On asymmetric distances (2013)*

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Abstract

In this paper we discuss asymmetric length structures and asymmetric metric spaces.

A length structure induces a (semi)distance function; by using the total variation formula, a (semi)distance function induces a length. In the first part we identify a topology in the set of paths that best describes when the above operations are idempotent. As a typical application, we consider the length of paths defined by a Finslerian functional in Calculus of Variations.

In the second part we generalize the setting of *General metric spaces* of Busemann, and discuss the newly found aspects of the theory: we identify three interesting classes of paths, and compare them; we note that a "geodesic segment" (as defined by Busemann) is not necessarily continuous in our setting; hence we present three different notions of "intrinsic metric space".

Keywords: asymmetric metric, general metric, quasi metric, ostensible metric, Finsler metric, run–continuity, intrinsic, path metric, length structure

1 Introduction

"Besides, one insists that the distance function be symmetric, that is, $d(x, y) = d(y, x)$. (This unpleasantly limits many applications $[\cdots]$)." M. Gromov ([12], Intr.)

The main purpose of this paper is to study an "asymmetric metric theory"; this theory naturally generalizes the metric part of Finsler Geometry, much as symmetric metric theory generalizes the metric part of Riemannian Geometry.

In order to do this, we will define the "asymmetric metric space" as a set M equipped with a positive $b : M \times M \to \mathbb{R}^+$ which satisfies the triangle inequality, but may fail to be symmetric (see Defn. 1.1 for the formal definition). We also generalize the theory of "length structures" to the asymmetric case.

The invention of a (possibly) asymmetric distance function is quite useful in some applied fields, and mainly in Calculus of Variation, as we will exemplify in the next section 1.1.3. It is then no surprise that this theory was invented and studied many times in the past.

It was indeed shown in fundamental studies that, even if a metric space does not have the "smooth" character of a Riemannian structure, still many definitions and results can be reformulated and carried on. Results date back to the work of Hopf and Rinow [14] for the symmetric case, and to Cohn–Vossen [9] for the asymmetric case [1]; more recently, to works by Busemann, see *e.g.* [5, 6, 7].

A different point of view is found in the theory of *quasi metrics* (or *ostensible metrics*), where much emphasis is given to the topological aspects of the theory; indeed, a quasi-metric will generate, in general, *three different topologies*. This brings forth many different and non-equivalent definitions of *"completeness"* (and of the other properties commonly stated in metric theory). In our definition of the "asymmetric metric space" (M, b), we choose a different topology than the one used in "quasi metric spaces". We will

*This 2013 version corrects some inaccuracies that were present in previous 2004 and 2007 version; and a few in the version published from AGMS, that are printed in red (see "errata" file for a description); and improves Prop.3.18.

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[1] Busemann [5] attributes the first such results in this respect to Cartan, in 1928.

compare the two fields in a forthcoming paper [16], since there we will present the different notions of *"complete"* and *"locally compact"*.

Our set of hypotheses is more similar to Busemann's theory of *general metric spaces* (as defined in [5]); we generalize his definition, in that we do not assume that the topologies generated by "forward balls" and "backward balls" coincide, but rather we associate to (M, b) the topology generated by both families of balls. This generalization has a peculiar effect on the theory: it is not guaranteed that an arc-parameterized rectifiable path is continuous (!) We will further detail differences and similarities in Section A.1.

Summarizing, this presentation of the theory of asymmetric metrics is more specific and more focused on "metric aspects" than what is seen in *ostensible metric theory*; at the same time it is less "geometric" compared to what is seen in Busemann's work.

In the rest of the introduction we will present shortly the main ideas in this paper, skipping many details and definitions.

1.1 Length and distance

1.1.1 The Asymmetric Metric Space

Let M be a non empty set.

Definition 1.1 $b : M \times M \to [0, \infty]$ *is an* ***asymmetric distance function*** *if* b *satisfies*

- $b \geq 0$ *and* $\forall x \in M, b(x, x) = 0$*;*
- $b(x, y) = b(y, x) = 0$ *implies* $x = y$*;*
- $b(x, z) \leq b(x, y) + b(y, z)\ \forall x, y, z \in M$*.*

The third condition is usually called "the triangle inequality". If the second condition does not hold, then we will call b an *asymmetric semidistance function*.[2]

We call the pair (M, b) an **asymmetric metric space**.

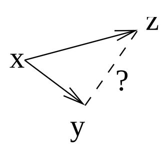
The space (M, b) is endowed with the topology generated by all forward balls $B^+(x, \varepsilon)$ and backward balls $B^-(x, \varepsilon)$ for $x \in M, \varepsilon > 0$, where

$$B^+(x, \varepsilon) \stackrel{\text{def}}{=} \{y \mid b(x, y) < \varepsilon\}, \quad B^-(x, \varepsilon) \stackrel{\text{def}}{=} \{y \mid b(y, x) < \varepsilon\} \quad ;$$

we postpone further details to Sec. 3.

There are subtle but important differences between *asymmetric* and *symmetric distance functions*; we just present here an initial remark.

Remark 1.2 *Let* $\varepsilon > 0$ *and* x, y, z *be such that* $b(x, y) < \varepsilon$ *and* $b(x, z) < \varepsilon$*. This does not imply, in general, that* $b(y, z) < 2\varepsilon$



[2] This is elsewhere called a "pseudo distance function"; in some texts the term "semimetric" is instead used when the triangle inequality does not hold, *cf* [2].

1.1.2 The induced length

Given a (semi)distance function b, we define from b the **length** Len^b of paths by using the **total variation**[3]

$$\mathrm{Len}^b(\gamma) \stackrel{\text{def}}{=} \sup_T \sum_{i=1}^n b\big(\gamma(t_{i-1}),\gamma(t_i)\big) \tag{1.1}$$

where, for a path $\gamma : [\alpha,\beta] \to M$, the sup is carried out over all finite subsets $T = \{t_0, \cdots, t_n\}$ of $[\alpha,\beta]$ such that $\alpha \le t_0 < \cdots < t_n \le \beta$. When $\mathrm{Len}^b(\gamma) < \infty$ we say that γ is **rectifiable**.

1.1.3 An example in Calculus of Variations

One possible origin of the idea of asymmetric distance functions is in this example from Calculus of Variations.

Example 1.3 *Suppose that M is a differential manifold [4]. Suppose that we are given a Borel function $F : TM \to [0,\infty]$, and that for all fixed $x \in M$, $F(x,\cdot)$ is positively 1-homogeneous. We define the* length $\mathrm{len}^F\,\xi$ *of an absolutely continuous path $\xi : [0,1] \to M$ as*

$$\mathrm{len}^F\,\xi = \int_0^1 F(\xi(s),\dot\xi(s))\; ds \quad . \tag{1.2}$$

We then define the asymmetric semidistance function $b^F(x,y)$ on M to be the infimum of this length $\mathrm{len}^F\,\xi$ in the class of all absolutely continuous ξ with given extrema $\xi(0) = x, \xi(1) = y$.

This example generalizes the definitions of "length of paths" and "distance function" that are found in Riemann or Finsler Manifolds (see *e.g.* section 6.2 in [1]); indeed it is called a *Finslerian Length* in Example 2.2.5 in [4]. So we will call b^F the *Finslerian distance function.*

1.1.4 Partially ordered sets

Another possible example is as follows.

Example 1.4 *Let $(M,\le)$ be a partially ordered set. Let $a_>, a_<, a_? \in [0,\infty]$ satisfying the following*

$$a_? \le a_< + a_> \quad , \quad a_> \le 2a_? \quad , \quad a_< \le 2a_? \quad , \quad a_? \ne 0 \quad .$$

We define b on $M \times M$ as

$$b(x,y) \stackrel{def}{=} \begin{cases} 0 & \text{if } x = y \\ a_< & \text{if } x < y \\ a_> & \text{if } x > y \\ a_? & \text{if } x,y \text{ are not comparable} \end{cases} \quad . \tag{1.3}$$

b is an asymmetric distance. (This is the most general b that depends only on the relative order of x,y). At the same time, if $a_> \ne a_<$ then, for any $x,y \in M$ with $x \ne y$, from the values of $b(x,y)$ and $b(y,x)$ we can recover the relative order of x,y.

Let us fix $a_< = 0, a_> = a_? = +\infty$; then the above example defines a functor from the category of partially ordered sets (with "the monotonic non decreasing maps" as morphisms) to the category of asymmetric metric spaces (with "the Lipschitz maps" as morphisms). Vice versa, given an asymmetric metric space (M,b), we may define a partial order $(M,\le)$ on M as

$$x \le y \iff b(x,y) = 0 \quad . \tag{1.4}$$

This functor is the left inverse of the one defined previously. See [13] and references therein for more on this categorical relationship.

Note that all this would not be possible if we required that b be a symmetric distance.

[3] According to Busemann [5], this definition goes back to Menger (1928).

[4] More precisely, let M be a finite dimensional connected smooth differential manifold, without boundary.

1.1.5 The length structure

Both the length len^F defined in (1.2) and the length Len^b defined in (1.1) share common properties. We will abstract them introducing the concept of *length structure*. (This is a commonly used concept in the symmetric case, see *e.g.* [12], [4]).

In Section 2.1 we will define a *length structure* $(\mathcal{C}, \mathrm{len})$ where $\mathcal{C}$ is an appropriate family of paths $\gamma : [\alpha, \beta] \to M$ (and α, β vary) and $\mathrm{len} : \mathcal{C} \to [0, \infty]$ is the "length" of the paths. We remark that $\mathcal{C}$ and len should satisfy some hypotheses, that we defer to Section 2.1 for ease of reading of the introduction.

In the case of len^F defined in (1.2), we would choose $\mathcal{C}$ to be the family of all absolutely continuous paths. In the case of Len^b defined in (1.1) we may choose $\mathcal{C}$ to be the family of all continuous paths; but we will see that this is not the only possible choice when b is asymmetric.

We can then induce a new semidistance function[5] from $(\mathcal{C}, \mathrm{len})$. We say that a path $\gamma : [a, b] \to M$ **"connects x to y"** when $\gamma(a) = x, \gamma(b) = y$.

Definition 1.5 *Given a length structure, the* ***induced semi distance*** *function $b^l(x, y)$ on M, is defined to be the infimum of the length $\mathrm{len}(\xi)$ in the class of all $\xi \in \mathcal{C}$ that connect x to y.*

1.1.6 Idempotency

We can then iterate the above ideas: indeed from a *length structure* we may induce a *distance* using Definition 1.5 above; whereas starting from a *distance* we may define a *length structure* using (1.1) (on an appropriate class $\mathcal{C}$ of paths). The above operations may be chained in two ways,

$$\boxed{\text{length}} \to \boxed{\text{distance}} \to \boxed{\text{length}} \to \dots$$
$$\boxed{\text{distance}} \to \boxed{\text{length}} \to \boxed{\text{distance}} \to \dots$$

and this leads to two different points of view to the problem, one discussed in Section 2, and the other one in Section 3; the two points of view essentially coincide in the symmetric case (as discussed in Sec. 3.5), but they diverge in the asymmetric case.

Under reasonable assumptions, we expect the above operations to stabilize after few iterations; indeed in the symmetric case the following propositions hold.

When we start from a length structure, the following results are known in the symmetric case.

Proposition 1.6 (Prop. 1.6 in [12]) *Suppose that $(\mathcal{C}, \mathrm{len})$ is a symmetric length structure satisfying the hypotheses in Sec. 1.A in [12]; let $b^l(x, y)$ be induced from $(\mathcal{C}, \mathrm{len})$ as explained in 1.5, and Len^{b^l} be defined as in (1.1). In general $\mathrm{Len}^{b^l} \le \mathrm{len}$. If len is lower semi continuous in the compact–open topology of continuous paths from $[0, 1]$ to M (considering M as a metric space with the distance function b^l), then $\mathrm{Len}^{b^l} \equiv \mathrm{len}$ in $\mathcal{C}$.*

Proposition 1.7 (Prop. 2.4.3 in [4]) *Suppose that $(\mathcal{C}, \mathrm{len})$ is a symmetric length structure satisfying the hypotheses in Sec. 2.1 in [4]; add to M a compatible topology. Let b^l and Len^{b^l} be defined as above. If len is lower semi continuous in the topology of pointwise convergence of continuous paths from $[0, 1]$ to M, then $\mathrm{Len}^{b^l} \equiv \mathrm{len}$ in $\mathcal{C}$.*

(For more details on the hypotheses, see Remark 2.5 and Section 2.3.1). We call "*idempotency*" the above property "$\mathrm{Len}^{b^l} \equiv \mathrm{len}$". This type of property will hold in the hypotheses of this paper as well: see Theorem 2.19; as an application in Section 2.5.3 we will prove *idempotency* for Example 1.3. In Section 2.5.2 we will argue that the *compact open* topology is not in general the correct topology to describe the problem. We will introduce (in Section 2.4.1) a specific topology, called "*DF topology*". We remark that the *DF* topology is not related to any choice of a "compatible" topology on M; see *e.g.* Proposition 3.18 and comments following. We will see in Sec. 2.5.4 that the proposed definitions and methods can deal with a larger class of problems than previously possible.

When we start from a metric space, the following property holds.

[5] See also Remark 2.8.

Proposition 1.8 *Suppose that (M,b) is a symmetric metric space; let Len^b be defined as in (1.1); let $\mathcal{C}$ be the class of all continuous and rectifiable paths in (M,b); then let $b^g(x,y)$ be defined as the infimum of length $\mathrm{Len}^b(\xi)$ in all $\xi \in \mathcal{C}$ connecting x to y; eventually define Len^{b^g} (using (1.1) with b^g in place of b). Then $\mathrm{Len}^{b^g} \equiv \mathrm{Len}^b$ in $\mathcal{C}$ and $b^g = (b^g)^g$ (and so on).*

(Note that the above statements hides a technical but important fact, see Proposition 3.11). It will turn out though that in the asymmetric case the matter is more complex, and in particular we will have to choose carefully the class $\mathcal{C}$ of admissible paths: we will present three interesting classes of paths, and compare them; curiously, we will see that the commonly used class of "continuous rectifiable paths" does not enjoy idempotency in the asymmetric case; whereas a newly defined class of "run–continuous paths" does.

This ends the introduction. In the following sections we will review all the above ideas, and provide detailed definitions, theorems and examples.

2 Length structures

Let M be a non empty set. Foremost we define the joining and reversal of paths.

