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# Maximal monotone operator in non-reflexive Banach space and the application to thin film equation in epitaxial growth on vicinal surface

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**Abstract** In this work we consider

$$w_t = [(w_{hh} + c_0)^{-3}]_{hh}, \qquad w(0) = w^0, \tag{1}$$ {abs}

which is derived from a thin film equation for epitaxial growth on vicinal surface. We establish a general framework for abstract evolution equations with maximal monotone operators in non-reflexive Banach space which can be used in a wide class of degenerate parabolic equations. Following this framework, we formulate a global strong solution to (1) which allows a Radon measure occurring. Then we prove the existence of such a global strong solution and obtain the almost everywhere positivity of $w_{hh} + c_0$.

## 1 Introduction

1.1 Background and motivation.

Below the roughening transition temperature, the crystal surface is not smooth and forms steps, terraces and adatoms on the substrate, which form solid films. Adatoms detach from steps, diffuse on the

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terraces until they meet other steps and reattach again, which lead to a step flow on the crystal surface. The evolution of individual steps is described mathematically by the Burton-Cabrera-Frank (BCF) type discrete models [5]. Although discrete models do have the advantage of reflecting physical principle directly, when we study the evolution of crystal growth from macroscopic view, continuum approximation for the discrete models involves fewer variables than discrete models and can briefly show the evolution of step flow. Many interesting continuum models can be found in the literature on surface morphological evolution; see [20,23,7,25,26,22,18,6,11] for one dimensional models and [17,27] for two dimensional models. Kohn clarified the evolution of surface height from the thermodynamic viewpoint in the book [14]. He considered the classical surface energy, which dates back to the pioneering work of Mullins [19], given by

$$F(h) := \int_{\Omega} (\beta_1 |\nabla h| + \beta_3 |\nabla h|^3)\,\mathrm{d}x, \tag{2}$$

where $\Omega$ is the "step locations area" we are concerned with. Then, by conservation of mass, we have the equation for surface height $h$

{heighteq}

$$\begin{aligned} h_t &= -\nabla\Big(M(\nabla h)\nabla \frac{\delta F}{\delta h}\Big) \\ &= -\nabla\Big(M(\nabla h)\nabla\Big(\nabla\cdot\Big(\beta_1 \frac{\nabla h}{|\nabla h|} + \beta_3 |\nabla h| \nabla h\Big)\Big)\Big), \end{aligned} \tag{3}$$

where $M(\nabla h)$ is a suitable "mobility" term depending on the dominating process of surface motion. Often two limit cases are considered. For diffusion-limited (DL) case, the dominated dynamics is diffusion across the terraces, we have $M(\nabla h) = 1$; while for attachment-detachment-limited (ADL) case, the dominating processes are the attachment and detachment of atoms at steps edges, and $M(\nabla h) = \frac{1}{|\nabla h|}$. In the DL regime, [13] obtained a fully understanding of the evolution and proved the finite-time flattening. However, in the ADL regime, due to the difficulty brought by mobility term $M(\nabla h) = \frac{1}{|\nabla h|}$, the dynamics of the solution to surface height equation (3), with both $\beta_1 = 0$ or $\beta_1 \neq 0$, is still an open question (see for instance [14]).

Although a general surface may have peaks and valleys, the analysis of step motion on the level of continuous PDE is complicated and we focus on a simpler situation first: a monotone one-dimensional step train. In this simpler case, $\beta_1 = 0$, and by taking $\beta_3 = \frac{1}{2}$, (3) becomes

$$h_t = -\Big[\frac{1}{h_x}\big(3h_x h_{xx}\big)_x\Big]_x. \tag{4}$$

Ozdemir, Zangwill [20] and Al Hajj Shehadeh, Kohn and Weare [1] realized that using the step slope as a new variable is a convenient way to study the continuum PDE model, i.e.,

{ueq}

$$u_t = -u^2 (u^3)_{hhhh}, \quad u(0) = u_0, \tag{5}$$

where $u$, considered as a $[0,1)$-periodic function of the step height $h$, is the step slope of the surface. [11] provided a method to rigorously obtain the convergence rate of discrete model to its corresponding continuum approximation.

Two questions then arise. One is how to formulate a proper solution to (5) and prove the well-posedness of its solution. The other one is the positivity of the solution. More explicitly, we want to know whether the sign of the solution $u$ to (5) is persistent. Our goal in this work is to validate the continuum slope PDE (5) by answering the above two questions. The equation (5) is a degenerate equation and we cannot prevent $u$ from touching zero, where singularity arise. We observe that we are able to rewrite (5) as an abstract evolution equation with maximal monotone operator using $\frac{1}{u}$. However, the main difficulty is that we have to work in a non-reflexive Banach space $L^1$, which does not possess weak compactness, so the classical theorem for maximal monotone operators in reflexive Banach space cannot be applied directly. In fact, due to the loss of weak compactness it is natural to allow a Radon measure being our solution $\frac{1}{u}$ and we do observe the singularity when $u$ approaches zero in numerical simulations [10]. Also see [15] for an example where a measure appears in the case of an exponential nonlinearity. Therefore, we devote ourselves to the establishment of a general abstract framework for problems associated with nonlinear monotone operators in non-reflexive Banach spaces and to solve our problem (5) by the abstract framework. Furthermore, the established abstract framework can be applied to a wide class of degenerate parabolic equations which can be recast as an abstract evolution equation with maximal monotone operator in some non-reflexive Banach space. We refer [2] for deep study of monotone operators in metric space and the existence of curves of maximal slope. However, we do not have $\lambda$-convex functionals in our problem. Let us introduce our framework in the next section.

### 1.2 Formal observations and abstract setup

Denote by $\varphi(h,t)$ as the step location when considered as a function of surface height $h$. Formally, we have

$$u(h,t) = h_x(\varphi(h,t),t) = \frac{1}{\varphi_h(h,t)},$$

and the $u$-equation (5) can be rewritten as $\varphi$-equation

$$\varphi_t = \left(\frac{1}{\varphi_h^3}\right)_{hhh}; \tag{6}$$

{eq:phin}

for further details we refer to the appendix of [12].

Motivated by the $\varphi$-equation, we want to recast (5) as an abstract evolution equation. If $u$ has a positive lower-bound $u \geq \alpha > 0$, then (5) can be rewritten as

$$\left(\frac{1}{u}\right)_t = (u^3)_{hhhh}, \quad u(0) = u_0. \tag{7}$$

{nueq}

Formally, if we take $w_{hh} = \frac{1}{u}$, then we have

$$w_t = (w_{hh}^{-3})_{hh}.$$

Since our problem (7) is in 1-periodic setting, i.e., one period $[0,1)$, we also want $w$ to be periodic. Denote by $\mathbb{T}$ the $[0,1)$-torus. For measure space, we can define periodic distributions as distributions on $\mathbb{T}$, i.e., linear functionals on $C(\mathbb{T})$. Let the $[0,1)$-periodic function $w$ be the solution of the Laplace equation

$$w_{hh} = \frac{1}{u} - c_0, \quad \int_{\mathbb{T}} w \, \mathrm{d}h = 0,$$

with compatibility condition

$$\int_{\mathbb{T}} w_{hh} \, \mathrm{d}h = \int_{\mathbb{T}} \frac{1}{u} - c_0 \, \mathrm{d}h = 0.$$

If (7) holds a.e., then we have

$$\int_{\mathbb{T}} \frac{1}{u} \, \mathrm{d}h \equiv \int_{\mathbb{T}} \frac{1}{u_0} \, \mathrm{d}h =: c_0$$

due to the periodicity of $u$. However, we cannot show that (7) holds almost everywhere. Actually, the possible existence of singular part for $w_{hh}$ or $\frac{1}{u}$ is intrinsic, since the equation (5) becomes degenerated when $u$ approaches zero. We cannot prevent $u$ from touching zero, and can only show $w_{hh} = \frac{1}{u} - c_0 \in$

$\mathcal{M}(\mathbb{T})$, where $\mathcal{M}$ is the set of finitely additive, finite, signed Radon measures. Hence the compatibility condition becomes

$$\int_{\mathbb{T}} \mathrm{d}\big(\frac{1}{u} - c_0\big) = 0.$$

Therefore, in this paper we consider the parabolic evolution equation

$$w_t = [(w_{hh} + c_0)^{-3}]_{hh}, \qquad w(0) = w^0, \tag{8}$$

under the assumption $w$ is periodic with period $\mathbb{T}$ and has mean value zero in one period, i.e., $\int_{\mathbb{T}} w \,\mathrm{d}h = 0$.

