ANALYTICAL VALIDATION OF A 2+1 DIMENSIONAL CONTINUUM MODEL FOR EPITAXIAL GROWTH WITH ELASTIC SUBSTRATE

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Abstract. We consider the evolution equation

$$h_t = \Delta[\mathcal{F}^{-1}(-aE\mathcal{F}(h)) - r/h^2 - \Delta h], \tag{1}$$

introduced in [21] by Tekalign and Spencer to describe the heteroepitaxial growth of a two-dimensional thin film on an elastic substrate. In the expression above, h denotes the surface height of the film, $\mathcal{F}$ is the Fourier transform, and a, E, r are positive material constants. For simplicity, we set $aE = r = 1$. As this equation does not have any particular structure, its analysis is quite challenging. Therefore, we introduce the auxiliary equation (with c being a given constant)

$$u_t = \nabla[-\nabla\cdot u - (\nabla\cdot u + c)^{-2} - \Delta\nabla\cdot u], \tag{2}$$

which has a variational structure. Equivalency between (1) and (2) will hold under sufficient regularity on the solution. The main aim of this paper is to provide an analytical validation to (2), by proving existence and regularity properties for weak solutions, under suitable assumptions on the initial datum.

1. Introduction

Epitaxial growth, a process in which a relatively thin film is deposited on a thick substrate, has gained interest in the recent years due to its applications to semiconductor electronics and quantum dots. Roughly speaking, the morphology of the film is known to be the result of a competition between the elastic energy associated to the mismatch between film and substrate, and the surface mass transport due to the film deposition. An extensive mathematical analysis of the mechanism associated to epitaxial growth has been carried out in the context of plane linear elasticity, and regularity results have been established for volume-constrained minimizers. For a non exhaustive list, we cite the papers by Bella, Goldman, Zwicknagl [3], Bonnetier, Chambolle [4], Fonseca, Pratelli, Zwicknagl [10], and Fonseca, Fusco, Leoni, Morini [5, 6]. Also related are the works by Gao, Liu, Lu [12, 11].

Short time existence for a surface diffusion type geometric evolution equation keeping into account elasticity has first been analyzed by Fonseca, Fusco, Leoni, Morini [7] for a two-dimensional setting (see also Piovano [19]). Subsequently, it has been recently extended in [8] to the three-dimensional case.

The central aim of this work is to study existence and regularity of solutions to the $2+1$ dimensional evolution equation

$$h_t = \Delta[\mathcal{F}^{-1}(-aE\mathcal{F}(h)) - r/h^2 - \Delta h] \tag{3}$$

derived in Tekalign, Spencer [21], with spatial domain $\mathbb{R}^2$, and time domain $[0,T]$ for given $T > 0$. Here $h : [0,T]\times\mathbb{R}^2 \longrightarrow \mathbb{R}$ denotes the surface height of the film, and $\mathcal{F}$ is the Fourier transform. The notation h_t denotes the derivative with respect to the time variable t, while Δ denotes the Laplacian with respect to the space coordinates. The quantities a, E, r are positive material constants. More precisely, a (resp. r) is the wavenumber (resp. wetting coefficient)

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associated to the equation, and $E := \dfrac{2\mu^F(1+\nu^F)(1-\nu^S)}{(1-\nu^F)\mu^S}$, where μ^F and ν^F (resp. μ^S and ν^S) are the elastic shear modulus and the Poisson's ratio of the film (resp. substrate). Equation (3) arises in the context of growth of an epitaxially strained, dislocation-free, thin solid film on a deformable substrate in the absence of vapor deposition, and under the assumption of thin film approximation. In equation (3) we wrote the Fourier transform (and its inverse) purely for fidelity with the original equation derived in [21]. Moreover, the constants a, E and r, while important from a modeling perspective, will play no role in our analysis. Hence, for simplicity we set $aE = r = 1$, and (3) reads

$$h_t = \Delta[-h - h^{-2} - \Delta h]. \tag{4}$$

Equation (4) is a fourth order nonlinear equation, with no particular structure, making its analysis quite challenging. To avoid the excessive technical complexities of working on a unbounded domain, where crucial inequalities (such as Poincaré inequality) do not hold, we restrict our spatial domain to be the unit disk

$$D := \{x \in \mathbb{R}^2 : |x| \le 1\},$$

and consider (4) with spatial domain D. Then we introduce the auxiliary parabolic equation

$$\begin{cases} u_t = \nabla[-\nabla\cdot u - (\nabla\cdot u + c)^{-2} - \Delta\nabla\cdot u] & \text{in } (0,T)\times D, \\ \langle u, \nu\rangle = \nabla\cdot u = 0 & \text{on } (0,T)\times \partial D, \\ u(0,\cdot) = u^0 & \text{on } \{t=0\}\times D, \end{cases} \tag{5}$$

where $c > 0$ is a given constant, u^0 is a given initial datum, and ν denotes the exterior unit normal to D. Here $u : [0,T]\times D \longrightarrow \mathbb{R}^2$ is a suitable function such that $h = \nabla\cdot u + c$, while u_t denotes the time derivative, and ∇, $\nabla\cdot$, Δ are respectively the gradient, divergence and Laplacian operators with respect to the spatial coordinates.

The main functional space is

$$\begin{aligned} V := \{&u \in L^2(D) : \nabla\cdot u \in L^2(D),\ \nabla\nabla\cdot u \in L^2(D), \\ &u = \nabla\psi \text{ for some function } \psi \in W^{1,2}(D),\ \langle u,\nu\rangle = \nabla\cdot u = 0 \text{ on } \partial D\}. \end{aligned} \tag{6}$$

We endow V with the norm

$$\|u\|_V := \|u\|_{L^2(D)} + \|\nabla\cdot u\|_{L^2(D)} + \|\nabla\nabla\cdot u\|_{L^2(D)}. \tag{7}$$

Here, and for future reference, for brevity we will use $L^2(D)$ to denote both $L^2(D;\mathbb{R})$ and $L^2(D;\mathbb{R}^2)$. There is no risk of confusion since clearly no function can belong to both spaces.

The relation between (4) and (5) is the following: if there exists a sufficiently regular solution u (e.g., if $u \in L^1(0,T;C_c^\infty(D))$ with $\nabla\cdot u + c$ *uniformly* bounded away from zero) such that (5) holds for a.e. x, t, then it can be straightforwardly checked that $h := \nabla\cdot u + c$ would be solution of (4). The presence of the constant c in $h = \nabla\cdot u + c$ is necessary due to the fact that we imposed $\int_D \nabla\cdot u \,\mathrm{d}x = 0$ in the definition of V to use Poincaré inequality (which allows us to use the norm (10) below, instead of (7)), while our physical model imposes $h > 0$ a.e.. Consequently, $c > 0$.

However, our main result (Theorem 1 below) does *not* prove that $\nabla\cdot u + c$ is uniformly bounded away from zero. Thus we cannot infer the full equivalence between (4) and (5). Nonetheless, (5) could be considered a "weaker" version of the original (4).