Definition 2.1 (Reverse path) *Given $\gamma : [\alpha,\beta] \to M$, the* ***reverse path*** *is $\hat\gamma : [-\beta,-\alpha] \to M$ defined by $\hat\gamma(t) \overset{def}{=} \gamma(-t)$.*

Definition 2.2 (Joining of paths) *Suppose we are given two paths*

$$\gamma' : [\alpha,\zeta] \to M \quad , \quad \gamma'' : [\zeta,\beta] \to M \quad .$$

Suppose $\gamma'(\zeta) = \gamma''(\zeta)$, then we may ***join***[6] *the two paths and obtain*

$$\gamma(t) = \begin{cases} \gamma'(t) & if\ t \in [\alpha,\zeta] \\ \gamma''(t) & if\ t \in [\zeta,\beta] \end{cases} \tag{2.1}$$

Clearly, if M is a topological set, and if γ',γ'' are continuous, then γ is continuous as well. Note that paths will not necessarily be continuous in the following.

2.1 Definition and properties

Let M be a non empty set. Let $\mathcal{C}$ be a family of paths $\gamma : [\alpha,\beta] \to M$ (note that α,β vary in the family). The *length functional* is a function $\mathrm{len} : \mathcal{C} \to [0,\infty]$. We define the pair $(\mathcal{C},\mathrm{len})$ to be a **length structure** on M when the following hypotheses are satisfied. [7]

Hypotheses 2.3

1. *Paths in $\mathcal{C}$ may be* splitted*, and* len *is* ***monotonic****: if $\gamma \in \mathcal{C}$ then its restriction $\gamma' = \gamma|_{[c,d]}$ is in $\mathcal{C}$, and $\mathrm{len}\,\gamma' \le \mathrm{len}\,\gamma$.*
2. *Paths in $\mathcal{C}$ may be* joined*, and* len *is* ***additive****. Suppose $\gamma',\gamma'' \in \mathcal{C}$ are as defined in 2.2 and γ is their join: then $\gamma \in \mathcal{C}$ and*
$$\mathrm{len}\,\gamma = \mathrm{len}\,\gamma' + \mathrm{len}\,\gamma'' \quad .$$
3. *The case $\alpha = \beta$ behaves as follows: $\forall x \in M$, $\forall \alpha \in \mathbb{R}$, the* singleton path *$\gamma : \{\alpha\} \to M$ with $\gamma(\alpha) = x$ is in $\mathcal{C}$ and $\mathrm{len}\,\gamma = 0$.*
4. *$(\mathcal{C},\mathrm{len})$ is* ***independent of linear reparameterization****: given $\gamma \in \mathcal{C}$, given $h > 0, k \in \mathbb{R}$, we define the linear reparameterization $\gamma' : [(\alpha-k)/h,(\beta-k)/h] \to M$ as $\gamma'(s) = \gamma(hs+k)$; then $\gamma' \in \mathcal{C}$, and $\mathrm{len}(\gamma) = \mathrm{len}(\gamma')$.* [8]

[6] For the record, [12] calls this operation "juxtaposition", and [4],[17] "concatenation".
[7] The following conditions are not independent, some of them imply some others; we do not detail.
[8] On this choice of hypothesis, see Sec. 2.5.1 and remark 2.26.

Some other properties that we may cite sometimes are the following.

Definition 2.4 *5.* $\mathcal{C}$ ***connects points****:* $\forall x, y \in M$ *there is at least one* $\gamma \in \mathcal{C}$*,* $\gamma : [\alpha, \beta] \to M$ *such that* $\gamma(\alpha) = x, \gamma(\beta) = y$*.*

6. len *is* ***run-continuous****: that is, the* running length

$$\ell^\gamma : [\alpha, \beta] \to [0, \infty] \quad , \quad \ell^\gamma(t) \stackrel{def}{=} \mathrm{len}(\gamma|_{[\alpha,t]}) \tag{2.2}$$

is continuous for all $\gamma \in \mathcal{C}$ s.t. $\mathrm{len}(\gamma) < \infty$*.*

7. $\mathcal{C}$ *is* ***reversible****: if* $\gamma \in \mathcal{C}$ *and* $\hat\gamma(t) \stackrel{def}{=} \gamma(-t)$ *is the reverse path, then* $\hat\gamma \in \mathcal{C}$*.*

8. $(\mathcal{C}, \mathrm{len})$ *is* ***symmetric****: if* $\mathcal{C}$ *is reversible and moreover* $\mathrm{len}(\gamma) = \mathrm{len}(\hat\gamma)$*.*

9. It is ***independent of homeomorphic reparameterization*** *when, given* $\gamma \in \mathcal{C}$*,* $\gamma : [\alpha, \beta] \to M$*,* $\varphi : [a, b] \to [\alpha, \beta]$ *an increasing homeomorphism such that* $\gamma \circ \varphi \in \mathcal{C}$*, then* $\mathrm{len}(\gamma) = \mathrm{len}(\gamma \circ \varphi)$*.*

In example 1.3, choosing $\mathcal{C} = \mathcal{C}_{\mathrm{AC}}$ to be the family of all absolutely continuous paths, we obtain that $(\mathcal{C}_{\mathrm{AC}}, \mathrm{len}^F)$ satisfies all the above properties in 2.3 and 2.4, [9] but for "symmetry" (unless $F(x, v) = F(x, -v) \forall x, v$); moreover ℓ^γ is absolutely continuous when $\mathrm{len}^F(\gamma)$ is finite.

Remark 2.5 Length structures *(at least in the symmetric case) are a well known mathematical object.The set of hypotheses we present here are though more general; indeed*

- *the definition in Sec. 1.A in [12] assumes all properties listed in 2.3 and 2.4.* (The *symmetric* property follows from the fact that [12] assumes that $\mathrm{len}(\gamma) = \mathrm{len}(\gamma \circ \phi)$ for all homeomorphisms ϕ.)
- *The definition in Sec. 2.1 in [4] assumes all properties listed in 2.3, and also 2.4 but for (9), that is proved to be a consequence.* (Note that in [4] "$\mathcal{C}$ connects points" is not explicitly assumed; but in the symmetric case the problem is addressed by working in path-connected components, as explained in Exercise 2.1.3 in [4].)

The reason we did not enforce all properties 2.4 as part of the definition 2.3 is that they are not needed in some following theorems, and are not satisfied in the examples we will see.

2.2 Length by total variation

Let b be an *asymmetric semi distance function* on M. We induce from b the length $\mathrm{Len}^b \gamma$ of a path $\gamma : [\alpha, \beta] \to M$, by using the total variation (seen in eqn. (1.1)).

We remark that Len^b is defined for all paths (and not only those in $\mathcal{C}$), and it satisfies all properties listed in the hypotheses 2.3 in the previous section. It moreover satisfies these properties.

Proposition 2.6 *Let* $b, \tilde b$ *be two asymmetric semidistance functions, then*

$$\mathrm{Len}^{b+\tilde b} = \mathrm{Len}^b + \mathrm{Len}^{\tilde b} \quad .$$

Proof. The supremum in the definition of Len^b is also the limit in the directed family of finite subsets T of $[\alpha, \beta]$ (the family is ordered by inclusion). □

The following property will be quite useful; it follows straightforwardly from the definition of the total variation length.

[9] Regarding property (9), see 2.26.

Proposition 2.7 (Change of variable) *Suppose b is an asymmetric semidistance function. Let $\gamma : [\alpha, \beta] \to M$, $\gamma_2 : [\alpha_2, \beta_2] \to M$ be linked by*

$$\gamma = \gamma_2 \circ \varphi$$

*where $\varphi : [\alpha, \beta] \to [\alpha_2, \beta_2]$ is monotone non decreasing, continuous, and $\varphi(\alpha) = \alpha_2$, $\varphi(\beta) = \beta_2$ (*i.e.* φ is surjective). Then*

$$\mathrm{Len}^b \gamma = \mathrm{Len}^b \gamma_2 \quad .$$

The above property holds also when φ is not bijective[10]; this is very useful, since it simplifies some proofs.

2.3 Induced semidistance

Let $(\mathcal{C}, \mathrm{len})$ to be a *length structure* on M.

As we did in 1.5, we again define the **induced semidistance function** $b^l(x, y)$ as the infimum of the length $\mathrm{len}(\xi)$ in the class of all $\xi \in \mathcal{C}$ connecting x to y. (If there is no such path, then $b^l(x, y) = \infty$).

Proposition 2.8 *b^l is an asymmetric semidistance function. Indeed b^l satisfies the triangle inequality, since* len *is* independent of reparameterization *and* additive. *Moreover $b^l \geq 0$, and $b^l(x, x) = 0$ since "singleton paths" are always in $\mathcal{C}$ (see property 3 in 2.3). Without additional hypotheses b^l may fail to be an asymmetric distance function, since we cannot be sure that $b^l(x, y) = b^l(y, x) = 0 \Longrightarrow x = y$.*

2.3.1 Compatible topology

[4] defines that

Definition 2.9 *A topology τ on M is **compatible** with $(\mathcal{C}, \mathrm{len})$ when, for any $x \in M$ and $U \in \tau$ with $x \in U$ there exists $\varepsilon > 0$ such that for any path $\gamma \in \mathcal{C}$ connecting $x = \gamma(\alpha)$ to a point $\gamma(\beta)$ not in U, we have $\mathrm{len}(\gamma) \geq \varepsilon$.*

In the symmetric case b^l is a symmetric semidistance function, so it induces a topology. (In the asymmetric case the topology induced by b^l is defined as explained in Sec. 3).

Proposition 2.10 *Suppose that* len *and b^l are symmetric.*

- *A topology τ is compatible with $(\mathcal{C}, \mathrm{len})$, if and only if*
- *τ is equal to, or coarser (has less open sets) than, the topology generated by b^l.*

So the topology induced by b^l is always compatible. (In the asymmetric case the above proposition needs some adjustments, in the second statement we need to consider the *forward topology* generated by b^l).

Remark 2.11 *[12] associates with M the (symmetric) distance function b^l; [4] assumes that M is endowed with a separated compatible topology; a consequence is that, in both approaches, any path $\gamma \in \mathcal{C}$ has to be continuous: see Lemma 2.21.*

Note that, if the compatible topology is separated, then b^l is necessarily a distance (and not a semidistance) function.

[10] This is sometimes overlooked in commonly found proofs, though it was noted already by Busemann in 1944.

2.4 Inducing a length is a relaxation

We now combine the ideas in the two previous subsections. Let $(\mathcal{C}, \mathrm{len})$ be a *length structure* on M. Let b^l be the *induced semidistance function*. Eventually define Len^{b^l} using the total variation (1.1) with $b = b^l$.

In the following lemmas we suppose that the length structure $(\mathcal{C}, \mathrm{len})$ satisfies all the hypotheses 2.3 in the previous section.

The reparameterization property of $(\mathcal{C}, \mathrm{len})$ and the Prop. 2.7 entail that we may consider only paths $\xi : [0, 1] \to M$ in the following.

Definition 2.12 *We call $\mathcal{C}_{01}$ the class of $\xi : [0, 1] \to M$ with $\xi \in \mathcal{C}$.*

Lemma 2.13 *Fix a path γ , $\gamma : [0, 1] \to M$ for simplicity. Given $T \subset [0, 1]$ a finite subset we define*

$$\Xi_{\mathcal{C}_{01},\gamma,T} \stackrel{def}{=} \{\xi \in \mathcal{C}_{01}, \forall t \in T, \xi(t) = \gamma(t)\} \quad ; \tag{2.3}$$

(we will write $\Xi_{\gamma,T}$ instead of $\Xi_{\mathcal{C}_{01},\gamma,T}$ in the following when no confusion would arise). Then

$$\mathrm{Len}^{b^l}(\gamma) = \sup_T \inf_{\xi \in \Xi_{\gamma,T}} \mathrm{len}(\xi) \tag{2.4}$$

where the sup is done on all choices of T as above, and the inf is done on the class $\Xi_{\gamma,T}$.

Proof. We insert the definition of b^l into the definition (1.1) of $\mathrm{Len}^{b^l} \gamma$:

$$\mathrm{Len}^{b^l}(\gamma) = \sup_T \sum_{i=1}^{n} \inf_{\xi_i} \mathrm{len}(\xi_i) \tag{2.5}$$

where the sup is done on all choices $T = \{t_0, t_1, \ldots t_{n-1}, t_n\}$ where we vary n and $t_1 \cdots < t_n$; and the inf is done on choices of paths $\xi_i \in \mathcal{C}_{01}$, connecting $\gamma(t_{i-1})$ to $\gamma(t_i)$. Since the paths are chosen independently, then

$$\sum_{i=1}^{n} \inf_{\xi_i} \mathrm{len}\, \xi_i = \inf_{\xi_1,\ldots,\xi_n} \sum_{i=1}^{n} \mathrm{len}\, \xi_i \tag{2.6}$$

Equivalently (since $(\mathcal{C}, \mathrm{len})$ is independent of reparameterization) we may choose $\xi_i : [t_{i-1}, t_i] \to M$; so we can join all the ξ_i and we obtain (2.4) $\square$

To anybody familiar with Calculus of Variations, the right hand side in (2.4) really looks like the *relaxation* len_* of len.

Definition 2.14 *Given a real function F defined on a topological space, the **relaxation** F_*,* a.k.a. *the **lower semi continuous envelope**, is the largest lower semi-continuous function less or equal to F. (See* e.g. *Sec. 2.3.4 in [10], or Sec. 1.3 in [8]). F_* is defined as*

$$F_*(x) = \sup_{U, x \in U} \inf_{y, y \in U} F(y)$$

where U are open sets in the domain of F; moreover

$$F_*(x) = \min\{F(x), \liminf_{y \to x} F(y)\} \ .$$

(In the following we will often abbreviate lower semi-continuous as "l.s.c.").

That is, (2.4) will be a *relaxation* as soon as we properly define the appropriate topology on $\mathcal{C}_{01}$.

2.4.1 The DF topology

Proposition 2.15 *The family $\Xi_{\gamma,T}$ with $\gamma \in \mathcal{C}_{01}$ and $T \subset [0,1]$ finite, is a base for a topology on $\mathcal{C}_{01}$.*

For lack of a better name, the topology generated by the above base will be called the **DF topology** on $\mathcal{C}_{01}$; the family of all open sets will be denoted by τ_{DF}. Note that τ_{DF} is the restriction to $\mathcal{C}_{01}$ of the product of discrete topologies on M.

Note that

$$\text{if } T \subseteq \tilde{T} \text{ then } \Xi_{\gamma,\tilde{T}} \subseteq \Xi_{\gamma,T} \tag{2.7}$$

so the family of neighborhoods of a path is a directed ordered set indexed by finite subsets T.

Note also that the DF topology may be equivalently defined on the family of all paths $M^{[0,1]}$ and then restricted to $\mathcal{C}_{01}$.

Properties The DF topology enjoys the following properties.

