On asymmetric distances

Andrea C. G. Mennucci ⋆

21st September 2004

Abstract

In this paper we discuss asymmetric metric spaces (that is, quasi-metric spaces) in an abstract setting, mimicking the usual theory of metric spaces, but adding ideas derived from Finsler geometry. As a typical application, we consider asymmetric metric spaces generated by functionals in Calculus of Variation.

KEYWORDS: ASYMMETRIC METRIC, QUASI METRIC, FINSLER METRIC, OSTENSIBLE METRIC, PATH METRIC, LENGTH, GEODESIC CURVE, HOPF–RINOW THEOREM

1 Introduction

"Besides, one insists that the distance function be symmetric, that is, $d(x, x') = d(x', x)$. (This unpleasantly limits many applications $[\cdots]$)." M. Gromov ([7], Intr.)

The main purpose of this paper is to extract a "metric theory" from Finsler Geometry, as Gromov extracted a metric theory from Riemannian Geometry, (see section I in [7]). In order to do this, we need to define "asymmetric metric spaces", and use "asymmetric definitions" such as "forward completeness", "forward boundedness", etc (see in §2). In §3 we study a typical application from Calculus of Variations.

§1.i Previous work

In section I in [7], Gromov models the "metric part" of Riemannian Geometry, as follows.

Consider a metric space (M, d): we can define the length $\operatorname{len}\gamma$ of a Lipschitz curve $\gamma : [\alpha, \beta] \to M$ using the total variation formula (see (2.4)); then we can define a new metric $d^g(x, y)$ as the inf of $\operatorname{len}\gamma$ in the class of all Lip curves connecting x to y.

Gromov defines that a metric space is "path-metric" if $d = d^g$; he then proves in §1.11 §1.12 in [7] that [(1)]

Theorem 1.1 (symmetric Hopf-Rinow) *Suppose that (M, d) is path-metric and locally compact; then the two following facts are equivalent*

i). (M, d) is complete

ii). closed bounded sets are compact

and they imply that each pair of points can be joined by a geodesic (i.e. a minimum of $\operatorname{len}\gamma$).

⋆ Scuola Normale Superiore Piazza dei Cavalieri 7, 56126 Pisa, Italy

[(1)] actually, we have added the easy implication *ii*$\Longrightarrow$*i* to the statement in [7]

The above is the "metric only" counterpart of the theorem of Hopf-Rinow in Riemannian Geometry: indeed, if (M, g) is a finite-dimensional Riemannian manifold, and d is the associated distance, then (M, d) is path-metric and locally compact.

Since there is a Hopf-Rinow theorem in Finsler Geometry, we would expect that there would be a corresponding theorem for "asymmetric metric spaces".

An "asymmetric metric spaces" (M, b) is a set M equipped with a metric $b : M \times M \to \mathbb{R}^+$ which satisfy the triangle inequality, but may fail to be symmetric (see §2 for the formal definition).

These metrics have already been studied in the past, with the name of "quasi metrics". A quasi-metric will generate, in general, *three different topologies* (see §2.v.1). In our definition of the "asymmetric metric space" (M, b), we chose a different topology associated to the space than the one used in "quasi metric spaces": this makes it difficult to compare the results in the two fields. We discuss further this issue in sec. §2.v.

In asymmetric metric spaces we can state the "metric only" Hopf-Rinow theorem:

Theorem 1.2 *Suppose that the* asymmetric metric space (M, b) *is path-metric and forward-locally compact; then the following two facts are equivalent*

- *forward-bounded closed sets are compact*
- (M, b) *is forward-complete*

and they imply that any two points can be joined by a geodesic.

This is the main result, proved as Theorem 2.37 in this paper. While trying to understand this result, we explored the "asymmetric metric spaces": they appear to be more complex[(2)] than it appears at first sight.

§1.ii Origin of the problem

Let M be a smooth connected differential manifold.

Consider an integrand $F : TM \to \mathbb{R}$ that is continuous and such that $v \mapsto F(x, v)$ is positively 1-homogeneous, and convex. Consider the length

$$\operatorname{len} \gamma = \int_0^1 F(\xi(s), \dot{\xi}(s)) \, ds \tag{1.3}$$

where $\gamma : [0, 1] \to M$ is a locally Lipschitz curve; this generates a "distance" $b(x, y)$, which is called a *Finsler metric*; but this distance will be, in general, asymmetric.

Consider also the Hamilton-Jacobi equation

$$\Big\{ H(x, Du(x)) = 0, \quad u(a) = 0 \tag{1.4}$$

where $H : T^*M \to \mathbb{R}$ is continuous and $p \mapsto H(x, p)$ is convex, and the solution $u : M \to \mathbb{R}$ is in the viscosity sense; and $a \in M$.

In the above settings, a broad number of ideas appear naturally:

- we may consider (M, b) as an *asymmetric metric space*; then we would expect to be able to imitate many results from the theory of *symmetric metric spaces*; for example, since (M, b) is path-metric and locally compact, then we may use the Hopf–Rinow theorem 1.2 (see below 1.5);

(2) and wild! see sec. §2.vi

- we may want to use theorems from the Calculus of Variations, Hamilton-Jacobi theory and Optimal control: indeed, we expect that the Hamilton-Jacobi equation (1.4) is naturally associated with the distance $b(x, y)$, so that $u(x) = b(a, x)$ is the viscosity solution of (1.4).

  Unfortunately, it is often assumed that $L \in C^2(TM)$ (where the $L(x, v) = F(x, v)^2$) and $H \in C^2(T^*M)$ (3); few results (see [12]) are known in absence of regularity assumptions on H.

  We believe that the theory of *asymmetric metric spaces* will provide a theorem of existence and uniqueness for Hamilton–Jacobi equations. This issue is explored in [11].

- The field of Finsler Geometry readily admits that b may be asymmetric: but in most common books, $L \in C^\infty$, and $v \mapsto L(x, v)$ is strongly convex for $v \neq 0$ (cf. 1.6); whereas we expect that many theorems would hold in less regular spaces.

The results regarding problem (1.3) are in §3; in particular, theorem 3.9 shows that the Hopf–Rinow theorem implies a purely geometric criteria for the existence of a minimum curve for problem (1.3), as follows:

Theorem 1.5 (4) *The two following are equivalent.*

- *(M, b) is forward-complete (resp. backward-complete) if and only if*
- *forward (resp. backward) bounded closed sets are compact;*

in both cases, for any $x, y \in M$, there is a locally Lipschitz curve ξ connecting them that minimizes (1.3).

§1.iii Notation

We conclude the introduction with a remark on definitions

Definition 1.6 *Let $f : \Omega \to \mathbb{R}$ where $\Omega \subset \mathbb{R}^n$ is convex. We recall that*

i). "f is strongly convex" when $f \in C^2$ and the Hessian $D^2 f(x) = \frac{\partial^2 f}{\partial x \partial x}(x)$ is positive definite (5) $\forall x$; whereas

ii). "f is strictly convex" when

$$f(\lambda x + (1 - \lambda)y) < \lambda f(x) + (1 - \lambda)f(y)$$

for all $x, y \in \Omega$, $x \neq y$, $0 < \lambda < 1$

Note that many authors use different definitions (defining (i) as "strictly convex").

2 Asymmetric metric spaces

We review some basilar concepts and results, commonly found in books on metric spaces (and we refer to ch. 4 in [1] and ch. 1 [7]), which we extend and discuss in the wider context of asymmetric metric spaces.

In all of this chapter, M will be a generic set.

(3) for example, in §2.4.2 [4]; or $L \in C^3$ and strongly convex in §1.8 in [5]

(4) the detailed statement is in Theorem 3.9.

(5) that is, $\alpha \cdot D^2 f(x)\alpha > 0$ for all $\alpha \in \mathbb{R}^n$, $\alpha \neq 0$

§2.i Asymmetric metric space

Definition 2.1 $b : M \times M \to \mathbb{R}^+$ *is an* asymmetric distance[6] *if* b *satisfies* [7]

- $b \geq 0$ *and* $b(x, y) = 0$ *iff* $x = y$
- $b(x, y) \leq b(x, z) + b(z, y)\ \forall x, y, z \in M$.

We call the pair (M, b) *an* asymmetric metric space.

