p)\big(\langle u_n, u_n - u\rangle_{s,p} - \langle u, u_n - u\rangle_{s,p}\big) + \int_{\mathbb{R}^N} V(x)(|u_n|^{p-2}u_n - |u|^{p-2}u)(u_n - u)dx\Bigg) = 0. \tag{4.16}$$

In particular, $M([u_n]_{s,p}^p)\big(\langle u_n, u_n-u\rangle_{s,p}-\langle u, u_n-u\rangle_{s,p}\big) \ge 0$ and similarly also $V(x)(|u_n|^{p-2}u_n-|u|^{p-2}u)(u_n-u) \ge 0$ by convexity, $(\mathcal{M}_1)$ and (V_1). Then, using (4.10) and (4.16), we get

$$\begin{aligned} &\lim_{n\to\infty}\big(\langle u_n, u_n - u\rangle_{s,p} - \langle u, u_n - u\rangle_{s,p}\big) = 0, \\ &\lim_{n\to\infty}\int_{\mathbb{R}^N} V(x)(|u_n|^{p-2}u_n - |u|^{p-2}u)(u_n - u)dx = 0. \end{aligned} \tag{4.17}$$

Let us recall the well known Simon inequalities

$$|\xi - \eta|^p \le \begin{cases} k_p\big(|\xi|^{p-2}\xi - |\eta|^{p-2}\eta\big)\cdot(\xi - \eta), & p \ge 2, \\ K_p\big[\big(|\xi|^{p-2}\xi - |\eta|^{p-2}\eta\big)\cdot(\xi - \eta)\big]^{p/2}(|\xi|^p + |\eta|^p)^{(2-p)/2}, & 1 < p < 2, \end{cases}$$

for all $\xi, \eta \in \mathbb{R}^N$, where k_p and K_p are positive constants depending only on p. Let us divide the proof into two cases.

Assume that $p \ge 2$. Taking $\xi = u_n(x) - u_n(y)$ and $\eta = u(x) - u(y)$ in the Simon inequality, we get by (4.17)

$$\begin{aligned} [u_n - u]_{s,p}^p &= \iint_{\mathbb{R}^{2N}} |u_n(x) - u_n(y) - u(x) + u(y)|^p|x - y|^{-(N+ps)}dxdy \\ &\le k_p\big(\langle u_n, u_n - u\rangle_{s,p} - \langle u, u_n - u\rangle_{s,p}\big) = o(1) \end{aligned}$$

as $n \to \infty$. Similarly, as $n \to \infty$

$$\|u_n - u\|_{p,V}^p \le k_p\int_{\mathbb{R}^N} V(x)(|u_n|^{p-2}u_n - |u|^{p-2}u)(u_n - u)dx = o(1),$$

thanks to (V_1) and (4.17). In conclusion, $\|u_n - u\|_W \to 0$ as $n \to \infty$, as required.

Assume that $1 < p < 2$. By (4.11) there exists $\sigma > 0$ such that $[u_n]_{s,p} \le \sigma$ for all $n \in \mathbb{N}$. Then, using the Simon, the Hölder inequalities and (4.17), we have as $n \to \infty$

$$\begin{aligned} [u_n - u]_{s,p}^p &\le K_p\big(\langle u_n, u_n - u\rangle_{s,p} - \langle u, u_n - u\rangle_{s,p}\big)^{p/2}\big([u_n]_{s,p}^p + [u]_{s,p}^p\big)^{(2-p)/2} \\ &\le K_p\big(\langle u_n, u_n - u\rangle_{s,p} - \langle u, u_n - u\rangle_{s,p}\big)^{p/2}\big([u_n]_{s,p}^{p(2-p)/2} + [u]_{s,p}^{p(2-p)/2}\big) \\ &\le \kappa\big(\langle u_n, u_n - u\rangle_{s,p} - \langle u, u_n - u\rangle_{s,p}\big)^{p/2} = o(1) \end{aligned}$$

where $\kappa = 2K_p\sigma^{p(2-p)/2}$ and where we have applied the following elementary inequality

$$(a + b)^{(2-p)/2} \le a^{(2-p)/2} + b^{(2-p)/2} \quad \text{for all } a, b \ge 0, \tag{4.18}$$