- This topology is separated. Indeed if $\gamma, \tilde{\gamma} \in \mathcal{C}_{01}$ and $\gamma \neq \tilde{\gamma}$ then let $t \in [0,1]$ be *s.t.* $\gamma(t) \neq \tilde{\gamma}(t)$: the neighborhoods
$$\Xi_{\gamma,\{t\}} \cap \Xi_{\tilde{\gamma},\{t\}} = \emptyset \quad .$$
- $\mathcal{C}_{01}$ is dense in all the paths $M^{[0,1]}$, *iff* for any $T = \{t_0, \cdots, t_n\} \subset [0,1]$ and choices $x_0, \ldots x_n \in M$ there is a $\gamma \in \mathcal{C}_{01}$ *s.t.* $\gamma(t_i) = x_i$, *iff* $\mathcal{C}$ connects points.
- Given $x, y \in M$, the set of all curves $\gamma \in \mathcal{C}_{01}$ that connect x to y is open and closed.
- Let $M = [0,1]^2$ (with Euclidean topology) and
$$\gamma_n : [0,1] \to M \ , \ \gamma_n(t) = \begin{cases} \big(t, nt(1-nt)\big) & t \in [0, 1/n] \\ (t, 0) & t \in [1/n, 1] \end{cases}$$
in this classical example we have that $\gamma_n \to \gamma_0$ pointwise and in the τ_{DF} topology, but $\gamma_n \not\to \gamma_0$ uniformly.
- Usually $\mathcal{C}_{01}$ is not compact in this topology. Indeed consider this simple example. Let again $M = [0,1]^2$ and the family of paths $\gamma_s(t) = (t, s)$ for $s \in [0,1]$: then the neighborhoods $\Xi_{\gamma_s,\{0\}}$ are all disjoint.
- The previous example does not satisfy the first and the second axiom of countability. So in general we do not expect that the DF topology be metrizable.
- In the previous example, we have that $\gamma_{1/n} \to_n \gamma_0$ uniformly, but $\gamma_{1/n} \not\to_n \gamma_0$ in the DF topology. So the topology of uniform convergence and the DF topology are in general incomparable. See though Lemma 2.22.
- The DF topology and the topology of pointwise convergence are comparable.

  Lemma 2.16 *Let M be a topological Hausdorff space. Let τ_{PW} be the topology of pointwise convergence (that is the product topology in $M^{[0,1]}$). Then $\tau_{PW} \subseteq \tau_{DF}$, that is, the DF is finer than the topology of pointwise convergence.*

  In particular if $\gamma_n \to \gamma_0$ in the τ_{DF} topology, then $\gamma_n \to \gamma_0$ pointwise; if len *is l.s.c. in the pointwise–convergence topology, it is l.s.c. in the DF topology.*

- "Reparameterization" is a homeomorphism.

  Lemma 2.17 *Suppose that $\varphi : [0,1] \to [0,1]$ is a bijection; suppose that forall $\gamma \in \mathcal{C}_{01}$, we have that $\gamma \circ \phi^{-1} \in \mathcal{C}_{01}$ and $\gamma \circ \phi \in \mathcal{C}_{01}$; define $\Phi : \mathcal{C}_{01} \to \mathcal{C}_{01}$ by $\Phi(\gamma) \stackrel{def}{=} \gamma \circ \varphi$; associate to $\mathcal{C}_{01}$ the DF topology; then Φ is a homeomorphism.*

This property is enjoyed also by the topology of pointwise convergence (when M is a topological space) and by the topology of uniform convergence (when M is a metric space); but is valid in Sobolev spaces only if φ is regular.

Note that the topology τ_{DF} itself is not related to any choice of "compatible topology" on M.

2.5 Idempotency in length structures

The length by total variation is l.s.c. in all the topologies listed in the previous section.

Proposition 2.18 *In general, let b be an asymmetric semidistance function on M. Consider the length Len^b, associated to all the paths $\gamma : [0,1] \to M$; that is, let's consider the length structure $(M^{[0,1]}, \mathrm{Len}^b)$. Len^b is always l.s.c. in the DF topology; if we endow M with a topology* s.t. *b is continuous, then Len^b is l.s.c. in the product topology τ_{PW}.*

The above is well known in the symmetric case (see *e.g.* Prop. 2.3.4 in [4]), and holds similarly in the asymmetric case as well.

The relaxation enjoys some well known properties; so we immediately obtain these results.

Theorem 2.19 *Let $(\mathcal{C}, \mathrm{len})$ be a length structure. Let b^l be the induced semidistance function.*

1. *If $\gamma \in \mathcal{C}$,*
$$\mathrm{Len}^{b^l}(\gamma) \le \mathrm{len}(\gamma) \ , \tag{2.8}$$
this immediately follows from (2.4) since $\gamma \in \Xi_{\gamma,T}, \forall T$; or, if we prefer, from the well known fact the relaxation is less or equal than the original function.

2. *Suppose $\gamma \in \mathcal{C}$;* len *is lower semi continuous at γ according to the* DF topology*, if and only if $\mathrm{Len}^{b^l}(\gamma) = \mathrm{len}(\gamma)$.*

   When $\mathrm{Len}^{b^l} \equiv \mathrm{len}$, that is, when len *is l.s.c. on all $\mathcal{C}$, we say that the length structure enjoys* idempotency*.*

   (For many practical purposes, usually this property is required only on rectifiable curves, that is $\mathcal{C} \cap \{\mathrm{len} < \infty\}$. Note that $(\mathcal{C} \cap \{\mathrm{len} < \infty\}, \mathrm{len})$ is again a length structure.)

3. *Suppose $\gamma \in \mathcal{C}$; if γ is a "limit point" for $\mathcal{C}$ then*
$$\mathrm{Len}^{b^l}(\gamma) = \min\{\mathrm{len}(\gamma), \liminf_{\xi \in \mathcal{C}, \xi \to \gamma} \mathrm{len}(\xi)\} \quad ;$$
if γ is not a "limit point" for $\mathcal{C}$, $\mathrm{Len}^{b^l}(\gamma) = \mathrm{len}(\gamma)$. [11]

4. *Let b^{ll} be induced from the length structure $(\mathcal{C}, \mathrm{Len}^{b^l})$; and then $\mathrm{Len}^{b^{ll}}$ defined using the total variation (1.1) with $b = b^{ll}$. Then $\mathrm{Len}^{b^{ll}} \equiv \mathrm{Len}^{b^l}$ on $\mathcal{C}$. Indeed* $\mathrm{len}_* \equiv (\mathrm{len}_*)_*$*. So $(\mathcal{C}, \mathrm{Len}^{b^l})$ always enjoys* idempotency*.*

5. *Suppose $\gamma \notin \mathcal{C}$; if γ is a "limit point" for $\mathcal{C}$ then*
$$\mathrm{Len}^{b^l}(\gamma) = \liminf_{\xi \in \mathcal{C}, \xi \to \gamma} \mathrm{len}(\xi) \quad ;$$
if γ is not a "limit point" for $\mathcal{C}$, $\mathrm{Len}^{b^l}(\gamma) = +\infty$.

[11] But recall that , if $\mathcal{C}$ *connects points*, then it is dense, so all paths are "limit points".

2.5.1 Homeomorphic reparameterization

We also note a curious fact. As soon as len is l.s.c. in the DF topology we obtain that len is *independent of homeomorphic reparameterization*, as defined in 2.4.(9), since Len^{b^l} enjoys this property.

In [4] the property 2.4.(9) is a consequence of the theory, since the hypotheses of the lemma 2.22 are all verified by the axioms of the length structures; whereas in [12] it is an axiom of the length structure.

If a length structure satisfies the hypotheses 2.3 but not the other properties, it may be the case that $\mathrm{len}(\gamma) \neq \mathrm{len}(\gamma \circ \varphi)$ for a generic homeomorphism.

Example 2.20 *Let $M = \mathbb{R}$ and $\mathcal{C}$ be the family of continuous non decreasing paths $\gamma : [\alpha, \beta] \to M$ that can be piecewise written as parabolas, that is $\exists t_0 = \alpha < t_1 \ldots < t_n = \beta$, $\forall t \in (t_i, t_{i+1})$, $\gamma(t) = a_{2,i}(t - t_i)^2 + a_{1,i}(t - t_i) + \gamma(t_i)$ with $a_{2,i} \geq 0, a_{1,i} > 0$; let*

$$\mathrm{len}(\gamma) \overset{def}{=} \int_\alpha^\beta \ddot{\gamma}(s)/\dot{\gamma}(s)\mathrm{d}s \ .$$

The pair $(\mathcal{C}, \mathrm{len})$ is a run–continuous length structure; 2.4.(9) does not hold for it; $(\mathcal{C}, \mathrm{len})$ is not l.s.c. in the DF topology, and $\mathrm{len}_* \equiv 0$.

2.5.2 Comparison to other approaches

We now compare our approach to previous settings in [12] and [4].

In all of the following, let $(\mathcal{C}, \mathrm{len})$ be a length structure, let b^l be the induced asymmetric semi distance function; let $d^l(x, y) = \max\{b^l(x, y), b^l(y, x)\}$, then d^l is a symmetric semi distance function (more details will be in Sec. 3).

Lemma 2.21 *Let $(\mathcal{C}, \mathrm{len})$ be a length structure. Suppose that len is run–continuous, that $\mathcal{C}$ is reversible; then any rectifiable path $\gamma \in \mathcal{C}$ is continuous w.r.t. d^l.*

Proof. Let $\gamma \in \mathcal{C}$, $\gamma : [0, 1] \to M$. Let ℓ be the run length of γ, so that

$$\ell(t) - \ell(s) = \mathrm{len}(\gamma|_{[s,t]}) \ \ ;$$

by hypotheses ℓ is continuous; note that, for $s < t$, $b^l(\gamma(s), \gamma(t)) \leq \ell(t) - \ell(s)$. Since $\mathcal{C}$ is reversible, the same holds for $\hat{\gamma}(t) = \gamma(-t)$. We conclude that γ is continuous w.r.t. d^l. □

This is the reason why curves are usually assumed to be continuous in classical definitions on length structures, [12] and [4].

We now propose a result that connects Prop 1.6 in [12] (see 1.6 here) to our present approach.

Proposition 2.22 *Suppose that* len *is run–continuous, that $\mathcal{C}$ is reversible; suppose moreover that* len *is l.s.c. in the uniform convergence w.r.t. d^l; then* len *is l.s.c. in the DF topology.*

Even if some of the methods employed in the proof are similar to those used in [12], we provide in Sec. B.1 the needed Lemmas, for convenience of the reader, since some adaptations are needed in the asymmetric cases.

Proof. Let $\gamma \in \mathcal{C}_{01}$ a path that is non isolated in $\mathcal{C}_{01}$ w.r.t. the DF topology. Let $l = \liminf_{\xi \to \gamma} \mathrm{len}(\xi)$ in the DF topology. If $l \geq \mathrm{len}(\gamma)$ the proof ends. Suppose instead that $l < \mathrm{len}(\gamma)$. We define $S_k = \{i/2^k, i = 0, \ldots 2^k\}$; by the Lemma B.3 in appendix B.1, there is a sequence $(\xi_k) \subset \mathcal{C}_{01}$ such that $\lim_n \mathrm{len}(\xi_n) = l$ and $\xi_k \to_k \gamma$ uniformly w.r.t. d^l, so by the hypothesis $l \geq \mathrm{len}(\gamma)$ (contradicting the previous assumption). □

As we before mentioned, in [12] and [4] a length structure is supposed to satisfy all the above properties: in this case the Lemma shows that lower–semi-continuity in the uniform convergence topology is sufficient to guarantee that $\mathrm{Len}^{b^l} \equiv \mathrm{len}$.

If we though drop the extra hypotheses, then we remarked that the uniform convergence topology is incomparable to the DF topology, so in general the DF topology is the correct topology to address the problem. This will be exemplified in Section 2.5.4.

2.5.3 Idempotency in Calculus of Variation

Let's go back to Example 1.3. The pair $(\mathcal{C}_{\mathrm{AC}}, \mathrm{len}^F)$ satisfies all hypotheses 2.3 and hence it is a *length structure*; it is also *run–continuous* and *reversible*.

Lemma 2.17 applies as follows. Let $\mathcal{C}_{\mathrm{AC01}}$ the family of absolutely continuous paths $\xi : [0,1] \to M$.

Lemma 2.23 *Suppose that $\varphi : [0,1] \to [0,1]$ is a homeomorphism with φ, φ^{-1} absolutely continuous; φ, φ^{-1} satisfy the Fichtenholz condition (see exercises 5.8.51, 5.8.61 in [3]) so for any $\xi : [0,1] \to M$, ξ is AC iff $\xi \circ \varphi$ is AC. Define $\Phi : \mathcal{C}_{AC01} \to \mathcal{C}_{AC01}$ by $\Phi(\gamma) \stackrel{def}{=} \gamma \circ \varphi$; associate to $\mathcal{C}_{AC01}$ the DF topology; then Φ is a homeomorphism.*

The above Lemma is useful when associated with the one following.

Lemma 2.24 *Let $\gamma \in \mathcal{C}_{AC}$, $\gamma : [a,b] \to M$ such that $\mathrm{len}^F(\gamma) < \infty$; forall $\varepsilon > 0$ there exists $\varphi : [a,b] \to [a,b]$ an increasing homeomorphism with φ absolutely continuous and φ^{-1} Lipschitz and* s.t. *setting $\theta = \gamma \circ \varphi$ we have $\mathrm{len}^F(\gamma) = \mathrm{len}^F(\theta)$ and*

$$\widetilde{\forall}\, s \in [a,b] \ , \quad (b-a) F\big(\theta(s), \dot\theta(s)\big) \le \varepsilon + \mathrm{len}^F(\gamma) \quad .$$

Proof. Let $\delta = \varepsilon/(b-a)$ and

$$\psi(t) = \int_a^t \max\{\delta, F(\gamma(s), \dot\gamma(s))\} \mathrm{d}s \quad , \quad \varphi(t) = \frac{\psi(t)(b-a)}{\psi(b)} + a$$

so φ is a homeomorphism; note that

$$\psi(b) \le \varepsilon + \mathrm{len}^F(\theta) = \varepsilon + \mathrm{len}^F(\gamma) < \infty \quad .$$

Let $\theta = \gamma \circ \varphi^{-1}$ then

$$(b-a) F(\theta(\tilde s), \dot\theta(\tilde s)) = \frac{F(\gamma(s), \dot\gamma(s))}{\max\{\delta, F(\gamma(s), \dot\gamma(s))\}} \psi(b) \le \psi(b)$$

setting $\tilde s = \varphi(s)$ for convenience. □

With this tool in hand, we now prove this *idempotency* result.