We agree that b defines a topology τ on M, [8] generated by the families of *forward* and *backward* open balls

$$B^+(x, \varepsilon) \doteq \{y \mid b(x, y) < \varepsilon\}, \quad B^-(x, \varepsilon) \doteq \{y \mid b(y, x) < \varepsilon\}$$

that is, the topology is generated by the symmetric distance

$$d(x, y) \doteq b(x, y) \vee b(y, x) \tag{2.2}$$

with balls

$$B(x, \varepsilon) \doteq \{y \mid d(x, y) < \varepsilon\} = B^+(x, \varepsilon) \cap B^-(x, \varepsilon)$$

By the above definition,

Proposition 2.3 $x_n \to x$ *iff* $d(x_n, x) \to 0$*, iff both* $b(x_n, x) \to 0$ *and* $b(x, x_n) \to 0$

We note that $(x, y) \mapsto b(x, y)$ is Lipschitz w.r.t d: see section §2.ii.

§2.ii Lipschitz maps

Let (N, δ) be a symmetric metric space, and $f : N \to M$, $g : M \to N$. We could define that f and g are "Lipschitz w.r.t. b", or "Lipschitz w.r.t. d". That is, in the first case

$$b(f(x), f(y)) \leq C\delta(x, y), \quad \delta(g(x), g(y)) \leq Cb(x, y)$$

for some constant $C > 0$. In the second case

$$d(f(x), f(y)) \leq C\delta(x, y), \quad \delta(g(x), g(y)) \leq Cd(x, y)$$

for some constant $C > 0$.

The two concepts are equivalent for f, but not equivalent for g.

§2.iii Length

We induce from b the length $\operatorname{len}^b \gamma$ of a continuous curve $\gamma : [\alpha, \beta] \to M$, by using the total variation

$$\operatorname{len}^b \gamma \doteq \sup_T \sum_{i=1}^{n} b\big(\gamma(t_{i-1}), \gamma(t_i)\big) \tag{2.4}$$

where the sup is carried out over all finite subsets $T = \{t_0, \cdots, t_n\}$ of $[\alpha, \beta]$ and $t_0 \leq \cdots \leq t_n$. If $\operatorname{len}^b \gamma < \infty$, then γ is called *rectifiable*.

(6) also known as *quasi metric* or *ostensible metric*

(7) If the first constraint is replaced by "$b \geq 0$ and $b(x, x) = 0$", then b is an *asymmetric semidistance*

(8) The forward (resp. backward) balls may generate a different topology: this issue is discussed in sec. §2.v

Remark 2.5 *The above length is* asymmetric*: if $\hat\gamma(t) \doteq \gamma(-t)$, then in general $\operatorname{len}^b \gamma$ will be different from $\operatorname{len}^b \hat\gamma$; hence there does not exist a measure $\mathcal{H}$ on M such that the length of a curve is the measure of its image, that is*

$$\mathcal{H}(\textit{image}(\gamma)) = \operatorname{len}^b \gamma \quad ;$$

indeed image(γ) = image$(\hat\gamma)$ which is incompatible in the general case in which $\operatorname{len}^b \gamma \neq \operatorname{len}^b \hat\gamma$.

We define b^g

$$b^g(x,y) = \inf \operatorname{len}^b \gamma \tag{2.6}$$

where the inf is taken in the class of all Lipschitz curves γ connecting x to y. If the infimum is a minimum, then the minimum curve is the *geodesic* connecting x to y.

Note that

$$\operatorname{len}^b \gamma \geq b(\gamma(\alpha), \gamma(\beta)) \tag{2.7}$$

and then $b^g \geq b$.

If the space (M, b) is Lipschitz-arcwise connected, then it is also easily proved that b^g is an asymmetric distance. b^g is indeed called *the geodesic (asymmetric) distance induced by* b. We may endow M with the distance $d^g(x,y) \doteq b^g(x,y) \vee b^g(y,x)$: then the resulting topology is finer than the one induced by b (has more open sets and less compact sets).

Suppose that γ is rectifiable. We define the *running length* [(9)] $\ell : [\alpha, \beta] \to \mathbb{R}^+$ of γ to be the length of γ restricted to $[\alpha, t]$, that is

$$\ell(t) \doteq \operatorname{len}^b \left(\gamma|_{[\alpha,t]}\right) \tag{2.8}$$

Note that:

- For $t \geq s$,

  $$b\big(\gamma(s), \gamma(t)\big) \leq \ell(t) - \ell(s) \tag{2.9}$$

  (by (2.7)) so, if ℓ is continuous, then γ is continuous; [(10)] if ℓ is Lipschitz, then γ is Lipschitz;

- whereas if γ is Lipschitz of constant L, then, by direct substitution in (2.4),

  $$\ell(t) - \ell(s) \leq (t-s)L$$

  so ℓ is Lipschitz.

  In this case the derivative $\frac{d\ell}{dt}$ exists for almost all t, and it is called the *metric derivative* of γ.

§2.iv Definitions

We add some definitions

- We say that the (asymmetric) metric space (M, b) is *a path-metric space* if $b = b^g$; [(11)]

[(9)]we may as well call it *partial total variation*

[(10)]when b is symmetric, if γ is rectifiable and continuous, then ℓ is continuous; this seems to be true also for asymmetric distances, but was not carefully checked

[(11)]if (M, b) is path-metric, we agree that M is Lipschitz-arc-connected, by definition and by 2.33

- and we say that it *admits geodesics*, or that *geodesics exist*, if, for all $x, y \in M$, the $\inf$ that defined $b^g(x, y)$ in (2.6) is attained by a minimizing curve, that is called a "geodesic".

- We say that (M, b) *forward-locally admits geodesics*, or that *geodesics exist forward-locally* if $\forall x \in M\ \exists \varepsilon > 0$ such that $\forall y \in B^+(x, \varepsilon)$, there is a geodesic connecting x to y (that is, (2.6) has a minimum)

- A sequence $(x_n) \subset M$ is called *forward Cauchy* (12) if

$$\forall \varepsilon > 0,\ \ \exists N \in \mathbb{N} \text{ such that } \forall n, m,\ \ m \geq n \geq N,\ \ b(x_n, x_m) < \varepsilon \qquad (2.10)$$

  We say that (M, b) is *forward complete* if any *forward Cauchy* sequence (x_n) converges to a point x. (13)

- We say that $A \subset M$ is *forward bounded* if these two equivalent propositions hold
  - $\exists x \in M, r > 0$ s.t. $A \subset B^+(x, r)$
  - $\forall x \in M, \exists r > 0$ s.t. $A \subset B^+(x, r)$

For any *forward* definition above there is a corresponding *backward* definition, obtained by exchanging the first and the second argument of b, or by using the *conjugate distance* $\bar{b}$ defined by

$$\bar{b}(x, y) = b(y, x) \quad ; \qquad (2.11)$$

and a corresponding *symmetric* definition, obtained by substituting d for b.

For the same reason, in this paper we will present mostly the *forward* versions of the theorems, since *backward* results are obtained as above.

We nonetheless emphasize these definitions.

- We say that (M, b) is *forward-locally compact* if $\forall x\ \exists \varepsilon > 0$ such that $\{y \mid b(x, y) \leq \varepsilon\}$ is compact.

- We say that (M, b) is *backward-locally compact* if $\forall x\ \exists \varepsilon > 0$ such that $\{y \mid b(y, x) \leq \varepsilon\}$ is compact.

- We say that (M, b) is *symmetrically-locally compact* if $\forall x\ \exists \varepsilon > 0$ such that $\{y \mid d(x, y) \leq \varepsilon\}$ is compact.

- We say that (M, b) is *locally compact* if $\forall x\ \exists \varepsilon > 0$ such that both $\{y \mid b(x, y) \leq \varepsilon\}$ and $\{y \mid b(y, x) \leq \varepsilon\}$ are compact.

The following implications hold:

$$\begin{array}{ccc} \text{locally compact} & \longrightarrow & \text{forward locally compact} \\ \downarrow & & \downarrow \\ \text{backward locally compact} & \longrightarrow & \text{symmetrically locally compact} \end{array}$$

(12) This definition agrees with the one used in Finsler Geometry (see ch. VI of [2]). See 2.13 for a different definition.

(13) according to the topology τ: cf. 2.3 and §2.v.1 on the notion of convergence

§2.v Comparison with quasi metric spaces

§2.v.1 Topology

As already pointed out, the definition 2.1 of the *asymmetric metric* coincides with the definition of a *quasi metric* that is found in the literature, cf. Kelly [9], Reilly, Subrahmanyam and Vamanamurthy [8] (14) Fletcher and Lindgren [6, (pp 176-181)], Künzi [10].