Similarly, by (4.11) there exists $U > 0$ such that $\|u_n\|_{p,V} \le U$ for all $n \in \mathbb{N}$. Moreover, by the Simon, the Hölder inequalities, (4.17) and (4.18), as $n \to \infty$

$$\|u_n - u\|_{p,V}^p \le L\left(\int_{\mathbb{R}^N} V(x)\big(|u_n|^{p-2}u_n - |u|^{p-2}u\big)(u_n - u)dx\right)^{p/2} = o(1),$$

where $L = 2K_pU^{p(2-p)/2}$. Hence, $\|u_n - u\|_W \to 0$ as $n \to \infty$ also for $1 < p < 2$. This completes the proof of the first case.

Case $\inf\limits_{n\in\mathbb{N}}[u_n]_{s,p} = 0$. If 0 is an isolated point for the real sequence $([u_n]_{s,p})_n$, then there is a subsequence $([u_{n_k}]_{s,p})_k$ such that $\inf\limits_{k\in\mathbb{N}}[u_{n_k}]_{s,p} = d > 0$, and we can proceed as before.

Otherwise, if 0 is an accumulation point of the sequence $([u_n]_{s,p})_n$, then thanks to (1.3) there is a subsequence, still relabeled $(u_n)_n$, such that

$$u_n \to 0 \quad \text{in } D^{s,p}(\mathbb{R}^N), \text{ in } L^{p_s^*}(\mathbb{R}^N) \text{ and a.e. in } \mathbb{R}^N. \tag{4.19}$$

We claim that $(u_n)_n$ converges strongly to 0 in W. To this aim, we need only to show that $\|u_n\|_{p,V} \to 0$ thanks to (4.19).

Now, (4.9) and (4.19) yield that as $n \to \infty$

$$C + C\|u_n\|_{p,V} + o(1) \ge \min\left\{\frac{1}{p} - \frac{1}{q}, \frac{1}{p} - \frac{1}{\mu}\right\}\|u_n\|_{p,V}^p - \ell\|u_n\|_{p,V} + o(1),$$

where now $\ell = C_\nu\|h\|_{\nu'}$ by Lemma 4.1. In particular, there exists a constant $\Lambda > 0$ such that for all $n \in \mathbb{N}$

$$\min\left\{\frac{1}{p} - \frac{1}{q}, \frac{1}{p} - \frac{1}{\mu}\right\}\|u_n\|_{p,V}^p - (C + \ell)\|u_n\|_{p,V} \le \Lambda.$$

Hence, $(u_n)_n$ is bounded in $L^p(\mathbb{R}^N, V)$ and so in W. Thus, by (4.19) and Lemma 4.1

$$u_n \rightharpoonup 0 \text{ in } W \text{ and in } L^\nu(\mathbb{R}^N). \tag{4.20}$$

By (4.20) and Lemma 4.2

$$\lim_{n\to\infty}\int_{\mathbb{R}^N} f(x,u_n)u_n dx = 0 \quad \text{and} \quad \lim_{n\to\infty}\int_{\mathbb{R}^N} h(x)u_n dx = 0, \tag{4.21}$$

since $h \in L^{\nu'}(\mathbb{R}^N)$.

Clearly (4.15) yields

$$\left|\int_{\mathbb{R}^N} g(x,u_n)u_n dx\right| \le \|u_n\|_{\theta p}^{\theta p} + K_r\|u_n\|_r^r = o(1) \quad \text{as } n \to \infty, \tag{4.22}$$

by virtue of (V_2), Lemma 4.4 and (4.19), since $p < \theta p < r < p_s^*$ by $(\mathcal{M})$, $(\mathcal{G})$ and Lemma 3.1, being $M(0) = 0$.