Proposition 2.25 *Let M, F and len^F be defined as in Example 1.3. Suppose that F is also a Caratheodory integrand, that is*

- *F is lower semi continuous,*
- *$F(x, \cdot)$ is convex and positively 1-homogeneous for all x.*

Suppose moreover that there is a Riemannian metric g on M such that $F(x,v)^2 \ge g_x(v,v)$ $\forall (x,v) \in TM$. Then len^F is l.s.c. in the DF topology, in all paths with $\mathrm{len}^F(\gamma) < \infty$.

Proof. Let b^F be the distance function induced by len^F and

$$d^F(x,y) = \max\{b^F(x,y), b^F(y,x)\} \ .$$

Let d_g be the distance function induced by (M, g); the second hypothesis implies that $d_g \le d^F$. Let $|v|_x = \sqrt{g_x(v,v)}$ when $(x,v) \in TM$.

Let $\gamma \in \mathcal{C}_{\mathrm{AC01}}$ with $\mathrm{len}^F(\gamma) < \infty$. (Note that γ is not isolated in $\mathcal{C}_{\mathrm{AC01}}$ w.r.t. the DF topology). If $F(\gamma, \dot\gamma) \notin L^\infty$, possibly using Lemma 2.23 and 2.24 and replacing γ with θ, we then assume *wlog* that $F(\gamma, \dot\gamma) \in L^\infty$ and we set $\|F(\gamma, \dot\gamma)\|_{L^\infty} = L$.

[12] [THIS FOOTNOTE WAS DELETED]

The image of γ is compact; we define $0 = s_1 < s_2, \ldots < s_I = 1$ so that the image of γ on $[s_i, s_{i+1}]$ is contained in a local chart Ψ_i, for $i = 1, \ldots I-1$; let $S = \{s_1, \ldots s_I\}$.

Let $l = \liminf_{\xi\to\gamma} \operatorname{len}(\xi)$ in the DF topology, with $\xi \in \mathcal{C}_{\mathrm{AC01}}$. If $l \geq \operatorname{len}^F(\gamma)$ the proof ends. Suppose instead that $l < \operatorname{len}(\gamma)$. We define $S_k = \{i/2^k, i = 0, \ldots 2^k\} \cup S$; by the Lemma B.3 in appendix B.1, there is a sequence $T_k \supseteq S_k$ of finite subsets of $[0,1]$ and a sequence $(\xi_k) \subset \mathcal{C}_{\mathrm{AC01}}$ satisfying: $\xi_k \in \Xi_{\gamma,T_k}$ and $\lim_n \operatorname{len}(\xi_n) = l$; $\xi_k \to_k \gamma$ uniformly w.r.t. d^F and *a fortiori* w.r.t. d_g; there exists K such that for $k > K$, for any $t, t' \in T_k$ consecutive points[13]

$$\operatorname{len}^F(\xi_k|_{[t,t']}) \leq \operatorname{len}^F(\gamma|_{[t,t']}) \ . \qquad ((B.2))$$

Moreover, for any $t, t' \in T_k$ consecutive points, we can reparameterize ξ_k in $[t, t']$ using the Lemma 2.24 above (with $\varepsilon = t' - t$), so that, after reparameterization,

$$\begin{aligned} \widetilde{\forall}\, s \in [t,t'] \quad , \quad & (t'-t)F(\xi_k(s), \dot\xi_k(s)) \leq (t'-t) + \operatorname{len}^F(\xi_k|_{[t,t']}) \leq \\ & \leq (t'-t) + \operatorname{len}^F(\gamma_k|_{[t,t']}) \leq (t'-t)(1+L) \end{aligned}$$

but then

$$|\dot\xi_k(s)|_{\xi_k(s)} \leq F(\xi_k(s), \dot\xi_k(s)) \leq 1 + L \qquad (2.9)$$

for almost all $s \in [t, t']$. This reparameterization does not affect all other properties of ξ_k.

We recall that, by Ioffe theorem [8], when $M = \mathbb{R}^n$, len^F is l.s.c. when $\xi_h \to_h \gamma$ strongly in L^1 and $\dot\xi_h \to_h \dot\gamma$ weakly in L^1.

We proved that $\xi_h \to_h \gamma$ uniformly, so for k large, for all $s_i, s_{i+1} \in S$ consecutive points, forall $s \in [s_i, s_{i+1}]$ both $\xi_h(s)$ and $\gamma(s)$ are contained in one of the aforementioned local charts Ψ_i; in that chart $\xi_h \to_h \gamma$ strongly in L^1. Moreover by (2.9) we obtain that $|\dot\xi_k|_{\xi_k}$ are uniformly bounded, so that, possibly up to a subsequence, $\dot\xi_h \to_h \dot\gamma$ weakly in L^1 (again in local charts).

So $\liminf_n \operatorname{len}^F(\xi_k) \geq \operatorname{len}^F(\xi)$, but $\liminf_n \operatorname{len}^F(\xi_k) = \lim_n \operatorname{len}^F(\xi_k) = l$. □

So we conclude that the Example 1.3 enjoys idempotency.

A further result will be in Prop. 3.19.

We moreover remark this fact.

Remark 2.26 *When verifying that* $(\mathcal{C}_{AC}, \operatorname{len}^F)$ *satisfies all hypotheses 2.3, we have to verify that* len^F *is* "independent of linear reparameterization"*: this is a trivial proof.*

At the end of discussion we then obtain that len^F *is also* "independent of homeomorphic reparameterization"*, that is,* $\operatorname{len}^F(\gamma) = \operatorname{len}^F(\gamma \circ \varphi)$ *for a generic increasing homeomorphism* φ *(as long as both* γ *and* $\gamma \circ \varphi$ *are absolutely continuous).*

2.5.4 A further example in Calculus of Variation

Suppose that M is a metric space; let $J : M \to [0, \infty)$ be a fixed function. Consider the family $\mathcal{C}^0$ of continuous paths $\gamma : [\alpha, \beta] \to M$, and the length defined (when $\alpha < \beta$) as

$$\operatorname{len}^J(\gamma) \stackrel{\text{def}}{=} \sum_{t\in(\alpha,\beta)} J(\gamma(t)) + c_{\mathrm{out}} J(\gamma(\alpha)) + c_{\mathrm{in}} J(\gamma(\beta))$$

with $c_{\mathrm{in}}, c_{\mathrm{out}} \geq 0$ and $c_{\mathrm{in}} + c_{\mathrm{out}} = 1$. $(\mathcal{C}^0, \operatorname{len}^J)$ satisfies all hypotheses 2.3 and hence it is a *length structure*; it is not *run–continuous.*

We present a possible application of the above framework.

Example 2.27 (International commerce) *Let* $M = \mathbb{R}^2$*. It is divided in finitely many "nations" that are open connected sets* $M_1, M_2, \ldots M_N$*, pairwise disjoint; the boundaries of the nations are contained in* $S \subseteq M$*, a closed set; moreover* S *is covered by finitely many smooth 1-dimensional submanifolds of* M*.*

[13] That is, $t < t'$ and $T_k \cap (t, t') = \emptyset$.

We define J as above, and require that $\{J > 0\} = S$. Let $\mathcal{C}_{\mathrm{Lip}}$ the class of Lipschitz paths; let F and len^F be defined as in the previous sections. The length

$$\mathrm{len}_7(\gamma) = \mathrm{len}^F(\gamma) + \mathrm{len}^J(\gamma)$$

is composed of two costs, the cost len^F of moving goods inside the nations, and the duty cost $\mathrm{len}^J(\gamma)$ of moving it across nation borders.

If we further assume that $c_{\mathrm{in}} = c_{\mathrm{out}} = 1/2$ and $F(x, v) = F(x, -v)$, we obtain that $(\mathcal{C}_{\mathrm{Lip}}, \mathrm{len}_7)$ is a *symmetric length structure*: this is a simple–looking problem, but it is not easy to attack it using standard tools in Calculus of Variations or (symmetric) Metric space theory. Indeed we note that len^J is not l.s.c. in the topology of uniform convergence or pointwise convergence (w.r.t. the Euclidean distance). We also remark that len^J is not *run–continuous*, so it does not fall in the classical hypotheses (as seen in [12] or [4]).

We can instead attack it using the framework proposed in this paper.

Proposition 2.28 len^J *is l.s.c. in the DF topology, at all $\gamma \in \mathcal{C}^0$ where $\mathrm{len}^J(\gamma) < \infty$.*

Proof. Let $\tilde{T} = \{t \in (\alpha, \beta), J(\gamma(t)) > 0\}$; when $\mathrm{len}^J(\gamma) < \infty$ we obtain $\tilde{T}$ is at most countable; suppose that it is countable, let $(t_k)_k$ be an injective enumeration of $\tilde{T}$; given $\varepsilon > 0$ then there exists K large such that $\sum_{k=K+1}^{\infty} J(\gamma(t_k)) < \varepsilon$; let $T = \{t_1, t_2, \dots t_K\} \cup \{\alpha, \beta\}$; for all $\xi \in \Xi_{\gamma,T}$ then

$$\mathrm{len}^J(\xi) \geq \sum_{k=1}^{K} J(\gamma(t_k)) \geq \mathrm{len}^J(\gamma) - \varepsilon \quad .$$

When T is finite we choose $T = \tilde{T} \cup \{\alpha, \beta\}$. □

Let then $\mathcal{C}_7$ be the class of all Lipschitz paths such that $\mathrm{len}_7(\gamma) < \infty$; suppose that F satisfies all properties in 2.25; then $(\mathcal{C}_7, \mathrm{len}_7)$ enjoys *idempotency.*

2.6 Changing admissible paths

It is important to remark that in all the discussion above the family of paths $\mathcal{C}$ was fixed and never changed. In the following section 3 we will instead deal with different families of paths. We anticipate an important idea.

Proposition 2.29 *Suppose that $\mathcal{C}_1, \mathcal{C}_2$ are two family of paths with $\mathcal{C}_1 \subseteq \mathcal{C}_2$, and $\mathrm{len} : \mathcal{C}_2 \to [0, \infty]$, so that $(\mathcal{C}_1, \mathrm{len})$ and $(\mathcal{C}_2, \mathrm{len})$ are two length structures using the same length functional. From each length structure we induce a semi distance function, so we obtain b_1 and respectively b_2.*

Obviously $b_2 \leq b_1$, and hence by the sheer definition $\mathrm{Len}^{b_2}(\gamma) \leq \mathrm{Len}^{b_1}(\gamma)$ on any path. By the previous theorem's eqn. (2.8), $\mathrm{Len}^{b_2} \leq \mathrm{Len}^{b_1} \leq \mathrm{len}$ on $\mathcal{C}_1$, and $\mathrm{Len}^{b_2} \leq \mathrm{len}$ on $\mathcal{C}_2$.

Suppose moreover that len is l.s.c. in the DF topology on $\mathcal{C}_2$ (and hence on $\mathcal{C}_1$ as well). Then by the previous theorem

$$\mathrm{len} \equiv \mathrm{Len}^{b_1} \equiv \mathrm{Len}^{b_2} \quad \text{on } \mathcal{C}_1$$

and

$$\mathrm{len} \equiv \mathrm{Len}^{b_2} \leq \mathrm{Len}^{b_1} \quad \text{in } \mathcal{C}_2$$

but there may be a $\gamma \in \mathcal{C}_2 \setminus \mathcal{C}_1$ where $\mathrm{len}(\gamma) = \mathrm{Len}^{b_2}(\gamma) < \mathrm{Len}^{b_1}(\gamma)$.

3 The Asymmetric Metric Space

Let (M, b) be an **asymmetric metric space**, as defined in 1.1.

3.1 Topology

We agree that b defines a topology τ on M, generated by the families of **forward** and **backward open balls**

$$B^+(x,\varepsilon) \stackrel{\text{def}}{=} \{y \mid b(x,y) < \varepsilon\}, \qquad B^-(x,\varepsilon) \stackrel{\text{def}}{=} \{y \mid b(y,x) < \varepsilon\}$$

that is, the topology is generated by the symmetric distance function

$$d(x,y) \stackrel{\text{def}}{=} b(x,y) \vee b(y,x) \tag{3.1}$$

with balls

$$B(x,\varepsilon) \stackrel{\text{def}}{=} \{y \mid d(x,y) < \varepsilon\} = B^+(x,\varepsilon) \cap B^-(x,\varepsilon) \ .$$

By the above definition,

Proposition 3.1 *$x_n \to x$ if and only if $d(x_n,x) \to 0$, if and only if both $b(x_n,x) \to 0$ and $b(x,x_n) \to 0$.*

We will sometimes denote by τ^+ (resp. τ^-) the topology generated by forward (resp. backward) balls alone; we remark that those may differ from τ.

It is important to remark that b is continuous.

Proposition 3.2 *The asymmetric metric b is continuous with respect to the symmetric topology τ. Indeed if we provide M_2 with the metric*

$$d_2((x,y),(x',y')) \stackrel{\text{def}}{=} d(x,x') + d(y,y') \ ,$$

then $(x,y) \mapsto b(x,y)$ is Lipschitz w.r.t. d_2, since

$$|b(x,y) - b(x',y')| \le |b(x,y) - b(x,y')| + |b(x,y') - b(x',y')| \le d(x,x') + d(y,y') \ .$$

Remark 3.3 *We say that*

$$\overline{b}(x,y) \stackrel{\text{def}}{=} b(y,x) \tag{3.2}$$

is the ***conjugate distance function****; we defined in 2.1 $\hat\gamma(t) \stackrel{\text{def}}{=} \gamma(t)$ to be the* reverse path*; we remark that all the theory we present is invariant under the joint transformation $\gamma \mapsto \hat\gamma$, $b \mapsto \overline{b}$.*

We conclude with this definition, due to Busemann.

Definition 3.4 *A* ***"General metric space"*** *is a space satisfying all requisites in Defn. 1.1, and moreover*

$$\forall x,y \in M, \quad b(x,y) = 0 \Longrightarrow x = y \ ;$$

and

$$\forall x \in M, \forall (x_n) \subset M, \quad b(x_n,x) \to 0 \text{ iff } b(x,x_n) \to 0 \quad . \tag{3.3}$$

This is equivalent to saying that $\tau^+ \equiv \tau^-$. We do not in general assume that (3.3) holds; the reason is that we do not need it, and that it clashes with some examples, see the discussion in Sec. A.1.

3.2 Length by total variation (again)

Let (M,b) an asymmetric metric space. We induce from b the length $\mathrm{Len}^b\,\gamma$ of a path $\gamma : [\alpha,\beta] \to M$, by using the total variation seen in eqn. (1.1). If $\mathrm{Len}^b(\gamma) < \infty$, then γ is called **rectifiable**. In Section 2.2 we saw many properties of Len^b. We add a further remark.