For $1 \le p < \infty$, $k \in \mathbb{Z}$, set

$$W^{k,p}_{\mathbb{T}_0}(\mathbb{T}) := \{u \in W^{k,p}(\mathbb{T}); u(h) = u(h+1), \text{a.e. and } u \text{ has mean value zero in one period}\},$$
$$L^{p}_{\mathbb{T}_0}(\mathbb{T}) := \{u \in L^{p}(\mathbb{T}); u(h) = u(h+1), \text{a.e. and } u \text{ has mean value zero in one period}\}.$$

Standard notations for Sobolev spaces are assumed above. If $k < 0$ and $(1/p) + (1/q) = 1$, $1 \le p < +\infty$, then it can be shown that $W^{k,q}_{\mathbb{T}_0}(\mathbb{T})$ is the dual of $W^{-k,p}_{\mathbb{T}_0}(\mathbb{T})$.

Our main functional spaces will be

$$V := \{v \in W^{2,1}_{\mathbb{T}_0}(\mathbb{T})\}, \tag{9}$$

and

$$\widetilde{V} := \{u \in W^{1,2}_{\mathbb{T}_0}(\mathbb{T}); u_{hh} \in \mathcal{M}(\mathbb{T})\}. \tag{10}$$

Define also

$$U := \{v \in L^{2}_{\mathbb{T}_0}(\mathbb{T})\}. \tag{11}$$

Endow $U$ and $V$ with the norms $\|u\|_U := \|u\|_{L^2(\mathbb{T})}$ and $\|v\|_V := \|v_{hh}\|_{L^1(\mathbb{T})}$ respectively. Note that the zero-mean conditions for functions of $V$ give the equivalence between $\|\cdot\|_V$ and $\|\cdot\|_{W^{2,1}(\mathbb{T})}$. Note also that the embeddings $V \hookrightarrow U \hookrightarrow V'$ are all dense and continuous.

The space $\widetilde{V}$.

Note also that any $\mathbb{T}$-periodic function $w$ who has mean value zero such that $w_{hh}$ is a finite Radon measure will belong to $W^{1,2}_{\mathbb{T}_0}(\mathbb{T})$, since the first derivative $w_h$ is a BV function (the total variation of $w_h$ is exactly the total mass of $w_{hh}$). Thus we can endow the space $\widetilde{V}$ with the norm

$$\|w\|_{W^{1,2}(\mathbb{T})} + \|w_{hh}\|_{\mathcal{M}(\mathbb{T})}.$$

Since $w$ is periodic and has mean value zero, we have

$$\|w_h\|_{L^2(\mathbb{T})} \le \|w_{hh}\|_{\mathcal{M}(\mathbb{T})}, \qquad \|w\|_{L^2(\mathbb{T})} \le \|w_h\|_{L^2(\mathbb{T})}.$$

So we can use the equivalent norm

$$\|w\|_{\widetilde{V}} := \|w_{hh}\|_{\mathcal{M}(\mathbb{T})} = \sup_{f \in C(\mathbb{T}),\ |f| \le 1} \int_{\mathbb{T}} f \,\mathrm{d}w_{hh}.$$

The weak -* convergence on $\widetilde{V}$ is then characterized as: a sequence $w^n$ converges weakly-* to $w$ in $\widetilde{V}$ if $w^n$ converges weakly to $w$ in $W^{1,2}(\mathbb{T})$, and $w^n_{hh}$ converges weakly -* to $w_{hh}$ in $\mathcal{M}(\mathbb{T})$, i.e.

$$\int_{\mathbb{T}} f \,\mathrm{d}w^n_{hh} \to \int_{\mathbb{T}} f \,\mathrm{d}w_{hh} \qquad \text{for any } f \in C(\mathbb{T}).$$

Relations between $V$ and $\widetilde{V}$.

Since $V$ is not reflexive, we first present a characterization of the bidual space $V''$. For any $v \in V$, we have $v_{hh} \in L^1_{\mathbb{T}_0}(\mathbb{T})$. Since also $C(\mathbb{T}) \subseteq L^\infty(\mathbb{T})$, we have:

(i) the dual space $V' = \{u \in (W^{2,1}_{\mathbb{T}_0}(\mathbb{T}))'\} = W^{-2,\infty}_{\mathbb{T}_0}(\mathbb{T})$ ;

(ii) for any $\xi \in V'$, $\eta \in V$, from the Riesz representation, there exists $\bar{\xi} \in L^\infty(\mathbb{T})$ such that

$$\langle \xi, \eta \rangle_{V',V} := \int_{\mathbb{T}} \bar{\xi} \eta_{hh} \, \mathrm{d}h,$$

and we denote $\bar{\xi}_{hh}$ as $\xi$ without risk of confusion;

(iii) the bidual space $V''$ is a subspace of $\widetilde{V}$.

Thus

$$V \subseteq V'' \subseteq \widetilde{V} \subseteq W^{1,2}_{\mathbb{T}_0}(\mathbb{T}) \subseteq U.$$

Therefore, we conclude that the canonical embedding $V \hookrightarrow V'' \hookrightarrow \widetilde{V} \hookrightarrow W^{1,2}_{\mathbb{T}_0}(\mathbb{T}) \hookrightarrow U$ is continuous and each one is a dense subset of the next, since $V$ is dense in $U$.

Observation 1.

From (8), one formal observation is that if we set

$$\phi(w) := \frac{1}{2} \int_{\mathbb{T}} (w_{hh} + c_0)^{-2} \, \mathrm{d}h, \tag{12}$$

then

$$w_t = -\frac{\delta \phi}{\delta w} = [(w_{hh} + c_0)^{-3}]_{hh}$$

is the first variation of $\phi$; see exact definition and calculation in (18) and Proposition 15. Hence we have

$$\frac{\mathrm{d}\phi}{\mathrm{d}t} = \int_{\mathbb{T}} \frac{\delta \phi}{\delta w} w_t \, \mathrm{d}h = \int_{\mathbb{T}} -w_t^2 \, \mathrm{d}h = \int_{\mathbb{T}} [(w_{hh} + c_0)^{-3}]^2_{hh} \, \mathrm{d}h \leq 0. \tag{13}$$

Besides, we also notice that $\phi(w) = \frac{1}{2} \int_{\mathbb{T}} (w_{hh} + c_0)^{-2} \, \mathrm{d}h$ is a convex functional. Recall that the sub-differential of a proper, convex, lower-semicontinuous function is a maximal monotone operator (see for instance [3]), which gives us the idea of using maximal monotone operator to formally rewrite our problem (8), i.e.,

$$w_t = -\partial \phi(w). \tag{14}$$

Observation 2.

Set also

$$E(w) := \frac{1}{2} \int_{\mathbb{T}} [(w_{hh} + c_0)^{-3}]^2_{hh} \, \mathrm{d}h = \int_{\mathbb{T}} w_t^2 \, \mathrm{d}h; \tag{15}$$

see exact definition in Definition 3. Taking the derivative $\partial_{hh}$ on the both side of (8), we have

$$[w_{hh} + c_0]_t = [(w_{hh} + c_0)^{-3}]_{hhhh}. \tag{16}$$ {whh}

Then another formal observation is that

$$\begin{aligned}\frac{\mathrm{d}E(w)}{\mathrm{d}t} &= \int_{\mathbb{T}} [(w_{hh}+c_0)^{-3}]_{hh}[(w_{hh}+c_0)^{-3}]_{hht}\,\mathrm{d}h \\ &= \int_{\mathbb{T}} [(w_{hh}+c_0)^{-3}]_{hhhh}[(w_{hh}+c_0)^{-3}]_{t}\,\mathrm{d}h = \int_{\mathbb{T}} [w_{hh}+c_0]_t[(w_{hh}+c_0)^{-3}]_{t}\,\mathrm{d}h \\ &= \int_{\mathbb{T}} -3\frac{[(w_{hh}+c_0)_t]^2}{(w_{hh}+c_0)^4}\,\mathrm{d}h \le 0.\end{aligned}$$

We point out the dissipation of $E(w)$ is important for the proof of existence result.

Observation 3.