Obviously, $\langle I'(u_n), u_n\rangle \to 0$ as $n \to \infty$, by (4.20) and the fact that $I'(u_n) \to 0$ in W'. Hence, using the continuity of M and (4.19)–(4.22) we have as $n \to \infty$

$$\begin{aligned} o(1) = \langle I'(u_n), u_n\rangle &= M([u_n]_{s,p}^p)[u_n]_{s,p}^p + \|u_n\|_{p,V}^p - \int_{\mathbb{R}^N} f(x,u_n)u_n dx \\ &\qquad - \int_{\mathbb{R}^N} g(x,u_n)u_n dx - \int_{\mathbb{R}^N} h(x)u_n dx \\ &= \|u_n\|_{p,V}^p + o(1). \end{aligned}$$

This shows the claim.

Therefore, I satisfies the Palais–Smale condition also in this second case and this completes the proof. $\square$

Proof of Theorem 1.3 The proof is divided into two steps.

Step 1. Let us prove that there exists $u_0 \in W$ such that $I'(u_0) = 0$ and $I(u_0) < 0$.

First, we claim that there exists a function $\psi \in C_0^\infty(\mathbb{R}^N)$ such that

$$\int_{\mathbb{R}^N} h(x)\psi(x)dx > 0. \tag{4.23}$$

Since $h \in L^{\nu'}(\mathbb{R}^N) \setminus \{0\}$, the function

$$\phi(x) = \begin{cases} |h(x)|^{\nu'-2}h(x), & \text{if } h(x) \neq 0, \\ 0, & \text{if } h(x) = 0 \end{cases} \in L^{\nu}(\mathbb{R}^N).$$

Then, there exists a sequence $(h_n)_n$ in $C_0^\infty(\mathbb{R}^N)$ such that $h_n \to \phi$ strongly in $L^\nu(\mathbb{R}^N)$, since $C_0^\infty(\mathbb{R}^N)$ is dense in $L^\nu(\mathbb{R}^N)$. Hence, there exists $n_0 \in \mathbb{N}$ such that

$$\|h_{n_0} - \phi\|_\nu \leq \frac{1}{2}\|h\|_{\nu'}^{\nu'-1}.$$

Thus, by the Hölder inequality, we have

$$\int_{\mathbb{R}^N} h_{n_0}(x)h(x)dx \geq -\|h_{n_0} - \phi\|_\nu \|h\|_{\nu'} + \int_{\mathbb{R}^N} |h(x)|^{\nu'}dx \geq \frac{1}{2}\|h\|_{\nu'}^{\nu'} > 0,$$

since $h \not\equiv 0$. The claim is proved taking $\psi = h_{n_0}$.

Now, putting $M_\rho = \max_{\xi \in [0,\rho]} M(\xi^p)$, where $\rho > 0$ is the number given in Lemma 4.3, by (4.3), (4.5), with $\varepsilon = 1$, and (4.23) we have

$$\begin{aligned} I(t\psi) &\leq \frac{1}{p}\mathscr{M}([t\psi]_{s,p}^p) + \frac{t^p}{p}\|\psi\|_{p,V}^p - \int_{\mathbb{R}^N} G(x,t\psi)dx - t\int_{\mathbb{R}^N} h(x)\psi(x)dx \\ &\leq M_\rho \frac{t^p}{p}[\psi]_{s,p}^p + \frac{t^p}{p}\|\psi\|_{p,V}^p + t^{\theta p}\|\psi\|_{\theta p}^{\theta p} + C_1 t^r \|\psi\|_r^r - t\int_{\mathbb{R}^N} h(x)\psi(x)dx < 0, \end{aligned}$$

for $t \in (0,1)$ small enough, since $1 < p < \theta p < r$ by $(\mathcal{M})$, $(\mathcal{G})$ and Lemma 3.1, being $M(0) = 0$. Thus, we obtain

$$c_0 = \inf\{I(u) : u \in \overline{B}_\rho\} < 0,$$

where $B_\rho = \{u \in W : \|u\|_W < \rho\}$. Then, by the Ekeland variational principle in $\overline{B}_\rho$ and Lemma 4.3, there exists a sequence $(v_n)_n \subset B_\rho$ such that