The main aim of this paper is to prove existence and regularity properties of solutions to (5), under suitable assumptions on the initial datum. For future reference, the notation $\langle , \rangle$ (without any subscript) will denote the Euclidean scalar product of $\mathbb{R}^2$. The main result is:

Theorem 1. *Given $T, c > 0$, let $u^0 \in V$ be an initial datum such that $\nabla\cdot u^0 + c > 0$ for a.e. $x \in D$, and there exists $z^0 \in L^2(D)$ such that*

$$\int_D [\langle z^0, v - u^0\rangle + \langle \nabla\nabla\cdot u^0, \nabla\nabla\cdot(v-u^0)\rangle - (\nabla\cdot u^0)(\nabla\cdot(v-u^0))]\,\mathrm{d}x + \phi(v) - \phi(u^0) \ge 0 \tag{8}$$

for any $v \in V$*. Here* ϕ *is the functional defined in* (11) *below. Then there exists a function*

$$u \in C^0([0,T]; L^2(D)) \cap L^\infty(0,T;V)$$

satisfying

$$u(0) = u^0, \qquad u_t \in L^\infty(0,T; L^2(D)),$$

and

$$\int_D \left[\langle u_t(t), \varphi \rangle + \langle \nabla\nabla \cdot u(t), \nabla\nabla \cdot \varphi \rangle - (\nabla \cdot u(t))(\nabla \cdot \varphi) - \frac{\nabla \cdot \varphi}{(\nabla \cdot u(t) + c)^2} \right] \mathrm{d}x = 0 \tag{9}$$

for a.e. $t \in [0,T]$ *and any* $\varphi \in V \cap C_c^\infty(D)$*.*

Further discussion about condition (8) will be carried in Lemma 8.

There are two main difficulties in our analysis, both due to the nonlinear term $\nabla(\nabla \cdot u + c)^{-2}$:

(1) first, $(\nabla \cdot u + c)^{-2}$ is unbounded as $\nabla \cdot u + c$ approaches zero.
(2) Second, $\nabla(\nabla\cdot u + c)^{-2}$ is *not* the highest order term in (5). This makes difficult, if not impossible, to characterize it as *unique* sub-gradient of some proper, convex, lower-semicontinuous functional. This prevents us from using the theory of maximal monotone operators in Hilbert spaces, and also the techniques in [13, 16, 17].

To overcome these issues, we will use the approach from [9], based on the theory of parabolic variational inequalities in reflexive Banach spaces. However, working in a two-dimensional spatial domain D makes the truncation argument from [9] much more delicate.

2. Preliminaries

The aim of this section is to present the setting of our problem, and to prove basic monotonicity estimates. We first check that V is a Banach space.

Lemma 2. *The space* V *is a Banach space.*

Proof. Consider an arbitrary Cauchy sequence $(v_n)_n \subseteq V$. It is straightforward to check that the space

$$X := \{u \in L^2(D) : \nabla \cdot u \in L^2(D),\ \nabla\nabla \cdot u \in L^2(D),\ \langle u, \nu \rangle = \nabla \cdot u = 0 \text{ on } \partial D\},$$

endowed with the norm

$$\|u\|_X := \|u\|_{L^2(D)} + \|\nabla \cdot u\|_{L^2(D)} + \|\nabla\nabla \cdot u\|_{L^2(D)},$$

is a Banach space. Note that a function $u \in X$ satisfies also $u \in V$ if and only if $u = \nabla\psi$ for some $\psi \in W^{1,2}(D)$. Hence there exists $v \in X$ such that $\|v_n - v\|_X \to 0$. Now to prove $v \in V$, it suffices to check $v = \nabla\psi$ for some $\psi \in W^{1,2}(D)$. For any n it holds $v_n \in V$, thus there exists ψ_n such that $v_n = \nabla\psi_n$. Without loss of generality, ψ_n can be assumed to have zero-average, hence, by Poincaré inequality, the sequence $(\psi_n)_n$ is bounded in $W^{1,2}(D)$. Thus there exists $\eta \in W^{1,2}(D)$ such that $\nabla\psi_n \rightharpoonup \nabla\eta$ in $L^2(D)$, i.e. $v_n = \nabla\psi_n \rightharpoonup \nabla\eta$. Since $v_n \to v$ strongly in X (and hence in $L^2(D)$), it follows $v = \nabla\eta$, hence $v \in V$. □

Lemma 3. *There exists a constant* $C > 0$ *such that*

$$\|\nabla \cdot v\|_{L^2(D)} \geq C\|v\|_{L^2(D)}$$

for any $v \in V$*.*

Proof. The thesis follows from the fact that we required all functions $u \in V$ to be gradients of some $\psi \in W^{1,2}(D)$, hence we can apply [1, Theorem 5.4]. □

As functions $u \in V$ are required to have zero-average, by Poincaré inequality and Lemma 3, it follows that the norm $\|u\|_V$ is equivalent to $\|\nabla\nabla\cdot u\|_{L^2(D)}$. Thus, for future reference, we will use the new norm (equivalent to (7), but formally simpler)

$$\|u\|_V := \|\nabla\nabla\cdot u\|_{L^2(D)}. \tag{10}$$

It is straightforward to check that V is reflexive. Moreover, the embeddings $V \hookrightarrow L^2(D) \hookrightarrow V'$ are both dense and continuous. Here we identified $L^2(D)$ with its dual. Endow $L^2(D)$ with the standard scalar product

$$(\forall f, g \in L^2(D)) \qquad \langle f, g\rangle_{L^2(D)} := \int_D fg\,\mathrm{d}x.$$

Then we can define, via integration by parts, a duality pairing between V' and V

$$(\forall v' \in V', v \in V) \qquad \langle v', v\rangle_{V',V} := \int_D v'v\,\mathrm{d}x.$$

We recall some useful definitions (see for instance [2]). Given a Banach space Y, denote by $\langle,\rangle_{Y',Y}$ the duality pairing between Y' and Y. Such a pairing can be defined by an inner product if there exists an Hilbert space Z such that $Y \subseteq Z = Z' \subseteq Y'$, and the embeddings $Y \hookrightarrow Z$ and $Z' \hookrightarrow Y'$ are dense and continuous. An operator $A : Y \longrightarrow Y'$ is:

(1) **monotone** if for any $u, v \in Y$, it holds
$$\langle Au - Av, u - v\rangle_{Y',Y} \geq 0.$$
Similarly, a set $G \subseteq Y \times Y'$ is *"monotone"* if for any pair (u, u'), $(v, v') \in G$, it holds
$$\langle u' - v', u - v\rangle_{Y',Y} \geq 0.$$

(2) **maximal monotone** if the graph
$$\Gamma_A := \{(u, Au) : u \in Y\} \subseteq Y \times Y'$$
is not a proper subset of any monotone set of $Y \times Y'$.

For future reference, the set $D_Y(A)$ will denote the domain of A, i.e.

$$D_Y(A) := \{u \in Y : Au \neq \emptyset\}.$$

The next two results prove that the right-hand side operator in (5) is maximal monotone.

Lemma 4. *Define the functions*

$$\Phi : \mathbb{R} \longrightarrow \mathbb{R} \cup \{+\infty\}, \qquad \Phi(\xi) := \begin{cases} +\infty & \text{if } \xi \leq 0, \\ \xi^{-1} & \text{if } \xi > 0, \end{cases} \tag{11}$$
$$\phi : V \longrightarrow \mathbb{R} \cup \{+\infty\}, \qquad \phi(u) := \int_D \Phi(\nabla\cdot u + c)\,\mathrm{d}x.$$

Then ϕ is proper, convex and lower-semicontinuous.