Remark 3.5 *The length* Len^b *is in general not symmetric: if* $\hat\gamma(t) \stackrel{def}{=} \gamma(-t)$*, then in general* $\mathrm{Len}^b\gamma$ *will be different from* $\mathrm{Len}^b\hat\gamma$*; hence there does not exist a measure* $\mathcal{H}$ *on* M *such that the length of a path is the measure of its image, that is*

$$\mathcal{H}(image(\gamma)) = \mathrm{Len}^b\gamma \quad ;$$

indeed $image(\gamma) = image(\hat\gamma)$ *which is incompatible with the general case in which* $\mathrm{Len}^b\gamma \neq \mathrm{Len}^b\hat\gamma$*.*

We now propose an asymmetric version of Example 1.4.b in [12], that immediately shows the above remark in action.

Example 3.6 *Let* $M = \mathbb{R}$ *and*

$$b(x,y) = \begin{cases} y - x & if\ y \geq x \\ \sqrt{x-y} & if\ y < x \end{cases}$$

then the topologies τ^+ *and* τ^- *(generated by forward and backward balls) are equal and coincide with the Euclidean topology. Let* $\varepsilon > 0$ *and* $\hat\gamma, \gamma : [-\varepsilon, \varepsilon] \to M$ *be defined simply as* $\gamma(t) = t$*,* $\hat\gamma(t) = -t$*; both paths are continuous, but* $\mathrm{Len}^b\gamma = 2\varepsilon$ *whereas* $\mathrm{Len}^b\hat\gamma = \infty$*.*

3.3 Running length

We suppose in this section that $\gamma : [\alpha,\beta] \to M$ is a rectifiable path. As in (2.2) we define the **running length** $\ell^\gamma : [\alpha,\beta] \to \mathbb{R}^+$ of γ to be the length of γ restricted to $[\alpha,t]$, that is

$$\ell^\gamma(t) \stackrel{\text{def}}{=} \mathrm{Len}^b\left(\gamma|_{[\alpha,t]}\right) \quad . \tag{3.4}$$

We will write ℓ instead of ℓ^γ in the following, to ease notation.

The following properties hold.

- $\ell(t)$ is monotonic. (In general is not left or right continuous).
- Let $\alpha \leq s \leq t \leq \beta$, since Len^b is additive, then the length of γ restricted to $[s,t]$ is $\ell(t) - \ell(s)$.
- So, by the definition (1.1) we obtain that

$$b\big(\gamma(s), \gamma(t)\big) \leq \ell(t) - \ell(s) \quad . \tag{3.5}$$

- We will see in Prop. 3.9 that, if γ is rectifiable and continuous, then ℓ is continuous.
- The fact that ℓ is continuous does not imply that γ is continuous, as shown in the example 4.1. This is counter-intuitive since it is contrary to what is seen in symmetric metric spaces, and in Busemann's *general metric spaces.*

We will call **run-continuous** a rectifiable path γ such that ℓ is continuous.

Remark 3.7 *Note that* "run–continuity" *cannot be expressed as* "continuity according to topology ..."*; indeed this would violate the invariance described in 3.3. Note also that if* γ *is run–continuous then it is right–continuous in the* τ^+ *topology and left–continuous in the* τ^- *topology; but it may fail to be continuous in the* τ^+ *or* τ^- *topology, as in Example 4.2.*

Proposition 3.8 *Let* $\bar b(x,y) = b(y,x)$ *as in (3.2), and* $d(x,y) \stackrel{def}{=} b(x,y) \vee b(y,x)$ *as in (3.1). The three following facts are equivalent.*

1. γ *is continuous and rectifiable, and the reverse path* $\hat\gamma(t) = \gamma(-t)$ *is rectifiable as well;*
2. *both* γ *and* $\hat\gamma$ *are run-continuous;*
3. γ *is continuous and* $\mathrm{Len}^d(\gamma) < \infty$ *(*ie *it is continuous and rectifiable in symmetric metric space* (M,d)*).*

The proof (1) $\Longrightarrow$ (2) *follows from Prop. 3.9 here following;* (2) $\Longrightarrow$ (1) *follows from eqn. (3.5);* (2) $\Longleftrightarrow$ (3) *from the relations* $b \le d$*,* $\bar{b} \le d$*,* $d \le b + \bar{b}$ *and Prop. 2.6.*

We conclude with this important proposition.

Proposition 3.9 *If* γ *is rectifiable and continuous, then* ℓ *is continuous.*

In the symmetric case, this is known, see Prop. 2.3.4 in [4] or 1.1.13 in [17]. The asymmetric proof needs some adjustments; we provide it for convenience of the reader, see Section B.2.

We will talk more about the relationship between *continuity* and *run-continuity* of γ in [16].

3.4 The induced distance

Suppose now that $\mathcal{C}$ is a family of paths such that $(\mathcal{C}, \mathrm{Len}^b)$ is a length structure. Let b^l be the induced semidistance function.

Proposition 3.10 *For* $x, y \in M$ *we have*

$$b(x,y) \le b^l(x,y) \quad . \tag{3.6}$$

Given $\gamma \in \mathcal{C}$*,* $\gamma : [\alpha, \beta] \to M$*, connecting* $x = \gamma(\alpha)$ *to* $y = \gamma(\beta)$*, we have that*

$$b^l(x,y) \le \mathrm{Len}^b(\gamma) \quad . \tag{3.7}$$

but

$$\mathrm{Len}^{b^l}(\gamma) = \mathrm{Len}^b(\gamma) \quad . \tag{3.8}$$

Proof. Indeed for any path $\xi \in C$ connecting x to y, by the sheer definition, we have that $b(x,y) \le \mathrm{len}(\xi)$ hence passing to the inf we obtain (3.6). For (3.7) we just use the definition of b^l. For (3.8) we use (3.6) and (2.8). □

In particular b^l is a distance (and not only a semidistance) function.

3.5 The symmetric case

In the symmetric case, there is one main natural choice for the family of paths, and that is the family $\mathcal{C}_g$ of all rectifiable and continuous paths[14]; moreover a rectifiable path is continuous *iff* it is run–continuous. So from $(\mathcal{C}_g, \mathrm{Len}^b)$ we induce a new (symmetric) distance b^g. When $b^g \equiv b$, b is said to be **intrinsic** or a **path metric** or a **length space**. In general $b^g \ge b$, the topology on (M, b^g) is finer than the topology in (M, b); but this result holds.

Proposition 3.11 *A path* $\gamma : [a, b] \to M$ *is continuous and rectifiable in* (M, b) iff *it is continuous and rectifiable in* (M, b^g)*.*

The above is found as Exercises 2.1.4 and 2.1.5 in [4], or Prop. 2.1.9 in [17]; the proof is simply based on Prop. 3.9 and 3.10. So when we consider the length Len^{b^g}, the natural choice for the family of paths is again the same family $\mathcal{C}_g$. So the discussion in the previous section applies. We thus easily prove that $\mathrm{Len}^{b^g} \equiv \mathrm{Len}^b$ and $b^g = (b^g)^g$, that is, b^g is always *intrinsic*. These facts are found *e.g.* in Prop. 2.3.12 and Prop. 2.4.1 in [4], or Prop. 2.1.10 and following in [17].

[14] Another commonly used family is the family of Lipschitz paths; but this choice does not significantly alter the discussion.

3.6 Length structures in (M, b)

In the asymmetric case, there is not only one natural choice for the family of paths. We propose three different structures, that use the same length functional Len^b, but employ three different admissible classes of paths:

- $\mathcal{C}_r$ is the class of all run-continuous paths;
- $\mathcal{C}_g$ is the class of all continuous rectifiable paths (that are also run-continuous, by Prop. 3.9);
- $\mathcal{C}_s$ is the class of all continuous paths such that both γ and $\hat\gamma(t) \stackrel{\text{def}}{=} \gamma(-t)$ are rectifiable (note that other equivalent definitions are in Prop. 3.8).

Obviously

$$\mathcal{C}_r \supseteq \mathcal{C}_g \supseteq \mathcal{C}_s \quad ;$$

in symmetric metric spaces the three classes coincide, but we see in Examples 3.6, 4.1 and 4.4 that in general in the asymmetric case these classes do not coincide.

By using the previous results such as 2.7 it is easy to see that the paths in the above classes may be reparameterized, joined and splitted; moreover constants are in the classes; so

$$(\mathcal{C}_r, \mathrm{Len}^b),\ (\mathcal{C}_g, \mathrm{Len}^b),\ (\mathcal{C}_s, \mathrm{Len}^b)$$

are three *length structures*. So they induce three new distance functions. Let then $b^r(x,y)$ (respectively $b^g(x,y)$, $b^s(x,y)$) be the infimum of $\mathrm{Len}^b(\xi)$ for all ξ connecting x, y and $\xi \in \mathcal{C}_r$ (respectively $\xi \in \mathcal{C}_g$, $\xi \in \mathcal{C}_s$). Obviously

$$b \le b^r \le b^g \le b^s \quad , \tag{3.9}$$

where the first inequality was already seen in (3.6). Moreover, by (3.9), we obtain that if b is a distance then b^r, b^g, b^s are distances (and not just semidistances), *cf* 2.8.

Remark 3.12 *A consequence of the above inequality (3.9) is that the identity maps*

$$(M, b^s) \to (M, b^g) \to (M, b^r) \to (M, b)$$

are continuous, that is, arrows go from finer to coarser topologies.[15] *The opposite arrows are in general not continuous (not even in the symmetric case) as we see in Example 4.7.*

At this point we obtain 3 new lengths $\mathrm{Len}^{b_r}, \mathrm{Len}^{b_g}, \mathrm{Len}^{b_s}$. The following is an application of the principles presented in Prop. 2.18, Theorem 2.19, Prop. 2.29 and Prop. 3.10.

Proposition 3.13 *From (3.9) and the sheer definition (1.1) we obtain that*

$$\mathrm{Len}^b(\gamma) \le \mathrm{Len}^{b_r}(\gamma) \le \mathrm{Len}^{b_g}(\gamma) \le \mathrm{Len}^{b_s}(\gamma) \quad ; \tag{3.10}$$

moreover, by eqn. (2.8)

$$\textit{if } \gamma \in \mathcal{C}_r \textit{ then } \mathrm{Len}^{b^r}\gamma = \mathrm{Len}^b\gamma \quad , \tag{3.11}$$

$$\textit{if } \gamma \in \mathcal{C}_g \textit{ then } \mathrm{Len}^{b^g}\gamma = \mathrm{Len}^{b^r}\gamma = \mathrm{Len}^b\gamma \quad , \tag{3.12}$$

$$\textit{if } \gamma \in \mathcal{C}_s \textit{ then } \mathrm{Len}^{b^s}\gamma = \mathrm{Len}^{b^g}\gamma = \mathrm{Len}^{b^r}\gamma = \mathrm{Len}^b\gamma \quad . \tag{3.13}$$

Remark 3.14 *The next step, in the spirit of Section 1.1.6, would be to derive new metrics from the 3 new lengths* $\mathrm{Len}^{b_r}, \mathrm{Len}^{b_g}, \mathrm{Len}^{b_s}$*; the problem now is in choosing the class of admissible paths; indeed at this point we have at hand three new asymmetric metric spaces*

$$(M, b^r)\ ,\ (M, b^g)\ ,\ (M, b^s) \quad ,$$

and for each one we would have the choice of three classes $\mathcal{C}_{*r}, \mathcal{C}_{*g}, \mathcal{C}_{*s}$.

Hence it becomes urgent to study the idempotency of the above operation i.e. *to see which choices provide the same results, to tame the combinatorial zoo (see figure 1 on the following page).*

[15] A finer topology has more open sets and less compact sets.

$$\begin{tikzcd}
 & & & (\mathcal{C}_{rr}, \mathrm{Len}^{b^r}) & \iddots \\
 & (\mathcal{C}_r, \mathrm{Len}^b) \arrow[r, dotted] & (M, b^r) \arrow[ur, dashed] \arrow[r, dashed] \arrow[dr, dashed] & (\mathcal{C}_{rg}, \mathrm{Len}^{b^r}) & \cdots \\
 & & & (\mathcal{C}_{rs}, \mathrm{Len}^{b^r}) & \ddots \\
 & & & (\mathcal{C}_{gr}, \mathrm{Len}^{b^g}) & \iddots \\
(M, b) \arrow[uuur, dashed] \arrow[r, dashed] \arrow[dddr, dashed] & (\mathcal{C}_g, \mathrm{Len}^b) \arrow[r, dotted] & (M, b^g) \arrow[ur, dashed] \arrow[r, dashed] \arrow[dr, dashed] & (\mathcal{C}_{gg}, \mathrm{Len}^{b^g}) & \\
 & & & (\mathcal{C}_{gs}, \mathrm{Len}^{b^g}) & \ddots \\
 & & & (\mathcal{C}_{sr}, \mathrm{Len}^{b^s}) & \iddots \\
 & (\mathcal{C}_s, \mathrm{Len}^b) \arrow[r, dotted] & (M, b^s) \arrow[ur, dashed] \arrow[r, dashed] \arrow[dr, dashed] & (\mathcal{C}_{sg}, \mathrm{Len}^{b^s}) & \cdots \\
 & & & (\mathcal{C}_{ss}, \mathrm{Len}^{b^s}) & \ddots
\end{tikzcd}$$

Figure 1: The combinatorial zoo.

3.6.1 Idempotency

The first such results are as follows.

Theorem 3.15 • *$\mathcal{C}_r = \mathcal{C}_{rr}$, hence $b^r \equiv (b^r)^r$.*

- *$\mathcal{C}_s = \mathcal{C}_{ss} = \mathcal{C}_{gs} = \mathcal{C}_{rs}$, hence $b^s \equiv (b^s)^s \equiv (b^g)^s \equiv (b^r)^s$.*
- *We add some inclusions.*

$$\mathcal{C}_{sg} \subseteq \mathcal{C}_{gg} \subseteq \mathcal{C}_{rg} \subseteq \mathcal{C}_g \quad , \quad \mathcal{C}_{sr} \subseteq \mathcal{C}_{gr} \subseteq \mathcal{C}_{rr} = \mathcal{C}_r \quad .$$

Proof. We first prove that

$$\mathcal{C}_{ss} \subseteq \mathcal{C}_{gs} \subseteq \mathcal{C}_{rs} \subseteq \mathcal{C}_s \quad , \quad \mathcal{C}_{sg} \subseteq \mathcal{C}_{gg} \subseteq \mathcal{C}_{rg} \subseteq \mathcal{C}_g \quad , \quad \mathcal{C}_{sr} \subseteq \mathcal{C}_{gr} \subseteq \mathcal{C}_{rr} \subseteq \mathcal{C}_r \quad .$$

Indeed, by remark 3.12, a path $\gamma : [a,b] \to M$ that is continuous in (M, b^s) is continuous in (M, b^g) as well; and so on. A consequence of the inequality (3.10) is moreover that if γ is run–continuous in (M, b^s) then it is run–continuous in (M, b^g) as well; and so on. So $\gamma \in \mathcal{C}_{ss}$ *iff* $\gamma, \hat{\gamma}$ are run–continuous in (M, b^s) (by point 2 in Prop. 3.8), *hence* $\gamma, \hat{\gamma}$ are run–continuous in (M, b^g), so $\gamma \in \mathcal{C}_{gs}$. All other cases are similar.