$$c_0 \leq I(v_n) \leq c_0 + \frac{1}{n} \quad \text{and} \quad I(v) \geq I(v_n) - \frac{1}{n}\|v - v_n\|_W \tag{4.24}$$

for all $n \in \mathbb{N}$ and for any $v \in \overline{B}_\rho$. Fixed $n \in \mathbb{N}$, for all $w \in S_W$, where

$$S_W = \{u \in W : \|u\|_W = 1\},$$

and for all $\sigma > 0$ so small that $v_n + \sigma\, w \in \overline{B}_\rho$, we have

$$I(v_n + \sigma\, w) - I(v_n) \geq -\frac{\sigma}{n},$$

by (4.24). Since I is Gâteaux differentiable in W, we get

$$\langle I'(v_n), w\rangle = \lim_{\sigma \to 0} \frac{I(v_n + \sigma\, w) - I(v_n)}{\sigma} \geq -\frac{1}{n}$$

for all $w \in S_W$. Hence

$$|\langle I'(v_n), w\rangle| \leq \frac{1}{n},$$

since $w \in S_W$ is arbitrary. Consequently, $I'(v_n) \to 0$ in W' as $n \to \infty$ and so $(v_n)_n$ is a bounded (PS) sequence of I in B_ρ. Thus, Lemmas 4.3 and 4.5 imply the existence of a function $u_0 \in B_\rho$ such that $I'(u_0) = 0$ and $I(u_0) = c_0 < 0$.

Step 2. Let us prove that there exists $u_1 \in W$ such that $I'(u_1) = 0$ and $I(u_1) > 0$.

By Lemma 4.3 and the mountain pass theorem, there exists a sequence, say again $(v_n)_n$, in W such that as $n \to \infty$

$$I(v_n) \to c_1 \quad \text{and} \quad I'(v_n) \to 0, \quad \text{where}$$
$$c_1 = \inf_{\gamma \in \Gamma} \max_{t \in [0,1]} I(\gamma(t)) > 0,$$
$$\Gamma = \{\gamma \in C([0,1], W) : \gamma(0) = 0, \gamma(1) = e\}$$

and $e \in W$ is the function constructed in Lemma 4.3. Then, Lemma 4.5 implies the existence of a function $u_1 \in W$ such that $I'(u_1) = 0$ and $I(u_1) = c_1 > 0$.

Finally, since $I(u_0) = c_0 < I(0) = 0 < I(u_1) = c_1$, the solutions u_0 and u_1 of (1.7) are nontrivial and independent. This concludes the proof.

5. The Schrödinger–Kirchhoff equation (1.7) in the radial case

In this section we prove the main existence result for the (1.7) in the radial case. To apply the mountain pass theorem and the Ekeland variational principle, we need the following embedding result obtained combining Theorem II.1 of [20] with Lemma 4.1.

Lemma 5.1. *Let $N \geq 2$. For any $p < \nu < p_s^*$, the embedding*

$$W_{\mathrm{rad}} \hookrightarrow\hookrightarrow L^{\nu}(\mathbb{R}^N)$$

is compact, where $W_{\mathrm{rad}} = \{u \in W : u$ is radially symmetric with respect $0\}$.

Throughout the section we assume that V, f, g and h are radially symmetric functions in x, and that $(\mathcal{M}_1)$, (V_1), $(\mathcal{F})$ and $(\mathcal{G})$ hold, without further mentioning. Again, the geometry stated in Lemma 4.3 continues to hold. The significant changes now occur in showing the validity of the Palais–Smale condition. To overcome the non–compactness of the embedding $W \hookrightarrow L^{\nu}(\mathbb{R}^N)$, $p < \nu < p_s^*$, we need to restrict the study searching solutions of (1.7) in W_{rad}.