Moreover, to ensure the surjectivity of the maximal monotone operator $\partial\phi$, we need to find a proper invariant ball. Another formal observation from (16) is that

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{T}}(w_{hh}+c_0)\,\mathrm{d}h = \int_{\mathbb{T}}[(w_{hh}+c_0)^{-3}]_{hhhh}\,\mathrm{d}h = 0.$$

So for a constant $C$ depending only on the initial value $w_0$, $\{\|w\|_V \le C\}$ could be an invariant ball provided $w_{hh}+c_0>0$ almost everywhere. But note that $V$ is not a reflexive space and that bounded sets in $L^1(\mathbb{T})$ do not have any compactness property. Actually we only obtain

$$w_{hh}+c_0>0, \text{ a.e. } (t,h)\in[0,T]\times\mathbb{T},$$

$$\int_{\mathbb{T}}\mathrm{d}(w_{hh}+c_0)\le C, \text{ for any } t\ge 0,$$

and choose $\{\|w\|_{\widetilde{V}}\le C\}$ to be the invariant ball. That is consistent with the prediction that $w_{hh}=\frac{1}{u}-c_0$ is possible to be a Randon measure.

After those formal observations, in order to rewrite our problem as an abstract problem precisely, we introduce the following definition.

**Definition 1** *For any $w\in\widetilde{V}$, from [8, p.42], we have the decomposition*

{decom}
$$w_{hh}=\eta+\nu \tag{17}$$

*with respect to the Lebesgue measure, where $\eta\in L^1(\mathbb{T})$ is the absolutely continuous part of $w_{hh}$ and $\nu$ is the singular part, i.e., the support of $\nu$ has Lebesgue measure zero. Recall $c_0$ is a constant in (8). Denote $g:=\eta+c_0$. Then $w_{hh}+c_0=g+\nu$ and $g\in L^1(\mathbb{T})$ is the absolutely continuous part of $w_{hh}+c_0$.*

*Define the proper, convex functional*

{defphi}
$$\phi:\widetilde{V}\longrightarrow\mathbb{R}\cup\{+\infty\},\quad \phi(w):=\begin{cases}\int_0^1\Phi(g)\,\mathrm{d}h & \text{if } w_{hh}+c\in\mathcal{M}^+(\mathbb{T}),\\ +\infty & \text{otherwise,}\end{cases}\qquad \Phi(x):=\begin{cases}+\infty & \text{if } x\le 0,\\ x^{-2}/2 & \text{if } x>0,\end{cases} \tag{18}$$

*where $g\in L^1(\mathbb{T})$ is the absolutely continuous part of $w_{hh}+c_0$. For some constant $C>0$ large enough, define the proper, convex functional*

{defpsi}
$$\psi:\widetilde{V}\longrightarrow\{0,+\infty\},\qquad \psi(w):=\begin{cases}0 & \text{if } \|w\|_{\widetilde{V}}\le C,\\ +\infty & \text{if } \|w\|_{\widetilde{V}}>C.\end{cases} \tag{19}$$

The domain of $\phi + \psi$ is

$$D(\phi + \psi) := \{w \in \widetilde{V};\ (\phi + \psi)(w) < +\infty\} \subseteq \widetilde{V} \cap \{\|w\|_{\widetilde{V}} \leq C\}.$$

Note that $\mathcal{K} := \{w \in \widetilde{V} : \|w\|_{\widetilde{V}} \leq C\}$ is closed and convex, hence its indicator (i.e., $\psi$) is convex, lower-semicontinuous and proper. Later, we will determine the constant $C$ by initial data $w_0$.

Now we can state two definitions of solutions we study in this work.

**Definition 2** *Given $\phi$, $\psi$ defined in Definition 1, for any $T > 0$, we call the function*

$$w \in L^\infty([0,T];\widetilde{V}) \cap C^0([0,T];U), \quad w_t \in L^\infty([0,T];U)$$

*a variational inequality solution to* (8) *if it satisfies*

$$\langle w_t, v - w\rangle_{U',U} + (\phi + \psi)(v) - (\phi + \psi)(w) \geq 0 \tag{20}$$ {vi}

*for a.e. $t \in [0,T]$ and all $v \in \widetilde{V}$.*

**Definition 3** *For any $T > 0$, let $\eta \in L^1(\mathbb{T})$ be the absolutely continuous part of $w_{hh}$ in* (17)*. Define*

$$E(t) := \frac{1}{2}\int_{\mathbb{T}} \left[((\eta + c_0)^{-3})_{hh}\right]^2 \mathrm{d}h. \tag{21}$$

*We call the function*

$$w \in L^\infty([0,T];\widetilde{V}) \cap C^0([0,T];U), \quad w_t \in L^\infty([0,T];U)$$

*a strong solution to* (8) *if*

*(i) it satisfies*

$$w_t = [(\eta + c_0)^{-3}]_{hh} \tag{22}$$ {slu}

*for a.e. $(t,h) \in [0,T] \times \mathbb{T}$;*

*(ii) we have $((\eta + c_0)^{-3})_{hh} \in L^\infty([0,T];U)$ and the dissipation inequality*

$$E(t) = \frac{1}{2}\int_{\mathbb{T}} \left[((\eta + c_0)^{-3})_{hh}\right]^2 \mathrm{d}h \leq E(0). \tag{23}$$

The main result in this work is to prove existence of the variational inequality solution and strong solution to (8), which is stated in Theorem 14 and Theorem 16 separately.

### 1.3 Overview of our method and related method

The key of our method is to rewrite the original problem as an abstract evolution equation $w_t = -\tilde{B}w$, where $\tilde{B}$ is the sub-differential of a proper, convex, lower semi-continuous function, i.e. $\tilde{B} = \partial(\phi + \psi)$. $\tilde{B}$ is a maximal monotone operator by classical results (see for instance [3]). $\psi$ is the indicator of the invariant ball $\mathcal{K}$ in (19). By constructing the proper invariant ball $\mathcal{K}$, we also obtain the restriction of $\tilde{B}$ to $L^2(\mathbb{T})$ is also a maximal monotone operator; see Lemma 11. Notice the definition of the functional $\phi$ involves only the absolutely continuous part of $w_{hh}$, so we need to prove that it is still lower semi-continuous on $\widetilde{V}$; see details in Proposition 7. Then by standard theorem for m-accretive operator (see Definition 6) in [3], we can prove the variational inequality solution to (8) in Theorem 14. Another key point is to prove the multi-valued operator $\partial(\phi + \psi)$ is actually single valued, which concludes that the variational inequality solution is also the strong solution defined in Definition 3. In order to prove $\partial(\phi + \psi)$ is single valued, we need to first determine $\partial\phi$ in Proposition 15, then make two *a-priori* assumptions and finally verify them using those dissipation observations in Section 1.2; see details in Theorem 16.

Actually, our definitions for variational inequality solution and strong solution in Definitions 2 and 3 hide a Radon measure in it. As we said before, this kind of fourth order degenerate equation has the intrinsic property of singular measure. We want to mention that [9] also used maximal monotone operator method for diffusion limited (DL) case. However, due to the mobility for DL model is $M = 1$ rather than $M = \frac{1}{h_x}$ in our case, DL model can be recast as an abstract evolution equation with maximal monotone operator using the anti-derivative of $h$. The coercivity of the this maximal monotone operator in DL case is natural and hence the operated space is a reflexive space. It is much easier than our case and singular part will not appear.

Recently, [12,16] also analyzed the positivity and the weak solution to the same equation (5) separately. They all considered this nonlinear fourth order parabolic equation, which comes from the same step flow model on vicinal surface. The aim is to answer the two questions in Section 1.1, which also are stated as open questions in [14]. The nonlinear structure of this equation, the key for both previous and current works, is important for the positivity of solution because we know that the sign changing is a general property for solutions to linear fourth order parabolic equations. For one dimensional case, following the regularized method in [4], [12] defined the weak solution on a subset, which has full measure, of $[0, 1]$ and proved positivity and existence. Using the method of approximating solutions, based on the implicit time-discretization scheme and carefully chosen regularization, [16] expanded the result in [12] to high dimensional case. Our results is consistent with theirs, but uses a totally different approach. Paper [16]'s method is delicate and subtle while our method seems to be more natural and we obtain the variational inequality solution to (8).

We point out that our method establishes a general structure for this kind of equation whose invariant ball exists in a non-reflexive Banach space. We believe this method can be applied to many similar degenerated problems as long as they can be reduced to an abstract evolution equation with maximal monotone operator which unfortunately is a non-reflexive Banach space.