Proof. It follows directly from the definition that ϕ is proper. Note that Φ is convex and lower semicontinuous. We need to check convexity and lower semicontinuity for ϕ.

Convexity. Consider arbitrary $u_1, u_2 \in V$. We need to check

$$(1-t)\phi(u_1) + t\phi(u_2) \geq \phi((1-t)u_1 + tu_2) \qquad \text{for all } t \in [0, 1]. \tag{12}$$

Without loss of generality, we assume $\phi(u_1), \phi(u_2) < +\infty$ (otherwise (12) is trivial). The function Φ is convex, hence

$$\begin{aligned}\Phi(\nabla\cdot((1-t)u_1 + tu_2) + c) &= \Phi((1-t)(\nabla\cdot u_1 + c) + t(\nabla\cdot u_2 + c)) \\ &\leq (1-t)\Phi(\nabla\cdot u_1 + c) + t\Phi(\nabla\cdot u_2 + c) \qquad \text{for a.e. } x \in D.\end{aligned} \tag{13}$$

Integrating on D gives

$$\begin{aligned}\phi((1-t)u_1+tu_2) &= \int_D \Phi(\nabla\cdot((1-t)u_1+tu_2)+c)\,\mathrm{d}x \\ &\le (1-t)\int_D \Phi(\nabla\cdot u_1+c)\,\mathrm{d}x + t\int_D \Phi(\nabla\cdot u_2+c)\,\mathrm{d}x = (1-t)\phi(u_1)+t\phi(u_2),\end{aligned}$$

hence (12).

Lower semicontinuity. Consider a sequence $(u_n)_n \subseteq V$, converging to $u \in V$. We need to check

$$\phi(u) \le \liminf_{n\to+\infty} \phi(u_n). \tag{14}$$

Without loss of generality, assume $\liminf_{n\to+\infty}\phi(u_n) < +\infty$, otherwise (14) is trivial. Upon removing finitely many elements from the sequence, we can also assume $\sup_n \phi(u_n) < +\infty$, which implies $\nabla\cdot u_n + c > 0$ for a.e. x and all n. Since $u_n \to u$ strongly in V, it follows $\nabla\cdot u_n \to \nabla\cdot u$ strongly in $L^2(D)$, hence (upon subsequence, which we do not relabel), $\nabla\cdot u_n \to \nabla\cdot u$ for a.e. $x \in D$, and $\Phi(\nabla\cdot u_n + c) \to \Phi(\nabla\cdot u + c)$ for a.e. $x \in D$. As by construction we have $\Phi \ge 0$, by Fatou's lemma we conclude

$$\int_D \Phi(\nabla\cdot u + c)\,\mathrm{d}x = \int_D \liminf_{n\to+\infty}\Phi(\nabla\cdot u_n + c)\,\mathrm{d}x \le \liminf_{n\to+\infty}\int_D \Phi(\nabla\cdot u_n + c)\,\mathrm{d}x,$$

hence (14). □

Lemma 5. *The operator*

$$B : V \longrightarrow V', \qquad Bu := \nabla\Delta\nabla\cdot u + \nabla\nabla\cdot u$$

is bounded, maximal monotone, and coercive in the sense that

$$(1+4/\pi^2)\|u-v\|_V^2 \ge \langle Bu - Bv, u-v\rangle_{V',V} \ge (1-4/\pi^2)\|u-v\|_V^2. \tag{15}$$

Proof. By construction, B is linear. Observe that given $u, v \in V$,

$$\begin{aligned}\langle Bu-Bv, u-v\rangle_{V',V} &= \langle B(u-v), u-v\rangle_{V',V} \\ &= \int_D [\langle \nabla\nabla\cdot(u-v), \nabla\nabla\cdot(u-v)\rangle - (\nabla\cdot u)(\nabla\cdot v)]\,\mathrm{d}x \\ &= \|\nabla\nabla\cdot(u-v)\|_{L^2(D)}^2 - \|\nabla\cdot(u-v)\|_{L^2(D)}^2.\end{aligned} \tag{16}$$

Since $\nabla\cdot(u-v)$ has zero-average (as $u, v \in V$), by Poincaré inequality it holds

$$\|\nabla\cdot(u-v)\|_{L^2(D)} \le C\|\nabla\nabla\cdot(u-v)\|_{L^2(D)}$$

for some constant C. In [18] it has been proven that such optimal Poincaré constant C does not exceed d/π, where d is the diameter of the (spatial) domain. Since in our case the domain is the unit disk D, we have $C \le 2/\pi$, hence (16) gives

$$\begin{aligned}\langle Bu-Bv, u-v\rangle_{V',V} &= \|\nabla\nabla\cdot(u-v)\|_{L^2(D)}^2 - \|\nabla\cdot(u-v)\|_{L^2(D)}^2 \\ &\ge (1-4/\pi^2)\|\nabla\nabla\cdot(u-v)\|_{L^2(D)}^2 = (1-4/\pi^2)\|u-v\|_V^2.\end{aligned}$$

To prove the boundedness of B, note that

$$\begin{aligned}|\langle Bu, v\rangle_{V',V}| &= \left|\int_D [\langle \nabla\nabla\cdot u, \nabla\nabla\cdot v\rangle - (\nabla\cdot u)(\nabla\cdot v)]\,\mathrm{d}x\right| \\ &\le \|\nabla\nabla\cdot u\|_{L^2(D)}\|\nabla\nabla\cdot v\|_{L^2(D)} + \|\nabla\cdot u\|_{L^2(D)}\|\nabla\cdot v\|_{L^2(D)} \\ &\le (1+4/\pi^2)\|\nabla\nabla\cdot u\|_{L^2(D)}\|\nabla\nabla\cdot v\|_{L^2(D)} = (1+4/\pi^2)\|u\|_V\|v\|_V \qquad (17) \\ &\Longrightarrow \|Bu\|_{V'} = \sup_{v\neq 0}\frac{|\langle Bu, v\rangle_{V',V}|}{\|v\|_V} \le (1+4/\pi^2)\|u\|_V,\end{aligned}$$

where (17) follows from Poincaré inequality, and the fact that our optimal Poincaré constant does not exceed $2/\pi$. Thus $B : V \longrightarrow V'$ is bounded.

As the operator B is well defined everywhere on V, monotone, continuous, linear, and by [2, Corollary 2.1] we infer that is B maximal monotone. □

3. Proof of Theorem 1

Now we are ready to prove existence and regularity of solutions to (5). We will use the following theorem, due to Kačur, from [14] (also available as [15, Theorem 2]).