Lemma 5.2. *Let $N \geq 2$ and $M(0) = 0$. Then I satisfies the Palais–Smale condition in W_{rad}.*

Proof. Fix $(u_n)_n$ any Palais–Smale sequence for I in W_{rad}. Then, proceeding exactly as in the proof of Lemma 4.5, we arrive at the same conclusion, replacing (V_2) and Lemma 4.4 by Lemma 5.1, being $1 < p < \theta p < \min\{q, \mu\}$ by $(\mathcal{M})$, $(\mathcal{F})$, $(\mathcal{G})$ and Lemma 3.1 since $M(0) = 0$. □

To prove the existence of radial solution for problem (1.7), we need the following result.

Lemma 5.3. *Let $N \geq 2$ and $M(0) = 0$. Then there exists a number $\delta > 0$ such that for all radial perturbation $h \in L^{\nu'}(\mathbb{R}^N)$, with $\|h\|_{\nu'} \leq \delta$, problem (1.7) admits a nontrivial mountain pass radial solution u_0 in W_{rad}. If furthermore h is nontrivial, then (1.7) possesses a second independent nontrivial radial solution u_1 in W_{rad}. More precisely, u_0 and u_1 are nontrivial critical points of the underlying functional I restricted to W_{rad}, that is*

$$\begin{aligned} M([u_k]_{s,p}^p)\langle u_k, \varphi\rangle_{s,p} &+ \int_{\mathbb{R}^N} V(x)|u_k(x)|^{p-2}u_k(x)\varphi(x)dx \\ &= \int_{\mathbb{R}^N} f(x, u_k(x))\varphi(x)dx + \int_{\mathbb{R}^N} g(x, u_k(x))\varphi(x)dx + \int_{\mathbb{R}^N} h(x)\varphi(x)dx \end{aligned} \tag{5.1}$$

for any $\varphi \in W_{\mathrm{rad}}$ and for $k = 0, 1$.

Proof. The proof of this lemma is divided into two steps.

Step 1. As in the proof of Theorem 1.3, since $h \in L^{\nu'}(\mathbb{R}^N)$ is a radial function and $h \neq 0$, we can choose a function $\psi \in W_{\mathrm{rad}}$ such that $\int_{\mathbb{R}^N} h(x)\psi(x)dx > 0$. Then

$$c_0 = \inf\{I(u) : u \in \overline{B}_\rho\} < 0,$$

where $\overline{B}_\rho = \{u \in W_{\mathrm{rad}} : \|u\|_W < \rho\}$ and $\rho > 0$ is the number given in Lemma 4.3. Thus, the Ekeland variational principle in $\overline{B}_\rho$, Lemma 4.3 and Lemma 5.2 imply the existence of a function $u_0 \in B_\rho$ such that $I(u_0) = c_0 < 0$ and $I'(u_0) = 0$ in W_{rad}.

Step 2. By Lemmas 4.3 and 5.2 and the mountain pass theorem, there exists a function $u_1 \in W_{\mathrm{rad}}$ such that $I(u_1) = c_1 > 0$ and $I'(u_1) = 0$ in W_{rad}, where

$$c_1 = \inf_{\gamma\in\Gamma} \max_{t\in[0,1]} I(\gamma(t)) > 0,$$
$$\Gamma = \{\gamma \in C([0, 1], W_{\mathrm{rad}}) : \gamma(0) = 0, \gamma(1) = e\}$$

and $e \in W_{\mathrm{rad}}$ is the function constructed in Lemma 4.3.

Hence, we have that $u_0, u_1 \neq 0$ are two independent radial solutions of (1.7) in the sense of definition (5.1). This concludes the proof. □

Observe that, up to this moment, the functions u_0 and u_1 given by Lemma 5.3 are solutions of problem (1.7) only in the $W_{\rm rad}$ sense. Now, let us show that these functions are solutions of (1.7) in the whole space W, that is in sense of definition (4.1). To this aim we use a version of the well known *principle of symmetric criticality*, due to Palais in [28], in the form proved in [11] which holds in reflexive strictly convex Banach spaces.