If $\gamma \in \mathcal{C}_r$ then by (3.11) $\mathrm{Len}^b(\gamma|_{[a,b]}) = \mathrm{Len}^{b^r}(\gamma|_{[a,b]})$ so γ is run–continuous in (M, b^r) as well, that is, $\gamma \in \mathcal{C}_{rr}$.

If $\gamma \in \mathcal{C}_s$ then we use (3.13), we apply the previous idea to both γ and $\hat{\gamma}$, and use point 2 in Prop. 3.8 and obtain that $\gamma \in \mathcal{C}_{ss}$. Alternatively we use the symmetric metric theory and use point 3 in Prop. 3.8. □

In the symmetric case (Prop. 3.11) we have that $\mathcal{C}_g = \mathcal{C}_{gg}$; in the asymmetric case this fails, see Example 4.4.

Currently, at this level of generality, the above are the only idempotencies and inclusions we could prove; the picture is not yet complete, see figure 2 on the next page.

3.6.2 Intrinsic asymmetric spaces

We thus propose this definition

Definition 3.16 *An asymmetric metric space (M, b) is called*

- ***r–intrinsic*** *when $b \equiv b^r$,*

$$\begin{tikzcd}
(M,b) & (M,b^r) & (M,b^g) & (M,b^s) \\
\mathcal{C}_r \arrow[r, leftrightarrow] & \mathcal{C}_{rr} & \mathcal{C}_{gr} \arrow[l] & \mathcal{C}_{sr} \arrow[l] \\
\mathcal{C}_g \arrow[u] & \mathcal{C}_{rg} \arrow[l] \arrow[u] & \mathcal{C}_{gg} \arrow[l] \arrow[u] & \mathcal{C}_{sg} \arrow[l] \arrow[u] \\
\mathcal{C}_s \arrow[u] \arrow[r, leftrightarrow] & \mathcal{C}_{rs} \arrow[u] \arrow[r, leftrightarrow] & \mathcal{C}_{gs} \arrow[u] \arrow[r, leftrightarrow] & \mathcal{C}_{ss} \arrow[u]
\end{tikzcd}$$

Figure 2: Inclusion between families of paths (arrows go from smaller to larger sets; arrows are bold when the inclusion is strict in some examples).

- ***g–intrinsic*** *when* $b \equiv b^g$*,*
- ***s–intrinsic*** *when* $b \equiv b^s$*.*

(By eqn. (3.9) the third implies the second, the second implies the first). The previous theorem shows that the induced metric space (M, b^r) is always r–intrinsic, and (M, b^s) is always s–intrinsic.

We will see in [16] that many important results valid in symmetric intrinsic metric spaces hold also in asymmetric r–intrinsic metric spaces; but not all!

Remark 3.17 *Let* $D^+(x,\varepsilon) \stackrel{def}{=} \{y \mid b(x,y) \le \varepsilon\}$*. In g–intrinsic spaces,* $\overline{B^+(x,\varepsilon)} = D^+(x,\varepsilon)$*. In r–intrinsic spaces, it does not (set* $x = \varepsilon = 1$ *in Example 4.1).*

3.6.3 Applications

We now cross the boundaries between the sections 3 and 2 and their point of views. Similarly to the symmetric case, we can state this.

Proposition 3.18 *Consider these two statements.*

1. (M,b) *is r-intrinsic.*
2. $(\mathcal{C}, \mathrm{len})$ *is a run-continuous length structure on* M*, and induces* b*.*
   [we do not need to assume " l.s.c. in the DF topology" here]

These two are equivalent, in the following sense.

If (1)*, then we set* $\mathcal{C} = \mathcal{C}_r, \mathrm{len} = \mathrm{Len}^b$ *and then* $b = b^r = b^l$ *and we obtain a length structure as specified in* (2)*; moreover we know that* $(\mathcal{C}, \mathrm{len}) = (\mathcal{C}_r, \mathrm{Len}^b)$ *is l.s.c. in the DF topology.*

Vice versa, suppose that a length structure as in (2) *exists, then* b *is r–intrinsic.*

Proof. Let $\mathcal{E}$ be the set of $\gamma \in \mathcal{C}$ such that $\mathrm{len}(\gamma) < \infty$. Note that b is also the distance induced by $(\mathcal{E}, \mathrm{len})$. Consider the class $\mathcal{C}_r$ in (M,b) and induce b^r from $(\mathcal{C}_r, \mathrm{Len}^b)$. Recalling from Theorem 2.19 that $\mathrm{len} \ge \mathrm{Len}^b$ on $\mathcal{C}$, we obtain that $\mathcal{E} \subseteq \mathcal{C}_r$. Consider the class $\mathcal{E}$ in (M,b) and b^l the distance induced by $(\mathcal{E}, \mathrm{Len}^b)$ then (by the point 4 in the aforementioned theorem) $b \equiv b^l$. Since $\mathcal{E} \subseteq \mathcal{C}_r$ so $b^l \ge b^r$; but also $b \le b^r$. (If moreover len is l.s.c. in the DF topology then $\mathrm{len} = \mathrm{Len}^b$ on $\mathcal{C}$.) □

We again remark that the second statement above does not require any "compatible" topology on M. We also remark that this proposition needs fewer hypotheses than lemma 2.22. (Note also that in the proof we obtain that $\mathcal{C} \subseteq \mathcal{C}_r$: this really needs the hypothesis that the length structure be run–continuous, indeed in the example in Sec. 2.5.4, $\mathcal{C}$ and $\mathcal{C}_r$ are incomparable).

As an application we propose this result.

Proposition 3.19 *Suppose that b is a Finslerian distance function, as defined in Example 1.3. Then $b \equiv b^r$, that is, (M, b) is* r–intrinsic*.*

The proof follows from the above proposition, letting $\mathcal{C}$ be the class of absolutely continuous paths ξ such that $\operatorname{len}^F(\xi) < \infty$.

Note that, even in Riemannian manifold, $\mathcal{C}_{\mathrm{AC}} \overset{\neq}{\subset} \mathcal{C}_r$, since the latter contains also rectifiable curves that are not absolutely continuous. Note that also that the Finslerian distance function may fail to be *g-intrinsic* and *s–intrinsic*, see Example 4.1.

4 Examples

We present some examples of length structures and asymmetric metric spaces that provide many interesting counterexamples.

Example 4.1 *Let $M = [-1, 1] \subset \mathbb{R}$ and let $f_1 : M \times \mathbb{R} \to \mathbb{R}$ be*

$$f_1(x, v) = \begin{cases} |v| & \text{if } v \geq 0 \text{ or } x \leq 0 \\ |v|/x & \text{if } v < 0 \text{ and } x > 0 \end{cases} . \tag{4.1}$$

We proceed as explained in the Example 1.3. From f_1 we define the length

$$\operatorname{len}^{f_1}(\xi) = \int_0^1 f_1(\xi(s), \dot{\xi}(s))\, ds$$

of an absolutely continuous path $\xi : [0, 1] \to M$; choosing $\mathcal{C}_{AC}$ to be the family of all absolutely continuous paths, we obtain that $(\mathcal{C}_{AC}, \operatorname{len}^{f_1})$ is a length structure; we induce the asymmetric distance function

$$b_1(x, y) = \begin{cases} |x - y| & \text{if } x \leq y \text{ or } y < x \leq 0 \\ \log(x/y) & \text{if } 0 < y < x \\ +\infty & \text{if } y \leq 0 < x \end{cases} . \tag{4.2}$$

Moreover, if ξ is monotonic, then

$$\operatorname{len}^{f_1}(\xi) = \operatorname{Len}^{b_1}(\xi) = b_1\big(\xi(0), \xi(1)\big) \quad .$$

(The first equality holds on all AC curves as discussed in Sec. 2.5.3).

We now concentrate on the space $([-1, 1], b_1)$. This example enjoys a lot of interesting features.

1. The topology τ^+ generated by forward balls is the usual Euclidean topology on $[-1, 1]$; whereas the topology τ^- generated by backward balls divides M into two disconnected components $[-1, 0]$ and $(0, 1]$, each one with the (induced) Euclidean topology; moreover $\tau = \tau^-$.

   [point 2 was deleted since it was a repetition of 7]

3. Since the lengths are invariant w.r.t. reparameterization, we will consider in the following only two family of paths $\gamma, \hat{\gamma}$: the paths $\gamma : [a, b] \to M$ with $-1 \leq a \leq b \leq 1$ and $\gamma(t) = t$, and their reverses $\hat{\gamma}(t) = -t$.

4. Let $\ell^\gamma(t) \stackrel{\text{def}}{=} \operatorname{Len}^{b_1}\left(\gamma|_{[a,t]}\right)$ be the *running length.* The path γ has $\ell^\gamma(t) = t - a$, so it is *run-continuous* but when $a < 0, b > 0$ it is not continuous, hence $\gamma \in \mathcal{C}_r, \gamma \notin \mathcal{C}_g$.

5. Let $a < b$. Let again $\gamma(t) = t$. Since $\gamma \in \mathcal{C}_r$ then $b_1^r(a, b) = b_1(a, b) = b - a$; since $\gamma \notin \mathcal{C}_g$, $\gamma \notin \mathcal{C}_s$ then $b_1^g(a, b) = b_1^s(a, b) = \infty$.

6. Consequently, $\operatorname{Len}^{b_1^r}(\gamma) = \operatorname{Len}^{b_1}(\gamma) = b - a$ (as proved by Prop. 3.13) but $\operatorname{Len}^{b_1^g}(\gamma) = \operatorname{Len}^{b_1^s}(\gamma) = \infty$.

7. By prop. 3.19 (M, b_1) is r–intrinsic *i.e.* $b_1 \equiv b_1^r$.

8. If we wish to use continuous rectifiable curves, that is the class $\mathcal{C}_g$, then we hit the problem that (M, b_1) is disconnected, so (M, b_1) is not g–intrinsic since so $b_1^g(-1,1) = \infty \neq b_1(-1,1) = 2$.

Note that, as remarked at the end of Sec. 2.1, in the case of Finslerian lengths, ℓ^ξ is absolutely continuous when it is finite; hence if $\xi \in \mathcal{C}_r, \xi \notin \mathcal{C}_g$ then $\mathrm{len}^{f_1}\left(\widehat{\xi}\right) = \infty$. In the example 4.6 we will see that this does not hold for generic length structures.

Example 4.2 *Let again $M = [-1,1]$ and let $f_2 : M \times \mathbb{R} \to \mathbb{R}$ be*

$$f_2(x,v) = \begin{cases} |v| & \textit{if } v \geq 0 \textit{ or } x = 0 \\ |v|/|x| & \textit{if } v < 0 \textit{ and } x \neq 0. \end{cases} \tag{4.3}$$

From this we induce

$$b_2(x,y) = \begin{cases} y - x & \textit{if } y \geq x \\ \log(x/y) & \textit{if } 0 < y < x \\ \log(y/x) & \textit{if } y < x < 0 \\ +\infty & \textit{if } y \leq 0,\ x \geq 0 \textit{ and } x \neq y \end{cases} \tag{4.4}$$

This example has another lot of interesting features.

1. The topology τ^+ makes $[-1,0)$ disconnected from $[0,1]$; whereas the topology τ^- makes $[-1,0]$ disconnected from $(0,1]$. The topology τ divides $[-1,1]$ in 3 connected components $[-1,0), \{0\}, (0,1]$, each one equipped with the Euclidean topology.
2. Again $\gamma : [-1,1] \to M$ defined by $\gamma(t) = t$ has $\ell^\gamma(t) = t - 1$, but is not continuous.

Other properties are as in the previous example.

The following examples are built by using a joining principle.

Lemma 4.3 (Joining of metric spaces) *Suppose that $(\widetilde{M}, \tilde{b})$ and (M, b) are asymmetric metric spaces, and that $\widetilde{M} \cap M = \{\overline{x}\}$; we can then define an asymmetric metric $\overline{b}$ on $\tilde{M} \cup M$ by*

$$\overline{b}(x,y) = \begin{cases} \tilde{b}(x,y) & \textit{if } x,y \in \widetilde{M}\ , \\ b(x,y) & \textit{if } x,y \in M\ , \\ \tilde{b}(x,\overline{x}) + b(\overline{x},y) & \textit{if } x \in \widetilde{M}, y \in M\ , \\ b(x,\overline{x}) + \tilde{b}(\overline{x},y) & \textit{if } x \in M, y \in \widetilde{M}\ . \end{cases} \tag{4.5}$$

Example 4.4 *Let $M = [-1,1]$, we define*

$$b_3(x,y) = \begin{cases} y - x & \textit{if } x \leq y\ , \\ x - y & \textit{if } y < x \leq 0\ , \\ \sqrt{x^2 + 2x} - x - y & \textit{if } y < 0 \leq x\ , \\ \sqrt{x^2 + 2(x-y)} - x & \textit{if } 0 \leq y < x\ . \end{cases} \tag{4.6}$$

This distance b_3 is the join of two asymmetric metrics defined on $[0,1]$ and $[-1,0]$. (See in the appendix B.3 for detailed proof). This metric has these features.

1. *With some straightforward computation, we prove that*

$$\frac{1}{3}|x-y| \leq b_3(x,y) \leq 2\sqrt{|x-y|}\ . \tag{4.7}$$

There follows that the topologies τ^+, τ^-, τ generated by b_3 are all equal to the Euclidean topology.

2. *Let* $\gamma : [0,1] \to M$ *be a continuous and monotonically decreasing path connecting* x *to* y *with* $0 < y < x$. *Then* $\mathrm{Len}^{b_3}\,\gamma = \log(x/y)$.