The main difference between our theory of *asymmetric metric spaces* and *quasi metric spaces* is in the choice of the associated topology. (15)

Indeed, we have three topologies at hand:

- the topology τ , generated by the families of forward and backward balls, or equivalently by the metric d defined in (2.2);
- the topology τ^+ generated by the families of forward balls;
- the topology τ^- generated by the families of backward balls;

it may happen that these three topologies are different.

This problem has been studied in [9]: there Kelly introduces the notion of a *bitopological space* (M, τ^+, τ^-), and extends many definition and theorems, (such as the Urysohn lemma, the Tietze's extension theorem, the Baire category theorem(16)) to these spaces. Unfortunately Kelly does not include the topology τ in his studies.

In many applications $\tau = \tau^+ = \tau^-$: see 2.17 and §3. We have chosen to associate the topology τ to the "asymmetric metric space". τ is a *symmetric kind of object*: as a consequence, we have only one notion of "*the sequence* (x_n) *converges to* x", and only one notion of "*the functions* $f : N \to M$ *and* $g : M \to N$ *are continuous*". Furthermore

Proposition 2.12 *If* $x_n \to x$ *(according to* τ*) then the sequence* (x_n) *is both a* forward Cauchy sequence *and a* backward Cauchy sequence,

in accordance with the symmetric case.

§2.v.2 Cauchy sequences, and completeness

In papers on *quasi metric spaces*, the quasi-metric space (M, b) is instead usually endowed with the topology τ^+: this entails a different notion of *convergence* and *compactness*, and poses the problem to find a good definition of "*Cauchy sequence*" (17) and "*complete space*".

This problem has been studied in [8], where 7 different notions of "*Cauchy sequence*" are presented. (18) Combining these 7 definition with the τ^+ topology, [8] presents 7 different definitions of "*complete space*". (19)

(14)[8] provides also a wide discussion of the references on *quasi metrics*

(15)This explain our choice of a different name.

(16)Another version of Baire theorem, using a better definition of completeness, is found in Theorem 2 in [8].

(17)What do we mean by "good definition"? We may need that the definition would satisfy a proposition similar to 2.12; see statement iv) below

(18)The list in [8] includes the three that we defined in §2.iv: a "forward Cauchy sequence" (resp. backward) is a "left K-Cauchy sequence" (resp. right); a "symmetrical Cauchy sequence" is a "b-Cauchy sequence"

(19)Actually, by combining 7 "*Cauchy sequences*" with all the above 3 topologies, we may reach a total of 14 (!) different definitions of "*complete space*" (considering the symmetry of using the $\bar{b}$ instead of b, see eq. (2.11)). To our knowledge, no one has taken the daunting task of examining all of them.

One of the notions of "*Cauchy sequence*" and "*complete space*" from [8] has been further studied by Künzi [10]; we present it here.

Definition 2.13 • *A sequence* $(x_n) \subset M$ *is a* "left b-Cauchy sequence" *when* $\forall \varepsilon > 0\ \exists x \in M$ *and* $\exists k \in \mathbb{N}$ *such that* $b(x, x_m) < \varepsilon$ *whenever* $m \geq k$*.*

- (M, b) *is a* "left b-sequentially complete space" *if any* left b-Cauchy sequence *converges to a point, according to the topology* τ^+*.*

It is easy to prove that

i). if $x_n \to x$ according to τ^+ then the sequence (x_n) is a "*left b-Cauchy sequence*".

ii). Any "*forward Cauchy sequence*" (as defined in (2.10)) is a "*left b-Cauchy sequence*" [(20)].

iii). If $\tau = \tau^+$, then any "*left b-sequentially complete space*" is a "*forward complete metric space*" as defined in this paper.

In case $\tau \neq \tau^+$, the implication may not hold.

iv). Whereas, if $x_n \to x$ according to τ^+ then the sequence (x_n) may fail to be either a "*forward Cauchy sequence*" or a "*backward Cauchy sequence*". [(21)] Cf. example 2.27.(iv) here.

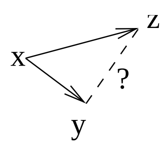
For those reasons, it is not easy to compare the results and examples in the above papers, with the result and examples here.

From here on, we return to our setting of "asymmetric metric spaces": we will always use the topology τ.

§2.vi Comparison with symmetric case

There are subtle but important differences between *asymmetric* and *symmetric distances*[(22)]. We start with a striking remark.

Remark 2.14 *Let* $\varepsilon > 0$ *and* x, y, z *be such that* $b(x, y) < \varepsilon$ *and* $b(x, z) < \varepsilon$*. This does not imply, in general, that* $b(y, z) < 2\varepsilon$



This has consequences such as

i). In example 2.27.(iv) there exists a sequence such that $b(x_n, x) \to 0$ but $x_n \not\to x$

[(20)]just choose $n = N = k$ and $x = x_n$ in the definition of *left b-Cauchy sequence*

[(21)]Indeed, Kelly [9] had encountered this problem, which was a motivation of [8]

[(22)]These are fundamental remarks: since the "*symmetric metric space*" is the metric space commonly studied in mathematics, mathematicians are *hardwired* to use their properties while proving theorems; so it is quite important to keep in mind that some of the basilar properties of "*symmetric metric spaces*" are lost in the asymmetric case ii). Fix a sequence $(x_n) \subset M$. Suppose that $\forall\varepsilon > 0$ there exists a converging sequence (y_n) such that $b(y_n, x_n) < \varepsilon$. If b is symmetric and (M, b) is complete, then (x_n) converges. If b is asymmetric and (M, b) is complete, then there is a counter-example in 2.27.(vi).

iii). In example 2.28.(iv) there is a ball $B^+(a, r)$ and points $a_i \notin B^+(a, r)$ such that $B^+(a, r) \subset \bigcup B^+(a_i, r/2)$: it is then difficult to find a useful definition of a "precompact set". (23)

Further examples are in §2.vii.3 and §2.viii.2.

To circumvent the above problems, we will use often the following propositions.

Proposition 2.15 *If $(x_n) \subset M$ is a sequence such that $b(x, x_n) \to 0$, and M is forward-locally compact, then $x_n \to x$.*

The above may be proved by using this Lemma

Lemma 2.16 (modulus of symmetrization) *Let $C \subset M$ be a compact set: then there exists a continuous monotonically (weakly) increasing function $\omega : \mathbb{R}^+ \to \mathbb{R}^+$, with $\omega(0) = 0$, such that*

$$\forall x, y \in C, \quad b(x, y) \le \omega(b(y, x))$$

Proof. Define

$$f(r) = \sup_{x,y \in C,\ b(x,y) \le r} b(y, x)$$

and then f is monotone. Since C is compact, then $f < \infty$ and $\lim_{r\to0} f(r) = 0$: otherwise we may find $\varepsilon > 0$ and x_n, y_n s.t. $b(x_n, y_n) \to 0$ while $b(y_n, x_n) > \varepsilon$: extracting converging subsequences, we obtain a contradiction.

From f we can define an ω as required, for example $\omega(r) = \frac{1}{r}\int_r^{2r} f(s)ds$ (note that $\omega \ge f$). □

This lemma may also be used as follows

Corollary 2.17 *If (M, b) is locally compact then $\forall x \in M, \varepsilon > 0\ \exists r > 0$ s.t.*

$$B^+(x, r) \subset B^-(x, \varepsilon), \quad B^-(x, r) \subset B^+(x, \varepsilon)$$

and then the topology on M can be equivalently generated by the family $\{B^+(x, r) \mid x, r\}$ of forward balls (or equivalently by $\{B^-(x, r) \mid x, r\}$ alone).

Therefore, in locally compact spaces, asymmetry is not important for *topological* questions (in the sense explained in §2.v.1); but we still care for asymmetry in *metric* questions (such as completeness). Cf. example 2.28 and sec. 3.

(23) An useful definition of "precompact set" is found in [10], where, though, a different completeness hypothesis is used (see 2.13 here).

§2.vii Properties

Proposition 2.18 *For any $A \subset M$, the following properties are equivalent*

- $\sup_{x,y\in A} b(x,y) < \infty$
- A *is* forward bounded *and* backward bounded
- A *is* symmetrically bounded

But note that in example 2.27.(v) there exists a set $A \subset M$ that is *forward bounded* but not *backward bounded* (and vice versa in 2.28.(i)).