Theorem 6. *Given a reflexive Banach space $(\tilde{V}, \|\cdot\|_{\tilde{V}})$, a Hilbert space $(\tilde{U}, \langle,\rangle_{\tilde{U}})$ such that the embeddings $\tilde{V} \hookrightarrow \tilde{U} \hookrightarrow \tilde{V}'$ are both dense and continuous, denote by $\langle,\rangle_{\tilde{V}',\tilde{V}}$ the duality pairing between $\tilde{V}'$ and $\tilde{V}$.*

Let $\tilde{A} : \tilde{V} \longrightarrow \tilde{V}'$ be a maximal monotone operator, let $\tilde{\phi} : \tilde{V} \longrightarrow (-\infty, +\infty]$ be a proper, convex, lower semi-continuous functional. Let $u^0 \in \tilde{U}$ be a given initial datum, and suppose there exist:

- *v^0 such that $\tilde{\phi}(v^0) < +\infty$ and*
$$\lim_{\|v\|_{\tilde{V}} \to +\infty} \frac{\langle \tilde{A}v, v - v^0\rangle_{\tilde{V}',\tilde{V}} + \tilde{\phi}(v)}{\|v\|_{\tilde{V}}} = +\infty, \tag{18}$$
- *$z^0 \in \tilde{U}$ such that for any $v \in \tilde{V}$*
$$\langle z^0, v\rangle_{\tilde{U}} + \langle Au^0, v\rangle_{\tilde{V}',\tilde{V}} + \tilde{\phi}(v) - \tilde{\phi}(u^0) \geq 0. \tag{19}$$

Then there exists a unique $u \in L^\infty(0,T;\tilde{V}) \cap C^0([0,T];\tilde{U})$ such that $u_t \in L^\infty(0,T;\tilde{U})$, $u(0) = u^0$, and
$$\langle u_t(t), v - u(t)\rangle_{\tilde{U}} + \langle \tilde{A}u(t), v - u(t)\rangle_{\tilde{V}',\tilde{V}} + \tilde{\phi}(v) - \tilde{\phi}(u(t)) \geq 0$$
for a.e. time $t \in (0,T)$, and all $v \in \tilde{V}$.

Applying Theorem 6 in our case gives:

Theorem 7. *Given $T > 0$ and an initial datum u^0 satisfying (8), there exists a unique function*
$$u \in L^\infty(0,T;V) \cap C^0([0,T];L^2(D))$$
such that
$$u_t \in L^\infty(0,T;L^2(D)), \qquad u(0) = u^0,$$
and
$$\langle u_t(t), v - u(t)\rangle_{L^2(D)} + \langle Bu(t), v - u(t)\rangle_{V',V} + \phi(v) - \phi(u(t)) \geq 0 \tag{20}$$
for a.e. time $t \in (0,T)$, and all $v \in V$.

Proof. We will apply Theorem 6 with $\tilde{A} = B$, $\tilde{\phi} = \phi$, $\tilde{V} = V$, $\tilde{U} = L^2(D)$. We need to check that hypotheses (18) and (19) are satisfied.

Lemma 4 proved that ϕ is proper, convex, lower semi-continuous, and Lemma 5 proved that B is linear, bounded, and maximal monotone. Hypothesis (8) gives (19).

To check (18), note that by construction we have $\Phi \geq 0$, hence $\phi \geq 0$. Thus taking $v^0 = 0$ (the function identically equal to 0), by (15) we have
$$\lim_{\|v\|_V \to +\infty} \frac{\langle Bv, v - v^0\rangle_{V',V} + \phi(v)}{\|v\|_V} \geq \lim_{\|v\|_V \to +\infty} \frac{(1 - 4/\pi^2)\|v\|_V^2}{\|v\|_V} = +\infty,$$
hence (18) is satisfied. Thus we are under the hypotheses of Theorem 7, hence there exists
$$u \in L^\infty(0,T;V) \cap C^0([0,T];L^2(D)) \tag{21}$$

such that

$$u_t \in L^\infty(0,T;L^2(D)), \qquad u(0) = u^0, \tag{22}$$

and

$$\langle u_t(t), v - u(t)\rangle_{L^2(D)} + \langle Bu(t), v - u(t)\rangle_{V',V} + \phi(v) - \phi(u(t)) \geq 0 \tag{23}$$

for a.e. time $t \in (0,T)$, and all $v \in V$. □

Now we can prove that the variational solution u given by Theorem 7 is a weak solution in the sense of (9).

Proof. (of Theorem 1) We prove that the function u given by Theorem 7 satisfies (9). The idea would be to consider an arbitrary $\varphi \in V \cap C_c^\infty(D)$, and test (20) with test functions $u \pm \varepsilon\varphi$. However, since a priori we have only $\nabla\cdot u + c > 0$ a.e., we cannot guarantee that $\nabla\cdot(u \pm \varepsilon\varphi) + c$ is also a.e. positive. Thus we test (20) with $v = u^\delta \pm \varepsilon\varphi$, for carefully chosen function u^δ and parameters $\varepsilon \ll 1$, $\delta = \delta(\varepsilon,\varphi)$, and then take the limit $\delta \to 0$. The main steps of the proof are:

(1) first, we prove that $(\nabla\cdot u(t) + c)^{-2} \in L^1(D)$.
(2) Second, we truncate $\nabla\cdot u(t)$ from below.
(3) Third, we test (20) with $v = u^\delta(t) \pm \varepsilon\varphi$, $\varepsilon > 0$, and take the limit $\delta \to 0^+$.

Step 1. Integrability of $(\nabla\cdot u(t) + c)^{-2}$. Fix an arbitrary t for which (23) holds. Note that for (23) to hold, it is required that

$$\phi(u(t)) = \int_D \Phi(\nabla\cdot u(t) + c)\,\mathrm{d}x < +\infty,$$

hence $\nabla\cdot u(t) + c > 0$ a.e., which in turn gives

$$\int_D \Phi(\nabla\cdot u(t) + c)\,\mathrm{d}x = \int_D (\nabla\cdot u(t) + c)^{-1}\,\mathrm{d}x < +\infty \qquad \text{for a.e. } t \in [0,T]. \tag{24}$$

Recall also that by hypothesis we have $c > 0$. We test (23) with $v = (1-\varepsilon)u(t)$ (with $0 < \varepsilon \ll 1$). Since $\nabla\cdot u(t) + c > 0$ a.e., it follows

$$\nabla\cdot[(1-\varepsilon)u(t)] + c = (1-\varepsilon)(\nabla\cdot u(t) + c) + \varepsilon c > \varepsilon c \qquad \text{for a.e. } x \in D. \tag{25}$$

Plugging $v = (1-\varepsilon)u(t)$ in (23) gives

$$\langle u_t(t), -\varepsilon u(t)\rangle_{L^2(D)} + \langle Bu(t), -\varepsilon u(t)\rangle_{V',V} \geq -[\phi((1-\varepsilon)u(t)) - \phi(u(t))],$$

which implies, in view of the regularity properties (21) and (22),

$$\begin{aligned}
+\infty &> -\varepsilon\Big[\langle u_t(t), u(t)\rangle_{L^2(D)} + \|\nabla\nabla\cdot u(t)\|_{L^2(D)} - \|\nabla\cdot u(t)\|_{L^2(D)}\Big] \\
&\geq -\phi((1-\varepsilon)u) + \phi(u(t)) = -\int_D \Phi(\nabla\cdot[(1-\varepsilon)u(t)] + c)\,\mathrm{d}x + \int_D \Phi(\nabla\cdot u(t) + c)\,\mathrm{d}x \\
&\overset{(25)}{=} -\int_D [(\nabla\cdot[(1-\varepsilon)u(t)] + c)^{-1} - (\nabla\cdot u(t) + c)^{-1}]\,\mathrm{d}x \\
&= -\varepsilon\int_D \frac{\nabla\cdot u(t)}{(\nabla\cdot[(1-\varepsilon)u(t)] + c)(\nabla\cdot u(t) + c)}\,\mathrm{d}x \\
&= -\varepsilon\int_D \frac{1}{\nabla\cdot[(1-\varepsilon)u(t)] + c}\,\mathrm{d}x + \varepsilon\int_D \frac{c}{(\nabla\cdot[(1-\varepsilon)u(t)] + c)(\nabla\cdot u(t) + c)}\,\mathrm{d}x.
\end{aligned} \tag{26}$$