3. *Again, since the lengths are invariant w.r.t. reparameterization, we will consider only the paths* $\gamma : [a,b] \to M$ *with* $-1 \le a \le b \le 1$ *and* $\gamma(t) = t$*, and their reverses* $\hat\gamma(t) = -t$ *in the following.*

4. *Let* $\ell^\gamma(t) \overset{def}{=} \mathrm{Len}^{b_3}\left(\gamma|_{[a,t]}\right)$ *be the* running length*. The path* γ *has* $\ell^\gamma(t) = t-a$*, so it is* run-continuous *and continuous but when* $a < 0, b > 0$ *the reverse path* $\hat\gamma$ *is not rectifiable, hence* $\gamma \in \mathcal{C}_g$ *but* $\gamma \notin \mathcal{C}_s$*.*

5. *There follows that* $(b_3)^r = (b_3)^g = b_1$*.*

   So, this example has a counterintuitive property: the space (M, b_3) *has* Euclidean *topological properties and is a* General Metric Space*, but the geodesic induced space* (M, b_3^g) *is quite non-Euclidean and is not a* General Metric Space*.(See also Remark A.1).*

6. *Moreover* $\gamma \in \mathcal{C}_g$ *but* $\gamma \notin \mathcal{C}_{rg}$*.*

7. *But then* $((b_3)^g)^g \neq (b_3)^g$*.*

8. *To conclude we remark that the class* $\mathcal{C}_s$ *contains only curves whose image is restricted to* $[-1,0]$*, hence* $\mathcal{C}_s$ *does not connect points.*

Example 4.5 *Let* $M = [-1,1]$*, we define*

$$b_4(x,y) = \begin{cases} y - x & \text{if } x \le y \ , \\ \sqrt{y^2 + 2(x-y)} + y & \text{if } y < x \le 0 \ , \\ \sqrt{x^2+2x} - x + \sqrt{y^2 - 2y} + y & \text{if } y < 0 \le x \ , \\ \sqrt{x^2 + 2(x-y)} - x & \text{if } 0 \le y < x \ . \end{cases} \tag{4.8}$$

This metric b_4 *is again the join of two asymmetric metrics defined on* $[0,1]$ *and* $[-1,0]$*.*

Again, b_4 *satisfies (4.7), so the topologies* τ^+, τ^-, τ *generated by* b_4 *are all equal to the Euclidean topology. But* $(b_4)^r = (b_4)^g = b_2$*. Similar comments as above follow.*

Example 4.6 *Let again* $M = [-1,1]$*. Let* $\mathcal{C}_H$ *be the family of all paths* $\xi : [\alpha,\beta] \to M$ *that are continuous and injective (hence monotonic). Let* len_5 *be defined as*

$$\mathrm{len}_5(\xi) = |\xi(\alpha) - \xi(\beta)| + \begin{cases} 1 & \xi(\beta) \le 0, \xi(\alpha) > 0 \\ 0 & \textit{otherwise} \end{cases} \tag{4.9}$$

the effect being that, when paths run rightwards the space looks as the usual Euclidean interval $[-1,1]$ *whereas when they run leftwards, it looks as if there is a gap of width 1 between* $[-1,0]$ *and* $(0,1]$*.*

We then define $\mathcal{C}_P$ *as the class of* $\xi : [\alpha,\beta] \to \mathbb{R}$ *that are continuous and piecewise injective (hence monotonic), and extend the length* len_5 *to* $\mathcal{C}_P$ *by addittivity.* $(\mathcal{C}_P, \mathrm{len}_5)$ *is a length structure. We then induce the distance function* b_5 *so we obtain that*

$$b_5(x,y) = |x-y| + \begin{cases} 1 & y \le 0, x > 0 \\ 0 & \textit{otherwise} \end{cases} \tag{4.10}$$

and the topologies are as in 4.1.

We now concentrate on the space $([-1,1], b_5)$*. It is easy to prove that* $\mathrm{Len}^{b_5} \equiv \mathrm{len}_5$ *on* $\mathcal{C}_P$ [16]*. Again in the following we will consider only the paths* $\gamma, \hat\gamma$ *with* $\gamma(t) = t$, $\hat\gamma(t) = -t$*. Let* $\ell^\gamma(t) \overset{def}{=} \mathrm{Len}^{b_5}\left(\gamma|_{[a,t]}\right)$ *be the* running length*. The path* γ *has* $\ell^\gamma(t) = t - a$*, so it is* run-continuous*, but when* $a < 0, b > 0$ *it is not continuous, hence* $\gamma \in \mathcal{C}_r, \gamma \notin \mathcal{C}_g$*; moreover* $\hat\gamma$ *is rectifiable so* $\ell^{\hat\gamma}(t)$ *is bounded but it is discontinuous.*

All other properties are as in the previous example 4.1.

[16] Use neighborhoods $\Xi_{\xi,T}$ with T containing all points where ξ changes monotonicity.

Example 4.7 *Consider*

$$M = \{x \in \mathbb{R}^2 \mid -1 \le x_1 \le 1, x_2 = 0\} \cup$$
$$\bigcup_{n \ge 1} \{x \in \mathbb{R}^2 \mid -1 \le x_1 \le 1, \quad x_2 = (x_1 - 1)/n\}$$

and b the Euclidean distance function (see fig. 3). In this case $b^r \equiv b^g \equiv b^s$. (M, b) is compact but (M, b^g) is not.

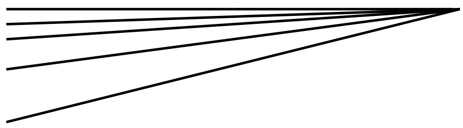


Figure 3: example 4.7

We tweak the example 4.7 a bit, to build this new example.

Example 4.8 *Consider*

$$M = \{x \in \mathbb{R}^2 \mid -1 \le x_1 \le 1, x_1 \ne 0, x_2 = 0\} \cup$$
$$\{x \in \mathbb{R}^2 \mid -1 \le x_1 \le 1, \quad 1/4 \ge x_2 > 0\} \cup$$
$$\bigcup_n \{x \in \mathbb{R}^2 \mid -1 \le x_1 \le 1, \quad x_2 = (x_1 - 1)/n\}$$

and b the Euclidean distance function (see fig. 4); let b^g be the distance induced by the length of continuous rectifiable curves. Let $A = (-1, 0)$, $B = (1, 0)$. Then (M, b) is locally compact, but is not path-metric and is not complete; (M, b) does not admit geodesics locally around A, that is, the points A and $x_n = (-1, -2/n)$ do not admit a minimal connecting geodesic.

The disc $\{y \mid b(B, y) \le 1/2\}$ is compact for b; but the discs $\{y \mid b^g(B, y) \le \varepsilon\}$ are never compact for b^g.

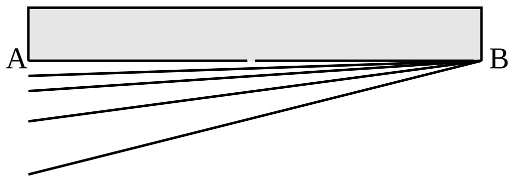


Figure 4: example 4.8

A Comparison with related works

To conclude, we compare our approach to "asymmetric metric spaces" to other approaches.

A.1 Comparison with Busemann's "General metric spaces"

The foundation of the theory of non-symmetric metric spaces was given by Busemann in his paper *Local Metric Geometry* [5], where it was christened as "General metric spaces".

It should be noted that, although Busemann has presented the foundation of that theory, most of his work makes the assumption that the distance function be symmetric; such is the case e.g. of the renowned work on *The Geometry of Geodesics* [6]. The study of non-symmetric "General metric spaces" was carried on for example in Phadke's [18] and Zaustinsky's [19].

We report now the theory of "General metric spaces" here, in a language that is more similar to our presentation and to [19], than to [5].

- A **"General metric space"** was defined in 3.4.
- A "General metric space" is **finitely compact** if any closed set that is contained in $B^+(x,r) \cup B^-(x,r)$ (for a choice of $x \in M$ and $r > 0$) is compact.
- A "General metric space" is **weakly finitely compact** if small closed forward balls are compact[17].

We add some comments that highlight the differences between the approach in the present paper and in [5].

- The additional hypothesis (3.3) used in defining "General metric spaces" holds in many applications; for example, when the space is locally compact, as in [15]. Still it is possible to find interesting examples of spaces where it does not hold, such as the examples 4.1 and 4.2, that are Finslerian Metrics, induced from a Calculus of Variations problem of the form discussed in introduction.
- Another problem we see in the definition of *General Metric Spaces* is that the induced space may not be.

  Remark A.1 *In the examples 4.4 and 4.5,* (M, b) *is a* General Metric Space*, it is topologically Euclidean,* [18] *but the induced geodesic space* (M, b^g) *does not satisfy (3.3), so it is an* asymmetric metric space *as defined in this paper, but not a* General Metric Space*.*

- One consequence of (3.3) is that any rectifiable run-continuous path is also continuous; so the classes $\mathcal{C}_r \equiv \mathcal{C}_s$; we instead need additional hypotheses for that implication (as "completeness" or "compactness", see [16]). (Note that, as far as we know, the class $\mathcal{C}_r$ was not contemplated at the time of [5]).
- In the aforementioned papers it was (sometimes silently) assumed that the metric b be intrinsic; [19] that is, no distinction was made between b and b^g; this is restrictive, since in many simple examples and situations we have considered the two do differ.

  A situation where we are led to consider non-path-metric spaces is as follows: suppose (M', b') is an asymmetric metric space, and (M, b) is a subset of it, with b being the restriction of b'; then it easy to devise examples where (M', b') is path-metric but (M, b) is not and vice versa. For example, in [11] a family of (symmetric) metric space are studied that are not (and cannot possibly) be path-metric; the reason being this theorem (adapted from [11]).

  Theorem A.2 *Suppose that* (M, d) *is a complete symmetric path-metric space, and that* $i : M \to E$ *is an isometric immersion in a uniformly convex Banach space* E*: then* $i(M)$ *is convex, and any two points in* M *can be joined by a unique minimal geodesic (unique up to reparameterization).*

  So in most of this paper we will carefully distinguish the two metrics b and b^g.

- In most of the cited works (with the exception of Phadke's [18]) the space is assumed to be locally compact (or even *finitely compact*).

  According to the notes at end of the introduction of [7], some work was carried on at that time to extend known results to *weakly finitely compact* spaces; but the results in this paper and in [16] do not appear in the announced papers.

[17] This is called *forward local compactness* in [16].

[18] It is also complete, as defined in [16].

[19] Note that the definition of *"intrinsic metric"* was not used at the time of [5]: it was in a sense replaced by the *"Menger convexity"* hypothesis; see [16] for further comments.

B Proofs

B.1 Lemmas for 2.22 and 2.25

This Lemma illustrates the general idea that would otherwise be hidden in the following Lemma.

Lemma B.1 *Let (X,τ) be a topological space and $f : X \to \mathbb{R}$ a function; let $\overline{x} \in X$ be a point that is not isolated; let eventually U_n be a family of open neighborhoods of $\overline{x}$ with $U_n \supseteq U_{n+1}$. Then there exists a sequence $(x_n) \subset X$ with $x_n \in U_n$ and $x_n \neq \overline{x}$ and such that*

$$\lim_{n\to\infty} f(x_n) = \liminf_{x\to\overline{x}} f(x) \quad .$$

Proof. Let $l = \liminf_{x\to\overline{x}} f(x)$; let

$$F(V) = \inf_{x\in V\setminus\{\overline{x}\}} f(x) \tag{B.1}$$

then $l = \sup_V F(V)$ where V are chosen in a fundamental family $\mathcal{V}$ of open neighborhoods of x; so there exists a decreasing sequence $(V_n) \subseteq \mathcal{V}$ such that $l = \sup_n F(V_n) = \lim_n F(V_n)$ but then a fortiori $l = \lim_n F(V_n \cap U_n)$; so by (B.1) there exists $x_n \in V_n \cap U_n$ with $x_n \neq x$ such that $f(x_n)$ is near $F(V_n \cap U_n)$ enough to obtain the thesis. □

This Lemma is useful when (X,τ) does not satisfy the first axiom of countability — indeed we do not claim neither expect that $x_n \to_n \overline{x}$. So we will exploit it to study the DF topology.

Definition B.2 *Given $T \subset [a,b]$ a finite subset $T = \{t_1,\ldots,t_n\}$ with $a \le t_1 < t_2 < \ldots t_n \le b$, the* ***"finess $|T|$ of T in $[a,b]$"*** *is*

$$|T| \stackrel{def}{=} \max_{i=0,\ldots n} (t_{i+1} - t_i)$$

(where $t_0 = a, t_{n+1} = b$ for convenience).

Lemma B.3 *Let $(\mathcal{C},\mathrm{len})$ be a length structure; let $\gamma \in \mathcal{C}_{01}$ a point that is non isolated in $\mathcal{C}_{01}$ w.r.t. the DF topology; let $l = \liminf_{\xi\to\gamma} \mathrm{len}(\xi)$ in the DF topology; fix an increasing sequence S_k of finite subsets of $[0,1]$.*

1. *There exists a sequence T_k of finite subsets of $[0,1]$ with $T_k \supseteq S_k$, and a sequence $(\xi_n) \subset \mathcal{C}_{01}$ with $\xi_n \in \Xi_{\gamma,T_n}$, $\xi_n \neq \gamma$ and $\lim_n \mathrm{len}(\xi_n) = l$.*
2. *If $l < \mathrm{len}(\gamma)$ and $0,1 \in S_k$, then we can assume moreover that $\exists K, \forall k > K, \forall t,s \in T_k$ consecutive points*[20]
$$\mathrm{len}(\xi_k|_{[t,s]}) \le \mathrm{len}(\gamma|_{[t,s]}) \ . \tag{B.2}$$
3. *Suppose that len is run–continuous, that $\mathcal{C}$ is reversible. Suppose that $0,1 \in S_k$ and $|S_k| \to_k 0$. Suppose that (B.2) holds. Let b^l be the induced asymmetric semi distance function from $(\mathcal{C},\mathrm{len})$. Let $d(x,y) = \max\{b^l(x,y), b^l(y,x)\}$* [21]*. Then $\xi_k \to_k \gamma$ uniformly w.r.t. d.*

The final claim is adapted from Prop. 1.6 in [12], we provide a detailed proof, for convenience of the reader, since some adaptations are needed in the asymmetric cases.

Proof.
- The first claim just follows from the general idea expressed in Lemma B.1; we reread the proof of B.1 while letting $U_n = \Xi_{\gamma,S_n}$, $F(V) = \inf_{x\in V\setminus\{\gamma\}} \mathrm{len}(x)$ using $\mathcal{V} = \{\Xi_{\gamma,T}\}$ as the fundamental family, setting $V_n = \Xi_{\gamma,\tilde{T}_n}$, eventually noting that $V_n \cap U_n = \Xi_{\gamma,T_n}$ by setting $T_n = S_n \cap \tilde{T}_n$ so that $l = \lim_n F(\Xi_{\gamma,T_n}) = \lim_n \mathrm{len}(\xi_n)$.