The rest of this work is devoted to first recall some useful definitions in Section 1.4. Then in Section 2, we rigorously study the sub-differential $\partial(\phi+\psi)$ and prove it is m-accretive on $U$, which leads to the existence result for variational inequality solution. In Section 3, we calculate the exact value of $\partial(\phi+\psi)$ and prove the variational inequality solution is actually a strong solution.

### 1.4 Preliminaries

In this section, we first recall the following classical definitions (see for instance [3]).

**Definition 4** *Given a Banach space $X$ with the duality pairing $\langle , \rangle_{X',X}$, an element $x \in X$, a functional $f : X \longrightarrow \mathbb{R} \cup \{+\infty\}$, the sub-differential of $f$ at $x$ is the set defined as*

$$\partial f(x) := \{x' \in X' : f(y) - f(x) \geq \langle x', y - x\rangle_{X',X} \ \forall y \in X\}.$$

*We denote the domain of $\partial f$ as usual by $D(\partial f)$, i.e. the set of all $x \in X$ such that $\partial f(x) \neq \emptyset$.*

**Definition 5** *Given a Banach space $X$ with the duality pairing $\langle , \rangle_{X',X}$, a multivalued operator $A : X \longrightarrow X'$ identified with its graph $\Gamma_A := \{(u, Au) : u \in X\} \subseteq X \times X'$ is:*

1. **monotone** *if for any pair $(u, u')$, $(v, v') \in \Gamma_A$, it holds*

$$\langle u' - v', u - v\rangle_{X',X} \geq 0;$$

2. **maximal monotone** *if the graph $\Gamma_A$ is not a proper subset of any monotone set;*

***Definition 6*** *Given a Hilbert space $X$, a multivalued operator $B : X \longrightarrow X$, with graph $\Gamma_B(X) := \{(x, Bx) : x \in X \text{ such that } Bx \in X)\}$. Denote $J_X : X \to X'$ as the canonical isomorphism of $X$ to $X'$. $B$ is*

1. **accretive** *if for any pair $(x, Bx)$, $(y, By)$, there exists an element $z \in J_X(x-y)$ such that $\langle z, Bx - By\rangle_{X',X} \geq 0$;*
2. **m-accretive** *if it is accretive and $R(I + B) = X$, where $R(I + B)$ denotes the range of $(I + B)$;*

*Remark:* For general Banach space, $J_X$ is the duality mapping of $X$; see details in [3, Section 1.1]. In our case, $X = U$, so $J_X$ is the identity operator $I$ in $U$.

## 2 Existence result for variational inequality solution

This section is devoted to obtain a variational inequality solution to (8). By restricting the operator in the non-reflexive Banach space $\widetilde{V}$ to $U$, we want to apply the classical result for m-accretive operator in $U$. However, since we do not have weak compactness for sequences in $V$, and a Radon measure may appear when taking the limit, we need to first prove weak-* lower semi-continuity for functional $\phi$ in $\widetilde{V}$.

### 2.1 Weak-* lower semi-continuity for functional in $\widetilde{V}$.

Since for any $w \in \widetilde{V}$, $\phi$ defined only on its absolutely continuous part, we need the following proposition to guarantee $\phi$ is lower-semi-continuous with respect to the weakly-* convergence in $\widetilde{V}$.

**Proposition 7** *The function $\phi$ defined in Definition 1 is lower semi-continuous with respect to the weakly-$*$ convergence in $\widetilde{V}$, i.e., if $w_n \overset{*}{\rightharpoonup} w$ in $\widetilde{V}$, we have*

$$\liminf_{n\to+\infty} \phi(w_n) \geq \phi(w).$$

For any $\mu \in \mathcal{M}(\mathbb{T})$, we denote $\mu \ll \mathcal{L}^1$ if $\mu$ is absolutely continuous with respect to Lebesgue measure and denote $\bar{\mu} := \frac{\mathrm{d}\mu}{\mathrm{d}\mathcal{L}^1}$ as the density of $\mu$. For notational simplification, denote $\mu_\parallel$ (resp. $\mu_\perp$) as the absolutely continuous part (resp. singular part) of $\mu$ with respect to Lebesgue measure. Before proving Proposition 7, we first state some lemmas. The following Lemma comes from the weak-* compactness of $L^\infty$.

**Lemma 8** *For any $N \geq 0$, given a sequence of measures $\mu_n$ in $\mathcal{M}(\mathbb{T})$, such that each $\mu_n \ll \mathcal{L}^1$ for any $n$, and the densities satisfy*

$$\sup_n \|\bar{\mu}_n\|_{L^\infty(\mathbb{T})} \leq N,$$

*then there exists a measure $\mu \ll \mathcal{L}^1$, $\|\bar{\mu}\|_{L^\infty(\mathbb{T})} \leq N$ and a subsequence $\mu_{n_k}$ such that $\mu_{n_k} \overset{*}{\rightharpoonup} \mu$ in $\mathcal{M}(\mathbb{T})$.*

From now on, we identify $\mu_n$ with its density $\bar{\mu}_n := \frac{\mathrm{d}\mu_n}{\mathrm{d}\mathcal{L}^1}$ and do not distinguish them for brevity. Given a sequence of measures $\mu_n$ such that $\mu_n \ll \mathcal{L}^1$, $\bar{\mu}_n \geq 0$ and $N > 0$, observe that

$$\mu_n = \min\{\mu_n, N\} + \max\{\mu_n, N\} - N. \tag{24}$$ {b}

From Lemma 8 we know, upon subsequence, $\min\{\mu_n, N\} \overset{*}{\rightharpoonup} \mu_-$ for some measure $\mu_-$ satisfying $\mu_- \ll \mathcal{L}^1$ and $N \geq \mu_- \geq 0$. We also need the following useful Lemma to clarify the relation between $\mu_-$ and the weak-$*$ limit of $\mu_n$.

**Lemma 9** *Given a sequence of measures $\mu_n$ such that $\mu_n \ll \mathcal{L}^1$, $\bar{\mu}_n \geq 0$, we assume moreover that $\mu_n \overset{*}{\rightharpoonup} \mu$, for some measure $\mu \geq 0$. Then for any $N > 0$, there exist $\mu_-$, $\mu_+ \in \mathcal{M}(\mathbb{T})$, such that*

$$\min\{\mu_n, N\} \overset{*}{\rightharpoonup} \mu_-, \quad \mu_- \ll \mathcal{L}^1, \quad \mu_- \leq \mu_\parallel, \tag{25}$$

{starlem1}

$$\max\{\mu_n, N\} \overset{*}{\rightharpoonup} \mu_+, \quad \mu_{+\parallel} \geq N, \tag{26}$$

{starlem2}

*where $\mu_\parallel$ (resp. $\mu_\perp$) is the absolutely continuous part (resp. singular part) of $\mu$. Moreover, for the function $\Phi$ defined in* (18)*, we have*

{PHI}

$$\Phi(\mu_\parallel) \leq \Phi(\mu_-). \tag{27}$$

*Proof* From Lemma 8 we know, upon subsequence, $\min\{\mu_n, N\} \overset{*}{\rightharpoonup} \mu_-$ for some measure $\mu_-$ satisfying $\mu_- \ll \mathcal{L}^1$ and $N \geq \mu_- \geq 0$. By Lebesgue decomposition theorem, there exist unique measures $\mu_\parallel \ll \mathcal{L}^1$ and $\mu_\perp \perp \mathcal{L}^1$ such that $\mu = \mu_\parallel + \mu_\perp$. The decomposition (24) then gives

$$0 \leq \mu_n - \min\{\mu_n, N\} = \max\{\mu_n, N\} - N \overset{*}{\rightharpoonup} \mu - \mu_-.$$

Taking $\mu_+ := \mu - \mu_- + N$, as the sequence $\max\{\mu_n, N\} - N \geq 0$, we obtain $\max\{\mu_n, N\} \overset{*}{\rightharpoonup} \mu_+$ and $(\mu - \mu_-)_\parallel = \mu_{+\parallel} - N \geq 0$. Besides, since $\Phi(\mu_\parallel)$ is decreasing with respect to $\mu_\parallel$, we obtain (27). □

Now we can start to prove Proposition 7.