In view of (25), we have $\nabla\cdot[(1-\varepsilon)u(t)] + c > (1-\varepsilon)(\nabla\cdot u(t) + c)$, hence

$$\int_D \frac{1}{\nabla\cdot[(1-\varepsilon)u(t)] + c}\,\mathrm{d}x \leq \frac{1}{1-\varepsilon}\int_D \frac{1}{\nabla\cdot u(t) + c}\,\mathrm{d}x < +\infty.$$

Dividing by ε in (26) then gives

$$+\infty > -[\langle u_t(t), u(t)\rangle_{L^2(D)} + \|\nabla\nabla\cdot u(t)\|_{L^2(D)} - \|\nabla\cdot u(t)\|_{L^2(D)}] + \frac{1}{1-\varepsilon}\int_D \frac{1}{\nabla\cdot u(t)+c}\,\mathrm{d}x$$
$$\geq \int_D \frac{c}{(\nabla\cdot[(1-\varepsilon)u(t)]+c)(\nabla\cdot u(t)+c)}\,\mathrm{d}x. \tag{27}$$

By monotone convergence theorem, we have

$$\begin{aligned}\lim_{\varepsilon\to 0}\int_D &\frac{c}{(\nabla\cdot[(1-\varepsilon)u(t)]+c)(\nabla\cdot u(t)+c)}\,\mathrm{d}x\\ &= \lim_{\varepsilon\to 0}\int_{\{\nabla\cdot u\geq 0\}} \frac{c}{(\nabla\cdot[(1-\varepsilon)u(t)]+c)(\nabla\cdot u(t)+c)}\,\mathrm{d}x\\ &\quad + \lim_{\varepsilon\to 0}\int_{\{\nabla\cdot u< 0\}} \frac{c}{(\nabla\cdot[(1-\varepsilon)u(t)]+c)(\nabla\cdot u(t)+c)}\,\mathrm{d}x\\ &= \int_D \frac{c}{(\nabla\cdot u(t)+c)^2}\,\mathrm{d}x,\end{aligned}$$

and taking the limit $\varepsilon\to 0$ in (27) gives

$$+\infty > -\Big[\langle u_t(t), u(t)\rangle_{L^2(D)} + \|\nabla\nabla\cdot u(t)\|_{L^2(D)} - \|\nabla\cdot u(t)\|_{L^2(D)}\Big] + \int_D \frac{1}{\nabla\cdot u(t)+c}\,\mathrm{d}x$$
$$\geq \int_D \frac{c}{(\nabla\cdot u(t)+c)^2}\,\mathrm{d}x. \tag{28}$$

Step 2. Truncating $\nabla\cdot u(t)$. Given $\delta>0$, we truncate $\nabla\cdot u(t)$ from below as follows: let

$$z^\delta := [\nabla\cdot u - (\delta - c)]^+ + \delta - c = \begin{cases} \nabla\cdot u & \text{on } D\backslash E_\delta(t),\\ \delta - c & \text{on } E_\delta(t),\end{cases}\qquad E_\delta(t) := \{\nabla\cdot u(t)\leq \delta - c\}.$$

Here $[f]^+ := \max\{f, 0\}$ denotes the positive part of f. As $\nabla\cdot u + c > 0$ a.e., we have $\mathcal{L}^2(E_\delta(t))\to 0$ as $\delta\to 0$, with $\mathcal{L}^2$ denoting the Lebesgue measure. Note that $\nabla\cdot u = 0 > \delta - c$ on ∂D, hence $z^\delta = 0$ on ∂D, and

$$z^\delta + c = \begin{cases} \nabla\cdot u + c & \text{on } D\backslash E_\delta(t),\\ \delta & \text{on } E_\delta(t).\end{cases} \tag{29}$$

Moreover, since $u\in V$,

$$0 = \int_D \nabla\cdot u(t)\,\mathrm{d}x \leq J_\delta \leq \delta\mathcal{L}^2(E_\delta(t)),\qquad J_\delta := \int_D z^\delta\,\mathrm{d}x.$$

However, z^δ has non-zero average, hence we introduce the function (expressed in polar coordinates)

$$w^\delta : D \longrightarrow \mathbb{R},\qquad w^\delta(r,\theta) := (2J_\delta/\pi)(r^2-1), \tag{30}$$

which is C^1 regular, and satisfies

$$w^\delta = 0 \text{ on } \partial D,\qquad -2J_\delta/\pi \leq w^\delta \leq 0,\qquad \int_D w^\delta\,\mathrm{d}x = 4\pi\int_0^1 rJ_\delta(r^2-1)/\pi\,\mathrm{d}r = -J_\delta \tag{31}$$

The function $z^\delta + w^\delta$ then satisfies

$$z^\delta + w^\delta = 0 \text{ on } \partial D,\qquad \int_D (z^\delta + w^\delta)\,\mathrm{d}x = 0,$$
$$z^\delta + w^\delta + c \geq \delta - (2/\pi)J_\delta \geq \delta[1-(2/\pi)\mathcal{L}^2(E_\delta(t))] \geq \delta/2, \tag{32}$$
$$\nabla(z^\delta + w^\delta) = \nabla([\nabla\cdot u - (\delta - c)]^+) + \nabla w^\delta = \begin{cases} \nabla\nabla\cdot u + \nabla w^\delta & \text{on } D\backslash E_\delta(t),\\ \nabla w^\delta & \text{on } E_\delta(t),\end{cases} \tag{33}$$

in view of [20]. Thus the Poisson equation (with Neumann boundary conditions)

$$\Delta\psi_\delta = z^\delta + w^\delta \text{ on } D, \qquad \langle \nabla\psi_\delta, \nu\rangle = 0 \text{ on } \partial D,$$

with ν denoting the exterior unit normal to D, admits a solution $\psi_\delta \in W^{1,2}(D)$, and $u^\delta := \nabla\psi_\delta \in V$ in view of (32) and (33).

Step 3. Testing (20) *with* $v = u^\delta \pm \varepsilon\varphi$, $\varepsilon > 0$. Consider an arbitrary test function $\varphi \in V \cap C_c^\infty(D)$. Fix an arbitrary t for which (23) holds. Note that (32) gives $\nabla\cdot u^\delta + c \geq \delta/2$, and we set

$$\varepsilon = \frac{\delta}{k}, \qquad k := 4\|\nabla\cdot\varphi\|_{L^\infty(D)}, \tag{34}$$

to get

$$\nabla\cdot(u^\delta + \varepsilon\varphi) + c \geq \delta/2 - \frac{\delta|\nabla\cdot\varphi|}{4\|\nabla\cdot\varphi\|_{L^\infty(D)}} \geq \delta/4. \tag{35}$$