Similarly

Proposition 2.19 *Let $(x_n) \subset M$ be a sequence; then the following are equivalent*

- (x_n) *is* forward Cauchy *and* backward Cauchy
- (x_n) *is* symmetrically Cauchy

From that we obtain: if (M,b) is either forward or backward complete, then it is symmetrically complete.

See examples 2.28, 2.29.

Proof. Suppose that (M,b) is forward complete; let (x_n) be *symmetrically Cauchy*: then it is *forward Cauchy*, and then, since (M,b) is forward complete, there is an x such that $x_n \to x$. Similarly if (M,b) is backward Cauchy. □

§2.vii.1 Mid-point properties

In path metric spaces, for any two points x, y, there is always a curve joining them with a quasi optimal distance (that is, very near to $b(x,y)$): this is used in the following proposition, in many different but equivalent fashions, to find *intermediate points* z at a prescribed distance b from x and y.

Proposition 2.20 *Suppose that the (asymmetric) metric space (M,b) is path-metric:*

i). let $\rho > 0$ and

$$S^+(a,\rho) \doteq \{y \mid b(a,y) = \rho\}$$
$$D^+(a,\rho) \doteq \{y \mid b(a,y) \le \rho\}$$

(which are closed, since b is continuous) then

$$\overline{B^+(a,\rho)} = D^+(a,\rho), \qquad \mathring{S}^+(a,\rho) = \emptyset$$

ii). $\forall \theta \in (0,1)$,

$$\forall x,y \in M, \forall \varepsilon > 0 \quad \exists z \in M \text{ such that}$$
$$b(x,z) < \theta b(x,y) + \varepsilon, \qquad b(z,y) < (1-\theta) b(x,y) + \varepsilon \tag{2.20.$\star$}$$

iii). let $x,y \in M$ and $\varepsilon > 0, \varepsilon \le b(x,y)$ then

$$\inf_{z\in S^+(x,\varepsilon)} b(z,y) = b(x,y) - \varepsilon \tag{2.20.$\star\star$}$$
$$\inf_{z\in D^+(x,\varepsilon)} b(z,y) = b(x,y) - \varepsilon \ . \tag{2.20.$\diamond$}$$

Note that, if there exists a minimum point $\overline{z} \in D^+(x,\varepsilon)$ to (2.20.$\diamond$), then $b(x,\overline{z}) = \varepsilon$, i.e. $\overline{z} \in S^+(x,\varepsilon)$.

On the other hand, if M is either forward or backward complete, and either

- *$\exists\theta \in (0,1)$ s.t. (2.20.$\star$) holds, or*
- *$\exists\theta \in (0,1)$ s.t. $\forall x,y$, when $\varepsilon = \theta b(x,y)$, (2.20.$\star\star$) holds, or*
- *$\exists\theta \in (0,1)$ s.t. $\forall x,y$, when $\varepsilon = \theta b(x,y)$, (2.20.$\diamond$) holds,*

then (M,b) is path-metric. (24)

If M admits geodesics, then, for any two points x,y, there is always a curve joining them with optimal distance (but not necessarily equal to $b(x,y)$): this is again used to find *intermediate points* z at a prescribed distance b^g from x and y, as shown in the following proposition.

Proposition 2.21 *If b admits geodesics then $\forall\theta \in (0,1)$*

$$\forall x,y \in M,\ \ \exists z \in M \text{ such that}$$
$$b^g(x,z) = \theta b^g(x,y), \quad b^g(z,y) = (1-\theta)b^g(x,y) \ . \tag{2.21.$\star$}$$

When $\theta = 1/2$, the above z is called a middle point *of x,y.*

Vice versa if M is either forward or backward complete, and the above holds for a $\theta \in (0,1)$, then b admits geodesics.

The proof of the two propositions above is essentially the same as in the symmetric case (cf. 1.8 [7]).

Remark 2.22 *Let $b(x,y) = |x-y|$, $M \subset \mathbb{R}^n$. If M is closed in $\mathbb{R}^n$ and $\forall x,y \in M \exists z \in M$ such that $b(x,z) = b(x,y)/2 = b(z,y)$, then M is convex.*

§2.vii.2 Properties of path metrics

For any asymmetric metric b, we may build many other asymmetric metrics $\tilde{b}$, as follows

Proposition 2.23 *Let $\varphi : \mathbb{R}^+ \to \mathbb{R}^+$ be continuous and concave, $\varphi(x) = 0$ only for $x = 0$; let $\tilde{b} \doteq \varphi \circ b$: then $\tilde{b}$ is an asymmetric distance.*

The above family $\tilde{b} \doteq \varphi \circ b$ is quite large; hence the following question arises: what happens if we induce $\tilde{b}^g$ from $\tilde{b}$: can we build a large family of path metrics on M, by this simple procedure? The answer is no.

Proposition 2.24 *Set everything as in the previous proposition. If moreover the derivative of φ exists and is finite at 0, then*

$$\varphi'(0) \operatorname{len}^b \gamma = \operatorname{len}^{\tilde{b}} \gamma$$

and

$$b^g(x,y)\varphi'(0) = \tilde{b}^g(x,y) \ .$$

(24) see example 2.25

Proof. Let $\varepsilon > 0$. In the definition (2.4) of $\operatorname{len}^b \gamma$ it is not restrictive to use only subsets T of $[\alpha, \beta]$ such that $b(\gamma(t_i), \gamma(t_{i+i})) \le \varepsilon \; \forall i \in \{1, \dots, n-1\}$.

Let now $\delta > 0$; then there exists a $\varepsilon > 0$ such that

$$x(a-\delta) \le \phi(x) \le x(a+\delta) \quad \forall x \in [0, \varepsilon]$$

with $a = \varphi'(0)$; we obtain that

$$(a-\delta) \operatorname{len}^b \gamma \le \operatorname{len}^{\tilde{b}} \gamma \le (a+\delta) \operatorname{len}^b \gamma$$

hence the conclusion □

If $\varphi'(0) = \infty$, wild things may happen: see for example 1.4.b in [7].

It is also possible to prove that we cannot generate other path metrics by iterating the operation (2.6): indeed $b^g = (b^g)^g$; see 2.44.

§2.vii.3 Examples

We consider several examples, for a better understanding of the above properties and definitions.

Example 2.25 *Let $b(x, y) = |x - y|$ and $M \subset \mathbb{R}^2$.*

- *If $M = \mathbb{Q}^2$ then (2.20.$\star$) is satisfied: but M is neither complete nor connected, and (M, b) is not path-metric, since $b^g \equiv \infty$.*
- *If $M = (\mathbb{R} \times \mathbb{Q}) \cup (\{0\} \times \mathbb{R})$ then (2.20.$\star$) is satisfied, and M is Lip-arc-connected, but (M, b) is not complete; and (M, b) is not path-metric, since $b^g(x, y) = |x_1| + |y_1| + |x_2 - y_2|$ when $x_2 \ne y_2$.*

Example 2.26 *Consider $M \subset \mathbb{R}^n$ to be an open set, and b to be the Euclidean distance; then*

- *(M, b) is locally compact and locally path-metric; whereas*
- *(M, b) admits geodesics iff M is convex,*
- *and (M, b) is complete iff $M = \mathbb{R}^n$;*
- *if moreover M is equal to the interior of the closure of M in $\mathbb{R}^n$, then (M, b) is path-metric iff M is convex.*

Example 2.27 *Let M be the disjoint union of segments*

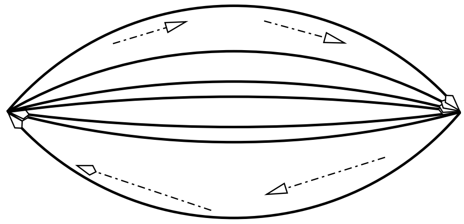
$$M \doteq \bigcup_{n \in \mathbb{Z}} (\{n\} \times [0, 1]) / \sim$$

with $\sim$ identifying all 0s into a class $[0]$, and 1s into $[1]$, i.e. $(n, 0) \sim (m, 0)$ and $(n, 1) \sim (m, 1)$. Let $l_n = \{n\} \times (0, 1)$. Let b be defined on segments as

$$b(x, y) \doteq \begin{cases} |x-y|(1+e^n) & \text{if } x \ge y \text{ and } x, y \in l_n \\ |x-y|(1+e^{-n}) & \text{if } x \le y \text{ and } x, y \in l_n \end{cases}$$

and

$$b([0], [1]) = b([1], [0]) = 1$$



and extend b *(geodesically) using the rule*

$$b(x,y) = \inf\big\{b(x,y),\ \ b(x,[0]) + b([0],y),\ \ b(x,[1]) + b([1],y)\big\}$$