*Proof (Proof of Proposition 7)* Without loss of generality we assume $\sup_{n\to+\infty} \phi(w_n) < +\infty$. This implies immediately that all $w_{nhh} + c_0$ are positive measures. Assume $w_n \overset{*}{\rightharpoonup} w$ in $\widetilde{V}$, thus we have $w_{nhh} \overset{*}{\rightharpoonup} w_{hh}$ in $\mathcal{M}$. Denote $f_n := w_{nhh} + c_0$ and $f := w_{hh} + c_0$. Since $\Phi(f_\parallel)$ is decreasing with respect to $f_\parallel$, we only concern the case $f_{n\parallel}$ may weakly-* converge to a singular measure. Thus without loss of generality, we assume $f_n \ll \mathcal{L}^1$, i.e., $f_{n\perp} = 0$. For any $M > 0$ large enough, denote $\phi_M(w_n) := \int_{\mathbb{T}} \Phi(\min\{f_n, M\})$. From the definition of $\Phi$ in (18), the truncated measures $\min\{f_n, M\}$ satisfy

$$\begin{aligned}
\phi_M(w_n) &= \int_{\mathbb{T}} \Phi(\min\{f_n, M\}) \,\mathrm{d}h \\
&= \int_{\{f_n \leq M\}} \Phi(\min\{f_n, M\}) \,\mathrm{d}h + \frac{1}{2M^2} \mathcal{L}^1(\{f_n > M\}) \\
&\geq \int_{\{f_n \leq M\}} \Phi(f_n) \,\mathrm{d}h + \int_{\{f_n > M\}} \Phi(f_n) \,\mathrm{d}h = \phi(w_n).
\end{aligned}$$

The second equality also shows

$$\begin{aligned}
\phi_M(w_n) - \frac{1}{2M^2} \mathcal{L}^1(\{f_n > M\}) &= \int_{\{f_n \leq M\}} \Phi(\min\{f_n, M\}) \,\mathrm{d}h \\
&\leq \int_{\mathbb{T}} \Phi(f_n) \,\mathrm{d}h = \phi(w_n).
\end{aligned}$$

Hence we obtain

{0104_28o}

$$|\phi(w_n) - \phi_M(w_n)| \leq \frac{1}{2M^2} \mathcal{L}^1(\{f_n > M\}) \leq \frac{1}{2M^2}. \tag{28}$$

From Lemma 8 and Lemma 9, we know the truncated sequence $\min\{f_n, M\}$ satisfies

{star0105_1}

$$\min\{f_n, M\} \overset{*}{\rightharpoonup} f_-, \quad f_- \ll \mathcal{L}^1, \quad \Phi(f_\parallel) \leq \Phi(f_-). \tag{29}$$

Hence by the convexity and lower semi-continuity of $\phi$ on $V$, we infer

{conv}

$$\liminf_{n\to+\infty} \int_{\mathbb{T}} \Phi(\min\{f_n, M\}) \,\mathrm{d}h \geq \int_{\mathbb{T}} \Phi(f_-) \,\mathrm{d}h \geq \int_{\mathbb{T}} \Phi(f_\parallel) \,\mathrm{d}h = \phi(w). \tag{30}$$

Combining this with (28), we obtain

$$\begin{aligned}\liminf_{n\to+\infty}\phi(w_n) &\geq \liminf_{n\to+\infty}\phi_M(w_n) - \frac{1}{2M^2}\\ &= \liminf_{n\to+\infty}\int_{\mathbb{T}}\Phi(\min\{f_n, M\})\,\mathrm{d}h - \frac{1}{2M^2}\\ &\geq \phi(w) - \frac{1}{2M^2},\end{aligned}\tag{31}$$

and thus we complete the proof of Proposition 7 by the arbitrariness of $M$. $\square$

## 2.2 Maximal monotone and m-accretive operator in $U$.

In this section, we first define the sub-differential of $\phi+\psi$ and then obtain some useful lemmas to ensure $\partial(\phi+\psi)$ is also a maximal monotone operator when restricted to $U$.

Let $\tilde{B} := \partial(\phi+\psi) : \widetilde{V} \to \widetilde{V}'$ be the sub-differential of $\phi+\psi$. Let us consider the operator $B$ as the restriction of $\tilde{B}$ from $U$ to $U'$.

**Definition 10** *Define the operator $B : D(B) \subseteq U \to U'$ such that*

$$Bw = \tilde{B}w, \text{ for any } w \in D(B) = \{w \in \widetilde{V}; \tilde{B}w \in U\}.$$

We first prove $B$ is maximal monotone in $U \times U'$, which is important to prove the existence result.

**Lemma 11** *The operator $B : D(B) \subseteq U \to U'$ in Definition 10 is maximal monotone in $U \times U'$.*

*Proof* It suffices to prove that $\phi+\psi$ is (i) proper, i.e. $D(\phi+\psi) \neq \emptyset$, (ii) convex and (iii) lower semi-continuous when considered as a functional from $(U, \|\cdot\|_U)$ to $\mathbb{R}\cup\{+\infty\}$. (i) First it is clear that $\phi+\psi$ is proper.

*(ii) Convexity.* Let $u_1, u_2 \in U$ be arbitrarily given, and we need to show

$$(1-t)(\phi+\psi)(u_1) + t(\phi+\psi)(u_2) \geq (\phi+\psi)((1-t)u_1 + tu_2).$$

If $u_1$ or $u_2$ does not belong to $D(\phi+\psi)$, then the left-hand side term is $+\infty$. If both $u_1$ and $u_2$ belong to $D(\phi+\psi)$, then $(1-t)u_1 + tu_2$ also belongs to $\widetilde{V} \cap \{\|\cdot\|_{\widetilde{V}} \leq C\}$, hence

$$\psi(u_1) = \psi(u_2) = \psi((1-t)u_1 + tu_2) = 0.$$

Notice the convexity of $\phi$, and the fact that the absolutely continuous part of $((1-t)u_1 + tu_2)_{hh}$ is $(1-t)((u_1)_{hh})_{\|} + t((u_2)_{hh})_{\|}$, where $((u_i)_{hh})_{\|}$ are notations representing the absolutely continuous parts of $(u_i)_{hh}$ separately. Then we obtain

$$(1-t)\phi(u_1) + t\phi(u_2) \geq \phi((1-t)u_1 + tu_2).$$

Thus $\phi+\psi$ is convex.

*(iii) Lower-semicontinuity.* Note that the lower-semicontinuity is here intended as with respect to the strong convergence in $U$ (less restrictive than the convergence in $\widetilde{V}$). Consider an arbitrary sequence $u^n \subseteq U$ converging to $u \in U$. We need to prove

$$(\phi+\psi)(u) \leq \liminf_{n\to+\infty}(\phi+\psi)(u^n).$$

If $\liminf_{n\to+\infty}(\phi+\psi)(u^n) = +\infty$ then the thesis is trivial. Thus assume (upon subsequence)

$$\liminf_{n\to+\infty}(\phi+\psi)(u^n) = \lim_{n\to+\infty}(\phi+\psi)(u^n) \le D < +\infty.$$

Without loss of generality we can further assume $(u^n)_n \subseteq \widetilde{V} \cap \{\|\cdot\|_{\widetilde{V}} \le C\}$. This implies $\|u^n_{hh}\|_{\mathcal{M}(\mathbb{T})} \le C$, so

$$\psi(u^n) = 0, \quad \forall n$$

and there exists $\xi \in \mathcal{M}(\mathbb{T})$ such that $u^n_{hh} \overset{*}{\rightharpoonup} \xi$ in $\mathcal{M}(\mathbb{T})$. Since from Proposition 7, $\phi+\psi$ is convex and lower-semicontinuous in $\widetilde{V}$, we infer

$$\phi(\xi) \le \liminf_{n\to+\infty} \phi(u^n). \tag{32}$$

The uniform boundedness of $\|u^n_{hh}\|_{\mathcal{M}(\mathbb{T})}$ also implies $(u^n)_n$ is bounded in $W^{1,\infty}(\mathbb{T})$ (and hence in $W^{1,p}(\mathbb{T})$ for any $p < +\infty$). Thus $(u^n)_n$ is (upon subsequence) weakly convergent in $W^{1,p}(\mathbb{T})$, and strongly convergent in $L^2(\mathbb{T})$ to $u$. Thus $u_{hh} = \xi$, and (32) now becomes

$$\phi(u_{hh}) = \phi(u) = \phi(\xi) \le \liminf_{n\to+\infty} \phi(u^n).$$

Therefore $\phi+\psi : U \longrightarrow \mathbb{R}\cup\{+\infty\}$ is proper, convex and lower-semicontinuous. Then by [3, Theorem 2.8] we have that $\partial(\phi+\psi) : U \longrightarrow U'$ is maximal monotone in $U \times U'$. $\square$

Notice $U' = U$. From Lemma 11 and the Definition 6 we deduce

**Proposition 12** *The operator $B : D(B) \subseteq U \to U'$ in Definition 10 is m-accretive from $D(B) \subseteq U$ to $U$.*

2.3 Existence result for variational inequality solution.

After those preliminary results, we can apply [3, Theorem 4.5] to obtain the existence of variational inequality solution to (8).