Plugging $v = u^\delta + \varepsilon\varphi$ in (23) gives

$$\begin{aligned}
&\langle u_t(t), u^\delta - u(t) + \varepsilon\varphi\rangle_{L^2(D)} + \langle Bu(t), u^\delta - u(t) + \varepsilon\varphi\rangle_{V',V} + \phi(u^\delta + \varepsilon\varphi) - \phi(u(t))\\
&\quad = \varepsilon[\langle u_t(t), \varphi\rangle_{L^2(D)} + \langle Bu(t), \varphi\rangle_{V',V}]\\
&\qquad + \langle u_t(t), u^\delta - u(t)\rangle_{L^2(D)} + \langle B(u(t) - u^\delta), u^\delta - u(t)\rangle_{V',V} + \langle Bu^\delta, u^\delta - u(t)\rangle_{V',V}\\
&\qquad + \phi(u^\delta + \varepsilon\varphi) - \phi(u(t))\\
&\quad \geq 0,
\end{aligned}$$

and using the monotonicity of B yields

$$\langle B(u(t) - u^\delta), u^\delta - u(t)\rangle_{V',V} \leq 0,$$

hence

$$\begin{aligned}
\varepsilon[\langle u_t(t), \varphi\rangle_{L^2(D)} + \langle Bu(t), \varphi\rangle_{V',V}] + \langle u_t(t), u^\delta - u(t)\rangle_{L^2(D)} + \langle Bu^\delta, u^\delta - u(t)\rangle_{V',V}&\\
+ \phi(u^\delta + \varepsilon\varphi) - \phi(u(t)) \geq 0.&
\end{aligned} \tag{36}$$

We claim

$$\lim_{\delta\to 0^+} \delta^{-1}\langle Bu^\delta, u^\delta - u(t)\rangle_{V',V} = 0, \tag{37}$$

$$\lim_{\delta\to 0^+} \delta^{-1}\langle u_t(t), u^\delta - u(t)\rangle_{L^2(D)} = 0, \tag{38}$$

$$\lim_{\delta\to 0^+} \delta^{-1}(\phi(u^\delta + \varepsilon\varphi) - \phi(u(t))) = -k^{-1}\int_D \frac{\nabla\cdot\varphi}{(\nabla\cdot u(t) + c)^2}\,\mathrm{d}x. \tag{39}$$

Sub-step 3.1. Proof of (37). Note that (33) gives

$$\nabla\nabla\cdot u^\delta = \nabla\nabla\cdot w^\delta \text{ on } E_\delta(t), \qquad \nabla\nabla\cdot(u^\delta - u) = \nabla\nabla\cdot w^\delta \text{ on } D\backslash E_\delta(t),$$

hence

$$\begin{aligned}
\langle Bu^\delta, u^\delta - u(t)\rangle_{V',V} &= \int_D \langle \nabla\nabla\cdot u^\delta, \nabla\nabla\cdot(u^\delta - u(t))\rangle\,\mathrm{d}x\\
&= \int_{E_\delta(t)} \langle \nabla w^\delta, \nabla(w^\delta - \nabla\cdot u(t))\rangle\,\mathrm{d}x + \int_{D\backslash E_\delta(t)} \langle \nabla(\nabla\cdot u + w^\delta), \nabla w^\delta\rangle\,\mathrm{d}x.
\end{aligned}$$

As by construction (in (30)) we have

$$\|\nabla w^\delta\|_{L^\infty(D)} \leq (4/\pi)J_\delta \leq (4/\pi)\delta\mathcal{L}^2(E_\delta(t)), \qquad \nabla\nabla\cdot u \in L^2(D),$$

we infer

$$\int_{E_\delta(t)} |\langle \nabla w^\delta, \nabla(w^\delta - \nabla\cdot u(t))\rangle|\,\mathrm{d}x \le \|\nabla w^\delta\|_{L^\infty(D)}\|\nabla\nabla\cdot u\|_{L^1(D)} + \|\nabla w^\delta\|^2_{L^\infty(D)},$$
$$\int_{D\setminus E_\delta(t)} |\langle \nabla(\nabla\cdot u + w^\delta), \nabla w^\delta\rangle\,\mathrm{d}x| \le \|\nabla w^\delta\|_{L^\infty(D)}\|\nabla\nabla\cdot u\|_{L^1(D)} + \|\nabla w^\delta\|^2_{L^\infty(D)},$$

hence

$$\begin{aligned}\Big|\int_{E_\delta(t)} \langle \nabla w^\delta, \nabla(w^\delta - \nabla\cdot u(t))\rangle\,\mathrm{d}x + \int_{D\setminus E_\delta(t)} \langle \nabla(\nabla\cdot u + w^\delta), \nabla w^\delta\rangle\,\mathrm{d}x\Big|&\\ \le (4/\pi)\delta\mathcal{L}^2(E_\delta(t))\|\nabla\nabla\cdot u\|_{L^1(D)} + (4/\pi)^2\delta^2\mathcal{L}^2(E_\delta(t))^2&,\end{aligned}$$

which in turn gives

$$\begin{aligned}&\lim_{\delta\to0^+}\delta^{-1}\langle Bu^\delta, u^\delta - u(t)\rangle_{V',V}\\ &\quad= \lim_{\delta\to0^+}\frac{1}{\delta}\Big(\int_{E_\delta(t)} \langle \nabla w^\delta, \nabla(w^\delta - \nabla\cdot u(t))\rangle\,\mathrm{d}x + \int_{D\setminus E_\delta(t)} \langle \nabla(\nabla\cdot u + w^\delta), \nabla w^\delta\rangle\,\mathrm{d}x\Big) = 0,\end{aligned}$$

thus (37) is proven.

Sub-step 3.2. Proof of (38). Theorem 7 gave that $u_t \in L^\infty(0,T;L^2(D))$, and since both u^δ, $u \in V$ (hence both are gradients of some $W^{1,2}(D)$ functions), we have by [1, Theorem 5.4]

$$|\langle u_t(t), u^\delta - u(t)\rangle_{L^2(D)}| \le \|u_t\|_{L^\infty(0,T;L^2(D))}\|u^\delta - u(t)\|_{L^2(D)} \le \|u_t\|_{L^\infty(0,T;L^2(D))}\|\nabla\cdot(u^\delta - u(t))\|_{L^2(D)}.$$

By construction, $\nabla\cdot u^\delta = z^\delta + w^\delta$, and (see for instance (29))

$$|\nabla\cdot(u^\delta - u(t))| = |w^\delta| \le (2/\pi)\delta\mathcal{L}^2(E_\delta(t)) \text{ on } D\setminus E_\delta(t), \qquad |\nabla\cdot(u^\delta - u(t))| \le \delta \text{ on } E_\delta(t).$$

Thus

$$\begin{aligned}\|\nabla\cdot(u^\delta - u(t))\|^2_{L^2(D)} &= \int_{D\setminus E_\delta(t)} |\nabla\cdot(u^\delta - u(t))|^2\,\mathrm{d}x + \int_{E_\delta(t)} |\nabla\cdot(u^\delta - u(t))|^2\,\mathrm{d}x\\ &\le \delta^2[(2/\pi)^2\mathcal{L}^2(E_\delta(t))^2 + \mathcal{L}^2(E_\delta(t))],\end{aligned}$$

hence

$$\begin{aligned}\lim_{\delta\to0^+}\delta^{-1}|\langle u_t(t), u^\delta - u(t)\rangle_{L^2(D)}| &\le \lim_{\delta\to0^+}\|u_t\|_{L^\infty(0,T;L^2(D))}\|\nabla\cdot(u^\delta - u(t))\|_{L^2(D)}\\ &\le \lim_{\delta\to0^+}\|u_t\|_{L^\infty(0,T;L^2(D))}[(2/\pi)^2\mathcal{L}^2(E_\delta(t))^2 + \mathcal{L}^2(E_\delta(t))]^{1/2} = 0,\end{aligned}$$

and (38) is proven.