We highlight the following properties

i). (M,b) *is a complete path-metric space.*

ii). (M,b) *is not locally compact.*

iii). There is no geodesic connecting $[0]$ *to* $[1]$

iv). Let $x_n = (-n, 1/n)$ *then*

$$b(x_n,[0]) = (1+e^{-n})/n \to 0$$

but

$$b([0],x_n) = (1+e^{n})/n \to \infty$$

v). Moreover the set $A = \{[0], x_n \mid n\}$ *is backward bounded but not forward bounded.*

vi). Moreover $\forall \varepsilon > 0$ *we can define*

$$y_n = \begin{cases} x_n & \text{if } n\varepsilon < 2 \\ [0] & \text{if } n\varepsilon \geq 2 \end{cases}$$

and then $b(x_n,y_n) < \varepsilon$ *and* $y_n \to [0]$*; but nonetheless* $x_n \not\to [0]$.

Example 2.28 *Let* $M = \mathbb{R}$ *and let*

$$b(x,y) = \begin{cases} e^{y} - e^{x} & \text{if } x < y \\ e^{-y} - e^{-x} & \text{if } x > y \end{cases} \tag{2.28.$\star$}$$

then b *generates on* $\mathbb{R}$ *the usual topology, and* (M,b) *is locally compact (this may be proved using 3.6 since* b *comes from a Finsler structure: see in sec. §10.iii.1 in [11]). The balls are the open intervals*

$$B^{+}(a,r) = \{y \mid b(a,y) < r\} = \big(-\log(r+e^{-a}), \log(r+e^{a})\big)$$

$$B^{-}(a,r) = \{x \mid b(x,a) < r\} = \big(\log(e^{a}-r), -\log(e^{-a}-r)\big)$$

where $\log(z) = -\infty$ *if* $z \leq 0$*. (see fig. 1 and 2)*

Figure 1: **f**orward and **b**ackward **b**alls, **l**eft and **r**ight extrema (M is vertical, $r = 1/2$, a in abscissa)

Figure 2: **f**orward and **b**ackward **b**alls, **l**eft and **r**ight extrema (M is vertical, $a = 1/2$, r in abscissa)

i). We immediately note that $B^-(0,1) = \mathbb{R}$, that is, M is backward bounded (but is not forward bounded).

ii). Let $x_n = n$: then for $m < n$, $b(x_n, x_m) = e^{-m} - e^{-n} < e^{-m}$ so this sequence is backward-Cauchy: then M is not backward complete.

iii). (M, b) is forward complete (and then is complete, by 2.19).

iv). Consider the ball $B^+(0,\rho)$ of extrema $(-R, R)$ with $R(\rho) = \log(\rho + 1)$, and then the two balls

$$B^+(R,r) = \big(-\log(r+e^{-R}), \log(r+e^R)\big), \quad B^+(-R,r) = \big(-\log(r+e^R), \log(r+e^{-R})\big)$$

then if $r \geq \rho/(\rho+1)$,

$$B^+(0,\rho) \subset B^+(R,r) \cup B^+(-R,r)$$

Example 2.29 *Consider two copies (M, b) of the above space, join them at the origin, reverse the metric on one: the resulting space $\tilde{M}, \tilde{b}$ is complete, but is neither forward nor backward complete.*

More precisely: let $M_+ = \mathbb{R} \times \{+\}$, $M_- = \mathbb{R} \times \{-\}$,

$$\tilde{M} = M_+ \cup M_- / \sim$$

where $(0,+) \sim (0,-)$. Let

$$\tilde{b}(x,y) = \begin{cases} b(x,y) & x,y \in M_+ \\ b(y,x) & x,y \in M_- \\ \tilde{b}(x,[0]) + \tilde{b}([0],y) & \textit{otherwise} \end{cases}$$

§2.viii Geodesics

Lemma 2.30 (change of variable) *Let $\gamma : [\alpha,\beta] \to M$, $\xi : [\alpha_1,\beta_1] \to M$ be linked by*

$$\gamma = \xi \circ \varphi$$

where $\varphi : [\alpha,\beta] \to [\alpha_1,\beta_1]$ is monotone increasing, and continuous, and $\varphi(\alpha) = \alpha_1$, $\varphi(\beta) = \beta_1$. Then

$$\operatorname{len}^b \gamma = \operatorname{len}^b \xi$$

Proof. For any partition $T = \{t_i\}$ of $[\alpha,\beta]$ we associate the partition $S = \{s_i = \varphi(t_i)\}$ of $[\alpha_1,\beta_1]$: in this case,

$$\sum_{i=1}^{n} b\big(\gamma(t_{i-1}), \gamma(t_i)\big) = \sum_{i=1}^{n} b\big(\xi(s_{i-1}), \xi(s_i)\big) \tag{2.31}$$

Similarly, for any partition $S = \{s_i\}$ of $[\alpha_1,\beta_1]$ we choose $t_i \in \varphi^{-1}(\{s_i\})$, and associate the partition $T = \{t_i\}$ of $[\alpha,\beta]$: then again (2.31) is verified regardless of the choice of t_i: indeed if $t'_i \in \varphi^{-1}(\{s_i\})$ then $\gamma(t_i) = \gamma(t'_i)$. □

Remark 2.32 *The above lemma holds also when φ is not bijective: this is not noted in commonly found proofs, but is very powerful, since it simplifies the next lemma, and leads to a very short and simple proof of 2.34 and the theorem of Buseman 2.35.*

Lemma 2.33 (reparametrization to arc parameter) [25] *For any curve* $\gamma : [\alpha, \beta] \to M$ *of length* L *such that* ℓ^γ *is continuous, there exists an unique Lipschitz curve* $\xi : [0, L] \to M$ *such that*

$$\gamma(t) = \xi(\ell^\gamma(t)), \quad \ell^\xi(t) = t, \quad t \in [0, L]$$

where ℓ^γ *is the* running length *of* γ*, and* ℓ^ξ *of* ξ *(see eq. (2.8)).*

ξ *is the* reparametrization to arc parameter *of* γ.

Proof. $\gamma(t) = \xi(\ell^\gamma(t))$ uniquely defines ξ: indeed, if $\ell^\gamma(t) = \ell^\gamma(t')$ then $\gamma(t) = \gamma(t')$ by (2.9). Let $\hat{l} = \ell^\gamma(\hat{t})$. If we restrict γ to $[a, \hat{t}]$ and ξ to $[0, \hat{l}]$, we can write $\gamma = \xi \circ \ell^\gamma$. By the previous lemma, $\ell^\gamma(\hat{t}) = \ell^\xi(\hat{l})$, that is $\hat{l} = \ell^\xi(\hat{l})$. □

Lemma 2.34 (existence of local geodesics) *Fix* x*. Let* $\varepsilon > 0$ *s.t.* $D = \{y \mid b^g(x, y) \leq \varepsilon\}$ *is compact. Then for any* y *such that* $b^g(x, y) < \varepsilon$*, there is a geodesic connecting* x *to* y*.*

Proof. Fix $x, \varepsilon > 0$ D, y as above. Let $L = b^g(x, y)$; suppose $y \neq x$ (otherwise $\gamma \equiv x$ is the geodesic).

Let $\gamma_n : [0, 1] \to M$ be a sequence of Lip paths from x to y such that $\operatorname{len} \gamma_n \to L$, and $\operatorname{len} \gamma_n \leq \varepsilon$. By (2.9), we know that all of γ_n is contained in D. By using the above lemma 2.33, we assume without loss of generality that the Lip constants of γ_n are bounded by ε.

Then by the Ascoli-Arzela theorem, we know that there is a Lip $\gamma : [0, 1] \to M$ such that, up to a subsequence, there is uniform convergence of $\gamma_n \to \gamma$; this uniform convergence is w.r.t the distance $d(x, y) = b(x, y) \vee b(y, x)$.