[20] That is, $t < s$ and $T_k \cap (t,s) = \emptyset$.
[21] We recall that d is a symmetric semi distance function, see in Section 3.

- If $l < \mathrm{len}(\gamma)$ then $\exists K, \forall k > K$ we have $\mathrm{len}(\xi_k) < \mathrm{len}(\gamma)$.

  Now fix for a moment $k > K$, let $0 = t_1 < \ldots < t_n = 1$ such that $T_k = \{t_1, \ldots, t_n\}$. For all $i = 1, \ldots n$, $\xi_k(t_i) = \gamma(t_i)$. We build $\tilde{\xi}_k$ as follows, for any $i = 1, \ldots n-1$ *s.t.*

  $$\mathrm{len}(\xi_k|_{[t_i,t_{t+1}]}) \le \mathrm{len}(\gamma|_{[t_i,t_{t+1}]})$$

  we set $\tilde{\xi}_k = \xi_k$ on $[t_i, t_{t+1}]$; when instead

  $$\mathrm{len}(\xi_k|_{[t_i,t_{t+1}]}) > \mathrm{len}(\gamma|_{[t_i,t_{t+1}]})$$

  we replace a piece of ξ_k by a piece of γ, that is we set $\tilde{\xi}_k = \gamma$ on $[t_i, t_{t+1}]$. At this point (B.2) is satisfied for $\tilde{\xi}_k$; note that $\tilde{\xi}_k \ne \gamma$ since $\mathrm{len}(\tilde{\xi}_k) \le \mathrm{len}(\xi_k) < \mathrm{len}(\gamma)$. We also obtain by construction that $\tilde{\xi}_n \in \Xi_{\gamma,T_n}$ so $F(\Xi_{\gamma,T_n}) \le \mathrm{len}(\tilde{\xi}_n) \le \mathrm{len}(\xi_n)$ hence $l = \lim_n \mathrm{len}(\tilde{\xi}_n)$ as well. So we can substitute ξ_n with $\tilde{\xi}_n$ to obtain the thesis.

- We write b for b^l for simplicity. By the first and second claim there exist $T_k \supseteq S_k$ and $\xi_k \in \Xi_{\gamma,T_k}$ *s.t.* $l = \lim_k \mathrm{len}(\xi_k)$ and (B.2) holds.

  By Lemma 2.21 any curve is continuous. Since γ is continuous then $\forall \varepsilon > 0 \exists \delta > 0$ such that $\forall s, t \in [0,1]$ with $0 \le s < t \le 1$, $t - s < \delta$ then $d\big(\gamma(s), \gamma(t)\big) < \varepsilon$, that is

  $$b\big(\gamma(s), \gamma(t)\big) < \varepsilon \quad , \quad b\big(\gamma(t), \gamma(s)\big) < \varepsilon \quad .$$

  Since len is run-continuous, possibly reducing $\delta > 0$ we obtain also that

  $$\ell(t) - \ell(s) = \mathrm{len}(\gamma|_{[s,t]}) < \varepsilon \quad .$$

  Consider any k with $|S_k| \le \delta$, then $|T_k| \le \delta$; for any point $s \in [0,1]$, there $\exists t, t' \in T_k$, with $|t' - t| \le \delta$ and $t \le s \le t'$:

  $$\begin{aligned} b(\xi_k(s), \gamma(s)) &\le b(\xi_k(s), \xi_k(t')) + b(\gamma(t'), \gamma(s)) \le \mathrm{len}(\xi_k|_{[s,t']}) + \varepsilon \le \\ &\le \mathrm{len}(\xi_k|_{[t,t']}) + \varepsilon \le \mathrm{len}(\gamma|_{[t,t']}) + \varepsilon \le 2\varepsilon \end{aligned}$$

  and similarly

  $$\begin{aligned} b(\gamma(s), \xi_k(s)) &\le b(\gamma(s), \gamma(t)) + b(\xi_k(t), \xi_k(s)) \le \varepsilon + \mathrm{len}(\xi_k|_{[t,s]}) \le \\ &\le \varepsilon + \mathrm{len}(\xi_k|_{[t,t']}) \le \varepsilon + \mathrm{len}(\gamma|_{[t,t']}) \le 2\varepsilon \end{aligned}$$

  Hence $\xi_k \to \gamma$ uniformly (w.r.t. the symmetric distance d)

□

B.2 Proof of 3.9

Proof. (The proof is depicted in figure 5 on the following page). We will prove that $\lim_{h\to 0+} \ell(t_0+h) = \ell(t_0)$ (the other limit being quite similar in proof). Since ℓ is monotonic, this is equivalent to $\inf_{h>0} \ell(t_0 + h) = \ell(t_0)$. Let then $\rho \stackrel{\text{def}}{=} \inf_{h>0} \ell(t_0 + h)$, obviously $\rho \ge \ell(t_0)$ so we will prove in the following that $\rho \le \ell(t_0)$. Let $\varepsilon > 0$ be fixed. There exists a $\delta > 0$ small enough so that $\ell(t_0 + \delta) \le \rho + \varepsilon$ and for all $0 \le h \le \delta$,

$$b(\gamma(t_0), \gamma(t_0 + h)) \le \varepsilon \tag{B.3}$$

(using the fact that γ is continuous, and 3.2, and by definition of ρ). Since the length of γ restricted to $[t_0, t_0 + \delta]$ is $\ell(t_0 + \delta) - \ell(t_0)$, then by the definition (1.1) we find $t_1 \ldots t_{n-1}$ with $t_0 < t_1 \cdots < t_n = t_0 + \delta$ such that

$$\sum_{i=1}^{n} b\big(\gamma(t_{i-1}), \gamma(t_i)\big) \ge \ell(t_0 + \delta) - \ell(t_0) - \varepsilon$$

so

$$b(\gamma(t_0), \gamma(t_1)) + \sum_{i=2}^{n} b\big(\gamma(t_{i-1}), \gamma(t_i)\big) \geq \ell(t_0 + \delta) - \ell(t_0) - \varepsilon$$

and then

$$\sum_{i=2}^{n} b\big(\gamma(t_{i-1}), \gamma(t_i)\big) \geq \ell(t_0 + \delta) - \ell(t_0) - 2\varepsilon$$

by (B.3). At the same time $t_1 > t_0$ so

$$\sum_{i=2}^{n} b\big(\gamma(t_{i-1}), \gamma(t_i)\big) \leq \ell(t_0 + \delta) - \ell(t_1) \leq \ell(t_0 + \delta) - \rho \quad ;$$

subtracting and comparing

$$\ell(t_0) + 2\varepsilon \geq \ell(t_0 + \delta) - \sum_{i=2}^{n} b\big(\gamma(t_{i-1}), \gamma(t_i)\big) \geq \rho$$

and by arbitrariness of ε this implies that $\rho = \ell(t_0)$. □

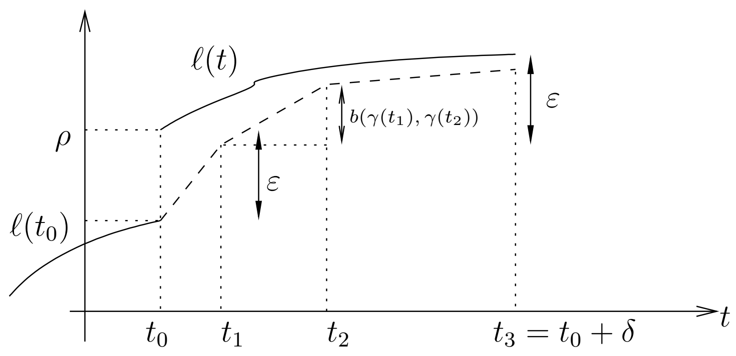


Figure 5: Proof of 3.9

B.3 Example 4.4

We now return to Example 4.4. We now prove that b_3 is an asymmetric metric on $M = [-1, 1]$.

Proof. For $x, y \in [0, 1]$ we define

$$b_3(x, y) = \begin{cases} y - x & \text{if } 0 \leq x \leq y \\ \sqrt{x^2 + 2(x - y)} - x & \text{if } 0 \leq y < x \end{cases} \tag{B.4}$$

whereas for $x, y \in [-1, 0]$ we define $b_3(x, y) = |x - y|$. If we join the two definitions by setting

$$b_3(x, y) = b_3(x, 0) + b_3(0, y)$$

in all cases when $xy < 0$, we arrive at the definition in (4.6).

We want to prove that the triangle inequality

$$b(x, y) + b(y, z) \geq b(x, z)$$

holds for all possible choices of $x, y, z \in [-1, 1]$. Since the joining process does preserve this inequality, we just need to prove its validity for $x, y, z \in [0, 1]$.

We divide two cases, dependent on the relative position of $\{x, z\}$; each case may have at most three subcases (depending on the position of y). (Note that we always omit the cases $x = z$ or $x = y$ or $y = z$, where the result is trivial).

Case $0 < x < z$**.** Three subcases.

Case $0 < x < z < y$**.**

$$(y - x) + (\sqrt{y^2 + 2(y - z)} - y) \geq (z - x)$$

that is

$$\sqrt{y^2 + 2(y - z)} \geq z \tag{B.5}$$

that is

$$(y + z + 2)(y - z) \geq 0 \ .$$

Case $0 < x < y < z$**.** Euclidean

Case $0 < y < x < z$**.**

$$(\sqrt{x^2 + 2(x - y)} - x) + (z - y) \geq (z - x)$$

that is

$$\sqrt{x^2 + 2(x - y)} \geq y$$

and then

$$(x + y + 2)(x - y) \geq 0 \ .$$

Case $0 \leq z < x$**.** Three subcases.

Case $0 < z < x < y$**.**

$$(y - x) + (\sqrt{y^2 + 2(y - z)} - y) \geq \sqrt{x^2 + 2(x - z)} - x$$

that is

$$(\sqrt{y^2 + 2(y - z)} \geq \sqrt{x^2 + 2(x - z)}$$

squaring

$$(y + x + 2)(y - x) \geq 0 \ .$$

Case $0 < z < y < x$**.** We prove that

$$(\sqrt{x^2 + 2(x - y)} - x) + (\sqrt{y^2 + 2(y - z)} - y) \geq \sqrt{x^2 + 2(x - z)} - x$$

that is

$$\sqrt{x^2 + 2(x - y)} + \sqrt{y^2 + 2(y - z)} \geq \sqrt{x^2 + 2(x - z)} + y$$

both sides are positive, so we square and simplify

$$\sqrt{(x^2 + 2(x - y))(y^2 + 2(y - z))} \geq y\sqrt{x^2 + 2(x - z)}$$

then again squaring and simplifying

$$(y - z)\big((2x + x^2) - (2y + y^2)\big) \geq 0 \quad .$$

Case $0 < y < z < x$**.**

$$(\sqrt{x^2 + 2(x - y)} - x) + (z - y) \geq \sqrt{x^2 + 2(x - z)} - x$$

that is

$$\sqrt{x^2 + 2(x - y)} + (z - y) \geq \sqrt{x^2 + 2(x - z)}$$

that is trivial, since $(x - y) \geq (x - z)$.

□

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References

[1] D. Bao, S. S. Chern, and Z. Shen. *An Introduction to Riemann-Finsler Geometry.* (Springer–Verlag), 2000. ISBN 978-0-387-98948-8.

[2] Leonard M. Blumenthal. *Theory and applications of distance geometry.* Second edition. Chelsea Publishing Co., New York, 1970.

[3] V. I. Bogachev. *Measure theory. Vol. I, II.* Springer-Verlag, Berlin, 2007. ISBN 978-3-540-34513-8; 3-540-34513-2. *DOI:* `10.1007/978-3-540-34514-5`. URL `http://dx.doi.org/10.1007/978-3-540-34514-5`.

[4] Dmitri Burago, Yuri Burago, and Sergei Ivanov. *A course in metric geometry*, volume 33 of *Graduate Studies in Mathematics.* American Mathematical Society, Providence, RI, 2001. ISBN 0-8218-2129-6.

[5] H. Busemann. Local metric geometry. *Trans. Amer. Math. Soc.*, 56:200–274, 1944.

[6] H. Busemann. *The geometry of geodesics*, volume 6 of *Pure and applied mathematics.* Academic Press (New York), 1955.

[7] H. Busemann. *Recent synthetic differential geometry*, volume 54 of *Ergebnisse der Mathematik und ihrer Grenzgebiete.* Springer Verlag, 1970.

[8] G. Buttazzo. *Semicontinuity, relaxation and integral representation in the calculus of variation.* Number 207 in scientific & technical. Longman, 1989.

[9] S. Cohn-Vossen. Existenz kürzester Wege. *Compositio math., Groningen,*, 3:441–452, 1936.

[10] Bernard Dacorogna. *Direct methods in the calculus of variations*, volume 78 of *Applied Mathematical Sciences.* Springer, New York, second edition, 2008. ISBN 978-0-387-35779-9.

[11] Alessandro Duci and Andrea Mennucci. Banach-like metrics and metrics of compact sets. eprint arXiv:0707.1174. 2007.

[12] M. Gromov. *Metric Structures for Riemannian and Non-Riemannian Spaces*, volume 152 of *Progress in Mathematics.* Birkhäuser Boston, 2007. ISBN 0-8176-3898-9. Reprint of the 2001 edition.

[13] Gonçalo Gutierres and Dirk Hofmann. Approaching metric domains. *Applied Categorical Structures*, pages 1–34, 2012. ISSN 0927-2852. *DOI:* `10.1007/s10485-011-9274-z`. URL `http://dx.doi.org/10.1007/s10485-011-9274-z`.

[14] H. Hopf and W. Rinow. Ueber den Begriff der vollständigen differentialgeometrischen Fläche. *Comment. Math. Helv.*, 3(1):209–225, 1931. ISSN 0010-2571. *DOI:* `10.1007/BF01601813`.

[15] A. C. G. Mennucci. Regularity and variationality of solutions to Hamilton-Jacobi equations. part ii: variationality, existence, uniqueness. *Applied Mathematics and Optimization*, 63(2), 2011. *DOI:* `10.1007/s00245-010-9116-7`.

[16] A. C. G. Mennucci. Geodesics in asymmetric metric spaces. URL `http://cvgmt.sns.it/person/109/`. in preparation, 2013.

[17] Athanase Papadopoulos. *Metric spaces, convexity and nonpositive curvature*, volume 6 of *IRMA Lectures in Mathematics and Theoretical Physics.* European Mathematical Society (EMS), Zürich, 2005. ISBN 3-03719-010-8.

[18] B. B. Phadke. Nonsymmetric weakly complete g-spaces. *Fundamenta Mathematicae*, 1974.

[19] E. M. Zaustinsky. *Spaces with non-symmetric distances.* Number 34 in Mem. Amer. Math. Soc. AMS, 1959.