Sub-step 3.3. Proof of (39). Note that by construction we have $u^\delta + \varepsilon\varphi > 0$, $u(t) > 0$ a.e., and

$$\phi(u^\delta + \varepsilon\varphi) - \phi(u(t)) = \int_D \frac{\nabla\cdot(u(t) - u^\delta - \varepsilon\varphi)}{(\nabla\cdot(u^\delta + \varepsilon\varphi) + c)(\nabla\cdot u(t) + c)}\,\mathrm{d}x. \tag{40}$$

Recalling that $\nabla\cdot u^\delta = z^\delta + w^\delta$, in view of estimates (35), (29) and (31) we have

$$\begin{aligned}&\frac{1}{\delta}\int_D \Big|\frac{\nabla\cdot(u(t) - u^\delta)}{(\nabla\cdot(u^\delta + \varepsilon\varphi) + c)(\nabla\cdot u(t) + c)}\Big|\,\mathrm{d}x\\ &\quad\le \frac{1}{\delta}\int_D \frac{|w^\delta|}{(\nabla\cdot(u^\delta + \varepsilon\varphi) + c)(\nabla\cdot u(t) + c)}\,\mathrm{d}x + \frac{1}{\delta}\int_D \frac{|z^\delta|}{(\nabla\cdot(u^\delta + \varepsilon\varphi) + c)(\nabla\cdot u(t) + c)}\,\mathrm{d}x\\ &\quad\le \int_D \frac{4\mathcal{L}^2(E_\delta(t))}{(\nabla\cdot u(t) + c)}\,\mathrm{d}x + \int_{E_\delta(t)} \frac{4}{(\nabla\cdot u(t) + c)}\,\mathrm{d}x \overset{\delta\to0^+}{\to} 0,\end{aligned} \tag{41}$$

since $\mathcal{L}^2(E_\delta(t)) \to 0$ and $(\nabla\cdot u(t)+c)^{-1} \in L^1(D)$ as proven in Step 1. For the other term

$$\frac{1}{\delta}\int_D \frac{\nabla\cdot(\varepsilon\varphi)}{(\nabla\cdot(u^\delta+\varepsilon\varphi)+c)(\nabla\cdot u(t)+c)}\,\mathrm{d}x = \frac{1}{k}\int_D \frac{\nabla\cdot\varphi}{(\nabla\cdot(u^\delta+\varepsilon\varphi)+c)(\nabla\cdot u(t)+c)}\,\mathrm{d}x,$$

note that clearly

$$\frac{\nabla\cdot\varphi}{(\nabla\cdot(u^\delta+\varepsilon\varphi)+c)(\nabla\cdot u(t)+c)} \to \frac{\nabla\cdot\varphi}{(\nabla\cdot u(t)+c)^2} \qquad \text{for a.e. } x\in D.$$

To use Lebesgue dominated convergence theorem, we need to find a dominating function. By (35), (29) and (31), on $E_\delta(t)$ it holds

$$\nabla\cdot(u^\delta+\varepsilon\varphi)+c \geq \delta/4 \geq (\nabla\cdot u(t)+c)/4,$$

while on $D\backslash E_\delta(t)$ it holds $\nabla\cdot u^\delta = \nabla\cdot u(t)+w^\delta$, and (due to the choice of ε in (34))

$$|\varepsilon\nabla\cdot\varphi| + w^\delta \leq \delta/3 \leq (\nabla\cdot u(t)+c)/3,$$

hence

$$\nabla\cdot(u^\delta+\varepsilon\varphi)+c = \nabla\cdot u(t)+c+\varepsilon\nabla\cdot\varphi+w^\delta \geq 2(\nabla\cdot u(t)+c)/3.$$

Thus in both cases (on $E_\delta(t)$ and $D\backslash E_\delta(t)$) it holds $\nabla\cdot(u^\delta+\varepsilon\varphi)+c \geq (\nabla\cdot u(t)+c)/4$, hence

$$\frac{\nabla\cdot\varphi}{(\nabla\cdot(u^\delta+\varepsilon\varphi)+c)(\nabla\cdot u(t)+c)} \leq \frac{4\|\varphi\|_{L^\infty(D)}}{(\nabla\cdot u(t)+c)^2} \qquad \text{for a.e. } x\in D,$$

and $\frac{4\|\varphi\|_{L^\infty(D)}}{(\nabla\cdot u(t)+c)^2} \in L^1(D)$ by Step 1. Thus by Lebesgue dominated convergence theorem we infer

$$\lim_{\delta\to 0^+}\frac{1}{\delta}\int_D \frac{\nabla\cdot(\varepsilon\varphi)}{(\nabla\cdot(u^\delta+\varepsilon\varphi)+c)(\nabla\cdot u(t)+c)}\,\mathrm{d}x = \frac{1}{k}\int_D \frac{\nabla\cdot\varphi}{(\nabla\cdot u(t)+c)^2}\,\mathrm{d}x,$$

and combining with (41) gives (39).

In view of (37), (38) and (39), dividing by δ and taking the limit $\delta\to 0^+$ in (36) gives

$$\begin{aligned}\lim_{\delta\to 0^+}\delta^{-1}\Big(&\varepsilon[\langle u_t(t),\varphi\rangle_{L^2(D)} + \langle Bu(t),\varphi\rangle_{V',V}] \\ &+ \langle u_t(t), u^\delta - u(t)\rangle_{L^2(D)} + \langle Bu^\delta, u^\delta - u(t)\rangle_{V',V} + \phi(u^\delta+\varepsilon\varphi) - \phi(u(t))\Big) \\ &= k^{-1}\Big(\langle u_t(t),\varphi\rangle_{L^2(D)} + \langle Bu(t),\varphi\rangle_{V',V} - \int_D \frac{\nabla\cdot\varphi}{(\nabla\cdot u(t)+c)^2}\,\mathrm{d}x\Big) \geq 0,\end{aligned}$$

hence (since $k>0$)

$$\langle u_t(t),\varphi\rangle_{L^2(D)} + \langle Bu(t),\varphi\rangle_{V',V} - \int_D \frac{\nabla\cdot\varphi}{(\nabla\cdot u(t)+c)^2}\,\mathrm{d}x \geq 0. \tag{42}$$

As the above argument can be repeated with $v = u^\delta - \varepsilon\varphi$, it follows also

$$\langle u_t(t),\varphi\rangle_{L^2(D)} + \langle Bu(t),\varphi\rangle_{V',V} - \int_D \frac{\nabla\cdot\varphi}{(\nabla\cdot u(t)+c)^2}\,\mathrm{d}x \leq 0. \tag{43}$$

Combining (42) and (43) gives (9). □

We finally check that condition (8) on the initial datum u^0 is satisfied by a wide range of functions.