The functional $\gamma \mapsto \operatorname{len}^b \gamma$ is lower semicontinuous w.r.t. uniform convergence, since it is the supremum of the continuous functionals

$$\sum_{i=1}^{n} b\big(\gamma(t_{i-1}), \gamma(t_i)\big)$$

□

Note that if $D \subset M$ is compact for b^g then it is compact for b; the opposite is not true, see 2.39

This lemma immediately implies the Buseman's theorem (see also thm. 4.3.1, [1]):

Theorem 2.35 (Buseman's theorem) *Suppose that* (M, b) *is compact, then* (M, b) *admits geodesics.*

As another consequence:

Proposition 2.36 *Suppose that* (M, b) *is forward-locally compact and path-metric, then* (M, b) *forward-locally admits geodesics.*

[25] cf. Theorem 4.2.1 in [1]: but we do not include here the *metric derivative* issue

§2.viii.1 Hopf-Rinow Theorem

We now restate and prove theorem 1.2

Theorem 2.37 (Hopf-Rinow) *Suppose that (M, b) is path-metric and forward-locally compact; then the following are equivalent*

- (M, b) *is forward complete*
- *forward-bounded and closed sets are compact*

and both imply that (M, b) admits geodesics

Proof. Let $D^+(a,\rho) \doteq \{y \mid b(a,y) \le \rho\}$.

We define the *radius of compactness* $R : M \to \mathbb{R}^+ \cup \{\infty\}$ as

$$R(x) \doteq \sup\{\rho \ge 0 \mid D^+(x,\rho) \text{ is compact }\} \quad .$$

Then, for any $\rho < R(x)$, $D^+(x,\rho)$ is compact.

Either $R \equiv \infty$, or $R < \infty$ and R is 1-Lipschitz w.r.t. d: indeed, for any $x, y \in M$, for any $\rho < R(x)$, we have that $D^+(y, \rho - b(x,y))$ is compact, since

$$D^+(y, \rho - b(x,y)) \subset D^+(x,\rho) \quad .$$

This implies that $R(y) > \rho - b(x,y)$, and then $R(y) \ge R(x) - b(x,y)$; if R is finite, the above entails $d(x,y) \ge |R(x) - R(y)|$, since x, y are arbitrary.

(M, b) is forward-locally compact iff $R(x) > 0$ at all points.

- Suppose that (M, b) is forward complete.

  Let y_n be a forward-bounded sequence; we want to extract a converging subsequence from it; to this end, we will define iteratively a function $n : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$.

  Fix $x_0 \in M$; up to a subsequence, we may assume that $b(x_0, y_n) \to L$.

  Let $\varepsilon_0 = \inf\{R(x_0)/2, L/2\}$: then $D^+(x_0, \varepsilon_0)$ is compact. By (2.20.◇),

  $$\min_{z \in D^+(x_0,\varepsilon_0)} b(z, y_n) = b(x_0, y_n) - \varepsilon_0 \quad . \tag{2.38}$$

  Let $z_{0,n}$ be corresponding minimum points. By 2.20.(iii), $b(x_0, z_{0,n}) = \varepsilon_0$. Since $D^+(x_0, \varepsilon_0)$ is compact, we extract a subsequence $n(0,k)$ so that

  $$z_{0,n(0,k)} \to_k x_1$$

  (and then $b(x_0, x_1) = \varepsilon_0$), and extract a corresponding subsequence $y_{n(0,k)}$ from y_n. Since $x_1 \in D^+(x_0, \varepsilon_0)$, by (2.38), $b(x_1, y_n) \ge b(x_0, y_n) - \varepsilon_0$ and then

  $$\liminf_n b(x_1, y_n) \ge L - \varepsilon_0 \quad .$$

  But, by triangle inequality,

  $$b(x_1, y_n) \le b(x_1, z_n) + b(z_n, y_n) = b(x_1, z_n) + b(x_0, y_n) - \varepsilon_0$$

  remembering that $z_{0,n(0,k)} \to_k x_1$ and $b(x_0, y_n) \to L$,

  $$\limsup_k b(x_1, y_{n(0,k)}) \le L - \varepsilon_0$$

and hence we obtain that

$$\lim_k b(x_1, y_{n(0,k)}) = L - \varepsilon_0 \quad .$$

We iterate the above reasoning to define x_m, ε_m, $n(m,k)$ such that

$$\begin{aligned} s_m &= \sum_{j=0}^{m} \varepsilon_j \\ \varepsilon_m &= \inf \left\{ R(x_m)/2, (L - s_{m-1})/2 \right\} \\ b(x_m, x_{m+1}) &= \varepsilon_m \\ \lim_k b(x_{m+1}, y_{n(m,k)}) &= L - s_m \end{aligned}$$

and $k \mapsto n(m,k)$ is a subsequence of $k \mapsto n(m-1,k)$.

By definition $\varepsilon_m \le (L - s_{m-1})/2$, then $\sum_{m=0}^{\infty} \varepsilon_m = \lim_m s_m \le L$ and $\varepsilon_m \to 0$. By the triangle inequality, for $h > m$,

$$b(x_m, x_h) \le s_{h-1} - s_{m-1}$$

we obtain that x_m is forward-Cauchy. Hence there exists $\overline{x}$ such that $x_m \to \overline{x}$.

We want to prove that $s_m \to L$; otherwise we would have that $\varepsilon_m = R(x_m)/2$ for m large, and then $R(x_m) \to 0$ whereas $x_m \to \overline{x}$ and $R(\overline{x}) > 0$.

Let $\overline{y}_k = y_{n(k,k)}$ be the diagonal sequence. Therefore, for any fixed m, $\overline{y}_k$ is definitely a subsequence of $y_{n(m,k)}$:

$$\lim_k b(x_{m+1}, \overline{y}_k) = L - s_m$$

We write

$$b(\overline{x}, \overline{y}_k) \le b(\overline{x}, x_{m+1}) + b(x_{m+1}, \overline{y}_k)$$

Fix $\varepsilon > 0$; then we can choose m large so that $b(\overline{x}, x_{m+1}) < \varepsilon$ and $L - s_m < \varepsilon$. Now choose h large so that $b(x_{m+1}, \overline{y}_k) < L - s_m + \varepsilon < 2\varepsilon$ for any $k > h$: therefore $b(\overline{x}, \overline{y}_k) < 3\varepsilon$ for $k > h$. This proves that $b(\overline{x}, \overline{y}_k) \to 0$, and then $\overline{y}_k \to \overline{x}$ (by 2.15).

- Suppose that forward-bounded closed sets are compact. If x_n is a forward-Cauchy sequence, then there exists N s.t. $b(x_N, x_m) \le 1$ for $m > N$, that is, $x_m \in D^+(x_N, 1)$ that is compact; then we can extract a converging subsequence, and use lemma 2.16 to obtain the result.
- Existence of geodesics is guaranteed by lemma 2.34.

□

We remark that the proof of the above equivalence cannot simply follow from the proof for metric spaces §1.11 in [7], since that proof uses the property (ii) in sec. §2.vi; neither it does follow from the proof in Finsler Geometry (see section VI of [2]), since the latter uses the exponential map.

Hence we devised a different proof, that combines the idea of the diagonalization of a sequence of subsequences, as in [7], and the idea of repeated application of (2.20.⋄) as in [2], plus some special ingredients such as the *radius of compactness*, and (2.38).

§2.viii.2 More examples

We show some examples to highlight the impact of the hypotheses in the above theorems 2.37 (a.k.a. 1.2) and 2.34.

Example 2.39 *Consider*

$$M = \{x \in \mathbb{R}^2 \mid -1 \le x_1 \le 1, x_1 \ne 0, x_2 = 0\} \cup \{x \in \mathbb{R}^2 \mid -1 \le x_1 \le 1, \ 1/4 \ge x_2 > 0\} \cup \bigcup_n \{x \in \mathbb{R}^2 \mid -1 \le x_1 \le 1, \ x_2 = (x_1 - 1)/n\}$$

and b the Euclidean distance (see fig. 3). Let $A = (-1, 0)$, $B = (1, 0)$. (M, b) is locally compact, but is not path-metric and is not complete; (M, b) does not admit geodesics locally around A.

The disc $\{y \mid b(B, y) \le 1/2\}$ is compact for b; but the discs $\{y \mid b^g(B, y) \le \varepsilon\}$ are never compact for b^g.