First let us recall [3, Theorem 4.5].

**Theorem 13** *([3, Theorem 4.5]) For any $T > 0$, let $U$ be a Hilbert space and let $B$ be a m-accretive operator from $D(B) \subseteq U$ to $U$. Then for each $y_0 \in D(B)$, the cauchy problem*

$$\begin{cases} \frac{\mathrm{d}y}{\mathrm{d}t}(t) + By(t) \ni 0, \quad t \in [0,T], \\ y(0) = y_0, \end{cases}$$

*has a unique strong solution $y \in W^{1,\infty}([0,T];U)$ in the sense that*

$$-\frac{\mathrm{d}y}{\mathrm{d}t}(t) \in By(t), \quad a.e.\, t \in (0,T), \qquad y(0) = y_0.$$

*Moreover, $y$ satisfies the estimate*

$$\|y_t\|_U \le |-By_0|_\star, \tag{33}$$

*where $|-By_0|_\star = \inf\{\|u\|_U; u \in -By_0\}$.*

Proposition 12 shows that $B$ defined in Definition 10 is m-accretive from $D(B) \subseteq U$ to $U$. Hence we can apply Theorem 13 to obtain

**Theorem 14** *Let $B : D(B) \subseteq U \to U$ be the operator defined in Definition 10. Given $T > 0$, initial datum $w^0 \in D(B)$, then* *(i) there exists a unique function* $w \in W^{1,\infty}([0,T];U)$ *such that*

{slu1} $$-w_t(t) \in Bw(t), \qquad w(0) = w^0, \quad \textit{for a.e.}\, t \in [0,T]. \tag{34}$$

*(ii)* $w$ *is also a variational inequality solution to* (8). *Moreover,*

{positive} $$\begin{aligned} &w_{hh} + c_0 \in \mathcal{M}^+(\mathbb{T}),\ \textit{for a.e.}\ (t,h) \in [0,T]\times\mathbb{T},\\ &\eta + c_0 > 0,\ \textit{for a.e.}\ (t,h) \in [0,T]\times\mathbb{T}, \end{aligned} \tag{35}$$

*with respect to the Lebesgue measure, where* $\eta$ *is the absolutely continuous part of* $w_{hh}$ *in* (17), *and* $\mathcal{M}^+(\mathbb{T})$ *denotes the set of positive Radon measures.*

*Proof* Proof of (i). From Proposition 12, we know $B$ is a m-accretive operator in $U \times U$. So (34) follows from Theorem 13 and $w \in W^{1,\infty}([0,T];U)$. From (33), we also have

$$\|w_t\|_U \le |-Bw_0|_\star, \tag{36}$$ {es1107_2}

where $|-Bw_0|_\star = \inf\{\|u\|_U; u \in -Bw_0\}$.

Proof of (ii). Since $-w_t(t) \in Bw(t) = \partial(\phi+\psi)(w)$ for $a.e.\, t \in [0,T]$ and $w \in W^{1,\infty}([0,T];U)$, by Definition 4, we have

$$\langle w_t, v-w\rangle_{U',U} + (\phi+\psi)(v) - (\phi+\psi)(w) \ge 0, \quad \text{a.e. } t \in [0,T], \tag{37}$$ {vi2}

for all $v \in U$, and

$$w \in C^0([0,T];U), \quad w_t \in L^\infty([0,T];U).$$

Let $v \in U$ such that $(\phi+\psi)(v) \le 1$. Then from (37), we also have

$$(\phi+\psi)(w) \le \|w_t\|_U \|w-v\|_U + 1 \quad \text{for a.e. } t \in [0,T]. \tag{38}$$ {45}

This implies

$$w \in L^\infty([0,T];\widetilde{V})$$

and with respect to the Lebesgue measure,

$$\eta + c_0 > 0 \ \text{ for a.e. } (t,h) \in [0,T]\times\mathbb{T},$$

$$w_{hh} + c_0 \in \mathcal{M}^+(\mathbb{T}) \ \text{ for a.e. } (t,h) \in [0,T]\times\mathbb{T},$$

where $\eta$ is the absolutely continuous part of $w_{hh}$ in (17). Therefore we obtain the variational inequality solution to (8) and $w$ satisfies the positivity property (35). □

## 3 Existence result for strong solution

Although we obtained a unique variational inequality solution in Theorem 14, we do not know whether $B$ is single-valued and which element belongs to $B$. We will determine $B$ in Section 3.1 and then prove the variational inequality solution is actually a strong solution to (8) in Section 3.2.

3.1 Sub-differential of $\phi$ is single-valued.

For any $f \in C(\mathbb{T})$, $f_{hh}$ induces a continuous linear form on $\widetilde{V}$ given by

$$f_{hh}(z) := \int_{\mathbb{T}} f \,\mathrm{d}z_{hh} < \infty \tag{39}$$ {n39}

for any $z \in \widetilde{V}$. Hence we can write $f_{hh} \in \widetilde{V}'$. In the following proposition, we calculate the sub-differential $\partial\phi(w)$, for $w \in D(\partial\phi) \cap \{w \in \widetilde{V}; g^{-3} \in C(\mathbb{T})\}$.

**Proposition 15** *Let $g \in L^1(\mathbb{T})$ be the absolutely continuous part of $w_{hh} + c_0$ from Definition 1. Let $\phi$ being the proper, convex, lower-semicontinuous functional defined in* (18). *Let*

$$\partial\phi : D(\partial\phi) \subseteq \widetilde{V} \longrightarrow \widetilde{V}'$$

*be the sub-differential of $\phi$. For any $w \in D(\partial\phi) \cap \{w \in \widetilde{V}; g^{-3} \in C(\mathbb{T})\}$, we have*

{sub21_2} $$\partial\phi(w) = \{-(g^{-3})_{hh}\} \in \widetilde{V}' \subseteq V', \tag{40}$$

*and*

$$\langle \partial\phi(w), v\rangle_{V',V} := \int_{\mathbb{T}} -g^{-3} v_{hh} \,\mathrm{d}h$$

*for any $v \in V$.*

*Proof* Step 1. Let

$$w \in D(\partial\phi) \cap \{w \in \widetilde{V}; g^{-3} \in C(\mathbb{T})\}$$

be given. We first prove that

{inc} $$\{-(g^{-3})_{hh}\} \subseteq \partial\phi(w). \tag{41}$$

Since $w \in D(\partial\phi)$, we know $\partial\phi(w) \neq \emptyset$ and $w \in D(\partial\phi) \subseteq D(\phi)$ by [3, Proposition 1.6]. Thus we can assume $\phi(w) < +\infty$, so $g > 0$ a.e. and $g^{-2} \in L^1(\mathbb{T})$.

Let $z \in V$ be given. If $\phi(w + \varepsilon z) = +\infty$, then the inequality

$$\phi(w + \varepsilon z) - \phi(w) \geq \langle -(g^{-3})_{hh}, \varepsilon z\rangle_{V',V}$$

is trivial.

If $\phi(w + \varepsilon z) < +\infty$, then $g + \varepsilon z_{hh} > 0$ a.e.. Using the convexity of $\Phi$, one can still have

{t08_1} $$\Phi(g + \varepsilon z_{hh}) - \Phi(g) \geq \Phi'(g)\varepsilon z_{hh} = -g^{-3}\varepsilon z_{hh} \qquad \text{for a.e. } h. \tag{42}$$

Since $g^{-3} \in C(\mathbb{T})$, the right-hand side of (42) is integrable, i.e.,

$$\int_{\mathbb{T}} \left|g^{-3}\varepsilon z_{hh}\right| \mathrm{d}h < +\infty. \tag{43}$$

Then by the definition for $\phi$, from (42), we have

$$\phi(w + \varepsilon z) - \phi(w) = \int_{\mathbb{T}} [\Phi(g + \varepsilon z_{hh}) - \Phi(g)] \,\mathrm{d}h \geq \int_{\mathbb{T}} -g^{-3}\varepsilon z_{hh} \,\mathrm{d}h \tag{44}$$

for any $z \in V$.