Lemma 8. *Condition (8) is satisfied for all $u^0 \in V\cap W^{4,2}(D)$ such that $\nabla\cdot u^0 + c$ is uniformly bounded away from zero.*

Proof. Lemma 4 proved that ϕ is proper, convex and lower semicontinuous. We claim that for any u^0 such that $\nabla \cdot u^0 + c \geq \alpha$ a.e. (for some $\alpha > 0$), it holds

$$\partial\phi(u^0) = \{\nabla(\nabla \cdot u^0 + c)^{-2}\},$$

where $\partial\phi(u^0) \subseteq V'$ denotes the sub-differential of ϕ at u^0, i.e.,

$$\xi \in \partial\phi(u^0) \Longleftrightarrow \phi(u^0 + v) - \phi(u^0) \geq \langle \xi, v\rangle_{V',V} \qquad \text{for any } v \in V.$$

We first prove $\partial\phi(u^0) \supseteq \{\nabla(\nabla \cdot u^0 + c)^{-2}\}$, provided that $\nabla(\nabla \cdot u^0 + c)^{-2} \in V'$. The goal is to show

$$\phi(u^0 + v) - \phi(u^0) \geq \langle \nabla(\nabla \cdot u^0 + c)^{-2}, v\rangle_{V',V} \qquad \text{for any } v \in V. \tag{44}$$

Consider an arbitrary $v \in V$. If $\nabla(u^0 + v) + c$ is not a.e. positive, then $\phi(u^0 + v) = +\infty$ and (44) is trivial. If $\nabla(u^0 + v) + c > 0$ a.e., then using the fact that $\Phi(\xi) = \xi^{-1}$ on $(0, +\infty)$ (which is convex) gives

$$\Phi'(\nabla \cdot u^0 + c)\nabla \cdot v \leq \Phi(\nabla \cdot (u^0 + v) + c) - \Phi(\nabla \cdot u^0 + c),$$

hence integrating on D gives

$$\int_D \Phi'(\nabla \cdot u^0 + c)\nabla \cdot v \,\mathrm{d}x \leq \int_D [\Phi(\nabla \cdot (u^0 + v) + c) - \Phi(\nabla \cdot u^0 + c)] \,\mathrm{d}x = \phi(u^0 + v) - \phi(u^0),$$

i.e.,

$$-\nabla\Phi'(\nabla \cdot u^0 + c) = \nabla(\nabla \cdot u^0 + c)^{-2} \in \partial\phi(u^0).$$

To prove that $\partial\phi(u^0)$ contains *only* $\nabla(\nabla \cdot u^0 + c)^{-2}$, we choose v such that $\nabla \cdot v \in L^\infty(D)$, and compute the Gâteaux derivative

$$\lim_{\varepsilon \to 0} \frac{\phi(u^0 + \varepsilon v) - \phi(u^0)}{\varepsilon} = \lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \int_D \frac{-\varepsilon\nabla \cdot v}{(\nabla \cdot (u^0 + \varepsilon v) + c)(\nabla \cdot u^0 + c)} \,\mathrm{d}x.$$

It is clear that

$$\frac{\nabla \cdot v}{(\nabla \cdot (u^0 + \varepsilon v) + c)(\nabla \cdot u^0 + c)} \to \frac{\nabla \cdot v}{(\nabla \cdot u^0 + c)^2} \qquad \text{for a.e. } x \in D.$$

To find a dominating function, note that since $\nabla \cdot u^0 + c \geq \alpha > 0$ a.e., and $\nabla \cdot v \in L^\infty(D)$, for all ε such that $\varepsilon\|\nabla \cdot v\|_{L^\infty(D)} < \alpha/2$ it holds $\nabla \cdot (u^0 + \varepsilon v) + c \geq \alpha/2$, hence

$$\left|\frac{\nabla \cdot v}{(\nabla \cdot (u^0 + \varepsilon v) + c)(\nabla \cdot u^0 + c)}\right| \leq \frac{2\|\nabla \cdot v\|_{L^\infty(D)}}{\alpha(\nabla \cdot u^0 + c)} \in L^1(D)$$

in view of (28). Thus by Lebesgue dominated convergence theorem, we infer

$$\lim_{\varepsilon \to 0} \frac{\phi(u^0 + \varepsilon v) - \phi(u^0)}{\varepsilon} = \lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \int_D \frac{-\varepsilon\nabla \cdot v}{(\nabla \cdot (u^0 + \varepsilon v) + c)(\nabla \cdot u^0 + c)} \,\mathrm{d}x = -\int_D \frac{\nabla \cdot v}{(\nabla \cdot u^0 + c)^2} \,\mathrm{d}x.$$

Thus the Gâteaux derivative is well defined for all $v \in V$ with $\nabla \cdot v \in L^\infty(D)$. If there were another element $\eta \in \partial\phi(u^0)$, then for all $v \in V$ with $\nabla \cdot v \in L^\infty(D)$ we have

$$\lim_{\varepsilon \to 0} \frac{\phi(u^0 + \varepsilon v) - \phi(u^0)}{\varepsilon} \geq \langle -\nabla\Phi'(\nabla \cdot u^0 + c), v\rangle_{V',V}, \qquad \lim_{\varepsilon \to 0} \frac{\phi(u^0 - \varepsilon v) - \phi(u^0)}{\varepsilon} \geq \langle \nabla\Phi'(\nabla \cdot u^0 + c), v\rangle_{V',V},$$

$$\lim_{\varepsilon \to 0} \frac{\phi(u^0 + \varepsilon v) - \phi(u^0)}{\varepsilon} \geq \langle \eta, v\rangle_{V',V}, \qquad \lim_{\varepsilon \to 0} \frac{\phi(u^0 - \varepsilon v) - \phi(u^0)}{\varepsilon} \geq \langle -\eta, v\rangle_{V',V},$$

hence $\eta = -\nabla\Phi'(\nabla \cdot u^0 + c)$ for all $v \in V$ with $\nabla \cdot v \in L^\infty(D)$. As

$$\{v \in V : \nabla \cdot v \in L^\infty(D)\} \supseteq V \cap C_c^\infty(D),$$

it is dense in V, thus we infer $\eta = -\nabla\Phi'(\nabla \cdot u^0 + c)$ as elements of V'.

Hence $\partial\phi(u^0) = \{\nabla(\nabla\cdot u^0 + c)^{-2}\}$. Thus, if u^0 has also sufficiently regularity (e.g. if $u^0 \in W^{4,2}(D)\cap V$) such that

$$\nabla\Delta\nabla\cdot u^0 + \nabla\nabla\cdot u^0 + \nabla(\nabla\cdot u^0 + c)^{-2} \in L^2(D),$$

choosing

$$z^0 := -[\nabla\Delta\nabla\cdot u^0 + \nabla\nabla\cdot u^0 + \nabla(\nabla\cdot u^0 + c)^{-2}]$$

gives

$$\begin{aligned}
0 &= \int_D \langle z^0 + \nabla\Delta\nabla\cdot u^0 + \nabla\nabla\cdot u^0 + \nabla(\nabla\cdot u^0 + c)^{-2}, v - u^0\rangle \,\mathrm{d}x \\
&= \int_D [\langle z^0, v-u^0\rangle + \langle \nabla\nabla\cdot u^0, \nabla\nabla\cdot(v-u^0)\rangle \\
&\qquad - (\nabla\cdot u^0)(\nabla\cdot(v-u^0)) - (\nabla\cdot(v-u^0))(\nabla\cdot u^0 + c)^{-2}]\,\mathrm{d}x \\
&\le \int_D [\langle z^0, v-u^0\rangle + \langle \nabla\nabla\cdot u^0, \nabla\nabla\cdot(v-u^0)\rangle - (\nabla\cdot u^0)(\nabla\cdot(v-u^0))]\,\mathrm{d}x + \phi(v) - \phi(u^0),
\end{aligned}$$

for any $v \in V$, i.e. (8). □

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