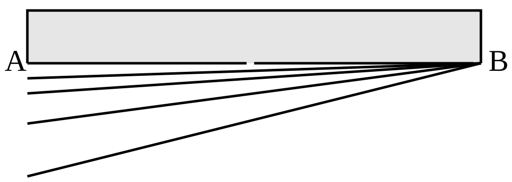


Figure 3: example 2.39

Example 2.40 *Let*

$$M \doteq \bigcup_n \{x \in \mathbb{R}^2 \mid x_1^2 + n^2 x_2^2 = 1, x_2 \ge 0\}$$

and b be the geodesic distance induced on M by the Euclidean distance; loosely speaking, M is the disjoint union of countable segments l_n, with length $\sim 2 + 1/n$, and with common end points.

Then (M, b) is path-metric, is complete, and is bounded, and it locally admits geodesics; but is not locally compact (and then it is not compact), and there is no geodesic curve connecting $(-1, 0)$ to $(1, 0)$.

§2.ix Length structure

We hereby define a *length structure* $\mathrm{len} : \mathcal{C} \to \mathbb{R}^+$, where $\mathcal{C}$ is a family of curves $\gamma : [\alpha, \beta] \to M$ (and α, β may vary); we say that [26]

- $\mathcal{C}$ *connects points*: $\forall x, y \in M$ there is at least one $\gamma \in \mathcal{C}$, $\gamma : [\alpha, \beta] \to M$ such that $\gamma(\alpha) = x, \gamma(\beta) = y$
- $\mathcal{C}$ is *reversible*: if $\gamma \in \mathcal{C}$ and $\hat\gamma(t) \doteq \gamma(-t)$ then $\hat\gamma \in \mathcal{C}$

[26] the following conditions are not independent: some of them imply some others; we do not detail

- len is *independent of reparametrization*: if $\gamma \in \mathcal{C}$ then, $\forall h > 0, k \in \mathbb{R}$, its linear reparametrization $\gamma'(s) = \gamma(hs + k)$ is in $\mathcal{C}$, and $\operatorname{len}\gamma = \operatorname{len}\gamma'$,

- len is *monotonic*: if $\gamma \in \mathcal{C}$ then its restriction $\gamma' = \gamma|_{[c,d]}$ is in $\mathcal{C}$, and $\operatorname{len}\gamma' \leq \operatorname{len}\gamma$

- len is *additive*: if $\gamma', \gamma'' \in \mathcal{C}$, $\gamma' : [\alpha', \beta'] \to M$, $\gamma : [\alpha'', \beta''] \to M$, and $\beta' = \alpha''$, then we may join the two curves and obtain γ: then γ is in $\mathcal{C}$, and

$$\operatorname{len}\gamma = \operatorname{len}\gamma' + \operatorname{len}\gamma''$$

- len is *run-continuous*: that is, the running length $t \mapsto \operatorname{len}(\gamma|_{[\alpha,t]})$ is continuous for all $\gamma \in \mathcal{C}$

- $\operatorname{len}\gamma = 0$ when γ is constant (and constants are in $\mathcal{C}$)

We then define $b(x, y)$ on M to be the infimum of this length $\operatorname{len}\gamma$ in the class of all $\xi \in \mathcal{C}$, $\xi : [0, 1] \to M$ with given extrema $\xi(0) = x, \xi(1) = y$.

Proposition 2.41 *b is an asymmetric semidistance: b satisfies the triangle inequality, since* len *is* independent of reparametrization *and* additive.

$b \geq 0$, and $b(x, x) = 0$ since len *is zero on constant curves.*

b may fail to be an asymmetric distance, since we cannot be sure that $b(x, y) = 0 \Longrightarrow x = y$

We also induce $\operatorname{len}^b \gamma$ from b using the total variation, as in (2.4).

Lemma 2.42 *Suppose* len *and $\mathcal{C}$ satisfy all the hypotheses above. If* len *is lower semi continuous w.r.t. uniform convergence, then $\operatorname{len}\gamma = \operatorname{len}^b \gamma$ for any $\gamma \in \mathcal{C}$*

The proof for the asymmetric case is not different from the symmetric case (cf. 1.6 [7]).

Corollary 2.43 *Suppose we are given an asymmetric space (M, b); let $\mathcal{C}$ be the class of rectifiable curves with continuous run-length, and $\operatorname{len} \doteq \operatorname{len}^b$: this is a length structure satisfying all above hypotheses. len^b induces b^g which induces $\operatorname{len}^{b^g} \gamma$. $\operatorname{len}^b \gamma$ is lower semi continuous since it is the supremum of the continuous functionals*

$$\sum_{i=1}^{n} b\big(\gamma(t_{i-1}), \gamma(t_i)\big)$$

By the lemma

$$\operatorname{len}^b \gamma = \operatorname{len}^{b^g} \gamma$$

This immediately entails the result:

Corollary 2.44 *The operation $b \mapsto b^g$ is idempotent, that is, b^g is path-metric, idem est $b^g = (b^g)^g$.*

3 Asymmetric metric manifold

Suppose now that M is a differential manifold [27], with an atlas $\mathcal{A}$ and a topology τ^M.

Definition 3.1 ($\mathcal{A}$-Lip) *We will say that a curve $\xi : [\alpha, \beta] \to M$ is locally Lipschitz w.r.t. $\mathcal{A}$, if for any local chart $\varphi : U \to \mathbb{R}^n$ in the atlas $\mathcal{A}$, $\varphi \circ \xi$ is locally Lipschitz for $t \in \xi^{-1}(U)$.*

Hypotheses 3.2 *Suppose that we are given a positive function $F : TM \to \mathbb{R}^+$; suppose*

- *F is lower-semi-continuous,*
- *$v \mapsto F(x, v)$ is positively 1-homogeneous, that is,*

$$F(x, \lambda v) = F(x, v)\lambda \quad \forall \lambda \geq 0$$

- *$F(x, v) = 0$ iff $v = 0$.*

We define the *length* $\operatorname{len}^L \gamma$ of a locally Lipschitz curve $\xi : [0, 1] \to M$ as

$$\operatorname{len}^L \gamma = \int_0^1 F(\xi(s), \dot{\xi}(s))\, ds \tag{3.3}$$

Remark 3.4 (The regular Finsler case) *If we would suppose that*

- *F is continuous, and C^∞ on the slit tangent bundle $TM \setminus 0$*
- *$v \mapsto F^2(x, v)$ is strongly convex for $v \neq 0$, (cf. 1.6)*

then this section would be exactly what is found in section 6.2 [2]; we will instead use less regular assumptions.

As in §2.ix, we then define the asymmetric distance $b(x, y)$ on M to be the infimum of this length $\operatorname{len}^L \gamma$ in the class of all locally Lipschitz ξ with given extrema $\xi(0) = x, \xi(1) = y$.

In the above hypotheses 3.2, b is an asymmetric distance; indeed, b is a semidistance by 2.41, and the relation $b(x, y) \implies x = y$ can be proved by the methods in the following Lemma.

This metric is also called a *Finsler metric*, and is naturally associated to Hamilton-Jacobi equations; see [12] and references therein.

There may be a confusion here, since we have two topologies at hand: the topology τ^M of the differential manifold M, and the topology τ^b induced by b; but we may use this lemma, that simply restates the lemma 6.2.1 in [2] (with less regularity assumptions on F)

Lemma 3.5 *Assume 3.2 and let F be locally bounded from above. Then for any $z \in M$ there exists U a compact neighborhood of z, associated with local coordinates $\varphi : U \to W \subset \mathbb{R}^n$, such that the push forward of b in local coordinates is bounded from above and below by the Euclidean norm: that is, $\exists c' > c > 0$*

$$c'|\hat{x} - \hat{y}| \geq b(x, y) \geq c|\hat{x} - \hat{y}| \tag{3.5.$\star$}$$

where $\hat{y} = \varphi(y)$ and $\hat{x} = \varphi(x)$.

---

[27] more precisely, let M be a finite dimensional connected smooth differential manifold, without boundary

Proof. Let $z \in M$ and Z be a neighborhood of z, with $\overline{Z}$ compact w.r.t. $\tau^{\mathcal{A}}$, associated to local coordinates $\varphi : Z \to W \subset \mathbb{R}^n$, with W convex bounded.