Since $g^{-3} \in C(\mathbb{T})$, from (39), we know

$$\{(g^{-3})_{hh}\} \subseteq \widetilde{V}' \subseteq V', \tag{45}$$

which implies that for any $v \in V$ we have

$$\int_{\mathbb{T}} (g^{-3})_{hh} v \, \mathrm{d}h = \int_{\mathbb{T}} g^{-3} v_{hh} \, \mathrm{d}h.$$

Hence $g^{-3} \in (L^1(\mathbb{T}))' \subseteq L^2(\mathbb{T})$.

Now in both cases we have

{Gvep}
$$\phi(w + \varepsilon z) - \phi(w) \geq \int_{\mathbb{T}} -g^{-3} \varepsilon z_{hh} \, \mathrm{d}h = \langle -(g^{-3})_{hh}, \varepsilon z \rangle_{V',V} \tag{46}$$

for any $z \in V$.

Note that for any $\xi \in \widetilde{V}'$, $z \in \widetilde{V}$, since $V$ is dense in $\widetilde{V}$, there exists a sequence $z^n \in V$ such that

$$\langle \xi, z \rangle_{\widetilde{V}',\widetilde{V}} = \lim_{n\to\infty} \langle \xi, z^n \rangle_{\widetilde{V}',\widetilde{V}} = \lim_{n\to\infty} \langle \xi, z^n \rangle_{V',V}. \tag{47}$$
{G2}

Taking $\xi = -(g^{-3})_{hh}$ in (47), together with (46), we obtain

$$\phi(w + \varepsilon z) - \phi(w) \geq \langle -(g^{-3})_{hh}, \varepsilon z \rangle_{\widetilde{V}',\widetilde{V}}, \tag{48}$$

and by the Definition 4, we have (41).

Step 2. Let

$$w \in D(\partial\phi) \cap \{w \in \widetilde{V}; g^{-3} \in C(\mathbb{T})\}$$

be given. We want to prove $\partial\phi$ is single-valued.

First we give a density argument. Assume for some $w$, there are two elements $\eta_1, \eta_2 \in \partial\phi(w)$. Assume the Gâteaux-derivative of $\phi$ exists for all $z$ in some dense set $G$ of $\widetilde{V}$. Then by the definition of sub-differential we have

$$\lim_{\varepsilon\to 0} \frac{\phi(w+\varepsilon z) - \phi(w)}{\varepsilon} \geq \langle \eta_1, z \rangle_{\widetilde{V}',\widetilde{V}}, \quad \lim_{\varepsilon\to 0} \frac{\phi(w-\varepsilon z) - \phi(w)}{\varepsilon} \geq -\langle \eta_1, z \rangle_{\widetilde{V}',\widetilde{V}},$$
$$\lim_{\varepsilon\to 0} \frac{\phi(w-\varepsilon z) - \phi(w)}{\varepsilon} \geq -\langle \eta_2, z \rangle_{\widetilde{V}',\widetilde{V}}, \quad \lim_{\varepsilon\to 0} \frac{\phi(w+\varepsilon z) - \phi(w)}{\varepsilon} \geq \langle \eta_2, z \rangle_{\widetilde{V}',\widetilde{V}}$$

for $z \in G$. Hence we have

$$\langle \eta_1 - \eta_2, z \rangle_{\widetilde{V}',\widetilde{V}} \leq 0, \quad \langle \eta_2 - \eta_1, z \rangle_{\widetilde{V}',\widetilde{V}} \leq 0, \tag{49}$$

and then

$$\langle \eta_1, z \rangle_{\widetilde{V}',\widetilde{V}} = \langle \eta_2, z \rangle_{\widetilde{V}',\widetilde{V}} \tag{50}$$
{dense21_1}

for $z \in G$. Since $G$ is dense in $\widetilde{V}$, we know (50) holds for any $z \in \widetilde{V}$, which shows $\partial\phi$ is single valued.

Then it is sufficient to prove for all functions $z$ belonging to some dense subset $G$ of $V$, which is a dense subset of $\widetilde{V}$, the Gâteaux-derivative of $\phi$ exists.

Second, we find the dense subset $G$. For any $c \geq 0$, define

$$D_{w,c} := \{\tilde{w} \in L^2(0,1); |\tilde{w}/g| \leq c \ a.e.\}, \qquad D_w := \bigcup_{c\geq 0} D_{w,c}.$$

The above sets are well-defined since $g > 0$ almost everywhere.

We claim that for any given $g$, such that $g > 0$, the set $D_w$ is dense in $L^2(\mathbb{T}) \cap L^\infty(\mathbb{T})$. Indeed, for any $f \in L^2(\mathbb{T}) \cap L^\infty(\mathbb{T})$, we consider the sequence in $D_w$ given by

$$f_n := \begin{cases} f & \text{if } |f/g| \leq n, \\ 0 & \text{otherwise.} \end{cases}$$

Then, note that, for any $p \in [1, \infty)$,

$$\|f_n - f\|_{L^p(\mathbb{T})} = \Big[ \int_{\{|f/g|>n\}} |f - f_n|^p \, \mathrm{d}h \Big]^{1/p},$$

and $\left|\frac{f}{g}\right| \le \frac{\|f\|_{L^\infty(\mathbb{T})}}{g}$. Hence

$$\Big\{ \Big|\frac{f}{g}\Big| > n \Big\} \subseteq \Big\{ g < \frac{\|f\|_{L^\infty(\mathbb{T})}}{n} \Big\},$$

which yields

$$\begin{aligned} \Big[ \int_{\{|\frac{f}{g}|>n\}} |f - f_n|^p \, \mathrm{d}h \Big]^{1/p} &\le \Big[ \int_{\{g<\frac{\|f\|_{L^\infty(\mathbb{T})}}{n}\}} |f - f_n|^p \, \mathrm{d}h \Big]^{1/p} \\ &\le \|f\|_{L^\infty(\mathbb{T})} \Big| \Big\{ g < \frac{\|f\|_{L^\infty(\mathbb{T})}}{n} \Big\} \Big|^{1/p} \overset{n\to+\infty}{\to} 0. \end{aligned}$$

Thus we have proven that $D_w$ is dense in $L^2(\mathbb{T}) \cap L^\infty(\mathbb{T})$, and hence dense in $L^2(\mathbb{T})$. Now define $G := \{z \in V;\, z_{hh} \in D_w\}$, which is a dense subset of $V$.

Third, we calculate the Gâteaux-derivative of $\phi$. Consider a test function $z \in G$ so that $z_{hh} \in D_w$. Since $D_w = \bigcup_{c \ge 0} D_{w,c}$, there exists $c$ such that $z_{hh} \in D_{w,c}$.

Direct computation gives

$$\phi(w + \varepsilon z) - \phi(w) = \int_{\mathbb{T}} \left( \frac{1}{2(g + \varepsilon z_{hh})^2} - \frac{1}{2g^2} \right) \mathrm{d}h = \int_{\mathbb{T}} \frac{-2\varepsilon g z_{hh} - \varepsilon^2 z_{hh}^2}{2g^2 (g + \varepsilon z_{hh})^2} \, \mathrm{d}h =: I_1 + I_2,$$

where

{I}

$$I_1 := -\varepsilon \int_{\mathbb{T}} \frac{z_{hh}}{g(g + \varepsilon z_{hh})^2} \, \mathrm{d}h, \qquad I_2 := -\varepsilon^2 \int_{\mathbb{T}} \frac{z_{hh}^2}{2g^2 (g + \varepsilon z_{hh})^2} \, \mathrm{d}h. \tag{51}$$

*Estimate for $I_2$.* Since we are interested in the behavior of $I_2$ in the limit $\varepsilon \to 0$, we will consider only $\varepsilon < 1/(2c)$. Thus

{w below}

$$g + \varepsilon z_{hh} \ge g - \varepsilon |z_{hh}/g| g \ge g(1 - c\varepsilon) \ge g/2 \qquad \text{for a.e. } h, \tag{52}$$

and

$$\begin{aligned} |I_2| =& \left| \varepsilon^2 \int_{\mathbb{T}} \frac{z_{hh}^2}{2g^2 (g + \varepsilon z_{hh})^2} \, \mathrm{d}h \right| \\ \le& \varepsilon^2 \frac{c^2}{2} \int_{\mathbb{T}} (g + \varepsilon z_{hh})^{-2} \, \mathrm{d}h \qquad (\text{due to } z_{hh} \in D_{w,c}) \\ \le& 2\varepsilon^2 c^2 \int_{\mathbb{T}} g^{-2} \, \mathrm{d}h = O(\varepsilon^2), \end{aligned}$$

where we also used (52) and $g^{-2} \in L^1(\mathbb{T})$.