Let $X = (X, \|\cdot\|_X)$ be a reflexive strictly convex Banach space. Then, thanks to the Hahn–Banach theorem, for any $f \in X'$ there exists a unique $u_0 \in X$ such that

$$\langle f, u_0\rangle = \|u_0\|_X^2 \quad \text{and} \quad \|f\|_{X'} = \|u_0\|_X, \tag{5.2}$$

where $\langle \cdot\,,\,\cdot\rangle$ denotes the the dual pairing between X' and X. Suppose that G is a subgroup of isometries $g : X \to X$, that is g is linear and $\|gu\|_X = \|u\|_X$ for all $u \in X$. Consider the G–invariant closed subspace of X

$$\Sigma = \{u \in X : gu = u \text{ for all } g \in G\}.$$

Lemma 5.4 (Proposition 3.1 of [11]). *Let X, G and Σ be as before and let $\mathcal{I}$ be a C^1 functional defined on X such that $\mathcal{I} \circ g = \mathcal{I}$ for all $g \in G$. Then $u \in \Sigma$ is a critical point of $\mathcal{I}$ if and only if u is a critical point of $\mathcal{I}|_\Sigma$.*

The proof of Lemma (5.4) is a consequence of (5.2) and of the arguments used in the proof of Proposition 3.1 of [11]. Finally, we have all the ingredients to complete the proof of Theorem 1.4.

Proof of Theorem 1.4 Let $SO(N)$ denote the special orthogonal group, that is

$$SO(N) = \{A \in \mathbb{R}^{N\times N} : A^t A = I_N \text{ and } \det A = 1\}.$$

Next, consider the following subgroup of linear operators of W in itself

$$G = \{a : W \to W : au = u \circ A, \text{ where } A \in SO(N)\}.$$

Observe that G is a subgroup of isometries of W. In fact fixed u in W, for all $a \in G$

$$\begin{aligned}\|au\|_W^p &= \iint_{\mathbb{R}^{2N}} \frac{|u(Ax) - u(Ay)|^p}{|x-y|^{N+ps}} dxdy + \int_{\mathbb{R}^N} V(x)|u(Ax)|^p dx \\ &= \iint_{\mathbb{R}^{2N}} \frac{|u(x') - u(y')|^p}{|x'-y'|^{N+ps}} dx'dy' + \int_{\mathbb{R}^N} V(x')|u(x')|^p dx' = \|u\|_W^p,\end{aligned}$$

since $|x-y| = |A(x-y)| = |Ax - Ay| = |x'-y'|$, $\det A = 1$ and V is a radial function. Moreover, it results that

$$W_{\rm rad} = \{u \in W : au = u \text{ for all } a \in G\}.$$

To apply Lemma 5.4 to the functional I, we need to show that $I \circ a = I$ for all $a \in G$. As before, fixed $u \in W$ for all $a \in G$ we have

$$\begin{aligned}(I \circ a)(u) &= \frac{1}{p}(\mathscr{M}([au]_{s,p}^p) + \|au\|_{p,V}^p) - \int_{\mathbb{R}^N} F(x, u(Ax))dx \\ &\quad - \int_{\mathbb{R}^N} G(x, u(Ax))dx - \int_{\mathbb{R}^N} h(x)u(Ax)dx \\ &= \frac{1}{p}(\mathscr{M}([u]_{s,p}^p) + \|u\|_{p,V}^p) - \int_{\mathbb{R}^N} F(x', u(x'))dx' \\ &\quad - \int_{\mathbb{R}^N} G(x', u(x'))dx' - \int_{\mathbb{R}^N} h(x')u(x')dx' = I(u),\end{aligned}$$

since V, f, g and h are radial functions in x. Hence, I satisfies Lemma 5.4, with $X = W$ and $\Sigma = W_{\rm rad}$.

By Lemma 5.3 we know that u_0, u_1 are critical points of $I|_{W_{\rm rad}}$, that is

$$\langle I'(u_k), \varphi\rangle = 0 \quad \text{for any } \varphi \in W_{\rm rad} \text{ and for } k = 0, 1.$$

Then, using Lemma 5.4, we get that u_0 and u_1 are critical points of I in the whole space W. Thus, u_0 and u_1 are solutions of the problem (1.7) in the sense of definition (4.1). $\square$

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