Let $\hat{F}$ be the push-forward of F; for any $x, y \in Z$, we may join $\hat{y} = \varphi(y)$ to $\hat{x} = \varphi(x)$ with a segment, and then $b(x, y) \le c'|\varphi(x) - \varphi(y)|$ where c' is the upper bound of $\hat{F}(w, v)$ for $w \in W$ and $|v| \le 1$.

In the above setting, let $U \subset\subset Z$, let c be the minimum of $\hat{F}(w, v)$ for $w \in \varphi(Z)$ and $|v| = 1$ (this minimum exists and is positive, since F is l.s.c); and let ε be the minimum of $|\varphi(x) - \varphi(z)|$ for $x \in \partial U, z \in \partial Z$; let $\hat{\xi} \doteq \varphi \circ \xi$ when $\xi \in Z$.

Then, for any ξ connecting x to y,

$$\operatorname{len}^L \xi = \int_0^1 F(\xi(s), \dot{\xi}(s))\, ds \ge \int_0^t c|\dot{\hat{\xi}}(s)|\, ds \ge c(\varepsilon \wedge |\hat{x} - \hat{y}|)$$

where $t = 1$ if ξ never exits from Z, otherwise it is the first time s such that $\xi(s) \in \partial Z$: then $b(x, y) \ge c(\varepsilon \wedge |\hat{x} - \hat{y}|)$, and we choose a smaller U. □

Corollary 3.6 *Under the same hypotheses, $\tau^b = \tau^M$, i.e. b generates the same topology that the atlas of M induces. Moreover both coincide with the topology generated by the forward balls; or respectively, by the backward balls.* (28)

This topology is locally compact: then

- $x_n \to x$ *iff* $b(x_n, x) \to 0$ *iff* $b(x, x_n) \to 0$ (29)*; and*
- (M, b) *locally admits geodesics (by 2.36).*

By (3.5.$\star$), a curve $\xi : [0, 1] \to M$ is locally Lipschitz w.r.t $\mathcal{A}$, as defined in 3.1, iff it is locally Lipschitz w.r.t b, as defined in §2.ii.

Proposition 3.7 *i). (M, b) is path-metric, that is, $b = b^g$*

ii). suppose 3.2, and $v \mapsto F(x, v)$ is convex; then for any Lipschitz γ, $\operatorname{len}^L \gamma$ coincides with the $\operatorname{len}^b \gamma$ defined in (2.4).

Proof. i) is a consequence of 2.20.

ii). We proceed as in lemma 2.42: we set $\mathcal{C}$ to be the class of Lipschitz curves, and $\operatorname{len} \doteq \operatorname{len}^L$. Fix γ, let $L \doteq \operatorname{len}^L \gamma$. If L is small, then the curve is contained in local coordinates.

By (3.5.$\star$), the uniform convergence happens also in local coordinates. By well-known theory (cf. thm. 2.3.3 in [3]), $\operatorname{len}^L \gamma$ is l.s.c, and we conclude as in the Lemma 3.5.

If we don't suppose that $\gamma(t) \in U\ \forall t \in [0, 1]$: we fix a very dense partition $\hat{T}$, so that we can associate to any point $t_i \in \hat{T}$ a local chart $\varphi_i : U_i \to \mathbb{R}^n$, so that $\bigcup_i U_i$ covers the image of γ, and more precisely $\gamma([t_i, t_{i+1}]) \subset U_i$. We repeat the above argument in any U_i. □

Lemma 3.8 (Reparametrization) *For any $\gamma : [\alpha, \beta] \to M$ Lipschitz, define $l : [\alpha, \beta] \to [0, L]$*

$$l(t) = \int_a^t F(\gamma(s), \dot{\gamma}(s))\, ds$$

(where $L = \operatorname{len} \gamma$).

Then there is a unique ξ such that $\gamma = \xi \circ l$, and

$$F(\xi(s), \dot{\xi}(s)) = 1 \qquad \tilde{\forall} s \in [0, L] \tag{3.8.$\star$}$$

(28)cf. 2.17 , or sec. 6.2C in [2] for the regular case
(29)cf. 6.2.5 in [2] for the regular case

Proof. We apply 2.33, and just note that $\ell^\xi(t) = t$ means that $\int_0^t F(\xi, \dot\xi) = t$ $\square$

We can then state this version of the *Hopf-Rinow* theorem

Theorem 3.9 *Assume that the function* $F : TM \to \mathbb{R}^+$

- *is locally bounded from above*
- $v \mapsto F(x, v)$ *is positively 1-homogeneous, and convex,*

and 3.2 holds.

Then the asymmetric metric space (M, b) *is path-metric, locally compact, and locally admits geodesics (that we can always reparametrize so that they satisfy (3.8.⋆)).*

- (M, b) *is forward-complete (resp. backward-complete) if and only if*
- *forward (resp. backward) bounded closed sets are compact.*

In both cases, for any $x, y \in M$*, there is a locally Lipschitz curve* ξ *connecting them that minimizes (3.3), satisfying (3.8.⋆).*

Further applications of these ideas are in [11]; in particular, in the appendix of [11], we consider a Lagrangian $L : TM \to \mathbb{R}^+$ (that is defined as $L(x, v) = F(x, v)^2$) and its Legendre-Fenchel dual Hamiltonian $H : T^*M \to \mathbb{R}^+$; we show that, if $H \in C^{1,1}$ and $p \mapsto H(x, p)$ is positively 2-homogeneous and strictly convex, we can define an exponential map and add it to the statement of the Hopf-Rinow theorem, to obtain a statement exactly as in ch. VI of [2].

Acknowledgements

The author thanks Prof. S. Mitter, who has reviewed and corrected various versions of this paper, and suggested many improvements.

Contents

1 Introduction **1**
- §1.i Previous work . . . 1
- §1.ii Origin of the problem . . . 2
- §1.iii Notation . . . 3

2 Asymmetric metric spaces **3**
- §2.i Asymmetric metric space . . . 4
- §2.ii Lipschitz maps . . . 4
- §2.iii Length . . . 4
- §2.iv Definitions . . . 5
- §2.v Comparison with quasi metric spaces . . . 7
  - §2.v.1 Topology . . . 7
  - §2.v.2 Cauchy sequences, and completeness . . . 7
- §2.vi Comparison with symmetric case . . . 8
- §2.vii Properties . . . 10
  - §2.vii.1 Mid-point properties . . . 10

§2.vii.2 Properties of path metrics 11
§2.vii.3 Examples 12
§2.viii Geodesics 15
§2.viii.1 Hopf-Rinow Theorem 17
§2.viii.2 More examples 19
§2.ix Length structure 19

3 Asymmetric metric manifold **21**

References

[1] L. Ambrosio and P. Tilli. *Selected topics in "analysis in metric spaces"*. appunti. Pisa, 2000.

[2] D. Bao, S. S. Chern, and Z. Shen. An introduction to Riemann-Finsler geometry. (October 1, 1999 version).

[3] G. Buttazzo. *Semicontinuity, relaxation and integral representation in the calculus of variation*. Number 207 in scientific & technical. Longman, 1989.

[4] A. Fathi. Weak KAM theorem in Lagrangian dynamics. (Preliminary version, 2 Feb 2001).

[5] W. H. Fleming and H. M. Soner. *Controlled Markov processes and viscosity solutions*. Springer Verlag, Berlin, 1993.

[6] P. Fletcher and W.F.Lindgren. *Quasi-uniform spaces*, volume 77 of *Lecture notes in pure and applied mathematics*. Marcel Dekker, 1982.

[7] M. Gromov. *Metric Structures for Riemannian and Non-Riemannian Spaces*. Birkhäuser, 1999.

[8] M.K.Vamanamurthy I.L.Reilly, P.V.Subrahmanyam. Cauchy sequences in quasi-pseudo-metric spaces. *Monat.Math.*, 93:127–140, 1982.

[9] J. C. Kelly. Bitopological spaces. *Proc. London Math. Soc.*, 13(3):71–89, 1963.

[10] H. P. Künzi. complete quasi-pseudo-metric spaces. *Acta Math. Hung.*, 59(1-2):121–146, 1992.

[11] A. C. G. Mennucci. Regularity and variationality of solutions to Hamilton-Jacobi equations. part ii: variationality, existence, uniqueness. in preparation, 2004.

[12] Antonio Siconolfi. Metric character of Hamilton-Jacobi equations. *transactions of the american mathematical society*, 355(5):1987–2009. S 0002-9947(03)03237-9, electronically published on January 8, 2003.