*Estimate for $I_1$.* Again, since we are interested in the behavior of $I_1$ in the limit $\varepsilon \to 0$, we will consider only $\varepsilon < 1/(2c)$. Then

$$I_1/\varepsilon = -\int_{\mathbb{T}} \frac{z_{hh}}{g(g + \varepsilon z_{hh})^2} \, \mathrm{d}h,$$

and

$$\frac{z_{hh}}{g(g + \varepsilon z_{hh})^2} \to \frac{z_{hh}}{g^3} \qquad \text{for a.e. } h.$$

Using (52), we have

$$\left|\frac{z_{hh}}{g(g+\varepsilon z_{hh})^2}\right| \le \left|\frac{4z_{hh}}{g^3}\right| \in L^1(\mathbb{T}),$$

with the last inclusion due to

$$\int_{\mathbb{T}} \left|\frac{z_{hh}}{g^3}\right| \mathrm{d}h \le \|z_{hh}\|_{L^2(\mathbb{T})} \|g^{-3}\|_{L^2(\mathbb{T})}.$$

Thus by Lebesgue's dominated convergence we infer

$$I_1/\varepsilon = -\int_{\mathbb{T}} \frac{z_{hh}}{g(g+\varepsilon z_{hh})^2}\,\mathrm{d}h \overset{\varepsilon\to 0}{\to} -\int_{\mathbb{T}} \frac{z_{hh}}{g^3}\,\mathrm{d}h = \langle -(g^{-3})_{hh}, z\rangle_{V',V}.$$

Since $(g^{-3})_{hh} \in \widetilde{V}'$, $z \in \widetilde{V}$, we have

$$\langle -(g^{-3})_{hh}, z\rangle_{V',V} = \langle -(g^{-3})_{hh}, z\rangle_{\widetilde{V}',\widetilde{V}}.$$

Therefore ,

$$\lim_{\varepsilon\to 0} \frac{\phi(w+\varepsilon z)-\phi(w)}{\varepsilon} = \langle -(g^{-3})_{hh}, z\rangle_{\widetilde{V}',\widetilde{V}} \tag{53}$$

for any $z \in G$. This together with the density argument (50), gives (40) and completes the proof. □

### 3.2 Existence result for strong solution.

Now combining Proposition 15 and Theorem 14, we can state existence result for strong solution as follows.

**Theorem 16** *Given $T > 0$, initial data $w^0 \in D(B)$, then the variational inequality solution $w$ obtained in Theorem 14 is also a strong solution to* (8)*, i.e.,*

$$w_t = (g^{-3})_{hh} = ((\eta + c_0)^{-3})_{hh}, \tag{54}$$ {main21_3}

*for a.e. $(t, h) \in [0, T] \times \mathbb{T}$. Besides, we have $((\eta+c_0)^{-3})_{hh} \in L^\infty([0, T]; U)$ and the dissipation inequality*

$$E(t) := \frac{1}{2}\int_{\mathbb{T}} \left[((\eta + c_0)^{-3})_{hh}\right]^2 \mathrm{d}h \le E(0), \tag{55}$$ {dissiE}

*where $\eta$ is the absolutely continuous part of $w_{hh}$ in* (17)*.*

*Proof* First we want to prove that

$$\partial\psi(w) = \{0\}, \tag{56}$$ {pd1}

and

$$g^{-3} \in C(\mathbb{T}) \text{ for a.e. } t \in [0, T]. \tag{57}$$ {pd2}

Indeed, if both (56) and (57) hold, Proposition 15 gives that

$$Bw = -(g^{-3})_{hh} = -((\eta + c_0)^{-3})_{hh}$$

is single-valued, which together with (34) implies (54).

Hence it is sufficient to prove

$$w \in \operatorname{Int} \mathcal{K}, \tag{58}$$ {int}

and (57). We choose the constant $C$ in Definition 1 as

$$C := \|w_0\|_{\widetilde{V}} + 2c_0 + 3.$$

We need to use two *a priori* assumptions

$$\|w\|_{\widetilde{V}} \le C - 1 < C, \tag{59}$$

$$\|(g^{-3})_{hh}\|_U \le \sqrt{2E(0)} + 1, \tag{60}$$

which will be verified later.

Under (59), we have $w \in \operatorname{Int}\mathcal{K}$ and $\partial\psi(w) = 0$. Under (60), we know for any $h_1, h_2 \in \mathbb{T}$,

$$\left|g^{-3}(h_1) - g^{-3}(h_2)\right| \le |h_1 - h_2| \|(g^{-3})_{hh}\|_U. \tag{61}$$

Notice that from (38) we also have $\int_{\mathbb{T}} \frac{1}{g^2}\,\mathrm{d}h < +\infty$. Hence

$$\left(\frac{1}{g^3}\right)^{2/3}_{\min} = \left(\frac{1}{g^2}\right)_{\min} \le \int_{\mathbb{T}} \frac{1}{g^2}\,\mathrm{d}h < +\infty. \tag{62}$$

This, together with (61), gives $g^{-3} \in C(\mathbb{T})$. Therefore, Proposition 15 shows that $Bw = -(g^{-3})_{hh}$. Then from assertion (i) in Theorem 14 we know

$$w_t = (g^{-3})_{hh} = ((\eta + c_0)^{-3})_{hh}, \tag{63}$$

for a.e. $(t, h) \in [0, T] \times \mathbb{T}$.

Next we verify the *a priori* assumption (59). Let $J_\delta$ be the standard $C^\infty(\mathbb{T})$ mollifier. Then we have

$$J_{\delta hh} \star w_t = J_{\delta hh} \star ((\eta + c_0)^{-3})_{hh}$$

for a.e. $(t, h) \in [0, T] \times \mathbb{T}$. Integrating on $\mathbb{T}$ shows that

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_{\mathbb{T}} J_\delta \star w_{hh}\,\mathrm{d}h = \int_{\mathbb{T}} \left[J_\delta \star ((\eta + c_0)^{-3})_{hh}\right]_{hh} \mathrm{d}h = 0. \tag{64}$$

Thus we obtain

$$\int_{\mathbb{T}} J_\delta \star w_{hh}\,\mathrm{d}h \le \int_{\mathbb{T}} J_\delta \star w_{0hh}\,\mathrm{d}h \le \|w_0\|_{\widetilde{V}},$$

and

$$\int_{\mathbb{T}} J_\delta \star (w_{hh} + c_0)\,\mathrm{d}h = \int_{\mathbb{T}} J_\delta \star w_{hh}\,\mathrm{d}h + c_0 \le \|w_0\|_{\widetilde{V}} + c_0.$$

Since we also have $w \in L^\infty([0, T]; \widetilde{V})$, we know that for $\delta$ small enough,

$$\begin{aligned} \|w\|_{\widetilde{V}} &\le \|w_{hh} + c_0\|_{\mathcal{M}(\mathbb{T})} + c_0 \\ &\le \int_{\mathbb{T}} J_\delta \star (w_{hh} + c_0)\,\mathrm{d}h + 1 + c_0 \\ &\le \|w_0\|_{\widetilde{V}} + 2c_0 + 1 \\ &= C - 2 < C - 1, \end{aligned} \tag{65}$$

which verifies the *a priori* assumption (59).

Now we turn to verify (60). Combining (63) and (36), we have the dissipation law

$$E(t) = \frac{1}{2}\|w_t\|_U^2 = \frac{1}{2}\|((\eta + c_0)^{-3})_{hh}\|_U^2 \le \frac{1}{2}\|Bw_0\|_U^2 = E(0) \tag{66}$$

for $E(t) = \frac{1}{2}\int_{\mathbb{T}} \left[((\eta + c_0)^{-3})_{hh}\right]^2 \mathrm{d}h$. Hence we know

$$\|(g^{-3})_{hh}\|_U = \|((\eta + c_0)^{-3})_{hh}\|_U \le \sqrt{2E(0)} < \sqrt{2E(0)} + 1, \tag{67}$$

which verifies the *a priori* assumption (60).

Therefore, we know (56) and (57), so $B$ is single-valued and (54) holds. The dissipation inequality (55) comes from (66) and we completes the proof of (iii). □

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